INTERMITTENCY ON CATALYSTS: VOTER MODEL[1]

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In this paper we study intermittency for the parabolic Anderson equation $\partial u/\partial t = \kappa \Delta u + \gamma \xi u$ with $u : \mathbb{Z}^d \times [0,\infty) \to \mathbb{R}$, where $\kappa \in [0,\infty)$ is the diffusion constant, Δ is the discrete Laplacian, $\gamma \in (0,\infty)$ is the coupling constant, and $\xi : \mathbb{Z}^d \times [0,\infty) \to \mathbb{R}$ is a space–time random medium. The solution of this equation describes the evolution of a "reactant" u under the influence of a "catalyst" ξ.

We focus on the case where ξ is the voter model with opinions 0 and 1 that are updated according to a random walk transition kernel, starting from either the Bernoulli measure ν_ρ or the equilibrium measure μ_ρ, where $\rho \in (0,1)$ is the density of 1's. We consider the annealed Lyapunov exponents, that is, the exponential growth rates of the successive moments of u. We show that if the random walk transition kernel has zero mean and finite variance, then these exponents are trivial for $1 \le d \le 4$, but display an interesting dependence on the diffusion constant κ for $d \ge 5$, with qualitatively different behavior in different dimensions.

In earlier work we considered the case where ξ is a field of independent simple random walks in a Poisson equilibrium, respectively, a symmetric exclusion process in a Bernoulli equilibrium, which are both reversible dynamics. In the present work a main obstacle is the nonreversibility of the voter model dynamics, since this precludes the application of spectral techniques. The duality with coalescing random walks is key to our analysis, and leads to a representation formula for the Lyapunov exponents that allows for the application of large deviation estimates.

1. Introduction and main results. The outline of this section is as follows. In Section 1.1 we provide motivation. In Sections 1.2–1.4 we recall some basic facts about the voter model. In Section 1.5 we define the annealed Lyapunov exponents, which are the main objects of our study. In Section 1.6 we prove a representation formula for these exponents in terms of coalescing random walks released at Poisson times along a random walk path. This representation formula is the starting

point for our further analysis. Our main theorems are stated in Section 1.7 (and proved in Sections 2–5). Finally, in Sections 1.8–1.9 we list some open problems and state a scaling conjecture.

1.1. *Reactant and catalyst.* The *parabolic Anderson equation* is the partial differential equation

$$(1.1) \qquad \frac{\partial}{\partial t} u(x,t) = \kappa \Delta u(x,t) + \gamma \xi(x,t) u(x,t), \qquad x \in \mathbb{Z}^d, t \geq 0.$$

Here, the u-field is $\mathbb{R}$-valued, $\kappa \in [0,\infty)$ is the diffusion constant, Δ is the discrete Laplacian, acting on u as

$$(1.2) \qquad \Delta u(x,t) = \sum_{\substack{y \in \mathbb{Z}^d \\ \|y-x\|=1}} [u(y,t) - u(x,t)]$$

($\|\cdot\|$ is the Euclidean norm), $\gamma \in [0,\infty)$ is the coupling constant, while

$$(1.3) \qquad \xi = \{\xi(x,t) : x \in \mathbb{Z}^d, t \geq 0\}$$

is an $\mathbb{R}$-valued random field that evolves with time and that drives the equation. As initial condition for (1.1) we take

$$(1.4) \qquad u(\cdot, 0) \equiv 1.$$

The PDE in (1.1) describes the evolution of a system of two types of particles, A and B, where the A-particles perform autonomous dynamics and the B-particles perform independent simple random walks that branch at a rate that is equal to γ times the number of A-particles present at the same location. The link is that $u(x,t)$ equals the average number of B-particles at site x at time t conditioned on the evolution of the A-particles. The initial condition in (1.4) corresponds to starting off with one B-particle at each site. Thus, the solution of (1.1) may be viewed as describing the evolution of a *reactant u* under the influence of a *catalyst ξ*. Our focus of interest will be on the *annealed Lyapunov exponents*, that is, the exponential growth rates of the successive moments of u.

In earlier work (Gärtner and den Hollander [5], Gärtner, den Hollander and Maillard [6, 8]) we treated the case where ξ is a field of independent simple random walks in a Poisson equilibrium, respectively, a symmetric exclusion process in a Bernoulli equilibrium. In the present paper we focus on the case where ξ is the *Voter Model* (VM), that is, ξ takes values in $\{0,1\}^{\mathbb{Z}^d \times [0,\infty)}$, where $\xi(x,t)$ is the opinion of site x at time t, and opinions are imposed according to a random walk transition kernel. We choose $\xi(\cdot, 0)$ according to either the Bernoulli measure ν_ρ or the equilibrium measure μ_ρ, where $\rho \in (0,1)$ is the density of 1's. We may think of 0 as a vacancy and 1 as a particle.

An overview of the main results in [5, 6, 8] and the present paper as well as further literature is given in Gärtner, den Hollander and Maillard [7]. Gärtner and Heydenreich [4] consider the case where the catalyst consists of a single random walk.

1.2. *Voter model.* Throughout the paper we abbreviate $\Omega = \{0, 1\}^{\mathbb{Z}^d}$ (equipped with the product topology), and we let $p : \mathbb{Z}^d \times \mathbb{Z}^d \to [0, 1]$ be the transition kernel of an irreducible random walk, that is,

$$\sum_{y \in \mathbb{Z}^d} p(x, y) = 1 \qquad \forall x \in \mathbb{Z}^d,$$

(1.5)
$$p(x, y) = p(0, y - x) \geq 0 \qquad \forall x, y \in \mathbb{Z}^d,$$

$$p(\cdot, \cdot) \text{ generates } \mathbb{Z}^d.$$

Occasionally we will need to assume that $p(\cdot, \cdot)$ has zero mean and finite variance. A special case is simple random walk

(1.6)
$$p(x, y) = \begin{cases} \dfrac{1}{2d}, & \text{if } \|x - y\| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

The VM is the Markov process on Ω whose generator L acts on cylindrical functions f as

(1.7)
$$(Lf)(\eta) = \sum_{x, y \in \mathbb{Z}^d} p(x, y)[f(\eta^{x \to y}) - f(\eta)], \qquad \eta \in \Omega,$$

where

(1.8)
$$\eta^{x \to y}(z) = \begin{cases} \eta(x), & \text{if } z = y, \\ \eta(z), & \text{if } z \neq y. \end{cases}$$

Under this dynamics, site x imposes its state on site y at rate $p(x, y)$. The states 0 and 1 are referred to as opinions or, alternatively, as vacancy and particle. The VM is a *nonconservative* dynamics: opinions are not preserved. We write $(S_t)_{t \geq 0}$ to denote the Markov semigroup associated with L.

Let $\xi_t = \{\xi(x, t); x \in \mathbb{Z}^d\}$ be the random configuration of the VM at time t. Let $\mathbb{P}_\eta$ denote the law of ξ starting from $\xi_0 = \eta$, and let $\mathbb{P}_\mu = \int_\Omega \mu(d\eta) \mathbb{P}_\eta$. We will consider two choices for the starting measure μ:

(1.9)
$$\begin{cases} \mu = \nu_\rho, & \text{the Bernoulli measure with density } \rho \in (0, 1), \\ \mu = \mu_\rho, & \text{the equilibrium measure with density } \rho \in (0, 1). \end{cases}$$

Let $p^*(\cdot, \cdot)$ be the *dual* transition kernel, defined by $p^*(x, y) = p(y, x)$, $x, y \in \mathbb{Z}^d$, and $p^{(s)}(\cdot, \cdot)$ the *symmetrized* transition kernel, defined by $p^{(s)}(x, y) = (1/2)[p(x, y) + p^*(x, y)]$, $x, y \in \mathbb{Z}^d$. The ergodic properties of the VM are qualitatively different for recurrent and for transient $p^{(s)}(\cdot, \cdot)$. In particular, when $p^{(s)}(\cdot, \cdot)$ is *recurrent* all equilibria are *trivial*, that is, $\mu_\rho = (1 - \rho)\delta_0 + \rho\delta_1$, while when $p^{(s)}(\cdot, \cdot)$ is *transient* there are also *nontrivial* equilibria, that is, ergodic measures μ_ρ. In the latter case, μ_ρ is taken to be the unique shift-invariant and ergodic equilibrium with density ρ. For both cases we have

(1.10)
$$\mathbb{P}_{\nu_\rho}(\xi_t \in \cdot) \to \mu_\rho(\cdot) \qquad \text{weakly as } t \to \infty,$$

with the same convergence for any starting measure μ that is stationary and ergodic with density ρ (see Liggett [10], Corollary V.1.13).

We will frequently use the measures $\nu_\rho S_T$, $T \in [0, \infty]$, where $\nu_\rho S_\infty = \mu_\rho$ by convention in view of (1.10). The VM is *attractive* (see Liggett [10], Definition III.2.1 and Theorem III.2.2). Consequently, since ν_ρ has positive correlations, the same is true for $\nu_\rho S_T$, that is, nondecreasing functions on Ω are positively correlated (see Liggett [10], Theorem II.2.14).

1.3. *Graphical representation and duality.* In the VM's *graphical representation* $\mathcal{G}_t$ from time 0 up to time t (see, e.g., Cox and Griffeath [3], Section 0), space is drawn sideward, time is drawn upward, and for each ordered pair of sites $x, y \in \mathbb{Z}^d$ arrows are drawn from x to y at Poisson rate $p(x, y)$. A path from $(x, 0)$ to (y, s), $s \in (0, t]$, in $\mathcal{G}_t$ (see Figure 1) is a sequence of space–time points $(x_0, s_0), (x_0, s_1), (x_1, s_1), \ldots, (x_n, s_n), (x_n, s_{n+1})$ such that:

(i) $x_0 = x$, $s_0 = 0$, $x_n = y$, $s_{n+1} = s$;
(ii) the sequence of times $(s_i)_{0 \leq i \leq n+1}$ is increasing;
(iii) for each $1 \leq i \leq n$, there is an arrow from (x_{i-1}, s_i) to (x_i, s_i);
(iv) for each $0 \leq i \leq n$, no arrow *points to* x_i at any time in (s_i, s_{i+1}).

Then ξ can be represented as

$$(1.11) \quad \xi(y, s) = \begin{cases} 1, & \text{if there exists a path from } (x, 0) \text{ to } (y, s) \text{ in } \mathcal{G}_t \\ & \text{for some } x \in \xi(0), \\ 0, & \text{otherwise,} \end{cases}$$
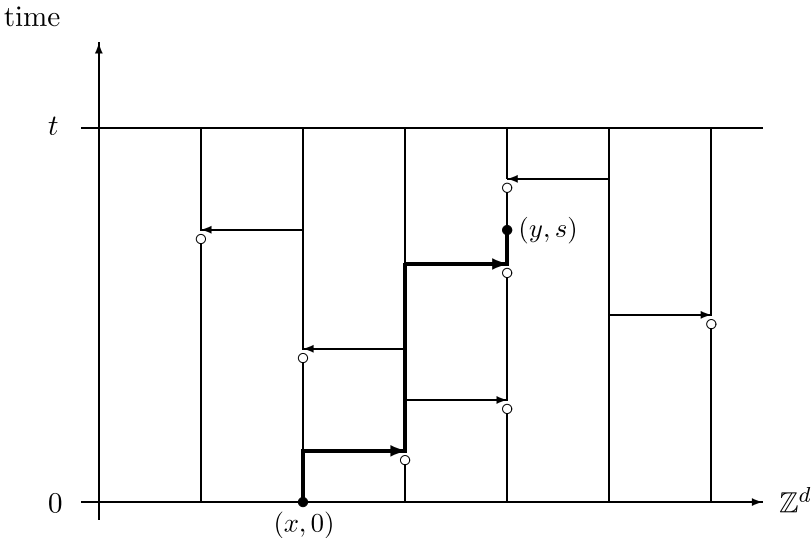

FIG. 1. *Graphical representation $\mathcal{G}_t$. Opinions propagate along paths.*

where $\xi(0) = \{x \in \mathbb{Z}^d : \xi(x, 0) = 1\}$ is the set of initial locations of the 1's. The graphical representation corresponds to binary *branching* with transition kernel $p(\cdot, \cdot)$ and step rate 1 and *killing* at the moment when an arrow comes in from another location. Figure 1 shows how opinions propagate along paths. An open circle indicates that the site adopts the opinion of the site where the incoming arrow comes from. The thick line from $(x, 0)$ to (y, s) shows that the opinion at site y at time s stems from the opinion at a unique site x at time 0.

We can define the *dual graphical representation* $\mathcal{G}_t^*$ by reversing time and direction of all the arrows in $\mathcal{G}_t$. The dual process $(\xi_s^*)_{0 \le s \le t}$ on $\mathcal{G}_t^*$ can then be represented as

$$(1.12) \quad \xi^*(x, t) = \begin{cases} 1, & \text{if there exists a path from } (y, t - s) \text{ to } (x, t) \text{ in } \mathcal{G}_t^* \\ & \text{for some } y \in \xi^*(t - s), \\ 0, & \text{otherwise,} \end{cases}$$

where $\xi^*(t - s) = \{x \in \mathbb{Z}^d : \xi^*(x, t - s) = 1\}$. The dual graphical representation corresponds to *coalescing random walks* with dual transition kernel $p^*(\cdot, \cdot)$ and step rate 1 (see Figure 2).

Figures 1 and 2 make it plausible that the equilibrium measure μ_ρ in (1.10) is *nonreversible*, because the evolution is not invariant under time reversal.

1.4. *Correlation functions.* A key tool in the present paper is the following *representation formula for the n-point correlation functions* of the VM, which is an immediate consequence of the dual graphical representation (see, e.g., Cox and Griffeath [3], Section 1). For $n \in \mathbb{N}$, $x_1, \ldots, x_n \in \mathbb{Z}^d$ and $-\infty < s_1 \le \cdots \le s_n \le t$,
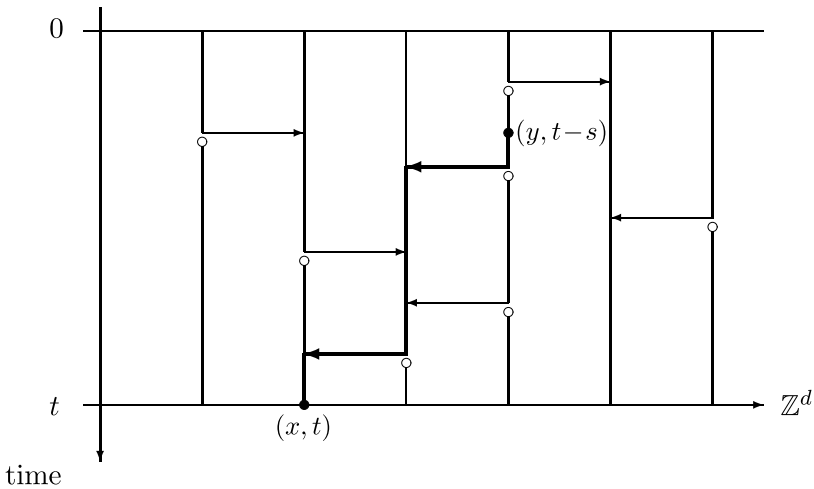

FIG. 2. *Dual graphical representation $\mathcal{G}_t^*$. Opinions propagate along time-reversed coalescing paths.*

let

$$\xi_t^* \{(x_1, s_1), \ldots, (x_n, s_n)\} \tag{1.13}$$

be the set of locations at time t of n coalescing random walks, with transition kernel $p^*(\cdot, \cdot)$ and step rate 1, when the mth random walk is born at site x_m at time s_m, $1 \leq m \leq n$, and let

$$\mathcal{N}_t \{(x_1, s_1), \ldots, (x_n, s_n)\} = |\xi_t^* \{(x_1, s_1), \ldots, (x_n, s_n)\}| \tag{1.14}$$

be the number of random walks alive at time t.

The following lemma gives us a handle on the n-point correlation functions.

LEMMA 1.1. *For all $n \in \mathbb{N}$, $T \in [0, \infty]$, $x_1, \ldots, x_n \in \mathbb{Z}^d$ and $-\infty < s_1 \leq \cdots \leq s_n \leq t < \infty$,*

$$\mathbb{P}_{\nu_\rho S_T} \big(\xi(x_m, t - s_m) = 1 \ \forall 1 \leq m \leq n\big) = \mathbb{E}^* \big(\rho^{\mathcal{N}_{T+t} \{(x_1, s_1), \ldots, (x_n, s_n)\}}\big), \tag{1.15}$$

where $\mathbb{E}^*$ denotes expectation with respect to the coalescing random walk dynamics.

PROOF. For $T < \infty$, we have

$$\begin{aligned}
\mathbb{P}_{\nu_\rho S_T} &\big(\xi(x_m, t - s_m) = 1 \ \forall 1 \leq m \leq n\big) \\
&= \mathbb{P}_{\nu_\rho} \big(\xi(x_m, T + t - s_m) = 1 \ \forall 1 \leq m \leq n\big).
\end{aligned} \tag{1.16}$$

The event in the right-hand side of (1.16) occurs if and only if $\xi(z, 0) = 1$ for all sites z in the set $\xi_{T+t}^* \{(x_1, s_1), \ldots, (x_n, s_n)\}$ (Figure 2), which under ν_ρ has probability $\rho^{\mathcal{N}_{T+t} \{(x_1, s_1), \ldots, (x_n, s_n)\}}$ and proves the claim. Since $t \mapsto \mathcal{N}_t$ is nonincreasing, we may let $T \to \infty$ in (1.15) and use (1.10) to get the formula for $T = \infty$. $\square$

Note that for $T = \infty$ the right-hand side of (1.15) does not depend on t, in accordance with the fact that $\nu_\rho S_\infty = \mu_\rho$ is an equilibrium measure.

1.5. *Lyapunov exponents.* By the Feynman–Kac formula, the formal solution of (1.1) and (1.4) reads

$$u(x, t) = \mathrm{E}_x \left(\exp\left[\gamma \int_0^t \xi\big(X^\kappa(s), t - s\big) \, ds \right] \right), \tag{1.17}$$

where X^κ is a simple random walk on $\mathbb{Z}^d$ with step rate $2d\kappa$, and E_x denotes expectation w.r.t. X^κ given $X^\kappa(0) = x$. Let μ be an arbitrary initial distribution. For $p \in \mathbb{N}$ and $t > 0$, the pth moment of the solution is then given by

$$\mathbb{E}_\mu \big([u(0, t)]^p\big) = (\mathbb{E}_\mu \otimes \mathrm{E}_0^{\otimes p}) \left(\exp\left[\gamma \int_0^t \sum_{q=1}^p \xi\big(X_q^\kappa(s), t - s\big) \, ds \right] \right), \tag{1.18}$$

where X_q^κ, $q = 1, \ldots, p$, are p independent copies of X^κ.

For $p \in \mathbb{N}$ and $t > 0$, define

$$(1.19) \qquad \Lambda_p^\mu(t) = \frac{1}{pt} \log \mathbb{E}_\mu([u(0, t)]^p).$$

Then

$$(1.20) \quad \Lambda_p^\mu(t) = \frac{1}{pt} \log(\mathbb{E}_\mu \otimes \mathrm{E}_0^{\otimes p}) \left(\exp \left[\gamma \int_0^t \sum_{q=1}^p \xi(X_q^\kappa(s), t - s) \, ds \right] \right).$$

We will see that for $\mu = \nu_\rho S_T$, $T \in [0, \infty]$, the last quantity admits a limit as $t \to \infty$,

$$(1.21) \qquad \lambda_p^\mu = \lim_{t \to \infty} \Lambda_p^\mu(t),$$

which is independent of T and which we call the pth *annealed Lyapunov exponent*. Note that $\Lambda_p^\mu(t) \in [\rho\gamma, \gamma]$ for all $t > 0$, as is immediate from (1.20) and Jensen's inequality. Hence,

$$(1.22) \qquad \lambda_p^\mu \in [\rho\gamma, \gamma].$$

From Hölder's inequality applied to (1.19), it follows that $\Lambda_p^\mu(t) \geq \Lambda_{p-1}^\mu(t)$ for all $t > 0$ and $p \in \mathbb{N} \setminus \{1\}$. Hence, $\lambda_p^\mu \geq \lambda_{p-1}^\mu$ for all $p \in \mathbb{N} \setminus \{1\}$. We say that the solution of the parabolic Anderson model is *p-intermittent* if $\lambda_p^\mu > \lambda_{p-1}^\mu$. In the latter case the solution is q-intermittent for all $q > p$ as well (see, e.g., Gärtner and Heydenreich [4], Lemma 3.1). We say that the solution is *intermittent* if it is p-intermittent for all $p \in \mathbb{N} \setminus \{1\}$. Intermittent means that the u-field develops sparse high peaks dominating the moments in such a way that each moment is dominated by its own collection of peaks (see Gärtner and König [9], Section 1.3, and Gärtner and den Hollander [5], Section 1.2).

1.6. *Representation formula.* In this section we derive a *coalescing random walk representation* for the Lyapunov exponents. Recall (1.14). For $n \in \mathbb{N}$, $x_1, \ldots, x_n \in \mathbb{Z}^d$ and $-\infty < s_1 \leq \cdots \leq s_n \leq t$, let

$$(1.23) \qquad \mathcal{N}_t^{\mathrm{coal}}\{(x_1, s_1), \ldots, (x_n, s_n)\} = n - \mathcal{N}_t\{(x_1, s_1), \ldots, (x_n, s_n)\}$$

be the number of random walks coalesced at time t. Let $\Pi_{\rho\gamma}$ and $\mathbb{P}_{\mathrm{Poiss}}$ denote the Poisson point process on $\mathbb{R}$ with intensity $\rho\gamma$ and its law, respectively. We consider $\Pi_{\rho\gamma}$ as a random subset of $\mathbb{R}$ and write $\Pi_{\rho\gamma}(B) = \Pi_{\rho\gamma} \cap B$ for Borel sets $B \subseteq \mathbb{R}$.

PROPOSITION 1.2. *For all $T \in [0, \infty]$, $t > 0$ and right-continuous paths $\varphi_q : [0, t] \to \mathbb{Z}^d$, $q = 1, \ldots, p$,*

$$e^{-\rho\gamma pt} \mathbb{E}_{\nu_\rho S_T} \left(\exp \left[\gamma \int_0^t \sum_{q=1}^p \xi(\varphi_q(s), t - s) \, ds \right] \right)$$

$$(1.24)$$

$$= (\mathbb{E}_{\mathrm{Poiss}}^{\otimes p} \otimes \mathbb{E}^*) \left(\rho^{-\mathcal{N}_{T+t}^{\mathrm{coal}} \{ \bigcup_{q=1}^p \{(\varphi_q(s), s) : s \in \Pi_{\rho\gamma}^{(q)}([0, t])\}\}} \right),$$

where $\Pi_{\rho\gamma}^{(q)}$, $q = 1, \ldots, p$, *are* p *independent copies of* $\Pi_{\rho\gamma}$. *In particular,*

$$
\begin{aligned}
(1.25) \quad & \exp\left[pt\left(\Lambda_p^{\nu_\rho S_T}(t) - \rho\gamma\right)\right] \\
& = \left(\mathbb{E}_0^{\otimes p} \otimes \mathbb{E}_{\text{Poiss}}^{\otimes p} \otimes \mathbb{E}^*\right)\left(\rho^{-\mathcal{N}_{T+t}^{\text{coal}}\{\bigcup_{q=1}^p \{(X_q^\kappa(s),s):s\in\Pi_{\rho\gamma}^{(q)}([0,t])\}\}}\right).
\end{aligned}
$$

PROOF. Fix φ_q, $q = 1, \ldots, p$. By a Taylor expansion of the factors $\exp[\gamma \times \int_0^t \xi(\varphi_q(s), t - s)\, ds]$, $q = 1, \ldots, p$, we have

$$
\begin{aligned}
& e^{-\rho\gamma pt}\mathbb{E}_{\nu_\rho S_T}\left(\exp\left[\gamma\int_0^t \sum_{q=1}^p \xi(\varphi_q(s), t-s)\, ds\right]\right) \\
& = e^{-\rho\gamma pt}\left[\prod_{q=1}^p \sum_{n_q=0}^\infty \frac{\gamma^{n_q}}{n_q!}\left(\prod_{m=1}^{n_q}\int_0^t ds_m^{(q)}\right)\right] \\
(1.26) \quad & \quad\times \mathbb{E}_{\nu_\rho S_T}\left(\prod_{q=1}^p \prod_{m=1}^{n_q} \xi(\varphi_q(s_m^{(q)}), t - s_m^{(q)})\right) \\
& = \left[\prod_{q=1}^p \sum_{n_q=0}^\infty \frac{(\rho\gamma t)^{n_q}}{n_q!} e^{-\rho\gamma t}\frac{1}{t^{n_q}}\left(\prod_{m=1}^{n_q}\int_0^t ds_m^{(q)}\right)\right] \\
& \quad\times \rho^{-\sum_{q=1}^p n_q}\mathbb{E}_{\nu_\rho S_T}\left(\prod_{q=1}^p \prod_{m=1}^{n_q} \xi(\varphi_q(s_m^{(q)}), t - s_m^{(q)})\right).
\end{aligned}
$$

For each $q = 1, \ldots, p$:

- $[(\rho\gamma t)^{n_q}/n_q!]\exp[-\rho\gamma t]$, $n_q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, is the Poisson distribution with parameter $\rho\gamma t$;
- $(1/t^{n_q})(\prod_{m=1}^{n_q}\int_0^t ds_m^{(q)})$ is the uniform distribution on $[0, t]^{n_q}$, coinciding with the distribution of the (unordered) points of $\Pi_{\rho\gamma}^{(q)}$ in $[0, t]$ given $|\Pi_{\rho\gamma}^{(q)}([0, t])| = n_q$, $n_q \in \mathbb{N}_0$.

Moreover, by Lemma 1.1, we have

$$
\begin{aligned}
(1.27) \quad & \mathbb{E}_{\nu_\rho S_T}\left(\prod_{q=1}^p \prod_{m=1}^{n_q} \xi(\varphi_q(s_m^{(q)}), t - s_m^{(q)})\right) \\
& = \mathbb{E}^*\left(\rho^{\mathcal{N}_{T+t}\{\bigcup_{q=1}^p \{(\varphi_q(s_m^{(q)}),s_m^{(q)}):m=1,\ldots,n_q\}\}}\right).
\end{aligned}
$$

Therefore, combining (1.26) and (1.27) and inserting (1.23), we get (1.24).

Recalling (1.20), we see that formula (1.25) follows from (1.24) by substituting $\varphi_q = X_q^\kappa$, $q = 1, \ldots, p$, and taking the expectation $\mathbb{E}_0^{\otimes p}$. $\quad\square$

What (1.25) in Proposition 1.2 says is that, for initial distribution $\mu = \nu_\rho S_T$, the pth Lyapunov exponent λ_p^μ can be computed by taking p simple random walks (with step rate $2d\kappa$), releasing coalescing random walks [with dual transition kernel $p^*(\cdot, \cdot)$ and step rate 1] from the paths of these p random walks at rate $\rho\gamma$ until time t, recording the total number of coalescences up to time $T + t$, and letting $t \to \infty$ afterward. The representation formula (1.25) will be the starting point of our large deviation analysis.

1.7. *Main theorems.*   Theorems 1.3–1.5 below are our main results. We write $\lambda_p^\mu(\kappa)$ to exhibit the κ-dependence of the Lyapunov exponents λ_p^μ. The dependence on the other parameters will generally be suppressed from the notation.

THEOREM 1.3.   *For all* $d \geq 1$, $p \in \mathbb{N}$, $\kappa \in [0, \infty)$, $\gamma \in (0, \infty)$ *and* $\rho \in (0, 1)$, *the limit* λ_p^μ *in* (1.21) *exists for* $\mu = \nu_\rho S_T$ *and is the same for all* $T \in [0, \infty]$ (*and is henceforth denoted by* λ_p).

THEOREM 1.4.   *For all* $d \geq 1$, $p \in \mathbb{N}$, $\gamma \in (0, \infty)$ *and* $\rho \in (0, 1)$:

(i) $\kappa \mapsto \lambda_p(\kappa)$ *is globally Lipschitz outside any neighborhood of* 0;
(ii) $\lambda_p(\kappa) > \rho\gamma$ *for all* $\kappa \in [0, \infty)$.

THEOREM 1.5.   *Fix* $p \in \mathbb{N}$, $\gamma \in (0, \infty)$ *and* $\rho \in (0, 1)$.

(i) *If* $1 \leq d \leq 4$ *and* $p(\cdot, \cdot)$ *has zero mean and finite variance, then* $\lambda_p(\kappa) = \gamma$ *for all* $\kappa \in [0, \infty)$.
(ii) *If* $d \geq 5$, *then*:
   (a) $\lim_{\kappa \downarrow 0} \lambda_p(\kappa) = \lambda_p(0)$;
   (b) $\lim_{\kappa \to \infty} \lambda_p(\kappa) = \rho\gamma$;
   (c) *if* $p(\cdot, \cdot)$ *has zero mean and finite variance, then there exists* $\kappa_0 > 0$ *such that* $p \mapsto \lambda_p(\kappa)$ *is strictly increasing for* $\kappa \in [0, \kappa_0)$.

Theorem 1.3 says that the Lyapunov exponents exist and do not depend on the choice of the starting measure μ. Theorem 1.4 says that the Lyapunov exponents are continuous functions of the diffusion constant κ away from 0 and that the system exhibits *clumping* for all κ: the Lyapunov exponents are strictly larger in the random medium than in the average medium. Theorem 1.5 shows that the Lyapunov exponents satisfy a *dichotomy* (see Figure 3): for $p(\cdot, \cdot)$ with zero mean and finite variance they are trivial when $1 \leq d \leq 4$, but display an interesting dependence on κ when $d \geq 5$. In the latter case (a) the Lyapunov exponents are continuous in κ at $\kappa = 0$; (b) the clumping vanishes in the limit as $\kappa \to \infty$: when the reactant particles move much faster than the catalyst particles, they effectively see the average medium; (c) the system is intermittent for small κ: when the reactant particles move much slower than the catalyst particles, the growth rates of their successive moments are determined by different piles of the catalyst.

Theorems 1.3 and 1.4 are proved in Sections 2 and 3, respectively. Section 4 contains block estimates for coalescing random walks, which are needed to ex-
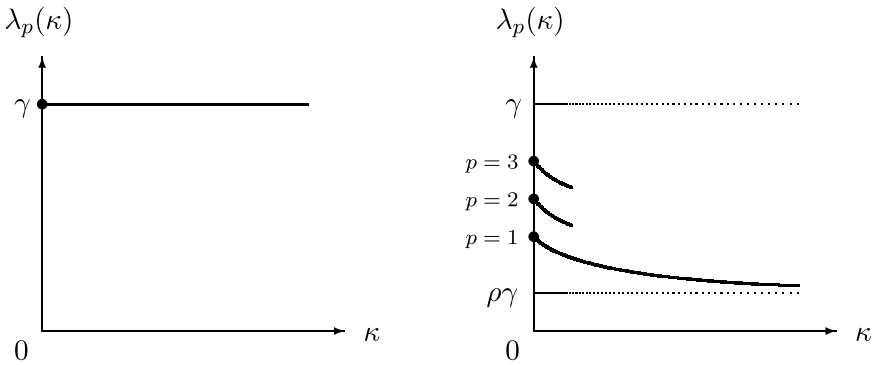
FIG. 3. $\kappa \mapsto \lambda_p(\kappa)$ *for $1 \leq d \leq 4$, respectively, $d \geq 5$, when $p(\cdot, \cdot)$ has zero mean and finite variance.*

ploit Proposition 1.2 in order to prove Theorem 1.5(ii)(a) and (b). Finally, Theorem 1.5(i) and (ii)(c) is proved in Section 5.

1.8. *Open problems*.   The following problems remain open:

(1) Show that $\lambda_p(\kappa) < \gamma$ for *all* $\kappa \in [0, \infty)$ when $d \geq 5$ and $p(\cdot, \cdot)$ has zero mean and finite variance.
(2) Show that $\kappa \mapsto \lambda_p(\kappa)$ is convex on $[0, \infty)$. Convexity, when combined with the properties in Theorems 1.4(ii) and 1.5(ii)(b), would imply that $\kappa \mapsto \lambda_p(\kappa)$ is strictly decreasing on $[0, \infty)$ when $d \geq 5$. Convexity was proved in [5] and [6] for the case where ξ is a field of independent simple random walks in a Poisson equilibrium, respectively, a symmetric exclusion process in a Bernoulli equilibrium.
(3) Show that the following extension of Theorem 1.5 is true: the Lyapunov exponents are nontrivial if and only if $p^{(s)}(\cdot, \cdot)$ is strongly transient, that is, $\int_0^\infty t p_t^{(s)}(0, 0)\, dt < \infty$. A similar *full dichotomy* was found in [6] for the case where ξ is a symmetric exclusion process in a Bernoulli equilibrium, namely, between recurrent and transient $p(\cdot, \cdot)$.

1.9. *A scaling conjecture*.   Let $p_t(x, y)$ be the probability for the random walk with transition kernel $p(\cdot, \cdot)$ [satisfying (1.5)] and step rate 1 to move from x to y in time t. The following conjecture is a refinement of Theorem 1.5(ii)(b).

CONJECTURE 1.6.   *Suppose that $p(\cdot, \cdot)$ is a simple random walk. Then for all $d \geq 5$, $p \in \mathbb{N}$, $\gamma \in (0, \infty)$ and $\rho \in (0, 1)$,*

(1.28)
$$\lim_{\kappa \to \infty} 2d\kappa[\lambda_p(\kappa) - \rho\gamma]$$
$$= \frac{\rho(1-\rho)\gamma^2}{G_d}G_d^* + 1_{\{d=5\}}(2d)^5\left[\frac{\rho(1-\rho)\gamma^2}{G_d}p\right]^2 \mathcal{P}_5$$

with

$$(1.29) \qquad G_d = \int_0^\infty p_t(0,0)\,dt, \qquad G_d^* = \int_0^\infty t p_t(0,0)\,dt$$

and

$$(1.30) \qquad \mathcal{P}_5 = \sup_{\substack{f \in H^1(\mathbb{R}^5) \\ \|f\|_2 = 1}} \left[\int_{\mathbb{R}^5} \int_{\mathbb{R}^5} dx\,dy\, \frac{f^2(x) f^2(y)}{16\pi^2 \|x - y\|} - \|\nabla f\|_2^2 \right] \in (0, \infty),$$

where $\| \cdot \|_2$ is the L^2-norm on $\mathbb{R}^5$, ∇ is the gradient operator, and $H^1(\mathbb{R}^5) = \{f : \mathbb{R}^5 \to \mathbb{R} : f, \nabla f \in L^2(\mathbb{R}^5)\}$.

A remarkable feature of (1.28) is the occurrence of a "polaron-type" term in $d = 5$. An important consequence of (1.28) is that in $d = 5$ there exists a $\kappa_1 < \infty$ such that $\lambda_p(\kappa) > \lambda_{p-1}(\kappa)$ for all $\kappa \in (\kappa_1, \infty)$ when $p = 2$ and, by the remark made after formula (1.22), also when $p \in \mathbb{N} \setminus \{1\}$, that is, the solution of the parabolic Anderson model is intermittent for all κ sufficiently large. For $d \geq 6$, Conjecture 1.6 does not allow to decide about intermittency for large κ.

The analogue of (1.28) for independent simple random walks and simple symmetric exclusion was proved in [5, 6] and [8] with quite a bit of effort (with $d = 3$ rather than $d = 5$ appearing as the critical dimension). We provide a heuristic explanation of (1.28) in the Appendix.

2. Proof of Theorem 1.3. Throughout this section we assume that $p(\cdot, \cdot)$ satisfies (1.5). The existence of the Lyapunov exponents for $\mu = \nu_\rho S_T$, $T \in [0, \infty]$, is proved in Section 2.1, the fact that they are equal is proved in Section 2.2. In what follows, $d \geq 1$, $p \in \mathbb{N}$, $\kappa \in [0, \infty)$, $\gamma \in (0, \infty)$ and $\rho \in (0, 1)$ are kept fixed. Recall (1.21).

2.1. *Existence of Lyapunov exponents.*

PROPOSITION 2.1. *For all $T \in [0, \infty]$, the Lyapunov exponent $\lambda_p^{\nu_\rho S_T}$ exists.*

PROOF.   The proof proceeds in 2 steps:
Step 1 (Bridge approximation argument). Let $Q_{t \log t} = \mathbb{Z}^d \cap [-t \log t, t \log t]^d$. As noted in Gärtner and den Hollander [5], Section 4.1, we have, for $\mu = \nu_\rho S_T$,

$$(2.1) \quad \underline{\Delta}_p^\mu(t) \leq \Lambda_p^\mu(t) \leq \frac{1}{pt} \log\bigl(|Q_{t \log t}|^p e^{pt \underline{\Delta}_p^\mu(t)} + p e^{\gamma p t} \mathrm{P}_0\bigl(X_1^\kappa(t) \notin Q_{t \log t}\bigr)\bigr)$$

with

$$(2.2) \quad \begin{aligned} \underline{\Delta}_p^\mu(t) = \frac{1}{pt} \log \max_{x \in \mathbb{Z}^d} (\mathbb{E}_\mu \otimes \mathrm{E}_0^{\otimes p}) \\ \times \left(\exp\left[\gamma \int_0^t \sum_{q=1}^p \xi(X_q^\kappa(s), t - s)\,ds \right] \prod_{q=1}^p \delta_x(X_q^\kappa(t)) \right). \end{aligned}$$

Since $\lim_{t\to\infty}(1/t)\log P_0(X_1^\kappa(t)\notin Q_{t\log t})=-\infty$, it follows that

$$(2.3) \qquad \lim_{t\to\infty}[\Lambda_p^\mu(t)-\underline{\Delta}_p^\mu(t)]=0.$$

Hence, to prove the existence of λ_p^μ, it suffices to prove the existence of

$$(2.4) \qquad \underline{\lambda}_p^\mu=\lim_{t\to\infty}\underline{\Delta}_p^\mu(t),$$

after which we can conclude from (2.3) that $\underline{\lambda}_p^\mu=\lambda_p^\mu$. We will prove (2.4) by showing that $t\mapsto t\underline{\Delta}_p^\mu(t)$ is superadditive, which will imply that

$$(2.5) \qquad \underline{\lambda}_p^\mu=\sup_{t>0}\underline{\Delta}_p^\mu(t).$$

Step 2 (Superadditivity). We first give the proof for $p=1$. To that end, abbreviate

$$(2.6) \qquad \mathcal{E}(t,y)=\exp\left[\gamma\int_0^t\xi\big(X^\kappa(s),t-s\big)\,ds\right]\delta_y(X^\kappa(t)), \qquad t>0, y\in\mathbb{Z}^d.$$

Using formula (1.24) in Proposition 1.2, we have, for all $t_1,t_2>0$ and $x,y\in\mathbb{Z}^d$,

$$e^{-\rho\gamma(t_1+t_2)}(\mathbb{E}_{\nu_\rho S_T}\otimes\mathrm{E}_0)\big(\mathcal{E}(t_1+t_2,x)\big)$$

$$= (\mathrm{E}_0\otimes\mathbb{E}_{\mathrm{Poiss}})\big(\delta_x\big(X^\kappa(t_1+t_2)\big)\mathbb{E}^*\big(\rho^{-\mathcal{N}_{T+t_1+t_2}^{\mathrm{coal}}\{(X^\kappa(s),s):s\in\Pi_{\rho\gamma}([0,t_1+t_2])\}}\big)\big)$$

$$\geq (\mathrm{E}_0\otimes\mathbb{E}_{\mathrm{Poiss}})\big(\delta_y(X^\kappa(t_1))\delta_x\big(X^\kappa(t_1+t_2)\big)$$

$$\times\mathbb{E}^*\big(\rho^{-\mathcal{N}_{T+t_1}^{\mathrm{coal}}\{(X^\kappa(s),s):s\in\Pi_{\rho\gamma}([0,t_1])\}}$$

$$(2.7)$$

$$\times\rho^{-\mathcal{N}_{T+t_1+t_2}^{\mathrm{coal}}\{(X^\kappa(s),s):s\in\Pi_{\rho\gamma}([t_1,t_1+t_2])\}}\big)\big)$$

$$= (\mathrm{E}_0\otimes\mathbb{E}_{\mathrm{Poiss}})\big(\delta_y(X^\kappa(t_1))\delta_{x-y}\big(X^\kappa(t_1+t_2)-X^\kappa(t_1)\big)$$

$$\times\mathbb{E}^*\big(\rho^{-\mathcal{N}_{T+t_1}^{\mathrm{coal}}\{(X^\kappa(s),s):s\in\Pi_{\rho\gamma}([0,t_1])\}}$$

$$\times\rho^{-\mathcal{N}_{T+t_1+t_2}^{\mathrm{coal}}\{(X^\kappa(s)-X^\kappa(t_1),s):s\in\Pi_{\rho\gamma}([t_1,t_1+t_2])\}}\big)\big),$$

where the inequality comes from inserting the extra factor $\delta_y(X^\kappa(t_1))$ under the expectation and ignoring coalescence between random walks that start before, respectively, after time t_1, and the last line uses the shift-invariance of $\mathcal{N}_{T+t_1+t_2}^{\mathrm{coal}}$. Because X^κ and $\Pi_{\rho\gamma}$ have independent stationary increments, we have

r.h.s. (2.7)

$$(2.8)$$

$$= (\mathrm{E}_0\otimes\mathbb{E}_{\mathrm{Poiss}})\big(\delta_y(X^\kappa(t_1))\mathbb{E}^*\big(\rho^{-\mathcal{N}_{T+t_1}^{\mathrm{coal}}\{(X^\kappa(s),s):s\in\Pi_{\rho\gamma}([0,t_1])\}}\big)\big)$$

$$\times(\mathrm{E}_0\otimes\mathbb{E}_{\mathrm{Poiss}})\big(\delta_{x-y}(X^\kappa(t_2))\mathbb{E}^*\big(\rho^{-\mathcal{N}_{T+t_2}^{\mathrm{coal}}\{(X^\kappa(s),s):s\in\Pi_{\rho\gamma}([0,t_2])\}}\big)\big)$$

$$= e^{-\rho\gamma t_1}(\mathbb{E}_{\nu_\rho S_T}\otimes\mathrm{E}_0)\big(\mathcal{E}(y,t_1)\big)\times e^{-\rho\gamma t_2}(\mathbb{E}_{\nu_\rho S_T}\otimes\mathrm{E}_0)\big(\mathcal{E}(x-y,t_2)\big),$$

where in the last line we again use formula (1.24). Taking the maximum over $x, y \in \mathbb{Z}^d$ in (2.7)–(2.8), we conclude that

$$(2.9) \quad \exp[(t_1 + t_2)\underline{\Lambda}_1^{v_\rho S_T}(t_1 + t_2)] \geq \exp[t_1\underline{\Lambda}_1^{v_\rho S_T}(t_1)] \times \exp[t_2\underline{\Lambda}_1^{v_\rho S_T}(t_2)],$$

which proves the superadditivity of $t \mapsto t\underline{\Lambda}_1^{v_\rho S_T}(t)$.

The same proof works for $p \in \mathbb{N} \setminus \{1\}$. Simply replace (2.6) by

$$\mathcal{E}_p(t, y) = \exp\left[\gamma \int_0^t \sum_{q=1}^p \xi(X_q^\kappa(s), t - s)\, ds\right] \prod_{q=1}^p \delta_y(X_q^\kappa(t)),$$

(2.10)
$$t \geq 0, y \in \mathbb{Z}^d,$$

and proceed in a similar manner. □

2.2. *Equality of Lyapunov exponents.*

PROPOSITION 2.2. $\lambda_p^{v_\rho} = \lambda_p^{v_\rho S_T}$ *for all* $T \in [0, \infty]$. *In particular,* $\lambda^{v_\rho} = \lambda^{\mu_\rho}$.

PROOF.   We first give the proof for $p = 1$.

$\lambda_1^{v_\rho} \leq \lambda_1^{v_\rho S_T}$: Since $t \mapsto \mathcal{N}_t^{\text{coal}}$ is nondecreasing, it is immediate from the representation formula (1.25) in Proposition 1.2 that

$$(2.11) \qquad \Lambda_1^{v_\rho}(t) \leq \Lambda_1^{v_\rho S_T}(t) \qquad \forall t > 0, T \in [0, \infty].$$

Since $\lambda_1^{v_\rho S_T} = \lim_{t \to \infty} \Lambda_1^{v_\rho S_T}(t)$, this implies the claim.

$\lambda_1^{v_\rho} \geq \lambda_1^{v_\rho S_T}$: We first assume that $T < \infty$. Recall (2.3) and (2.4)–(2.6), and estimate, for $T, t > 0$,

$$\lambda_1^{v_\rho} = \Lambda_1^{v_\rho}(\infty) = \underline{\Lambda}_1^{v_\rho}(\infty)$$

(2.12)
$$\geq \underline{\Lambda}_1^{v_\rho}(T + t) = \frac{1}{T + t} \log \max_{x \in \mathbb{Z}^d} (\mathbb{E}_{v_\rho} \otimes \mathrm{E}_0)(\mathcal{E}(T + t, x)).$$

In the right-hand side of (2.12), drop the part $s \in [t, T + t]$ from the integral over $s \in [0, T + t]$ in definition (2.6) of $\mathcal{E}(T + t, x)$, insert an extra factor $\delta_x(X^\kappa(t))$ under the expectation, and use the Markov property of ξ and X^κ at time t. This gives

$$(2.13) \quad \text{r.h.s. (2.12)} \geq \frac{1}{T + t} \log \max_{x \in \mathbb{Z}^d} \{(\mathbb{E}_{v_\rho S_T} \otimes \mathrm{E}_0)(\mathcal{E}(t, x))\mathrm{P}_0(X^\kappa(T) = 0)\}.$$

Combine (2.12) with (2.13) to get

$$(2.14) \qquad \lambda_1^{v_\rho} \geq \frac{t}{T + t}\underline{\Lambda}_1^{v_\rho S_T}(t) + \frac{1}{T + t} \log \mathrm{P}_0(X^\kappa(T) = 0).$$

Let $t \to \infty$ to get $\lambda_1^{\nu_\rho} \geq \underline{\Lambda}_1^{\nu_\rho S_T}(\infty) = \lambda_1^{\nu_\rho S_T}$, which proves the claim.

Next, for $T, t > 0$ and $x \in \mathbb{Z}^d$,

$$(2.15) \quad \lambda_1^{\nu_\rho} \geq \lambda_1^{\nu_\rho S_T} = \underline{\Lambda}_1^{\nu_\rho S_T}(\infty) \geq \underline{\Lambda}_1^{\nu_\rho S_T}(t) \geq \frac{1}{t} \log(\mathbb{E}_{\nu_\rho S_T} \otimes \mathrm{E}_0)(\mathcal{E}(t, x)),$$

where we have used (2.5). The weak convergence of $\nu_\rho S_T$ to μ_ρ implies that we can take the limit as $T \to \infty$ to obtain

$$(2.16) \qquad\qquad \lambda_1^{\nu_\rho} \geq \frac{1}{t} \log(\mathbb{E}_{\mu_\rho} \otimes \mathrm{E}_0)(\mathcal{E}(t, x)).$$

Finally, taking the maximum over x and letting $t \to \infty$, we arrive at $\lambda_1^{\nu_\rho} \geq \lambda_1^{\mu_\rho}$, which is the claim for $T = \infty$.

The same proof works for $p \in \mathbb{N} \setminus \{1\}$ by using (2.10) instead of (2.6). $\quad\square$

3. Proof of Theorem 1.4. Throughout this section we assume that $p(\cdot, \cdot)$ satisfies (1.5). In Section 3.1 we show that $\kappa \mapsto \lambda_p(\kappa)$ is globally Lipschitz outside any neighborhood of 0. In Section 3.2 we show that $\lambda_p(\kappa) > \rho\gamma$ for all $\kappa \in [0, \infty)$. In what follows, $d \geq 1$, $p \in \mathbb{N}$, $\gamma \in (0, \infty)$ and $\rho \in (0, 1)$ are kept fixed.

3.1. *Lipschitz continuity.* In this section we prove Theorem 1.4(i).

PROOF OF THEOREM 1.4(i). In what follows, μ can be any of the initial distributions $\nu_\rho S_T$, $T \in [0, \infty]$ (recall Proposition 2.2). We write $\Lambda_p^\mu(\kappa; t)$ to indicate the κ-dependence of $\Lambda_p^\mu(t)$ given by (1.20). We give the proof for $p = 1$.

Pick $\kappa_1, \kappa_2 \in (0, \infty)$ with $\kappa_1 < \kappa_2$ arbitrarily. By a standard application of Girsanov's formula,

$$\exp[t \Lambda_1^\mu(\kappa_2; t)]$$

$$= (\mathbb{E}_\mu \otimes \mathrm{E}_0)\left(\exp\left[\gamma \int_0^t \xi(X^{\kappa_2}(s), t - s) \, ds \right] \right)$$

$$(3.1) \qquad = (\mathbb{E}_\mu \otimes \mathrm{E}_0)\left(\exp\left[\gamma \int_0^t \xi(X^{\kappa_1}(s), t - s) \, ds \right] \right.$$

$$\left. \times \exp[J(X^{\kappa_1}; t) \log(\kappa_2/\kappa_1) - 2d(\kappa_2 - \kappa_1)t] \right)$$

$$= I + II,$$

where $J(X^{\kappa_1}; t)$ is the number of jumps of X^{κ_1} up to time t, I and II are the contributions coming from the events $\{J(X^{\kappa_1}; t) \leq M2d\kappa_2 t\}$, respectively, $\{J(X^{\kappa_1}; t) > M2d\kappa_2 t\}$, and $M > 1$ is to be chosen. Clearly,

$$(3.2) \qquad I \leq \exp\left[(M2d\kappa_2 \log(\kappa_2/\kappa_1) - 2d(\kappa_2 - \kappa_1))t \right] \exp[t \Lambda_1^\mu(\kappa_1; t)],$$

while

$$(3.3) \qquad\qquad II \leq e^{\gamma t} \mathrm{P}_0\big(J(X^{\kappa_2}; t) > M2d\kappa_2 t \big)$$

because we may estimate $\int_0^t \xi(X^{\kappa_1}(s), t - s)\, ds \leq t$ and afterward use Girsanov's formula in the reverse direction. Since $J(X^{\kappa_2}; t) = J^*(2d\kappa_2 t)$ with $(J^*(t))_{t\geq 0}$ a rate-1 Poisson process, we have

$$(3.4) \qquad \lim_{t\to\infty} \frac{1}{t} \log \mathrm{P}_0\big(J(X^{\kappa_2}; t) > M2d\kappa_2 t\big) = -2d\kappa_2 \mathcal{I}(M)$$

with

$$(3.5) \qquad \mathcal{I}(M) = \sup_{u\in\mathbb{R}}[Mu - (e^u - 1)] = M \log M - M + 1.$$

Since $\lambda_1(\kappa) = \lim_{t\to\infty} \Lambda_1^\mu(\kappa; t)$, it follows from (3.1)–(3.4) that

$$(3.6) \qquad \begin{aligned} \lambda_1(\kappa_2) &\leq [M2d\kappa_2 \log(\kappa_2/\kappa_1) - 2d(\kappa_2 - \kappa_1) + \lambda_1(\kappa_1)] \\ &\quad \vee [\gamma - 2d\kappa_2 \mathcal{I}(M)]. \end{aligned}$$

On the other hand, estimating $J(X^{\kappa_1}; t) \geq 0$ in (3.1), we have

$$(3.7) \qquad \exp[t\Lambda_1^\mu(\kappa_2; t)] \geq \exp[-2d(\kappa_2 - \kappa_1)t] \exp[t\Lambda_1^\mu(\kappa_1; t)],$$

which gives the lower bound

$$(3.8) \qquad \lambda_1(\kappa_2) - \lambda_1(\kappa_1) \geq -2d(\kappa_2 - \kappa_1).$$

Next, for $\kappa \in (0, \infty)$, define

$$(3.9) \qquad \begin{aligned} D^+\lambda_1(\kappa) &= \limsup_{\delta\to 0} \delta^{-1}[\lambda_1(\kappa + \delta) - \lambda_1(\kappa)], \\ D^-\lambda_1(\kappa) &= \liminf_{\delta\to 0} \delta^{-1}[\lambda_1(\kappa + \delta) - \lambda_1(\kappa)]. \end{aligned}$$

Then, picking $\kappa_1 = \kappa$ and $\kappa_2 = \kappa + \delta$ (resp., $\kappa_1 = \kappa - \delta$ and $\kappa_2 = \kappa$) in (3.6) and letting $\delta \downarrow 0$, we get

$$(3.10) \quad D^+\lambda_1(\kappa) \leq (M - 1)2d \qquad \forall M > 1: 2d\kappa\mathcal{I}(M) - (1 - \rho)\gamma \geq 0$$

[with the latter together with $\lambda_1(\kappa) \geq \rho\gamma$ guaranteeing that the first term in the right-hand side of (3.6) is the maximum], while (3.8) gives

$$(3.11) \qquad D^-\lambda_1(\kappa) \geq -2d.$$

We may pick

$$(3.12) \qquad M = M(\kappa) = \mathcal{I}^{-1}\left(\frac{(1 - \rho)\gamma}{2d\kappa}\right)$$

with $\mathcal{I}^{-1}$ the inverse of $\mathcal{I}: [1, \infty) \to \mathbb{R}$. Since $\mathcal{I}(M) = \frac{1}{2}(M - 1)^2[1 + o(1)]$ as $M \downarrow 1$, it follows that

$$(3.13) \qquad [M(\kappa) - 1]2d = 2d\sqrt{\gamma \frac{1 - \rho}{d\kappa}}[1 + o(1)] \qquad \text{as } \kappa \to \infty.$$

By (3.10), the latter implies that $\kappa \mapsto D^+\lambda_1(\kappa)$ is bounded from above outside any neighborhood of 0. Since, by (3.11), $\kappa \mapsto D^-\lambda_1(\kappa)$ is bounded from below, the claim follows.

The extension to $p \in \mathbb{N} \setminus \{1\}$ is straightforward and is left to the reader. $\square$

3.2. *Clumping.* In this section we prove Theorem 1.4(ii).

PROOF OF THEOREM 1.4(ii). Fix $d \geq 1$, $\kappa \in [0, \infty)$, $\gamma \in (0, \infty)$ and $\rho \in (0, 1)$. Since $p \mapsto \lambda_p(\kappa)$ is nondecreasing, it suffices to give the proof for $p = 1$. In what follows, μ can be any of the measures $\nu_\rho S_T$, $T \in [0, \infty]$ (recall Proposition 2.2).

Abbreviate

$$(3.14) \qquad I(X^\kappa; T) = \gamma \int_0^T ds \, [\xi(X^\kappa(s), T - s) - \rho], \qquad T > 0.$$

For any $T > 0$ we have, recalling (2.2)–(2.5),

$$(3.15) \qquad \begin{aligned} \lambda_1(\kappa) &= \Lambda_1^\mu(\infty) = \underline{\Delta}_1^\mu(\infty) \geq \underline{\Delta}_1^\mu(T) \\ &\geq \rho\gamma + \frac{1}{T} \log(\mathbb{E}_\mu \otimes \mathrm{E}_0)(\exp[I(X^\kappa; T)]\delta_0(X^\kappa(T))) \\ &\geq \rho\gamma + \frac{1}{T} \log(\mathbb{E}_\mu \otimes \mathrm{E}_0)\left(\left[1 + I(X^\kappa; T) \right.\right. \\ &\qquad\qquad \left.\left. + \frac{1}{2} I(X^\kappa; T)^2 e^{-\gamma T} \right] \delta_0(X^\kappa(T))\right), \end{aligned}$$

where in the third line we use that $e^x \geq 1 + x + \frac{1}{2} x^2 e^{-|x|}$, $x \in \mathbb{R}$.

As $T \downarrow 0$, we have

$$(3.16) \qquad \begin{aligned} (\mathbb{E}_\mu \otimes \mathrm{E}_0)\left(\left[\frac{1}{T} I(X^\kappa; T)\right]^2 \delta_0(X^\kappa(T))\right) &\to \gamma^2 \int_\Omega \mu(d\eta)[\eta(0) - \rho]^2 \\ &= \rho(1 - \rho)\gamma^2 \end{aligned}$$

and

$$(3.17) \qquad (\mathbb{E}_\mu \otimes \mathrm{E}_0)\left(\left[\frac{1}{T} I(X^\kappa; T)\right] \delta_0(X^\kappa(T))\right) \geq -O(T^2).$$

The claim in (3.16) is obvious, the claim in (3.17) will be proven below. Combining (3.15)–(3.17), we have

$$(3.18) \qquad \lambda_1(\kappa) - \rho\gamma \geq \frac{1}{4} T \rho(1 - \rho)\gamma^2, \qquad 0 < T \leq T_0(\kappa),$$

for some $T_0(\kappa) < \infty$, showing that $\lambda_1(\kappa) > \rho\gamma$.

To prove (3.17), let $J(X^\kappa; T)$ denote the number of jumps by X^κ up to time T. Then

$$(3.19) \qquad \begin{aligned} &(\mathbb{E}_\mu \otimes \mathrm{E}_0)\left(\left[\frac{1}{T} I(X^\kappa; T)\right] \delta_0(X^\kappa(T))\right) \\ &\quad = (\mathbb{E}_\mu \otimes \mathrm{E}_0)\left(\left[\frac{1}{T} I(X^\kappa; T)\right] \delta_0(X^\kappa(T)) \right. \\ &\qquad\qquad \left. \times \left(\mathbb{1}\{J(X^\kappa; T) = 0\} + \mathbb{1}\{J(X^\kappa; T) \geq 1\}\right)\right). \end{aligned}$$

The first term in the right-hand side of (3.19) equals

$$(3.20) \qquad \mathrm{P}_0\bigl(J(X^\kappa; T) = 0\bigr) \frac{\gamma}{T} \int_0^T ds\, \mathbb{E}_\mu\bigl(\xi(0, s) - \rho\bigr) = 0,$$

while the second term is bounded below by

$$
\begin{aligned}
(3.21) \qquad -\rho\gamma\mathrm{P}_0\bigl(J(X^\kappa; T) \geq 1,\, X^\kappa(T) = 0\bigr) &\geq -\rho\gamma\mathrm{P}_0\bigl(J(X^\kappa; T) \geq 2\bigr) \\
&= -O(T^2),
\end{aligned}
$$

as $T \downarrow 0$. Combine (3.19)–(3.21) to get the claim in (3.17). $\quad\square$

4. Proof of Theorem 1.5(ii)(a) and (b).   Throughout this section we assume that $p(\cdot, \cdot)$ satisfies (1.5) and that $d \geq 5$. In Section 4.1 we state an estimate for blocks of coalescing random walks. In Section 4.2 we formulate two lemmas, and in Section 4.3 we use these lemmas to prove the block estimate. The block estimate is used in Sections 4.4 and 4.5 to prove Theorem 1.5(ii)(a) and (b), respectively.

4.1. *Block estimate.*   We call a collection of subsets $S_1, \ldots, S_N$ of $\mathbb{R}$ *ordered*, if $s < t$ for all $s \in S_i$, $t \in S_j$ and $i < j$. Given a path $\psi : \mathbb{R} \to \mathbb{Z}^d$ and a collection of disjoint finite subsets $S_1, \ldots, S_N$ of $\mathbb{R}$, we are going to estimate the moment generating function of $\mathcal{N}_\infty^{\mathrm{coal}}\{(\psi(s), s) : s \in \bigcup_{j=1}^N S_j\}$, the number of random walks starting from sites $\psi(s)$ at times $s \in \bigcup_{j=1}^N S_j$ that coalesce eventually [recall (1.23)]. Let $d(S_i, S_j)$ denote the Euclidean distance between S_i and S_j.

Our key estimate, which will be proved in Section 4.3, is the following proposition.

PROPOSITION 4.1.   *Let $d \geq 5$. Then there exist $\delta : (0, \infty) \to (0, \infty)$ with $\lim_{K \to \infty} \delta(K) = 0$ and, for each $\epsilon \in (0, (d-4)/2)$, $C_\epsilon > 0$ such that the following holds. For all $\rho \in (0, 1)$, $\psi : \mathbb{R} \to \mathbb{Z}^d$, all ordered collections of disjoint finite subsets $S_1, \ldots, S_N$ of $\mathbb{R}$, all $\epsilon \in (0, (d-4)/2)$, $K > 0$ and $r, r' > 1$ with $1/r + 1/r' = 1$,*

$$\mathbb{E}^*\bigl(\rho^{-\mathcal{N}_\infty^{\mathrm{coal}}\{(\psi(s), s)\,:\,s \in \bigcup_{j=1}^N S_j\}}\bigr)$$

$$
\begin{aligned}
(4.1) \qquad &\leq \exp\left[\frac{\delta(K)}{\rho} \sum_{j=1}^N |S_j| + C_\epsilon K \frac{\rho^{-r'} - 1}{r'} \sum_{1 \leq j < k \leq N} \frac{|S_j||S_k|}{d(S_j, S_k)^{1+\epsilon}}\right] \\
&\quad \times \left[\prod_{j=1}^N \mathbb{E}^*\bigl(\rho^{-r\mathcal{N}_\infty^{\mathrm{coal}}\{(\psi(s), s)\,:\,s \in S_j\}}\bigr)\right]^{1/r}.
\end{aligned}
$$

Let $I_1', I_1'', \ldots, I_N', I_N''$ be a finite collection of adjacent time intervals and assume that $S_j \subset I_j'$ for $j = 1, \ldots, N$. What the above proposition does is *decouple* the coalescing random walks that start in disjoint *time-blocks* I_j' separated by *time-gaps* I_j''.

4.2. *Preparatory lemmas.* To prove Proposition 4.1, we need Lemmas 4.2–4.3 below. To this end, fix a path $\psi : \mathbb{R} \to \mathbb{Z}^d$ arbitrarily. Let $(Y^u)_{u \in \mathbb{R}}$ be a family of independent random walks Y^u with transition kernel $p^*(\cdot, \cdot)$ and step rate 1 starting from $\psi(u)$ at time u. Set $Y^u(s) = \psi(u)$ for $s < u$. We write $\mathbb{P}^*$ for the joint law of these random walks.

Given $u \in \mathbb{R}$ and $j \in \mathbb{Z}$, let

$$(4.2) \qquad R^u_j = \{Y^u(s) : s \in [j, j+1]\}$$

denote the *range* of Y^u in the time interval $[j, j+1]$. For $u \in \mathbb{R}$ and $K > 0$, define the event that Y^u is K-*good* by

$$(4.3) \qquad G^u_K = \bigcap_{j=\lfloor u \rfloor}^{\infty} \{|R^u_j| \le K \log(j - \lfloor u \rfloor + 5)\}.$$

For $u, v \in \mathbb{R}$ with $u < v$, define the event that Y^u and Y^v *meet* by

$$(4.4) \qquad M^{u,v} = \{\exists s \ge v : Y^u(s) = Y^v(s)\}.$$

Our two lemmas stated below give bounds for the probabilities of random walks not to be K-good, respectively, to meet given that the random walk that starts later is K-good.

LEMMA 4.2. *For all $u \in \mathbb{R}$ and $K > 0$,*

$$(4.5) \qquad \mathbb{P}^*([G^u_K]^c) \le \delta(K)$$

with

$$(4.6) \qquad \delta(K) = \sum_{j=5}^{\infty} \exp\big[-\lfloor K \log j \rfloor \big(\log(\lfloor K \log j \rfloor) - 1\big) - 1\big] < \infty$$

satisfying $\lim_{K \to \infty} \delta(K) = 0$.

PROOF. Recalling (4.3) and taking into account that Y^u has stationary increments, we have

$$(4.7) \quad \mathbb{P}^*([G^u_K]^c) \le \sum_{j=0}^{\infty} \mathbb{P}^*\big(|R^0_j| > K \log(j+5)\big) \le \sum_{j=5}^{\infty} \mathbb{P}^*\big(N_1 \ge \lfloor K \log j \rfloor\big),$$

where N_1 denotes the Poisson number of jumps of Y^0 during a time interval of length 1. An application of Chebyshev's exponential inequality yields, for $\beta > 0$,

$$(4.8) \qquad \begin{aligned} \mathbb{P}^*(N_1 \ge \lfloor K \log j \rfloor) &\le e^{-\beta \lfloor K \log j \rfloor} \mathbb{E}^*(e^{\beta N_1}) \\ &= \exp[-\beta \lfloor K \log j \rfloor + e^\beta - 1] \\ &= \exp\big[-\lfloor K \log j \rfloor \big(\log(\lfloor K \log j \rfloor) - 1\big) - 1\big], \end{aligned}$$

where in the last line we optimize over the choice of β by taking $\beta = \log(\lfloor K \times \log j \rfloor)$. Combining (4.7) and (4.8), we get the claim. $\square$

LEMMA 4.3.   *Let $d \geq 5$. Then for all $\epsilon \in (0, (d-4)/2)$ there exists $C_\epsilon > 0$ such that for all $K > 0$ and all $u, v \in \mathbb{R}$ with $u < v$,*

$$(4.9) \qquad \mathbb{P}^*(M^{u,v} \mid Y^v) \leq \frac{C_\epsilon K}{(v-u)^{1+\epsilon}} \qquad \text{on } G_K^v.$$

PROOF.   Fix $u, v \in \mathbb{R}$ with $u < v$. Recall (4.2)–(4.4) to see that

$$(4.10) \qquad M^{u,v} \subseteq \bigcup_{j=\lfloor v \rfloor}^{\infty} \bigcup_{z \in R_j^v} \{\exists s \in [j, j+1] : Y^u(s) = z\}.$$

Hence,

$$(4.11) \qquad \mathbb{P}^*(M^{u,v} \mid Y^v) \leq \sum_{j=\lfloor v \rfloor}^{\infty} \sum_{z \in R_j^v} \mathbb{P}^*(\exists s \in [j, j+1] : Y^u(s) = z).$$

Since the transition kernel $p^*(\cdot, \cdot)$ generates $\mathbb{Z}^d$ [recall (1.5)], there exists a constant $C > 0$ such that

$$(4.12) \qquad p_t^*(x, y) \leq \frac{C}{(t+7)^{d/2}} \qquad \forall t \geq 0, \ \forall x, y \in \mathbb{Z}^d$$

(see Spitzer [12], Proposition 7.6). Let Y be a random walk on $\mathbb{Z}^d$ with transition kernel $p^*(\cdot, \cdot)$ and jump rate 1. Let $\mathbb{P}_y^Y$ denote its law when starting at y and $\tau_z = \inf\{s \geq 0 : Y(s) = z\}$ its first hitting time of z. Then, since Y^u and Y have the same independent and stationary increments, we have, for $j \geq \lfloor v \rfloor$,

$$\mathbb{P}^*(\exists s \in [j, j+1] : Y^u(s) = z) \leq \sum_{y \in \mathbb{Z}^d} p_{(j \vee u)-u}^*(\psi(u), y) \mathbb{P}_y^Y(\tau_z \leq 1)$$

$$(4.13) \qquad\qquad\qquad \leq \frac{C}{(j-u+6)^{d/2}} \sum_{y \in \mathbb{Z}^d} \mathbb{P}_0^Y(\tau_y \leq 1)$$

$$\qquad\qquad\qquad = \frac{C}{(j-u+6)^{d/2}} \mathbb{E}_0^Y(|R|),$$

where $R = \{Y(s) : s \in [0, 1]\}$ is the range of Y in the time interval $[0, 1]$. Since $|R| \leq 1 + N_1$ with N_1 the Poisson number of jumps of Y in $[0, 1]$, we have $\mathbb{E}_0^Y(|R|) \leq 2$. Now assume that Y^v is K-good [recall (4.3)]. Then, combining (4.11) with (4.13), we obtain

$$\mathbb{P}^*(M^{u,v} \mid Y^v) \leq 2CK \sum_{j=\lfloor v \rfloor}^{\infty} \frac{\log(j - \lfloor v \rfloor + 5)}{(j - u + 6)^{d/2}}$$

$$(4.14) \qquad\qquad\qquad \leq 2CK \sum_{j=\lfloor v \rfloor}^{\infty} \frac{\log(j - \lfloor u \rfloor + 5)}{(j - \lfloor u \rfloor + 5)^{d/2}}$$

$$\qquad\qquad\qquad \leq 2CK \frac{\log(\lfloor v \rfloor - \lfloor u \rfloor + 4)}{(\lfloor v \rfloor - \lfloor u \rfloor + 4)^{(d-2)/2}}.$$

Since $d \geq 5$, this clearly implies (4.9). □

4.3. *Proof of block estimate.* In this section we use Lemmas 4.2 and 4.3 to prove Proposition 4.1.

PROOF OF PROPOSITION 4.1. Fix a path $\psi : \mathbb{R} \to \mathbb{Z}^d$ and an ordered collection of disjoint finite subsets $S_1, \ldots, S_N$ of $\mathbb{R}$ arbitrarily. Assume that the coalescing random walks starting from sites $\psi(s)$ at times $s \in \bigcup_{j=1}^N S_j$ are constructed from the independent random walks Y^u, $u \in \bigcup_{j=1}^N S_j$, introduced in Section 4.2, in the obvious recursive manner: if two walks meet for the first time, then the random walk that started earlier is killed and the random walk that started later survives.

Now recall (4.3). Distinguishing between all possible ways to distribute the good and the bad events and using the independence of the random walks Y^u, we estimate

$$\mathbb{E}^* \big(\rho^{-\mathcal{N}_\infty^{\mathrm{coal}} \{ (\psi(s), s) : s \in \bigcup_{j=1}^N S_j \}} \big)$$

$$= \sum_{\substack{A_i \subseteq S_i \\ 1 \leq i \leq N}} \mathbb{E}^* \bigg(\rho^{-\mathcal{N}_\infty^{\mathrm{coal}} \{ (\psi(s), s) : s \in \bigcup_{j=1}^N S_j \}}$$

(4.15)

$$\times \mathbb{1} \bigg\{ \bigcap_{j=1}^N \bigcap_{u \in A_j} G_K^u \bigg\} \mathbb{1} \bigg\{ \bigcap_{j=1}^N \bigcap_{u \in S_j \setminus A_j} [G_K^u]^c \bigg\} \bigg)$$

$$\leq \sum_{\substack{A_i \subseteq S_i \\ 1 \leq i \leq N}} \mathbb{E}^* \bigg(\rho^{-\mathcal{N}_\infty^{\mathrm{coal}} \{ (\psi(s), s) : s \in \bigcup_{j=1}^N A_j \}} \mathbb{1} \bigg\{ \bigcap_{j=1}^N \bigcap_{u \in A_j} G_K^u \bigg\} \bigg)$$

$$\times \rho^{-\sum_{j=1}^N |S_j \setminus A_j|} \prod_{j=1}^N \prod_{u \in S_j \setminus A_j} \mathbb{P}^* ([G_K^u]^c).$$

To estimate the expectation in the right-hand side of (4.15), we note that

$$\mathcal{N}_\infty^{\mathrm{coal}} \bigg\{ (\psi(s), s) : s \in \bigcup_{j=1}^N A_j \bigg\}$$

(4.16)

$$\leq \sum_{j=1}^N \mathcal{N}_\infty^{\mathrm{coal}} \{ (\psi(s), s) : s \in A_j \} + \sum_{j=1}^{N-1} \sum_{u \in A_j} \mathbb{1} \bigg\{ \bigcup_{k=j+1}^N \bigcup_{v \in A_k} M^{u,v} \bigg\}.$$

Here we overestimate the number of coalescences of random walks starting in one "time-block" A_j with random walks starting in later "time-blocks" A_k by the

number of them that meet at least one random walk starting in a later "time-block."
Together with Hölder's inequality with $r, r' > 1$ and $1/r + 1/r' = 1$, this yields

$$\mathbb{E}^*\left(\rho^{-\mathcal{N}_\infty^{\text{coal}}\{(\psi(s),s)\,:\,s\in\bigcup_{j=1}^N A_j\}}\mathbb{1}\left\{\bigcap_{j=1}^N\bigcap_{u\in A_j}G_K^u\right\}\right)$$

$$\le \mathbb{E}^*\left(\rho^{-\sum_{j=1}^N\mathcal{N}_\infty^{\text{coal}}\{(\psi(s),s)\,:\,s\in A_j\}}\right.$$

$$\left. \times\,\rho^{-\sum_{j=1}^{N-1}\sum_{u\in A_j}\mathbb{1}\{\bigcup_{k=j+1}^N\bigcup_{v\in A_k}(M^{u,v}\cap G_K^v)\}}\right)$$

$$(4.17)\quad \le\left[\prod_{j=1}^N\mathbb{E}^*\left(\rho^{-r\mathcal{N}_\infty^{\text{coal}}\{(\psi(s),s)\,:\,s\in S_j\}}\right)\right]^{1/r}$$

$$\times\left[\mathbb{E}^*\left(\prod_{j=1}^{N-1}\prod_{u\in S_j}\rho^{-r'\mathbb{1}\{\bigcup_{k=j+1}^N\bigcup_{v\in S_k}M^{u,v}\cap G_K^v\}}\right)\right]^{1/r'}$$

$$=\left[\prod_{j=1}^N\mathbb{E}^*\left(\rho^{-r\mathcal{N}_\infty^{\text{coal}}\{(\psi(s),s)\,:\,s\in S_j\}}\right)\right]^{1/r}$$

$$\times\left[\mathbb{E}^*\left(\prod_{j=1}^{N-1}\prod_{u\in S_j}\left(1+(\rho^{-r'}-1)\mathbb{1}\left\{\bigcup_{k=j+1}^N\bigcup_{v\in S_k}(M^{u,v}\cap G_K^v)\right\}\right)\right)\right]^{1/r'}.$$

In the last step we use the identity $\rho^{-r'\mathbb{1}\{A\}}=1+(\rho^{-r'}-1)\mathbb{1}\{A\}$. Now, by
conditional independence and Lemma 4.3, we have, for $\epsilon\in(0,(d-4)/2)$ and
$1\le j\le N-1$,

$$\mathbb{E}^*\left(\prod_{u\in S_j}\left(1+(\rho^{-r'}-1)\mathbb{1}\left\{\bigcup_{k=j+1}^N\bigcup_{v\in S_k}(M^{u,v}\cap G_K^v)\right\}\right)\middle|Y^w, w\in\bigcup_{l>j}S_l\right)$$

$$(4.18)\quad \le\prod_{u\in S_j}\left(1+(\rho^{-r'}-1)\sum_{k=j+1}^N\sum_{v\in S_k}\mathbb{P}^*(M^{u,v}|Y^v)\mathbb{1}\{G_K^v\}\right)$$

$$\le\exp\left[C_\epsilon K(\rho^{-r'}-1)\sum_{u\in S_j}\sum_{k=j+1}^N\sum_{v\in S_k}\frac{1}{(v-u)^{1+\epsilon}}\right].$$

Clearly,

$$(4.19)\qquad \sum_{u\in S_j}\sum_{k=j+1}^N\sum_{v\in S_k}\frac{1}{(v-u)^{1+\epsilon}}\le\sum_{k=j+1}^N\frac{|S_j||S_k|}{d(S_j,S_k)^{1+\epsilon}}.$$

Substituting this into the right-hand side of (4.18) and using the resulting deterministic bounds successively for $j = 1, \ldots, N-1$, we find that

(4.20)
$$\mathbb{E}^* \left(\prod_{j=1}^{N-1} \prod_{u \in S_j} \left(1 + (\rho^{-r'} - 1) \mathbb{1} \left\{ \bigcup_{k=j+1}^{N} \bigcup_{v \in S_k} (M^{u,v} \cap G_K^v) \right\} \right) \right)$$
$$\leq \exp \left[C_\epsilon K (\rho^{-r'} - 1) \sum_{1 \leq j < k \leq N} \frac{|S_j||S_k|}{d(S_j, S_k)^{1+\epsilon}} \right].$$

It remains to estimate the second factor in the right-hand side of (4.15). By Lemma 4.2,

(4.21)
$$\rho^{-\sum_{j=1}^{N} |S_j \setminus A_j|} \prod_{j=1}^{N} \prod_{u \in S_j \setminus A_j} \mathbb{P}^*([G_K^u]^c) \leq \left(\frac{\delta(K)}{\rho} \right)^{\sum_{j=1}^{N} |S_j \setminus A_j|}.$$

Observe that, by the binomial formula,

(4.22)
$$\sum_{\substack{A_i \subseteq S_i \\ 1 \leq i \leq N}} \left(\frac{\delta(K)}{\rho} \right)^{\sum_{j=1}^{N} |S_j \setminus A_j|} = \left(1 + \frac{\delta(K)}{\rho} \right)^{\sum_{j=1}^{N} |S_j|}$$
$$\leq \exp \left[\frac{\delta(K)}{\rho} \sum_{j=1}^{N} |S_j| \right].$$

Proposition 4.1 now follows by combining (4.15) with (4.17), (4.20) and (4.21), and afterward applying (4.22). $\quad \square$

4.4. *Continuity at $\kappa = 0$.* In this section we prove Theorem 1.5(ii)(a). We pick $\mu = \mu_\rho$ as the starting measure (recall Proposition 2.2).

By requiring that the p random walks in (1.20) do not step until time t, we have, for any $\kappa \in [0, \infty)$,

(4.23)
$$\Lambda_p^{\mu_\rho}(t; \kappa) \geq \Lambda_p^{\mu_\rho}(t; 0) + \frac{1}{pt} \log \mathrm{P}_0^{\otimes p}(X_q^\kappa(s) = 0 \; \forall s \in [0, t] \; \forall 1 \leq q \leq p)$$
$$= \Lambda_p^{\mu_\rho}(t; 0) - 2d\kappa.$$

Let $t \to \infty$ to obtain

(4.24)
$$\lambda_p(\kappa) \geq \lambda_p(0) - 2d\kappa.$$

Therefore, the continuity at $\kappa = 0$ reduces to proving that, for all $d \geq 5$, $p \in \mathbb{N}$, $\gamma \in (0, \infty)$ and $\rho \in (0, 1)$,

(4.25)
$$\limsup_{\kappa \downarrow 0} \lambda_p(\kappa) \leq \lambda_p(0).$$

PROOF OF THEOREM 1.5(ii)(a). We first give the proof for $p = 1$. Fix $L > 0$ and $\vartheta \in (0, 1)$ arbitrarily. For $j \in \mathbb{N}$, let

$$
(4.26) \quad \begin{aligned}
I_j &= \big[(j-1)L, jL\big), \qquad I'_j = \big[(j-1)L, (j-\vartheta)L\big), \\
I''_j &= \big[(j-\vartheta)L, jL\big)
\end{aligned}
$$

be the jth *time-interval*, *time-block* and *time-gap*, respectively. Fix r, r' with $1/r + 1/r' = 1$ arbitrarily and set

$$
(4.27) \qquad M = \frac{\rho\gamma(\rho^{-2r'} - 1)}{r' \log(1/\rho)}.
$$

For any Borel set $B \subseteq \mathbb{R}$, let

$$
(4.28) \qquad \widetilde{\Pi}_{\rho\gamma}(B) = \begin{cases} \Pi_{\rho\gamma}(B), & \text{if } |\Pi_{\rho\gamma}(B)| \leq LM, \\ \varnothing, & \text{otherwise.} \end{cases}
$$

Since

$$
(4.29) \quad \Pi_{\rho\gamma}([0, t]) \subseteq \bigcup_{j=1}^{\lceil t/L \rceil} \big(\widetilde{\Pi}_{\rho\gamma}(I'_j) \cup (\Pi_{\rho\gamma}(I'_j) \setminus \widetilde{\Pi}_{\rho\gamma}(I'_j)) \cup \Pi_{\rho\gamma}(I''_j) \big),
$$

we have

$$
\begin{aligned}
\mathcal{N}_\infty^{\text{coal}}\{(X^\kappa(s), s) : s \in \Pi_{\rho\gamma}([0, t])\} &\leq \mathcal{N}_\infty^{\text{coal}}\left\{ (X^\kappa(s), s) : s \in \bigcup_{j=1}^{\lceil t/L \rceil} \widetilde{\Pi}_{\rho\gamma}(I'_j) \right\} \\
(4.30) \qquad\qquad &+ \sum_{j=1}^{\lceil t/L \rceil} |\Pi_{\rho\gamma}(I'_j)| \mathbb{1}\{|\Pi_{\rho\gamma}(I'_j)| > LM\} \\
&+ \sum_{j=1}^{\lceil t/L \rceil} |\Pi_{\rho\gamma}(I''_j)|.
\end{aligned}
$$

Combining the representation formula (1.25) for $p = 1$ and $T = \infty$ with (4.30) and applying Hölder's inequality, we find that

$$
(4.31) \qquad \exp\big[t\big(\Lambda_1^{\mu_\rho}(t; \kappa) - \rho\gamma\big)\big] \leq \mathcal{E}_1 \mathcal{E}_2 \mathcal{E}_3,
$$

where

$$
(4.32) \qquad \mathcal{E}_1 = \big((\mathrm{E}_0 \otimes \mathbb{E}_{\text{Poiss}} \otimes \mathbb{E}^*)\big(\rho^{-r\mathcal{N}_\infty^{\text{coal}}\{(X^\kappa(s), s) : s \in \bigcup_{j=1}^{\lceil t/L \rceil} \widetilde{\Pi}_{\rho\gamma}(I'_j)\}}\big)\big)^{1/r},
$$

$$
(4.33) \qquad \mathcal{E}_2 = \left(\prod_{j=1}^{\lceil t/L \rceil} \mathbb{E}_{\text{Poiss}}\big(\rho^{-r'|\Pi_{\rho\gamma}(I'_j)|\mathbb{1}\{|\Pi_{\rho\gamma}(I'_j)| > LM\}}\big) \right)^{1/r'},
$$

$$
(4.34) \qquad \mathcal{E}_3 = \prod_{j=1}^{\lceil t/L \rceil} \mathbb{E}_{\text{Poiss}}\big(\rho^{-|\Pi_{\rho\gamma}(I''_j)|}\big) = \exp[\vartheta(1 - \rho)\gamma L \lceil t/L \rceil].
$$

To estimate $\mathcal{E}_1$ in (4.32), we apply Proposition 4.1 with $\psi(s) = X^\kappa(s)$, $N = \lceil t/L \rceil$, $S_j = \widetilde{\Pi}_{\rho\gamma}(I'_j)$ and ρ replaced by ρ^r. Then we obtain, for arbitrary $\epsilon \in (0, (d-4)/2)$ and $K > 0$,

$$(4.35) \qquad \mathbb{E}^*\big(\rho^{-r\mathcal{N}_\infty^{\mathrm{coal}}\{(X^\kappa(s),s)\,:\,s\in\bigcup_{j=1}^{\lceil t/L \rceil}\widetilde{\Pi}_{\rho\gamma}(I'_j)\}}\big) \leq \mathcal{E}'_1\mathcal{E}''_1$$

with

$$(4.36) \qquad \mathcal{E}'_1 = \left(\prod_{j=1}^{\lceil t/L \rceil}\mathbb{E}^*\big(\rho^{-r^2\mathcal{N}_\infty^{\mathrm{coal}}\{(X^\kappa(s),s)\,:\,s\in\widetilde{\Pi}_{\rho\gamma}(I'_j)\}}\big)\right)^{1/r}$$

and

$$(4.37) \qquad \begin{aligned} \mathcal{E}''_1 = \exp\Bigg[&\frac{\delta(K)}{\rho^r}\sum_{j=1}^{\lceil t/L \rceil}|\widetilde{\Pi}_{\rho\gamma}(I'_j)| \\ &+ C_\epsilon K\frac{\rho^{-rr'}-1}{r'}\sum_{1\leq j<k\leq\lceil t/L \rceil}\frac{|\widetilde{\Pi}_{\rho\gamma}(I'_j)||\widetilde{\Pi}_{\rho\gamma}(I'_k)|}{d(I_j,I_k)^{1+\epsilon}}\Bigg]. \end{aligned}$$

To estimate $\mathcal{E}'_1$, we write

$$(4.38) \qquad \Pi_{\rho\gamma} = \Pi^{(1)}_{\rho^{r^2}\gamma} \cup \Pi^{(2)}_{(\rho-\rho^{r^2})\gamma},$$

where $\Pi^{(1)}_{\rho^{r^2}\gamma}$ and $\Pi^{(2)}_{(\rho-\rho^{r^2})\gamma}$ are independent Poisson processes on $\mathbb{R}$ with intensity $\rho^{r^2}\gamma$ and $(\rho-\rho^{r^2})\gamma$, respectively, and we use that [recall (4.28)]

$$(4.39) \qquad \begin{aligned} &\mathcal{N}_\infty^{\mathrm{coal}}\{(X^\kappa(s),s)\,:\,s\in\widetilde{\Pi}_{\rho\gamma}(I'_j)\} \\ &\leq \mathcal{N}_\infty^{\mathrm{coal}}\{(X^\kappa(s),s)\,:\,s\in\Pi^{(1)}_{\rho^{r^2}\gamma}(I_j)\} + \big|\Pi^{(2)}_{(\rho-\rho^{r^2})\gamma}(I_j)\big|. \end{aligned}$$

This leads to

$$(4.40) \qquad \begin{aligned} \mathcal{E}'_1 \leq& \left(\prod_{j=1}^{\lceil t/L \rceil}\mathbb{E}^*\big(\rho^{-r^2\mathcal{N}_\infty^{\mathrm{coal}}\{(X^\kappa(s),s)\,:\,s\in\Pi_{\rho^{r^2}\gamma}(I_j)\}}\big)\right)^{1/r} \\ &\times \exp\left[(\rho-\rho^{r^2})\gamma\frac{\rho^{-r^2}-1}{r}L\lceil t/L \rceil\right]. \end{aligned}$$

To estimate $\mathcal{E}''_1$, note that $|\widetilde{\Pi}_{\rho\gamma}(I'_j)| \leq LM$ for all j and $d(I'_j,I'_k) \geq \vartheta L(k-j)$ for $k > j$, so that

$$(4.41) \qquad \mathcal{E}''_1 \leq \exp\left[\left(\frac{\delta(K)}{\rho^r}M + C'_\epsilon K\frac{\rho^{-rr'}-1}{r'}\frac{M^2}{\vartheta^{1+\epsilon}L^\epsilon}\right)L\lceil t/L \rceil\right],$$

where $C'_\epsilon = C_\epsilon \sum_{j=1}^\infty j^{-(1+\epsilon)}$. Since the distribution of $\mathcal{N}_\infty^{\mathrm{coal}}$ is invariant w.r.t. spatial shifts of the coalescing random walks, and X^κ and $\Pi_{\rho^{r^2}\gamma}$ have independent and stationary increments, we obtain

$$
(E_0 \otimes \mathbb{E}_{\mathrm{Poiss}}) \left(\prod_{j=1}^{\lceil t/L \rceil} \mathbb{E}^* \left(\rho^{-r^2 \mathcal{N}_\infty^{\mathrm{coal}}\{(X^\kappa(s),s)\,:\,s\in\Pi_{\rho^{r^2}\gamma}(I_j)\}} \right) \right)
$$

$$
= (E_0 \otimes \mathbb{E}_{\mathrm{Poiss}}) \left(\prod_{j=1}^{\lceil t/L \rceil} \mathbb{E}^* \left(\rho^{-r^2 \mathcal{N}_\infty^{\mathrm{coal}}\{(X^\kappa(s)-X^\kappa((j-1)L),s)\,:\,s\in\Pi_{\rho^{r^2}\gamma}(I_j)\}} \right) \right)
$$

(4.42)

$$
= \left((E_0 \otimes \mathbb{E}_{\mathrm{Poiss}} \otimes \mathbb{E}^*) \left(\rho^{-r^2 \mathcal{N}_\infty^{\mathrm{coal}}\{(X^\kappa(s),s)\,:\,s\in\Pi_{\rho^{r^2}\gamma}([0,L])\}} \right) \right)^{\lceil t/L \rceil}
$$

$$
= \exp\left[\left(\Lambda_1^{\mu_{\rho^{r^2}}}(L;\kappa) - \rho^{r^2}\gamma \right) L \lceil t/L \rceil \right],
$$

where in the last line we have used the representation formula (1.25) for $p = 1$, $T = \infty$ and ρ and t replaced by ρ^{r^2} and L, respectively. Now substitute (4.40) and (4.41) into (4.35), substitute the obtained inequality into (4.32) and use (4.42) to arrive at

$$
\mathcal{E}_1 \leq \exp\left[\frac{1}{r^2} \left(\Lambda_1^{\mu_{\rho^{r^2}}}(L;\kappa) - \rho^{r^2}\gamma \right) L \lceil t/L \rceil \right]
$$

(4.43)

$$
\times \exp\left[\left((\rho - \rho^{r^2})\gamma \frac{\rho^{-r^2} - 1}{r^2} + \frac{\delta(K)}{r\rho^r}M \right.\right.
$$

$$
\left.\left. + C'_\epsilon K \frac{\rho^{-rr'} - 1}{rr'} \frac{M^2}{\vartheta^{1+\epsilon}L^\epsilon} \right) L \lceil t/L \rceil \right].
$$

We next estimate $\mathcal{E}_2$ in (4.33). Using Chebyshev's exponential inequality, we obtain, for $j = 1, \ldots, \lceil t/L \rceil$,

$$
\mathbb{E}_{\mathrm{Poiss}} \left(\rho^{-r'|\Pi_{\rho\gamma}(I'_j)|} \mathbb{1}\{|\Pi_{\rho\gamma}(I'_j)| > LM\} \right)
$$

$$
\leq 1 + \mathbb{E}_{\mathrm{Poiss}} \left(\rho^{-r'|\Pi_{\rho\gamma}(I'_j)|} \mathbb{1}\{|\Pi_{\rho\gamma}(I'_j)| > LM\} \right)
$$

(4.44)

$$
\leq 1 + \rho^{r'LM} \mathbb{E}_{\mathrm{Poiss}} \left(\rho^{-2r'|\Pi_{\rho\gamma}(I'_j)|} \right)
$$

$$
\leq 1 + \rho^{r'LM} \mathbb{E}_{\mathrm{Poiss}} \left(\rho^{-2r'|\Pi_{\rho\gamma}(I_j)|} \right)
$$

$$
= 1 + \exp\left[\left(\rho\gamma(\rho^{-2r'} - 1) - r'M\log(1/\rho) \right) L \right].
$$

By our choice of M in (4.27), the expression in the right-hand side equals 2, and we conclude that

(4.45)
$$
\mathcal{E}_2 \leq e^{\lceil t/L \rceil}.
$$

Finally, substitute (4.43), (4.45) and (4.34) into (4.31), take the logarithm on both sides of the resulting inequality, divide by t, pass to the limit as $t \to \infty$ and recall (1.21). Then we obtain

$$
\begin{aligned}
(4.46) \quad \lambda_1^{\mu_\rho}(\kappa) - \rho\gamma &\le \frac{1}{r^2}\big(\Lambda_1^{\mu_{\rho^{r^2}}}(L;\kappa) - \rho^{r^2}\gamma\big) + (\rho - \rho^{r^2})\gamma\frac{\rho^{-r^2}-1}{r^2} \\
&\quad + \frac{\delta(K)}{r\rho^r}M + C'_\epsilon K\frac{\rho^{-rr'}-1}{rr'}\frac{M^2}{\vartheta^{1+\epsilon}L^\epsilon} + \frac{1}{L} + \vartheta(1-\rho)\gamma.
\end{aligned}
$$

As can be seen from (1.20), $\kappa \mapsto \Lambda_1^{\mu_{\rho^{r^2}}}(L;\kappa)$ is continuous at $\kappa = 0$. Hence, passing in (4.46) to the limits as $\kappa \downarrow 0$, $L \to \infty$, $K \to \infty$ and $\vartheta \downarrow 0$ (in this order), we find that

$$
(4.47) \quad \limsup_{\kappa \downarrow 0}\big(\lambda_1^{\mu_\rho}(\kappa) - \rho\gamma\big) \le \frac{1}{r^2}\big(\lambda_1^{\mu_{\rho^{r^2}}}(0) - \rho^{r^2}\gamma\big) + (\rho - \rho^{r^2})\gamma\frac{\rho^{-r^2}-1}{r^2}.
$$

Expanding the exponential function in the right-hand side of (1.20) into a Taylor series and using (1.15), we see that $\rho \mapsto \Lambda_1^{\mu_\rho}(t;0)$ is nondecreasing. Hence, the same is true for $\rho \mapsto \lambda_1^{\mu_\rho}(0)$. Taking this into account, we may finally pass to the limit as $r \downarrow 1$ in (4.47) to arrive at

$$
(4.48) \quad \limsup_{\kappa \downarrow 0}\big(\lambda_1^{\mu_\rho}(\kappa) - \rho\gamma\big) \le \lambda_1^{\mu_\rho}(0) - \rho\gamma.
$$

This is the desired inequality (4.25) for $p = 1$.

The extension to $p \in \mathbb{N} \setminus \{1\}$ is straightforward. The proof follows the same arguments with X^κ and $\Pi_{\rho\gamma}$ replaced by p independent copies X_q^κ and $\Pi_{\rho\gamma}^{(q)}$, $q = 1, \ldots, p$, of X^κ and $\Pi_{\rho\gamma}$, respectively. $\quad\square$

4.5. *Large κ.* In this section we prove Theorem 1.5(ii)(b). We again pick $\mu = \mu_\rho$ as the starting measure (recall Proposition 2.2).

PROOF OF THEOREM 1.5(ii)(b). Recall (1.22). We first give the proof for $p = 1$. We show that, for all $\rho \in (0, 1)$, $\gamma > 0$ and $L > 0$,

$$
(4.49) \quad \lim_{\kappa \to \infty} \Lambda_1^{\mu_\rho}(L;\kappa) = \rho\gamma.
$$

Then the claim for $p = 1$ follows from (4.46) by passing to the limits as $\kappa \to \infty$, $L \to \infty$, $K \to \infty$, $\vartheta \downarrow 0$ and $r \downarrow 1$ (in this order).

To prove (4.49), we use the representation formula (1.25):

$$
\begin{aligned}
(4.50) \quad &\Lambda_1^{\mu_\rho}(L;\kappa) - \rho\gamma \\
&= \frac{1}{L}\log(\mathrm{E}_0 \otimes \mathbb{E}_{\mathrm{Poiss}} \otimes \mathbb{E}^*)\big(\rho^{-\mathcal{N}_\infty^{\mathrm{coal}}\{(X^\kappa(s),s)\,:\,s\in\Pi_{\rho\gamma}([0,L])\}}\big).
\end{aligned}
$$

Recall that we are in a transient situation ($d \geq 5$) and write $X^\kappa(s) = X^1(\kappa s)$. Then, $P_0 \otimes \mathbb{P}_{\text{Poiss}}$-a.s.

$$(4.51) \qquad \lim_{\kappa \to \infty} \min_{\substack{s_1, s_2 \in \Pi_{\rho\gamma}([0,L]) \\ s_1 \neq s_2}} |X^\kappa(s_1) - X^\kappa(s_2)| = \infty,$$

and, consequently,

$$(4.52) \quad \lim_{\kappa \to \infty} \mathcal{N}_\infty^{\text{coal}}\{(X^\kappa(s), s) : s \in \Pi_{\rho\gamma}([0, L])\} = 0 \qquad \text{in probability w.r.t. } \mathbb{P}^*.$$

Since, moreover, $\mathcal{N}_\infty^{\text{coal}}\{(X^\kappa(s), s) : s \in \Pi_{\rho\gamma}([0, L])\} \leq |\Pi_{\rho\gamma}([0, L])|$, we may apply Lebesgue's dominated convergence theorem to see that the expression on the right of (4.50) converges to 0 as $\kappa \to \infty$. This proves (4.49).

The extension to $p \in \mathbb{N} \setminus \{1\}$ is easy. Indeed, by (1.17)–(1.19) and Jensen's inequality,

$$\exp[pt\Lambda_p^{\mu_\rho}(t; \kappa, \gamma)] = \mathbb{E}_{\mu_\rho}\left(\left[\mathrm{E}_0\left(\exp\left[\gamma \int_0^t \xi(X^\kappa(s), t-s)\, ds\right]\right)\right]^p\right)$$

$$(4.53) \qquad\qquad \leq \mathbb{E}_{\mu_\rho}\left(\mathrm{E}_0\left(\exp\left[p\gamma \int_0^t \xi(X^\kappa(s), t-s)\, ds\right]\right)\right)$$

$$= \exp[t\Lambda_1^{\mu_\rho}(t; \kappa, p\gamma)].$$

Let $t \to \infty$ to get

$$(4.54) \qquad\qquad\qquad \lambda_p(\kappa; \gamma) \leq \frac{1}{p}\lambda_1(\kappa; p\gamma).$$

This together with the assertion for $p = 1$ and (1.22) implies the claim for arbitrary $p \in \mathbb{N}$. $\quad\square$

5. Proof of Theorem 1.5(i) and (ii)(c).
Throughout this section we assume that $p(\cdot, \cdot)$ satisfies (1.5) and has zero mean and finite variance. Theorem 1.5(i) is proved in Section 5.1 and Theorem 1.5(ii)(c) in Section 5.2. As a starting measure we pick $\mu = \nu_\rho$ (recall Proposition 2.2).

5.1. *Triviality in low dimensions.* The proof of Theorem 1.5(i) is similar to that of Theorem 1.3.2(i) in Gärtner, den Hollander and Maillard [6]. The key observation is the following:

LEMMA 5.1. *If* $1 \leq d \leq 4$, *then for any finite* $Q \subset \mathbb{Z}^d$ *and* $\rho \in (0, 1)$,

$$(5.1) \qquad\qquad \lim_{t \to \infty} \frac{1}{t} \log \mathbb{P}_{\nu_\rho}\big(\xi(x, s) = 1 \; \forall x \in Q \; \forall s \in [0, t]\big) = 0.$$

PROOF. In the spirit of Bramson, Cox and Griffeath [1], Section 1, we argue as follows. The graphical representation of the VM (recall Section 1.3) allows us to write down a suitable expression for the probability in (5.1). Indeed, let

$$(5.2) \qquad H_t^Q = \{x \in \mathbb{Z}^d : \text{there is a path from } (x, 0) \text{ to } Q \times [0, t] \text{ in } \mathcal{G}_t\},$$

where, as in Section 1.3, $\mathcal{G}_t$ is the graphical representation of the voter model up to time t (see Figure 4).

Note that $H_0^Q = Q$ and that $t \mapsto H_t^Q$ is nondecreasing. Denote by $\mathcal{P}$ and $\mathcal{E}$, respectively, probability and expectation associated with the graphical representation $\mathcal{G}_t$. Then

$$(5.3) \qquad \mathbb{P}_{\nu_\rho}\big(\xi(x, s) = 1 \ \forall x \in Q \ \forall s \in [0, t]\big) = (\mathcal{P} \otimes \nu_\rho)\big(H_t^Q \subseteq \xi(0)\big),$$

where $\xi(0) = \{x \in \mathbb{Z}^d : \xi(x, 0) = 1\}$ is the set of initial locations of 1's. Indeed, (5.3) holds because if $\xi(x, 0) = 0$ for some $x \in H_t^Q$, then this 0 will propagate into Q prior to time t (see Figure 4).

By Jensen's inequality,

$$(5.4) \qquad (\mathcal{P} \otimes \nu_\rho)\big(H_t^Q \subseteq \xi(0)\big) = \mathcal{E}\big(\rho^{|H_t^Q|}\big) \geq \rho^{\mathcal{E}|H_t^Q|}.$$

Moreover, $H_t^Q = \bigcup_{y \in Q} H_t^{\{y\}}$, implying

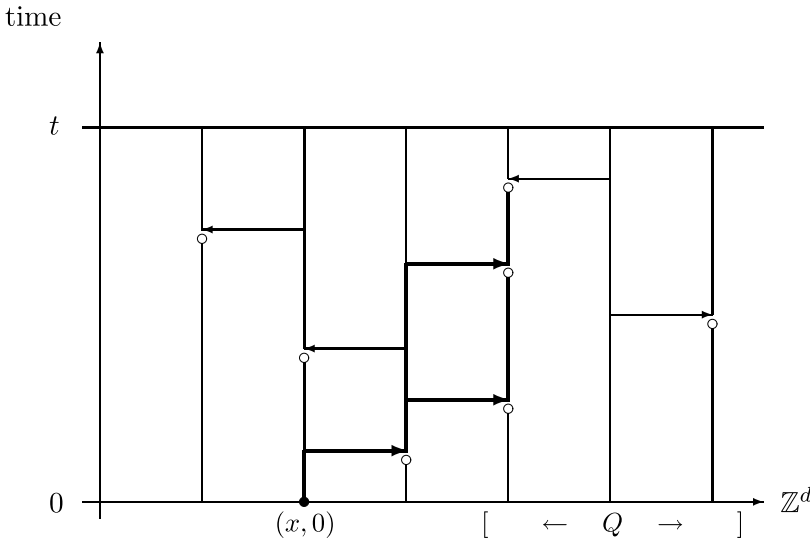$$(5.5) \qquad \mathcal{E}|H_t^Q| \leq |Q|\mathcal{E}|H_t^{\{0\}}|.$$

FIG. 4. *Some paths from $(x, 0)$ to $Q \times [0, t]$ in $\mathcal{G}_t$.*

By the dual graphical representation, $|H_t^{\{0\}}|$ coincides in distribution with the number of coalescing random walks alive at time t when starting at site 0 at times generated by a rate 1 Poisson stream. As shown in Bramson, Cox and Griffeath [1], Theorem 2, if $p(\cdot, \cdot)$ is a simple random walk, then

$$(5.6) \qquad \mathcal{E}|H_t^{\{0\}}| = o(t) \qquad \text{as } t \to \infty \text{ when } 1 \le d \le 4,$$

in which case (5.1) follows from (5.3)–(5.5). As noted in Bramson, Cox and Le Gall [2], Lemma 2, and its proof, the key ingredient in the proof of (5.6) extends from a simple random walk to a random walk with zero mean and finite variance. $\qquad \square$

We are now ready to give the proof of Theorem 1.5(i).

PROOF OF THEOREM 1.5(i).   Fix $1 \le d \le 4$, $\kappa \in [0, \infty)$, $\gamma \in (0, \infty)$ and $\rho \in (0, 1)$. Since $p \mapsto \lambda_p(\kappa)$ is nondecreasing and $\lambda_p(\kappa) \le \gamma$ for all $p \in \mathbb{N}$ [recall (1.22)], it suffices to give the proof for $p = 1$. For $p = 1$, (1.20) reads

$$(5.7) \qquad \Lambda_1^{\nu_\rho}(t) = \frac{1}{t} \log(\mathbb{E}_{\nu_\rho} \otimes \mathrm{E}_0) \left(\exp\left[\gamma \int_0^t \xi(X^\kappa(s), t - s)\, ds \right] \right).$$

By restricting X^κ to stay inside a finite box $Q \subset \mathbb{Z}^d$ around 0 up to time t and requiring ξ to be 1 in the entire box up to time t, we obtain

$$(5.8) \quad \begin{aligned} &(\mathbb{E}_{\nu_\rho} \otimes \mathrm{E}_0) \left(\exp\left[\gamma \int_0^t \xi(X^\kappa(s), t - s)\, ds \right] \right) \\ &\qquad \ge e^{\gamma t} \mathbb{P}_{\nu_\rho}\big(\xi(x, s) = 1\ \forall x \in Q\ \forall s \in [0, t]\big) \mathrm{P}_0\big(X^\kappa(s) \in Q\ \forall s \in [0, t]\big). \end{aligned}$$

The first factor is $e^{o(t)}$ by Lemma 5.1. For the second factor, we have

$$(5.9) \qquad \lim_{t \to \infty} \frac{1}{t} \log \mathrm{P}_0\big(X^\kappa(s) \in Q\ \forall s \in [0, t]\big) = \lambda^\kappa(Q),$$

with $\lambda^\kappa(Q) < 0$ the principal Dirichlet eigenvalue on Q of $\kappa \Delta$, the generator of X^κ. Combining (5.1) and (5.7)–(5.9), we arrive at

$$(5.10) \qquad \lambda_1(\kappa) = \lim_{t \to \infty} \Lambda_1^{\nu_\rho}(t) \ge \gamma + \lambda^\kappa(Q).$$

Finally, let $Q \uparrow \mathbb{Z}^d$ and use that $\lim_{Q \uparrow \mathbb{Z}^d} \lambda^\kappa(Q) = 0$ (see, e.g., Spitzer [12], Section 21) to arrive at $\lambda_1(\kappa) \ge \gamma$. Since, trivially, $\lambda_1(\kappa) \le \gamma$, we get $\lambda_1(\kappa) = \gamma$. $\quad \square$

5.2. *Intermittency for small κ.*   We start this section by recalling some large deviation results for the VM that will be needed to prove Theorem 1.5(ii)(c). Cox and Griffeath [3] showed that for the VM with a simple random walk transition kernel given by (1.6), the occupation time of the origin up to time $t \ge 0$,

$$(5.11) \qquad T_t = \int_0^t \xi(0, s)\, ds,$$

satisfies a strong law of large numbers and a central limit theorem for $d \geq 2$. For $d = 1$ there is no law of large numbers: T_t / t has a nontrivial limiting law. These results carry over to a random walk with zero mean and finite variance.

The following proposition gives *large deviation bounds*.

PROPOSITION 5.2 (Bramson, Cox and Griffeath [1], Theorem 1; Bramson, Cox and Le Gall [2], Lemma 2 and its proof; Maillard and Mountford [11], Theorem 1.3.2). *Suppose that $p(\cdot, \cdot)$ has zero mean and finite variance. Then for every $\alpha \in (\rho, 1)$ there exist $0 < I^-(\alpha) < I^+(\alpha) < \infty$ such that, for t sufficiently large (depending on α),*

$$(5.12) \qquad e^{-I^+(\alpha)b_t} \leq \mathbb{P}_{\nu_\rho}\left(\frac{T_t}{t} \geq \alpha\right) \leq e^{-I^-(\alpha)b_t}$$

with

$$(5.13) \qquad b_t = \begin{cases} \log t, & \text{if } d = 2, \\ \sqrt{t}, & \text{if } d = 3, \\ \dfrac{t}{\log t}, & \text{if } d = 4, \\ t, & \text{if } d \geq 5. \end{cases}$$

By interchanging the opinions 0 and 1, similar bounds are obtained for $\mathbb{P}_{\nu_\rho}(T_t / t \leq \alpha)$, $\alpha \in (0, \rho)$. The case $\alpha = 1$ may be included in $d \geq 3$ but not in $d = 2$, for which it is shown in Maillard and Mountford [11], Theorem 1.3.1, that $\mathbb{P}(T_t = t)$ is of order $\exp[-(\log t)^2]$. A full large deviation principle is expected to hold for $d \geq 3$, but this has not been established. Inspection of the proof in Bramson, Cox and Griffeath [1] shows that for $d \geq 5$ there exists a $C > 0$ such that

$$(5.14) \qquad I^-(\alpha) \geq C(\sqrt{\alpha} - \sqrt{\rho})^2, \qquad \alpha \in (\rho, 1).$$

No comparable upper bound on I^+ is given.

We are now ready to give the proof of Theorem 1.5(ii)(c).

PROOF OF THEOREM 1.5(ii)(c). We first give the proof for $\kappa = 0$. Fix $d \geq 5$, $p \in \mathbb{N}$, $\gamma \in (0, \infty)$ and $\rho \in (0, 1)$, and recall that $\lambda_p(0) > \rho\gamma$ by Theorem 1.4(ii). Pick $\alpha \in (\rho, \gamma^{-1}\lambda_p(0))$ and define

$$(5.15) \qquad I(\alpha) = -\limsup_{t \to \infty} \frac{1}{t} \log \mathbb{P}_{\nu_\rho}\left(\frac{1}{t}T_t \geq \alpha\right) > 0,$$

where the positivity of the limit comes from the upper bound in (5.12), which implies $I(\alpha) \geq I^-(\alpha) > 0$. Put

$$(5.16) \qquad \beta = \gamma^{-1}\left[\lambda_p(0) + \frac{1}{2p}I(\alpha)\right]$$

and split

$$(5.17) \qquad \Lambda_p^{\nu_\rho}(t) = \frac{1}{pt} \log \mathbb{E}_{\nu_\rho}(e^{p\gamma T_t}) = \frac{1}{pt} \log(A_t + B_t + C_t)$$

with

$$
\begin{aligned}
A_t &= \mathbb{E}_{\nu_\rho}\left(e^{p\gamma T_t} \mathbb{1}\left\{0 \le \frac{1}{t}T_t < \alpha\right\}\right), \\
(5.18) \qquad B_t &= \mathbb{E}_{\nu_\rho}\left(e^{p\gamma T_t} \mathbb{1}\left\{\alpha \le \frac{1}{t}T_t < \beta\right\}\right), \\
C_t &= \mathbb{E}_{\nu_\rho}\left(e^{p\gamma T_t} \mathbb{1}\left\{\frac{1}{t}T_t \ge \beta\right\}\right).
\end{aligned}
$$

Next, note that

$$(5.19) \qquad A_t \le e^{p\gamma\alpha t}, \qquad B_t \le e^{p\gamma\beta t}\mathbb{P}_{\nu_\rho}\left(\frac{1}{t}T_t \ge \alpha\right).$$

Thus, in (5.17) both A_t and B_t are negligible as $t \to \infty$, because $\lim_{t\to\infty} \Lambda_p^{\nu_\rho}(t) = \lambda_p(0)$ while $\gamma\alpha < \lambda_p(0)$ and $\gamma\beta - \frac{1}{p}I(\alpha) = \lambda_p(0) - \frac{1}{2p}I(\alpha) < \lambda_p(0)$. Hence,

$$(5.20) \qquad \lambda_p(0) = \lim_{t\to\infty} \frac{1}{pt} \log C_t.$$

Now, by (5.16) and (5.20), we have

$$
\begin{aligned}
(5.21) \qquad \lambda_{p+1}(0) &= \lim_{t\to\infty} \frac{1}{(p+1)t} \log \mathbb{E}_{\nu_\rho}(e^{(p+1)\gamma T_t}) \\
&\ge \limsup_{t\to\infty} \frac{1}{(p+1)t} \log \mathbb{E}_{\nu_\rho}\left(e^{(p+1)\gamma T_t}\mathbb{1}\left\{\frac{1}{t}T_t \ge \beta\right\}\right) \\
&\ge \frac{1}{p+1}\gamma\beta + \lim_{t\to\infty} \frac{1}{(p+1)t} \log C_t \\
&= \frac{1}{p+1}\gamma\beta + \frac{p}{p+1}\lambda_p(0) \\
&= \lambda_p(0) + \frac{1}{2p(p+1)}I(\alpha) > \lambda_p(0),
\end{aligned}
$$

which proves the gap between $\lambda_p(0)$ and $\lambda_{p+1}(0)$.

By the continuity of $\kappa \mapsto \lambda_p(\kappa)$ at $\kappa = 0$ in Theorem 1.5(ii)(a), it follows that there exists $\kappa_0 > 0$ such that $\lambda_p(\kappa) > \lambda_{p-1}(\kappa)$ for all $\kappa \in [0, \kappa_0)$ when $p = 2$ and, by the remark made after formula (1.22), also when $p \in \mathbb{N} \setminus \{1\}$. $\qquad \square$

APPENDIX: HEURISTIC EXPLANATION OF CONJECTURE 1.6

In this appendix we give a heuristic explanation of (1.28). We only consider the case $p = 1$. A similar argument works for $p \in \mathbb{N} \setminus \{1\}$. As starting measure we pick $\mu = \mu_\rho$ (recall Proposition 2.2).

1. *Pair correlation.* Lemma 1.1 for $n = 2$ yields the following representation for the pair correlation function of the VM in equilibrium.

LEMMA A.1. *Suppose that $p(\cdot, \cdot)$ is symmetric and transient. Then, for all $x_1, x_2 \in \mathbb{Z}^d$ and $s \geq 0$,*

$$\text{(A.1)} \quad \mathbb{E}_{\mu_\rho}\big([\xi(x_1, s) - \rho][\xi(x_2, 0) - \rho]\big) = \frac{\rho(1 - \rho)}{G_d} \int_0^\infty p_{s+t}(x_1, x_2) \, dt$$

with $G_d = \int_0^\infty p_t(0, 0) \, dt$.

PROOF. The proof is standard. By (1.15) with $T = \infty$ and $n = 2$, we have

$$\text{(A.2)} \quad \begin{aligned} \mathbb{E}_{\mu_\rho}\big([\xi(x_1, s) &- \rho][\xi(x_2, 0) - \rho]\big) \\ &= \rho(1 - \rho)\mathbb{P}^*\big(\mathcal{N}_\infty\{(x_1, 0), (x_2, s)\} = 1\big). \end{aligned}$$

The probability in the right-hand side of (A.2) can be computed as follows. The first random walk starts from site x_1 at time 0, moves freely until time s, and reaches some site y at time s. The second random walk starts from site x_2 at time s and has to eventually coalesce with the first random walk. This gives

$$\text{(A.3)} \quad \mathbb{P}^*\big(\mathcal{N}_\infty\{(x_1, 0), (x_2, s)\} = 1\big) = \sum_{y \in \mathbb{Z}^d} p_s(x_1, y) w(y - x_2)$$

with

$$\text{(A.4)} \quad w(z) = \mathbb{P}_z(Z_t = 0 \text{ for some } 0 \leq t < \infty), \qquad z \in \mathbb{Z}^d.$$

Here we use that, by the symmetry of $p(\cdot, \cdot)$, the difference between the two random walks is a single random walk Z running at double the speed. By a renewal argument (see Spitzer [12], Section 4), for transient $p(\cdot, \cdot)$ we have

$$\text{(A.5)} \quad w(z) = \frac{1}{G_d} \int_0^\infty p_t(z, 0) \, dt.$$

Combining (A.2), (A.3) and (A.5), we obtain (A.1). $\square$

2. *Green term.* From now on let $p(\cdot, \cdot)$ be a simple random walk. Fix $d \geq 5$, $\gamma \in (0, \infty)$ and $\rho \in (0, 1)$. Scaling time by κ in (1.20), we have $\lambda_1(\kappa) = \kappa \lambda_1^*(\kappa)$ with

$$\text{(A.6)} \quad \lambda_1^*(\kappa) = \lim_{t \to \infty} \Lambda_1^*(\kappa; t)$$

and

$$(A.7) \qquad \Lambda_1^*(\kappa; t) = \frac{1}{t} \log(\mathbb{E}_{\mu_\rho} \otimes E_0) \left(\exp\left[\frac{\gamma}{\kappa} \int_0^t ds\, \xi\left(X(s), \frac{t-s}{\kappa} \right) \right] \right),$$

where $X = X^1$. For large κ, the ξ-field in (A.7) evolves slowly and therefore does not manage to cooperate with the X-process in determining the growth rate. As a result, the expectation over the ξ-field can be computed via a *Gaussian approximation*, which we expect to become sharp in the limit as $\kappa \to \infty$, that is,

$$\Lambda_1^*(\kappa; t) - \frac{\rho\gamma}{\kappa}$$

$$= \frac{1}{t} \log(\mathbb{E}_{\mu_\rho} \otimes E_0) \left(\exp\left[\frac{\gamma}{\kappa} \int_0^t ds \left(\xi\left(X(s), \frac{t-s}{\kappa} \right) - \rho \right) \right] \right)$$

$$(A.8) \quad \approx \frac{1}{t} \log E_0 \left(\exp\left[\frac{\gamma^2}{2\kappa^2} \int_0^t ds \int_0^t du\, \mathbb{E}_{\mu_\rho} \left(\left[\xi\left(X(s), \frac{t-s}{\kappa} \right) - \rho \right] \right. \right. \right.$$

$$\left. \left. \left. \times \left[\xi\left(X(u), \frac{t-u}{\kappa} \right) - \rho \right] \right) \right] \right).$$

(In essence, what happens here is that the asymptotics for $\kappa \to \infty$ is driven by moderate deviations of the ξ-field, which fall in the Gaussian regime.) Next, by Lemma A.1, for any $0 \le s \le u \le t$ we have

$$(A.9) \qquad \begin{aligned} &\mathbb{E}_{\mu_\rho} \left(\left[\xi\left(X(s), \frac{t-s}{\kappa} \right) - \rho \right] \left[\xi\left(X(u), \frac{t-u}{\kappa} \right) - \rho \right] \right) \\ &\qquad = C \int_0^\infty dv\, p_{(u-s)/\kappa+v}(X(s), X(u)), \end{aligned}$$

where $C = \rho(1-\rho)/G_d$. Hence,

$$(A.10) \qquad \begin{aligned} \lim_{\kappa \to \infty} 2d\kappa[\lambda_1(\kappa) - \rho\gamma] &= \lim_{\kappa \to \infty} 2d\kappa^2 \left[\lambda_1^*(\kappa) - \frac{\rho\gamma}{\kappa} \right] \\ &= \lim_{\kappa \to \infty} 2d\kappa^2 \lim_{t \to \infty} \left[\Lambda_1^*(\kappa; t) - \frac{\rho\gamma}{\kappa} \right] \\ &= \lim_{\kappa \to \infty} 2d\kappa^2 \lim_{t \to \infty} I(\kappa; t) \end{aligned}$$

with

$$I(\kappa; t)$$

$$(A.11) \quad = \frac{1}{t} \log E_0 \left(\exp\left[\frac{C\gamma^2}{\kappa^2} \int_0^t ds \int_s^t du \int_0^\infty dv\, p_{(u-s)/\kappa+v}(X(s), X(u)) \right] \right)$$

$$\approx \frac{C\gamma^2}{t\kappa^2} \int_0^t ds \int_s^t du \int_0^\infty dv\, E_0\big(p_{(u-s)/\kappa+v}(X(s), X(u))\big).$$

In the last line of (A.11), a *linear approximation* is made in the expectation over the random walk X, which we expect to become sharp in the limit as $\kappa \to \infty$ in $d \geq 6$. Next, for any $0 \leq s \leq u \leq t$ and $T \geq 0$,

$$E_0(p_T(X(s), X(u))) = \sum_{x,y \in \mathbb{Z}^d} p_{2ds}(0, x) p_{2d(u-s)}(x, y) p_T(x, y)$$

(A.12)
$$= \sum_{x \in \mathbb{Z}^d} p_{2ds}(0, x) p_{2d(u-s)+T}(x, x)$$

$$= p_{2d(u-s)+T}(0, 0).$$

Here, we use that $p(\cdot, \cdot)$ is a simple random walk, so that ξ fits with X. We therefore have

(A.13) $\quad$ r.h.s. (A.11) $= \dfrac{C\gamma^2}{t\kappa^2} \displaystyle\int_0^t ds \int_s^t du \int_0^\infty dv \, p_{2d(u-s)1[\kappa]+v}(0, 0),$

where we abbreviate $1[\kappa] = 1 + \frac{1}{2d\kappa}$. Rewriting

(A.14)
$$\frac{1}{t} \int_0^t ds \int_s^t du \int_0^\infty dv \, p_{2d(u-s)1[\kappa]+v}(0, 0)$$

$$= \int_0^t dw \int_0^\infty dv \left(\frac{t-w}{t}\right) p_{2dw1[\kappa]+v}(0, 0),$$

we get from (A.11)–(A.13) that

(A.15)
$$\lim_{t \to \infty} I(\kappa; t) = \frac{C\gamma^2}{2d\kappa^2 1[\kappa]} \int_0^\infty dw \int_0^\infty dv \, p_{w+v}(0, 0)$$

$$= \frac{C\gamma^2}{2d\kappa^2 1[\kappa]} G_d^*.$$

Recalling (A.10), we arrive at (1.28) for $d \geq 6$.

3. *Polaron term*. Where does the term with $\mathcal{P}_5$ come from? We expect this term to arise from the part of the integral in the exponent in the first line of (A.11) with $(u-s)/\kappa$ and v of order κ^2, as we will argue next. Put $\mathbb{Z}_\kappa^d = \kappa^{-1}\mathbb{Z}^d$ and, for $t \geq 0$ and $x, y \in \mathbb{Z}_\kappa^d$, define

(A.16) $\quad X^\kappa(t) = \kappa^{-1} X(\kappa^2 t), \qquad p_t^\kappa(x, y) = \kappa^d p_{2d\kappa^2 t}(\kappa x, \kappa y).$

In the limit as $\kappa \to \infty$, $(X^\kappa(t))_{t \geq 0}$ converges weakly to Brownian motion, while $(p_t^\kappa(\cdot, \cdot))_{t \geq 0}$ converges to the corresponding family of Gaussian transition kernels $(p_t^G(\cdot, \cdot))_{t \geq 0}$ given by

(A.17) $\quad p_t^G(x, y) = (4\pi t)^{-d/2} \exp[-\|x - y\|^2/4t], \qquad x, y \in \mathbb{R}^d.$

After scaling, the part we are after is approximately

(A.18) $\quad C\gamma^2 \kappa^{4-d} \displaystyle\int_0^{\kappa^{-2}t} ds \int_{s+\varepsilon\kappa}^{s+K\kappa} du \int_0^K dv \, p_{1/(2d)((u-s)/\kappa+v)}^G(X^\kappa(s), X^\kappa(u)),$

where $0 < \varepsilon \ll 1 \ll K < \infty$. For $\delta > 0$, divide the first and the second integral in (A.18) into pieces of length $\delta\kappa$, and define the occupation time measures

$$(A.19) \quad \Xi_w^\kappa(A) = \frac{1}{\delta\kappa} \int_w^{w+\delta\kappa} 1_A(X^\kappa(u)) \, du, \qquad w \geq 0, A \subset \mathbb{R}^d \text{ Borel.}$$

Then, when $\delta \ll \varepsilon$, $(u-s)/\kappa$ is almost constant on time intervals of length $\delta\kappa$ and, consequently,

$$(A.18) \approx C\gamma^2\kappa^{4-d} \int_0^{\kappa^{-2}t} ds \int_{s+\varepsilon\kappa}^{s+K\kappa} du \int_0^K dv$$

$$(A.20)$$

$$\times \int_{\mathbb{R}^d} \Xi_s^\kappa(dx) \int_{\mathbb{R}^d} \Xi_u^\kappa(dy) \, p_{1/(2d)((u-s)/\kappa+v)}^G(x, y).$$

Using the large deviation principle for $\Xi_{(\cdot)}^\kappa$ as $\kappa \to \infty$, we find that the contribution of (A.20) to $I(\kappa; t)$ for large κ is approximately

$$\frac{1}{t} \sup_{\mu_{(\cdot)}} \left[C\gamma^2\kappa^{4-d} \int_0^{\kappa^{-2}t} ds \int_{s+\varepsilon\kappa}^{s+K\kappa} du \int_0^K dv \int_{\mathbb{R}^d} \mu_s(dx) \int_{\mathbb{R}^d} \mu_u(dy) \right.$$

$$(A.21)$$

$$\left. \times p_{1/(2d)((u-s)/\kappa+v)}^G(x, y) - \int_0^{\kappa^{-2}t} J(\mu_s) \, ds \right],$$

where the supremum is taken over all probability measure-valued paths $\mu_{(\cdot)}$ and

$$(A.22) \qquad\qquad J(\nu) = \begin{cases} \|\nabla\sqrt{d\nu/d\lambda}\|_2^2, & \text{if } \nu \ll \lambda, \\ \infty, & \text{otherwise,} \end{cases}$$

with λ the Lebesgue measure on $\mathbb{R}^d$. By the convexity of the large deviation rate function J, the supremum in (A.21) diagonalizes and reduces to

$$(A.21) = \frac{1}{\kappa^2} \sup_\nu \left[C\gamma^2\kappa^{4-d} \int_{\mathbb{R}^d} \nu(dx) \int_{\mathbb{R}^d} \nu(dy) \int_{\varepsilon\kappa}^{K\kappa} du \int_0^K dv \right.$$

$$(A.23)$$

$$\left. \times p_{1/(2d)(u/\kappa+v)}^G(x, y) - J(\nu) \right].$$

Putting $u = 2d\kappa\tilde{u}$, $v = 2d\tilde{v}$ and letting $\varepsilon \downarrow 0$ and $K \to \infty$, we end up with a contribution to $\lim_{\kappa\to\infty} 2d\kappa^2 \lim_{t\to\infty} I(\kappa; t)$ of the form

$$2d \sup_\nu \left[(2d)^2 C\gamma^2 \int_{\mathbb{R}^d} \nu(dx) \int_{\mathbb{R}^d} \nu(dy) \int_0^\infty d\tilde{u} \int_0^\infty d\tilde{v} \, p_{\tilde{u}+\tilde{v}}^G(x, y) \right.$$

$$(A.24)$$

$$\left. - J(\nu) \right]$$

in $d = 5$ and zero in $d \geq 6$. In $d = 5$ we have from (A.17)

$$(A.25) \quad \int_0^\infty d\tilde{u} \int_0^\infty d\tilde{v} \, p_{\tilde{u}+\tilde{v}}^G(x, y) = \int_0^\infty dt \, t p_t^G(x, y) = \frac{1}{16\pi^2\|x-y\|}.$$

Substituting this into (A.24), putting $\nu = f^2 \lambda$ and recalling (A.22), we get

$$(A.26) \quad (A.24) = 2d \sup_{\|f\|_2=1} \left[(2d)^2 C \gamma^2 \int_{\mathbb{R}^5} \int_{\mathbb{R}^5} dx\,dy\, \frac{f^2(x)f^2(y)}{16\pi^2 \|x-y\|} - \|\nabla f\|_2^2 \right].$$

Scaling of f shows that the supremum with the prefactor $(2d)^2 C \gamma^2$ equals $((2d)^2 C \gamma^2)^2$ times the supremum without this prefactor. Hence, we get

$$(A.27) \qquad\qquad (A.24) = 2d((2d)^2 C \gamma^2)^2 \mathcal{P}_5,$$

where we recall (1.30). This is precisely the "polaron-type" term in (1.28) for $p = 1$.

The heuristic argument in parts 2 and 3 follows a line of thought that was made rigorous in Gärtner and den Hollander [5] and Gärtner, den Hollander and Maillard [6, 8] for the case where ξ is a field of independent simple random walks in a Poisson equilibrium, respectively, a simple symmetric exclusion process in a Bernoulli equilibrium. We refer to these papers for further details. There it is also explained why for $p \in \mathbb{N} \setminus \{1\}$ the polaron term is p^2 times that for $p = 1$.

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