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COVERS OF SURFACES WITH FIXED BRANCH LOCUS

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Given a connected smooth projective surface X over $\mathbb{C}$, together with a simple normal crossings divisor D on it, we study finite normal covers $Y \to X$ that are unramified outside D. Given further a fibration of X onto a curve C, we prove that the "height" of Y over C is bounded linearly in terms of the degree of $Y \to X$. We indicate how an arithmetic analog of this result, if true, can be auxiliary in proving the existence of a polynomial time algorithm that computes the mod-ℓ Galois representations associated to a given smooth projective geometrically connected surface over $\mathbb{Q}$. A precise conjecture is formulated.

Keywords: Branched covers; intersection theory; heights; Galois representations.

Mathematics Subject Classification 2000: 14E20, 14E22, 14E15

1. Introduction

In this paper, we suppose that the following data are given:

- a connected smooth projective surface X over $\mathbb{C}$;
- a simple normal crossings divisor D on X (i.e. all components of D are smooth, and they intersect transversally);
- a connected smooth projective curve C over $\mathbb{C}$;
- a flat morphism $h\colon X \to C$.

We emphasize that we do not require the fibers of h to be connected. We denote by U the complement of D in X. We are interested in finite étale covers $V \to U$ that are not necessarily connected; these are considered to be the "variable" in our set-up. Given a finite étale cover $V \to U$ denote by $\pi\colon Y \to X$ the normalization of X in the product of the function fields of the connected components of V. By "finiteness of integral closure," the map π is finite. As the topological fundamental group of U is finitely generated, cf. [12, Exposé II, Théorème 2.3.1], there are only finitely many $V \to U$ of a given degree. In particular, for a fixed degree, the height over C of the associated covers $Y \to X$ is bounded. Our aim is to prove an effective version of this result. Let $\rho\colon Y' \to Y$ be a minimal resolution of singularities of Y, and denote by $f\colon Y' \to C$ the composed morphism $h\pi\rho$. Note that Y' and Y are projective and flat over C (C being a Dedekind scheme).

Theorem 1.1. *Let $h\colon X \to C$ and $U \subset X$ be given as above. Then there is an integer c such that for all finite étale $\pi\colon V \to U$ we have, in the notation as above*:

$$|\deg \det \mathrm{R}^{\cdot} f_* O_{Y'}| \leq c \cdot \deg(\pi)\,.$$

Here $\det \mathrm{R}^{\cdot} f_* O_{Y'}$ stands for the determinant of cohomology of $O_{Y'}$, cf. [3]. This is an invertible sheaf on C with $c_1(\det \mathrm{R}^{\cdot} f_* O_{Y'}) = c_1(\mathrm{R}^0 f_* O_{Y'}) - c_1(\mathrm{R}^1 f_* O_{Y'})$ in the Chow ring of C. According to [16, Theorem 3.6(v)], the degree $\deg \det \mathrm{R}^{\cdot} f_* O_{Y'}$ is non-negative if the fibers of f are connected and the arithmetic genus of the fibers of f is positive.

Our proof uses the Grothendieck–Riemann–Roch theorem, intersection theory on the normal surface Y and precise information about the minimal resolution of singularities of Y. In the last section we state an arithmetic analog of our result as a conjecture, motivated by a possible application to the complexity of counting points on reductions over finite fields of a fixed surface over $\mathbb{Q}$.

In a previous version of this text, our bound in Theorem 1.1 was quadratic in $\deg(\pi)$. Esnault and Viehweg showed us how to deduce a linear upper bound, with a precise constant, from Arakelov's inequality. This result is now included as Theorem 4.1. A closer inspection of our proof of Theorem 1.1 also gave a linear bound.

A good reason to include several proofs of the main result of this article is that this may help to find a proof that can be made to work in the arithmetic context. Since more than 25 years from now, attempts were made to prove an arithmetic analog of Arakelov's inequality, without success so far. Such an analog would have led to an effective version of Faltings's theorem (previously Mordell's conjecture). The same observations apply to the Bogomolov–Miyaoka–Yau inequality. On the other hand, we do believe that our conjectural analog (Conjecture 5.1) of Theorem 1.1 is not too hard to prove.

2. Preliminaries

Let $\pi\colon V \to U$ be as in the statement of Theorem 1.1, and let $V = \coprod_i V_i$ be the decomposition of V into connected components. Then also $Y = \coprod_i Y_i$, and $Y' = \coprod_i Y_i'$. Let $f_i := f|_{Y_i'}$. It follows that $\deg \det \mathrm{R}^{\cdot} f_* O_{Y'} = \sum_i \deg \det \mathrm{R}^{\cdot} f_{i,*} O_{Y_i'}$. Hence, in order to prove Theorem 1.1, we can and do assume that V is connected.

Denote by d the degree of $V \to U$. By [15, Lemma 2], the map $\pi\colon Y \to X$ is finite locally free of rank d. We write $D^{\rm sing}$ for the singular locus of D.

Lemma 2.1. *The singularities of Y occur in the inverse image under π of $D^{\rm sing}$. Furthermore, the map $\pi^{-1}(D - D^{\rm sing}) \to D - D^{\rm sing}$ is étale.*

Proof. We base our argument on a consideration of fundamental groups, as in [1, pp. 102–103]. Let x be a closed point of X lying on D but not on $D^{\rm sing}$. Locally for the analytic topology we identify a neighborhood W of x in X with the bi-disk $Z = \{(z_1, z_2) \in \mathbb{C}^2\colon |z_1| < 1, |z_2| < 1\}$, identifying x with the origin and D locally with the zero set of z_1. Let y be a point of Y mapping to x, and consider the connected component B of $\pi^{-1}W$ that contains y. We have then that $B - \pi^{-1}(D) \to W - D$ is a connected finite degree topological covering. Thus $\Gamma = \pi_*(\pi_1(B - \pi^{-1}(D)))$ is a subgroup of finite index of $\pi_1(W - D)$. The latter is infinite cyclic; let e be the index of the subgroup Γ. As the map $W \to W$ given by $(z_1, z_2) \mapsto (z_1^e, z_2)$ is a connected cover of W, homeomorphic above $W - D$ to the covering $B - \pi^{-1}(D) \to W - D$, we have by a theorem of Grauert–Remmert (cf. [11, Exposé XII, Théorème 5.4]) that B itself is analytically isomorphic to W and the holomorphic map $B \to W$ equivalent to the given map $W \to W$. We deduce that Y is regular above $D - D^{\rm sing}$ and that $\pi^{-1}(D - D^{\rm sing}) \to D - D^{\rm sing}$ is étale. $\square$

Write $D = \sum_{i \in I} D_i$ for the decomposition of D into prime components, and write $\pi^{-1}(D_i) = \sum_{j \in J_i} D_{ij}$ for the decomposition into prime components of the inverse image with reduced structure under π of a D_i. For $i \in I$ and $j \in J_i$ denote by e_{ij} the ramification index of π along D_{ij} (i.e. the ramification index of π at the generic point of D_{ij}) and denote by f_{ij} the degree of D_{ij} over D_i. For each $i \in I$ we have $\sum_{j \in J_i} e_{ij} f_{ij} = d$. If x is a closed point on X and y is a point of Y mapping to x we denote by d_y the rank of the completed local ring $\widehat{O}_{Y,y}$ as a free module over $\widehat{O}_{X,x}$. For all closed points x on X we have $\sum_{y\colon y \mapsto x} d_y = d$.

A point on a complex surface is said to have type $A_{n,q}$ if the complete local ring at that point is isomorphic as a $\mathbb{C}$-algebra to the complete local ring at the image of $(0,0)$ of the quotient of $\mathbb{C}^2$ under the action $(w_1, w_2) \mapsto (\zeta_n w_1, \zeta_n^q w_2)$, where n and q are integers with $n > 0$, $\gcd(n, q) = 1$ and where $\zeta_n = \exp(2\pi i/n)$. For $n > 1$ this type is known as the cyclic quotient singularity of type $A_{n,q}$. For $n = 1$ such a point is nonsingular; this case is included for notational convenience.

Lemma 2.2. *Let y be a point of Y mapping to $D^{\rm sing}$, say $\pi(y) = x \in D_i \cap D_{i'}$.*

- (i) *There are unique $j \in J_i$ and $j' \in J_{i'}$ such that $y \in D_{ij} \cap D_{i'j'}$.*
- (ii) *Let e_{ij} be the ramification index of π along D_{ij}, and let $e_{i'j'}$ be the ramification index of π along $D_{i'j'}$. Then there are positive integers n, m_1, m_2 depending on y such that the following holds: $e_{ij} = nm_1$, $e_{i'j'} = nm_2$, the rank d_y of $\widehat{O}_{Y,y}$ over $\widehat{O}_{X,x}$ equals nm_1m_2. If y is a singular point of Y then y is a cyclic quotient singularity of type $A_{n,q}$ for some positive integer q with $\gcd(n, q) = 1$.*

Proof. As in the previous lemma we base our argument on a consideration of fundamental groups, following [1, pp. 102–103]. We identify a local neighborhood W of x with the bi-disk $Z = \{(z_1, z_2) \in \mathbb{C}^2 \colon |z_1| < 1, |z_2| < 1\}$, letting D_i correspond to the zero set of z_1 and $D_{i'}$ to the zero set of z_2. Let $Z^* = Z - \{z_1 z_2 = 0\}$ and W^* the corresponding open subset of W. If $\gamma_i \in \pi_1(Z^*)$ is the class of a positively oriented little loop around the z_i axis, then $\pi_1(Z^*) \cong \mathbb{Z} \times \mathbb{Z}$ with generators $\gamma_1 = (1, 0)$ and $\gamma_2 = (0, 1)$. Let B be the connected component of $\pi^{-1}W$ that contains y. Put $B^* = B - \pi^{-1}D$. We have then that $B^* \to W^*$ is a connected finite degree topological covering. Let $\Gamma = \pi_*(\pi_1(B^*))$ be the image of the topological fundamental group of B^* in $\pi_1(W^*) \cong \mathbb{Z} \times \mathbb{Z}$. Then Γ is of finite index in $\pi_1(W^*)$. We pick generators of Γ as follows: $\Gamma \cap (\mathbb{Z} \times 0)$ is nontrivial, so there is a unique $n' > 0$ such that $(n', 0)$ generates this intersection. As the quotient $\Gamma/\mathbb{Z}(n', 0)$ is isomorphic to $\mathbb{Z}$ there is a unique $(q', m_2) \in \Gamma$ with $0 \leq q' < n'$ and $m_2 > 0$ such that $(n', 0)$ and (q', m_2) generate Γ. Let $m_1 = \gcd(n', q')$ and write $n' = nm_1$, $q' = qm_1$. Thus $\Gamma = \mathbb{Z}(nm_1, 0) + \mathbb{Z}(qm_1, m_2)$ which is of index nm_1m_2 in $\mathbb{Z} \times \mathbb{Z}$. Let $\Gamma' = \mathbb{Z}(nm_1, 0) + \mathbb{Z}(0, nm_2) \subset \Gamma$. Then Γ' is the largest subgroup of Γ of the form $\mathbb{Z}(*, 0) + \mathbb{Z}(0, *)$, and the quotient $\Gamma/\Gamma' \subset m_1\mathbb{Z}/nm_1\mathbb{Z} \times m_2\mathbb{Z}/nm_2\mathbb{Z}$ is cyclic of order n and projects isomorphically to both factors. Let $\widetilde{W}^* \to W^*$ be a universal cover. Then $B^* = \widetilde{W}^*/\Gamma$. Now $\widetilde{W}^*/\Gamma' \to W^*$ is isomorphic to the cover $Z^* \to Z^*$, which is given by $(z_1, z_2) \mapsto (z_1^{nm_1}, z_2^{nm_2})$. Via normalization this induces the ramified cover $Z \to Z$, given by the same formula. The group Γ/Γ' acts on Z, with quotient B; this action is free outside $(0, 0)$. Hence y is a singular point of Y if and only if $n > 1$. We conclude that $e_{ij} = nm_1$ and $e_{i'j'} = nm_2$, and we have natural inclusions of $\mathbb{C}$-algebras $\mathbb{C}[[z_1, z_2]] \cong \widehat{O}_{X,x} \rightarrowtail \widehat{O}_{Y,y} = \mathbb{C}[[z_1^{1/nm_1}, z_2^{1/nm_2}]]^{\Gamma/\Gamma'}$, where the generator given by (qm_1, m_2) acts as $z_1^{1/nm_1} \mapsto \zeta_n^q z_1^{1/nm_1}$, $z_2^{1/nm_2} \mapsto \zeta_n z_2^{1/nm_2}$. This realizes $\widehat{O}_{Y,y}$ as a direct summand of the free $\widehat{O}_{X,x}$-module $\mathbb{C}[[z_1^{1/nm_1}, z_2^{1/nm_2}]]$. Statement (i) follows. We see that $\widehat{O}_{Y,y}$ is free of rank nm_1m_2 as an $\widehat{O}_{X,x}$-module whereas $\mathbb{C}[[z_1^{1/nm_1}, z_2^{1/nm_2}]]$ is finite free of rank $n^2m_1m_2$ as an $\widehat{O}_{X,x}$-module. We get $d_y = nm_1m_2$, and if $n > 1$, the singularity y is a cyclic quotient singularity of type $A_{n,q}$. The lemma is proved. ∎

Let $\rho \colon Y' \to Y$ be a minimal resolution of singularities of Y and denote by $E_1, \ldots, E_s$ the exceptional components of $Y' \to Y$. Let K_Y be the Weil divisor obtained by taking the closure in Y of a canonical divisor on the nonsingular locus of Y. Since Y has only cyclic quotient singularities, each Weil divisor on Y is $\mathbb{Q}$-Cartier, i.e. has the property that a certain integer multiple of it is a Cartier divisor on Y. Let ρ^*K_Y be the pull-back of the $\mathbb{Q}$-Cartier divisor K_Y. On $Y' - \cup_i E_i$, ρ^*K_Y is a canonical divisor. Hence for a canonical divisor $K_{Y'}$ of Y' there is a linear equivalence of $\mathbb{Q}$-Cartier divisors:

$$K_{Y'} \equiv \rho^*K_Y + \sum_{i=1}^{s} a_i E_i,$$

where the a_i are rational numbers. Such a_i are unique, because any rational function on Y' whose divisor is a linear combination of the E_i is a rational function on Y that is regular outside the singular locus, and hence constant.

We have the following local statement. Let y in Y be a singular point, hence a cyclic quotient singularity, say of type $A_{n,q}$. Then for any Weil divisor W on Y, $n \cdot W$ is Cartier at y (see [8, Proposition 5.15] and its proof). The minimal resolution of y is described in [1, pp. 100–101]. We have $n > 1$, and can and do assume that $q < n$. Write:

$$\frac{n}{q} = b_1 - \frac{1}{b_2 - \frac{1}{b_3 - \cdots}} = [b_1, \dots, b_\lambda]$$

with $b_i \in \mathbb{Z}_{>0}$ for the Hirzebruch–Jung continued fraction of n/q. Then the reduced exceptional locus $\rho^{-1}(y)$ is a chain of λ $\mathbb{P}^1$'s with self-intersections $-b_1, -b_2, \dots, -b_\lambda$. We will need some estimates related to this resolution of singularities.

Lemma 2.3. *Let y be a singular point of Y of type $A_{n,q}$. Assume that $E_1, \dots, E_\lambda$ are the E_i above y, numbered as they appear in the chain. For i in $\{1, \dots, \lambda\}$, let $b_i = -(E_i, E_i)$. Then $2 \le b_i \le n$ for each $i = 1, \dots, \lambda$. The number λ of components is bounded by n as well. The a_i are determined by the recursion $b_i a_i - a_{i-1} - a_{i+1} = 2 - b_i$ with boundary conditions $a_0 = 0 = a_{\lambda+1}$. We have $a_i \in (-1, 0]$ for $i = 1, \dots, \lambda$ and the rational number $\sum_{i=1}^{\lambda} a_i(b_i - 2)$ is bounded from above by 2 and from below by $-n$.*

Proof. That $b_i \ge 2$ is clear from the way the b_i are defined. Spelling out the definition, the integers b_i are determined by the following recursion (variant of the Euclidean algorithm) for integers c_i: $c_i = b_{i+2}c_{i+1} - c_{i+2}$, $0 \le c_{i+2} < c_{i+1}$ for $i = -1, \dots, \lambda - 2$ with initial conditions $c_{-1} := n, c_0 := q$. It follows that $b_i \le c_{i-2} \le c_{-1} = n$. Further we have $n = c_{-1} > c_0 > c_1 > \cdots > c_\lambda = 0$ so that the number λ is bounded from above by n.

By the adjunction formula we have $(K_{Y'} + E_i, E_i) = -2$ for all i so we see that the a_i form the unique solution to the recursion $b_i a_i - a_{i-1} - a_{i+1} = 2 - b_i$ for $i = 1, \dots, \lambda$ with boundary conditions $a_0 = 0 = a_{\lambda+1}$.

Suppose that there is an i in $\{1, \dots, \lambda\}$ with $a_i \ge 0$. Let j with $1 \le j \le \lambda$ be an index with $a_j = \max_i a_i$. We find

$$\begin{aligned} a_j &= \frac{a_{j-1} + a_{j+1}}{b_j} + \frac{2 - b_j}{b_j} = \frac{2a_j}{b_j} + \frac{a_{j-1} - a_j + a_{j+1} - a_j}{b_j} + \frac{2 - b_j}{b_j} \\ &\le a_j + \frac{a_{j-1} - a_j}{b_j} + \frac{a_{j+1} - a_j}{b_j} \end{aligned}$$

whence $a_{j-1} = a_j = a_{j+1}$ and $b_j = 2$. Hence the maximum of the a_i is also attained at $j - 1$ and at $j + 1$. Continuing with the same reasoning we find that all $b_i = 2$ and all $a_i = 0$. Hence all $a_i \le 0$.

Let j with $1 \leq j \leq \lambda$ be an index with $a_j = \min_i a_i$. Our recursion can be written as $(b_i - 2)(a_i + 1) = (a_{i-1} - a_i) + (a_{i+1} - a_i)$, so we see that if $a_j \leq -1$, then $a_{j-1} = a_j = a_{j+1}$, and a_{j-1} and a_{j+1} are also minimal and ≤ -1, and we get the contradiction $0 = a_0 \leq -1$. Hence for all i we have $a_i > -1$.

By adding the equalities $c_i = b_{i+2}c_{i+1} - c_{i+2}$ for $i = -1, \ldots, \lambda - 2$ we find

$$n + q + 1 + 2\sum_{i=1}^{\lambda-2} c_i = \sum_{i=1}^{\lambda} b_i c_{i-1},$$

and hence

$$n + q + 1 = \sum_{i=1}^{\lambda}(b_i - 2)c_{i-1} + 2c_{\lambda-1} + 2c_0 = \sum_{i=1}^{\lambda}(b_i - 2)c_{i-1} + 2 + 2q\,.$$

Since $c_{i-1} \geq 1$ and $b_i \geq 2$ for $i = 1, \ldots, \lambda$ we find $\sum_{i=1}^{\lambda}(b_i - 2) \leq n - q - 1 < n$. Adding the equalities $b_i a_i - a_{i-1} - a_{i+1} = 2 - b_i$ for $i = 1, \ldots, \lambda$ we find that $\sum_{i=1}^{\lambda} a_i(b_i - 2) = \sum_{i=1}^{\lambda}(-b_i + 2) - (a_1 + a_\lambda)$. Since $\sum_{i=1}^{\lambda}(-b_i + 2) > -n$ and $a_1 + a_\lambda \leq 0$ we find that $\sum_{i=1}^{\lambda} a_i(b_i - 2) > -n$. The upper bound $\sum_{i=1}^{\lambda} a_i(b_i - 2) \leq 2$ follows since $b_i \geq 2$ always and $a_1 + a_\lambda > -2$. □

We will need to compare the topological Euler characteristics of X and Y. The following general lemma is useful for this. We denote by $\mathrm{H}_c^{\cdot}(-, \mathbb{Q})$ the cohomology with compact supports and with rational coefficients on the category of paracompact Hausdorff spaces. We use the notation $e_c(-)$ for the compactly supported Euler characteristic $e_c(-) = \sum_i (-1)^i \dim \mathrm{H}_c^i(-, \mathbb{Q})$; this is a well-defined integer for separated $\mathbb{C}$-schemes of finite type.

Lemma 2.4. *Let M, N be separated $\mathbb{C}$-schemes of finite type.*

(i) *If Z is a closed subscheme of M, then $e_c(M) = e_c(Z) + e_c(M - Z)$.*
(ii) *If $M \to N$ is a finite étale cover of degree n then $e_c(M) = n \cdot e_c(N)$.*

Proof. The first statement follows from the long exact sequence of compactly supported cohomology:

$$\cdots \to \mathrm{H}_c^i(M - Z) \to \mathrm{H}_c^i(M) \to \mathrm{H}_c^i(Z) \to \mathrm{H}_c^{i+1}(M - Z) \to \cdots$$

As to the second statement, we may assume first of all that M and N are connected. Second, we may reduce to the case that $M \to N$ is Galois. Indeed, let $P \to N$ be a Galois closure of $M \to N$, and denote by G the group of automorphisms of P such that $N = P/G$. Let H be the subgroup of G such that $M = P/H$. If the result is true for Galois covers, we find

$$e_c(N) = \frac{1}{\#G} e_c(P) = \frac{\#H}{\#G} e_c(M) = \frac{1}{n} e_c(M)$$

and the result also follows in the general case. So let us assume that $M \to N$ is Galois, with group G. If V is a $\mathbb{Q}[G]$-module of finite type, let $[V]$ be the class of

V in the Grothendieck group of such modules. More generally, if V is a $\mathbb{Z}$-graded $\mathbb{Q}[G]$-module of finite type, like $\mathrm{H}_c(M)$ for example, then we denote by $[V]$ the class of $\sum_i (-1)^i V^i$. Now remark that G acts freely on M, hence by the Lefschetz trace formula for compactly supported cohomology (see [4, Theorem 3.2]) we have for all nontrivial $g \in G$ that $\sum_i (-1)^i \mathrm{trace}(g, \mathrm{H}_c^i(M)) = 0$. By character theory it follows that $[\mathrm{H}_c(M)]$ is a multiple of $[\mathbb{Q}[G]]$, the class of the regular representation of G, say $[\mathrm{H}_c(M)] = m \cdot [\mathbb{Q}[G]]$ with $m \in \mathbb{Z}$. Since we also have that $\mathrm{H}_c(N) = \mathrm{H}_c(M)^G$ we get $e_c(N) = \dim_{\mathbb{Q}} \mathrm{H}_c(M)^G = m$. As $e_c(M) = \dim_{\mathbb{Q}} \mathrm{H}_c(M) = m \cdot \#G$ the result follows. $\square$

Finally we want to work with the Grothendieck–Riemann–Roch theorem. We recall the statement and all notions that go into it. Let M, N be smooth quasi-projective varieties over $\mathbb{C}$. One has a Grothendieck group $K_0(M)$ for coherent sheaves on M. This group is isomorphic to its analog for locally free O_M-modules of finite rank, and therefore, it has a natural ring structure. There is also a Chow ring $\mathrm{CH}(M)$, coming with a natural grading. For $p\colon M \to N$ a projective morphism one has a map $p_!\colon K_0(M) \to K_0(N)$ given by $p_!([\mathcal{F}]) = \sum_i (-1)^i [\mathrm{R}^i p_* \mathcal{F}]$. Also one has a map $p_*\colon \mathrm{CH}(M) \to \mathrm{CH}(N)$ given by proper push-forward of cycles. The Chern character ch gives a ring homomorphism $\mathrm{ch}\colon K_0(M)_{\mathbb{Q}} \to \mathrm{CH}(M)_{\mathbb{Q}}$. Each coherent sheaf $\mathcal{F}$ on M has a Todd class $\mathrm{td}(\mathcal{F})$ in $\mathrm{CH}(M)_{\mathbb{Q}}$. The Todd class $\mathrm{td}(M)$ of M is by definition the Todd class of the tangent bundle T_M of M.

The Grothendieck–Riemann–Roch theorem reads as follows.

Theorem 2.1. *Let M, N be smooth quasi-projective varieties over $\mathbb{C}$. Let $p\colon M \to N$ be a projective morphism and let $\mathcal{F}$ be a coherent sheaf on M. Then the equality*

$$\mathrm{ch}(p_! \mathcal{F}) \cdot \mathrm{td}(N) = p_*(\mathrm{ch}(\mathcal{F}) \cdot \mathrm{td}(M))$$

holds in $\mathrm{CH}(N)_{\mathbb{Q}}$.

We recall the formulas

$$\mathrm{ch}(\mathcal{F}) = c_0(\mathcal{F}) + c_1(\mathcal{F}) + \frac{1}{2}(c_1^2(\mathcal{F}) - 2c_2(\mathcal{F})) + \text{h.o.t.}$$

and

$$\mathrm{td}(\mathcal{F}) = 1 + \frac{1}{2}c_1(\mathcal{F}) + \frac{1}{12}(c_1^2(\mathcal{F}) + c_2(\mathcal{F})) + \text{h.o.t.}$$

We have $c_0(\mathcal{F}) = \mathrm{rank}(\mathcal{F})$ if $\mathcal{F}$ is locally free. Finally $c_1(\mathcal{F}) = c_1(\det \mathcal{F})$. In particular $c_1(\det \mathrm{R}^{\cdot} f_* O_{Y'}) = c_1(f_! O_{Y'})$.

3. Proof of Theorem 1.1

We recall that we assumed V to be connected. We start by deriving a useful expression for $c_1(f_! O_{Y'})$. We recall that the singular points of Y are cyclic quotient singularities. According to [1, Proposition III.3.1] such singularities are rational, i.e. we have

$$\rho_* O_{Y'} = O_Y, \quad \mathrm{R}^i \rho_* O_{Y'} = 0 \quad \text{for } i > 0\,.$$

Using the Leray spectral sequence we find, writing $\bar\pi = \pi\rho$, that $\mathrm{R}^i\bar\pi_* O_{Y'} = \mathrm{R}^i\pi_* O_Y$ for all i. As π is finite we obtain

$$\bar\pi_* O_{Y'} = \pi_* O_Y\,, \quad \mathrm{R}^i\bar\pi_* O_{Y'} = 0 \quad \text{for } i > 0\,.$$

Applying then the Leray spectral sequence to the diagram

$$\begin{tikzcd} Y' \arrow[r, "\bar\pi"'] \arrow[dr, "f"'] & X \arrow[d, "h"] \\ & C \end{tikzcd}$$

we obtain $\mathrm{R}^i f_* O_{Y'} = (\mathrm{R}^i h_*)(\bar\pi_* O_{Y'}) = (\mathrm{R}^i h_*)(\pi_* O_Y)$ for all i and hence

$$f_! O_{Y'} = h_!(\pi_* O_Y)\,.$$

The Grothendieck–Riemann–Roch theorem then gives

$$\mathrm{ch}(f_! O_{Y'}) \cdot \mathrm{td}(C) = h_*(\mathrm{ch}(\pi_* O_Y) \cdot \mathrm{td}(X))\,.$$

We recall that we write d for the degree of π. Also we recall that the sheaf $\pi_* O_Y$ is locally free of rank d. Comparing terms in degree 0 therefore yields

$$c_0(f_! O_{Y'}) = h_*(d \cdot \mathrm{td}(X)_{(1)} + c_1(\pi_* O_Y))\,.$$

On the other hand the Grothendieck–Riemann–Roch theorem applied directly to f gives

$$\mathrm{ch}(f_! O_{Y'}) \cdot \mathrm{td}(C) = f_*(\mathrm{ch}(O_{Y'}) \cdot \mathrm{td}(Y')) = f_*(\mathrm{td}(Y'))\,.$$

Comparing terms in degree 1 we find

$$c_1(f_! O_{Y'}) + c_0(f_! O_{Y'}) \cdot \mathrm{td}(C)_{(1)} = f_*(\mathrm{td}(Y')_{(2)}).$$

Comparing with our previous expression for $c_0(f_! O_{Y'})$ we get

$$c_1(f_! O_{Y'}) = f_*(\mathrm{td}(Y')_{(2)}) - h_*(d \cdot \mathrm{td}(X)_{(1)} + c_1(\pi_* O_Y)) \cdot \mathrm{td}(C)_{(1)}$$

hence $\deg\det \mathrm{R}^{\cdot} f_* O_{Y'}$ equals the degree of

$$\frac{1}{12} f_*\left(c_1^2(T_{Y'}) + c_2(T_{Y'})\right) - h_*(d \cdot \mathrm{td}(X)_{(1)} + c_1(\pi_* O_Y)) \cdot \mathrm{td}(C)_{(1)}$$

in $\mathrm{CH}^1(C)_{\mathbb{Q}}$. We are done once we show that the degrees of $c_1^2(T_{Y'})$, $c_2(T_{Y'})$ and of $h_*(c_1(\pi_* O_Y)) \cdot \mathrm{td}(C)_{(1)}$ are bounded from above and below by linear polynomials in d with coefficients depending only on D and h. We start by considering the term involving $c_1(\pi_* O_Y)$. As before, write $D = \sum_{i\in I} D_i$ for the decomposition of D into its prime components. Define R to be the Weil divisor, supported on $\pi^{-1}(D)$, given as follows: let D_{ij} be a component of $\pi^{-1}(D)$ mapping onto D_i, then the multiplicity of D_{ij} in R is $(e_{ij} - 1)$. Put $B := \pi_* R$. Note that we have a trace pairing $\pi_* O_Y \otimes_{O_X} \pi_* O_Y \to O_X$. This induces a monomorphism $(\det \pi_* O_Y)^{\otimes 2} \rightarrowtail O_X$, identifying $(\det \pi_* O_Y)^{\otimes 2}$ with the ideal sheaf $O_X(-B)$ of B, as a local computation shows (see e.g. [14, III, Sec. 6, Proposition 13]). We obtain $c_1(\pi_* O_Y) = -\frac{1}{2}[B]$ in

$\mathrm{CH}(X)_{\mathbb{Q}}$ so we are done for this term if we could show that the multiplicity of each D_i in B is bounded linearly in d. But this multiplicity is $\sum_{j\in J_i}(e_{ij}-1)f_{ij}$ with f_{ij} the degree of D_{ij} over D_i, which is bounded by d.

Next we consider the term $c_2(T_{Y'})$. We recall that by a version of the Gauss–Bonnet formula (see e.g. [7, p. 416]) we have $\deg c_2(T_{Y'}) = e_c(Y')$, the topological Euler characteristic of Y'. For each $i \in I$ write $d_i := \sum_{j\in J_i} f_{ij}$. By Lemma 2.1 the map $\pi^{-1}(D - D^{\mathrm{sing}}) \to D - D^{\mathrm{sing}}$ is étale, so we have, invoking Lemma 2.4

$$\begin{aligned} e_c(Y) &= e_c(\pi^{-1}U) + e_c(\pi^{-1}D) \\ &= de_c(U) + \sum_{i\in I} d_i e_c(D_i - D^{\mathrm{sing}}) + e_c(\pi^{-1}D^{\mathrm{sing}}) \\ &= de_c(U) + \sum_{i\in I} d_i e_c(D_i - D^{\mathrm{sing}}) + \#\pi^{-1}D^{\mathrm{sing}} \end{aligned}$$

with D^{sing} the singular locus of D. This shows that $e_c(Y)$ is bounded from above and below by linear polynomials in d with coefficients depending only on D. Now $e_c(Y') = e_c(Y) + s$, where s is the total number of exceptional components $E_1, \ldots, E_s$ of $Y' \to Y$. If y is a singular point of Y of type $A_{n,q}$, say, and mapping to x on X, then by Lemma 2.3 the number of exceptional components above y is bounded from above by n. By Lemma 2.2 this is again bounded from above by the local degree d_y of y over x. Since for any x on X we have $\sum_{y\colon y\mapsto x} d_y = d$ we obtain that $e_c(Y')$ is bounded from above and below by linear polynomials in d with coefficients depending only on D.

Finally we consider $c_1^2(T_{Y'})$. Note that we can write $\deg c_1^2(T_{Y'}) = (K_{Y'}, K_{Y'})$, the self-intersection number of the divisor $K_{Y'}$ on Y'. We compute this self-intersection number. By [17, Theorem 4.1] the normal surface Y is an Alexander scheme, implying (cf. [17, Note 2.4]) among other things that for the proper maps $\rho\colon Y' \to Y$ and $\pi\colon Y \to X$ one has a projection formula for Weil divisors, provided that one works on Y with the intersection theory with $\mathbb{Q}$-coefficients as in [10, Sec. IIb]. Thus we compute:

$$\begin{aligned} (K_{Y'}, K_{Y'}) &= \left(\rho^* K_Y + \sum_i a_i E_i, K_{Y'}\right) \\ &= (\rho^* K_Y, K_{Y'}) + \sum_i a_i(b_i - 2) \\ &= (K_Y, K_Y) + \sum_i a_i(b_i - 2)\,. \end{aligned}$$

But $K_Y = \pi^* K_X + R$, so

$$\begin{aligned} (K_Y, K_Y) &= d\cdot(K_X, K_X) + 2(\pi^* K_X, R) + (R, R) \\ &= d\cdot(K_X, K_X) + 2(K_X, B) + (R, R)\,. \end{aligned}$$

We are done once we show that $\sum_i a_i(b_i - 2)$, (K_X, B) and (R, R) are bounded from above and below by linear polynomials in d with coefficients depending only on D. We start with the term $\sum_i a_i(b_i-2)$. By Lemma 2.3 the contribution coming from one singularity y is bounded from above by 2 and from below by $-n$ with n determined as usual by the type of y. Again, since n is bounded by d_y and $\sum_{y\colon y\mapsto x} d_y = d$ for all x on X we get in total that the sum $\sum_i a_i(b_i-2)$ is bounded by at most linear polynomials in d with coefficients depending only on D.

The intersection number (K_X, B) is bounded linearly in d by our description of B given earlier in this proof. As for (R,R), we obtain from Lemmas 2.1 and 2.2 that the irreducible components of the inverse image under π of a D_i are disjoint and hence we can write:

$$(R,R) = \sum_{i,j}(e_{ij}-1)^2(D_{ij},D_{ij}) + \sum_{\substack{(i,i'),j,j'\\ i\neq i'}}(e_{ij}-1)(e_{i'j'}-1)(D_{ij},D_{i'j'})\,.$$

We have, for each $i\in I$, that $\pi^*D_i = \sum_j e_{ij}D_{ij}$, so on the one hand for a given j_0

$$(D_{ij_0},\pi^*D_i) = \sum_j e_{ij}(D_{ij},D_{ij_0}) = e_{ij_0}(D_{ij_0},D_{ij_0})$$

by the disjointness of the D_{ij}, and on the other hand

$$(D_{ij_0},\pi^*D_i) = (\pi_*D_{ij_0},D_i) = f_{ij_0}(D_i,D_i)$$

by the projection formula. Thus

$$(D_{ij_0},D_{ij_0}) = \frac{f_{ij_0}}{e_{ij_0}}(D_i,D_i)$$

and hence for a given i

$$\sum_j (e_{ij}-1)^2(D_{ij},D_{ij}) = \sum_j (e_{ij}-1)^2\frac{f_{ij}}{e_{ij}}(D_i,D_i)\,.$$

As $0\le \sum_j(e_{ij}-1)^2\frac{f_{ij}}{e_{ij}} < d$ we are done for the first term $\sum_{i,j}(e_{ij}-1)^2(D_{ij},D_{ij})$ in our formula for (R,R). The second term

$$\sum_{\substack{(i,i'),j,j'\\ i\neq i'}}(e_{ij}-1)(e_{i'j'}-1)(D_{ij},D_{i'j'})$$

can be written as

$$\sum_{\substack{(i,i')\\ i\neq i'}}\ \sum_{x\in D_i\cap D_{i'}}\ \sum_{y\mapsto x}\sum_{j,j'}(e_{ij}-1)(e_{i'j'}-1)(D_{ij},D_{i'j'})_y\,.$$

But as is stated in Lemma 2.2 for each $y\mapsto x$ with $x\in D_i\cap D_{i'}$ there is exactly one pair (j,j') such that $(D_{ij},D_{i'j'})_y\neq 0$. So, the summation over j and j' can be replaced by a single term with indices $j(y)$ and $j'(y)$. We can compute $(D_{ij(y)},D_{i'j'(y)})_y$ as follows. Let $A_{n,q}$ be the type of y. By what we said before Lemma 2.3, $n\cdot D_{ij(y)}$ is a Cartier divisor at y. But then as $D_{i'j'(y)}$ is smooth the intersection number $(D_{ij(y)},D_{i'j'(y)})_y$ is just given as $1/n$ times the valuation in

$O_{D_{i'j'(y)},y}$ of a local function defining $n \cdot D_{ij(y)}$ around y on Y. Since this function is a local coordinate around y on $D_{i'j'(y)}$ we find $(D_{ij(y)}, D_{i'j'(y)})_y$ to be equal to $1/n$. By Lemma 2.2 there exist positive integers m_1, m_2 such that $e_{ij(y)} = nm_1$, $e_{i'j'(y)} = nm_2$, $d_y = nm_1m_2$, hence

$$\begin{aligned}(e_{ij(y)} - 1)(e_{i'j'(y)} - 1)(D_{ij(y)}, D_{i'j'(y)})_y &\leq e_{ij(y)}e_{i'j'(y)}(D_{ij(y)}, D_{i'j'(y)})_y \\ &= (nm_1)(nm_2)/n = nm_1m_2 = d_y\,.\end{aligned}$$

Keeping in mind that $\sum_{y\colon y\mapsto x} d_y = d$ for all x on X we find that

$$\sum_{\substack{(i,i')\\ i\neq i'}} \sum_{x\in D_i\cap D_{i'}} \sum_{y\mapsto x} (e_{ij(y)} - 1)(e_{i'j'(y)} - 1)(D_{ij(y)}, D_{i'j'(y)})_y$$

is bounded by a linear polynomial in d with coefficients depending only on D. This finishes the proof.

4. Alternative Proofs

Esnault and Viehweg have proposed another proof of (the upper bound of) Theorem 1.1, based on Arakelov's inequality. In fact, their method leads to a more precise version.

Theorem 4.1 (Esnault–Viehweg). *Take the assumptions of Theorem 1.1, and assume moreover that $h\colon X \to C$ is semi-stable (i.e. the singularities in its fibers are ordinary double points), with connected fibers, and that $D = D^{\mathrm{hor}} + h^{-1}D_C$ with $D^{\mathrm{hor}} \to C$ étale and with D_C a divisor on C. Let $g(C)$ be the genus of C, and $g(F)$ the genus of any smooth fiber F of $h\colon X \to C$. Let $S \subset C$ be the set of $s \in C$ such that the fiber X_s of h is singular. Then, for every finite étale $\pi\colon V \to U$ we have:*

$$\deg\det \mathrm{R}^{\cdot} f_*O_{Y'} \leq \left(g(F) + \frac{1}{2}(D^{\mathrm{hor}}, F)\right)\cdot\left(g(C) + 2\#D_C + \frac{1}{2}(1 + \#S)\right)\cdot \deg(\pi)\,.$$

Proof. We may and do assume that V is connected.

The results of the beginning of Sec. 2 show the following statements. The reduced fibers of $f\colon Y' \to C$ are normal crossings divisors on Y'. For $s \in C$ with $s \notin D_C$, the fiber $f^{-1}s$ is semi-stable. Let $\mathrm{Sing}(f)$ denote the singular set of f, i.e. the set where the tangent map of f vanishes. Then $f\mathrm{Sing}(f) \subset S \cup D_C$.

We write $\omega_{Y'} := \Omega^2_{Y'}$ and $\omega_C := \Omega^1_C$ for the dualizing sheaves of Y' and C. We define $\omega_{Y'/C} := \omega_{Y'} \otimes (f^*\omega_C)^\vee$, the relative dualizing sheaf for f. We note that $\omega_{Y'/C}$ coincides with $\Omega^1_{Y'/C}$ on the complement of $\mathrm{Sing}(f)$. By [9, Theorem 5.1] we have $(\mathrm{R}^1f_*O_{Y'})^\vee = f_*\omega_{Y'/C}$. On the other hand, $f_*O_{Y'}$ is the O_C-algebra corresponding to the Stein factorization $Y' \to \widetilde{C} \to C$ of f. As Y' is reduced, the same is true for $\widetilde{C}$, hence the trace form gives an injection of $(\det O_{\widetilde{C}})^{\otimes 2}$ into O_C, hence $\deg\det f_*O_{Y'} \leq 0$. We get:

$$\deg\det \mathrm{R}^{\cdot} f_*O_{Y'} = \deg c_1\mathrm{R}^0 f_*O_{Y'} - \deg c_1\mathrm{R}^1 f_*O_{Y'} \leq -\deg c_1\mathrm{R}^1 f_*O_{Y'}. \quad (4.1)$$

The coherent O_C-module $\mathrm{R}^1 f_* O_{Y'}$ sits in an exact sequence

$$0 \to (\mathrm{R}^1 f_* O_{Y'})_{\mathrm{tors}} \to \mathrm{R}^1 f_* O_{Y'} \to \overline{\mathrm{R}^1 f_* O_{Y'}} \to 0\,,$$

given by its torsion submodule and its locally free quotient. This implies:

$$\begin{aligned} -\deg c_1(\mathrm{R}^1 f_* O_{Y'}) &= -\deg c_1(\overline{\mathrm{R}^1 f_* O_{Y'}}) - \deg c_1(\mathrm{R}^1 f_* O_{Y'})_{\mathrm{tors}} \\ &\le -\deg c_1(\overline{\mathrm{R}^1 f_* O_{Y'}})\,. \end{aligned} \tag{4.2}$$

As $(\mathrm{R}^1 f_* O_{Y'})^\vee = (\overline{\mathrm{R}^1 f_* O_{Y'}})^\vee$, Eqs. (4.1) and (4.2) give:

$$\deg \det \mathrm{R}^{\cdot} f_* O_{Y'} \le \deg \det f_* \omega_{Y'/C}\,.$$

The main idea of the proof is now to use Arakelov's inequality (to be explained below), but for that we need a finite base change $b\colon C' \to C$, with C' a smooth connected projective complex curve, after which Y' admits a semi-stable model $Y'' \to C'$. This precisely means that the ramification indices of b at the $s \in D_C$ must be sufficiently divisible. We pick any s_0 in $C - D_C$. Then $\pi_1(C - (\{s_0\} \cup D_C))$ is freely generated by the usual kind of generators a_i, b_i with $1 \le i \le g(C)$, and γ_s for $s \in D_C$. This shows that a $b\colon C' \to C$ does exist as it is required, unramified outside $E_C := \{s_0\} \cup D_C$. We pick such a b.

Let $f'\colon Y'' \to C'$ be obtained by pull-back via b of $f\colon Y' \to C$, then normalization, and then minimal resolution of singularities. Then f' is semi-stable (see [13; 6, p. 83]), and we have a commutative diagram:

$$\begin{tikzcd} Y'' \arrow[r, "b'"] \arrow[d, "f'"] & Y' \arrow[d, "f"] \\ C' \arrow[r, "b"] & C \end{tikzcd}$$

Lemma 4.1. *In this situation, there is an injection of $b^* f_* \omega_{Y'/C}$ into the locally free $O_{C'}$-module $b^* O_C(E_C) \otimes f'_* \omega_{Y''/C'}$.*

Proof. To start with, the projection formula gives $f_* \omega_{Y'/C} = (f_* \omega_{Y'}) \otimes \omega_C^\vee$, and, similarly, $f'_* \omega_{Y''/C'} = (f'_* \omega_{Y''}) \otimes \omega_{C'}^\vee$. The pull-back of two-forms along b' gives a morphism of $O_{Y'}$-modules $(b')^* \omega_{Y'} \to \omega_{Y''}$, which is generically an isomorphism. Applying f'_* to this morphism, and composing with the natural morphism $b^* f_* \omega_{Y'} \to f'_* (b')^* \omega_{Y'}$ gives a morphism of locally free $O_{C'}$-modules $b^* f_* \omega_{Y'} \to f'_* \omega_{Y''}$ which is generically an isomorphism. Pull-back of one-forms via b gives a morphism of invertible $O_{C'}$-modules $b^* \omega_C \to \omega_{C'}$, which is an isomorphism outside $b^{-1} E_C$. If P is in C' and ω is a generating one-form at $b(P)$, then $b^* \omega$ has a zero of order $e(P) - 1$ at P, where $e(P)$ is the ramification index of b at P. It follows that we have an inclusion of $b^* O_C(-E_C) \otimes \omega_{C'}$ into $b^* \omega_C$. Dually, this gives an inclusion of $b^* \omega_C^\vee$ into the coherent $O_{C'}$-module $b^* O_C(E_C) \otimes \omega_{C'}^\vee$. Combining all this gives an inclusion:

$$b^* f_* \omega_{Y'/C} = b^* f_* \omega_{Y'} \otimes b^* \omega_C^\vee \to f'_* \omega_{Y''} \otimes b^* O_C(E_C) \otimes \omega_{C'}^\vee = b^* O_C(E_C) \otimes f'_* \omega_{Y''/C'},$$

as required. $\square$

We are now ready to invoke Arakelov's inequality (see [13, p. 58]):

$$\deg\det(f'_*\omega_{Y''/C'}) \leq \frac{1}{2}\mathrm{rank}(f'_*\omega_{Y''/C'}) \cdot (2g(C') - 2 + \#f'\mathrm{Sing}f') \,. \tag{4.3}$$

The rank of $f'_*\omega_{Y''/C'}$ equals that of $f_*\omega_{Y'/C}$, both being the sum of the genera of the connected components of the geometric generic fiber of f. Combining this with our previous inequalities, we obtain:

$$\begin{aligned}
\deg\det \mathrm{R}^{\cdot} f_*O_{Y'} &\leq \deg\det f_*\omega_{Y'/C} = \frac{1}{\deg b}\deg\det(b^*f_*\omega_{Y'/C}) \\
&\leq \frac{1}{\deg b}\deg\det(b^*O_C(E_C)\otimes f'_*\omega_{Y''/C'}) \\
&= \frac{1}{\deg b}(\deg(b^*O_C(E_C))\cdot \mathrm{rank} f'_*\omega_{Y''/C'} + \deg\det f'_*\omega_{Y''/C'}) \\
&\leq (\#E_C)\mathrm{rank} f_*\omega_{Y'/C} \\
&\quad + \frac{1}{2\deg b}(\mathrm{rank} f_*\omega_{Y'/C})\cdot(2g(C') - 2 + \#(f'\mathrm{Sing}f')) \,.
\end{aligned} \tag{4.4}$$

Finally, we bound the quantities in the last term. Letting d_j be the degrees of the connected components Z_j of the geometric generic fiber of f over the geometric generic fiber of h, and g_j the genera of the Z_j, we have $\sum_i d_i = \deg(\pi)$, and Hurwitz's formula gives

$$2g_i - 2 = d_i\cdot(2g(F) - 2) + \deg R_i \,, \quad \deg R_i \leq (d_i - 1)(D^{\mathrm{hor}}, F) \,.$$

This leads to:

$$\mathrm{rank}\, f_*\omega_{Y'/C} \leq (g(F) + (D^{\mathrm{hor}}, F)/2)\deg(\pi) \,. \tag{4.5}$$

For $g(C')$, we note that $b\colon C' \to C$ is unramified outside $E_C = \{s_0\}\cup D_C$. Hurwitz's formula gives:

$$\frac{1}{\deg b}(2g(C') - 2) \leq 2g(C) - 2 + 1 + \#D_C \,. \tag{4.6}$$

At the beginning of the proof we noticed that $f\mathrm{Sing}f$ is contained in $S\cup D_C$. Therefore, $f'\mathrm{Sing}f'$ is contained in $b^{-1}(S\cup D_C)$, so

$$\frac{1}{\deg b}\#(f'\mathrm{Sing}f') \leq \#S + \#D_C \,. \tag{4.7}$$

Combining (4.4)–(4.7) we get:

$$\begin{aligned}
\deg\det \mathrm{R}\cdot f_*O_{Y'} \leq &\left(g(F) + \frac{1}{2}(D^{\mathrm{hor}}, F)\right) \\
&\cdot\left(g(C) + 2\#D_C + \frac{1}{2}(1 + \#S)\right)\cdot\deg(\pi).
\end{aligned}$$

This ends our proof of Theorem 4.1.

We remark that instead of invoking Arakelov's inequality it is also possible, again at least for the upper bound implied by Theorem 1.1, to invoke the

Bogomolov–Miyaoka–Yau inequality. Indeed, our work done at the beginning of Sec. 3 showed that Theorem 1.1 can be reduced to providing linear bounds in $\deg(\pi)$ for the degrees of each of the three terms $c_1^2(T_{Y'})$, $c_2(T_{Y'})$ and $h_*(c_1(\pi_* O_Y)) \cdot \mathrm{td}(C)_{(1)}$. The latter two were relatively easily to deal with, whereas the term $c_1^2(T_{Y'})$ required significantly more work. Now instead of calculating the c_1^2 directly, one can also remark that there exist *a priori* inequalities relating the c_2 and the c_1^2. [1, Chapter VI, Table 10] first of all shows that for smooth compact connected complex surfaces which are not of general type c_1^2 is bounded absolutely from above by 9. The Bogomolov–Miyaoka–Yau inequality [1, Theorem VII.4.1] says that for a smooth compact connected complex surface which is of general type, the inequality $c_1^2 \leq 3c_2$ holds. Invoking these results one obtains yet another proof of Theorem 1.1.

5. An Arithmetic Analog

In [5] an algorithm is given that computes the $\mathrm{GL}_2(\mathbb{F}_\lambda)$ Galois representations associated to a given normalized Hecke eigenform f of level 1 in time polynomial in $\#(\mathbb{F}_\lambda)$. Here λ runs through the finite degree 1 places of the field of coefficients of the form. By a famous argument due to Schoof, this leads to an algorithm that on input a prime number p computes the p-th coefficient of the Fourier development of f, in time polynomial in $\log p$. As a consequence, the number of vectors with half length-squared equal to p in a fixed even unimodular lattice can be computed in time polynomial in $\log p$.

Generalizations of the above results seem possible in various different directions. For example, one could look at the case of mod-ℓ Galois representations occurring in the étale cohomology of a given smooth, projective and geometrically connected surface S over $\mathbb{Q}$. Letting ℓ be a prime number, one has the cohomology groups $\mathrm{H}^i(S_{\overline{\mathbb{Q}},\mathrm{et}}, \mathbb{F}_\ell)$ for $0 \leq i \leq 4$, being finite dimensional $\mathbb{F}_\ell$-vector spaces with $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action. It seems reasonable to suspect that, again, there is an algorithm that on input a prime ℓ computes these cohomology groups, with their $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action, in time polynomial in ℓ. Once such an algorithm is known, one also has an algorithm that, on input a prime p of good reduction of S gives the number of points $\#S(\mathbb{F}_p)$ of S over $\mathbb{F}_p$ in time polynomial in $\log p$. This result would be of interest because the known p-adic algorithms for finding such numbers have running time exponential in $\log p$.

The idea in [5] to compute mod-ℓ étale cohomology is to trivialize the sheaves involved, using suitable covers of (modular) curves, and to reduce to computing in the ℓ-torsion of their Jacobians. In our case, using a Lefschetz fibration, one can first reduce to computing cohomology groups $\mathrm{H}^1(U_{\overline{\mathbb{Q}}}, \mathcal{F}_l)$, where U is a nonempty open subscheme of $\mathbb{P}^1_{\mathbb{Q}}$ determined by S and the chosen Lefschetz fibration, and where $\mathcal{F}_l$ are certain étale locally constant sheaves of $\mathbb{F}_\ell$-vector spaces of a fixed dimension, say r. For each ℓ let $V_\ell := \underline{\mathrm{Isom}}_U(\mathbb{F}_\ell^r, \mathcal{F}_l)$. Then each cover $V_\ell \to U$ is finite Galois with group $G \cong \mathrm{GL}_r(\mathbb{F}_\ell)$, and the group $\mathrm{H}^1(U_{\overline{\mathbb{Q}}}, \mathcal{F}_l)$ can be related to $\mathrm{H}^1(V_{\ell,\overline{\mathbb{Q}}}, \mathbb{F}_\ell^r)$

which sits in the ℓ-torsion of the Jacobian of the smooth projective model $\overline{V_\ell}$ of V_ℓ. It is our hope that methods as in [5] can show that we have a polynomial algorithm for computing these cohomology groups once we have a bound for the Faltings height of $\overline{V_\ell}$ that is polynomial in ℓ.

A recent result of Bilu and Strambi, see [2, Theorem 1.2], suggests at least the existence of an exponential bound: it implies that if $Y_{\overline{\mathbb{Q}}} \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ is any connected degree $d \geq 2$ cover, unramified outside a finite subset B of $\mathbb{P}^1_{\overline{\mathbb{Q}}}$, then $Y_{\overline{\mathbb{Q}}}$ has a plane model given by an equation $F(u,v)$ with coefficients in $\overline{\mathbb{Q}}$ such that $\deg_u F = d$, $\deg_v F \leq d \cdot (\#B/2)$, and such that the affine logarithmic height of F is bounded from above by

$$(h+1)(d^3 \cdot \#B)^{5d^2 \cdot \#B + 12d}.$$

Here h is the maximum of the logarithmic projective heights of the elements of B.

Note, however, that in our situation we can take more restrictive assumptions than in the result of Bilu and Strambi. For one thing, our covers $V_\ell \to U$ are defined over $\mathbb{Q}$. For another, our covers $V_\ell \to U$ have the property that there is a nonempty open subscheme U' of $\mathbb{P}^1_{\mathbb{Z}}$ containing U, such that each V_ℓ extends to a finite étale cover of $U'_{\mathbb{Z}[1/\ell]}$. We would like therefore to propose the following arithmetic analog of the main theorem of this note.

Conjecture 5.1. *Let $U \subset \mathbb{P}^1_{\mathbb{Z}}$ be a nonempty open subscheme. Then there are integers a and b with the following property. For any prime number ℓ, and for any connected finite étale cover $\pi\colon V \to U_{\mathbb{Z}[1/\ell]}$, the absolute value of the Faltings height of the normalization of $\mathbb{P}^1_{\mathbb{Q}}$ in the function field of V is bounded by* $\deg(\pi)^a \cdot \ell^b$.

As an example, let us mention that for the family of modular curves $X_1(\ell)$, all covers of the j-line, where one can take $U = \mathbb{P}^1_{\mathbb{Z}} - \{0, 1728, \infty\}$, it is proved in [5] that the Faltings height is bounded above as $O(\ell^2 \log \ell)$. It is tempting to interpret ℓ^2 (up to a constant factor) as the degree of $X_1(\ell)$ over the j-line, and the factor $\log \ell$ as coming from the ramification at ℓ. As in our application the degree of π itself depends polynomially on ℓ, it is irrelevant for this application if the bound in the conjecture is $\deg(\pi)^a \cdot \ell^b$ or $\deg(\pi)^a \cdot (\log \ell)^b$. Of course, one may also ask about the conjecture with this stronger bound.

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