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Ideal gas approximation for a two-dimensional rarefied gas under Kawasaki dynamics

A. Gaudillière[a], F. den Hollander[b,c], F.R. Nardi[a,d,c], E. Olivieri[e], E. Scoppola[a,*]

[a] Dipartimento di Matematica, Università di Roma Tre, Largo S. Leonardo Murialdo 1, 00146 Rome, Italy
[b] Mathematical Institute, Leiden University, P.O. Box 9512, 2300 RA Leiden, The Netherlands
[c] EURANDOM, P.O. Box 513, 5600 MB Eindhoven, The Netherlands
[d] Department of Mathematics and Computer Science, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands
[e] Dipartimento di Matematica, Università di Roma Tor Vergata, Via della Ricerca Scientifica, 00133 Rome, Italy

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Abstract

In this paper we consider a two-dimensional lattice gas under Kawasaki dynamics, i.e., particles hop around randomly subject to hard-core repulsion and nearest-neighbor attraction. We show that, at fixed temperature and in the limit as the particle density tends to zero, such a gas evolves in a way that is close to an ideal gas, where particles have no interaction. In particular, we prove three theorems showing that particle trajectories are non-superdiffusive and have a diffusive spread-out property. We also consider the situation where the temperature and the particle density tend to zero simultaneously and focus on three regimes corresponding to the stable, the metastable and the unstable gas, respectively.

Our results are formulated in the more general context of systems of "Quasi-Random Walks", of which we show that the low-density lattice gas under Kawasaki dynamics is an example. We are able to deal with a large class of initial conditions having no anomalous concentration of particles and with time horizons that are much larger than the typical particle collision time. The results will be used in two forthcoming papers, dealing with metastable behavior of the two-dimensional lattice gas in large volumes at low temperature and low density.
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* Corresponding author. Tel.: +39 0657338217; fax: +39 06 57338080.
E-mail address: scoppola@mat.uniroma3.it (E. Scoppola).

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1. Introduction

1.1. Ideal gas approximation

In this paper we consider a two-dimensional lattice gas at low density evolving under *Kawasaki dynamics*: particles hop around randomly subject to hard-core repulsion and nearest-neighbor attraction. More precisely, we consider a square box $\Lambda \subset \mathbb{Z}^2$ with periodic boundary conditions, and a Markov process $(\eta_t)_{t\geq 0}$ on $\{0,1\}^\Lambda$ given by the standard Metropolis algorithm

$$\eta \to \eta' \quad \text{with rate } \exp[-\beta\{0 \vee [H(\eta') - H(\eta)]\}] \tag{1.1}$$

with Hamiltonian

$$\mathrm{H}(\eta) := -U \sum_{\{x,y\}\in\Lambda^*} \eta(x)\eta(y), \tag{1.2}$$

where $\beta \geq 0$ is the inverse temperature, $-U \leq 0$ is the binding energy, Λ^* is the set of unordered pairs of nearest-neighbor sites in Λ, and η' is any configuration obtainable from η via an exchange of occupation numbers between a pair of sites in Λ^*.

Our goal is to prove an *ideal gas approximation*, i.e., we want to show that the dynamics is well approximated by a process of *Independent Random Walks* (IRWs). Indeed, if the lattice gas is sufficiently rarefied, then each particle spends most of its time moving like a random walk. When two particles are occupying nearest-neighbor sites, the binding energy inhibits their random walk motion, and these pauses are long when the temperature is low. However, if the time intervals in which a particle is interacting with the other particles are short compared to the time intervals in which it is free, then we may hope to represent the interaction as a small perturbation of a free random walk motion. We prove that this is indeed the case in the low-density limit $\rho \downarrow 0$, for any $U \geq 0$ and any $\beta \geq 0$. Note that the case $U = 0$, corresponding to the simple exclusion process, is included.

More difficult is the situation when $\beta \to \infty$ and $\rho \downarrow 0$ *simultaneously*, linked as $\rho := \mathrm{e}^{-\beta\Delta}$ with $\Delta > 0$ an activity parameter. The reason is that low temperature corresponds to strong interaction, so that the ideal gas approximation is far from trivial. This is also the more interesting situation from a physical point of view. Indeed, it is easy to see that $\mathrm{e}^{-2U\beta}$ is the density of the saturated gas at the condensation point. For densities smaller than this, namely, $\Delta \in (2U, \infty)$, we have a *stable gas* so rarefied that it behaves like an ideal gas *up to very large times*. If we increase the density further, picking $\Delta \in (U, 2U)$, *avoiding* however the appearance of droplets of the liquid phase, then we get a *metastable gas*. This regime, which is the most interesting and which motivated the present paper, will be addressed in Gaudillière, den Hollander, Nardi, Olivieri and Scoppola [8,9], two forthcoming papers that rely on the results presented below. In this regime we still have a rarefied gas, and we will prove that it behaves like an ideal gas *up to relatively large times*. If we increase the density still further, picking $\Delta \in (0, U)$, then we get an *unstable gas*, which behaves like a rarefied gas only *up to short times*. The heuristics of these three regimes will be discussed in Section 1.2.

The main focus of the present paper is to address the more interesting and challenging regime where $\beta \to \infty$ with particle density $\rho = \mathrm{e}^{-\Delta\beta}$ before the formation of large clusters.

At exponentially small density we have a non-trivial dynamics in an exponentially large volume only. Consequently, we will assume that our Markov process $(\eta_t)_{t\geq 0}$ takes values in $\{0,1\}^{\Lambda_\beta}$ with $\Lambda_\beta \subset \mathbb{Z}^2$ a square box with periodic boundary conditions and volume

$$|\Lambda_\beta| = \mathrm{e}^{\Theta\beta} \quad \text{for some } \Theta > \Delta. \tag{1.3}$$

The ideal gas is represented by a process of IRWs. We need a process of *Quasi-Random Walks* (QRWs) to describe the "ideal gas approximation". By a process of QRWs we denote a process of N labelled particles that can be coupled to a process of N IRWs in such a way that the two processes follow the same paths outside rare time intervals, called *pause intervals*, in which the paths of the QRW-process remain confined to small regions. We will show that the low-density Kawasaki dynamics with labelled particles is a QRW-process. Moreover, we will generalize to QRWs the following three well-known properties valid for a system of N continuous-time IRW trajectories observed over a time T,

$$\zeta_i : t \in [0,T] \mapsto \zeta_i(t), \quad i \in \{1,\ldots,N\}, T \geq 2. \tag{1.4}$$

For proofs of these properties, see e.g. Jain and Pruitt [12] and Révèsz [15].

THREE PROPERTIES OF IRW. Uniformly in N and T, the following properties hold:

(i) *Non-superdiffusivity:*

$$\forall i \in \{1,\ldots,N\}, \forall \delta > 0: \\ \lim_{\beta\to\infty} \frac{1}{\beta} \ln P\left(\exists t \in [0,T] : \|\zeta_i(t) - \zeta_i(0)\|_2 > \sqrt{T}\mathrm{e}^{\delta\beta}\right) = -\infty. \tag{1.5}$$

(ii) *Spread-out property, upper bound:*

$$\forall I \subset \{1,\ldots,N\}, \forall (z_i)_{i\in I} \in (\mathbb{Z}^2)^I : \quad P\left(\forall i \in I : \zeta_i(T) = z_i\right) \leq \left(\frac{\mathrm{cst}}{T}\right)^{|I|}. \tag{1.6}$$

(iii) *Spread-out property, lower bound:*

$$\forall I \subset \{1,\ldots,N\}, \forall (z_i)_{i\in I} \in (\mathbb{Z}^2)^I:$$

$$\begin{cases} \left(\forall i \in I : 0 \leq \|z_i - \zeta_i(0)\|_2 \leq \sqrt{T}\right) \Rightarrow P\left(\forall i \in I : \zeta_i(T) = z_i\right) \geq \left(\dfrac{\mathrm{cst}}{T}\right)^{|I|}, \\ \left(\forall i \in I : 0 < \|z_i - \zeta_i(0)\|_2 \leq \sqrt{T}\right) \\ \quad \Rightarrow P\left(\forall i \in I : \lfloor \tau_{z_i}(\zeta_i) \rfloor = T\right) \geq \left(\dfrac{\mathrm{cst}}{T\ln^2 T}\right)^{|I|}, \end{cases} \tag{1.7}$$

with

$$\tau_{z_i}(\zeta_i) := \inf\{t > 0 \colon \zeta_i(t) = z_i\}. \tag{1.8}$$

(Throughout the paper, 'cst' will denote a positive constant independent of the model parameters, the value of which may change from line to line.)

Remark. The statements in (1.4)–(1.8) are partially redundant because the independence of the trajectories of IRWs trivially implies factorization. However, in our generalization of these properties to QRWs, whose trajectories are not independent, the factorization is an essential ingredient of the statements.

While the non-superdiffusivity will be proven for all particles, the spread-out property will be proven for "non-sleeping" particles only, i.e., those particles for which the pause intervals are not too long. Note that on the time scale $1/\rho = e^{\Delta\beta}$ we may expect the gas to behave like a gas of IRWs, because $1/\sqrt{\rho}$ is the average distance between particles. We will, however, show that the *ideal gas approximation extends well beyond this time scale*.

The main difficulty in analyzing the metastable behavior for Kawasaki dynamics at low density and low temperature is the description of the *interaction between the droplets and the gas*. As part of the nucleation process, droplets grow and shrink by exchanging particles with the gas that surrounds them, as is typical for a *conservative dynamics*. It was precisely for the need to control this droplet–gas interaction why the notion of QRWs was introduced in den Hollander, Olivieri and Scoppola [11].

Two models played an important role in earlier work:

(1) The *local model*: In den Hollander, Olivieri and Scoppola [11], a model of Kawasaki dynamics on a finite box Λ_0 with open boundary was considered. Particles move according to Kawasaki dynamics *inside* Λ_0, and are created and annihilated at the *boundary* $\partial\Lambda_0$, at rates $e^{-\Delta\beta}$ and 1, respectively. The "open boundary" replaces a *gas reservoir* surrounding Λ_0, with density $\rho = e^{-\Delta\beta}$. For this local model, the metastable behavior for $\beta \to \infty$ (and Λ_0 fixed) was described in full detail in Gaudillière, Olivieri and Scoppola [10] and in Bovier, den Hollander and Nardi [2]. In this model there is no effect of the droplets in Λ_0 on the gas outside Λ_0.
(2) The *local interaction model*: In den Hollander, Olivieri and Scoppola [11], an extension of the local model was considered in which the gas reservoir consists of IRWs. The total number of particles was fixed and it was shown that, for $\beta \to \infty$, this model is well approximated by the local model as far as its metastable behavior is concerned. Note that in the local interaction model even though the gas outside Λ_0 is an "ideal gas", it influences the Kawasaki gas inside Λ_0, and vice versa. This mutual influence was described by means of QRWs: the gas particles perform random walks, interspersed with pause intervals during which they interact with the other particles, and interspersed with jumps corresponding to the difference between the positions of the particle at the end and at the beginning of a pause interval. Due to the fact that Λ_0 is finite, the jumps are small w.r.t. the displacement of the random walks on time scales that are exponentially large in β. Moreover, the number of pause intervals is controlled by the rare returns of the random walk to Λ_0. These two ingredients – few pause intervals and small jumps – were sufficient to control the dynamics.

As we will show, the QRW-approximation continues to hold for Kawasaki dynamics in an *exponentially large box*. As long as the clusters are small, we may expect the jumps in the QRWs to be small: at most of the order of the size of the clusters. The crucial obstacle in approximating the gas particles by QRWs is the fact that the interaction acts everywhere. Therefore we need to replace the control on the rare returns of a random walk to a fixed finite box by a control on the number of particle–particle and particle–cluster collisions. This will be achieved with the help of non-collision estimates developed in Gaudillière [7], which is our main tool in the present paper.

Related literature. Our approach is different from that followed by Kipnis and Varadhan [13] to analyze the trajectory of a tagged particle in reversible interacting particle systems. Using martingale arguments, they proved that in *infinite* volume at *any density* and *starting from equilibrium*, if $X(t)$ denotes the position at time t of the tagged particle, then the process $(\sqrt{\epsilon}X(t/\epsilon))_{t\geq 0}$ converges to a rescaled Brownian motion $(D_{\text{self}}B(t))_{t\geq 0}$ in the limit as $\epsilon \downarrow 0$.

This is an invariance principle, where "cumulative chaos" leads to Gaussian behavior. Our approach is complementary, because we use the *low-density* limit to view Kawasaki dynamics as a small perturbation of an IRW-process and prove large deviation and local occupation bounds, and this perturbation also works *away from equilibrium*. This will lead us to introduce a time scale beyond which our results no longer apply. This time scale will be much longer than the typical particle inter-collision time (collision time), namely, it will be of the order of the *minimum* of $1/\rho^2$, the *square* of the typical particle collision time, and the time of the *first anomalous concentration* of particles.

We mention two other papers where a coupling between the one-dimensional simple exclusion process (for which $D_{\text{self}} = 0$; see [1]) and an IRW-process was constructed. In Ferrari, Galves and Presutti [5] and in De Masi, Ianiro, Pellegrinotti and Presutti [3], Chapter 3, a hierarchy on the particles is introduced, which leads to a coupling with strong symmetry properties. This hierarchy is used to prove non-superdiffusivity. Unfortunately, in higher dimensions and as soon as $U > 0$, these symmetry properties are lost.

Higher dimension. Our analysis will be restricted to the two-dimensional lattice and our proofs will be based on a lower bound for the two-dimensional non-collision probability of random walks in the presence of obstacles. Since this lower bound works as well in dimension three or higher, the results in the present paper carry over to higher dimension. A legitimate question is the following. Would it be possible to obtain a better ideal gas approximation in higher dimension based on a better estimate for the non-collision probability? We believe that the answer is no. Even though the random walk is transient in dimension three and higher, the number of collisions undergone by a particle on time scale $1/\rho$ is still of order 1. For a QRW with such a frequency of pause intervals, properties like the non-superdiffusivity property cannot hold beyond the time horizon $1/\rho^2$, the square of the typical particle collision time.

1.2. Three regimes

We have to compare the average particle collision time $1/\rho$ with the pauses caused by the binding energy. We distinguish three cases.

(1) If $\Delta > 2U$ (stable gas), then the pauses are typically much shorter than $e^{\Delta\beta}$. On this time scale the gas will essentially behave like a gas of IRWs, i.e., the probabilities at time T to find a given set of particles in a given set of sites are similar to those for IRWs. We will be able to prove that this is true up to time scale $e^{2\Delta\beta}$, provided the gas starts from equilibrium, and up to time scale $e^{\frac{3}{2}\Delta\beta} \wedge e^{(2\Delta-2U)\beta}$ for a much wider class of starting configurations, namely, those that exclude anomalous concentrations of particles.
(2) If $\Delta < U$ (unstable gas), then the pauses are typically much longer than $e^{\Delta\beta}$. For this case we will only have very weak results, limited to time scale $e^{\Delta\beta}$.
(3) If $U < \Delta < 2U$ (metastable gas), then typically some pauses are much shorter than $e^{\Delta\beta}$ while others are much longer. For $D \in (U, \Delta)$, as close to U as we want, we will say that a particle "falls asleep" when it makes a pause longer than $e^{D\beta}$. We will say that non-sleeping particles are *active* and we will be able to obtain results for active particles up to time scale $e^{2\Delta\beta}$, provided the system starts from a "metastable equilibrium" and Θ is not too large.

In what follows we will deal with these three regimes simultaneously. To that end, we introduce a constant $D \in (0, \Delta)$, as close to 0, U, $2U$ as we want in the unstable, metastable and stable regimes, respectively. The different regimes will be discussed separately in Section 6 only.

1.3. Outline

In Section 2 we build the Kawasaki dynamics as a dynamics of labelled particles $\hat{\eta}$ coupled to an IRW-process ζ and we introduce the main notions necessary to build up the definition of a QRW-process. Our main results are stated in Section 3. In Section 4 the non-superdiffusivity and the spread-out property are proved for QRWs. In Section 5 we prove that the low-density limit of Kawasaki dynamics with labelled particles is a QRW-process and prove some stronger estimates for the lower bound of the spread-out property as well. In Section 6 these results are applied to the three different regimes of Section 1.2. Some of the proofs in this paper do not rely on the notion of QRW-process, and therefore are placed in Appendices A and B.

2. Building QRWs

In Section 2.1 we introduce some basic notation. In Section 2.2 we construct the Kawasaki dynamics in a way that will be needed for the proofs of our main results formulated in Section 3. In Section 2.3 we introduce free, active and sleeping particles. In Section 2.4 we introduce the notion of Quasi-Random Walk on which most of the present paper is built.

2.1. Notation

1. Apart from the parameters that define the dynamics $(U, \Delta, \Theta, \beta)$, we need three further parameters: $D \in (0, \Delta)$ (see Section 1.2), $\alpha > 0$ small, and $\beta \mapsto \lambda(\beta)$, a slowly increasing unbounded function that satisfies

$$\lambda(\beta) \ln \lambda(\beta) = o(\ln \beta), \tag{2.1}$$

e.g. $\lambda(\beta) = \sqrt{\ln \beta}$. Given $\alpha > 0$, we define a *reference time* almost of order $\mathrm{e}^{\Delta\beta}$

$$T_\alpha := \mathrm{e}^{(\Delta-\alpha)\beta}, \tag{2.2}$$

and we assume that α is small enough so that $T_\alpha > \mathrm{e}^{D\beta}$.

2. We denote by N the total number of particles in Λ_β:

$$N := \rho|\Lambda_\beta| = \mathrm{e}^{(\Theta-\Delta)\beta}. \tag{2.3}$$

We call $\mathcal{X}_N$ the subset of $\{0, 1\}^{\Lambda_\beta}$ in which $(\eta_t)_{t\geq 0}$ evolves:

$$\mathcal{X}_N := \left\{ \eta \in \{0, 1\}^{\Lambda_\beta} \colon \sum_{x\in\Lambda_\beta} \eta(x) = N \right\}. \tag{2.4}$$

We will frequently identify a configuration $\eta \in \mathcal{X}_N$ with its support $\mathrm{supp}(\eta) = \{x \in \mathbb{Z}^2 \colon \eta(x) = 1\}$.

3. For $\Lambda \subset \Lambda_\beta$, we write $\Lambda \sqsubset \Lambda_\beta$ if Λ is a square box, i.e., there are $a, b, c \in \mathbb{R}$ such that

$$\Lambda = ([a, a + c] \times [b, b + c]) \cap \Lambda_\beta. \tag{2.5}$$

For $\Lambda \subset \Lambda_\beta$ and $\eta \in \{0, 1\}^{\Lambda_\beta}$, we denote by $\eta|_\Lambda$ the restriction of η to Λ, and put

$$|\eta|_\Lambda| := \sum_{x\in\Lambda} \eta(x). \tag{2.6}$$

We denote by $\mathcal{T}_{\alpha,\lambda}$ the *first time of anomalous concentration*:

$$\mathcal{T}_{\alpha,\lambda} := \inf\left\{t \ge 0\colon |\eta_t|_\Lambda| \ge \frac{\lambda(\beta)}{4} \text{ for some } \Lambda \sqsubset \Lambda_\beta \text{ with } |\Lambda| \le \mathrm{e}^{(\Delta-\frac{\alpha}{4})\beta}\right\}. \tag{2.7}$$

4. For $p \ge 1$, the p-norm on $\mathbb{R}^2$ is

$$\|\cdot\|_p\colon (x,y) \in \mathbb{R}^2 \mapsto \begin{cases} \left(|x|^p + |y|^p\right)^{1/p} & \text{if } p < \infty, \\ |x| \vee |y| & \text{if } p = \infty. \end{cases} \tag{2.8}$$

We denote by $B_p(z,r)$, $z \in \mathbb{R}^2$, $r > 0$, the open ball with center z and radius r in the p-norm. The closure of $A \subset \mathbb{R}^2$ is denoted by $\overline{A}$.

5. For $\eta \in \mathcal{X}$, we denote by η^{cl} the *clusterized part of* η:

$$\eta^{cl} := \left\{z \in \eta\colon \|z - z'\|_1 = 1 \text{ for some } z' \in \eta\right\}. \tag{2.9}$$

We call *clusters of* η the connected components of the graph drawn on η^{cl} obtained by connecting nearest-neighbor sites. For $A \subset \mathbb{Z}^2$, we denote by ∂A its external border, i.e.,

$$\partial A := \left\{z \in \mathbb{Z}^2 \setminus A\colon \|z - z'\|_1 = 1 \text{ for some } z' \in A\right\}. \tag{2.10}$$

For $r > 0$, we put

$$[A]_r := \bigcup_{z \in A} \overline{B_\infty(z,r)} \cap \mathbb{Z}^2. \tag{2.11}$$

We say that A is a rectangle on $\mathbb{Z}^2$ if there are $a, b, c, d \in \mathbb{R}$ such that

$$A = [a,b] \times [c,d] \cap \mathbb{Z}^2. \tag{2.12}$$

We write $\mathrm{RC}(A)$, called the *circumscribed rectangle* of A, to denote the intersection of all the rectangles on $\mathbb{Z}^2$ containing A.

6. The hitting time of A for a generic random process ξ_0 is denoted by

$$\tau_A(\xi_0) := \inf\{t \ge 0\colon \xi_0(t) \in A\}. \tag{2.13}$$

7. A function $\beta \mapsto f(\beta)$ is called superexponentially small (SES) if

$$\lim_{\beta\to\infty} \frac{1}{\beta} \ln f(\beta) = -\infty. \tag{2.14}$$

If $(A_j(\beta))_{j\in J}$ is a family of events, we say that "A_j occurs with probability $1-$ SES uniformly in j" when there is an SES-function f independent of j such that

$$P(A_j(\beta)^c) = 1 - P(A_j(\beta)) \le f(\beta) \quad \forall j \in J,\ \beta > 0. \tag{2.15}$$

For example, by Brownian approximation and scaling, for ζ_0 a simple random walk in continuous time and $\delta > 0$ we have

$$P\left(\exists t \in [0, m+1]\colon \|\zeta_0(t) - \zeta_0(0)\|_2 > \mathrm{e}^{\delta\beta}\sqrt{m}\right) \le \text{ SES} \quad \text{uniformly in } m \in \mathbb{N}. \tag{2.16}$$

Note that, in general, the dependence on β of $P(A_j(\beta))$ will be deeper than in this simple example: for the process $(\eta_t)_{t\ge0}$, β is a parameter of the dynamics itself.

2.2. Kawasaki dynamics

Kawasaki dynamics is naturally defined as a "dynamics of configurations", in the sense that it describes the evolution of a set of occupied sites rather than of individual particles occupying these sites. We will construct a process $\hat{\eta} = (\hat{\eta}_1, \ldots, \hat{\eta}_N)$ with state space

$$\hat{\mathcal{X}}_N := \left\{ (z_1, \ldots, z_N) \in \Lambda_\beta^N \colon z_i \neq z_j \; \forall i, j \in \{1, \ldots, N\}, i \neq j \right\} \tag{2.17}$$

that describes the trajectories $\hat{\eta}_i \colon t \mapsto \hat{\eta}_i(t)$ of N labelled particles such that the Kawasaki dynamics is defined by setting

$$(\eta_t)_{t \geq 0} := (\mathcal{U}(\hat{\eta}(t)))_{t \geq 0} \tag{2.18}$$

with $\mathcal{U}$ the natural *unlabelling application* that sends $\hat{\mathcal{X}}_N$ onto $\mathcal{X}_N$. We will couple $\hat{\eta}$ with an IRW-process $\zeta = (\zeta_1, \ldots, \zeta_N)$ on Λ_β^N by starting from ζ and building $\hat{\eta}$ out of ζ via random labels.

Given N Poisson processes $\theta_1, \ldots, \theta_N$ of intensity 1 and N families

$$(e_{1,k})_{k \in \mathbb{N}}, (e_{2,k})_{k \in \mathbb{N}}, \ldots, (e_{N,k})_{k \in \mathbb{N}} \tag{2.19}$$

of independent unit random vectors equally distributed in the four directions (north, south, east, west), all mutually independent, we define a process $\zeta = (\zeta_1, \ldots, \zeta_N)$ of N IRWs starting from $z = (z_1, \ldots, z_N) \in \Lambda_\beta^N$ by putting

$$\zeta_i(t) := z_i + \sum_{k=1}^{\theta_i(t)} e_{i,k}, \quad i \in \{1, \ldots, N\}, \; t \geq 0. \tag{2.20}$$

Suppose that $\zeta(0) = z \in \hat{\mathcal{X}}_N$ (recall (2.17)). To build a Kawasaki dynamics with labelled particles $\hat{\eta} = (\hat{\eta}_1, \ldots, \hat{\eta}_N)$ starting from z, we introduce N families

$$(U_{1,k})_{k \in \mathbb{N}}, (U_{2,k})_{k \in \mathbb{N}}, \ldots, (U_{N,k})_{k \in \mathbb{N}} \tag{2.21}$$

of independent marks, uniformly distributed in $[0, 1]$, mutually independent and independent of the families in (2.19), and apply the following three-step updating rule each time the process ζ changes position, i.e., at each t with $\zeta(t_-) \neq \zeta(t)$:

1. Define a first candidate $\hat{\eta}'$ for the new configuration:
$$\hat{\eta}' := \hat{\eta}(t_-) + \zeta(t) - \zeta(t_-) \in \Lambda_\beta^N. \tag{2.22}$$
2. Test $\hat{\eta}'$ to define a second candidate $\hat{\eta}''$ as follows:
   - If $\hat{\eta}' \notin \hat{\mathcal{X}}_N$, then $\hat{\eta}'' := \hat{\eta}(t_-)$.
   - If $\hat{\eta}' \in \hat{\mathcal{X}}_N$ and for some $i \in \{1, \ldots, N\}$
   $$\exp\left[-\beta\left(\mathrm{H}(\mathcal{U}(\hat{\eta}')) - \mathrm{H}(\mathcal{U}(\hat{\eta}(t_-)))\right)\right] \geq U_{i,\theta_i(t)} \quad \text{and} \quad \theta_i(t) \neq \theta_i(t_-), \tag{2.23}$$
   then $\hat{\eta}'' := \hat{\eta}'$.
   - If $\hat{\eta}' \in \hat{\mathcal{X}}_N$ and for all $i \in \{1, \ldots, N\}$
   $$\exp\left[-\beta\left(\mathrm{H}(\mathcal{U}(\hat{\eta}')) - \mathrm{H}(\mathcal{U}(\hat{\eta}(t_-)))\right)\right] < U_{i,\theta_i(t)} \quad \text{or} \quad \theta_i(t) = \theta_i(t_-), \tag{2.24}$$
   then $\hat{\eta}'' := \hat{\eta}(t_-)$.
3. Define $\hat{\eta}(t)$ as the configuration obtained from $\hat{\eta}''$ by an appropriate local permutation of the positions of the particles (so that $\mathcal{U}(\hat{\eta}(t)) = \mathcal{U}(\hat{\eta}'')$).

Two permutation rules are often considered in the literature:

- $\hat{\eta}(t) := \hat{\eta}''$ (no permutation at all);
- if the first candidate did not violate the exclusion (i.e., if $\hat{\eta}' \in \hat{\mathcal{X}}_N$), then $\hat{\eta}(t) := \hat{\eta}''$, while if $\hat{\eta}' \notin \hat{\mathcal{X}}_N$, then $\hat{\eta}(t)$ is obtained from $\hat{\eta}'' = \hat{\eta}(t_-)$ by exchanging the position of the particles responsible for the violation of the exclusion.

The latter rule is often used when the dynamics is built from Poisson processes associated with bonds rather than sites. In De Masi, Ianiro, Pellegrinotti and Presutti [3] standard permutation rules are combined on the basis of particle hierarchy. All these permutation rules satisfy a *local permutation hypothesis* according to the following definitions:

Definition 2.2.1. Associate with each $\hat{\eta} \in \hat{\mathcal{X}}_N$ the *cluster partition* on $\{1, \ldots, N\}$ induced by the following equivalence relation: two particles labelled i and j are equivalent when $i = j$ or when they are in the same cluster of $\mathcal{U}(\hat{\eta})^{cl}$.

Local Permutation Hypothesis: *The permutation performed respects the cluster partition of* $\hat{\eta}(t_-)$.

In the next section we will introduce a new permutation rule for further application to metastability. In the following we will work under the general assumption that our Local Permutation Hypothesis is satisfied.

It is easy to see that the process defined by

$$(\eta_t)_{t\geq 0} := \big(\mathcal{U}(\hat{\eta}(t))\big)_{t\geq 0} \tag{2.25}$$

evolves according to the rules defined in (1.1) and (1.2). This process is reversible with respect to the *canonical Gibbs measure* defined by

$$\nu_N(\eta) := \frac{\mathrm{e}^{-\beta \mathrm{H}(\eta)} 1_{\mathcal{X}_N}(\eta)}{Z_N}, \quad \eta \in \mathcal{X}, \tag{2.26}$$

where Z_N is the normalizing partition sum.

2.3. Free, active and sleeping particles

Definition 2.3.1. We say that a particle $i \in \{1, \ldots, N\}$ is *free* at time $t_0 \geq 0$ if there exists a trajectory starting from $\hat{\eta}(t_0)$,

$$\hat{\eta} : t \in [t_0, t_0 + T] \longmapsto \hat{\eta}(t) \in \hat{\mathcal{X}}_N, \tag{2.27}$$

that respects the rules of the dynamics and satisfies

(i) $\|\hat{\eta}_i(t_0 + T) - \hat{\eta}_i(t_0)\|_2 > T_\alpha^{1/2}$,
(ii) $\forall t \in [t_0, t_0 + T]$: $\mathcal{U}(\hat{\eta}(t))^{cl} = \mathcal{U}(\hat{\eta}(t_0))^{cl}$.

Remark. Here, $T_\alpha^{1/2}$ plays the role of a reference distance that is almost of the order of the typical inter-particle distance. Note that the definition of a free particle only depends on the moves that are allowed by the dynamics, and that T has no role to play other than that of being positive in order to make (i) possible. Whether or not a particle is free depends on the present configuration only. For $t < \mathcal{T}_{\alpha,\lambda}$ (i.e., prior to the first anomalous concentration; recall (2.7)) the clusterized part of η_t consists of small islands (the clusters of η_t) surrounded by a sea (the single

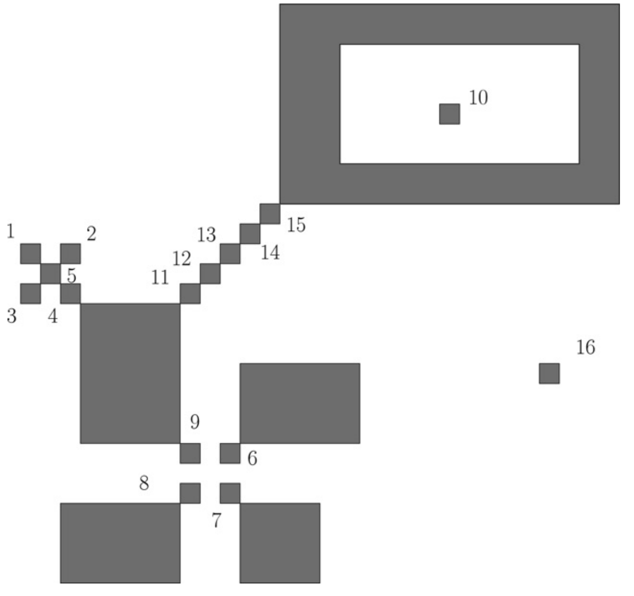

Fig. 1. Each particle is represented by a unit square. Particles 1–5 and 16 are free, particles 6–9, 10 and 11–15 are not free, while the other particles are clusterized.

connected component of $\Lambda_\beta \setminus \eta_t^{cl}$ that wraps around the torus). If these islands are frozen, then the free particles are those that can travel anywhere in the sea without attaching themselves (this is a consequence of our Local Permutation Hypothesis), possibly in a cooperative way only (we look at the existence of a trajectory of the whole process $\hat{\eta}$ on $\hat{\mathcal{X}}_N$ and not at the trajectory of a single particle on Λ_β). If we denote by η_t^f the set of sites occupied by the free particles, then we have $\eta_t^f \subset \eta_t \setminus \eta_t^{cl}$, in some cases with strict inclusion (see Fig. 1).

We can now define *active* and *sleeping* particles. Unlike for free particles, here we do need to know the history of the particle.

Definition 2.3.2. For $t > \mathrm{e}^{D\beta}$, we say that a particle is *sleeping* at time t if it never was free between times $t - \mathrm{e}^{D\beta}$ and t. We call a non-sleeping particle *active*. By convention, we say that prior to time $\mathrm{e}^{D\beta}$ all particles are active. With each particle i we associate, at any time t, its *wake-up time*

$$w_i(t) := \inf\{s \in [0, t): \text{ particle } i \text{ is active during the whole time interval } [s, t]\}. \tag{2.28}$$

By convention, for a sleeping particle at time t we put $w_i(t) = \inf \emptyset = \infty$.

As announced in Section 1, we will derive a spread-out property for active particles only. Consequently, the fewer sleeping particles there are, the stronger are our results. That is why we introduce a last example of local permutation rule, intended to minimize their number. At each time t, we define a hierarchy on the particles in all the clusters C of η_t: the later the particles lose their freedom, the higher they are in the hierarchy.

Special permutation rule: *If some particles were in some cluster C at time t_- and were free in* $\hat{\eta}''$ at time t_-, *then $\hat{\eta}(t)$ is obtained from $\hat{\eta}''$ by exchanging randomly their positions with those of the higher particles in the hierarchy of C at time t_-.*

2.4. Random walk with pauses and QRWs

We define a new process $Z = (Z_1, \ldots, Z_N)$ on Λ_β^N, coupled to $\hat\eta$ and ζ. To do so, we start from $Z(0) := \hat\eta(0) = \zeta(0)$ and apply the following rule each time the process ζ changes position:

$$\forall i \in \{1, \ldots, N\}: \quad Z_i(t) := \begin{cases} Z_i(t_-) + \zeta_i(t) - \zeta_i(t_-) & \text{if } i \text{ was free at time } t_-, \\ Z_i(t_-) & \text{if } i \text{ was not free at time } t_-. \end{cases} \tag{2.29}$$

Then Z is a process of "random walks with pauses" according to the following definition.

Definition 2.4.1. A process $Z = (Z_1, \ldots, Z_N)$ on Λ_β^N is called a *random walk with pauses* (RWP) associated with the stopping times

$$0 = \sigma_{i,0} = \tau_{i,0} \le \sigma_{i,1} \le \tau_{i,1} \le \sigma_{i,2} \le \tau_{i,2} \le \cdots, \quad i \in \{1, \ldots, N\}, \tag{2.30}$$

if, for any $i \in \{1, \ldots, N\}$, Z_i is constant on all time intervals $[\sigma_{i,k}, \tau_{i,k}]$, $k \in \mathbb{N}_0$, and if the process $\tilde Z = (\tilde Z_1, \ldots, \tilde Z_N)$ obtained by cutting off, for each i, these pause intervals, i.e.,

$$\tilde Z_i(s) := Z_i\left(s + \sum_{k<j_i(s)} (\tau_{i,k} - \sigma_{i,k})\right)$$
$$\text{with } j_i(s) := \inf\left\{ j \in \mathbb{N}: s + \sum_{k<j} (\tau_{i,k} - \sigma_{i,k}) \le \sigma_{i,j} \right\}, \tag{2.31}$$

is an IRW-process by law.

Indeed, for fixed $i \in \{1, \ldots, N\}$, define by induction the sequence of stopping times

$$0 = \sigma_{i,0} = \tau_{i,0} \le \sigma_{i,1} \le \tau_{i,1} \le \sigma_{i,2} \le \tau_{i,2} \le \cdots \tag{2.32}$$

with

$$\forall k \in \mathbb{N}: \begin{cases} \sigma_{i,k} := \inf\{t > \tau_{i,k-1}: \ i \text{ is not free at time } t\}, \\ \tau_{i,k} := \inf\{t > \sigma_{i,k}: \ i \text{ is free at time } t\}. \end{cases} \tag{2.33}$$

Then Z_i is a Markov process that does not move during the time intervals $[\sigma_{i,k}, \tau_{i,k}]$, $k \in \mathbb{N}_0$ (these are the pause intervals), and outside these time intervals moves exactly like a simple random walk in continuous time. $\tilde Z$ is an IRW-process as a consequence of the independence of the Poisson processes $\theta_1, \ldots, \theta_N$ and the increments $(e_{i,k})_{i\in\{1,\ldots,N\},k\in\mathbb{N}}$ in (2.19). Note that, for the same reasons, $Z - \zeta$ is a process of random walks with pauses during the time intervals $[\tau_{i,k}, \sigma_{i,k+1}]$, $k \in \mathbb{N}_0$. Note also that during any of these time intervals $\hat\eta_i$, Z_i and ζ_i evolve jointly, i.e., the pair differences are constant.

Apart from the length of these pauses – for which we introduced the distinction between active and sleeping particles – we need to control two quantities to prove our ideal gas approximation:

- The number of pauses of the processes Z_i prior to time T.
- The distance between the processes $\hat\eta$ and Z.

The smaller these are, the closer are $\hat\eta$ and ζ. This is the idea that leads us to introduce the notion of *Quasi-Random Walks*.

Definition 2.4.2. We say that a process $\xi = (\xi_1, \ldots, \xi_N)$ on Λ_β^N is a Quasi-Random Walk process with parameter $\alpha > 0$ up to stopping time $\mathcal{T}$, written as QRW$(\alpha, \mathcal{T})$, if there exists a coupling between ξ and an RWP-process Z associated with the stopping times

$$0 = \sigma_{i,0} = \tau_{i,0} \le \sigma_{i,1} \le \tau_{i,1} \le \sigma_{i,2} \le \tau_{i,2} \le \cdots, \quad i \in \{1, \ldots, N\}, \tag{2.34}$$

such that: $\xi(0) = Z(0)$, for any $i \in \{1, \ldots, N\}$, ξ_i and Z_i evolve jointly ($\xi_i - Z_i$ is constant) outside the pause intervals $[\sigma_{i,k}, \tau_{i,k}]$, $k \in \mathbb{N}_0$, and for any $t_0 \ge 0$ the following events occur with probability $1-$ SES uniformly in i and t_0:

$$\begin{aligned} F_i(t_0) &:= \left\{ \left| \left\{ k \in \mathbb{N}\colon \sigma_{i,k} \in [t_0 \wedge \mathcal{T}, (t_0 + T_\alpha) \wedge \mathcal{T}] \right\} \right| \le l(\beta) \right\}, \\ G_i(t_0) &:= \left\{ \forall k \in \mathbb{N}, \forall t \ge t_0\colon \sigma_{i,k} \in [t_0 \wedge \mathcal{T}, (t_0 + T_\alpha) \wedge \mathcal{T}] \right. \\ &\qquad \left. \Rightarrow \left\| \xi(t \wedge \tau_{i,k} \wedge \mathcal{T}) - \xi(t \wedge \sigma_{i,k} \wedge \mathcal{T}) \right\|_2 \le l(\beta) \right\}, \end{aligned} \tag{2.35}$$

for some $\beta \mapsto l(\beta)$ satisfying

$$\lim_{\beta \to \infty} \frac{1}{\beta} \ln l(\beta) = 0. \tag{2.36}$$

Remarks. 1. In other words, ξ is a QRW$(\alpha, \mathcal{T})$-process if "up to time $\mathcal{T}$" it can be coupled to an RWP-process Z (Definition 2.4.1) with few pause intervals on time scale T_α and such that in each of these pause intervals ξ has a small variation. More precisely, both the number of pause intervals and the variation of ξ are bounded by the same quantity $l(\beta)$, which by (2.36) is exponentially negligible. Outside these pause intervals ξ behaves like an IRW-process. We use the expression "up to time $\mathcal{T}$" because the QRW-property does not imply anything about the process after time $\mathcal{T}$. If $t_0 \ge \mathcal{T}$, then the events described in (2.35) are trivially verified. The parameter α determines the reference time T_α, which has to be thought of as a time smaller than but close to $1/\rho$ (recall (2.2)).

2. Any RWP-process is a QRW(α, ∞)-process provided the pauses are few. For example, a system of random walks in a random environment with local traps, where the particles get stuck during random times, is a QRW(α, ∞)-process as soon as the traps are sufficiently sparse (typically with density $\le e^{-\Delta\beta}$).

3. The first RWP-process Z we constructed at the beginning of this section was also coupled to an IRW-process $\zeta = (\zeta_1, \ldots, \zeta_N)$ such that $\zeta(0) = Z(0)$ and, for any $i \in \{1, \ldots, N\}$, ζ_i evolves jointly with Z_i (and $\hat{\eta}_i$) outside the pause intervals $[\sigma_{i,k}, \tau_{i,k}]$, $k \in \mathbb{N}_0$. It is easy to show that *any* RWP-process Z can be coupled to an IRW-process ζ with such properties. This implies, in particular, that $Z - \zeta$ is an RWP-process with pauses during the time intervals $[\tau_{i,k}, \sigma_{i,k+1}]$, $k \in \mathbb{N}_0$. In the following we will assume that a generic QRW$(\alpha, \mathcal{T})$-process ξ is not only coupled to an RWP-process Z associated with the stopping times

$$0 = \sigma_{i,0} = \tau_{i,0} \le \sigma_{i,1} \le \tau_{i,1} \le \sigma_{i,2} \le \tau_{i,2} \le \cdots, \quad i \in \{1, \ldots, N\}, \tag{2.37}$$

but also to such an IRW-process ζ. In addition, for any QRW$(\alpha, \mathcal{T})$-process ξ there is a natural generalization of the concepts of free, active and sleeping particles. We say that particle i is free outside the pause intervals of the coupled process Z_i, and define sleeping and active particles as in Definition 2.3.2.

3. Main results

In Section 3.1 we state that Kawasaki dynamics is a QRW-process. In Section 3.2 we formulate some consequences of the QRW-property. In Section 3.3 we formulate a lower bound for the spread-out property of Kawasaki dynamics that is stronger than the one implied by the QRW-property. Proofs are given in Sections 4 and 5.

3.1. Kawasaki dynamics as a QRW-process

Theorem 3.1.1. *For any increasing unbounded function λ satisfying (2.1) and any $\alpha \in (0, \Delta)$, $\hat{\eta}$ is a $QRW(\alpha, \mathcal{T}_{\alpha,\lambda})$-process. Moreover, the associated function l can be taken to be*

$$l(\beta) := (\Delta\beta)^{\mathrm{cst}\,\lambda(\beta)^8}. \tag{3.1}$$

Remark. As will become clear from the proof of Theorem 3.1.1 in Section 5.2, the role of the random time $\mathcal{T}_{\alpha,\lambda}$ is crucial. The fact that the QRW-property holds only up to this time is not a technical restriction: we are describing the Kawasaki dynamics prior to anomalous concentration and we may expect that its behavior changes beyond $\mathcal{T}_{\alpha,\lambda}$, for instance when the dynamics has grown a large cluster. In Section 6 we will give estimates on $\mathcal{T}_{\alpha,\lambda}$ in the three regimes mentioned in Section 1.2 (stable, metastable and unstable). In particular, we will see that in the stable regime the QRW-property itself in some sense preserves the absence of anomalous concentration. This is because an IRW-process produces anomalous concentration with a small probability, and hence so does a QRW-process in the stable regime.

3.2. Consequences of the QRW-property

We can now generalize the non-superdiffusivity and the spread-out property stated in (1.4)–(1.8) to general QRW-processes. To that end, we introduce a *standard behavior event* $\Omega^{(\delta)}$ of probability $1-$ SES and we prove both with respect to $P^{(\delta)}$, the conditional probability given $\Omega^{(\delta)}$ defined by

$$P^{(\delta)}(\cdot) := P(\cdot|\Omega^{(\delta)}). \tag{3.2}$$

We recall that a generic QRW-process ξ is assumed to be coupled to an RWP-process Z, but also to an IRW-process ζ (third remark after Definition 2.4.2).

Definition 3.2.1 (*Standard Behavior Event*). For $\delta > 0$, let

$$\Omega^{(\delta)} := \bigcap_{i=1}^{N} \left(\bigcap_{k=1}^{T_\alpha} F_i(kT_\alpha) \cap G_i(kT_\alpha) \right) \cap \left(\bigcap_{m=1}^{T_\alpha^2} J_{i,m}^1 \cap J_{i,m}^2 \right), \tag{3.3}$$

where $F_i(t_0)$ and $G_i(t_0)$ are defined in Definition 2.4.2 and

$$\begin{aligned} J_{i,m}^1 &:= \left\{ \forall t \in [0, m+1]\colon \|Z_i(t) - Z_i(0)\|_2 \le \mathrm{e}^{\frac{\delta}{10}\beta}\sqrt{m} \right\}, \\ J_{i,m}^2 &:= \Bigg\{ \forall t \in [0, m+1]\colon \|(Z_i - \zeta_i)(t) - (Z_i - \zeta_i)(0)\|_2 \\ &\quad \le \mathrm{e}^{\frac{\delta}{10}\beta} \sqrt{\sum_{\sigma_{i,k} \le m} \mathcal{T} \wedge \tau_{i,k} \wedge m - \mathcal{T} \wedge \sigma_{i,k}} \Bigg\}. \end{aligned} \tag{3.4}$$

In other words, $\Omega^{(\delta)}$ is the event that excludes: (1) a number of pauses larger than $l = l(\beta)$ for any particle in any time interval $[kT_\alpha, (k+1)T_\alpha]$ before time $\mathcal{T}$; (2) trajectories longer than l for any unfree particle before time $\mathcal{T}$; (3) superdiffusive behavior for the RWP-processes Z and $Z - \zeta$. (Since, for any i, $Z_i - \zeta_i$ takes its pauses when Z_i does not, the sum that appears in the definition of $J^2_{i,m}$ is the difference between m and the total length of the pause intervals of $Z_i - \zeta_i$ up to time m.)

Proposition 3.2.2. *For any $\delta > 0$, $P(\Omega^{(\delta)}) \geq 1 -$ SES uniformly in $\hat{\eta}(0)$.*

Proof. Note that $\Omega^{(\delta)}$ is the intersection of an exponential number of events each of which occurring with probability $1 -$ SES uniformly in i, k and m. As far as the events $F_i(kT_\alpha)$ and $G_i(kT_\alpha)$ are concerned, this is a consequence of Definition 2.4.2. Since Z and $Z - \zeta$ are RWP-processes, the events $J^1_{i,m}$ and $J^2_{i,m}$ occur with probability $1 -$ SES, uniformly in i and m, as a consequence of the obvious extension of (2.16) to RWP-processes. $\square$

Theorems 3.2.3–3.2.5 below are our main results for QRW-processes and will be proven in Section 4.

Theorem 3.2.3 (*Non-Superdiffusivity*). *Let ξ be a $QRW(\alpha, \mathcal{T})$-process and $\delta > 0$. Then there exists a $\beta_0 > 0$ such that, for all $T = T(\beta) \in [2, T_\alpha^2]$ and all $i \in \{1, \ldots, N\}$,*

$$\forall \beta > \beta_0\colon \quad P^{(\delta)}\left(\mathcal{T} > T \text{ and } \exists t \in [0, T)\colon \|\xi_i(t) - \xi_i(0)\|_2 > e^{\delta\beta}\sqrt{T}\right) = 0. \tag{3.5}$$

Consequently,

$$P\left(\mathcal{T} > T \text{ and } \exists t \in [0, T)\colon \|\xi_i(t) - \xi_i(0)\|_2 > e^{\delta\beta}\sqrt{T}\right) \leq \text{ SES} \tag{3.6}$$

uniformly in $\hat{\eta}(0)$, $i \in \{1, \ldots, N\}$ and $T = T(\beta) \in [2, T_\alpha^2]$.

Theorem 3.2.4 (*Spread-out Property, Upper Bound*). *Let ξ be a $QRW(\alpha, \mathcal{T})$-process and $\delta > 0$. Then there exists a $\beta_0 > 0$ such that, for all $T = T(\beta) \in [2, T_\alpha^2]$ and all $I \subset \{1, \ldots, N\}$, if $(\Lambda_i)_{i \in I}$ is a family of square boxes contained in Λ_β such that*

$$\forall i \in I\colon |\Lambda_i| \geq \left\lceil \frac{T}{T_\alpha} \right\rceil \left(\left\lceil \frac{T}{T_\alpha} \right\rceil \vee e^{D\beta} \right), \tag{3.7}$$

then

$$\forall \beta > \beta_0\colon \quad P^{(\delta)}\left(\mathcal{T} > T \text{ and } \forall i \in I\colon \xi_i(T) \in \Lambda_i \text{ and } w_i(T) = 0\right) \leq \prod_{i \in I} \left(\frac{|\Lambda_i| e^{\delta\beta}}{T} \right). \tag{3.8}$$

Theorem 3.2.5 (*Spread-out Property, Lower Bound*). *Let ξ be a $QRW(\alpha, \mathcal{T})$-process, $\delta > 0$, and I a finite subset of $\mathbb{N}$. Then there exists a $\beta_0 > 0$ such that the following holds for any $T = T(\beta) \in [2, T_\alpha^2]$ and any family $(\Lambda_i)_{i \in I}$ of square boxes contained in Λ_β:*

(i) *If*

$$\forall i \in I\colon \quad |\Lambda_i| \geq e^{\delta\beta} \left\lceil \frac{T}{T_\alpha} \right\rceil \left(\left\lceil \frac{T}{T_\alpha} \right\rceil \vee e^{D\beta} \right) \quad \text{and} \quad \Lambda_i \subset B_2\left(\xi_i(0), \sqrt{T}\right), \tag{3.9}$$

then

$$\forall \beta > \beta_0\colon \quad P^{(\delta)}\left(T \geq \mathcal{T} \text{ or } \forall i \in I\colon \xi_i(T) \in \Lambda_i \text{ or } w_i(T) > 0\right) \geq \prod_{i \in I} \left(\frac{\text{cst}\, |\Lambda_i|}{T} \right). \tag{3.10}$$

(ii) *If, in addition,*

$$\epsilon := \sup_{i\in I} \frac{4|\Lambda_i|}{T} \le 1 \quad \textit{and} \quad \forall i \in I\colon \xi_i(0) \notin [\Lambda_i]_{\sqrt{|\Lambda_i|}}, \tag{3.11}$$

then

$$\forall \beta > \beta_0\colon P^{(\delta)} \begin{pmatrix} T + \epsilon T \ge \mathcal{T} \\ \textit{or } \forall i \in I\colon \tau_{\Lambda_i}(\xi_i) \in [T, T+\epsilon T] \textit{ or } w_i(T+\epsilon T) > 0 \end{pmatrix}$$
$$\ge \prod_{i\in I} \left(\frac{|\Lambda_i|}{T e^{\delta\beta}}\right). \tag{3.12}$$

Remarks. 1. Theorem 3.2.5 generalizes the spread-out property in (1.7) for particles that are active during the whole time interval $[0, T]$, i.e., particles for which $w_i(T) = 0$. For an active particle at time T with $w_i(T) > 0$, by time translation, we get the same estimate with $\frac{1}{T - w_i(T)}$ replacing T.

2. Since we control the number of pause intervals and the behavior of the QRW$(\alpha, \mathcal{T})$-process during these intervals on the reference scale T_α only, we need to distinguish two cases: (1) $T \le T_\alpha$; (2) $T > T_\alpha$. In case (1), condition (3.7) reads ($\forall i \in I\colon |\Lambda_i| \ge e^{D\beta}$) and we have a result "at resolution $1 : e^{D\beta}$": instead of considering the probability to be at a site z at time T we consider the probability to be in a square box with volume of order $e^{D\beta}$ at time T. In case (2), we have a result at lower resolution. Similar considerations hold in both cases for the interpretation of condition (3.9).
3. In (3.12) the quantity ϵT plays the role of a temporal indetermination on $\tau_{\Lambda_i}(\xi_i)$. This temporal indetermination is of order $\sup_{i\in I} |\Lambda_i|$: the temporal and spatial resolutions are of the same order.
4. In Theorem 3.2.4, $|I|$ may grow with β, while β_0 is independent of I. In Theorem 3.2.5, $|I|$ is a finite number independent of β, while β_0 depends on I. If we would be able to prove (3.10) and (3.12) for any set of indices I such that $|I|$ is an increasing unbounded function of β, then we would have SES *lower* bounds for a conditional probability given an event of probability $1-$ SES: estimates with a limited relevance. This is not the case for the SES *upper* bounds given for such sets I in Theorem 3.2.4. We will make use of these bounds in Section 6.

3.3. Stronger lower bounds for Kawasaki dynamics

As far as Kawasaki dynamics is concerned, for further application to the study of metastability we need some lower bounds to get a spread-out property at higher resolution—typically at a resolution of order $1 : 1$ or $1 : \lambda$. In Section 5 we will prove the following.

Theorem 3.3.1. *Let I be a finite subset of $\mathbb{N}$, $\hat\eta(0) \in \hat{\mathcal{X}}_N$ such that $\mathcal{T}_{\alpha,\lambda} > 0$, $T = e^{C\beta}$ for some $C > 0$ different from U and $2U$, and $(z_i)_{i\in I} \in (\Lambda_\beta)^{|I|}$ such that, for all i in I, $\|z_i - \hat\eta_i(0)\|_2 \le \frac{1}{2}\sqrt{T}$.*

(i) *If $T \le T_\alpha$, all the particles with label $i \in I$ are free at time $t = 0$ and*

$$\forall i \in I\colon \inf_{1\le j\le N} \|z_i - \hat\eta_j(0)\|_1 > 13\lambda \quad \textit{and} \quad \inf_{j\in I, j\ne i} \|z_i - z_j\|_1 > 11, \tag{3.13}$$

then, for any $\delta > 0$,

$$P\left(\forall i \in I\colon \left\lfloor \tau_{\{z_i\}}(\hat\eta_i) \right\rfloor = \lfloor T \rfloor\right) \ge \left(\frac{1}{T e^{\delta\beta}}\right)^{|I|} - \textit{SES} \tag{3.14}$$

uniformly in $\hat\eta(0)$, T and $(z_i)_{i\in I}$.

(ii) *If* $T \le T_\alpha$, $T > \mathrm{e}^{D\beta}$ *and* (3.13) *is satisfied, then, for any* $\delta > 0$,

$$P\left(\forall i \in I\colon \left\lfloor \tau_{\{z_i\}}(\hat\eta_i)\right\rfloor = \lfloor T\rfloor \text{ or } w_i(T) > 0\right) \ge \left(\frac{1}{T\mathrm{e}^{\delta\beta}}\right)^{|I|} - SES \tag{3.15}$$

uniformly in $\hat\eta(0)$, T *and* $(z_i)_{i\in I}$.

(iii) *If* $T_\alpha < T < T_\alpha^2(T_\alpha^{-1/2} \wedge \mathrm{e}^{-D\beta})$ *and*

$$\forall i \in I\colon \inf_{1\le j\le N} \|z_i - \hat\eta_j(0)\|_1 > 17\lambda \quad \text{and} \quad \inf_{j\in I, j\ne i} \|z_i - z_j\|_1 > 9\lambda, \tag{3.16}$$

then, for any $\delta > 0$,

$$P\left(T > \mathcal{T}_{\alpha,\lambda} \text{ or } \forall i \in I\colon \left\lfloor \tau_{[z_i]_{4\lambda}}(\hat\eta_i)\right\rfloor = \lfloor T\rfloor \text{ or } w_i(T) > 0\right)$$

$$\ge \left(\frac{1}{T\mathrm{e}^{\delta\beta}}\right)^{|I|} - SES \tag{3.17}$$

uniformly in $\hat\eta(0)$, T *and* $(z_i)_{i\in I}$.

Remark. The condition $C \ne U, 2U$ is not actually necessary. In order to remove it, some of the estimates in Section 5.3 (e.g. the last estimate of Lemma 5.3.2) would need to be derived at a higher order of precision. We will not insist on this point.

4. Consequences of QRW-property: Proofs

In Sections 4.1–4.3 we prove Theorems 3.2.3–3.2.5, respectively.

4.1. Non-superdiffusivity

Proof of Theorem 3.2.3. Fix $\delta > 0$ and $i \in \{1, \ldots, N\}$. By (3.3), on $\Omega^{(\delta)}$, Z_i will not have more than $\lceil T/T_\alpha\rceil l$ pauses up to time $T \wedge \mathcal{T}$, and during each of these pauses the distance between Z_i and ξ_i will not increase by more than ℓ. Consequently (recall that $T \le T_\alpha^2$)

$$\sup_{t\le T\wedge\mathcal{T}} \|\xi_i(t) - Z_i(t)\|_2 \le \left\lceil \frac{T}{T_\alpha}\right\rceil l^2 \le \mathrm{e}^{\frac{\delta}{10}\beta}\sqrt{T} \quad \text{on } \Omega^{(\delta)} \tag{4.1}$$

for all $\beta \ge \beta_1(l, \delta)$. In addition,

$$\sup_{t\le T} \|Z_i(t) - Z_i(0)\|_2 \le \mathrm{e}^{\frac{\delta}{10}\beta}\sqrt{T} \quad \text{on } \Omega^{(\delta)}. \tag{4.2}$$

Consequently (by the triangular inequality),

$$P^{(\delta)}\left(\exists t \in [0, T]\colon \|\xi_i(t) - \xi_i(0)\|_2 > \mathrm{e}^{\delta\beta}\sqrt{T}\right) = 0 \tag{4.3}$$

for all $\beta \ge \beta_0(l, \delta)$. □

4.2. Spread-out property, upper bound

Proof of Theorem 3.2.4. On $\Omega^{(\delta)}$, for any $i \in I$, $\|\xi_i - Z_i\|_2$ can be estimated from the above as in Section 4.1, to get

$$\sup_{t\le T\wedge\mathcal{T}} \|\xi_i(t) - Z_i(t)\|_2 \le \left\lceil \frac{T}{T_\alpha}\right\rceil l^2 \le \mathrm{e}^{\frac{\delta}{9}\beta}\sqrt{|\Lambda_i|} \quad \text{on } \Omega^{(\delta)} \tag{4.4}$$

for all $\beta \geq \beta_1(l,\delta)$. In addition, if i never falls asleep during the whole time interval $[0, T \wedge \mathcal{T}]$, then, by using that

$$\Omega^{(\delta)} \subset \bigcap_{i=1}^{N} \bigcap_{m=1}^{T_\alpha^2} J_{i,m}^2, \tag{4.5}$$

we also get

$$\sup_{t \leq T \wedge \mathcal{T}} \|Z_i(t) - \zeta_i(t)\|_2 \leq \mathrm{e}^{\frac{\delta}{10}\beta} \sqrt{\left\lceil \frac{T}{T_\alpha} \right\rceil l \mathrm{e}^{D\beta}} \leq \mathrm{e}^{\frac{\delta}{9}\beta} \sqrt{|\Lambda_i|} \quad \text{on } \Omega^{(\delta)} \tag{4.6}$$

for all $\beta \geq \beta_2(l,\delta) > \beta_1(l,\delta)$. Consequently (by the triangular inequality),

$$\xi_i(T) \in \Lambda_i \Rightarrow \zeta_i(T) \in [\Lambda_i]_{\mathrm{e}^{\frac{\delta}{8}\beta}\sqrt{|\Lambda_i|}} \tag{4.7}$$

for all $\beta \geq \beta_3(l,\delta) > \beta_2(l,\delta)$. If we choose β_3 large enough so that also

$$\left| [\Lambda_i]_{\mathrm{e}^{\frac{\delta}{8}\beta}\sqrt{|\Lambda_i|}} \right| \leq |\Lambda_i| \mathrm{e}^{\frac{\delta}{2}\beta} \quad \text{and} \quad P(\Omega^{(\delta)}) \geq \frac{1}{2}, \tag{4.8}$$

then it follows, for all $\beta \geq \beta_3(l,\delta)$ and by the spread-out property for the IRW-process, that

$$\begin{aligned} &P^{(\delta)}\left(\mathcal{T} > T \text{ and } \forall i \in I\colon \xi_i(T) \in \Lambda_i \text{ and } w_i(T) = 0\right) \\ &\quad \leq 2P\left(\Omega^{(\delta)} \cap \{\mathcal{T} > T \text{ and } \forall i \in I\colon \xi_i(T) \in \Lambda_i \text{ and } w_i(T) = 0\}\right) \\ &\quad \leq 2P\left(\forall i \in I\colon \zeta_i(T) \in [\Lambda_i]_{\mathrm{e}^{\frac{\delta}{8}\beta}\sqrt{|\Lambda_i|}}\right) \\ &\quad \leq \prod_{i \in I}\left(\mathrm{cst}\, \frac{|\Lambda_i| \mathrm{e}^{\frac{\delta}{2}\beta}}{T}\right), \end{aligned} \tag{4.9}$$

so that we get (3.8) for some $\beta_0 \geq \beta_3$ large enough to make $\mathrm{e}^{\delta\beta}$ an upper bound for the factors $\mathrm{cst}\,\mathrm{e}^{\frac{\delta}{2}\beta}$ of the latter product. □

4.3. Spread-out property, lower bound

Proof of Theorem 3.2.5. Let

$$q := \frac{1}{8} \inf_{i \in I} \sqrt{|\Lambda_i|}, \tag{4.10}$$

and observe that

$$\left\lceil \frac{T}{T_\alpha} \right\rceil l^2 c + \mathrm{e}^{\frac{\delta}{10}\beta} \sqrt{\left\lceil \frac{T}{T_\alpha} \right\rceil l \mathrm{e}^{D\beta}} \leq q \tag{4.11}$$

for all $\beta \geq \beta_1(l,\delta)$, so that, as in Section 4.2, for the particles $i \in I$ that never fall asleep during the whole time interval $[0, T \wedge \mathcal{T}]$,

$$\sup_{t \leq T \wedge \mathcal{T}} \|\xi_i(t) - \zeta_i(t)\|_2 \leq q \quad \text{on } \Omega^{(\delta)}. \tag{4.12}$$

(i) For $i \in I$, let Λ'_i be the largest square box in Λ_β such that

$$\left[\Lambda'_i\right]_q \subset [\Lambda_i]. \tag{4.13}$$

On the one hand, we have, for all $\beta \geq \beta_1$,

$$P\left(\forall i \in I\colon \zeta_i(T) \in \Lambda'_i\right) \leq P^{(\delta)}\left(\mathcal{T} \leq T \text{ or } \forall i \in I\colon \xi_i(T) \in \Lambda_i \text{ or } w_i(T) > 0\right) + (1 - P(\Omega^{(\delta)})). \tag{4.14}$$

On the other hand, by the spread-out property for the IRW-process, we have

$$P\left(\forall i \in I\colon \zeta_i(T) \in \Lambda'_i\right) \geq \prod_{i \in I} \frac{\text{cst}\,|\Lambda'_i|}{T} \geq \prod_{i \in I} \frac{\text{cst}\,\left(|\Lambda_i| - 4q\sqrt{|\Lambda_i|}\right)}{T} \geq \prod_{i \in I} \frac{\text{cst}\,|\Lambda_i|}{T}. \tag{4.15}$$

Since $|I|$ is finite, does not depend on β, and $T \leq T_\alpha^2$, the latter product is not SES. Consequently,

$$1 - P(\Omega^{(\delta)}) \leq \frac{1}{2} P\left(\forall i \in I\colon \zeta_i(T) \in \Lambda'_i\right) \tag{4.16}$$

for all $\beta \geq \beta_2(l, \delta) > \beta_1(l, \delta)$ that depend on the law of ξ only. This proves (3.10) for all $\beta_0 \geq \beta_2$.

(ii) Assume now that $\xi_i(0) \notin [\Lambda_i]_{\sqrt{|\Lambda_i|}}$ for all $i \in I$ and define, for any $i \in I$,

$$\Lambda''_i := [\Lambda_i]_q. \tag{4.17}$$

On the one hand, by Brownian approximation and scaling, we have

$$P\left(\forall i \in I\colon \tau_{\Lambda''_i}(\zeta_i) \in [T, T + |\Lambda_i| \mathrm{e}^{-\frac{\delta}{20}\beta}]\right) \geq \prod_{i \in I} \frac{\text{cst}\,|\Lambda_i|}{T \mathrm{e}^{\delta 10 \beta}}. \tag{4.18}$$

On the other hand, for all $\beta \geq \beta_3(l, \delta)$ that depend on the law of ξ only, we can show as previously that

$$\begin{aligned} &P\left(\forall i \in I\colon \tau_{\Lambda''_i}(\zeta_i) \in [T, T + |\Lambda_i| \mathrm{e}^{-\frac{\delta}{20}\beta}]\right) \\ &\quad \leq P^{(\delta)}\left(\mathcal{T} \leq T \text{ or } \forall i \in I\colon w_i(T) > 0 \text{ or } \begin{cases} \tau_{\Lambda_i}(\xi_i) > T \\ \Lambda_i \subset B_2\left(\xi_i(T), 2\sqrt{|\Lambda_i|}\right) \end{cases}\right) \\ &\quad + \frac{1}{2} P\left(\forall i \in I\colon \tau_{\Lambda''_i}(\zeta_i) \in [T, T + |\Lambda_i| \mathrm{e}^{-\frac{\delta}{20}\beta}]\right). \end{aligned} \tag{4.19}$$

Since

$$|\Lambda_i| \geq \mathrm{e}^{\delta\beta} \left\lceil \frac{T}{T_\alpha} \right\rceil \left(\left\lceil \frac{T}{T_\alpha} \right\rceil \vee \mathrm{e}^{D\beta}\right) \quad \forall i \in I, \tag{4.20}$$

we also have

$$|\Lambda_i| \geq \mathrm{e}^{\delta\beta} \left\lceil \frac{\epsilon T}{T_\alpha} \right\rceil \left(\left\lceil \frac{\epsilon T}{T_\alpha} \right\rceil \vee \mathrm{e}^{D\beta}\right) \quad \forall i \in I, \tag{4.21}$$

provided that

$$\epsilon := \sup_{i \in I} \frac{4|\Lambda_i|}{T} \leq 1. \tag{4.22}$$

We may now conclude the proof by using (4.18) and (4.19), the Markov property at time T, and (3.10) with ϵT instead of T. $\square$

5. Back to Kawasaki dynamics

In Section 5.1 we state a key result from Gaudillière [7] for the non-collision probability of a random walk with obstacles. In Section 5.2 we prove Theorem 3.1.1. In Section 5.3 we prove Theorem 3.3.1.

5.1. Preliminaries

Let $\mathcal{R}$ be the collection of all finite sets of rectangles on $\mathbb{Z}^2$. We begin by defining a family of transformations $(g_r)_{r\geq 0}$ on $\mathcal{R}$ grouping into single rectangles those rectangles that have a distance smaller than r between them. To do so, with $r \geq 0$ and

$$\underline{S} = \left\{R_1, R_2, \ldots, R_{|\underline{S}|}\right\} \in \mathcal{R} \tag{5.1}$$

we associate a graph $G = (V, E)$ with vertex set

$$V := \left\{1, 2, \ldots, |\underline{S}|\right\} \tag{5.2}$$

and edge set

$$E := \left\{\{i, j\} \subset V : i \neq j \text{ and } \inf_{s\in R_i} \inf_{s'\in R_j} \|s - s'\|_\infty \leq r\right\}. \tag{5.3}$$

Calling C the set of the connected components of G, we define

$$\bar{g}_r : \underline{S} \in \mathcal{R} \longmapsto \left\{\mathrm{RC}\left(\bigcup_{i\in c} R_i\right)\right\}_{c\in C} \in \mathcal{R}, \tag{5.4}$$

where RC denotes the circumscribed rectangle, and $g_r(\underline{S}) \in \mathcal{R}$ is defined as the limit set of the iterates of $\underline{S}$ under $\bar{g}_r$ (which clearly exists because $|\underline{S}|$ is finite). Note that $g_r(\underline{S}) = \underline{S}$ means that $\|R - R'\|_\infty > r$ for all $R, R' \in \underline{S}$ that are distinct.

We associate with $\underline{S} \in \mathcal{R}$ its *perimeter*

$$prm(\underline{S}) := \sum_{R\in\underline{S}} |\partial R| \tag{5.5}$$

and we use the notation

$$S := \operatorname{supp} \underline{S} := \bigcup_{R\in\underline{S}} R \subset \mathbb{Z}^2. \tag{5.6}$$

For $\underline{S} \in \mathcal{R}$, $n \in \mathbb{N}$ and $\zeta = (\zeta_1, \ldots, \zeta_n)$ an IRW-process on $(\mathbb{Z}^2)^n$, we define the *first collision time*

$$\mathcal{T}_c := \inf\left\{t \geq 0 : \exists R \in \underline{S}, \exists (i, j) \in \{1, \ldots, n\}^2, \inf_{s\in R} \|\zeta_i(t) - s\|_1 = 1 \right.$$
$$\left. \text{or } \|\zeta_i(t) - \zeta_j(t)\|_1 = 1\right\}. \tag{5.7}$$

Proposition 5.1.1 (*Gaudillière [7]*). *There exists a constant $c_0 \in (0, \infty)$ such that, for all $n \geq 2$ and $p \geq 2$ the following holds. If $\underline{S} \in \mathcal{R}$ is such that*

$$\begin{cases} g_3(\underline{S}) = \underline{S}, \\ prm(\underline{S}) \leq p, \end{cases} \tag{5.8}$$

and $\zeta(0) \in (\mathbb{Z}^2)^n$ is such that

$$\begin{cases} \inf_{i \neq j} \|\zeta_i(0) - \zeta_j(0)\|_1 > 1, \\ \inf_i \inf_{s \in S} \|\zeta_i(0) - s\|_\infty > 3, \end{cases} \tag{5.9}$$

then, for the IRW-process on $(\mathbb{Z}^2)^n$ that starts from $\zeta(0)$,

$$\forall T \geq T_0, \quad P(\mathcal{T}_c > T) \geq \frac{1}{(\ln T)^\nu} \tag{5.10}$$

with

$$\begin{cases} \nu := c_0 n^4 p^2 \ln p, \\ T_0 := \exp\{\nu^2\}. \end{cases} \tag{5.11}$$

We will need two other results derived in [7], namely, the estimate

$$\forall \underline{S} \in \mathcal{R}, \forall r \geq 0 : prm(g_r(\underline{S})) \leq prm(\underline{S}) + 4r(|\underline{S}| - |g_r(\underline{S})|), \tag{5.12}$$

and the following corollary of Proposition 5.1.1:

Proposition 5.1.2. *There is a constant $c_0' \in (0, \infty)$ such that, if $n \geq 2$, $S \in \mathcal{R}$, ζ an IRW-process on $(\mathbb{Z}^2)^n$ verifying (5.8) and (5.9) for some $p \geq 2$, and $z \in (\mathbb{Z}^2)^n$ and $T > 0$ satisfy*

$$\begin{cases} \inf_{i \neq j} \|z_i - z_j\|_1 > 1, \\ \inf_i \inf_{s \in S} \|z_i - s\|_\infty > 3, \\ \sup_i \|z_i - \zeta_i(0)\|_2 \leq \sqrt{T}, \\ T \geq c_0' T_0, \end{cases} \tag{5.13}$$

with (ν, T_0) defined in (5.11), then

$$P(\mathcal{T}_c > T \text{ and } \forall i \in \{1, \ldots, n\}, \quad \zeta_i(T) = z_i) \geq \frac{L_{n,p}(T)}{T^n}, \tag{5.14}$$

where $L_{n,p}$ is the slowly varying function defined as

$$L_{n,p}(T) = \exp\{-c_0' n^4 p^3 \nu^2 (\ln \ln T)^2\}, \quad T > 1. \tag{5.15}$$

5.2. QRW-property

Proof of Theorem 3.1.1. We give the proof by showing that the RWP-process Z constructed in Section 2.4 fits with Definition 2.4.2. If $\hat{\eta}(0)$ is such that $\mathcal{T}_{\alpha,\lambda} = 0$, then there is nothing to prove. We therefore assume $\mathcal{T}_{\alpha,\lambda} > 0$.

We associate with each particle i a ball centered at its initial position with radius

$$r := \mathrm{e}^{\frac{\alpha}{4}\beta}\sqrt{T_\alpha}, \tag{5.16}$$

and we call B_0 their union:

$$B_0 := \bigcup_{i=1}^{N} B_2\left(\hat{\eta}_i(0), r\right). \tag{5.17}$$

Since r is much larger than the diffusive distance associated with time T_α, this suggests a partition of $\{1, \dots, N\}$ into *clouds of potentially interacting particles on time scale T_α*. We say that two particles are in the same cloud when they belong to the same connected component of B_0. We call τ_e the first time when one of the particles leaves B_0 (B_0 is fixed and does not change with time) and observe that before τ_e each cloud evolves independently of the others. With these definitions we can divide the proof into 4 steps:

1. We estimate from the above the number of particles in each cloud using $\mathcal{T}_{\alpha,\lambda} > 0$.
2. Given $i \in \{1, \dots, N\}$, for any $s \le T_\alpha$ and conditionally on $\{s < \tau_e\}$, we estimate from below the probability that in the cloud to which i belongs and during the time interval $[s, T_\alpha \wedge \tau_e]$ no particle loses its freedom. After Step 1, this can be done by using the estimates on the non-collision probability (Proposition 5.1.1).
3. We deduce from the previous estimates that, with
$$\mathcal{T} := T_\alpha \wedge \tau_e, \tag{5.18}$$
$\hat{\eta}$ is a $\mathrm{QRW}(\alpha, \mathcal{T})$-process associated with the function l defined in (3.1).
4. We use the non-superdiffusivity of the QRW-process (Theorem 3.2.3) to get first that $\hat{\eta}$ is a $\mathrm{QRW}(\alpha, T_\alpha)$-process and second that it is a $\mathrm{QRW}(\alpha, \mathcal{T}_{\alpha,\lambda})$-process associated with the same function l.

Step 1. We divide Λ_β into $|\Lambda_\beta|/V$ square cells of volume

$$V := \mathrm{e}^{(\Delta - \frac{\alpha}{4})\beta}. \tag{5.19}$$

It follows from $\mathcal{T}_{\alpha,\lambda} > 0$ that no cell contains more than $\lambda/4$ particles at time $t = 0$. Since

$$\frac{\sqrt{V}}{r} = \mathrm{e}^{\frac{\alpha}{8}\beta}, \tag{5.20}$$

no connected component of B_0 can move from one side to the opposite side in any domino made of two contiguous cells (for β large enough). Consequently, each of these connected components is contained in a union of four cells, and each cloud contains at most λ particles.

Step 2. Given $i \in \{1, \dots, N\}$ and $s \le T_\alpha$, we call $\mathcal{C}_0$ the family of the clusters of $\hat{\eta}(s)$ that contain (at time s) some particle of the cloud (defined at time $t = 0$) to which i belongs. We define (recall (2.11) and the definition of g_r after (5.4))

$$\begin{aligned} &\underline{S}_0 := \{\mathrm{RC}(c)\}_{c \in \mathcal{C}_0} \quad \in \mathcal{R}, \\ &\underline{S} := g_5\left(\underline{S}_0\right) \quad \in \mathcal{R}, \\ &\left[\underline{S}\right]_1 := \{[R]_1\}_{R \in \underline{S}} \quad \in \mathcal{R}. \end{aligned} \tag{5.21}$$

We note that $g_3(\left[\underline{S}\right]_1) = \left[\underline{S}\right]_1$, and that at time s the gas surrounding $[S]_1$ is made of free particles only.

Definition 5.2.1 (*Enrichment and Collision Times*). Given $\underline{S} \in \mathcal{R}$, we say that the gas surrounding $[S]_1$ is enriched each time a particle arrives into ∂S from S, and we say that a collision occurs each time two particles collide outside $[S]_1$ or one particle arrives in $\partial\,[S]_1$ from $\Lambda_\beta \setminus ([S]_1 \cup \partial\,[S]_1)$.

For the system restricted to the cloud to which i belongs (recall that before time τ_e each cloud evolves independently of the other ones) we call $A(s)$ the sequence of the following events defined on the time interval $[s, T_\alpha \wedge (\tau_e \vee s)]$:

A_1: All the free particles inside $[[S]_1]_3$ (if any) leave $[[S]_1]_3$ without collision before η^{cl} changes. (Note that, after A_1 is completed, S contains all the unfree particles and $\partial\,[S]_1$, like $[S]_1 \setminus S$, does not contain particles.)

A_2: The gas surrounding $[S]_1$ evolves without collision up to the first of the following three stopping times: the next enrichment time, $\tau_e \vee s$ and T_α.

A_3: After each enrichment, the particle responsible for the enrichment moves outside $[[S]_1]_3$ without collision and before $\eta|_S$ changes. Subsequently, the gas surrounding $[S]_1$ evolves without collision up to the next enrichment, and so on, up to time $T_\alpha \wedge (\tau_e \vee s)$.

Note that $A(s)$ implies that there is no loss of freedom of particles in the cloud to which i belongs in the time interval $[s, T_\alpha \wedge (\tau_e \vee s)]$.

To estimate $P(A(s)|s < \tau_e)$ from below, we need some estimates on $|\partial[S]_1|$. Since there are not more than λ particles in the cloud, we have

$$\operatorname{prm}\left(\underline{S}_0\right) \le 4\lambda \quad \text{and} \quad \left|\underline{S}_0\right| \le \lambda, \tag{5.22}$$

so that, via (5.12),

$$\operatorname{prm}(\underline{S}) \le 24\lambda \quad \text{and} \quad \left|\underline{S}\right| \le \lambda. \tag{5.23}$$

Moreover,

$$|\partial[S]_1| \le prm\left(\underline{S}\right) + 8\left|\underline{S}\right| \le 32\lambda. \tag{5.24}$$

It is then easy to get

$$P\left(A_1|s < \tau_e\right) \ge \left(\frac{1}{4\lambda}\right)^{\mathrm{cst}\,\lambda}. \tag{5.25}$$

Then, using the strong Markov property at the time when the last free particle inside $[[S]_1]_3$ leaves it together with Proposition 5.1.1:

$$P\left(A_1 \cap A_2|\, s < \tau_e\right) \ge \left(\frac{1}{4\lambda}\right)^{\mathrm{cst}\,\lambda} \left(\frac{1}{\ln T_\alpha}\right)^{\mathrm{cst}\,\lambda^6 \ln\lambda} \tag{5.26}$$

as soon as

$$T_\alpha = \mathrm{e}^{(\Delta-\alpha)\beta} \ge \exp\left\{\mathrm{cst}\,(\lambda^6 \ln\lambda)^2\right\}, \tag{5.27}$$

i.e., β is larger than some β_0 that depends on Δ, α and λ only.

Finally, if $A(s)$ occurs, then no particle that exits S can come back. Consequently, there cannot be more than λ enrichments and we find, using repeatedly the strong Markov property at the successive enrichment times, that

$$P\left(A(s)|\, s < \tau_e\right) \ge \left[\left(\frac{1}{4\lambda}\right)^{\mathrm{cst}\,\lambda} \left(\frac{1}{\ln T_\alpha}\right)^{\mathrm{cst}\,\lambda^6 \ln\lambda}\right]^{\lambda} \ge \left(\frac{1}{\ln T_\alpha}\right)^{\mathrm{cst}\,\lambda^7 \ln\lambda}. \tag{5.28}$$

Step 3. We denote by $(\tau_m)_{m\geq 1}$ the increasing sequence of stopping times when a particle loses its freedom in the cloud i it belongs to. By the strong Markov property and the previous estimate, we have, for $\beta \geq \beta_0$ and any a,

$$P\left(|\{m \geq 1 : \tau_m \leq \mathcal{T}\}| \geq a\right) \leq \left[1 - \left(\frac{1}{\ln T_\alpha}\right)^{\mathrm{cst}\,\lambda^7 \ln\lambda}\right]^a. \tag{5.29}$$

We also have (recall the definition of $\sigma_{i,k}$, $\tau_{i,k}$ in Section 2.4)

$$\left|\left\{k \in \mathbb{N} : \sigma_{i,k} \in [0, \mathcal{T}]\right\}\right| \leq 1 + |\{m \geq 1 : \tau_m \leq \mathcal{T}\}|, \tag{5.30}$$

and it is easy to see that, under our Local Permutation Hypothesis (see Section 2.2), for all $k \in \mathbb{N}$ and $t \geq 0$,

$$\left|\hat{\eta}_i(t \wedge \tau_{i,k} \wedge \mathcal{T}) - \hat{\eta}_i(t \wedge \sigma_{i,k} \wedge \mathcal{T})\right| \leq 24\lambda\left(1 + |\{m \geq 1 : \tau_m \leq \mathcal{T}\}|\right). \tag{5.31}$$

(Define S_m at time τ_m like S at time s, observe that any unfree particle is contained in S_m up to time $\tau_{m+1} \wedge \mathcal{T}$, and recall that $|\partial S_m| \leq 24\lambda$ if $\tau_m \leq \tau_e$.) Finally, we choose

$$a := \frac{(\ln T_\alpha)^{\lambda^8} - 1}{24\lambda} \tag{5.32}$$

in (5.29) to get that $\hat{\eta}$ is a QRW$(\alpha, \mathcal{T})$-process for which the function l in Definition 2.4.2 can be taken as in (3.1). Note that if (2.1) holds, then (2.36) follows.

Step 4. By Theorem 3.2.3, the particles are non-superdiffusive on time scale T_α and up to time $\mathcal{T}$. This gives

$$P\left(T_\alpha = T_\alpha \wedge \tau_e\right) = 1 - \text{SES} \tag{5.33}$$

and implies that $\hat{\eta}$ is a QRW(α, T_α)-process associated with the same function l.

To prove that $\hat{\eta}$ is a QRW$(\alpha, \mathcal{T}_{\alpha,\lambda})$-process associated with the RWP-process Z, it suffices to prove that, for any $i \in \{1, \dots, N\}$ and $t_0 \geq 0$, and *conditionally on* $\{\mathcal{T}_{\alpha,\lambda} > t_0\}$, the inequalities that appear in Definition 2.4.2 hold with probability $1 -$ SES uniformly in i and t_0. Since, on the one hand, η and Z are Markov processes and, on the other hand, $Z - Z(t_0) + \hat{\eta}(t_0)$ and Z have the same pause intervals and evolve jointly, this is a direct consequence of the fact that $\hat{\eta}$ is a QRW(α, T_α)-process. □

As a byproduct of this proof we get the following.

Proposition 5.2.2. *If λ satisfies (2.1), $\alpha < \Delta$, and $\hat{\eta}(0)$ is such that $\mathcal{T}_{\alpha,\lambda} > 0$, then $\hat{\eta}$ is a QRW(α, T_α)-process.*

5.3. Stronger lower bounds for the spread-out property

In this section we prove Theorem 3.3.1 using as a key estimate Proposition 5.1.2. Like in Section 5.2, this estimate cannot be applied directly because of the gas enrichment phenomenon. There we dealt with this difficulty by observing that, in the absence of collisions in the gas surrounding some $\left[\underline{S}\right]_1 \in \mathcal{R}$, there are at most λ effective enrichments in each cloud of potentially interacting particles. Here, however, we need more information. To get this, we extend to the present situation a few easy results on the local Kawasaki model in den Hollander, Olivieri and Scoppola [11] coming from the standard cycle and cycle-path theory introduced by Freidlin and Wentzell [6] (see also Olivieri and Vares [14]). We need to extend this theory because it only applies to a finite state space, while in the present situation we have a state space with cardinality of order $\lambda^{2\kappa\lambda}$, with $\kappa > 0$ and λ our growing unbounded function of β. However, since λ is only *slowly* growing, the situation we face is not so qualitatively different from the standard one. In addition, we do not generalize the full theory to our context: we only give the definitions and prove the lemmas that we need in order to complete the proof of Theorem 3.3.1. The further study of metastability will require a much more complete analysis of cycles and cycle paths. This will be the object of the forthcoming paper, Gaudillière, den Hollander, Nardi, Olivieri and Scoppola [8].

For $\underline{S} \in \mathcal{R}$ such that $g_5(\underline{S}) = \underline{S}$, we define the associated *local Hamiltonian* $\mathrm{H}_{\underline{S}}$ by (recall (1.2))

$$\mathrm{H}_{\underline{S}}(\eta) = \sum_{R\in\underline{S}} \mathrm{H}(\eta|_R) + \Delta|\eta|_{R\cup\partial R}|, \quad \forall\, \eta \in \mathcal{X} = \{0,1\}^{\Lambda_\beta}. \tag{5.34}$$

Definition 5.3.1. Given $\underline{S} \in \mathcal{R}$ with $g_5(\underline{S}) = \underline{S}$ and $k \in \{0, 1, 2\}$ such that $kU < \Delta$, we say that a configuration $\eta \in \mathcal{X}$ is kU-reducible if there is a sequence of configurations $\eta = \eta_0, \eta_1, \ldots, \eta_n$ in $\mathcal{X}$, each of them obtained from the previous one by a displacement of a single particle to a nearest-neighbor vacant site, such that

$$\begin{cases} \mathrm{H}_{\underline{S}}(\eta_n) < \mathrm{H}_{\underline{S}}(\eta), \\ \sup\limits_j \mathrm{H}_{\underline{S}}(\eta_j) \leq \mathrm{H}_{\underline{S}}(\eta) + kU. \end{cases} \tag{5.35}$$

We say that a labelled configuration $\hat\eta$ is kU-reducible when $\mathcal{U}(\hat\eta)$ is (recall (2.18)).

Remark. If $2U < \Delta$, then the only $2U$-irreducible configurations are the configurations without particles inside S. Indeed, any cluster carries at least four particles that can only be separated at cost $2U$.

Lemma 5.3.2. *Let $\lambda = \lambda(\beta)$ satisfy (2.1), $\kappa > 0$, $\underline{S} \in \mathcal{R}$ such that $g_5(\underline{S}) = \underline{S}$ and $prm(\underline{S}) \leq \lambda^\kappa$, and let the initial labelled configuration $\hat\eta(0)$ be such that at time $t = 0$ there are no particles inside $\partial\,[S]_1$, no particles inside $[S]_1 \setminus S$, and not more than λ particles inside S. Let τ_c be the first collision time for the gas surrounding $[S]_1$ and τ_+ its first enrichment time.*

(1) *If $\hat\eta(0)$ is kU-reducible, then, for any $\delta > 0$,*

$$P\left(\exists t \leq \mathrm{e}^{(kU+\delta)\beta}, \hat\eta(t) \text{ is } kU\text{-irreducible or } \tau_+ = t \,\middle|\, \tau_c > \mathrm{e}^{(kU+\delta)\beta} \wedge \tau_+\right) \geq 1 - \mathit{SES} \tag{5.36}$$

uniformly in $\underline{S}$ and $\hat\eta(0)$.

(2) *If $\hat\eta(0)$ is kU-irreducible, then, for any $\delta > \delta' > 0$,*

$$P\left(\tau_+ \leq \mathrm{e}^{((k+1)U-\delta)\beta} \,\middle|\, \tau_c > \mathrm{e}^{((k+1)U-\delta)\beta} \wedge \tau_+\right) \leq \mathrm{e}^{-\delta'\beta} + \mathit{SES} \tag{5.37}$$

uniformly in $\underline{S}$ and $\hat\eta(0)$.

Proof. See Appendix A. □

We are now ready to prove (i), (ii) and (iii) of Theorem 3.3.1.

Proof of Theorem 3.3.1. Proof of (i). Let $\hat{\eta}(0)$, I, T and $(z_i)_{i\in I}$ satisfy the required hypotheses. We define the clouds of potentially interacting particles on time scale T_α as in Section 5.2. By Theorem 3.2.3 and Proposition 5.2.2, with probability $1-$ SES and uniformly in $\hat{\eta}(0)$ and I, each cloud evolves independently of the others up to time T_α. Consequently, it suffices to prove the result when all the particles $i \in I$ belong to the same cloud. We have seen in Section 5.2 that each cloud contains at most λ particles and we can now restrict ourselves to considering a single cloud of $n \le \lambda$ particles.

We call $\underline{S}_0$ the set of the circumscribed rectangles of the clusters of the initial configuration, and we define

$$\begin{aligned} &\underline{S}' := g_5(\underline{S}_0), \\ &\underline{S} := \underline{S}' \cup \{\{z_i\} : i \in I\}. \end{aligned} \tag{5.38}$$

As in (5.23),

$$prm(\underline{S}') \le 24\lambda, \tag{5.39}$$

and it follows from (3.13) that $g_5(\underline{S}) = \underline{S}$. In addition, for β large enough, we have

$$|I| \le \lambda \quad \text{and} \quad |\underline{S}'| \le \lambda, \tag{5.40}$$

and so

$$\begin{aligned} &\mathrm{prm}(\underline{S}) \le 24\lambda + 4\lambda = 28\lambda, \\ &|\partial\, [S]_1\,| \le 28\lambda + 8(\lambda+\lambda) = 44\lambda. \end{aligned} \tag{5.41}$$

For the largest $k \in \{0, 1, 2\}$ that satisfies $kU < C$, we call τ_r the first time when $\hat{\eta}$ is not kU-reducible with respect to $\underline{S}$, and we consider, for the system restricted to the cloud to which each $i \in I$ belongs, the sequence of events A_1, A_2, A_3 defined on the time interval $[0, T_\alpha \wedge \tau_r]$ as in Section 5.2 replacing $(\tau_e \vee s)$ with τ_r. As Section 5.2, the probability of this sequence of events can be estimated from below by a non-exponentially small quantity p_1:

$$p_1 \ge \left(\frac{1}{\ln T_\alpha}\right)^{\mathrm{cst}\,\lambda^7 \ln \lambda}. \tag{5.42}$$

By Lemma 5.3.2, we make only an SES-error by assuming that the time between each of the enrichments in this sequence of events and $\tau_r \wedge T_\alpha$ or the successive enrichment is smaller than $\mathrm{e}^{(kU+\delta_0/2)\beta}$, with $\delta_0 > 0$ such that

$$kU + \delta_0 < C. \tag{5.43}$$

Since in such a sequence of events there cannot be more than λ enrichments, we get, in particular, for $\delta > 0$ and β large enough,

$$P\left(\tau_r < \mathrm{e}^{(kU+\delta_0)\beta} \text{ and } \tau_r < \tau_c\right) \ge \mathrm{e}^{-\frac{\delta}{3}\beta} \tag{5.44}$$

with τ_c the first collision time in the gas surrounding $[S]_1$ for the system restricted to the cloud we consider.

We next choose $|I|$ distinct and non-nearest-neighbor sites $(z_i')_{i\in I}$ such that

$$\forall i \in I, \qquad \inf_{s\in[S]_1} \|z_i' - s\|_\infty = \inf_{s\in[\{z_i\}]_1} \|z_i' - s\|_\infty = 4. \tag{5.45}$$

Condition (3.13) ensures that we can find such a family $(z_i')_{i\in I}$. We claim that

$$P\left(\tau_c \geq T-4 \text{ and } \forall i \in I, \hat{\eta}_i(T-4) = z_i'\right) \geq \frac{\mathrm{e}^{-\frac{\delta}{2}\beta}}{T^{|I|}}. \tag{5.46}$$

To prove this estimate, we distinguish two cases: $k=2$ and $k=0,1$.

$k=2$: At time τ_r, S does not contain particles. The estimate is then a consequence of the Markov property applied at time τ_r, the estimate (5.44), and Proposition 5.1.2.

$k=0,1$: There exists a $\delta_1>0$ such that $(k+1)U-\delta_1=C$. For all $i\in I$, the probability that $\hat{\eta}_i(T-4)=z_i'$, without collision or enrichment for the gas surrounding $[S]_1$ between times τ_r and $T-4$, can be estimated from below by Proposition 5.1.2 and Lemma 5.3.2. Once again (5.46) follows from the Markov property applied at time τ_r.

Finally, using the Markov property at time $T-4$ and "driving by hand" the particles after time $T-4$, we obtain

$$P\left(\forall i\in I, \left\lfloor \tau_{\{z_i\}}(\hat{\eta}_i)\right\rfloor = \lfloor T\rfloor\right) \geq \frac{\mathrm{e}^{-\delta\beta}}{T^{|I|}} \tag{5.47}$$

for the restricted system. □

Proof of (ii). We can follow the proof of (i) up to (5.46), which we change into

$$P\left(\tau_c \geq T-4, \forall i \in I\colon \hat{\eta}_i(T-4) = z_i' \text{ or } w_i(T-4)>0\right) \geq \frac{\mathrm{e}^{-\frac{2\delta}{3}\beta}}{T^{|I|}}. \tag{5.48}$$

$k=2$: We still have (5.46), which implies (5.48).

$k=0,1$: The previous arguments no longer give (5.46), because there can be some particles $i\in I$ in S at time τ_r. The arguments now give

$$P\left(\tau_c \geq T-4, \forall i \in I\colon \hat{\eta}_i(T-4) = z_i' \text{ or } \hat{\eta}_i(t)\in S\, \forall t\in[0,T-4]\right) \geq \frac{\mathrm{e}^{-\frac{\delta}{2}\beta}}{T^{|I|}}. \tag{5.49}$$

However, a particle i that is confined to S up to time $T-4>\mathrm{e}^{(D+\delta_2)\beta}$ for $\delta_2>0$ and β large enough falls asleep before time $T-4$ with probability $1-$ SES. This can be seen as an application of Theorem 3.2.4: assume that $w_i(T-4)=0$, choose a square box Λ_i of volume $\mathrm{e}^{D\beta}$ that contains the connected component of S to which i remains confined, divide the time interval $[0,\mathrm{e}^{(D+\delta_2)\beta}]$ into $\mathrm{e}^{\delta_2\beta/2}$ intervals of length $\mathrm{e}^{(D+\delta_2/2)\beta}$, and apply the proposition $\mathrm{e}^{\delta_2\beta/2}$ times with $\delta=\delta_2/3$. Consequently, we get (5.48) for β large enough, and we again conclude with the Markov property applied at time $T-4$. □

Proof of (iii). Let $\hat{\eta}(0)$, I, T and $(z_i)_{i\in I}$ satisfy the required hypotheses. We will work on two time scales: T_α, which allows for "high resolution estimates" (because on this time scale the cloud of potentially interacting particles contains a small number of particles), and T, for which we can use the lower resolution estimates. We will use different tools to deal with different time scales. The proof will be divided into five steps: the first two steps are relevant only for starting configurations in which the initial positions $\hat{\eta}_i(0)$ of some particles $i\in I$ are "close" to their associated targets $[z_i]_{4\lambda}$.

Step 1. We begin by estimating from below the probability that none of the particles $i\in I$ enters $[z_i]_{4\lambda}$ before time T_α. To do so, we consider the clouds of potentially interacting particles on time scale T_α, we call, for $i \in I$, $\underline{S}''_{0,i}$ the set of the circumscribed rectangles of the clusters of $\hat{\eta}(0)$ made of particles contained in the cloud to which i belongs, and we define

$$\begin{aligned}&\underline{S}'_{0,i} := g_5(\underline{S}''_{0,i}),\\&\underline{S}_{0,i} := \underline{S}'_{0,i} \cup \left\{\left[z_j\right]_{4\lambda} : j \in I\right\}.\end{aligned} \tag{5.50}$$

Observe that, as previously, $prm(\underline{S}'_{0,i}) \le 24\lambda$ and note that, by (3.16), $g_5(\underline{S}_{0,i}) = \underline{S}_{0,i}$. In addition, $|\underline{S}'_{0,i}| \le \lambda$ and $|I| \le \lambda$, so that

$$prm(\underline{S}_{0,i}) \le 24\lambda + \lambda \times 33\lambda \le 34\lambda^2 \tag{5.51}$$

and

$$|\partial \left[S_{0,i}\right]_1| \le 34\lambda^2 + 8(\lambda + \lambda) \le 35\lambda^2 \tag{5.52}$$

for β large enough.

Let $\tau_{c,0,i}$ be the collision time associated with $\left[S_{0,i}\right]_1$ for the system restricted to the cloud that contains i. Using the fact that, with probability $1-$ SES, the various clouds do not interact with each other up to time T_α, following the arguments that led to (5.28) or (5.42), and taking into account (5.52), we conclude that, for any $\delta > 0$,

$$\begin{aligned}P\left(\forall i \in I : \tau_{[z_i]_{4\lambda}}(\hat{\eta}_i) > T_\alpha\right) &\ge \prod_i P\left(\tau_{c,0,i} > T_\alpha\right) - \text{ SES}\\&\ge \prod_i \left(\frac{1}{\ln T_\alpha}\right)^{\text{cst}\,\lambda^9 \ln \lambda} - \text{ SES}\\&\ge (\mathrm{e}^{-\delta\beta})^{|I|} - \text{ SES}\end{aligned} \tag{5.53}$$

uniformly in $\hat{\eta}(0)$, T and $(z_i)_{i\in I}$.

Step 2. We deduce from this last estimate a lower bound for the probability that τ_1, the first time when all the particles $i \in I$ that never fell asleep are outside $[z_i]_{3\mathrm{e}^{-\delta\beta}\sqrt{T_\alpha}}$ satisfies

$$\tau_1 \le T_\alpha \wedge \inf\left\{\tau_{[z_i]_{4\lambda}}(\hat{\eta}_i) : i \in I, w_i(T_\alpha) > 0\right\}. \tag{5.54}$$

To do so, we assume without loss of generality that $\mathrm{e}^{-2\delta\beta}T_\alpha$ is larger than $\mathrm{e}^{D\beta}$ and we divide the time interval $[0, T_\alpha]$ into $\mathrm{e}^{\delta\beta/2}$ subintervals of length $\mathrm{e}^{-\delta\beta/2}T_\alpha$. By Theorem 3.2.4 applied at the end of each of these subintervals, a particle i that does not fall asleep during the time interval $[0, T_\alpha]$ is in $[z_i]_{3\mathrm{e}^{-\delta\beta}\sqrt{T_\alpha}}$ with a probability smaller than $\mathrm{e}^{-\delta\beta}$, so that, by the Markov property applied at the end of each of the subintervals,

$$\begin{aligned}&P\left(\forall i \in I, w_i(T_\alpha) > 0 \text{ or } \tau_1 < \tau_{[z_i]_{4\lambda}}(\hat{\eta}_i) \wedge T_\alpha\right)\\&\quad\ge P\left(\forall i \in I, \tau_{[z_i]_{4\lambda}}(\hat{\eta}_i) > T_\alpha\right) - \left(|I|\mathrm{e}^{-\delta\beta}\right)^{\mathrm{e}^{\delta\beta/2}}\\&\quad\ge (\mathrm{e}^{-\delta\beta})^{|I|} - \text{ SES.}\end{aligned} \tag{5.55}$$

Using the non-superdiffusivity property (Theorems 3.1.1 and 3.2.3) and the fact that $\|z_i - \hat{\eta}_i(0)\|_2 \le \frac{1}{2}\sqrt{T}$, we have also the stronger result

$$P\left(\forall i \in I: \begin{array}{l} w_i(T_\alpha) > 0 \text{ or } \tau_1 < \tau_{[z_i]_{4\lambda}}(\hat{\eta}_i) \wedge T_\alpha \\ \text{and } [z_i]_{\mathrm{e}^{-\delta\beta}\sqrt{T_\alpha}} \subset B_2\left(\hat{\eta}_i(\tau_1), \dfrac{3}{4}\sqrt{T}\right)\end{array}\right) \ge (\mathrm{e}^{-\delta\beta})^{|I|} - \text{ SES} \tag{5.56}$$

uniformly in $\hat{\eta}(0)$, T and $(z_i)_{i\in I}$.

Step 3. We next give a lower bound for the probability that all the particles $i \in I$ that never fell asleep are in $[z_i]_{e^{-\delta\beta/2}\sqrt{T_\alpha}}$ at some time T_2 smaller than $\tau_{[z_i]_{4\lambda}}$ and contained during the time interval $[T - 2e^{-\delta\beta/2}T_\alpha, T - e^{-\delta\beta/2}T_\alpha]$, provided that $\mathcal{T}_{\alpha,\lambda} > T$. To do so, we will use the Markov property at time τ_1 and Theorem 3.2.5 with

$$\begin{aligned} &\delta' := \delta, \\ &T' := T - e^{-\delta\beta/2}T_\alpha - 16e^{-2\delta\beta}T_\alpha - \tau_1, \\ &(\Lambda'_i) := \left([z_i]_{e^{-\delta\beta}\sqrt{T_\alpha}}\right), \end{aligned} \tag{5.57}$$

instead of δ, T and (Λ_i). Conditionally on

$$A := \left\{\forall i \in I\colon [z_i]_{e^{-\delta\beta}\sqrt{T_\alpha}} \subset B_2\left(\hat{\eta}_i(\tau_1), \frac{3}{4}\sqrt{T}\right)\right\} \tag{5.58}$$

and for δ small enough, the hypotheses (3.9) and (3.11) are easily verified at time τ_1 instead of 0, and we get, with θ_{τ_1} denoting the usual shift in the trajectories of the Markov process,

$$\begin{aligned} &P\left(T - e^{-\delta\beta/2}T_\alpha > \mathcal{T}_{\alpha,\lambda} \text{ or } \forall i \in I\colon \begin{array}{l} w_i(T - e^{-\delta\beta/2}T_\alpha) > 0 \quad \text{or} \\ \tau_1 + \tau_{\Lambda'_i}(\hat{\eta}_i) \circ \theta_{\tau_1} \in \left[T' + \tau_1, T - e^{-\delta\beta/2}T_\alpha\right] \end{array} \;\middle|\; A\right) \\ &\quad \geq \left(\frac{4T_\alpha}{e^{3\delta\beta}T}\right)^{|I|} - \text{SES} \end{aligned} \tag{5.59}$$

uniformly in $\hat{\eta}(0)$, T and $(z_i)_{i\in I}$. We then define τ_2 as the first time after time τ_1 when one of the particles $i \in I$ that never fall asleep reaches $[z_i]_{e^{-\delta\beta}\sqrt{T_\alpha}}$, and we use the non-superdiffusivity property to get

$$\begin{aligned} &P\left(\tau_2 > \mathcal{T}_{\alpha,\lambda} \text{ or } \forall i \in I\colon \begin{array}{l} w_i(\tau_2) > 0 \text{ or} \\ \left\{\begin{array}{l} \tau_2 \in \left[T - 2e^{-\delta\beta/2}T_\alpha, T - e^{-\delta\beta/2}T_\alpha\right] \\ \tau_2 < \tau_1 + \tau_{[z_i]_{4\lambda}}(\hat{\eta}_i) \circ \theta_{\tau_1} \\ \hat{\eta}_i(\tau_2) \in [z_i]_{e^{-\delta\beta/2}\sqrt{T_\alpha}} \end{array}\right. \end{array} \;\middle|\; A\right) \\ &\quad \geq \left(\frac{4T_\alpha}{e^{3\delta\beta}T}\right)^{|I|} - \text{SES}. \end{aligned} \tag{5.60}$$

Together with (5.56) and the Markov property at time τ_1, this gives

$$\begin{aligned} &P\left(\tau_2 > \mathcal{T}_{\alpha,\lambda} \text{ or } \forall i \in I\colon \begin{array}{l} w_i(\tau_2) > 0 \quad \text{or} \\ \left\{\begin{array}{l} \tau_2 \in \left[T - 2e^{-\delta\beta/2}T_\alpha, T - e^{-\delta\beta/2}T_\alpha\right] \\ \tau_2 < \tau_{[z_i]_{4\lambda}}(\hat{\eta}_i) \\ \hat{\eta}_i(\tau_2) \in [z_i]_{e^{-\delta\beta/2}\sqrt{T_\alpha}} \end{array}\right. \end{array}\right) \\ &\quad \geq \left(\frac{4T_\alpha}{e^{4\delta\beta}T}\right)^{|I|} - \text{SES} \end{aligned} \tag{5.61}$$

uniformly in $\hat{\eta}(0)$, T and $(z_i)_{i\in I}$. This means that, with a probability of order $e^{-4\delta\beta}T_\alpha/T$, at time τ_2 the particles i that never fell asleep are at a diffusive distance (on a time scale of order T_α) of their targets $[z_i]_{4\lambda}$, have never reached these targets before, and still have ahead a time of order T_α until time T.

Step 4. We will be working once again on time scale T_α. We define at time τ_2 the clouds of potentially interacting particles on time scale T_α, we call, for $i \in I$, $S''_{2,i}$ the set of the circumscribed rectangles of the clusters of $\hat\eta(\tau_2)$ made of particles contained in the cloud to which i belongs, and we set

$$\begin{aligned}\underline{S}'_{2,i} &:= \underline{S}''_{2,i} \cup \left\{[z_j]_{4\lambda} : j \in I\right\},\\ \underline{S}_{2,i} &:= g_5(\underline{S}'_{0,i}).\end{aligned} \tag{5.62}$$

Here, the union with the targets is made *before* applying the operator g_5, which is different from what was done previously, for example, in Step 1. Provided $\mathcal{T}_{\alpha,\lambda} > \tau_2$, we have

$$|\underline{S}_{2,i}| \le 2\lambda \quad \text{and} \quad prm(\underline{S}'_{2,i}) \le 4\lambda + \lambda \times 33\lambda \le 34\lambda^2, \tag{5.63}$$

so that, by (5.12),

$$prm(\underline{S}_{2,i}) \le 34\lambda^2 + 4 \times 5 \times 2\lambda \le 35\lambda^2 \tag{5.64}$$

and

$$|\partial\left[S_{2,i}\right]_1| \le 35\lambda^2 + 8 \times 2\lambda \le 36\lambda^2 \tag{5.65}$$

for β large enough. We can then choose $|I|$ sites $(z'_i)_{i\in I}$ such that

$$\begin{cases}\inf\limits_{i\in I}\inf\limits_{s\in S_{2,i}} \|z'_i - s\|_\infty > 3,\\ \inf\limits_{i\neq j} \|z'_i - z'_j\|_1 > 1,\\ \sup\limits_{i\in I} \|z_i - z'_i\|_\infty \le 19\lambda^2,\end{cases} \tag{5.66}$$

use the Markov property at time τ_2, and follow the arguments that led to (5.46) and (5.48), to get

$$\begin{aligned}&P\left(T - 19\lambda^2 > \mathcal{T}_{\alpha,\lambda} \text{ or } \forall i \in I\colon \begin{matrix} w_i(T-19\lambda^2) > 0 \quad \text{or} \\ \begin{cases} T - 19\lambda^2 < \tau_{[z_i]_{4\lambda}}(\hat\eta_i) \\ \hat\eta_i(T-19\lambda^2) = z'_i \end{cases}\end{matrix}\right)\\ &\quad \ge \left(\frac{1}{e^{5\delta\beta}T}\right)^{|I|} - \text{ SES}\end{aligned} \tag{5.67}$$

uniformly in $\hat\eta(0)$, T and $(z_i)_{i\in I}$.

Step 5. Finally, consider the clouds of potentially interacting particles defined at time $T_3 := T - 19\lambda^2$, call, for $i \in I$, $S'_{3,i}$ the set of the circumscribed rectangles of the clusters made of particles that are in the cloud containing i, and define $\underline{S}_{3,i} := g_5(\underline{S}'_{3,i})$. Since $prm(\underline{S}_{3,i}) \le 24\lambda$ (provided $\mathcal{T}_{\alpha,\lambda} > T_3$) and $|\partial\,[z_i]_{4\lambda}| \ge 32\lambda$, the rectangles in $\underline{S}_{3,i}$ cannot cover $[z_i]_{4\lambda}$. Consequently, the particles i in z'_i at time T_3 can bypass these separated rectangles to reach their targets $[z_i]_{4\lambda}$ at time T with a non-exponentially small probability. Together with (5.67) and the Markov property at time T_3, this implies that, uniformly in $\hat\eta(0)$, T and $(z_i)_{i\in I}$,

$$P\left(T > \mathcal{T}_{\alpha,\lambda} \text{ or } \forall i \in I\colon \left\lfloor \tau_{[z_i]_{4\lambda}}(\hat\eta_i)\right\rfloor = \lfloor T\rfloor \text{ or } w_i(T) > 0\right) \ge \left(\frac{1}{e^{6\delta\beta}T}\right)^{|I|} - \text{ SES} \tag{5.68}$$

concluding the proof. $\square$

6. Application to the three regimes

Both the stable and the unstable regimes can be seen as limit cases of the metastable regime. Indeed, in the metastable regime $U < \Delta < 2U$ small clusters (say, squares representing droplets of liquid) in a low-density background (rarefied gas) have a tendency to shrink and disappear whereas the large clusters have a tendency to grow. This behavior defines a critical droplet size $\ell_c = \lceil U/(2U - \Delta)\rceil$, giving rise to interesting phenomena. The cases of the unstable gas ($\Delta < U$) and the stable gas ($\Delta > 2U$) correspond to the limiting cases where $\ell_c = 1$ and $\ell_c = \infty$, respectively.

Both the unstable and the metastable regimes have a "stable equilibrium", given by a single large droplet of liquid (with an equilibrium Wulff shape) in a sea of vapor (see Dobrushin, Kotecký and Shlosman [4]). Indeed, if the total number of particles is fixed (i.e., an equilibrium given by the canonical ensemble), then for values of the thermodynamic parameters corresponding to the coexistence of the two phases there is a segregation phenomenon. On the other hand, in the stable regime $\Delta > 2U$ (corresponding to a non-vanishing magnetic field in the spin-language), the equilibrium is given by a weakly correlated gaseous phase. At first sight, a paradoxical behavior occurs. The larger the inverse temperature β (which means the stronger the attractive interaction), the more uncorrelated the unique equilibrium phase. It is not difficult, however, to understand the reason behind this behavior. Large β implies that a good description of our system *at equilibrium* can be given in terms of an ensemble of very rarefied contours (in the spirit of low-temperature expansions). It is clear that in the metastable regime, even though initially we have a very rarefied gas, during the evolution we have to face the problem of strong attractive interaction because β is large. In the regime $\Delta < U$, on the other hand, no kind of quasi-stationary situation can be established and we expect that the system goes fast to the segregated situation.

The absence of superdiffusivity for Kawasaki dynamics has been established up to time $\mathcal{T}_{\alpha,\lambda} \wedge T_\alpha^2$ in Theorems 3.1.1 and 3.2.3. As far as the spread-out estimates are concerned, Theorems 3.2.4, 3.2.5 and 3.3.1 can be applied only to active particles up to time $\mathcal{T}_{\alpha,\lambda}$. Indeed, we derived upper (lower) estimates of intersections (unions) of events involving anomalous concentration, activity and localization of particles. Since activity is a notion that depends on the parameter D, which assumes different values in the three different regimes (as explained in Section 1.2), we need to discuss the applicability and the consequences of our results in each of these regimes. This is done in Sections 6.1–6.3.

6.1. Stable gas

When $\Delta > 2U$ we choose $D \in (2U, \Delta)$. The simple exclusion process ($U = 0$) is part of this regime.

Proposition 6.1.1. *For $t > 0$, let $\mathcal{A}(t)$ be the event that all particles are active up to time t. Then*

$$P(\mathcal{A}(t) \text{ or } \mathcal{T}_{\alpha,\lambda} < t) = 1 - \text{ SES}. \tag{6.1}$$

Proof. By Definition 2.3.2, prior to time $e^{D\beta}$ all particles are active. Assume now that some particle i loses its freedom at some time $t < \mathcal{T}_{\alpha,\lambda}$. Then it suffices to show that i will recover its freedom with probability $1 -$ SES before time $t + e^{D\beta}$. By the Markov property, we can restrict ourselves to the special case $t = 0$. By Proposition 5.2.2, which states that $\hat{\eta}$ is a QRW(α, T_α)-process, and by the non-superdiffusivity property, we can further restrict ourselves to considering the system reduced to the cloud of potentially interacting particles on time scale T_α to which i belongs.

Pick $\delta, \delta_0 > 0$ such that $D - (2U + \delta_0) = \delta$, set $t_n := n\mathrm{e}^{(2U+\delta_0)\beta}$ and $0 \le n \le \mathrm{e}^{\delta\beta} - 1$. Consider, at any time t_n, the set $\underline{S}'_n$ of the circumscribed rectangles of the clusters of the cloud, define $\underline{S}_n := g_5(\underline{S}'_n)$, let $\tau_{r,n}$ be the first time after time t_n when $\underline{S}_n$ does not contain any particles, i.e., the first time after time t_n when η is not $2U$-reducible with respect to $\underline{S}_n$, and denote by $\tau_{c,n}$ the associated collision time. Then, by (5.44) (established uniformly in the initial configuration for the special case $t_n = t_0 = 0$, but valid for any t_n by the Markov property), we have

$$P\left(\tau_{r,n} < t_n + \mathrm{e}^{(2U+\delta_0)\beta} \text{ and } \tau_{r,n} < \tau_{c,n}\right) \ge \mathrm{e}^{-\delta\beta/3}, \tag{6.2}$$

and so, by the Markov property applied at $t_0, t_1, t_2, \ldots$, we obtain

$$P\left(i \text{ is not free on the whole time interval } [0, \mathrm{e}^{D\beta}]\right) \le \left(1 - \mathrm{e}^{-\delta\beta/3}\right)^{\mathrm{e}^{\delta\beta}} = \text{SES}. \quad \Box \tag{6.3}$$

Proposition 6.1.1 implies that, in the stable regime, our spread-out estimates can be stated in a stronger version: the intersection with $\{w_i(T) = 0\}$ can be removed from the statement of Theorem 3.2.4 and the unions with $\{w_i(T) > 0\}$ can be removed from the statements of Theorems 3.2.5 and 3.3.1.

Next, applying the spread-out estimates, we can control the first time of anomalous concentration that limits our time horizon. We denote by $\mathcal{X}_N(\alpha, \lambda)$ the set of configurations without α-anomalous concentration, so that $\mathcal{T}_{\alpha,\lambda}$ is the hitting time of the complement of $\mathcal{X}_N(\alpha, \lambda)$.

Proposition 6.1.2. *If $\hat{\eta}(0) \in \mathcal{X}_N(\frac{\alpha}{5}, \lambda)$, then*

$$P\left(\mathcal{T}_{\alpha,\lambda} \ge T_\alpha^2 (T_\alpha^{-1/2} \wedge \mathrm{e}^{-D\beta})\right) = 1 - \mathit{SES}. \tag{6.4}$$

Proof. Note that $\hat{\eta}(0) \in \mathcal{X}_n(\frac{\alpha}{5}, \lambda)$ implies that $|\mathcal{U}(\hat{\eta}(0))|_\Lambda|| < \frac{\lambda}{4}$ for any box Λ with $|\Lambda| < \mathrm{e}^{(\Delta - \frac{\alpha}{20})\beta}$, and that $\mathcal{T}_{\alpha,\lambda} > \mathcal{T}_{\frac{\alpha}{5},\lambda} > 0$. Consequently,

$$P\left(\mathcal{T}_{\alpha,\lambda} < T_{\frac{\alpha}{19}}\right) = \text{SES}, \tag{6.5}$$

since such an event implies that there is at least one particle with superdiffusive behavior before $\mathcal{T}_{\alpha,\lambda}$.

For larger T such that

$$T \le T_\alpha^2 (T_\alpha^{-1/2} \wedge \mathrm{e}^{-D\beta}), \tag{6.6}$$

the event $\{\lfloor \mathcal{T}_{\alpha,\lambda} \rfloor = \lfloor T \rfloor\}$ has probability SES. This follows from the upper bound in the spread-out property in Theorem 3.2.4 applied to a single box $|\Lambda| = \mathrm{e}^{(\Delta - \frac{\alpha}{4})\beta}$. Indeed, the event $\{\lfloor \mathcal{T}_{\alpha,\lambda} \rfloor = \lfloor T \rfloor\}$ implies that with probability $1-$ SES at time $T - 1$ there are $\frac{\lambda}{4}$ particles in the box $[\Lambda]_\lambda$. On the one hand, the n particles that have a non- SES probability to be in $[\Lambda]_\lambda$ at time $T - 1$ are contained in a box $[\Lambda]_{\sqrt{T}\mathrm{e}^{\delta\beta}}$, for δ arbitrarily small, and so they are at most

$$n \le \frac{\lambda}{4} \left\lceil \frac{T\mathrm{e}^{\delta\beta}}{\mathrm{e}^{(\Delta - \frac{\alpha}{20})\beta}} \right\rceil, \tag{6.7}$$

since $\hat{\eta}(0) \in \mathcal{X}_n(\frac{\alpha}{5}, \lambda)$. On the other hand, the probability p that $\frac{\lambda}{4}$ given particles are all in $[\Lambda]_\lambda$ at time $T - 1$ is estimated, via Theorem 3.2.4 for $T > T_{\frac{\alpha}{5}}$, by

$$p \leq \left(\frac{|\Lambda| e^{\delta\beta}}{T}\right)^{\frac{\lambda}{4}}, \tag{6.8}$$

since $T \leq T_\alpha^2 (T_\alpha^{-1/2} \wedge e^{-D\beta})$ implies that condition (3.7) is satisfied for $|\Lambda| = e^{(\Delta - \frac{\alpha}{4})\beta}$. We have

$$\binom{n}{\frac{\lambda}{4}} \left(\frac{|\Lambda| e^{\delta\beta}}{T}\right)^{\frac{\lambda}{4}} \leq \left(\frac{\lambda}{4} \left\lceil \frac{T e^{\delta\beta}}{e^{(\Delta - \frac{\alpha}{20})\beta}} \right\rceil \frac{|\Lambda| e^{\delta\beta}}{T}\right)^{\frac{\lambda}{4}} = \text{SES}, \tag{6.9}$$

and so we conclude that

$$P\left(T_{\alpha/5} \leq \mathcal{T}_{\alpha,\lambda} \leq T_\alpha^2 (T_\alpha^{-1/2} \wedge e^{-D\beta})\right) = \text{SES}, \tag{6.10}$$

since Λ_β is only exponentially large in β. Together with (6.5) and the fact that $T_{\frac{\alpha}{5}} < T_{\frac{\alpha}{19}}$, this completes the proof. □

Finally, if the starting configuration is chosen according to the equilibrium measure ν_N defined in (2.26), then the first time of anomalous concentration is larger than any exponential in β:

Proposition 6.1.3. *For all $C > 0$,*

$$P_{\nu_N}(\mathcal{T}_{\alpha,\lambda} \geq e^{C\beta}) = 1 - \mathit{SES}. \tag{6.11}$$

Proof. See Appendix B. □

There are interesting problems for the stable gas regime that are not in the range of applications of our results. An example is the evolution of configurations with anomalous concentration, such as the evaporation of a large droplet.

6.2. Unstable gas

When $\Delta < U$ we choose $D \in (0, U)$. This is the regime in which the density is so high that the condensation starts immediately and all the clusterized particles fall asleep. We expect to see in a time $e^{(\Delta+\delta)\beta}$ an anomalous concentration, after which our claims are empty. Actually, our estimates only describe the gas in this initial transient period, i.e., in the short time of initial condensation. However, we note that starting from any configuration without anomalous concentration (i.e., such that $\mathcal{T}_{\alpha,\lambda} > 0$) our spread-out estimates hold up to time T_α (for active particles), and so does the non-superdiffusivity property, by Proposition 5.2.2.

6.3. Metastable gas

When $U < \Delta < 2U$ we choose $D \in (U, \Delta)$. This is the more interesting regime where active and sleeping particles are both present. In this case a situation with a homogeneous and rarefied but supersaturated vapor (i.e., with density larger than the value $\exp(-2U\beta)$ at the condensation point) can be established representing an apparent equilibrium. It is easy to see that $\mu_{\mathcal{R}}$, the Gibbs measure conditioned on a suitable set of configurations $\mathcal{R}$ without large clusters, is quasi-stationary, in the sense that at times much prior to the exiting time of $\mathcal{R}$ the corresponding averages of key observables are almost invariant (see Olivieri and Vares [14], Theorem 6.28). Consequently, the central problem when $U < \Delta < 2U$ is describing starting from an initial configuration distributed according to $\mu_{\mathcal{R}}$, the escape from metastability, i.e., the first exit from $\mathcal{R}$. In particular, in order to get a good upper bound on the escape time, we have to construct a suitable escape event. This will involve a nucleation pattern corresponding to the appearance of a 2×2 droplet, which subsequently grows along a sequence of quasi-square droplets. The construction of this escape event is based on the notion of QRWs, in particular, on their properties as given in Theorems 3.2.3–3.2.5 and 3.3.1. This will be done in the forthcoming papers, Gaudillière, den Hollander, Nardi, Olivieri and Scoppola [8,9], combining the analysis of the sleeping particles developed for the local interaction model in den Hollander, Olivieri and Scoppola [11] with the estimates obtained in the present paper for the active particles of the gas.

In Section 2.3 we introduced the special permutation rule in order to minimize the number of sleeping particles: once again, the fewer they are, the stronger are our results. As far as the first time of anomalous concentration is concerned, we will prove an *a priori* estimate analogous to (6.11): starting from $\mu_{\mathcal{R}}$ it will be possible to exclude any anomalous concentration up to the escape time from metastability, which is much larger than T_α^2.

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Appendix A. Freidlin–Wentzell theory for a slowly growing state space

Proof of Lemma 5.3.2. We observe that up to time $\tau_+ \wedge \tau_c$ the evolution of the system inside S is independent of its evolution outside S. We distinguish between the cases $k = 0, 1, 2$.

• CASE $k = 0$.

If $\hat{\eta}(0)$ is 0-reducible, then there is a sequence of configurations $\mathcal{U}(\hat{\eta}) = \eta_0, \eta_1, \ldots, \eta_n$ in $\mathcal{X}$, each of them obtained from the previous one by a displacement of a single particle to a nearest-neighbor vacant site, such that

$$\begin{cases} \mathrm{H}_{\underline{S}}(\eta_n) < \mathrm{H}_{\underline{S}}(\eta), \\ \sup_j \mathrm{H}_{\underline{S}}(\eta_j) \le \mathrm{H}_{\underline{S}}(\eta). \end{cases} \tag{A.1}$$

Without loss of generality we may assume that $\mathrm{H}_{\underline{S}}(\eta_{j+1}) \le \mathrm{H}_{\underline{S}}(\eta_j)$ for all $j < n$. This implies that the number of particles inside S does not increase along this sequence. We may further assume that η_n is the first configuration along this sequence where a 0-irreducible configuration is reached or the gas surrounding $[S]_1$ is enriched. This implies that the number of particles inside S is a constant $a \le \lambda$ from η_0 to η_n. Finally, we may assume that n is smaller than or equal to the total number of configurations with $a \le \lambda$ particles inside S, so that, using the isoperimetric inequality, we get

$$n \le \binom{|S|}{a} \le \binom{\lambda^{2\kappa}}{a} \le \binom{\lambda^{2\kappa}}{\lambda} \le \lambda^{2\kappa\lambda}. \tag{A.2}$$

Under these assumptions, the probability that the conditioned process restricted to S follows this sequence in a time $e^{\frac{\delta}{2}\beta}$ is larger than or equal to (recall (2.1))

$$\exp\left\{-\mathrm{cst}\,\lambda^{2\kappa\lambda}\right\} \geq e^{-\frac{\delta}{4}\beta} - \text{ SES} \tag{A.3}$$

uniformly in $\underline{S}$ and $\hat{\eta}(0)$. We now divide the time interval $[0, e^{\delta\beta}]$ into $e^{\frac{\delta}{2}\beta}$ intervals of length $e^{\frac{\delta}{2}\beta}$. By the Markov property, we get

$$\begin{aligned} &P\left(\forall t \leq e^{\delta\beta},\ \hat{\eta}(t) \text{ is 0-reducible and } \tau_+ \neq t \,\middle|\, \tau_c > e^{\delta\beta} \wedge \tau_+\right) \\ &\quad \leq \left(1 - e^{-\frac{\delta}{4}\beta}\right)^{e^{\frac{\delta}{2}\beta}} + \text{ SES} \leq \text{ SES} \end{aligned} \tag{A.4}$$

uniformly in $\underline{S}$ and $\hat{\eta}(0)$.

If $\hat{\eta}(0)$ is 0-irreducible and $\tau_+ > 0$, then, conditionally on $\{\tau_c \geq \tau_+\}$, the system has to perform a move inside S of cost at least U to enrich the gas surrounding $[S]_1$. Since, up to time τ_c, the particles inside S cannot be more than λ, this move occurs within a time $e^{(U-\delta)\beta}$ with probability larger than or equal to

$$\lambda e^{-\delta\beta} \leq e^{-\delta'\beta} + \text{ SES} \tag{A.5}$$

uniformly in $\underline{S}$ and $\hat{\eta}(0)$.

Before proceeding with the proof for the cases $k = 1$ and $k = 2$, we define the U-cycles associated with $\underline{S}$ and prove some of their properties.

Definition A.0.1. Let $\underline{S} \in \mathcal{R}$ be such that $g_5(\underline{S}) = \underline{S}$. Suppose that C is a set of configurations satisfying:

(i) each η in C is 0-irreducible;
(ii) with positive probability C can be completely visited by the process $\mathcal{U}(\hat{\eta})$ without it going outside C;
(iii) there is a configuration $\eta' \notin C$ that can be obtained from some configuration in C by a displacement at cost U of a single particle inside S to a nearest-neighbor site;
(iv) any set of configurations that contains C and satisfies (i)–(iii) is equal to C.

Then we say that C is a U-cycle with exit η'.

Remark. We will not actually need the maximality property (iv). We added it here to recover, in our special case, the analogue of the cycles for the local model of den Hollander, Olivieri and Scoppola [11].

Lemma A.0.2. *Let $\underline{S} \in \mathcal{R}$ be such that $g_5(\underline{S}) = \underline{S}$ and $prm(\underline{S}) \leq \lambda^\kappa$, and let C be a U-cycle with exit η'. Then, for any $\hat{\eta}(0)$ in C such that there are no particles inside $\partial\,[S]_1$, no particles inside $[S]_1 \setminus S$, and not more than λ particles inside S at time $t = 0$, and for any $\delta > 0$,*

$$\begin{aligned} &P\left(\eta|_S(\tau) = \eta'|_S, \tau \leq e^{(U+\delta)\beta} \,\middle|\, \tau_c \geq e^{(U+\delta)\beta} \wedge \tau_+\right) \\ &\quad \geq \exp\left\{-\mathrm{cst}\,\lambda^{2\kappa\lambda} \ln\lambda\right\} - \mathit{SES} \end{aligned} \tag{A.6}$$

uniformly in $\hat{\eta}(0)$, $\underline{S}$, C and η', with τ the exit time from C.

Proof. On the one hand, conditionally on $\{\tau_c \geq e^{(U+\delta)\beta} \wedge \tau_+\}$, the probability of the event $\{\tau \leq e^{\frac{\delta}{2}\beta}\}$ is larger than or equal to

$$\left(\frac{\mathrm{cst}}{4\lambda}\right)^{\lambda^{2\kappa\lambda}} e^{-U\beta} \geq e^{-(U+\frac{\delta}{3})\beta} - \text{ SES} \tag{A.7}$$

uniformly in $\hat{\eta}(0)$ $\underline{S}$, C and η'. Hence, dividing the time interval $[0, e^{(U+\delta)\beta}]$ into $e^{(U+\frac{\delta}{2})\beta}$ intervals of length $e^{\frac{\delta}{2}\beta}$ and using the Markov property, we get

$$P\left(\tau > e^{(U+\delta)\beta} \,\middle|\, \tau_c > e^{(U+\delta)\beta} \wedge \tau_+\right) \leq \left(1 - e^{-(U+\frac{\delta}{3})\beta}\right)^{e^{(U+\frac{\delta}{2})\beta}} \leq \text{ SES.} \tag{A.8}$$

On the other hand, conditionally on $\{\tau_c \geq \tau_+\}$, the probability of the event $\{\eta|_S(\tau) = \eta'|_S\}$ is larger than or equal to

$$\sum_{j\geq 0}\left(1 - \lambda e^{-U\beta}\right)^j \left(\frac{\mathrm{cst}}{4\lambda}\right)^{\lambda^{2\kappa\lambda}} e^{-U\beta} \geq \exp\left\{-\mathrm{cst}\,\lambda^{2\kappa\lambda}\ln\lambda\right\} - \text{ SES} \tag{A.9}$$

uniformly in $\hat{\eta}(0)$, $\underline{S}$, C and η'. □

We are now ready to conclude the proof of Lemma 5.3.2 for the cases $k = 1$ and $k = 2$.

• Case $k = 1$. If $\hat{\eta}(0)$ is U-reducible, then it is easy to see that there exists a sequence of not more than $\lambda^{2\kappa\lambda}$ U-cycles and configurations such that:

(1) $H_{\underline{S}}$ does not increase between two successive configurations;
(2) each cycle C is preceded by a configuration it contains and followed by an exit configuration η;
(3) the sequence ends in a configuration η_n that is the first along this sequence where a U-irreducible configuration is reached or the gas surrounding $[S]_1$ is enriched.

Using Lemma A.0.2, we can estimate the probability that the conditioned process restricted to S follows this sequence in time $e^{(U+\frac{\delta}{2})\beta}$ from below by (recall (2.1))

$$\exp\left\{-\mathrm{cst}\,\lambda^{4\kappa\lambda}\ln\lambda\right\} \geq e^{-\frac{\delta}{4}\beta} - \text{ SES} \tag{A.10}$$

uniformly in $\underline{S}$ and $\hat{\eta}(0)$. After dividing the time interval $[0, e^{(U+\delta)\beta}]$ into $e^{\frac{\delta}{2}\beta}$ intervals of length $e^{(U+\frac{\delta}{2})\beta}$ and using the Markov property, we get

$$P\left(\forall t \leq e^{(U+\delta)\beta},\ \hat{\eta}(t) \text{ is } U\text{-reducible and } \tau_+ \neq t \,\middle|\, \tau_c > e^{(U+\delta)\beta} \wedge \tau_+\right)$$
$$\leq \left(1 - e^{\frac{\delta}{4}\beta}\right)^{e^{\frac{\delta}{2}\beta}} \leq \text{ SES} \tag{A.11}$$

uniformly in $\underline{S}$ and $\hat{\eta}(0)$.

If $\hat{\eta}(0)$ is U-irreducible, then, conditionally on $\{\tau_c \geq \tau_+\}$, to enrich the gas surrounding $[S]_1$ the system has to perform a move inside S of cost at least $2U$, or (by the result for the case $k = 0$) two moves of cost U in a time smaller than $e^{\delta''\beta}$ for a given $\delta'' > 0$. Since, up to time τ_c, the particles inside S cannot be more than λ, this occurs within time $e^{(2U-\delta)\beta}$ with probability less than or equal to

$$\lambda e^{-(\delta-\delta'')\beta} \leq e^{-\delta'\beta} + \text{ SES} \tag{A.12}$$

uniformly in S and $\hat{\eta}(0)$, provided we choose δ'' such that $\delta - \delta'' > \delta'$.

• CASE $k = 2$. Using the fact that any cluster carries at least four particles that can only be separated at cost $2U$, the first probability estimate is once again obtained after dividing the time interval $[0, \mathrm{e}^{(2U+\delta)\beta}]$ into $\mathrm{e}^{(2U+\frac{\delta}{2})\beta}$ intervals of length $\mathrm{e}^{\frac{\delta}{2}\beta}$ and using the Markov property. The second probability estimate is obvious: the gas cannot be enriched if there are no particles inside S. □

Appendix B. An estimate on the canonical Gibbs measure

Proof of Proposition 6.1.3. Since ν_N is the invariant measure of the dynamics, and Λ_β is only exponentially large in β, it is enough to prove that

$$\nu_N\left(\eta \in \mathcal{X}_N : |\eta|_\Lambda| = a\right) \leq \text{ SES} \tag{B.1}$$

uniformly in $a \geq \lambda$ and for Λ a square box of volume $|\Lambda| = \mathrm{e}^{(\Delta-\frac{\alpha}{4})\beta}$.

Pick any such a and Λ. For any $\eta \in \mathcal{X} = \{0, 1\}^{\Lambda_\beta}$ and x in η, we let $\mathrm{cc}(x)$ be the connected component of x in η, i.e., either the cluster of η that contains x if $x \in \eta^{cl}$ or the singleton $\{x\}$ if $x \in \eta \setminus \eta^{cl}$. Let

$$A(\eta) := \{x \in \eta : \mathrm{cc}(x) \cap \Lambda \neq \emptyset\}. \tag{B.2}$$

We will show that, uniformly in a and Λ,

$$\nu_N\left(\eta \in \mathcal{X}_N : |A(\eta)| = a\right) \leq \text{ SES}, \tag{B.3}$$

which implies (B.1).

To prove (B.3), define

$$\begin{aligned} Z_N &:= \sum_{|\eta|=N} \exp\{-\beta \mathrm{H}(\eta)\}, \\ Z_{out} &:= \sum_{|\eta|=N-a} \exp\{-\beta \mathrm{H}(\eta)\}\, \mathbb{1}_{\{|A|=0\}}(\eta), \\ Z_{in} &:= \sum_{|\eta|=a} \exp\{-\beta \mathrm{H}(\eta)\}\, \mathbb{1}_{\{|A|=a\}}(\eta). \end{aligned} \tag{B.4}$$

Then, clearly,

$$\nu_N\left(\eta \in \mathcal{X}_N : |A(\eta)| = a\right) \leq \frac{Z_{out} Z_{in}}{Z_N} \tag{B.5}$$

and

$$\begin{aligned} Z_N &\geq \frac{Z_{out}\,(N-a)!\,\big(|\Lambda_\beta| - (N-a)\big) \times \cdots \times \big(|\Lambda_\beta| - (N-a) - (a-1)\big)}{N!} \\ &\geq Z_{out} \left(\frac{|\Lambda_\beta| - N}{N}\right)^a, \end{aligned} \tag{B.6}$$

i.e.,

$$Z_N \geq Z_{out} \left(\frac{1 - \mathrm{e}^{-\Delta\beta}}{\mathrm{e}^{-\Delta\beta}}\right)^a. \tag{B.7}$$

Next, we derive an upper bound on Z_{in} by making the following observations:

- given a positive integer n_i, any cluster of n_i particles that intersects Λ is covered by a tree with one leaf in Λ and with n_i vertices that are connected by edges linking nearest-neighbor occupied sites;
- a random walker on such a tree can visit the whole tree, starting from that leaf, in at most $3n_i - 3$ steps;
- there are $4^{3(n_i-1)}$ random walks of length $3n_i - 3$ on $\mathbb{Z}^2$ that start from a given point;
- a single cluster η of volume n_i has an energy $-2U|\eta| + \frac{U}{2}|\partial\eta| \geq -2U(n_i - 1)$;
- there are $\binom{a-1}{k-1}$ ways of writing $a - k$ as a sum of k integers.

In view of these observations, we choose U' such that $2U < \Delta - \frac{\alpha}{4} < 2U' < \Delta$, so that, for β large enough,

$$\begin{aligned} Z_{in} &\leq \sum_{k=1}^{a} \sum_{\substack{n_1+\cdots+n_k=a \\ n_1,\ldots,n_k\geq 1}} |\Lambda|^k \prod_i 4^{3(n_i-1)} \exp\{2U(n_i-1)\beta\} \\ &\leq \sum_{k=1}^{a} \sum_{\substack{n_1+\cdots+n_k=a-k \\ n_1,\ldots,n_k\geq 0}} \exp\left\{k\left(\Delta - \frac{\alpha}{4}\right)\beta + (a-k)\left(\frac{3\ln 4}{\beta} + 2U\right)\beta\right\} \\ &\leq \sum_{k=1}^{a} \binom{a-1}{k-1} \exp\left\{\left(\Delta - \frac{\alpha}{4}\right) a\beta\right\} \\ &\leq 2^{(a-1)} \exp\left\{\left(\Delta - \frac{\alpha}{4}\right) a\beta\right\} \\ &\leq \exp\left\{2U'a\beta\right\}. \end{aligned} \tag{B.8}$$

Together with (B.5) and (B.7), this last estimate gives

$$\nu_N\left(\eta \in \mathcal{X}_N : |A(\eta)| = a\right) \leq \left(\frac{\exp\{(2U' - \Delta)\beta\}}{1 - e^{-\Delta\beta}}\right)^{\lambda} = \text{SES} \tag{B.9}$$

and concludes the proof. □

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