Duality, Bosonic Particle Systems and Some Exactly Solvable Models of Non-Equilibrium

About the cover: the picture is a density plot of data from computer simulation of the Symmetric Inclusion Process with 20000 particles on 2000 horizontal sites with periodic boundary conditions. The parameter m in the model is dynamically changed during the simulation starting from m=100, corresponding to almost pure diffusion and reduced to m=0.3 at the end, corresponding to a significant increase in inclusion moves versus diffusion. The time on the vertical axis runs from 0 to 116 on a linear scale from up to down.

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Contents

0 Introduction

The fundamental quest of statistical mechanics is to understand the macroscopic laws of thermodynamics from the microscopic world of interacting particles. In equilibrium statistical mechanics, the transition from the microworld to the macroworld is conceptually well understood. The macroscopic equilibrium properties can be obtained by studying the Boltzmann-Gibbs distribution as a function of temperature and other parameters such as external fields.

The Gibbs formalism, i.e., the study of Boltzmann-Gibbs distributions in the thermodynamic limit has been rigorously formulated in the so-called DLR (Dobrushin-Lanford-Ruelle) formalism. Within this framework, one can rigorously understand macroscopic equilibrium phenomena such as phase transitions and the laws of equilibrium thermodynamics. Even if the equilibrium formalism is well-established, it is still rarely the case that models in this framework are exactly solvable, i.e., that one has e.g. explicit expressions for the free energy. Exactly solvable models such as the Ising model serve as paradigmatic examples where fine details such as correlation functions even at the critical point can be computed.

In non-equilibrium statistical mechanics, there is no analogue of the Gibbs formalism, i.e., there is no general formalism that gives the distribution of microstates even for a "simple" non-equilibrium scenario such as a system in contact with two heat reservoirs at different temperatures or with two particle reservoirs with different chemical potentials. Only close to equilibrium there is the general theory of linear response that relates currents to equilibrium correlation functions.

One problem with the theory of non-equilibrium is the diversity of phenomena it is supposed to describe, as John Von Neumann once put it: "theory of non-elephants". In this work, we therefore want to focus on the simplest possible non-equilibrium systems, which are systems in contact with two different reservoirs. The aim is to derive rigorous and exact properties of the so-called non-equilibrium steady state (NESS). This is the stationary measure of such a system, which, although stationary, is non-equilibrium because of the non-equilibrium constraints imposed by the reservoirs. In other words, the stationary measure will be non-reversible, and the system will have a strictly positive stationary entropy production. Typically the non-reversible character is clearly visible in the presence of a stationary current.

The nature of NESS is quite different from that of an equilibrium measure. E.g. quite generically long-range correlations are expected (see [4],[7]), whereas in equilibrium systems, they usually appear only at the critical point. These long-range correlations are also manifest in the large deviations from the NESS temperature or density profile. Generically, the associated free energy is a non-local function [4]. From a macroscopic point of view, i.e., starting from the hydrodynamic limit and associated large deviations, Bertini, Jona-Lasinio, Landim et al [1, 2, 3, 4] developed a quite general theory predicting the non-equilibrium density or temperature profile, as well as large deviations, i.e., the leading order of the exponentially small probability of deviations from this profile.

Our aim is to study models where in the NESS the profile can be computed exactly, as well as correlation functions, such as the two-point function. The obtained expressions can then be used to test general non-equilibrium theories, such as the formalism developed in [1], or the theory of McLennan ensembles [18]. The models studied in this thesis belong to the class of interacting particle systems, or systems of interacting diffusions. Interacting particle systems (IPS) are systems of particles moving on a lattice and interacting with each other according to stochastic rules. Their study started in the early seventies in papers by Spitzer [25] and Dobrushin [5]. A standard reference is Liggett [15]. A famous and thoroughly studied example of IPS is the exclusion process (EP) where particles move on a lattice according to independent random walks with the additional constraint that each lattice site is occupied by at most one particle. Interacting diffusion models come up naturally if one wants to model heat conduction, or energy transport.

The basic technical tool developed to study the models in this thesis is *duality*. Via duality, we connect models of interacting diffusions to simpler interacting particle systems, both in equilibrium and non-equilibrium setting. Because duality is such a powerful method, part of the thesis is also devoted to develop a general formalism that can be used to produce dual processes and associated duality functions or self-duality functions. In the following we give an overview of the results and models introduced and studied in this thesis. In the first section we define and shortly review the Brownian Momentum Process (aka BMP) and its dual, the Symmetric Inclusion Process (aka SIP, which is a new interacting particle system). Although we also consider several other models in detail in later chapters and give many statements and theorems that are equally applicable to a wider class of models, these two models and their generalizations are an essential starting point in our work and are used extensively as illustrating examples. After that we give a review of the chapters in the thesis in a way to emphasize the link between the different chapters rather than the details. Finally we give some future research directions.

0.1 BMP and its relation with SIP

Heat conduction is an example of a non-equilibrium phenomenon closely related to mass transport. In a given microscopic model, it is of interest to know the temperature profile in the non-equilibrium steady state (NESS) for specific boundary conditions. One aim is to derive the Fourier's law from the microscopic model. Fourier's law is a macroscopic phenomenological law which tells that the heat current is proportional to the temperature difference across the boundaries and the proportionality constant is independent of the temperature. Besides showing the Fourier's law one wants to understand better the correlation structure of the microscopic degrees of freedom in the NESS. It is expected that non-equilibrium systems exhibit generically long range correlations in the steady state, related to the inverse of the Laplacian (Dirichlet Green's function), see [4], [7] and [17]. Therefore it is important to have microscopic models where the two-point function and possibly higher order correlation functions can be computed explicitly.

The Brownian Momentum Process (aka BMP) is a model of heat conduction with stochastic diffusion of energy analyzed in [8]. To each site i of a lattice we associate a continuous degree of freedom x_i which has to be thought of as momentum. Between every two adjacent sites $(i, i+1)$ and for every small time interval there is a random exchange of momentum that leaves the total energy of the two sites $\{x_i^2 + x_{i+1}^2\}$ invariant.

More precisely, the model is defined as a Markov diffusion process on the configuration space of N-dimensional vectors $(x_1, \ldots, x_N) \in \mathbb{R}^N$, interpreted as the momenta associated to the lattice sites $\{1, \ldots, N\}$. The boundary sites 1 and N are in contact with heat baths at temperatures T_L and T_R respectively.

The generator of BMP working on the core of smooth functions is:

$$L = B_1 + B_N + \sum_{i=1}^{N-1} L_{i,i+1}. \tag{0.1.1}$$

Here

$$L_{i,j} = (x_i \partial_j - x_j \partial_i)^2$$

represents the exchange of momentum in the bulk part of the system. The operators B_1 and B_N are the generator of the Ornstein-Uhlenbeck processes representing the coupling to the heat baths at temperatures T_L and T_R and are given by

$$B_1 = T_L \partial_1^2 - x_1 \partial_1$$

$$B_N = T_R \partial_N^2 - x_N \partial_N.$$

One way to intuitively understand the effect of the bulk part of the generator is to consider the operator

$$\mathscr{A} = \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)^2 \tag{0.1.2}$$

In the polar coordinates where $x = r \cos \theta$ and $y = r \sin \theta$, this operator reduces to

$$\mathscr{A} = \frac{\partial^2}{\partial \theta^2}$$

which is the generator of a Brownian process for the variable θ. This means that in the process $(x(t), y(t))$ with generator (0.1.2) we will have $r(t) = r(0)$ unchanged and $\theta(t)$ will be a Brownian motion on the interval $[0, 2\pi]$. This provides for the mixing of the values of the $(x(t), y(t))$ while the value of $r(t)^2$ (total energy) remains preserved, thus providing an energy-conserving mechanism for the transport of momentum in the model.

This diffusive exchange of momentum between adjacent sites is different from the energy transport mechanism in the well known KMP model [6]. The later is a model of *energy* transport where energy is exchanged randomly at discrete random times. The two models are however closely related. One can obtain KMP via the 'instantaneous thermalization' limit [9] of the Brownian Energy Process which is directly related to BMP (see chapter 1 for more details).

0.2 Duality

Duality is a powerful tool in the study of Markov processes. It has played a fundamental role in the study of interacting particle systems and in models of population dynamics [19]. For example in the context of the Symmetric Exclusion Process (aka

SEP) it has been the crucial tool in order to obtain the complete ergodic theory of this process (see [15], chapters 2 and 8).

Two Markov processes $\{x_t : t \geq 0\}$ and $\{\xi_t : t \geq 0\}$ with state spaces Ω, resp. Ω' and with generators L, resp. $\mathscr{L}$ are called dual to each other if there exist a duality function $D : \Omega' \times \Omega \to \mathbb{R}$ such that

$$LD(\xi, \vec{x}) = \mathscr{L}D(\xi, \vec{x}) \tag{0.2.1}$$

where in the lhs of (0.2.1) the operator L is working on the $\vec{x}$ variable, and in the rhs the operator $\mathscr{L}$ is working on the ξ variable (here we implicitly assume that $D(\xi, .)$ is in the domain of L and $D(., \vec{x})$ is in domain of $\mathscr{L}$). This relation then lifts to the semigroups (which arise by exponentiation of the generator) and to the processes. This then yields the duality relation between the processes:

$$\mathbb{E}_{\vec{x}}[D(\xi, \vec{x}(t))] = \hat{\mathbb{E}}_{\xi}[D(\xi(t), \vec{x})]. \tag{0.2.2}$$

This relation is useful in the case that the $\{x_t : t \geq 0\}$ is 'complicated' and the $\{\xi_t : t \geq 0\}$ is 'easy' and the set of dual functions is sufficiently rich. For instance one can think of ξ being discrete objects indexing polynomials in $\vec{x}$.

In case that $\Omega = \mathbb{R}^N$, if the equations for the evolution of correlation functions of degree n for the x process are closed (i.e. there is no polynomial of higher order than n involved), that can be a hint to the existence of the duality property where the dual process will then be a particle system where the number of particles is not increasing.

0.3 Duality between BMP and SIP

In the study of BMP, a crucial ingredient is that it is dual to a discrete particle system with absorbing left and right boundaries. The configuration space of this particle system is $\Omega = \mathbb{N}^{N+2}$. We interpret $\xi \in \Omega = \mathbb{N}^{N+2}$ as prescribing the number of particles in each lattice site $i \in \{0, \ldots, N+1\}$.

The dual process is as follows. A configuration $\xi = (\xi_0, \ldots, \xi_{N+1})$ represents K particles (or walkers) on $\{0, 1, \ldots, N+1\}$ with $K = \sum_{i=0}^{N+1} \xi_i$. The walkers can only jump to neighboring sites and are stuck when arriving to sites 0 or $N+1$. The rate at which there is a jump of a walker depends on how many walkers there are at neighboring sites. If we have ξ_i walkers at site i, ξ_{i-1} walkers at site $i-1$ and ξ_{i+1}

walkers at site $i+1$ (for $i = 2, \dots N-1$) then each of the walkers at site i jumps to site $i-1$ at rate $2(2\xi_{i-1}+1)$ and to site $i+1$ at rate $2(2\xi_{i+1}+1)$.

The duality function relating BMP to this dual particle system is a polynomial indexed by the particle configuration $\xi = (\xi_0, \dots, \xi_{N+1})$, $\xi_i \in \mathbb{N}$ explicitly given by

$$D(\xi, \vec{x}) = T_L^{\xi_0} T_R^{\xi_{N+1}} \prod_{i=1}^{N} \frac{x^{2\xi_i}}{(2\xi_i - 1)!!} \tag{0.3.1}$$

where $k!! = \prod_{j=1}^{k}(2j-1)$. Due to the symmetry of the generator only even powers of x_i need to be considered here.

In the dual process particles tend to jump with higher rates to the neighbors which contain more particles. This causes an attractive interaction between the particles, hence we choose the name Symmetric Inclusion Process (aka SIP) for this process. This has to be seen in contrast to the repulsive interaction in the exclusion process (SEP) where there is at most one particle per site..

At the boundaries each of the ξ_1 walkers at site 1 is absorbed at site 0 at rate 2 and it jumps to site 2 at rate $2(2\xi_2+1)$; each of the ξ_N walkers at site N is absorbed at site $N+1$ at rate 2 and it jumps to site $N-1$ at rate $2(2\xi_{N-1}+1)$. So particles that are absorbed at the 0 and $N+1$ boundary sites do not interact with each other and with other particles. An important property of this process is that it conserves the total number of particles and that starting from any initial configuration, all of the particles will be ultimately absorbed at either one of the boundaries. Duality between BMP and SIP has been used to obtain the temperature profile, exact expressions for the two-point correlation functions, proofs that the equilibrium ($T_L = T_R$) is Gaussian and of the existence of a unique stationary measure in the non-equilibrium case ($T_L \neq T_R$) [8].

0.4 Duality and symmetry

If two Markov process are dual to each other, then the probabilistic properties of one can be obtained through the study of the other, given that the duality functions constitute a sufficiently rich (e.g. measure determining) class. This is specially useful if one of the processes is easier to study than the other.

Duality has been used in the probabilistic literature and particularly in interacting particle systems since Spitzer [25] used it to study symmetric exclusion process (SEP) and independent random walkers. Ligget [15] used duality systematically for studying the ergodic properties of spin systems, the SEP and the voter model. Duality has also been useful in the context of transport models and non-equilibrium statistical mechanics. For instance, Spohn used duality in the study of SEP in contact with particle reservoirs at different chemical potentials [26], showing the existence of long-range correlations. Further applications of duality are in models of energy transport like the Kipnis-Marchioro-Presutti (KMP) model [14] for heat conduction and also for other models like BMP and BEP [8]. Duality has also been used in the study of biological population models, see for example [19].

However, in general there has been no systematic way to show that there is a duality between two Markov processes, neither a method to construct a (new) dual process for a given Markov process. Duality between two Markov processes is usually obtained in an ad-hoc manner, i.e. by an explicit ansatz for a duality function.

We consider two different cases of duality. The duality between two different Markov processes as introduced before, but also the duality of a Markov process with itself, called self-duality. The use of self-duality comes from the fact that the dual process (which is just a copy of of the original process) is often running on a smaller portion of the state space than the original process, which means that probabilistic properties of a larger system can be obtained via study of a smaller system. This is most manifest in the case that the original state space is infinite and the dual state space is finite, which allows to fully understand the behavior of a system of possibly infinitely many particles in terms of the behavior of the same system with only finitely many particles.

In chapter one we show that self-duality is directly related to the non-abelian symmetries of the generator of the Markov process (we say that an operator S is a symmetry of the generator L if they commute $S.L = L.S$). In fact for every symmetry of the generator there is a duality function associated and for every duality function there is a corresponding symmetry of the generator. In the case of duality between two different Markov processes, duality requires a conjugacy relation between the two corresponding generators. So duality between two different processes can be viewed as a change of representation of the generator.

One way to think about duality and symmetry is to think of a generator L as being composed of 'abstract operators' (like for example creation and annihilation operators) which generate an algebra with specific commutation relations. Then for every different representation of this algebra we can obtain different time evolutions, not necessarily

Markov processes, which are dual to each other.

So it turns out that duality is directly related to different representations of an algebra. Notice however that such a change of representation of the algebra does not necessarily transform the generator to a new Markov generator. In the case of finite state spaces, one can already see that a change of basis does not necessarily preserve the fact that off-diagonal elements are non-negative which is a necessary property of a Markov generator. Only when after a change of representation the Markov generator is transformed into a Markov generator, we are in the situation of two Markov processes related by duality.

Sandow and Schutz [23] were the first to notice the relation between $SU(2)$ symmetry of the SEP and its self-duality, by rewriting its generator in terms of quantum spin operators. In chapter one we show in much greater generality the relation between self-dualities and symmetries and give several new examples. For interacting particle systems used as transport models such as BMP we show how to modify the duality functions in order to include the effect of the reservoirs at the boundaries. For energy transport models we uncover a hidden $SU(1,1)$ symmetry in a large class of models (including BMP, KMP model) which explains their duality property, as the $SU(2)$ does for the SEP process. We also show the $SU(1,1)$ symmetry of SIP and the corresponding self-duality.

0.5 SIP and its comparison to SEP; correlation inequalities

Particles in the SIP perform two distinct motions. In addition to a symmetric and independent random walk, they jump to neighboring sites with a rate which is proportional to the number of particles at that site (inclusion jumps, or jumps by 'invitation'). The jump rate for a particle from site i to $i+1$ is $2\xi_i(1+2\xi_{i+1})$ which can be interpreted as follows. Every particle at site i performs a random walk jump to site $i+1$ at rate 2 and additionally every particle at site $i+1$ invites every particle at i at rate 4 (the inclusion jumps from i to $i+1$).

These inclusion moves result in a net attractive interaction between particles. This has to be compared to SEP where particles tend to effectively repel each other (by not being allowed to be at the same site) in addition to their symmetric random walk.

In physical terminology, one can therefore think of SIP as the bosonic counterpart of the fermionic SEP. Intuitively, particles in SIP starting from any configuration tend to gather and to be less spread out than independent symmetric random walks starting from the same configuration. Comparison inequalities (as introduced in Liggett [15] for SEP versus independent random walks) are a rigorous way of describing this idea.

In chapter 2 we analyze SIP in detail and prove the analogue of Liggets comparison inequality for it. From the comparison inequality, we deduce a series of correlation inequalities. As expected intuitively, the correlations turn from negative in SEP to positive in SIP. This is from another point of view quite remarkable because since the SIP is not a monotone process and positive correlations are in no way related to a FKG property, such as in the case of ferromagnetic Glauber dynamics. Since the SIP is dual to the heat conduction model it is immediate to extend those correlation inequalities to the Brownian momentum process and the Brownian energy process. We also consider the more general non-equilibrium case in which the system is in contact with boundary particle reservoirs where we use the self-duality property of SIP to obtain a correlation inequality.

0.6 Condensation in SIP and other models

Condensation phenomena in particle systems can be described as follows; in a given finite system we take the limit as the number of particles goes to infinity, if in the steady state almost all of the particles get concentrated on a finite number of sites , i.e. if all sites have a finite number of particles except a few (these few turn out to be the site(s) where the marginal of the reversible measure has the heaviest tail) , then the system exhibits condensation.

The attractive interaction between the particles in the SIP makes it a natural candidate to study for condensation phenomena. Condensation can arise due to the presence of sub-exponential tails resulting from a strong particle attraction, as has been shown in detail in the context of zero-range processes [12] .

In chapter 3 we show that SIP exhibits exponential tails, and thus the attraction between particles alone is not strong enough and a second contributing factor is required for condensation. One such factor can be spatial inhomogeneities (or also an asymmetry in a finite or semi-infinite system). Another possibility for condensation in SIP is to introduce a parameter m defined as the rate of random walks jumps while the rates of inclusion jumps are kept unchanged. Thus for example setting $m = 0$ would result in a pure inclusion process. We show that in the limit as $m \to 0$, SIP exhibits condensation. We also show parallel condensation phenomena in the Brownian Energy Process (derived from BMP and thus related to SIP), which gives an interesting example of condensation for continuous variables.

0.7 Weak coupling to the heat bath of BMP

In chapter 4 we study the BMP in close-to-equilibrium conditions. One way of achieving such conditions is to make the system in contact with two heat baths at the boundaries such that the temperatures of the two baths are different but very close. In this case we show that the distance between the local equilibrium measure and the true non- equilibrium steady state is of order at most the square of the temperature difference between the two baths, which is in agreement with the theory of McLennan ensembles [16].

An alternative way to achieve close to equilibrium conditions is to fix the temperatures of the two heat baths to arbitrary non-equal temperatures but modify and weaken the coupling of the bulk system to the heat bath with a parameter λ. We then study the behavior of the non-equilibrium steady state measure for small values of coupling constant λ. In particular we show which equilibrium measure is selected as $\lambda \to 0$.

For both cases the temperature profile turn out to be linear in the bulk system. We also give exact computations for the two-point correlation functions for some small finite size systems and discuss their generic form and we show that they are generally not multi-linear.

0.8 (Self)-dualities with $SU(3)/SU(n)$ symmetry; future plans

An interesting future line of research is to find new particle systems or Markov processes that exhibit new kinds of symmetries and corresponding (self-)duality properties. Natural examples are symmetric exclusion type processes with several types of particles. In the case of two type of particles its natural to expect $SU(3)$ symmetry for appropriate choices of jump rates. More generally if one considers $n-1$ types of particles we expect having $SU(n)$ symmetries.

It is an interesting problem to find the necessary and sufficient conditions on the rates and allowed transitions in a specific process such that it will have a particular symmetry and be thus an ‘exactly solvable’ model.

In search for new processes and their corresponding dualities, the the idea of the abstract generator we discussed earlier will be useful. One can start from an abstract generator of a Markov process that is composed of operators that obey a particular algebra. Different representations of the operators in the algebra will then yield different process interrelated via duality.

Moreover, as is the case for symmetric exclusion process, one can hope that appropriate asymmetric modifications of such processes are associated to the deformations of the corresponding algebras, as has been established in the case of the asymmetric exclusion process, [24] and [13].

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1 Duality and hidden symmetries in interacting particle systems

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1.0 Abstract

In the context of Markov processes, both in discrete and continuous setting, we show a general relation between duality functions and symmetries of the generator. If the generator can be written in the form of a Hamiltonian of a quantum spin system, then the "hidden" symmetries are easily derived. We illustrate our approach in processes of symmetric exclusion type, in which the symmetry is of $SU(2)$ type, as well as for the Kipnis-Marchioro-Presutti (KMP) model for which we unveil its $SU(1,1)$ symmetry. The KMP model is in turn an instantaneous thermalization limit of the energy process associated to a large family of models of interacting diffusions, which we call Brownian energy process (BEP) and which all possess the $SU(1,1)$ symmetry. We treat in detail the case where the system is in contact with reservoirs and the dual process becomes absorbing.

1.1 Introduction

Duality is a technique developed in the probabilistic literature that allows to obtain elegant and general solutions of some problems in interacting particle systems. One transforms the evaluation of a correlation function in the original model to a simpler quantity in the dual one.

The basic idea of duality in interacting particle systems goes back to Spitzer [11] who introduced it for symmetric exclusion process (SEP) and independent random walkers to characterize the stationary distribution. Later, Ligget [8] systematically introduced duality for spin systems and used it, among others, for the complete characterization of ergodic properties of SEP, voter model, etc. Duality property might also be useful in the context of transport models and non-equilibrium statistical mechanics, that is when the bulk particle systems is in contact at its boundaries with reservoirs working at different values of their parameters. For instance, considering again the symmetric exclusion process in contact with particle reservoirs at different chemical potentials, Spohn used duality to compute the 2-point correlation function [12], showing the existence of long-range correlations in non-equilibrium systems. In the case of energy transport, i.e. interacting particle systems with a continuous dynamical variable (the energy) connected at their boundaries to thermal reservoirs working at different temperatures, duality has been constructed for the Kipnis-Marchioro-Presutti

(KMP) model [7] for heat conduction and also for other models [6]. Consequences of duality include the possibility to express the n-point energy correlation functions in terms of n (interacting) random walkers. Duality has also been used in the study of biological population models, see [9] and references therein.

One should notice that the construction of a dual process is usually performed with an ad-hoc procedure which requires the ansatz of a proper duality function on which the duality property can be established. The closure of n-point correlations functions at each order might be an indication that a dual process exists. However in the general case the closure property is neither sufficient nor necessary to construct the dual process. In this paper we present a general procedure to derive a duality function and a dual process from the symmetries of the original process. When applied to transport models, our theorems allow to identify the source of the existence of a dual process with the non-abelian symmetries of the evolution operator. The idea is simple: transport models have in the bulk a symmetry associated with a conserved quantity, the one that is transported. It may happen in some cases that this symmetry is a subgroup of a larger group, i.e. that extra (less obvious) symmetry are present. In that case, one can describe the same physical situation as the transport of another quantity (another element of the group), and in some cases this makes the problem simpler. In the physics literature Sandow and Schutz [10] realized that this is case for the SEP process, whose $SU(2)$ symmetry they made explicit by writing the evolution operator in quantum spin notation. In this paper we study in full generality the relation between duality and symmetries. We give a general scheme for constructing duality for continuous time Markov processes whose generator has a symmetry. For interacting particle systems used as transport models we detail the effect of the reservoirs. For particle transport models we generalize the symmetric exclusion process to a situation where each site can accommodate up to $2j$ particles, with $j \in \mathbb{N}/2$. For energy transport models we uncover a hidden $SU(1,1)$ symmetry in a large class of models for energy transport (including KMP model) which explains their duality property, as the $SU(2)$ does for the SEP process.

1.2 Definitions and Results

1.2.1 Generalities

Let $(\eta_t)_{t\geq 0}$ denote a Markov process on a state space Ω. Elements of the state space are denoted by η, ξ, ζ,.. The probability measure on path space starting from η is called $\mathbb{P}_\eta$, and $\mathbb{E}_\eta$ denotes expectation with respect to $\mathbb{P}_\eta$. In the whole of this paper, we will restrict to Feller processes. In that case, to the process $(\eta_t)_{t\geq 0}$ there corresponds a strongly continuous, positivity-preserving, contraction semigroup $A_t : \mathscr{C}(\Omega) \to \mathscr{C}(\Omega)$ with domain the set $\mathscr{C}(\Omega)$ of continuous functions $f : \Omega \to \mathbb{R}$

$$A_t f(\eta) := \mathbb{E}_\eta f(\eta_t) = \mathbb{E}(f(\eta_t)|\eta_0 = \eta) = \int f(\eta')p_t(\eta, d\eta') \tag{1.2.1}$$

where $p_t(\eta, d\eta')$ is the transition kernel of the process. The infinitesimal generator of the semigroup is denoted by L,

$$Lf = \lim_{t\to 0} \frac{A_t f - f}{t}$$

and is defined on its natural domain, i.e. the set of functions $f : \Omega \to \mathbb{R}$ for which the limit in the r.h.s. exists in the uniform metric. We also consider the adjoint of the semigroup, with domain $\mathscr{M}(\Omega)$ the set of signed finite Borel measures, $A_t^* : \mathscr{M}(\Omega) \to \mathscr{M}(\Omega)$, defined by

$$< f, A_t^*\mu > = < A_t f, \mu >$$

where the pairing $< \cdot, \cdot >: \mathscr{C}(\Omega) \times \mathscr{M}(\Omega) \to \mathbb{R}$ is given by

$$< f, \mu >= \int f d\mu$$

The processes which appear in our applications will always be either jump process or diffusions.

Example 1.2.1. *In the case that the Markov process $(\eta_t)_{t\geq 0}$ is a pure jump process and the state space Ω is finite or countable then the generator is of the form*

$$Lf(\eta) = \sum_{\eta'\in\Omega} c(\eta, \eta')(f(\eta') - f(\eta))$$

where $c(\eta, \eta') \geq 0$ is the rate for a transition from configuration η to configuration η'. Equivalently we can write

$$Lf(\eta) = \sum_{\eta'\in\Omega} L(\eta, \eta')f(\eta')$$

where $L(\eta, \eta')$ is a matrix having positive off-diagonal elements and rows sum equal to zero, namely

$$L(\eta, \eta') = \begin{cases} c(\eta, \eta') & \text{if} \quad \eta \neq \eta' \\ -\sum_{\eta'' \neq \eta} c(\eta, \eta'') & \text{if} \quad \eta = \eta' \end{cases}$$

In the context of a countable state space Ω we have the usual exponential of a matrix, so that

$$A_t = e^{tL} = \sum_{i=0}^{\infty} (tL)^i / i!$$

and $A_t^* = A_t^T$ where the superscript ^{T} denotes transposition.

Example 1.2.2. *General diffusion processes with state space $\Omega = \mathbb{R}^N$ are also considered here. In this case the generator take the form of a differential operator of the second order*

$$Lf = \sum_{i,j=1}^{N} a(x_i, x_j) \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b(x_i) \frac{\partial f}{\partial x_i}$$

(see [13] for general conditions which guarantees that L satisfy the maximum principle and thus generate a positivity preserving semigroup).

1.2.2 Duality and Self-duality

Definition 1.2.3 (self-duality)**.** *Consider two independent copies $(\eta_t)_{t\geq 0}$ and $(\xi_t)_{t\geq 0}$ of a continuous time Markov processes on a state space Ω. We say that the process is self-dual with self-duality function $D : \Omega \times \Omega \to \mathbb{R}$ if for all $(\eta, \xi) \in \Omega \times \Omega$, we have*

$$\mathbb{E}_\eta D(\eta_t, \xi) = \mathbb{E}_\xi D(\eta, \xi_t) \tag{1.2.2}$$

Definition 1.2.4 (duality)**.** *Consider two continuous time Markov processes: $(\eta_t)_{t\geq 0}$ on a state space Ω and $(\xi_t)_{t\geq 0}$ on a state space Ω_{dual}. We say that $(\xi_t)_{t\geq 0}$ is the dual of $(\eta_t)_{t\geq 0}$ with duality function $D : \Omega \times \Omega_{dual} \to \mathbb{R}$ if for all $\eta \in \Omega$, $\xi \in \Omega_{dual}$ we have*

$$\mathbb{E}_\eta D(\eta_t, \xi) = \mathbb{E}_\xi^{dual} D(\eta, \xi_t) \tag{1.2.3}$$

If A_t denotes the semigroup of the original process $(\eta_t)_{t\geq 0}$ and A_t^{dual} denotes the semigroup of the related dual process $(\xi_t)_{t\geq 0}$ then, using Eq. (1.2.1), the definition 1.2.4 is equivalent to

$$A_t D(\eta, \xi) = A_t^{dual} D(\eta, \xi) \tag{1.2.4}$$

where it is understood that on the l.h.s. of (1.2.4) the operator A_t works on the η variable, while on the r.h.s. the operatorA_t^{dual} works on the ξ variable.

If the original process $(\eta_t)_{t\geq 0}$ and the dual process $(\xi_t)_{t\geq 0}$ are Markov processes with finite or countably infinite state space Ω, resp. Ω_{dual}, (cfr. Example 1.2.1) property (1.2.4) is equivalent with its "infinitesimal version" in terms of the generators

$$\sum_{\eta'\in\Omega} L(\eta,\eta')D(\eta',\xi) = \sum_{\xi'\in\Omega} L_{dual}(\xi,\xi')D(\eta,\xi') \tag{1.2.5}$$

In matrix notation, this reads

$$LD = DL_{dual}^T \tag{1.2.6}$$

where D is the matrix with elements $D(\eta,\xi)$ and $(\eta,\xi)\in\Omega\times\Omega_{dual}$. Remark that in this case D is not necessarily a square matrix, because the state spaces Ω and Ω_{dual} are not necessarily equal and or of equal cardinality.

When $\Omega=\Omega_{dual}$ and $A_t = A_t^{dual}$, then an equivalent condition for self-duality (cfr. (1.2.2)) is

$$LD = DL^T \tag{1.2.7}$$

1.2.3 Duality and Symmetries

We first discuss self-duality and then duality. We consider the simple context of finite or countably infinite state space Markov processes. In many cases of interacting particle systems, the generator is a sum of operators working only on a finite set of coordinates of the configuration. Therefore, showing (self)-duality reduces to showing (self)-duality for the individual terms appearing in this sum, which is a finite state space situation.

Definition 1.2.5. *Let A and B be two matrices having the same dimension. We say that A is a symmetry of B if A commutes with B, i.e.*

$$AB = BA \tag{1.2.8}$$

The first theorem shows that self-duality functions and symmetries are in one-to-one correspondence, provided L and L^T are similar matrices, which is automatically the case in the finite state space context.

Theorem 1.2.6. *Let L be the generator of a finite or countable state space Markov process. Let Q be a matrix such that*

$$L^T = QLQ^{-1}\,. \tag{1.2.9}$$

Then we have

1. *If S is a symmetry of the generator, then SQ^{-1} is a self-duality function.*
2. *If D is a self-duality function, then DQ is a symmetry of the generator.*
3. *If S is a symmetry of L^T, then $Q^{-1}S$ is a self-duality function*
4. *If D is a self-duality function, then QD commutes with L^T.*

Proof. The proof is elementary. We show items 1 and 2 (item 3 and 4 are obtained in a similar manner). Combining (1.2.9) with (1.2.8), we find

$$L(SQ^{-1}) = (SQ^{-1})L^T \tag{1.2.10}$$

i.e., $D = SQ^{-1}$ is a self-duality function (see Eq. (1.2.7)). Conversely, if D is a self-duality function, then combining (1.2.9) with (1.2.7) one proves (1.2.8) for $S = DQ$. □

Remark 1.2.7. *Self-duality functions are not unique, i.e. there might exist several self-duality functions for a process. This is evident from the fact that if D is a duality function for self-duality, and S is a symmetry, then SD is also a duality function for self-duality. An interesting question is to study the vector space of self-duality functions, its dimension, etc. However this question is not addressed in this paper. See [9] for a discussion of this issue and some examples in the context of Markov processes with discrete state space.*

Remark 1.2.8. *In the finite state space context, L and L^T are always similar matrices [14], i.e., there exists a conjugation matrix Q such that $L^T = QLQ^{-1}$. In interacting particle system the matrix Q can usually be easily constructed. As an example, in the case that $\mathscr{L}$ has a reversible measure, i.e., a probability measure μ on Ω such that*

$$\mu(\eta)L(\eta,\eta') = \mu(\eta')L(\eta',\eta) \tag{1.2.11}$$

for all $\eta,\eta' \in \Omega$, then a diagonal conjugation matrix Q is given by

$$Q(\eta,\eta') = \mu(\eta)\delta_{\eta,\eta'} \tag{1.2.12}$$

In general, if μ is a stationary measure then

$$L_{rev}(\eta,\xi) := \frac{L(\xi,\eta)\mu(\eta)}{\mu(\xi)}$$

is the generator of the time-reversed process, which is clearly similar to L^T. Therefore, the similarity of L and L^T is equivalent with the similarity of the generator and the time-reversed generator.

Self-duality is a particular case of duality. To generalize Theorem 1.2.6 to the context of (general) duality we need the notion of conjugation between two matrices.

Definition 1.2.9. *Let A be a matrix of dimension $m \times m$ and let B be a matrix of dimension $n \times n$. A and B are called* conjugate *if there exist matrices C of dimension $m \times n$ and $\tilde{C}$ of dimension $n \times m$ such that*

$$AC = CB, \qquad \tilde{C}A = B\tilde{C} \tag{1.2.13}$$

We then have the following analogue of Theorem 1.2.6.

Theorem 1.2.10. *Let L and L_{dual} be generators of finite or countable state space Markov chains. Then we have the following.*

1. *If Q is the matrix that gives the similarity*

$$L_{dual}^T = QL_{dual}Q^{-1} \tag{1.2.14}$$

   and C and $\tilde{C}$ are the matrices giving the conjugacy between L and L_{dual} in the sense of definition 1.2.9, then:

   a) *For any symmetry S of the generator L, $D = SCQ^{-1}$ is a duality function.*

   b) *If D is a duality function, then $S = DQ\tilde{C}$ is a symmetry of L.*

2. *If Q is the matrix that gives the similarity*

$$L^T = QLQ^{-1} \tag{1.2.15}$$

   and C and $\tilde{C}$ are the matrices giving the conjugacy between L^T and L_{dual}^T in the sense of definition 1.2.9, then:

   a) *For any symmetry S of the transposed generator L^T, $Q^{-1}SC$ is a duality function.*

b) If D is a duality function, then $QD\tilde{C}$ commutes with L^T.

Proof. The proof of item 1(a) is given by the following series of equalities

$$L(SCQ^{-1}) = SLCQ^{-1} = SCL_{dual}Q^{-1} = (SCQ^{-1})L^T_{dual} \tag{1.2.16}$$

The first equality uses the hypothesis of S being a symmetry of the generator L, the second comes from the conjugation of the generators, the third is obtained from the similarity transformation (1.2.14). If one recall (1.2.6) then Eq.(1.2.16) shows that $D = SCQ^{-1}$ is a duality function. The proof of the other items follow from a similar argument. □

1.3 Examples with two sites

In this section we present a series of examples where particles jump on two lattice sites. We wish to show how (self)-duality can be established by making use of the previous theorems. To identify the symmetries we will rewrite the stochastic generator, or its adjoint, in terms of generators of some symmetry group. Some of the examples will be useful later for the study of transport models. In fact, many transport models such as the exclusion process have a generator that is written as the sum of operators working on two sites.

1.3.1 Self-duality for symmetric exclusion

We first recover the classical self-duality for symmetric exclusion [8]. One has two sites (labeled $1, 2$) and configurations have at most one particle at each site. Particles hop at rate one from one site to another, and jumps leading to more than one particle at a site are suppressed. As usual we write $0, 1$ for absence resp. presence of particle. The state space is then $\Omega = \{00, 01, 10, 11\}$. Elements in the state space are denoted as $\eta = (\eta_1\eta_2)$. The matrix elements of the generator are given by $L_{01,10} = L_{10,01} = 1 = -L_{01,01} = -L_{10,10}$, and all other elements are zero.

To apply Theorem 1.2.6 we need to identify a symmetry S of the generator. The transposed of the generator can be written as

$$L^T = J_1^+ \otimes J_2^- + J_1^- \otimes J_2^+ + 2J_1^0 \otimes J_2^0 - \frac{1}{2}\mathbb{1}_1 \otimes \mathbb{1}_2 \tag{1.3.1}$$

where the operators J_i^a with $i \in \{1,2\}$ and $a \in \{+,-,0\}$ act on a 2-dimensional Hilbert space, with basis $|0\rangle = \binom{1}{0}, |1\rangle = \binom{0}{1}$, as

$$J_i^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad J_i^- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad J_i^0 = \begin{pmatrix} -1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \tag{1.3.2}$$

and $\mathbb{1}_i$ is the identity matrix. The operators J_i^a with $a \in \{+,-,0\}$ satisfy the $SU(2)$ commutation relations:

$$\begin{aligned} [J_i^0, J_i^{\pm}] &= J_i^{\pm} \\ [J_i^-, J_i^+] &= -2J_i^0 \end{aligned} \tag{1.3.3}$$

from which we deduce (cfr (1.3.1)) that L^T commutes with the three generators of the $SU(2)$ group, $J^a = J_1^a \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes J_2^a$ for $a \in \{+,-,0\}$. A possible choice for the symmetry of L^T is then obtained by considering the creation operator J^+ and exponentiating in order to have a factorized form

$$S = e^{J^+} = e^{J_1^+ \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes J_2^+} = e^{J_1^+} \otimes e^{J_2^+} = S_1 \otimes S_2$$

More explicitly, in the basis $|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle$, the matrix S is

$$S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

We also need the similarity transformation between L and L^T. The matrix Q, relating L to its transposed, is the identity since L is symmetric. A duality function for self-duality is thus given by $D = Q^{-1}S = S$. Notice that D can also be written as

$$D(\eta_1\eta_2, \xi_1\xi_2) = \prod_{i \in \{1,2\}: \xi_i = 1} \eta_i$$

which is the usual self-duality function of [8].

1.3.2 Self-duality for $2j$-symmetric exclusion

Now we consider two sites with at most $2j$ particles on each site, with $j \in \mathbb{N}/2$. The state space is $\Omega = \Omega_1 \times \Omega_2$ where $\Omega_i = \{0, 1, \ldots 2j\}$. The rates for transitions are the following: if there are η_1 particles at site 1 and η_2 particles at site 2, a particle is moved from 1 to 2 at rate $\eta_1(2j-\eta_2)$ and from 2 to 1 at rate $\eta_2(2j-\eta_1)$. So in this case the generator is given by

$$\begin{aligned} L(\eta_1\eta_2, \eta_1'\eta_2') &= \eta_1(2j-\eta_2)\delta_{\eta_1-1,\eta_1'}\delta_{\eta_2+1,\eta_2'} + \eta_2(2j-\eta_1)\delta_{\eta_1+1,\eta_1'}\delta_{\eta_2-1,\eta_2'} \\ &\quad -(\eta_1(2j-\eta_2)+\eta_2(2j-\eta_1))\delta_{\eta_1,\eta_1'}\delta_{\eta_2,\eta_2'} \end{aligned}$$

The transposed of this generator can also be expressed as the scalar product between two spin operators satisfying the $SU(2)$ algebra, namely

$$L^T = J_1^+ \otimes J_2^- + J_1^- \otimes J_2^+ + 2J_1^0 \otimes J_2^0 - 2j^2 \mathbb{1}_1 \otimes \mathbb{1}_2 \tag{1.3.4}$$

where the J_i^a, $i \in \{1,2\}$ and $a \in \{+,-,0\}$, act on a $(2j+1)$-dimensional Hilbert space with orthonormal basis $|0\rangle, |1\rangle, \ldots |2j\rangle$ as

$$\begin{aligned} J_i^+|\eta_i\rangle &= (2j-\eta_i)|\eta_i+1\rangle \\ J_i^-|\eta_i\rangle &= \eta_i|\eta_i-1\rangle \\ J_i^0|\eta_i\rangle &= (\eta_i-j)|\eta_i\rangle \end{aligned} \tag{1.3.5}$$

The standard symmetric exclusion process of the previous section is recovered when $j = 1/2$. Reasoning as above, a symmetry of the generator is

$$S = S_1 \otimes S_2 = e^{J_1^+} \otimes e^{J_2^+}$$

which has matrix elements $S(\eta_1\eta_2, \xi_1\xi_2) = S_1(\eta_1, \xi_1)S_2(\eta_2, \xi_2)$ with

$$S_i(\eta_i, \xi_i) = \langle \eta_i | e^{J_i^+} | \xi_i \rangle = \binom{2j-\eta_i}{\eta_i-\xi_i} \tag{1.3.6}$$

where we adopt the convention $\binom{n}{m} = 0$ for $m > n$.

To detect the matrix Q giving the similarity transform between L and L^T (notice that L is not symmetric anymore for $j \neq 1/2$) we make use of remark 1.2.8 and use the fact that the invariant measures of the $2j$-symmetric exclusion process are products of binomials $Bin(2j, \rho)$, with a free parameter $0 < \rho < 1$ (this will be proved in Theorem 1.4.2). Therefore, if we choose $\rho = 1/2$ then a possible choise is $Q = Q_1 \otimes Q_2$ with

$$Q_i(\eta_i, \eta_i') = \delta_{\eta_i,\eta_i'}\binom{2j}{\eta_i} \tag{1.3.7}$$

Combining (1.3.6) and (1.3.7), Theorem 1.2.6 then implies that a duality function for self-duality is given by

$$D = D_1 \otimes D_2 = Q_1^{-1} S_1 \otimes Q_2^{-1} S_2$$

with

$$D_i(\eta_i, \xi_i) = (Q_i^{-1} S_i)(\eta_i, \xi_i) = \frac{\binom{\eta_i}{\xi_i}}{\binom{2J}{\xi_i}} \tag{1.3.8}$$

Later, in Theorem 1.4.2, we will give a probabilistic interpretation of this function.

1.3.3 Self-duality for the dual-BEP

This is a process that can be viewed as a "bosonic" analogue of the SEP (particles attract each other rather than repel with the exclusion hard core constraint). The state space is $\Omega = \Omega_1 \times \Omega_2$ with $\Omega_i = \mathbb{N}$, i.e. we have two sites each of which can accommodate an unlimited number of particles. For η_1 particles at site 1, η_2 particles at site 2, the rate of putting a particle from 1 to 2 is given by $2\eta_1(2\eta_2+1)$ and the rate of moving a particle from 2 to 1 is given by $2\eta_2(2\eta_1+1)$. We will see later how this process arises naturally as a dual of the Brownian Energy Process (BEP), see Section 1.5 below.

The matrix of the generator is given by

$$\begin{aligned} L(\eta_1\eta_2, \eta_1'\eta_2') \quad = \quad & 2\eta_1(2\eta_2+1)\delta_{\eta_1',\eta_1-1}\delta_{\eta_2',\eta_2+1} + 2\eta_2(2\eta_1+1)\delta_{\eta_1',\eta_1+1}\delta_{\eta_2',\eta_2-1} \\ & -(8\eta_1\eta_2 + 2\eta_1 + 2\eta_2)\delta_{\eta_1,\eta_1'}\delta_{\eta_2,\eta_2'}. \end{aligned} \tag{1.3.9}$$

The transposed of the generator can be written in terms of generators of a $SU(1,1)$ algebra as follows. On each site $i \in \{1,2\}$ we consider operators K_i^a with $a \in \{+,-,0\}$ given by

$$\begin{aligned} K_i^+|\eta_i\rangle &= (\eta_i + 1/2)|\eta_i + 1\rangle \\ K_i^-|\eta_i\rangle &= \eta_i|\eta_i - 1\rangle \\ K_i^0|\eta_i\rangle &= (\eta_i + 1/4)|\eta_i\rangle \end{aligned} \tag{1.3.10}$$

They satisfy the commutation relations of $SU(1,1)$:

$$\begin{aligned} [K_i^0, K_i^\pm] &= \pm K_i^\pm \\ [K_i^-, K_i^+] &= 2K_i^0 \end{aligned} \tag{1.3.11}$$

The transposed of the generator then reads

$$L^T = 4\left(K_1^+ \otimes K_2^- + K_1^- \otimes K_2^+ - 2K_1^0 \otimes K_2^0 + \frac{1}{8}\mathbb{1}_1 \otimes \mathbb{1}_2\right) \tag{1.3.12}$$

From the commutation relations, it is easy to see that L^T commutes with $K^a = K_1^a \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes K_2^a$, for $a \in \{+, -, 0\}$. A possible symmetry is then given by the matrix

$$S = S_1 \otimes S_2 = e^{K_1^+} \otimes e^{K_2^+}$$

which has matrix elements $S(\eta_1\eta_2, \xi_1\xi_2) = S_1(\eta_1, \xi_1)S_2(\eta_2, \xi_2)$ with

$$S_i(\eta_i, \xi_i) = \langle \eta_i | e^{K_1^+} | \xi_i \rangle = \frac{(2\eta_i - 1)!!}{(2\xi_i - 1)!!(\eta_i - \xi_i)!\ 2^{\eta_i - \xi_i}} \tag{1.3.13}$$

A similarity transformation $L^T = Q^{-1}LQ$ to pass to the transposed is suggested (remark 1.2.8) by the knowledge of the stationary measure of the dual-BEP (see Theorem 1.5.1)

$$Q(\eta_1\eta_2, \eta_1'\eta_2') = Q_1(\eta_1, \eta_1')Q_2(\eta_2, \eta_2')$$

with

$$Q_i(\eta_i, \xi_i) = \delta_{\eta_i, \xi_i}\left(\frac{\eta_i!}{(2\eta_i - 1)!!}2^{\eta_i}\right)^{-1} \tag{1.3.14}$$

The self-duality function corresponding to S of (1.3.13) and Q of (1.3.14) then reads

$$D(\eta_1\eta_2, \xi_1\xi_2) = D_1(\eta_1, \xi_1)D_2(\eta_2, \xi_2)$$

$$D_i(\eta_i, \xi_i) = Q^{-1}(\eta_i, \eta_i)S_i(\eta_i, \xi_i) = 2^{\xi_i}\frac{\eta_i!}{(\eta_i - \xi_i)!(2\xi_i - 1)!!} \tag{1.3.15}$$

1.3.4 Self-duality for independent random walkers

This is a classical example which is included here for the sake of completeness. We have two site 1 and 2, and particles hop independently from 1 to 2 and from 2 to 1 at rate one. So the rate to put a particle from 1 to 2 in a configuration with η_1 particles at 1 and η_2 particles at 2 is simply η_1. The generator is given by the matrix

$$L(\eta_1\eta_2, \eta_1'\eta_2') = \eta_2\delta_{\eta_1, \eta_1'-1}\delta_{\eta_2, \eta_2'+1} + \eta_1\delta_{\eta_1, \eta_1'+1}\delta_{\eta_2, \eta_2'-1} + (-\eta_1 - \eta_2)\delta_{\eta_1, \eta_1'}\delta_{\eta_2, \eta_2'}$$

A self-duality function is $D = D_1 \otimes D_2$ with

$$D(\eta_i, \xi_i) = \frac{\eta_i!}{(\eta_i - \xi_i)!}$$

The invariant measures are product of Poisson distributions and a possible conjugation is thus given by $Q = Q_1 \otimes Q_2$ with

$$Q_i(\eta_i, \xi_i) = \delta_{\eta_i,\xi_i} \frac{1}{\eta_i!}$$

As a consequence, a symmetry of the generator is given by $S = S_1 \otimes S_2$ with

$$S_i(\eta_i, \xi_i) = (D_i Q_i)(\eta_i, \xi_i) = \frac{1}{(\eta_i - \xi_i)!} \tag{1.3.16}$$

This symmetry comes once more from an underlying structure of creation and annihilation operators satisfying the Heisenberg algebra. Indeed, if one defines for $i \in \{1, 2\}$ operators a_i^+ and a_i^- which are represented on an Hilbert space with basis $|0\rangle, |1\rangle, |2\rangle, \ldots$ by operators working as

$$\begin{aligned} a_i^+ |\eta_i\rangle &= |\eta_i + 1\rangle \\ a_i^- |\eta_i\rangle &= \eta_i |\eta_i - 1\rangle \end{aligned} \tag{1.3.17}$$

then one easily verifies the commutation relation

$$[a_i^-, a_i^+] = \mathbb{1}_i \ .$$

In terms of these matrices, the transposed of the generator reads

$$L^T = -(a_1^+ \otimes \mathbb{1}_2 - \mathbb{1}_1 \otimes a_2^+)(a_1^- \otimes \mathbb{1}_2 - \mathbb{1}_1 \otimes a_2^-) \tag{1.3.18}$$

which commutes with $a^+ = a_1^+ \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes a_2^+$. The symmetry S in (1.3.16) is then recognized as $S = \exp(a_1^+) \otimes \exp(a_2^+)$.

1.3.5 Duality between independent random walkers and a deterministic system

As an example of application of Theorem 1.2.10 we consider again a system of independent random walkers jumping between sites 1 and 2. We show that this system is dual to a deterministic system evolving according to ordinary differential equations.

We consider the "abstract" operator $\mathscr{L}$

$$\mathscr{L} = -(a_1^+ - a_2^+)(a_1^- - a_2^-) \tag{1.3.19}$$

where a_i^+, a_j^- are operators satisfying the canonical commutation relations

$$[a_i^-, a_j^+] = \delta_{i,j} \mathbb{1} \, . \tag{1.3.20}$$

One way to represent the previous operator is by considering

$$a_i^- = \frac{\partial}{\partial x_i}, \qquad a_i^+ = x_i, \qquad i \in \{1, 2\}$$

which obviously satisfy (1.3.20). In this case the operator (1.3.19) takes the form

$$L = -(x_1 - x_2) \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right)$$

which is the generator of the deterministic system of differential equations

$$\begin{aligned} \frac{dx_1(t)}{dt} &= -(x_1(t) - x_2(t)) \\ \frac{dx_2(t)}{dt} &= (x_1(t) - x_2(t)) \end{aligned} \tag{1.3.21}$$

with solutions

$$\begin{aligned} x_1(t) &= \frac{x_1(0) + x_2(0)}{2} + \frac{x_1(0) - x_2(0)}{2} e^{-2t} \\ x_2(t) &= \frac{x_1(0) + x_2(0)}{2} - \frac{x_1(0) - x_2(0)}{2} e^{-2t} \end{aligned} \tag{1.3.22}$$

Another possible way to represent the operator (1.3.19) has just been seen in the previous paragraph. In this case the creation and annihilation operators are represented as matrices with elements given by (1.3.17) and then the operator (1.3.19) can be seen as the transposed of the generator for a system on independent random walkers

$$L_{dual}^T = -(a_1^+ \otimes \mathbb{1}_2 - \mathbb{1}_1 \otimes a_2^+) \circ (a_1^- \otimes \mathbb{1}_2 - \mathbb{1}_1 \otimes a_2^-) \, .$$

It is immediately checked that the function

$$D(x, \xi) = D(x_1, \xi_1) D(x_2, \xi_2)$$

with

$$D(x_i, \xi_i) = x_i^{\xi_i}$$

gives a conjugation between L and L_{dual}^T, namely

$$L D(x, \xi) = D L_{dual}^T (x, \xi)$$

In this case, this relation reads more explicitly,

$$\begin{aligned} & -(x_1 - x_2)\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)(x_1^{n_1} x_2^{n_2}) \\ = \;& n_1 x_1^{n_1-1} x_2^{n_2+1} + n_2 x_1^{n_1+1} x_2^{n_2-1} - (n_1+n_2) x_1^{n_1} x_2^{n_2} \end{aligned} \tag{1.3.23}$$

This implies that $x(t)$ and $\xi(t)$ are each other's dual with duality function D and the following relation holds

$$x_1(t)^{\xi_1} x_2(t)^{\xi_2} = \mathbb{E}_{\xi_1,\xi_2}(x_1(0)^{\xi_1(t)} x_2(0)^{\xi_2(t)}) \tag{1.3.24}$$

where the expectation in the rhs is over the independent random walkers starting from initial configuration with η_1 particles at 1 and η_2 particles at 2. We will come back to this example in section 1.6.4.

Remark 1.3.1. *In the last example, we can still use other representations of the operators a_i^-, a_i^+, satisfying the commutation relation $[a_i^-, a_j^+] = \delta_{ij}$, such that the abstract operator* (1.3.19) *is the generator of a Markovian diffusion process. E.g., if we choose*

$$\begin{aligned} a_i^+ &= -\frac{\partial}{\partial x_i} + x_i \\ a_i^- &= \frac{\partial}{\partial x_i} \end{aligned} \tag{1.3.25}$$

then the abstract operator (1.3.19) *reads*

$$\mathscr{L} = \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)^2 + (x_1 - x_2)\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)$$

which is the generator of the (degenerate) diffusion: the "coordinate $(x_1 - x_2)/2$ undergoes a Brownian motion and $(x_1 + x_2)/2$ remains constant. So in that case we also have duality

$$\mathbb{E}_{x_1,x_2}(D(x_1(t), n_1) D(x_2(t), n_2)) = \mathbb{E}_{n_1,n_2}(D(x_1, n_1(t)) D(x_2, n_2(t)))$$

where $D(x,n)$ can be found by the recursion

$$D(x, n+1) = a^+ D(x,n)$$

e.g. the first five polynomials are

$D(x,0) = 1, D(x,1) = x, D(x,2) = x^2 - 1, D(x,3) = -3x + x^3, D(x,4) = 3 - 6x^2 + x^4$

1.4 Symmetric exclusion processes

In this section we study the $2j$-SEP (with $j \in \mathbb{N}/2$), i.e. exclusion processes with at most $2j$ particles per site, on a graph S. We show how we can understand self-duality for the $2j$-SEP from "classical duality" (in the sense of [8]) of the symmetric exclusion process on special graphs. We also consider two limits $j \to \infty$ leading to a deterministic process or a system of independent random walkers. Finally, we consider the boundary driven case, and show that we have a dual with absorbing boundaries.

1.4.1 Symmetric exclusion on ladder graphs

Consider a countable set $\mathscr{S}$, to be thought of as the underlying lattice, and a finite set $\mathscr{I}$ with cardinality $2j \in \mathbb{N}$. The set $\mathscr{I}$ is to be thought of as a "ladder" on each site with $2j$ levels.

The state space of SEP on the ladder graph $\mathscr{S} \times \mathscr{I}$ is $\Omega = \{0,1\}^{\mathscr{S}\times\mathscr{I}}$. A configuration $\zeta \in \Omega$ is called finite if it contains a finite number of particles, i.e., if $\sum_{i\in\mathscr{S},\alpha\in\mathscr{I}} \zeta(i,\alpha) < \infty$. The process is described as follows. Let $p(i,l)$ denote a symmetric random walk kernel on $\mathscr{S}$, i.e., $p(i,l) = p(l,i) \geq 0$, $\sum_{l\in\mathscr{S}} p(i,l) = 1$. At each site $i \in \mathscr{S}$ and level $\alpha \in \mathscr{I}$, there is at most one particle. Each particle attempts to jump at rate $p(i,l)$ to a site $l \in \mathscr{S}$ and level $\beta \in \mathscr{I}$.

More formally, the SEP on a ladder graph $\mathscr{S} \times \mathscr{I}$ is the process with the following generator on local functions $f : \Omega \to \mathbb{R}$

$$Lf(\zeta) = \sum_{i,l\in\mathscr{S}} \sum_{\alpha,\beta\in\mathscr{I}} p(i,l)(\zeta(i,\alpha)(1-\zeta(l,\beta))(f(\zeta^{(i,\alpha),(l,\beta)}) - f(\zeta)) \tag{1.4.1}$$

where $\zeta^{(i,\alpha),(l,\beta)}$ denotes the configuration obtained from η by removing a particle at site i level α and placing it at site l level β. Since this process is a symmetric exclusion process on a special graph, then it is self-dual in the following sense:

Proposition 1.4.1. *Define for $\zeta, \tilde{\zeta} \in \Omega$, $\tilde{\zeta}$ finite,*

$$D(\zeta,\tilde{\zeta}) = \prod_{i,\alpha:\tilde{\zeta}(i,\alpha)=1} \zeta(i,\alpha)$$

then we have the duality relation from [8]

$$\mathbb{E}_{\zeta} D(\zeta_t, \tilde{\zeta}) = \mathbb{E}_{\tilde{\zeta}} D(\zeta, \tilde{\zeta}_t) \tag{1.4.2}$$

where ζ_t, $\tilde{\zeta}_t$ *are independent copies of the ladder SEP with generator* (1.4.1) *starting from* ζ, *resp.* $\tilde{\zeta}$.

1.4.2 From the ladder SEP to the $2j$-SEP

To define the $2j$-SEP on a graph $\mathscr{S}$ we consider, for a given SEP on a ladder graph $\mathscr{S} \times \mathscr{I}$ with $2j$ levels, the process which consists of giving at time $t > 0$, and each site $i \in \mathscr{S}$ the number of levels (i, α) which are occupied in ζ_t. More precisely, define the map $\pi : \Omega \to \Omega^{(j)} = \{0, 1, \ldots, 2j\}^{\mathscr{S}}$

$$\zeta \mapsto \pi(\zeta) = \eta \qquad \text{with} \quad \eta_i = \sum_{\alpha \in \mathscr{I}} \zeta(i, \alpha) \tag{1.4.3}$$

Then we have the following theorem.

Theorem 1.4.2. *Let* ζ_t *be the ladder SEP with generator* (1.4.1)*. Then the following holds:*

a) $\eta_t = \pi(\zeta_t)$ *is a Markov process on* $\Omega^{(j)}$ *with generator*

$$L^{(j)} f(\eta) = \sum_{i,l \in S} p(i,l) \eta_i (2j - \eta_l)(f(\eta^{i,l}) - f(\eta)) \tag{1.4.4}$$

This process will be called the $2j$*-SEP or reduced ladder SEP with* $2j$ *levels.*

b) The process η_t *with generator* $L^{(j)}$ *is self-dual with duality function*

$$D(\eta, \xi) = \prod_{i \in S} \frac{\binom{\eta_i}{\xi_i}}{\binom{2j}{\xi_i}} \tag{1.4.5}$$

for $\xi \leq \eta$ *a configuration with a finite number of particles (*D *is defined to be zero in other cases). More precisely, we have*

$$\mathbb{E}_\eta D(\eta_t, \xi) = \mathbb{E}_\xi D(\eta, \xi_t) \tag{1.4.6}$$

c) The extremal invariant measures are of the form

$$\nu_\rho^{(j)} = \otimes_{i \in S} \mathrm{Bin}(2j, \rho_i)$$

where ρ_i *is harmonic for* $p(i,l)$*, i.e., such that*

$$\sum_l p(i,l) \rho_l = \rho_i$$

In particular, if the only bounded harmonic functions are constants, then the only extremal invariant measures are products of binomials with constant density.

Proof. For point (a) remark that the jump rates in the generator (1.4.4) only depend on the number of particles at a site, and not on the levels. Therefore, if $f : \Omega \to \mathbb{R}$ depends on ζ only through $\eta = \pi(\zeta)$, i.e., if $f(\zeta) = \psi(\pi(\zeta)) = \psi(\eta)$, then

$$Lf(\zeta) = L^{(j)}\psi(\eta) \tag{1.4.7}$$

Therefore, for every local function $\psi : \Omega^{(j)} \to \mathbb{R}$,

$$\psi(\pi(\zeta_t)) - \psi(\pi(\zeta_0)) - \int_0^t L^{(j)}(\psi)(\pi(\zeta_s))ds = M_t \tag{1.4.8}$$

is a martingale w.r.t. the filtration $\mathscr{F}_t = \sigma\{\pi(\zeta)_s : 0 \leq s \leq t\}$. This shows that $\pi(\zeta_t)$ is a solution of the martingale problem associated to the generator $L^{(j)}$, and hence coincides with the unique Markov process generated by $L^{(j)}$ (see Th. 4.1, page 182, of [4]).

Now we turn to point (b). At each site $i \in S$ we choose ξ_i levels at random. For a given configuration $\eta \in \Omega_j$, we choose $\zeta \in \Omega$ consistent, i.e., such that $\pi(\zeta) = \eta$. Then the probability (w.r.t. to the random choices) that all chosen levels are occupied in ζ is exactly equal to $D(\eta, \xi)$. As $\pi(\zeta_t) = \eta_t$, the probability that the chosen levels are occupied at time t (i.e., in ζ_t) is given by $\mathbb{E}_\eta D(\eta_t, \xi)$. By self-duality of the ladder SEP (Prop. 1.4.1), the event that at time $t > 0$ the chosen levels are occupied is the same as the probability that the particles evolving from the chosen levels during a time t find themselves at positions which are occupied in ζ. The latter probability equals $\mathbb{E}_\xi D(\xi_t, \eta)$.

Point c) follows from the fact that for the ladder SEP with generator (1.4.1), the product Bernoulli measures indexed by harmonic functions of $p(i, j)$ are the extremal invariant measures (see [8] for details) and the image measure of a product of Bernoulli measure under π is a product of Binomial measures. $\square$

1.4.3 Limiting processes as $j \to \infty$

In this section we show that for large j the $2j$-SEP converges, when considered on an appropriate time scale, either to a system of independent random walkers or to a deterministic limit, depending on the initial density. We remind the reader that for independent random walkers on a graph S, the configuration space is $\Omega_\infty = \mathbb{N}^{\mathscr{S}}$ and the generator is given by

$$L_{irw}f(\eta) = \sum_{i,l\in\mathscr{S}} \eta(i)p(i,l)(f(\eta^{i,l}) - f(\eta)) \tag{1.4.9}$$

The stationary measures are products of Poisson measures, and the process with generator (1.4.9) is self-dual with self-duality function

$$D_{irw}(\eta,\xi) = \prod_{i\in S} \frac{\eta_i!}{(\eta_i - \xi_i)!} \tag{1.4.10}$$

for finite configurations $\xi \leq \eta$, and $D = 0$ otherwise.

The relation with the reduced ladder SEP for large j is given in the following theorem.

Theorem 1.4.3. *Consider the process* $\{\eta_t^{(j)} : t \geq 0\}$ *with generator* (1.4.4) *started from initial configuration* $\eta^{(j)} \in \Omega^{(j)}$. *Suppose that, as* $j \to \infty$, $\eta^{(j)} \to \eta \in \Omega_\infty$, *then the process* $\{\eta_{t/2j}^{(j)} : t \geq 0\}$ *converges weakly in path space to a system of independent random walkers with generator* (1.4.9) *and initial configuration* η.

Proof. The process $\{\eta_{t/2j}^{(j)} : t \geq 0\}$ has generator

$$\mathscr{L}_j' = \frac{1}{2j} L^{(j)} \tag{1.4.11}$$

In order to have a sequence of processes all defined on the same sample space $\Omega^{(\infty)}$ we consider the auxiliary process on $\mathbb{N}^S$ with generator

$$\mathscr{L}_j'' f(\eta) = \sum_{i,l\in S} p(i,l)\eta_i \left(1 - \frac{\eta_l}{2j}\right) I(\eta_i \leq 2j) I(\eta_l \leq 2j) \left(f(\eta^{i,l}) - f(\eta)\right) \tag{1.4.12}$$

This auxiliary process behaves as the process with generator $\mathscr{L}_j'$ except for sites which have more than $2j$ particles, which are frozen. Started from an initial configuration with all sites having at most $2j$ particles, this process coincides with the process $\eta_{t/2j}$. For any local function $f : \mathbb{N}^S \to \mathbb{R}$, we then have

$$\begin{aligned} \lim_{j\to\infty} \mathscr{L}_j'' f(\eta) &= \lim_{j\to\infty} \sum_{i,l\in S} p(i,l)\eta_i \left(1 - \frac{\eta_l}{2j}\right) I(\eta_i \leq 2j) I(\eta_l \leq 2j) \left(f(\eta^{i,l}) - f(\eta)\right) \\ &= \mathscr{L}_{irw} f(\eta) \end{aligned} \tag{1.4.13}$$

Therefore, by the Trotter-Kurtz theorem (see Theorem 2.5 of [4]), this implies that the corresponding processes $\eta_{t/2j}$ converge weakly on path space as $j \to \infty$ to the process with generator $\mathscr{L}_{irw}$. $\square$

To see that (1.4.10) is (up to a multiplicative consant) a limit of duality functions of the $2j$-SEP, we start from the symmetry (1.3.6) and use the similarity transformation with $Q^{(j)}$ given by

$$Q^{(j)}(\eta,\xi) = \prod_{i\in\mathscr{S}} Q_i^{(j)}(\eta_i,\xi_i)$$

with

$$Q_i^{(j)}(\eta_i,\xi_i) = \delta_{\eta_i,\xi_i}\binom{2j}{\eta_i}\left(\frac{1}{2j}\right)^{\eta_i}\left(1-\frac{1}{2j}\right)^{2j-\eta_i}$$

Then the duality functions

$$\tilde{D}^{(j)} = {Q^{(j)}}^{-1} S$$

with S defined in (1.3.16), satisfy

$$\lim_{j\to\infty} \tilde{D}^{(j)} = eD_{irw} \ .$$

Another possible limit is obtained when the initial condition has a number of particles that diverges proportional to j. This limit, as can be understood from the law of large numbers, is deterministic.

Theorem 1.4.4. *Consider the process*

$$x_i^{(j)}(t) = \frac{\eta_{t/2j}^{(j)}}{2j}$$

and suppose that $x_i^{(j)}(0) \to x_i(0) \in [0,1]$ for all $i \in \mathscr{S}$ as $j \to \infty$. Then we have that $x_i^{(j)}(t)$ converges to a deterministic system of coupled differential equations with generator

$$Lf(x) = \sum_{i,l\in S} p(i,l)x_i(1-x_l)\left(\frac{\partial}{\partial x_l} - \frac{\partial}{\partial x_i}\right) \tag{1.4.14}$$

Proof. The generator of the process $x_i^{(j)}(t)$ reads

$$\mathscr{L}_j' f(x) = 2j\sum_{i,l} p(i,l)x_i(1-x_l)(f(x^{(j);i,l}) - f(x^{(j)}))$$

where $x^{(j),i,l}$ arises from $x^{(j)}$ by removing a unit $1/2j$ from $i \in \mathscr{S}$ and putting it at $l \in \mathscr{S}$. Therefore, for a local smooth function $f : [0,1]^S \to \mathbb{R}$ we have, by Taylor expansion

$$\mathscr{L}'_j f(x) = \sum_{i,l} p(i,l) x_i (1 - x_l) \left(\frac{\partial}{\partial x_l} - \frac{\partial}{\partial x_i} \right) f(x) + O\left(\frac{1}{2j}\right)$$

The result then follows once more from the Trotter Kurtz theorem. Since the limiting generator is a first order differential operator, the corresponding process is deterministic. $\square$

1.4.4 Boundary driven case

We first discuss a duality theorem for standard (i.e. $j = 1/2$) symmetric exclusion with extra creation and annihilation of particles at the boundary. The context is a countable set $\mathscr{S}$, of which we distinguish a subset $\partial\mathscr{S} \subseteq \mathscr{S}$ called the boundary. We then consider the generator

$$\begin{aligned} \mathscr{L} f(\eta) &= \sum_{i,l \in \mathscr{S}} p(i,l)(\eta(i)(1-\eta(l))(f(\eta^{i,l}) - f(\eta)) \\ &+ \sum_{i \in \partial\mathscr{S}} (1-\rho_i)\eta(i) \left(f(\eta^i) - f(\eta)\right) + \rho_i (1 - \eta(i)) \left(f(\eta^i) - f(\eta)\right) \end{aligned} \tag{1.4.15}$$

where $0 < \rho_i < 1$ represent the densities of the particle reservoirs with which the system is in contact at the boundary sites, and where η^i is the configuration obtained from η by flipping the occupancy at i.

The first part of the generator represents the hopping of particles on $\mathscr{S}$ according to a symmetric exclusion process, whereas the second part represents creation and annihilation of particles at the boundary sites.

To introduce duality for this process, we introduce a set $\partial_e\mathscr{S}$ of sink sites, and a bijection $i \mapsto i_e$ which associates each site $i \in \partial\mathscr{S}$ to a sink site. The dual process will then be a process that behaves as the exclusion process in the bulk, but particles can leave the system via boundary sites to sink sites, and will then be stuck at sink sites. More precisely, a configuration of the dual process is a map

$$\xi : \mathscr{S} \cup \partial_e\mathscr{S} \to \mathbb{N}$$

such that $\xi(i) \in \{0,1\}$ for $i \in \mathscr{S}$ (only the sink sites can contain more than one particle). We call Ω_{dual} the set of all configurations of the dual process.

The duality function is defined as follows: for $\eta \in \{0,1\}^{\mathscr{S}}$, $\xi \in \Omega_{dual}$

$$D(\eta,\xi) = \left(\prod_{i\in\mathscr{S}:\xi(i)=1} \eta(i) \right) \left(\prod_{i\in\partial_e\mathscr{S}} \rho_i^{\xi_i} \right) \tag{1.4.16}$$

The idea here is that we have the ordinary duality function for the sites $i \in \mathscr{S}$ and for the sink sites, we replace the variable η_i by its expectation ρ_i, corresponding to the stationary measure of the boundary generator L_i.

The generator of the dual process is then defined as follows:

$$\mathscr{L}_{dual} f(\xi) = \sum_{i,l\in\mathscr{S}} p(i,l)(\xi(i)(1-\xi(l))(f(\xi^{i,l}) - f(\xi)) + \sum_{i\in\partial\mathscr{S}} \xi(i)\left(f(\xi^{i,i_e}) - f(\xi)\right) \tag{1.4.17}$$

We then have

Theorem 1.4.5. *The boundary driven exclusion process* $(\eta_t)_{t\geq 0}$ *with generator* $\mathscr{L}$ *in* (1.4.15) *and the process* $(\xi_t)_{t\geq 0}$ *with generator* $\mathscr{L}_{dual}$ *in* (1.4.17) *are dual with duality function* $D(\eta,\xi)$ *given by* (1.4.16)*, i.e.,*

$$\mathbb{E}_\eta D(\eta_t,\xi) = \mathbb{E}_\xi D(\eta,\xi_t) \tag{1.4.18}$$

Proof. Abbreviate

$$L_i f(\eta) = (1-\rho_i)\eta(i)\left(f(\eta^i) - f(\eta)\right) + \rho_i(1-\eta(i))\left(f(\eta^i) - f(\eta)\right) \tag{1.4.19}$$

and

$$L_i^{dual} f(\xi) = \xi(i)\left(f(\xi^{i,i_e}) - f(\xi)\right) \tag{1.4.20}$$

For $f(\eta) = \eta(i)$ one sees that

$$L_i f(\eta) = \rho_i - \eta(i) \tag{1.4.21}$$

and hence for ξ a dual configuration which is non-zero only on the sites $i \in \partial\mathscr{S}$ and on the corresponding sink site $i_e \in \partial_e\mathscr{S}$, we find

$$\begin{aligned} L_i D(\eta,\xi) &= \rho_i^{\xi_{i_e}} \left(L_i\left(\eta(i)\xi(i) + (1-\xi(i))\right)\right) \\ &= \xi(i)\rho_i^{\xi_{i_e}}(\rho_i - \eta(i)) \\ &= \xi(i)\left(\rho_i^{\xi_{i_e}+1} - \rho_i^{\xi_{i_e}}\eta(i)\right) = L_i^{dual} D(\xi,\eta) \end{aligned}$$

From that and the self-duality of the symmetric exclusion process, it follows that

$$\mathscr{L} D(\eta, \xi) = \mathscr{L}_{dual} D(\eta, \xi) \tag{1.4.22}$$

□

In order to apply this duality result for the boundary driven process with generator (1.4.4), we first look at the boundary driven exclusion process on a ladder graph. More precisely, for $\zeta \in \{0,1\}^{\mathscr{S} \times \mathscr{I}}$ we consider the process $(\zeta_t)_{t \geq 0}$ with generator

$$\begin{aligned} \mathscr{L} f(\zeta) &= \sum_{i,l \in \mathscr{S}} \sum_{\alpha, \beta \in \mathscr{I}} p(i,l)(\zeta(i,\alpha)(1-\zeta(l,\beta))(f(\zeta^{(i,\alpha),(l,\beta)}) - f(\zeta)) \\ &+ \sum_{i \in \partial \mathscr{S}} \sum_{\alpha \in \mathscr{I}} (1-\rho_i)\zeta(i,\alpha) \Big(f(\zeta^{(i,\alpha)}) - f(\zeta)\Big) + \rho_i (1-\zeta(i,\alpha)) \Big(f(\zeta^{(i,\alpha)}) - f(\zeta)\Big) \end{aligned} \tag{1.4.23}$$

In words, this process is the ladder SEP, with additional boundary driving, where the creation and annihilation rate of particles at the boundary sites does not depend on the level. If we consider the reduced process, consisting of counting at each site $i \in \mathscr{S}$ how many levels in $\mathscr{I}$ are occupied, i.e. $\eta_t = \pi(\zeta_t)$ then we recover once again a Markov process (cfr. Theorem 1.4.2). This process, defined on the state space $\Omega^{(j)} = \{0, 1, \ldots 2j\}^{\mathscr{S}}$ and called the boundary driven $2j$-SEP, has generator

$$\begin{aligned} \mathscr{L}_j f(\eta) &= \sum_{i,l \in \mathscr{S}} p(i,l)(\eta(i)(2j - \eta(l))(f(\eta^{i,l}) - f(\eta)) \\ &+ \sum_{i \in \partial \mathscr{S}} (1-\rho_i)\eta(i) \Big(f(\eta^{(i,\alpha)}) - f(\eta)\Big) + \rho_i (2j - \eta_i) \big(f(\eta^i) - f(\eta)\big) \end{aligned} \tag{1.4.24}$$

It turns out that the boundary driven $2j$-SEP has a nice dual too. To introduce this dual, we consider admissible dual configurations as maps $\xi : \mathscr{S} \cup \partial_e \mathscr{S} \to \mathbb{N}$ such that $0 \leq \xi(i) \leq 2j$ for $i \in \mathscr{S}$ (only the sink sites can contain more that $2j$ particles). The generator of the dual of the boundary driven $2j$-SEP is a process on admissible dual configurations, defined by

$$\begin{aligned} \mathscr{L}_j^{dual} f(\xi) &= \sum_{i,l \in \mathscr{S}} p(i,l)(\xi(i)(2j - \xi(l))(f(\xi^{i,l}) - f(\xi)) \\ &+ \sum_{i \in \partial \mathscr{S}} \xi(i) \big(f(\xi^{i,i_e}) - f(\xi)\big) \end{aligned} \tag{1.4.25}$$

From Theorem 1.4.5 we then infer, in the same way as we derived Theorem 1.4.2 the following.

Theorem 1.4.6. *Let $(\eta_t)_{t\geq 0}$ denote the boundary driven $2j$-SEP with generator* (1.4.24). *Then $(\eta_t)_{t\geq 0}$ is dual to the process $(\xi_t)_{t\geq 0}$ with generator* (1.4.25) *with duality function given by*

$$D(\eta,\xi) = \prod_{i\in\partial_e\mathscr{S}} \rho_i^{\xi_i} \prod_{i\in\mathscr{S}} \frac{\binom{\eta_i}{\xi_i}}{\binom{2j}{\xi_i}} \tag{1.4.26}$$

Proof. Denote for $i \in \partial\mathscr{S}$,

$$\mathscr{L}_i f(\eta) = (1-\rho_i)\eta(i)\left(f(\eta^{(i,\alpha)}) - f(\eta)\right) + \rho_i(2j-\eta(i))\left(f(\eta^i) - f(\eta)\right)$$

and

$$\mathscr{L}_i^{dual} f(\xi) = \xi(i)\left(f(\xi^{i,i_e}) - f(\xi)\right)$$

One then easily computes that for ξ a dual configuration which is non-zero only on the sites $i \in \partial\mathscr{S}$ and on the corresponding sink site $i_e \in \partial_e\mathscr{S}$,

$$\begin{aligned}
\mathscr{L}_i D(\eta,\xi) &= \rho_i^{\xi_{i_e}} \frac{1}{\binom{2j}{\xi_i}} \mathscr{L}_i \left(\binom{\eta_i}{\xi_i}\right) \\
&= \rho_i^{\xi_{i_e}} \frac{1}{\binom{2j}{\xi_i}} \left[(1-\rho_i)\eta_i \left(\binom{\eta_i - 1}{\xi_i} - \binom{\eta_i}{\xi_i}\right) + (2j-\eta_i)\rho_i \left(\binom{\eta_i+1}{\xi_i} - \binom{\eta_i}{\xi_i}\right)\right] \\
&= \xi_i(\rho_i^{\xi_{i_e}+1} - \rho_i)\frac{\binom{\eta_i}{\xi_i}}{\binom{2j}{\xi_i}} = \mathscr{L}_i^{dual} D(\eta,\xi)
\end{aligned}$$

The result then follows from combination of this fact and the duality relation (1.4.6). □

1.5 The Brownian Momentum Process and $SU(1,1)$ symmetry

In this section we study the Brownian momentum process that was introduced in [5, 2]. We will recover duality [6] in the context of our main Theorems and study the reversible measures of the dual process.

1.5.1 Generator and Quantum spin chain

Let $\mathscr{S}$ be a countable set and $p(i,j)$ a symmetric random walk transition probability on $\mathscr{S}$. The Brownian momentum process is a Markov process $(x_t)_{t\geq 0}$ on $\mathbb{R}^{\mathscr{S}}$, with generator

$$L = \sum_{i,j\in\mathscr{S}} p(i,j)L_{ij} \tag{1.5.1}$$

where

$$L_{ij} = \left(x_i\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_i}\right)^2 \tag{1.5.2}$$

and $p(i,j)$ is a symmetric random walk kernel on $\mathscr{S}$.

The generator L_{ij} conserves the energy $x_i^2 + x_j^2$ and generates a Brownian rotation of the angle $\theta_{ij} = \arctan(x_j/x_i)$. The interpretation of the generator (1.5.1) is then as follows: each bond independently, at rate $p(i,j)$ undergoes a Brownian rotation of its angle $\theta_{ij} = \arctan(x_j/x_i)$. An important example to keep in mind is $\mathscr{S} = \mathbb{Z}^d$, and $p(i,j)$ the nearest neighbor symmetric random walk.

The processes with generator L can be related to quantum spin chains [6]. Consider the operators

$$\begin{aligned} K_i^+ &= \frac{1}{2}x_i^2 \\ K_i^- &= \frac{1}{2}\frac{\partial^2}{\partial x_i^2} \\ K_i^0 &= \frac{1}{4}\left(\frac{\partial}{\partial x_i}(x_i\cdot) + x_i\frac{\partial}{\partial x_i}\right) \end{aligned} \tag{1.5.3}$$

which satisfy the commutation relations of $SU(1,1)$:

$$\begin{aligned} [K_i^0, K_i^\pm] &= \pm K_i^\pm \\ [K_i^-, K_i^+] &= 2K_i^0 \end{aligned} \tag{1.5.4}$$

Then the negative of the adjoint of the generator L can be seen as the quantum "Hamiltonian"

$$H = -L^* = -4\sum_{ij\in\mathscr{S}} p(i,j)\left(K_i^+K_j^- + K_i^-K_j^+ - 2K_i^0K_j^0 + \frac{1}{8}\right) \tag{1.5.5}$$

with spin satisfying the $SU(1,1)$ algebra (in a representation with spin value 1/4).

1.5.2 Dual process

In [6] we showed that the process with generator L in (1.5.1) and (1.5.2) has a dual, which is a system of interacting random walkers on $\mathscr{S}$. We show here how this dual process comes out of the structure of the Hamiltonian (1.5.5).

We notice that the $SU(1,1)$ group admits a discrete (infinite dimensional) representation as (unbounded) operators on $l_2(\mathbb{N})$:

$$\begin{aligned} \mathscr{K}_i^+|\xi_i\rangle &= \left(\frac{1}{2}+\xi_i\right)|\xi_i+1\rangle \\ \mathscr{K}_i^-|\xi_i\rangle &= \xi_i|\xi_i-1\rangle \\ \mathscr{K}_i^0|\xi_i\rangle &= \left(\xi_i+\frac{1}{4}\right)|\xi_i\rangle \end{aligned} \tag{1.5.6}$$

where $i \in \mathscr{S}$ and $\xi_i \in \mathbb{N}$ and $|0\rangle, |1\rangle, |2\rangle, \ldots$ denotes the canonical basis on $l_2(\mathbb{N})$. It is immediately checked that the (unbounded) operators in (1.5.6) satisfy the $SU(1,1)$ commutation relations in (1.5.4). We then define a new generator via the same Hamiltonian as in (1.5.5), but now in the representation (1.5.6):

$$H_{dual} = -L^*_{dual} = -4\sum_{ij\in\mathscr{S}} p(i,j)\left(\mathscr{K}_i^+\mathscr{K}_j^- + \mathscr{K}_i^-\mathscr{K}_j^+ - 2\mathscr{K}_i^0\mathscr{K}_j^0 + \frac{1}{8}\right) \tag{1.5.7}$$

From the previous equation and using the representation (1.5.6) we deduce that the Hamiltonian above defines a Markov process $(\xi_t)_{t\geq 0}$ with state space $\mathbb{N}^{\mathscr{S}}$ and generator

$$L_{dual} = \sum_{i,j\in\mathscr{S}} p(i,j)L_{ij}^{dual} \tag{1.5.8}$$

where

$$\begin{aligned} L_{ij}^{dual}f(\xi) &= 2\xi_i(2\xi_j+1)(f(\xi^{i,j})-f(\xi)) + \\ & \quad 2\xi_j(2\xi_i+1)(f(\xi^{j,i})-f(\xi)) \end{aligned} \tag{1.5.9}$$

with $\xi^{i,j}$ the configuration obtained from ξ by removing a particle from i and putting it at j. Note that, in general, changing a representation does not imply that a generator continues to be a generator: the fact that H and H_{dual} are well-defined as a Hamiltonian is conserved by similarity transformations (change of representation) but their property of being (minus) the adjoint of the generator of a Markov process is dependent on the representation, and needs to be verified by hand.

1.5.3 The duality function explained

In [6] we found that L and L_{dual} are dual processes, with duality function

$$D(x,\xi) = \prod_{i\in\mathscr{S}} D_i(x_i,\xi_i) \tag{1.5.10}$$

with

$$D_i(x_i,\xi_i) = \frac{x_i^{2\xi_i}}{(2\xi_i-1)!!} \tag{1.5.11}$$

We now show how these functions arise from the change of representation. Suppose that we find a function

$$C = C(x,\xi) = \prod_{i\in\mathscr{S}} C_i(x_i,\xi_i)$$

such that

$$K_i^a C_i = C_i \mathscr{K}_i^a \tag{1.5.12}$$

for $a \in \{+,-,0\}$, $i \in \mathscr{S}$ and K_i^a, resp. $\mathscr{K}_i^a$ defined in (1.5.3), (1.5.6). The "matrix product" in the lhs of (1.5.12) is defined as the differential operator K_i^a working on the x_i-variable of $C_i(x_i,\xi_i)$, and in the rhs $C_i\mathscr{K}_i^a(x_i,\xi_i) = \sum_{\xi_i'} C(x_i,\xi_i')\mathscr{K}_i^a(\xi_i',\xi_i)$.

Then for the generators L in (1.5.1),(1.5.2) and the generator L_{dual} in (1.5.8), (1.5.9) we find, as a consequence of (1.5.12) and using that $L^* = L$,

$$LC = L^*C = CL_{dual}^* \tag{1.5.13}$$

i.e., such a function C is a duality function (cfr. (1.2.6)).

The equation (1.5.12) is most easy if $a = +$, it then reads

$$\frac{1}{2}x_i^2 C_i(x_i,\xi_i) = \left(\frac{1}{2}+\xi_i\right) C_i(x_i,\xi_i+1) \tag{1.5.14}$$

To find $C_i(x_i,0)$ we use $K_i^- C_i(x_i,0) = 0$ (that follows from (1.5.6), (1.5.12)) so we can choose $C_i(x_i,0) = 1$ and then find, via (1.5.14)

$$C_i(x_i,\xi_i) = \frac{x_i^{2\xi_i}}{(2\xi_i-1)!!} \tag{1.5.15}$$

which is exactly the duality function that we found in [6]. It is then easy to verify that (1.5.12) is also satisfied for $a \in \{-,0\}$ with the choice (1.5.15).

1.5.4 Reversible measures of the dual-BEP

The dual of the BEP, with generator L_{dual} in (1.5.8) and (1.5.9), is in itself an interacting particle system (particles attract each other), and it can therefore be considered as a model of independent interest. In some sense, it can be viewed as “the bosonic counterpart” of the SEP. Surprisingly, despite the interaction, the process has reversible product measures, as is shown below. Remark that, due to the attractive interaction between the particles, this process does not fall in the class of “misantrope processes” considered in [1], [3] (where one also has in particular cases stationary product measures, despite interaction).

Theorem 1.5.1. *Consider, for $\lambda < 1/2$ the translation invariant product measure ν_λ on $\mathbb{N}^{\mathscr{S}}$ with marginals*

$$\nu_\lambda(\eta_0 = k) = \frac{1}{Z_\lambda}\frac{(2k-1)!!}{k!}\lambda^k \tag{1.5.16}$$

where the normalization is

$$Z_\lambda = \sum_{k=0}^{\infty}\frac{(2k-1)!!}{k!}\lambda^k = \frac{1}{\sqrt{(1-2\lambda)}} \; . \tag{1.5.17}$$

Then ν_λ is reversible for the process with generator L_{dual} in (1.5.8) *and* (1.5.9).

Proof. From the generator (1.5.8), (1.5.9), we infer that

$$\alpha_i\alpha_j 2i(2j+1) = 2(j+1)(2i-1)\alpha_{j+1}\alpha_{i-1}$$

is a sufficient condition for detailed balance of a product measure μ with marginals $\mu(\eta_0 = k) = \alpha_k$ for the generator L_{ij}^{dual}, which is sufficient for detailed balance for L_{dual}. This leads to

$$\frac{\alpha_{j+1}}{\alpha_j} = \frac{2j+1}{2j+2}(2c)$$

for some positive constant c. This in turn gives

$$\alpha_j = \frac{(2j-1)!!}{j!}(2c)^j\alpha_0$$

which is (1.5.16). The explicit expression for Z_λ can be obtained easily from the identity

$$\int_{-\infty}^{\infty} x^{2k}e^{-x^2/2}dx = \sqrt{2\pi}(2k-1)!!$$

□

1.6 The Brownian Energy Process

As it was done for SEP, it is interesting to consider the Brownian Momentum Process on ladder graph $\mathscr{S} \times \mathscr{I}$ with $|\mathscr{I}| = m \in \mathbb{N}$ levels and look at the induced process which gives the energy at each site.

1.6.1 Generator

Consider the generator, working on local functions $f : \mathbb{R}^{\mathscr{S}\times\mathscr{I}} \to \mathbb{R}$

$$L = \sum_{i,j\in\mathscr{S}} \sum_{\alpha,\beta=1}^{m} p(i,j) L_{i\alpha,j\beta} \tag{1.6.1}$$

where

$$L_{i\alpha,j\beta} = \left(x_{i,\alpha} \frac{\partial}{\partial x_{j,\beta}} - x_{j,\beta} \frac{\partial}{\partial x_{i,\alpha}} \right)^2 . \tag{1.6.2}$$

In this section we show that for the process with the generator above, the total energy per site defined via

$$z_i = \sum_{\alpha=1}^{m} x_{i,\alpha}^2 \tag{1.6.3}$$

is again a Markov process

Theorem 1.6.1. *Consider the process $x(t) = (x_{i,\alpha}(t))_{i\in\mathscr{S},\alpha=1,\ldots,m}$ with generator L of* (1.6.1) *and* (1.6.2)*. Consider the corresponding process $z(t) = (z_i(t))_{i\in\mathscr{S}}$ defined via the mapping* (1.6.3)*. This is a Markov process on $\mathbb{R}_+^{\mathscr{S}}$ with generator*

$$L^{(m)} = \sum_{ij\in\mathscr{S}} p(i,j) L_{ij}^{(m)} \tag{1.6.4}$$

with

$$L_{ij}^{(m)} = 4 z_i z_j \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right)^2 - 2m(z_i - z_j) \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right) \tag{1.6.5}$$

and with stationary measures which are product measures with chi-squared marginals.

Proof. Denote $\pi : (x_{i,\alpha})_{i\in\mathscr{S},\alpha=1,\ldots,m} \mapsto (z_i)_{i\in\mathscr{S}}$. Denote by ∂_i partial derivative w.r.t. z_i and by $\partial_{i,\alpha}$ partial derivative w.r.t. $x_{i,\alpha}$. Then, using the identities

$$\begin{aligned} \partial_{i,\alpha} &= 2x_{i,\alpha}\partial_i \\ \partial_{i,\alpha}^2 &= 2\partial_i + 4x_{i\,\alpha}^2 \partial_i^2 \\ \partial_{i,\alpha}\partial_{j,\beta} &= 4x_{i,\alpha}x_{j,\beta}\partial_i\partial_j \end{aligned} \tag{1.6.6}$$

we find for a function $f : \mathbb{R}^{\mathscr{S}\times\mathscr{I}} \to \mathbb{R}$ depending on x only through $z = \pi(x)$

$$Lf \circ \pi(x) = (L^{(m)} f)(\pi(x)) \tag{1.6.7}$$

The proof then proceeds via the martingale problem as in the proof of Theorem 1.4.2. The stationary measure of the process with generator $L^{(m)}$ is deduced from the knowledge of the stationary measure for the process with generator L. Indeed, it is easy to check that for the process $x_{i,\alpha}(t)$, products of Gaussian measures $\otimes_{i\in\mathscr{S}_1,\alpha} N(0,\sigma^2)$ are invariant and ergodic. The image measure under the transformation $\pi(x) = z$ are products $\otimes_{i\in\mathscr{S}_1} \chi^2_m(\sigma)$ where for $\sigma = 1$, $\chi^2_m(1)$ is the chi-squared distribution with m degrees of freedom, and for general σ, follows from scaling $\chi^2_m(\sigma^2) = \sigma^2 \chi^2_m(1)$. $\square$

1.6.2 Duality

In this section we show that the BEP defined above has a dual process which is again a jump process.

To construct the dual we follow a procedure similar to the one of the previous section. Remark that the generator L in (1.6.1) and (1.6.2) can be written in terms of the operators

$$K_i^{a,(m)} = \sum_{\alpha=1}^{m} K_{i,\alpha}^{a} \tag{1.6.8}$$

with $a \in \{+,-,0\}$, where

$$\begin{aligned} K_{i,\alpha}^{+} &= \frac{1}{2} x_{i,\alpha}^2 \\ K_{i,\alpha}^{-} &= \frac{1}{2} \frac{\partial^2}{\partial x_{i,\alpha}^2} \\ K_{i,\alpha}^{0} &= \frac{1}{4} \left(\frac{\partial}{\partial x_{i,\alpha}} (x_{i,\alpha} \cdot) + x_{i,\alpha} \frac{\partial}{\partial x_{i,\alpha}} \right) \end{aligned} \tag{1.6.9}$$

In other words L is related to a quantum spin chain with Hamiltonian

$$H^{(m)} = -L^* = -4 \sum_{ij\in\mathscr{S}} p(i,j) \left(K_i^{+,(m)} K_j^{-,(m)} + K_i^{-,(m)} K_j^{+,(m)} - 2K_i^{0,(m)} K_j^{0,(m)} + \frac{m^2}{8} \right) \tag{1.6.10}$$

where the K-operators in (1.6.8) and (1.6.9) satisfy the commutation relations of $SU(1,1)$. Moreover the K operators defined in (1.6.8), (1.6.9), can be rewritten in z-variables as follows:

$$\begin{aligned} K_i^{+,(m)} &= \frac{1}{2}z_i \\ K_i^{-(m)} &= 2z_i\partial_i^2 + m\partial_i \\ K_i^{0,(m)} &= z_i\partial_i + \frac{m}{4} \end{aligned} \tag{1.6.11}$$

The generator $L^{(m)}$ of (1.6.4) is then simply minus the adjoint of the Hamiltonian $H^{(m)}$ in (1.6.10), rewritten with K-operators in z-variables.

At this point it is important to remark that the $SU(1,1)$ group admits a family of discrete infinite dimensional discrete representation labeled by $m \in \mathbb{N}$ given by

$$\begin{aligned} \mathscr{K}_i^{+,(m)}|\xi_i\rangle &= \left(\frac{m}{2} + \xi_i\right)|\xi_i + 1\rangle \\ \mathscr{K}_i^{-,(m)}|\xi_i\rangle &= \xi_i|\xi_i - 1\rangle \\ \mathscr{K}_i^{0,(m)}|\xi_i\rangle &= \left(\frac{m}{4} + \xi_i\right)|\xi_i\rangle \end{aligned} \tag{1.6.12}$$

where $i \in \mathscr{S}$ and $\xi_i \in \mathbb{N}$ and $|0\rangle, |1\rangle, |2\rangle, \dots$ denotes the canonical basis on $l_2(\mathbb{N})$. We then define a new generator via the same Hamiltonian as in (1.6.10), but now in the representation (1.6.12):

$$\begin{aligned} H_{dual}^{(m)} &= -L_{dual}^{*} \qquad (1.6.13) \\ = - & \quad 4 \sum_{ij\in\mathscr{S}} p(i,j)\left(\mathscr{K}_i^{+,(m)}\mathscr{K}_j^{-,(m)} + \mathscr{K}_i^{-,(m)}\mathscr{K}_j^{+,(m)} - 2\mathscr{K}_i^{0,(m)}\mathscr{K}_j^{0,(m)} + \frac{m^2}{8}\right) \end{aligned}$$

Using the representation (1.6.12) we deduce that the Hamiltonian above defines a Markov process $(\xi_t)_{t\geq 0}$ with state space $\mathbb{N}^{\mathscr{S}}$ and generator

$$L_{dual}^{(m)} = \sum_{i,j\in\mathscr{S}} p(i,j)L_{ij}^{(m),dual} \tag{1.6.14}$$

where

$$\begin{aligned} L_{ij}^{(m),dual}f(\xi) &= 2\xi_i(2\xi_j + m)(f(\xi^{i,j}) - f(\xi)) + \\ & \quad 2\xi_j(2\xi_i + m)(f(\xi^{j,i}) - f(\xi)) \end{aligned} \tag{1.6.15}$$

The duality function is given in the following theorem.

Theorem 1.6.2. *The processes with generator $L^{(m)}$ and $L^{(m)}_{dual}$ are each others dual, with duality function*

$$D(z,\xi) = \prod_{i\in\mathscr{S}} D_i(z_i,\xi_i) \tag{1.6.16}$$

where

$$D_i(z_i,\xi_i) = z_i^{\xi_i}\frac{\Gamma\left(\frac{m}{2}\right)}{2^{\xi_i}\Gamma\left(\frac{m}{2}+\xi_i\right)} \tag{1.6.17}$$

with $\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx$ the gamma function.

Proof. Let

$$C_i(z_i,\xi_i) = z_i^{\xi_i}\frac{\Gamma\left(\frac{m}{2}\right)}{2^{\xi_i}\Gamma\left(\frac{m}{2}+\xi_i\right)}$$

One verifies easily that

$$K_i^{a,(m)}C_i = C_i\mathscr{K}_i^{a,(m)}$$

for $a = +,-,0$. The proof then continues as in section 1.5.3. $\square$

1.6.3 The instantaneous thermalization limit and the KMP process

In the KMP model, introduced in [7], the energies E_i of different sites $i \in \mathscr{S}$ are updated by selecting a pair of lattice sites (i,j) and uniformly redistributing the energy under the constraint of conserving $E_i + E_j$. In this section we show that the KMP model arises by taking what we call here an instantaneous thermalization limit of the process with generator $L^{(m)}$, *for the case* $m = 2$.

We start by computing the stationary measure of the process with generator $L_{ij}^{(m)}$.

Lemma 1.6.3. *Let $(z_i(t), z_j(t))$ be the Markov process with generator $L_{ij}^{(m)}$, starting from an initial condition $(z_i(0), z_j(0))$ with $z_i(0) + z_j(0) = E$. Then in the limit $t \to \infty$ the distribution of $(z_i(t), z_j(t))$ converges to the distribution of the couple $((E+\epsilon)/2, (E-\epsilon)/2)$ where ϵ has probability density*

$$f(\epsilon) = C_m(E^2-\epsilon^2)^{\frac{m}{2}-1} \tag{1.6.18}$$

$-E \le \epsilon \le E$ and $f = 0$ otherwise, and where C_m is the normalizing constant.

Proof. Define $(E(t), \epsilon(t)) = (z_i(t) + z_j(t), z_i(t) - z_j(t))$. Then simple rewriting of $L_{ij}^{(m)}$ in the new variables yields that $(E(t), \epsilon(t))$ is a Markov process with generator

$$\mathscr{L}' = 4(E^2 - \epsilon^2)\partial_\epsilon^2 - 4m\epsilon\partial_\epsilon \tag{1.6.19}$$

From the form of $\mathscr{L}'$ we see immediately that E is conserved and that for given E, $\epsilon(t)$ is an ergodic diffusion process with stationary measure solving

$$\partial_\epsilon^2(4(E^2 - \epsilon^2)f) + \partial_\epsilon(4m\epsilon f) = 0 \tag{1.6.20}$$

Now notice that the rhs of (1.6.18) solves

$$\partial_\epsilon(4(E^2 - \epsilon^2)f) + (4m\epsilon f) = 0$$

and hence (1.6.20). □

Denote by γ_m the distribution of $((E+\epsilon)/2, (E-\epsilon)/2)$. We can now define what we mean by instantaneous thermalization.

Definition 1.6.4. *Let $f : [0,\infty)^{\mathscr{S}} \to \mathbb{R}$. For $e = (e_i)_{i\in\mathscr{S}}$ a configuration of energies, $(i,j) \in \mathscr{S} \times \mathscr{S}$, $(e_i', e_j') \in [0,\infty) \times [0,\infty)$ we denote by $t(e, e_i', e_j')$ the configuration obtained from e by replacing e_i by e_i' and e_j by e_j'. The instantaneous thermalization of a pair $(i,j) \in \mathscr{S} \times \mathscr{S}$ is defined by*

$$\mathscr{T}_{ij}^{(m)} f(e) = \int f(t(e, e_i', e_j'))d\gamma_m(e_i', e_j') \tag{1.6.21}$$

The instantaneously thermalized version of the Brownian energy process is then defined as the process with generator

$$\mathscr{L}_m^{IT} f(e) = \sum_{ij\in\mathscr{S}} p(i,j)(\mathscr{T}_{ij}^{(m)} f(e) - f(e)) \tag{1.6.22}$$

This means that with rate $p(ij)$ a pair $(i,j) \in \mathscr{S} \times \mathscr{S}$ is chosen and the energy is instantaneously thermalized according to the measure γ_m. From (1.6.18) one sees that, for $m = 2$, the uniform redistribution of the KMP model is recovered.

It is interesting to consider the dual of the instantaneous thermalization process for general $m \in \mathbb{N}$. From the previous discussion one knows that in the case $m = 2$ this is just the dual of the KMP model. However, the model with generator has for general m a dual with different duality functions as is shown in Theorem 1.6.6 below. To introduce this dual, we remind the reader that the Brownian energy process with generator $L^{(m)}$ is dual to the discrete particle jump process with generator $L^{(m)}_{dual}$. The following lemma gives the stationary measure of the dual BEP, which is needed in the construction of the instantaneous thermalized version of the dual BEP.

Lemma 1.6.5. *Let (k_t, l_t) evolve according to the generator $\mathscr{L}^{ij}_{dual}$, and suppose that initially $k_0 + l_0 = N$, then in the limit $t \to \infty$, (k_t, l_t) converges in distribution to $((N+\Delta)/2, (N-\Delta)/2)$ where Δ has distribution μ on $\{-N, -N+2, \ldots, N\}$ with*

$$\frac{\mu(\Delta)}{\mu(\Delta-2)} = \frac{(N-\Delta+1)(N+\Delta-1+m)}{(N+\Delta)(N-\Delta+m)} \tag{1.6.23}$$

In particular for $m = 2$, (k_t, l_t) converges to the uniform measure on the set $\{(a,b) \in \{0,\ldots,N\} : a+b = N\}$.

Proof. The process $(N_t, \Delta_t) := (k_t + l_t, k_t - l_t)$ performs transitions $(N, \Delta) \to (N, \Delta - 2)$ at rate $\frac{1}{4}(N+\Delta)(N-\Delta+m)$ and $(N,\Delta) \to (N, \Delta+2)$ at rate $\frac{1}{4}(N-\Delta)(N+\Delta+m)$. The marginal Δ_t is then an irreducible continuous-time Markov chain on the set $\{-N, -N+2, \ldots, N\}$, and hence has a unique stationary measure. Since it is a pure birth and death chain, this measure is also reversible. The recursion (1.6.23) then follows from detailed balance. □

We denote by $\hat{\gamma}_m(k,l)$ the stationary distribution of lemma (1.6.5). For $\xi \in \mathbb{N}^{\mathscr{S}}$, and $(i,j) \in \mathscr{S} \times \mathscr{S}$, $(\xi'_i, \xi'_j) \in \mathbb{N} \times \mathbb{N}$ we denote by $t(\xi, \xi'_i, \xi'_j)$ the configuration obtained from ξ by replacing the value at i by ξ'_i and at j by ξ'_j. We then define the dual thermalization by

$$\mathscr{T}^{dual,(m)}_{ij} f(\xi) = \sum_{\xi'_i, \xi'_j:\ \xi'_i + \xi'_j = \xi_i + \xi_j} f(t(\xi, \xi'_i, \xi'_j)) \hat{\gamma}_m(\xi'_i, \xi'_j) \tag{1.6.24}$$

and the dual instantaneously thermalized energy process as the process with generator

$$\mathscr{L}^{IT,(m)}_{dual} f(\xi) = \sum_{ij \in \mathscr{S}} (\mathscr{T}^{dual,(m)}_{ij} f(\xi) - f(\xi)) \tag{1.6.25}$$

Theorem 1.6.6. *Consider the instantaneously thermalized version of the Brownian energy process, with generator $\mathscr{L}^{IT}_m$. This process is dual to the process with generator $\mathscr{L}^{IT,(m)}_{dual}$ with duality function given by*

$$D(e,\xi) = \prod_i D_i(e_i, \xi_i) \tag{1.6.26}$$

$$D_i(e_i, \xi_i) = e_i^{\xi_i} \frac{\Gamma(m/2)}{2^{\xi_i}\Gamma(m/2+\xi_i)} \tag{1.6.27}$$

Proof. By the duality result for the Brownian energy process, Theorem 1.6.2, we have for all $(i,j) \in \mathscr{S} \times \mathscr{S}$

$$L_{ij}^{(m)} D(e,\xi) = L_{ij}^{(m),dual} D(e,\xi) \tag{1.6.28}$$

therefore,

$$\lim_{t\to\infty} (e^{tL_{ij}^{(m)}} - id) D(e,\xi) = \lim_{t\to\infty} (e^{tL_{ij}^{(m),dual}} - id) D(e,\xi) \ . \tag{1.6.29}$$

The result then follows from the definition of the processes, together with lemma 1.6.3 and lemma 1.6.5. $\square$

1.6.4 Limiting processes as $m \to \infty$

As it was done for the $2j$-SEP, we study here the limiting behavior of the m-BEP process for large m.

Theorem 1.6.7. *Consider the process $\{z_t^{(m)} : t \geq 0\}$ with generator $L^{(m)}$ and initial condition $z^{(m)} \in \mathbb{R}_+^{\mathscr{S}}$ and its dual $\{\xi_t^{(m)} : t \geq 0\}$ with generator $L_{dual}^{(m)}$ and initial condition $\xi^{(m)} \in \mathbb{N}^{\mathscr{S}}$. Suppose that, as $m \to \infty$, $z^{(m)} \to z \in \mathbb{R}_+^{\mathscr{S}}$ and $\xi^{(m)} \to \xi \in \mathbb{N}^{\mathscr{S}}$. Then:*

1. *the process $\{z_{t/m}^{(m)} : t \geq 0\}$ converges to the process, $(z_t)_{t\geq 0}$ started from z, with generator*

$$L = \sum_{ij\in\mathscr{S}} p(i,j) L_{ij}$$

$$L_{ij} = -2(z_i - z_j)\left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j}\right)$$

2. *the process $\{\xi_{t/m}^{(m)}\}$ converges to a system of independent random walkers $(\xi_t)_{t\geq 0}$ started from ξ, with generator*

$$L_{dual} = \sum_{ij\in\mathscr{S}} p(i,j) L_{ij}^{dual}$$

$$L_{ij}^{dual} = 2\xi_i(f(\xi^{ij}) - f(\xi)) + 2\xi_j(f(\xi^{ji}) - f(\xi))$$

3. *The two limiting processes* $(x_t)_{t\geq 0}$ *and* $(\xi_t)_{t\geq 0}$ *above are each other's dual, with duality function*

$$D(x,\xi) = \prod_{i\in\mathscr{S}} x_i^{\xi_i}$$

Proof. The proof of items 1. and 2. proceeds like in Theorem 1.4.3. For item 3. compare to example in section 1.3.5. □

1.6.5 Boundary driven process

In this last section we consider the m-BEP process in contact at its boundary to energy reservoirs of the Ornstein-Uhlenbeck type. A duality result for the Brownian Momentum Process with reservoirs was already proven in [6]. Here we generalize this result to the general Brownian energy process for arbitrary $m \in \mathbb{N}$. We start from the momentum process $\{(x(t)_{i,\alpha}) : i \in \mathscr{S}, \alpha = 1,\dots,m, t \geq 0\}$ on a ladder graph with m levels and all levels at sites $i \in \partial\mathscr{S}$ connected to a thermalizing Ornstein-Uhlenbeck process which parameter T_i, to be thought as the temperature. The generator reads

$$\begin{aligned} L &= \sum_{i,j\in\mathscr{S}} p(i,j) \sum_{\alpha,\beta=1}^{m} \left(x_{i,\alpha}\frac{\partial}{\partial x_{j,\beta}} - x_{j,\beta}\frac{\partial}{\partial x_{i,\alpha}} \right)^2 \\ &+ \sum_{i\in\partial\mathscr{S}} \sum_{\alpha=1}^{m} T_i \frac{\partial^2}{\partial x_{i,\alpha}^2} - x_{i,\alpha}\frac{\partial}{\partial x_{i,\alpha}} \end{aligned} \tag{1.6.30}$$

If we now consider the induced process $\{z_i(t) : i \in \mathscr{S}, t \geq 0\}$ measuring the energy at each site via the map

$$z_i = \sum_{\alpha=1}^{m} x_{i,\alpha}^2 \ ,$$

then, using the identities (1.6.6), we find the generator

$$\begin{aligned} L &= \sum_{i,j\in\mathscr{S}} p(i,j) 4 z_i z_j \left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right)^2 - 2m(z_i - z_j)\left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial z_j} \right) \\ &+ \sum_{i\in\partial\mathscr{S}} 2T_i \left(m\frac{\partial}{\partial z_i} + 2z_i \frac{\partial^2}{\partial z_i^2} \right) - 2z_i \frac{\partial}{\partial z_i} \end{aligned} \tag{1.6.31}$$

Introducing as usual a set $\partial_e\mathscr{S}$ of sink sites and a bijection $i \mapsto i_e$ which associate each boundary site $i \in \partial\mathscr{S}$ to a sink site $i_e \in \partial_e\mathscr{S}$, we have the following duality theorem:

Theorem 1.6.8. *Let* $(z_t)_{t\geq 0}$ *denote the boundary driven* m*-BEP with generator* (1.6.31). *Then* $(z_t)_{t\geq 0}$ *is dual to the process* $(\xi_t)_{t\geq 0}$ *with generator*

$$\begin{aligned} L_{dual}f(\xi) &= \sum_{i,j\in\mathscr{S}} p(i,j)2\xi_i(2\xi_j+m)(f(\xi^{i,j})-f(\xi))+2\xi_j(2\xi_i+m)(f(\xi^{j,i})-f(\xi)) \\ &+ \sum_{i\in\partial\mathscr{S}} 2\xi_i(f(\xi^{i,i_e})-f(\xi)) \end{aligned} \tag{1.6.32}$$

with duality function given by

$$D(z,\xi)=\prod_{i\in\partial_e\mathscr{S}} T_i^{\xi_i}\prod_{i\in\mathscr{S}} z_i^{\xi_i}\frac{\Gamma\left(\frac{m}{2}\right)}{2^{\xi_i}\Gamma\left(\frac{m}{2}+\xi_i\right)} \tag{1.6.33}$$

Proof. The bulk part of the duality function coincides with the one of Theorem 1.6.2; the boundary part is easily checked with an explicit computation. □

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2 Correlation inequalities for interacting particle systems with duality

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2.0 Abstract

We prove a *comparison inequality* between a system of independent random walkers and a system of random walkers which either interact by *attracting each other* – a process which we call here the symmetric inclusion process (SIP) – or *repel each other* – a generalized version of the well-known symmetric exclusion process. As an application, new *correlation inequalities* are obtained for the SIP, as well as for some interacting diffusions which are used as models of heat conduction, – the so-called Brownian momentum process, and the Brownian energy process. These inequalities are counterparts of the inequalities (in the opposite direction) for the symmetric exclusion process, showing that the SIP is a natural bosonic analogue of the symmetric exclusion process, which is fermionic. Finally, we consider a boundary driven version of the SIP for which we prove duality and then obtain correlation inequalities.

2.1 Introduction

In Liggett [14], Chapter VIII, proposition 1.7, a comparison inequality between independent symmetric random walkers and corresponding exclusion symmetric random walkers is obtained. This inequality plays a crucial role in the understanding of the exclusion process (SEP); it makes rigorous the intuitive picture that symmetric random walkers interacting by exclusion are more spread out than the corresponding independent walkers, as a consequence of their repulsive interaction (exclusion), or in more physical terms, because of the fermionic nature of the exclusion process. The comparison inequality is a key ingredient in the ergodic theory of the symmetric exclusion process, i.e., in the characterization of the invariant measures, and the measures which are in the course of time attracted to a given invariant measure. The comparison inequality has been generalized later on by Andjel [1], Liggett [15], and recently in the work of Borcea, Brändén and Liggett [3].

In the search of a natural conservative particle system where the opposite inequality holds, i.e., where the particles are *less spread out* than corresponding independent random walkers, it is natural to think of a "bosonic counterpart" of the exclusion process. In fact, such a process was introduced in [9] and [10] as the *dual* of the Brownian momentum process, a stochastic model of heat conduction (similar models of heat conduction were introduced in [4] and [8], see also [5] for the study of the structure function in a natural asymmetric version).

In the present paper we analyze this "bosonic counterpart" of the exclusion process. We will call this process (as will be motivated by a Poisson clock representation) the "symmetric inclusion process" (SIP). In the SIP, jumps are performed according to independent random walks, and on top of that particles "invite" other particles to join their site (inclusion). For this process we prove the analogue of the comparison inequality for the symmetric exclusion process. From the comparison inequality, using the knowledge of the stationary measure and the self-duality property of the process, we deduce a series of correlation inequalities. Again, in going from exclusion to inclusion process the correlations turn from negative to positive. We remark however that these positive correlation inequalities are different from the ordinary preservation of positive correlations for monotone processes [12], because the SIP is not a monotone process. Since the SIP is dual to the heat conduction model it is immediate to extend those correlation inequalities to the Brownian momentum process and the Brownian energy process.

We also introduce the non-equilibrium versions of the SIP, i.e., we consider the boundary driven version of SIP. In this case, for appropriate choice of the boundary generators, we prove duality of the process to a SIP model with absorbing boundary condition. We then deduce a correlation inequality, explaining and generalizing the positivity of the covariance in the non-equilibrium steady state of the heat conduction model in [9].

All the results will be stated in the context of a family of SIP(m) models, which are labeled by parameter $m \in \mathbb{N}$. As the SEP model can be generalized to the situation where there are at most $n \in \mathbb{N}$ particles per site (this corresponds to a quantum spin chain with SU(2) symmetry and spin value $j = n/2$), in the same way the SIP model can be extended to represent the situation of a quantum spin chain with SU(1,1) symmetry and spin value $k = m/4$ [9].

The paper is organized as follows. In Section 2 we define the SIP(m) process, restricting to a context where its existence can be immediately established. The main comparison inequality, which allows to compare SIP walkers to independent walkers (by a suitable generalization of Liggett comparison inequality) is proved in Section 3. Correlation inequalities for the SIP(m) process that can be deduced from the comparison inequality are proved in Section 5 (the necessary knowledge of the stationary measure and the self-duality property are presented in Section 4). In particular, in

Section 5 it is proved that when the SIP(m) process is started from its stationary measure then correlations are always positive, while when the process is initialized with a general product measure then positivity of correlations is recovered in the long time limit. Further correlation inequalities for systems similar to the SIP(m) process are discussed in the subsequent Sections. Attractive interaction (the SEP(n), which generalize the standard SEP) is presented in Section 6. Some interacting diffusions dual to the SIP(m) process are studied in Section 7. Finally the boundary driven SIP(m) process is analyzed in Section 8.

2.2 Definition

In the whole of the paper, S will denote either a finite set, or $S = \mathbb{Z}^d$. Next, $p(x, y)$ denotes an irreducible (discrete-time) symmetric random walk transition probability on S, i.e., $p(x, y) = p(y, x) \geq 0$, $\sum_y p(x, y) = 1$, and $p(x, x) = 0$. In the case $S = \mathbb{Z}^d$, we suppose furthermore that $p(x, y)$ is finite range and translation invariant, i.e., $p(x, y) = \pi(y - x)$, and there exists $R > 0$ such that $p(x, y) = 0$ for $|x - y| > R$. This assumption for the infinite-volume case avoids technical problems for the existence of the $SIP(m)$ which for the subject of this paper are irrelevant. The proof of existence of the $SIP(m)$ in our infinite-volume context (with the process started from a "tempered" initial configuration, i.e. $\eta(y) \leq ||y||^k$ for some k and for all y) follows from self-duality, along the lines of [6], Chapter 2.

The symmetric inclusion process with parameter $m \in (0, \infty)$ associated to the transition kernel p is the Markov process on $\Omega := \mathbb{N}^S$ with generator defined on the core of local functions by

$$Lf(\eta) = \sum_{x,y \in S} p(x, y) 2\eta_x (m + 2\eta_y) \left(f(\eta^{x,y}) - f(\eta)\right) \tag{2.2.1}$$

where, for $\eta \in \Omega$, $\eta^{x,y}$ denotes the configuration obtained from η by removing one particle from x and putting it at y.

In [9], for $m = 1$ this model was introduced as the dual of a model of heat conduction, the so-called Brownian momentum process, see also [10], and [4] for generalized and or similar models of heat conduction.

The process with generator (2.2.1) can be interpreted as follows. Every particle has two exponential clocks: one clock -the so-called random walk clock- has rate $2m$, the other clock -the so-called inclusion clock- has rate 4. When the random walk clock of a particle at site $x \in S$ rings, the particle performs a random walk jump with probability $p(x, y)$ to site $y \in S$. When the inclusion process clock rings at site $y \in S$, with probability $p(y, x) = p(x, y)$ a particle from site $x \in S$ is selected and joins site y.

From this interpretation, we see that besides jumps of a system of independent random walkers, this system of particles has the tendency to bring particles together at the same site (inclusion), and can therefore be thought of as a "bosonic" counterpart of the symmetric exclusion process.

To make the analogy with the exclusion process even more transparent, in an exclusion process with at most n particles ($n \in \mathbb{N}$) per site (notation $SEP(n)$), the jump rate is $\eta_i(n - \eta_j)p(i, j)$. Apart from a global factor 4, the $SIP(m)$ is obtained by changing the minus into a plus and choosing $n = m/2$.

Notice that the rates in (2.2.1) are increasing both in the number of particles of the departure and in the number of particles of the arrival site (the rate is $p(x, y)2\eta_x(m + 2\eta_y)$ for a particle to jump from x to y). Therefore, by the necessary and sufficient conditions of [11], Theorem 2.21, the SIP is not a monotone process. It is also easy to see that due to the attraction between particles in the SIP, there cannot be a coupling that preserves the order of configurations, i.e., in any coupling starting from an unequal ordered pair of configurations, the order will be lost in the course of time with positive probability.

2.2.1 Assumptions on the transition probability kernel

In this section we introduce the assumptions that we need to prove the positivity of correlations of stationary measures obtained as limits of general initial product measures (see later for precise definitions). This assumptions are only relevant in the infinite volume case $S = \mathbb{Z}^d$ and they are indeed satisfied in the context of finite-range translation-invariant underlying random walk kernel $p(x, y) = \pi(y - x)$. However, all our results on correlation inequalities for stationary measures depend only on one or both of the assumptions below, i.e., if on more general graphs, or on $\mathbb{Z}^d$ with more general $p(x, y)$, existence of $SIP(m)$ would be established, then the corresponding correlation inequalities hold under one or both of the assumptions A1, A2 below.

We define the associated continuous-time random walk transition probabilities of random walk jumping at rate $2m$:

$$p_t(x,y) = \sum_{n=0}^{\infty} \frac{(2mt)^n}{n!} e^{-2mt} p^{(n)}(x,y) \tag{2.2.2}$$

where $p^{(n)}$ denotes the n^{th} power of the transition matrix p. Denote by $\mathbb{P}^{IRW(m)}_{x,y}$ the probability measure on path space associated to two independent random walkers X_t, Y_t started at x, y and jumping according to (2.2.2) and by $\mathbb{P}^{SIP(m)}_{x,y}$ the corresponding probability for two SIP walkers $X_t^{'}, Y_t^{'}$ jumping with the rates of generator (2.2.1).

We consider two assumptions

- Assumption (A1)

$$\lim_{t\to\infty} \sup_{x,y} \mathbb{P}^{IRW(m)}_{x,y}(X_t = Y_t) = 0 \tag{2.2.3}$$

- Assumption (A2)

$$\lim_{t\to\infty} \sup_{x,y} \mathbb{P}^{SIP(m)}_{x,y}(X_t^{'} = Y_t^{'}) = 0 \tag{2.2.4}$$

The assumption (A1) amounts to requiring that for large $t > 0$, two independent random walkers walking according to the continuous time random walk probability (2.2.2) will be at the same place with vanishing probability. The assumption (A1) follows immediately if we have

$$\lim_{t\to\infty} \sup_{x,y} p_t(x,y) = 0 \tag{2.2.5}$$

since then

$$\lim_{t\to\infty} \sup_{x,y} \mathbb{P}^{IRW(m)}_{x,y}(X_t = Y_t) = \lim_{t\to\infty} \sup_{x,y} \sum_{u\in S} p_t(x,u)p_t(y,u) = \lim_{t\to\infty} \sup_{x,y} p_{2t}(x,y) = 0 \tag{2.2.6}$$

Notice also that, by simple rescaling of time, (A1) holds for all $m > 0$ as soon as it holds for some $m > 0$.

Assumption (A2) guarantees that two walkers evolving with the SIP dynamics will be typically at different positions at large times. Notice that in the case we consider, i.e., the translation invariant finite-range case $S = \mathbb{Z}^d$, $p(x,y) = p(0, y-x) =: \pi(y-x)$, this is automatically satisfied, as the difference walk $X_t^{'} - Y_t^{'}$ of two SIP particles is a random walk Z_t on $\mathbb{Z}^d$ with generator

$$L^Z f(z) = 8\pi(z)(f(0) - f(z)) + \sum_y 4m\pi(y)(f(z+y) - f(z)) \tag{2.2.7}$$

which is clearly not positive recurrent.

Assumption (A2) implies that any finite number of SIP particles will eventually be at different locations. This is made precise in Lemma 2.5.3 in section 2.5.

2.3 Comparison of the SIP with independent random walks

We will first consider the SIP process with a finite number of particles in subsection 2.3.1 and then state the comparison inequality in subsection 2.3.2.

2.3.1 The finite SIP

If we start the SIP with n particles at positions $x_1, \ldots, x_n \in S$, we can keep track of the labels of the particles. This gives then a continuous-time Markov chain on S^n with generator

$$\begin{aligned} \mathscr{L}_n f(x_1, \ldots, x_n) &= \sum_{i=1}^{n} \sum_{y \in S} 2p(x_i, y) \left(m + 2 \sum_{j=1}^{n} I(y = x_j) \right) (f(x^{x_i,y}) - f(x)) \\ &= \mathscr{L}_{1,n} f(x) + \mathscr{L}_{2,n} f(x) \end{aligned} \tag{2.3.1}$$

where $x^{x_i,y}$ denotes the n-tuple $(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n)$. Further, $\mathscr{L}_{1,n}$, resp. $\mathscr{L}_{2,n}$ denote the random walk resp. inclusion part of the generator and are defined as follows

$$\mathscr{L}_{1,n} f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \sum_{y \in S} 2mp(x_i, y)(f(x^{x_i,y}) - f(x)) \tag{2.3.2}$$

$$\mathscr{L}_{2,n} f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} 4p(x_i, x_j)(f(x^{x_i,x_j}) - f(x)) \tag{2.3.3}$$

2.3.2 Comparison inequality

From the description above, it is intuitively clear that in the SIP, particle tend to be less spread out than in a system of independent random walkers. Theorem 2.3.1 below formalizes this intuition and is the analogue of a comparison inequality of the SEP ([14], Chapter VIII, Proposition 1.7).

To formulate it, we need the notion of a positive definite function. A function $f : S \times S \to \mathbb{R}$ is called positive definite if for all $\beta : S \to \mathbb{R}$ such that $\sum_x |\beta(x)| < \infty$

$$\sum_{x,y} f(x,y)\beta(x)\beta(y) \geq 0$$

A function $f : S^n \to \mathbb{R}$ is called positive definite if it is positive definite in every pair of variables.

We first introduce a slightly more general generator with parameters $a > 0$, $b \in \mathbb{R}$ that includes both process of exclusion and inclusion type.

$$\mathscr{L}_n^{a,b} f(x_1,\ldots,x_n) = \sum_{i=1}^{n}\sum_{y\in S} p(x_i,y)\left(a + b\sum_{j=1}^{n} I(y=x_j)\right)(f(x^{x_i,y}) - f(x)) \quad (2.3.4)$$

so

$$\mathscr{L}_n^{a,b} = \mathscr{L}_{1,n}^{a} + \mathscr{L}_{2,n}^{b}$$

where

$$\mathscr{L}_{1,n}^{a} f(x_1,\ldots,x_n) = a\sum_{i=1}^{n}\sum_{y\in S} p(x_i,y)(f(x^{x_i,y}) - f(x)) \quad (2.3.5)$$

is the independent random walk part (random walks jumping at rate a) and

$$\mathscr{L}_{2,n}^{b} f(x_1,\ldots,x_n) = b\sum_{i=1}^{n}\sum_{j=1}^{n} p(x_i,x_j)(f(x^{x_i,x_j}) - f(x)) \quad (2.3.6)$$

is the “clumping” part, i.e., when $b < 0$ clumping is discouraged, and $b > 0$ clumping is favored.

We call $T_n^{a,b}(t)$ the semigroup on functions $f : S^n \to \mathbb{R}$ associated to the generator (2.3.4), and $U_n^a(t)$ the semigroup of a system of independent continuous-time random walkers (jumping at rate a), i.e., the semigroup associated to the generator $\mathscr{L}_{1,n}^{a}$ in (2.3.5). Notice that when $b < 0$, $T_n^{a,b}(t)$ is not always a Markov semigroup. However, for the applications of negative b, we have in mind generalized exclusion process (see Section 6) in which case a/b is an integer and in this case $T_n^{a,b}(t)$ is a Markov semigroup.

Theorem 2.3.1. *Let $f : S^n \to \mathbb{R}$ be positive definite and symmetric. Then we have for $b > 0$*

$$U_n^a(t)f \leq T_n^{a,b}(t)f \quad (2.3.7)$$

and for $b < 0$, if $(T^{a,b}(t))_{t\geq 0}$ is a Markov semigroup, we have

$$U_n^a(t)f \geq T_n^{a,b}(t)f \quad (2.3.8)$$

Proof. The proof follows the proof in [14], but for the sake of self-constistency we prefer to give it explicitly. Suppose $b > 0$.

Start with the decomposition (2.3.1) and use the symmetry of $p(x, y)$ and f to write

$$
\begin{aligned}
& (\mathscr{L}_n^{a,b} f - \mathscr{L}_{1,n}^a f)(x) = (\mathscr{L}_{2,n}^b f)(x) \\
= \quad & b \sum_{i=1}^n \sum_{j=1}^n p(x_i, x_j)(f(x^{x_i,x_j}) - f(x)) \\
= \quad & \frac{b}{2} \sum_{i=1}^n \sum_{j=1}^n p(x_i, x_j)(f(x^{x_i,x_j}) + f(x^{x_j,x_i}) - 2f(x)) \\
= \quad & \frac{b}{2} \sum_{i=1}^n \sum_{j=1}^n p(x_i, x_j) \\
\times \quad & \sum_{u,v} f(x_1, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_{j-1}, v, x_{j+1}, \ldots, x_n)(\delta_{x_i,u} - \delta_{x_j,u})(\delta_{x_i,v} - \delta_{x_j,v}) \\
\geq \quad & 0
\end{aligned}
\tag{2.3.9}
$$

where in the last step we used that f is positive definite.

Since $U_n^a(t)$ is the semigroup of independent walks, it maps positive definite functions into positive definite functions, and so we have

$$
\left(\mathscr{L}_n U_n^a(t) f - \mathscr{L}_{1,n}^a U_n^a(t) f\right) = \mathscr{L}_{2,n}^b U_n^a(t) f \geq 0
$$

We can then use the variation of constants formula

$$
T_n^{a,b}(t) f - U_n^a(t) f = \int_0^t ds\; T_n^{a,b}(t-s) \left(\mathscr{L}_{2,n}^b U_n^a(s) f\right) \geq 0 \tag{2.3.10}
$$

and remember that $T_n^{a,b}(t)$ is a Markov semigroup which therefore maps non-negative functions into non-negative functions.

The proof for $b < 0$, under the assumption that $T_n^{a,b}(t)$ is a Markov semigroup is identical. □

2.4 Stationary measures and self-duality for the $SIP(m)$

The stationary measures of $SIP(m)$ are product measures of "discrete gamma distributions"

$$
\nu_\lambda(d\eta) = \otimes_{x \in S} \nu_\lambda^m(d\eta_x)
$$

where for $n \geq 0$

$$\nu_\lambda^m(n) = \frac{1}{Z_{\lambda,m}} \frac{\lambda^n}{n!} \frac{\Gamma(\frac{m}{2}+n)}{\Gamma(\frac{m}{2})}, \qquad n \in \mathbb{N} \tag{2.4.1}$$

with $0 \leq \lambda < 1$ a parameter, $\Gamma(r)$ the gamma-function and

$$Z_{\lambda,m} = \left(\frac{1}{1-\lambda}\right)^{m/2}$$

Notice that for $m = 2$, ν_λ^m is a geometric distribution (starting from zero), i.e., $\nu_\lambda^2(n) = \lambda^n(1-\lambda), n \in \mathbb{N}$ and for $m/2$ an integer ν_λ^m is negative binomial distribution $NB(m/2, \lambda)$. Moreover, the measures ν^m have the following convolution property

$$\nu_\lambda^m * \nu_\lambda^l = \nu_\lambda^{m+l} \tag{2.4.2}$$

where $*$ denotes convolution, i.e., a sample from $\nu_\lambda^m * \nu_\lambda^l$ is obtained by site-wise addition of a sample from ν_λ^m and an independent sample from ν_λ^l.

The $SIP(m)$ process is self-dual [10] with duality functions given by $D(\xi, \eta) = \prod_x d(\xi_x, \eta_x)$, with

$$d(k,l) = \frac{l!}{(l-k)!} \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}+k\right)} \tag{2.4.3}$$

where $k \leq l$. Self-duality means that

$$\mathbb{E}_\eta^{SIP(m)} D(\xi, \eta_t) = \mathbb{E}_\xi^{SIP(m)} D(\xi_t, \eta) \tag{2.4.4}$$

where $\mathbb{E}_\eta^{SIP(m)}$ denotes expectation in the SIP process started from the configuration η.

The relation between the polynomials D and the measure ν_λ^m reads

$$\int D(\xi, \eta) \nu_\lambda^m(d\eta) = \left(\frac{\lambda}{1-\lambda}\right)^{|\xi|} \tag{2.4.5}$$

as follows from a simple computation using the definition of the Γ-function, $\Gamma(r) = \int_0^\infty x^{r-1} e^{-x} dx$.

From conservation of particles in the dual process, we see that self-duality and the relation (2.4.5) gives stationarity of the measure ν_Λ.

The relation (2.4.5) can be generalized to "local stationary measure", i.e. the product measures that are obtained from the stationary measure (2.4.1) by allowing a site-dependent parameter. More precisely, given

$$\overline{\lambda} : S \to [0,1)$$

we define the local stationary measure associated to the profile $\overline{\lambda}$ by

$$\nu_{\overline{\lambda}} = \otimes_{x\in S}\nu^m_{\lambda(x)}(d\eta_x) \tag{2.4.6}$$

For $x_1,\ldots,x_n \in S$ we denote by $\sum_{i=1}^n \delta_{x_i}$ the particle configuration $\xi \in \mathbb{N}^S$ obtained by putting a particles at locations x_i, i.e., $\xi(x) = \sum_{i=1}^n I(x_i = x)$. We then have the following relation between the duality functions and the local stationary measures

$$\int D\left(\sum_{i=1}^n \delta_{x_i}, \eta\right)\nu_{\overline{\lambda}}(d\eta) = \prod_{i=1}^n \rho(x_i) \tag{2.4.7}$$

where

$$\rho(x_i) = \frac{\lambda(x_i)}{1-\lambda(x_i)}$$

For a constant profile $\overline{\lambda}(x) = \lambda\ , \forall x \in S$, we recover (2.4.5).

By Lemma 2.5.3 below, in the case $S = \mathbb{Z}^d$ and translation invariant finite-range $p(x,y)$, any number of dual particles in the $SIP(m)$ will eventually diffuse away to infinity. From that it is easy to deduce that the measures ν_λ are extremal invariant. To see this, we denote for two finite-particle configurations $\xi \perp \xi'$, if their supports are disjoint, i.e., there are no site $x \in S$ where there are ξ and ξ' particles. If $\xi \perp \xi'$ then $D(\xi+\xi',\eta) = D(\xi,\eta)D(\xi',\eta)$. Since at large $t > 0$, assumption (A2) implies that, in the SIP started with a finite number of particles, particles are with probability close to one at different locations (see Lemma 2.5.3 for a proof of this), we have that for ξ' a fixed configuration, the event $\xi_t \perp \xi'$ has probability close to one as $t \to \infty$. Therefore

$$\begin{aligned}
& \lim_{t\to\infty} \int \mathbb{E}^{SIP(m)}_\eta \left(D(\xi,\eta_t)\right) D(\xi',\eta)\nu_\lambda(d\eta) \\
= \;& \lim_{t\to\infty} \mathbb{E}^{SIP(m)}_\xi \int D(\xi_t,\eta)D(\xi',\eta)\nu_\lambda(d\eta) \\
= \;& \lim_{t\to\infty} \mathbb{E}^{SIP(m)}_\xi \int D(\xi_t,\eta)D(\xi',\eta)I(\xi_t\perp\xi')\nu_\lambda(d\eta) \\
= \;& \lim_{t\to\infty} \rho_\lambda^{|\xi_t|+|\xi'_t|} \\
= \;& \rho_\lambda^{|\xi|+|\xi'|} \\
= \;& \int D(\xi,\eta)\nu_\lambda(d\eta)\int D(\xi',\eta)\nu_\lambda(d\eta)
\end{aligned} \tag{2.4.8}$$

which shows that time-dependent correlations of (linear combinations of) $D(\xi,\cdot)$ polynomials decay in the course of time to zero, and hence, by standard arguments, ν_λ is mixing and thus ergodic.

2.5 Correlation inequalities in the $SIP(m)$

For a probability measure μ on the configuration space $\mathbb{N}^S$, we denote its "duality moment function" $K_\mu : S^n \to \mathbb{R}$ by

$$K_\mu(x_1, \ldots, x_n) = \int D\left(\sum_{i=1}^{n} \delta_{x_i}, \eta\right) \mu(d\eta) \tag{2.5.1}$$

If $\mu = \nu_{\overline{\lambda}}$ is a local stationary measure with profile $\overline{\lambda}$, then

$$K_{\nu_{\overline{\lambda}}}(x_1, \ldots, x_n) = \prod_{i=1}^{n} \rho(x_i) \tag{2.5.2}$$

which is clearly positive definite and symmetric. We can therefore apply Theorem 2.3.1 and obtain the following result.

Proposition 2.5.1. *For all $t \geq 0$, for all profiles $\overline{\lambda} : S \to [0,1)$ and for all $x_1, \ldots, x_n \in S$ we have*

$$K_{\nu_{\overline{\lambda}} S_t}(x_1, \ldots, x_n) \geq \prod_{i=1}^{n} K_{\nu_{\overline{\lambda}} S_t}(x_i) \tag{2.5.3}$$

where S_t denotes the semigroup of the $SIP(m)$ process. In particular, when the $SIP(m)$ is started from $\nu_{\overline{\lambda}}$, the random variables $\{\eta_t(x), x \in S\}$ are positively correlated, i.e., for $(x,y) \in S \times S$

$$\int \mathbb{E}_\eta^{SIP(m)} \left(\eta_t(x)\eta_t(y)\right) \nu_{\overline{\lambda}}(d\eta) \geq \int \mathbb{E}_\eta^{SIP(m)} \left(\eta_t(x)\right) \nu_{\overline{\lambda}}(d\eta) \int \mathbb{E}_\eta^{SIP(m)} \left(\eta_t(y)\right) \nu_{\overline{\lambda}}(d\eta)$$

Proof. Denote by $\mathbb{E}^{SIP(m)}_{x_1,\ldots,x_n}$ expectation in the $SIP(m)$ process started with n particles at positions $(x_1, \ldots, x_n)$, by $\mathbb{E}^{IRW(m)}$ expectation in the process of independent random walkers (jumping at rate $2m$) and $\mathbb{E}^{RW(m)}$ a single random walker expectation. We then have the following chain of inequalities, which is obtained by using sequentially the following: self-duality property (2.4.4), the comparison inequality (2.3.7), the relation between the measure $\nu_{\overline{\lambda}}$ and the duality function D (2.4.7), the independence between random walkers, the fact that a single SIP particle moves as a continuous time random walk, and finally again self-duality (2.4.4)

$$\begin{aligned}
& \int \mathbb{E}_\eta^{SIP(m)} D\left(\sum_{i=1}^n \delta_{x_i}, \eta_t\right) \nu_{\overline{\lambda}}(d\eta) \\
= \;& \mathbb{E}_{x_1,\ldots,x_n}^{SIP(m)} \int D\left(\sum_{i=1}^n \delta_{X_i(t)}, \eta\right) \nu_{\overline{\lambda}}(d\eta) \\
\geq \;& \mathbb{E}_{x_1,\ldots,x_n}^{IRW(m)} \int D\left(\sum_{i=1}^n \delta_{X_i(t)}, \eta\right) \nu_{\overline{\lambda}}(d\eta) \\
= \;& \mathbb{E}_{x_1,\ldots,x_n}^{IRW(m)} \left(\prod_{i=1}^n \rho(X_i(t))\right) \\
= \;& \prod_{i=1}^n \mathbb{E}_{x_i}^{RW(m)} \rho(X_i(t)) \\
= \;& \prod_{i=1}^n \int \mathbb{E}_{x_i}^{SIP(m)} \big(D(\delta_{X_i(t)}, \eta)\big) \nu_{\overline{\lambda}}(d\eta) \\
= \;& \prod_{i=1}^n \int \mathbb{E}_\eta^{SIP(m)} \left(D(\delta_{x_i}, \eta_t)\right) \nu_{\overline{\lambda}}(d\eta)
\end{aligned} \tag{2.5.4}$$

$\square$

This proposition shows that starting from a local stationary measure $\nu_{\overline{\lambda}}$, the density profile $\rho_t(x) = \mathbb{E}_x^{RW(m)} \rho_t(x)$ predicts (by duality) correctly the density at time $t > 0$ but the true measure at time $t > 0$, $\nu_{\overline{\lambda}} S_t$, lies above (in the sense of expectations of D-functions) the product measure with density profile $\rho_t(x)$.

From the analogy with the SEP emphasized above, one could think that (2.5.3) extends to the case when the SIP process is started from a general product measure. However, for general probability measures μ on Ω, the duality moment function $K_\mu : S^n \to R$ defined in (2.5.1) is not necessarily positive definite (as is the case for the special product measures $\nu_{\overline{\lambda}}$), since we do not have the equality $D\left(\sum_{i=1}^n \delta_{x_i}, \eta\right) = \prod_{i=1}^n D(\delta_{x_i}, \eta)$ in general. Notice that this problem does not appear in the context of the standard SEP, as for that model, the self-duality functions are

$$D_{SEP}\left(\sum_{i=1}^n \delta_{x_i}, \eta\right) = \prod_{i=1}^n \eta_{x_i} = \prod_{i=1}^n D_{SEP}\left(\delta_{x_i}, \eta\right)$$

and hence automatically, for any measure μ, the function K_μ is positive definite in that model.

If however *all x_i are different*, we have $D(\sum_{i=1}^n \delta_{x_i}, \eta) = \prod_{i=1}^n D(\delta_{x_i}, \eta)$. For every probability measure μ on Ω, the function $\Psi_\mu : S^n \to \mathbb{R}$ defined by

$$\Psi_\mu(x_1, \ldots, x_n) = \int \prod_{i=1}^n D(\delta_{x_i}, \eta)\mu(d\eta) \tag{2.5.5}$$

is clearly positive definite. This, together with the fact that under assumption (A2), a finite number of $SIP(m)$ particles diffuse and therefore eventually will be typically at different positions, suggests that in a stationary measure, the variables η_{x_i} are positively correlated.

To state this result we introduce the class of probability measures with uniform finite moments

$$\mathscr{P}_f =: \{\mu : \forall n \in \mathbb{N}, \sup_{|\xi|=n} \int D(\xi, \eta)\mu(d\eta) =: M_\mu^n < \infty\} \tag{2.5.6}$$

For a sequence of measures $\mu_n \in \mathscr{P}_f$, and $\mu \in \mathscr{P}_f$, we define that $\mu_n \to \mu$ if for all ξ finite particle configuration,

$$\lim_{n\to\infty} \int D(\xi, \eta)\mu_n(d\eta) = \int D(\xi, \eta)\mu(d\eta)$$

We can then formulate our next result.

Proposition 2.5.2. *Assume (A1) and (A2). Let $\nu \in \mathscr{P}_f$ be a product measure. Let $S(t)$ denote the semigroup of the SIP(m). Suppose that*

$$\mu = \lim_{n\to\infty} \nu S(t_n) \tag{2.5.7}$$

for a subsequence $t_n \uparrow \infty$. Then we have $\mu \in \mathscr{P}_f$, μ is invariant and

$$K_\mu(x_1, \ldots, x_n) \geq \prod_{i=1}^n K_\mu(x_i) \tag{2.5.8}$$

Proof. First, by duality we have, referring to the definition of $\mathscr{P}_f$, for all $t > 0$,

$$\int \mathbb{E}_\eta^{SIP(m)} D(\xi, \eta_t)\nu(d\eta) = \mathbb{E}_\xi^{SIP(m)} \int D(\xi_t, \eta)\nu(d\eta) \leq M_\nu^{|\xi|} < \infty$$

which shows that both $\nu S(t_n)$ and μ are elements of $\mathscr{P}_f$. The invariance of μ follows from duality, $\nu \in \mathscr{P}_f$ and Lemma 1.26 in [14], chapter V.

To proceed with the proof of the proposition, we start with the following lemma, which ensures that, under condition (A2), any number of $SIP(m)$ particles will eventually be at different locations.

Lemma 2.5.3. *Assume (A2). Start the finite $SIP(m)$ with particles at locations $\{x_1, \dots, x_n\}$, then*

$$\lim_{t\to\infty} \mathbb{P}^{SIP(m)}_{x_1,\dots,x_n} (\exists i \neq j : X_i(t) = X_j(t)) = 0 \tag{2.5.9}$$

Proof. We give the proof for $m = 1$. The general case is a straightforward extension. Put $\eta := \sum_{i=1}^n \delta_{x_i}$. Using self-duality we can write

$$\begin{aligned}
\mathbb{P}^{SIP(1)}_{\eta} (\exists i \neq j : X_i(t) = X_j(t)) &\leq \sum_z \mathbb{P}^{SIP(1)}_{\eta} \big(\eta_t^2(z) - \eta_t(z) > 1\big) \\
&\leq \sum_z \mathbb{E}^{SIP(1)}_{\eta} (\eta_t^2(z) - \eta_t(z)) \\
&= \frac{3}{4} \sum_z \mathbb{E}^{SIP(1)}_{\eta} (D(2\delta_z, \eta_t)) \\
&= \frac{3}{4} \sum_z \mathbb{E}^{SIP(1)}_{z,z} (D(\delta_{X_t} + \delta_{Y_t}, \eta)) \\
&\leq 3 \sum_z \mathbb{E}^{SIP(1)}_{z,z} (\eta(X_t)\eta(Y_t)) \\
&= 3 \sum_z \sum_{i,j=1}^n \mathbb{E}^{SIP(1)}_{z,z} (I(X_t = x_i) I(Y_t = x_j)) \\
&\leq 3n^2 \sup_{x,y} \mathbb{P}^{SIP(m)}_{x,y} (X_t = Y_t)
\end{aligned} \tag{2.5.10}$$

where in the last step we used the symmetry of the transition probabilities of the $SIP(1)$ (with two particles). $\square$

We now proceed with the proof of the proposition. For $x_1, \dots, x_n \in S$ we define

$$\left| D(\sum_{i=1}^n \delta_{x_i}, \eta) - \prod_{i=1}^n D(\delta_{x_i}, \eta) \right| = \Delta(x_1, \dots, x_n, \eta) \tag{2.5.11}$$

We have that $\Delta(x_1, \dots, x_n, \eta) = 0$ if all x_i are different, i.e., if $|\{x_1, \dots, x_n\}| = n$. Since by assumption (A2) and Lemma 2.5.3, the probability that two $SIP(m)$ walkers out of a finite number n of them occupy the same position, i.e. $X_i(t) = X_j(t)$ for some $i \neq j$, vanishes in the limit $t \to \infty$, we conclude, using $\nu \in \mathscr{P}_f$, for any $x_1, \ldots, x_n \in S$,

$$\lim_{t\to\infty} \int \mathbb{E}^{SIP(m)}_{x_1,\ldots,x_n} \Delta(X_1(t), \ldots, X_n(t), \eta)\nu(d\eta) = 0 \tag{2.5.12}$$

Moreover from the comparison inequality (2.3.7) we have, using the notation (2.5.5)

$$\begin{aligned}
\mathbb{E}^{SIP(m)}_{x_1,\ldots,x_n} \Psi_\nu(X_1(t), \ldots, X_n(t)) &\geq \mathbb{E}^{IRW(m)}_{x_1,\ldots,x_n} \Psi_\nu(X_1(t), \ldots, X_n(t)) \qquad (2.5.13)\\
&= \mathbb{E}^{IRW(m)}_{x_1,\ldots,x_n} \int \prod_{i=1}^n D\left(\delta_{X_i(t)}, \eta\right) \nu(d\eta)\\
&= \prod_{i=1}^n \mathbb{E}^{RW(m)}_{x_i} \int D(\delta_{X_i(t)}, \eta)\nu(d\eta) + \epsilon(t)
\end{aligned}$$

where $\epsilon(t) \to 0$ as $t \to \infty$ by assumption (A1), i.e., for large $t > 0$, independent random walkers are at different locations with probability close to one. Therefore, using the definition (2.5.7), the self-duality property (2.4.4), the equation (2.5.12), the equation (2.5.13), and taking limits along the subsequence t_n we have

$$\begin{aligned}
K_\mu(x_1, \ldots, x_n) &= \lim_{t\to\infty} \int \mathbb{E}^{SIP(m)}_\eta D\left(\sum_{i=1}^n \delta_{x_i}, \eta_t\right) \nu(d\eta)\\
&= \lim_{t\to\infty} \int \mathbb{E}^{SIP(m)}_{x_1,\ldots,x_n} D\left(\sum_{i=1}^n \delta_{X_i(t)}, \eta\right) \nu(d\eta)\\
&= \lim_{t\to\infty} \mathbb{E}^{SIP(m)}_{x_1,\ldots,x_n} \Psi_\nu(X_1(t), \ldots, X_n(t))\\
&\geq \lim_{t\to\infty} \prod_{i=1}^n \mathbb{E}^{RW(m)}_{x_i} \int D(\delta_{X_i(t)}, \eta)\nu(d\eta)\\
&= \prod_{i=1}^n K_\mu(x_i) \qquad (2.5.14)
\end{aligned}$$

$\square$

2.6 Correlation inequalities in the SEP(n)

We now consider the application of the generalized Liggett inequality for negative b. The $SEP(n)$ is the Markov process on $\Omega = \{0, 1, \ldots, n\}^S$ with generator

$$Lf(\eta) = \sum_{x,y\in S} \eta(x)(n - \eta(y))p(x,y)\left(f(\eta^{xy}) - f(\eta)\right) \tag{2.6.1}$$

The stationary measures of this process are products of binomial distributions, i.e., for $\rho \in [0,1]$,

$$\nu_\rho = \otimes_{x\in S} Bin(n,\rho) \tag{2.6.2}$$

Similar to the case of the inclusion process, for a profile $\overline{\rho} : S \to [0,1]$ we define the local stationary measure

$$\nu_{\overline{\rho}} = \otimes_{x\in S} Bin(n, \overline{\rho}(x))$$

The duality functions for self-duality are given by (see [10])

$$D(\xi,\eta) = \prod_x d(\xi_x, \eta_x) \tag{2.6.3}$$

for $\xi \in \Omega$ a configuration with finitely many particles (at most n per site) and with

$$d(k,l) = \frac{\binom{l}{k}}{\binom{n}{k}} \tag{2.6.4}$$

The relation between the duality functions and the local stationary measures is, as usual, i.e., for $\xi = \sum_{i=1}^n \delta_{x_i} \in \Omega$ (i.e., at most n particles per site), and $\overline{\rho}$ a profile:

$$\int D(\xi,\eta)\nu_{\overline{\rho}}(d\eta) = \prod_{i=1}^n \rho(x_i) \tag{2.6.5}$$

We define, for a probability measure μ on Ω, its duality moment function

$$K_\mu(x_1, \ldots, x_n) = \int D\left(\sum_{i=1}^n \delta_{x_i}, \eta\right)\mu(d\eta) \tag{2.6.6}$$

The following proposition is then the analogue of Proposition 2.5.1 in this context (with inequality in the other direction since $b < 0$).

Proposition 2.6.1. *For $\overline{\rho} : S \to [0,1]$ a density profile and $t > 0$,*

$$K_{\nu_{\overline{\rho}} S_t}(x_1, \ldots, x_n) \le \prod_{i=1}^{n} K_{\nu_{\overline{\rho}} S_t}(x_i) \tag{2.6.7}$$

In particular, for starting from $\nu_{\overline{\rho}}$, the variables $\{\eta_t(x) : x \in S\}$ are negatively correlated.

2.7 Correlation inequalities for some interacting diffusions

2.7.1 The Brownian Momentum Process

The Brownian momentum process is a system of interacting diffusions, initially introduced as a model of heat conduction in [8], and analyzed via duality in [9]. It is defined as a Markov process on $X = \mathbb{R}^S$ via the formal generator on local functions:

$$L_{BMP} f(\eta) = \left(\sum_{x,y \in S} p(x,y) \left(\eta_x \frac{\partial}{\partial \eta_y} - \eta_x \frac{\partial}{\partial \eta_y} \right)^2 \right) f(\eta) \tag{2.7.1}$$

The variable η_x has to be thought of as momentum of an "oscillator" associated to the site $x \in S$. The local kinetic energy η_x^2 has to be thought of as the analogue of the number of particles at site x in the $SIP(m)$ with $m = 1$. The expectation of η_x^2 is interpreted as the local temperature at x.

Defining the polynomials

$$D(n,z) = \frac{z^{2n}}{(2n-1)!!}$$

we have the duality function $D(\xi, \cdot)$ defined on X and indexed by finite particle configurations $\xi \in \mathbb{N}^S$, $\sum_x \xi_x < \infty$:

$$D(\xi, \eta) = \prod_{x \in S} D(\xi_x, \eta_x) \tag{2.7.2}$$

In [9], [10], we proved the duality relation

$$\mathbb{E}_\eta^{BMP} \left(D(\xi, \eta_t) \right) = \mathbb{E}_\xi^{SIP(1)} \left(D(\xi_t, \eta) \right) \tag{2.7.3}$$

As before, for $x_1, \ldots, x_n \in S$ we denote by $\sum_{i=1}^n \delta_{x_i}$ the particle configuration obtained by putting a particle at each x_i.

Let μ be a product of Gaussian measures on X, with site-dependent variance, i.e., for a function $\rho : S \to [0, \infty)$, we define

$$\mu_\rho = \otimes_{x \in S} \nu_{\rho(x)}(d\eta_x) \tag{2.7.4}$$

where

$$\nu_{\rho(x)}(d\eta_x) = \frac{e^{-\eta_x^2/2\rho(x)}}{\sqrt{2\pi\rho(x)}} d\eta_x$$

is the Gaussian measure on $\mathbb{R}$ with mean zero and variance $\rho(x)$. Then we have

$$\int D\left(\sum_{i=1}^n \delta_{x_i}, \eta\right) \mu_\rho(d\eta) = \prod_{i=1}^n \rho(x_i) \tag{2.7.5}$$

From this expression, it is obvious that the map

$$S^n \to \mathbb{R} : (x_1, \ldots, x_n) \mapsto \int D\left(\sum_{i=1}^n \delta_{x_i}, \eta\right) \mu_\rho(d\eta) \tag{2.7.6}$$

is positive definite. Therefore, combining the duality property between BMP process and $SIP(m)$ process, (2.7.3), with Theorem 2.3.1 we have the inequality

$$
\begin{aligned}
& \int \mathbb{E}_\eta^{BMP} D\left(\sum_{i=1}^n \delta_{x_i}, \eta_t\right) \mu_\rho(d\eta) \\
= \ & \mathbb{E}_{x_1,\ldots,x_n}^{SIP(1)} \int D\left(\sum_{i=1}^n \delta_{X_i(t)}, \eta\right) \mu_\rho(d\eta) \\
\geq \ & \mathbb{E}_{x_1,\ldots,x_n}^{IRW(m)} \int D\left(\sum_{i=1}^n \delta_{X_i(t)}, \eta\right) \mu_\rho(d\eta) \\
= \ & \mathbb{E}_{x_1,\ldots,x_n}^{IRW(m)} \left(\prod_{i=1}^n \int D\left(\delta_{X_i(t)}, \eta\right) \mu_\rho(d\eta)\right) \\
= \ & \mathbb{E}_{x_1,\ldots,x_n}^{IRW(m)} \left(\prod_{i=1}^n \rho(X_i(t))\right) \\
= \ & \prod_{i=1}^n \mathbb{E}_{x_i}^{RW(m)} \rho(X_i(t)) \\
= \ & \prod_{i=1}^n \int \mathbb{E}_{x_i}^{SIP(1)} \left(D\left(\delta_{X_i(t)}, \eta\right)\right) \mu_\rho(d\eta) \\
= \ & \prod_{i=1}^n \int \mathbb{E}_\eta^{BMP} \left(D\left(\delta_{x_i}, \eta_t\right)\right) \mu_\rho(d\eta)
\end{aligned} \tag{2.7.7}
$$

which is the analogue of Proposition 2.5.1 for the BMP process.

In words, it means that the "non-equilibrium temperature profile" is above the temperature profile predicted from the discrete diffusion equation. It also implies that the variables $\{\eta_x^2 : x \in S\}$ are positively correlated under the measure $(\mu_\rho)_t$ for all choices of ρ, $t > 0$.

More precisely, if we denote

$$\rho_t(x) = \mathbb{E}_x^{RW(m)} \rho(X_t)$$

then we have that η_x^2 at time t has expectation $\rho_t(x)$ when the starting measure is μ_ρ (since a single particle in the $SIP(1)$ moves as a continuous time random walk). The correlation inequality for the BMP which we just derived shows that the true measure at time $t > 0$ when started from a product of Gaussian measures lies stochastically above the Gaussian product measure with mean zero and variance $\rho_t(x)$.

Similarly, we obtain an analogous correlation inequality for the BMP for a measure obtained as a limit of product measures. We define

$$\mathscr{P}_f(X) = \{\mu : \forall n \in \mathbb{N} : \sup_{|\xi|=n} \int D(\xi,\eta)\mu(d\eta) < \infty\}$$

Proposition 2.7.1. *Assume (A1) and (A2). Suppose $\nu \in \mathscr{P}_f(X)$ is a product measure and μ is a limit point of the set $\{\nu S(t) : t \geq 0\}$, where $S(t)$ denotes the semigroup of the BMP process. Then we have the inequality*

$$K_\mu(x_1,\ldots,x_n) \geq \prod_{i=1}^n K_\mu(x_i)$$

2.7.2 The Brownian Energy Process

The Brownian energy process with parameter $m > 0$ (notation $BEP(m)$) is introduced in [10] as the process on state space $X = [0,\infty)^S$, with generator

$$L = \sum_{x,y\in S} p(x,y) L^m_{xy} \tag{2.7.8}$$

with

$$L^m_{xy} f(\eta) = 4\eta_x\eta_y \left(\frac{\partial}{\partial\eta_x} - \frac{\partial}{\partial\eta_x}\right)^2 f(\eta) - 2m(\eta_x - \eta_y)\left(\frac{\partial}{\partial\eta_x} - \frac{\partial}{\partial\eta_x}\right) f(\eta) \tag{2.7.9}$$

This process is dual to the $SIP(m)$ in the following sense. Define, for $\xi \in \mathbb{N}^S$ a finite particle configuration, and $\eta \in X$ the polynomials

$$D(\xi,\eta) = \prod_{x\in S} d(\xi_x,\eta_x) \tag{2.7.10}$$

with, for $k \in \mathbb{N}, y \in [0,\infty)$

$$d(k,y) = y^k \frac{\Gamma\left(\frac{m}{2}\right)}{2^k \Gamma\left(\frac{m}{2}+k\right)} \tag{2.7.11}$$

then we have

$$\mathbb{E}^{BEP(m)}_\eta D(\xi,\eta_t) = \mathbb{E}^{SIP(m)}_\xi D(\xi_t,\eta) \tag{2.7.12}$$

As a consequence, extremal invariant measure of the $BEP(m)$ are products of Γ-distributions with shape parameters $m/2$ and scale parameter $\theta > 0$:

$$\nu_\theta(d\eta) = \otimes_{x\in S}\nu_\theta(d\eta_x) \tag{2.7.13}$$

with

$$\nu_\theta(dz) = \frac{1}{\theta^{m/2}\Gamma\left(\frac{m}{2}\right)} z^{\frac{m}{2}-1} e^{-z/\theta} \tag{2.7.14}$$

Similarly we define the local stationary measures

$$\nu_{\overline{\theta}} = \otimes_{x\in S} \nu_{\overline{\theta}(x)}(d\eta_x) \tag{2.7.15}$$

with $\overline{\theta} : S \to [0,\infty)$, and the duality moment function of a probability measure μ on X:

$$K_\mu(x_1,\ldots,x_n) = \int D\left(\sum_{i=1}^n \delta_{x_i}, \eta\right) \mu(d\eta) \tag{2.7.16}$$

As a consequence of the correlation inequalities derived for the $SIP(m)$, we derive the following.

Proposition 2.7.2. *1. For all $\overline{\theta} : S \to [0,\infty)$, $t > 0$, and $x_1,\ldots,x_n \in S$ we have*

$$K_{\nu_{\overline{\theta}} S_t^{BEP(m)}}(x_1,\ldots,x_n) \geq \prod_{i=1}^n K_{\nu_{\overline{\theta}} S_t^{BEP(m)}} \mu(x_i) \tag{2.7.17}$$

2. If for some product measure ν on X with finite moments, and a sequence of $t_n \uparrow \infty$ the limit

$$\mu = \lim_{n\to\infty} \nu S^{BEP(m)}(t_n)$$

exists, then

$$K_\mu(x_1,\ldots,x_n) \geq \prod_{i=1}^n K_\mu(x_i) \tag{2.7.18}$$

2.8 The boundary driven $SIP(m)$

In this section we consider the non-equilibrium one-dimensional model that is obtained by considering particle reservoirs attached to the first and last sites of the chain. We will show that, if one requires reversibility w.r.t. the measure ν_λ^m *and* duality with absorbing boundaries, this uniquely fixes the birth and death rates at the boundaries.

2.8.1 Duality for the the boundary driven $SIP(m)$

The generator of the boundary driven $SIP(m)$ on a chain $\{1,\ldots,N\}$ driven at the end points, reads

$$\mathscr{L} = \mathscr{L}_1 + \mathscr{L}_N + \mathscr{L}_{bulk} \tag{2.8.1}$$

where $\mathscr{L}_{bulk}$ denotes the $SIP(m)$ generator, with nearest neighbor random walk as underlying kernel, i.e.,

$$\begin{aligned}\mathscr{L}_{bulk} f(\eta) &= \sum_{x\in\{1,\ldots,N-1\}} 2\eta_x(m+2\eta_{x+1})\left(f(\eta^{x,x+1})-f(\eta)\right) \\ &+ \sum_{x\in\{1,\ldots,N-1\}} 2\eta_{x+1}(m+2\eta_x)\left(f(\eta^{x+1,x})-f(\eta)\right)\end{aligned} \tag{2.8.2}$$

and where $\mathscr{L}_1, \mathscr{L}_N$ are birth and death processes on the first and N-th variable respectively, i.e.,

$$\mathscr{L}_1 f(\eta) = d_L(\eta_1)(f(\eta-\delta_1)-f(\eta)) + b_L(\eta_1)(f(\eta+\delta_1)-f(\eta))$$

and

$$\mathscr{L}_N f(\eta) = d_R(\eta_N)(f(\eta-\delta_N)-f(\eta)) + b_R(\eta_N)(f(\eta+\delta_N)-f(\eta))$$

These generators model contact with respectively the left and right particle reservoir.

The rates d_L, b_L, d_R, b_R are chosen such that detailed balance is satisfied w.r.t. the measure ν_λ^m, with $\lambda = \lambda_L$ for d_L, b_L, and $\lambda = \lambda_R$ for d_R, b_R. More precisely, this means that these rates satisfy

$$b_\alpha(k)\nu_{\lambda_\alpha}^m(k) = d_\alpha(k+1)\nu_{\lambda_\alpha}^m(k+1) \tag{2.8.3}$$

for $\alpha \in \{L, R\}$.

To state our duality result, we consider functions $\mathscr{D}(\xi,\eta)$ indexed by particle configurations ξ on $\{0,\ldots,N+1\}$ defined by

$$\mathscr{D}(\xi,\eta) = \rho_L^{|\xi_0|} D(\xi_{\{1,\ldots,N\}},\eta)\rho_R^{|\xi_{N+1}|} \tag{2.8.4}$$

where $\rho_\alpha = \rho_{\lambda_\alpha} = \lambda_\alpha/(1-\lambda_\alpha)$, and where we remember that

$$D(k,n) = \frac{n!}{(n-k)!}\frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}+k\right)}$$

is the duality function for the $SIP(m)$. I.e., for the "normal" sites $\{1,\ldots,N\}$ we simply have the old duality functions, and for the "extra added" sites $\{0, N+1\}$ we have the expectation of the duality function over the measure ν_λ^m.

We now want duality to hold with duality functions $\mathscr{D}$, and with a dual process that behaves in the bulk as the $SIP(m)$, and which has *absorbing boundaries* at $\{0, N+1\}$. More precisely, we want the generator of the dual process to be

$$\hat{\mathscr{L}} = \mathscr{L}_{bulk} + \hat{\mathscr{L}}_1 + \hat{\mathscr{L}}_N \tag{2.8.5}$$

with $\mathscr{L}_{bulk}$ given by (2.8.2), and

$$\hat{\mathscr{L}}_1 f(\xi) = \xi_1 \left(f(\xi^{1,0}) - f(\xi)\right)$$
$$\hat{\mathscr{L}}_N f(\xi) = \xi_N \left(f(\xi^{N,N+1}) - f(\xi)\right)$$

for $\xi \in \mathbb{N}^{\{0,1\dots,N+1\}}$. The duality relation then reads, as usual,

$$\left(\mathscr{L}\mathscr{D}(\xi,\cdot)\right)(\eta) = \left(\hat{\mathscr{L}}\mathscr{D}(\cdot,\eta)\right)(\xi) \tag{2.8.6}$$

Since self-duality is satisfied for the bulk generator with the choice (2.8.4), i.e., since

$$\left(\mathscr{L}_{bulk}\mathscr{D}(\xi,\cdot)\right)(\eta) = \left(\mathscr{L}_{bulk}\mathscr{D}(\cdot,\eta)\right)(\xi)$$

(2.8.6) will be satisfied if we have the following relations at the boundaries: for all $k \le n$:

$$\begin{aligned} & b_\alpha(n)(D(k,n+1) - D(k,n)) + d_\alpha(n)(D(k,n-1) - D(k,n)) \\ = \quad & k(D(k-1,n)\rho_\alpha - D(k,n)) \end{aligned} \tag{2.8.7}$$

where $\alpha \in \{L, R\}$.

From detailed balance (2.8.3) we obtain

$$d_\alpha(n) = \frac{1}{\lambda_\alpha}\left(\frac{n}{\frac{m}{2}+n-1}\right) b_\alpha(n-1) \tag{2.8.8}$$

Working out (2.8.7) gives, using (2.4.3),

$$\begin{aligned} & b_\alpha(n)\left(\frac{n+1}{n+1-k} - 1\right) + d_\alpha(n)\left(\frac{n-k}{n} - 1\right) \\ = \quad & k\left(\frac{\left(\frac{m}{2}+k-1\right)\rho_\alpha}{n-k+1} - 1\right) \end{aligned} \tag{2.8.9}$$

which simplifies to

$$\frac{b_\alpha(n)}{n+1-k} - \frac{d_\alpha(n)}{n} = \left(\frac{\left(\frac{m}{2}+k-1\right)\rho_\alpha}{n-k+1} - 1\right) \tag{2.8.10}$$

Choosing

$$d_\alpha(n) = \frac{n}{1-\lambda_\alpha} \tag{2.8.11}$$

and by the detailed balance condition (2.8.8),

$$b_\alpha(n) = \left(\frac{m}{2} + n\right) \frac{\lambda_\alpha}{1-\lambda_\alpha} \tag{2.8.12}$$

it is then an easy computation to see that (2.8.7) is satisfied with the choices (2.8.11), (2.8.12). Indeed, (2.8.10) reduces to the simple identity

$$\left(\frac{m}{2} + n\right)\left(\frac{\lambda}{1-\lambda}\right)\frac{1}{n+1-k} - \frac{1}{1-\lambda} = \frac{\frac{m}{2}+k-1}{n+1-k}\left(\frac{\lambda}{1-\lambda}\right) - 1$$

We remark that the requirement of detailed balance alone is not sufficient to fix the rates uniquely. However, the additional duality constraint (2.8.7) does fix the rates to the unique expression given by (2.8.11) and (2.8.12).

As a consequence of duality with duality functions (2.8.4), we have that the boundary driven $SIP(m)$ with generator (2.8.1) has a unique stationary measure $\mu_{L,R}$ for which expectations of the polynomials $D(\xi,\eta)$ are given in terms of absorption probabilities:

$$\begin{aligned}\int D(\xi,\eta)\mu_{L,R}(d\eta) &= \lim_{t\to\infty} \mathbb{E}_\eta \mathscr{D}(\xi,\eta_t) \\ &= \lim_{t\to\infty} \hat{\mathbb{E}}_\xi \mathscr{D}(\xi_t,\eta) \\ &= \sum_{k,l:k+l=|\xi|} \rho_L^k \rho_R^l \hat{\mathbb{P}}_\xi\left(\xi_\infty = k\delta_0 + l\delta_{N+1}\right) \end{aligned} \tag{2.8.13}$$

Here, $\hat{\mathbb{E}}_\xi$ denotes expectation in the dual process (which is absorbing at $\{0, N+1\}$) starting from ξ. In particular, since a single $SIP(m)$ particle performs continuous time simple random walk (at rate $2m$) we have a linear density profile, i.e.,

$$\int D(\delta_i,\eta)\mu_{L,R}(d\eta) = \rho_L\left(1 - \frac{i}{N+1}\right) + \rho_R \frac{i}{N+1} \tag{2.8.14}$$

2.8.2 Correlation inequality for the boundary driven $SIP(m)$

For $x_1,\ldots,x_n \in \{1,\ldots,N\}$ let us denote by $(X_1(t),\ldots,X_n(t))$ the positions of particles at time t evolving according to the $SIP(m)$ with absorbing boundary sites at $\{0,N+1\}$, i.e., according to the generator (2.8.5), and initially at positions $x_1,\ldots,x_n$. Let $(Y_1(t),\ldots,Y_n(t))$ denote the positions at time t of independent random walkers (jumping at rate $2m$) absorbed (at rate 1) at $\{0,N+1\}$, initially at positions $x_1,\ldots,x_n$. Since the absorption parts of the generators of $(X_1(t),\ldots,X_n(t))$ and $(Y_1(t),\ldots,Y_n(t))$

are the same, we have the same inequality for expectations of positive definite functions as in Theorem 2.3.1. Therefore, we have the following result on positivity of correlations in the stationary state. This has once more to be compared to the analogous situation of the boundary driven exclusion process, where the stationary covariances of site-occupations are negative.

Proposition 2.8.1. *Let $\mu_{L,R}$ denote the unique stationary measure of the process with generator* (2.8.1)*. Let $x_1, \dots, x_n \in \{1, \dots, N\}$, then we have*

$$\int D\left(\sum_{i=1}^n \delta_{x_i}, \eta\right) \mu_{L,R}(d\eta) \geq \prod_{i=1}^n \int D(\delta_{x_i}, \eta) \mu_{L,R}(d\eta) \tag{2.8.15}$$

In particular, $\eta_x, x \in \{1, \dots, N\}$ are positively correlated under the measure $\mu_{L,R}$.

Proof. Start from the measure ν_λ^m. Define the map $\{0, \dots, N+1\}^n \to \mathbb{R}$:

$$(x_1, \dots, x_n) \mapsto \int \mathscr{D}\left(\sum_{i=1}^n \delta_{x_i}, \eta\right) \nu_\lambda^m(d\eta) = \prod_{i=1}^n \rho(x_i) \tag{2.8.16}$$

where $\rho(x) = \frac{\lambda}{1-\lambda}$ for $x \in \{1, \dots, N\}$ and $\rho(0) = \rho_L, \rho(N+1) = \rho_R$. This is clearly positive definite. Therefore, for $x_1, \dots, x_n \in \{1, \dots, N\}$, we have

$$\begin{aligned}
\int D\left(\sum_{i=1}^n \delta_{x_i}, \eta\right) \mu_{L,R}(d\eta) &= \lim_{t\to\infty} \int \mathbb{E}_\eta \mathscr{D}\left(\sum_{i=1}^n \delta_{x_i}, \eta_t\right) \nu_\lambda^m(d\eta) \\
&= \lim_{t\to\infty} \int \hat{\mathbb{E}}^{SIP(m),abs}_{x_1,\dots,x_n} \left(\mathscr{D}(\sum_{i=1}^n \delta_{X_i(t)}, \eta)\right) \nu_\lambda^m(d\eta) \\
&\geq \lim_{t\to\infty} \hat{\mathbb{E}}^{IRW(m),abs}_{x_1,\dots,x_n} \left(\int \mathscr{D}(\sum_{i=1}^n \delta_{X_i(t)}, \eta) \nu_\lambda^m(d\eta)\right) \\
&= \prod_{i=1}^n \lim_{t\to\infty} \hat{\mathbb{E}}^{IRW(m),abs}_{x_i} \rho(X_i(t)) \\
&= \prod_{i=1}^n \int D\left(\delta_{x_i}, \eta\right) \mu_{L,R}(d\eta)
\end{aligned} \tag{2.8.17}$$

where we denoted $\hat{\mathbb{E}}^{SIP(m),abs}$ for expectation over $SIP(m)$ particles absorbed at $\{0, N+1\}$, and $\hat{\mathbb{E}}^{IRW(m),abs}$ for expectation over a system of independent random walkers (jumping at rate $2m$) absorbed (at rate 1) at $\{0, N+1\}$. □

Remark 2.8.2. *1. Proposition 2.8.1 is in agreement with the findings of [9], where the covariance of η_i, η_j in the measure $\mu_{L,R}$ was computed explicitly, and turned out to be positive.*

2. For the nearest neighbor SEP on $\{1, \ldots, N\}$ driven at the boundaries, we have self-duality with absorption of dual particles at $\{0, N+1\}$ and duality function

$$\mathscr{D}_{SEP}\left(\sum_{i=1}^{n} \delta_{x_i}, \eta\right) = \prod_{i=1}^{n} \eta_{x_i}$$

where $\eta_0 := \rho_L, \eta_{N+1} = \rho_R$. Since for SEP particles we have the comparison inequality of Liggett, we have as an analogue of (2.8.15) in the SEP context,

$$\int \prod_{i=1}^{n} \eta_{x_i} \ \mu_{L,R}(d\eta) \leq \prod_{i=1}^{n} \int \eta_{x_i} \ \mu_{L,R}(d\eta)$$

i.e., η_{x_i} are negatively correlated. The same holds for the non-equilibrium $SEP(n)$ driven by appropriate boundary generators. This is in agreement with the results in [16], where the two-point function of the measure $\mu_{L,R}$ is computed, and with the work of [7], where some multiple correlations are explicitly computed.

3. We expect the KMP-model, a model of heat conduction introduced and studied in [13] to also have positive correlations. Indeed, the KMP and the $BEP(2)$ model are related by a so-called instantaneous thermalization limit [10]. Therefore, it is natural to think that similar correlation inequalities should hold for the KMP as we have derived for the BEP. The limit to obtain the KMP from the BEP is however difficult to perform on the level of the n-particle representation and it is thus not clear (to us) how to prove that the KMP preserves the positive correlation structure of the BEP. A positive hint in this direction comes from the explicit expression of the two point function which has been computed for the KMP in the non-equilibrium context in [2].

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3 Condensation in the inclusion process and related models

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3.0 Abstract

We study condensation in several particle systems related to the inclusion process. For an asymmetric one-dimensional version with closed boundary conditions and drift to the right, we show that all but a finite number of particles condense on the rightmost site. This is extended to a general result for independent random variables with different tails, where condensation occurs for the index (site) with the heaviest tail, generalizing also previous results for zero-range processes. For inclusion processes with homogeneous stationary measures we establish condensation in the limit of vanishing diffusion strength in the dynamics, and give several details about how the limit is approached for finite and infinite systems. Finally, we consider a continuous model dual to the inclusion process, the so-called Brownian energy process, and prove similar condensation results.

3.1 Introduction

In [1], [2], an interacting particle system was introduced, where particles perform random walks and interact by "inclusion", i.e., every particle at site i can attract particles from a site j to its site at rate $p(i,j) = p(j,i)$. This particle system, the so-called symmetric inclusion process (SIP), is "exactly solvable" by self-duality, and its ergodic stationary measures are products of discrete gamma distributions, indexed by the density. The inclusion process also turns out to be dual to a system of interacting diffusions, the so-called Brownian energy process (BEP). More details on duality, self-duality, and the precise relations between SIP and BEP can be found in [1]. In the present paper we only need the explicit form of the stationary measures of these models.

We prove existence of stationary product measures for inclusion processes under rather general conditions, in analogy to classical results for exclusion processes [16]. We introduce asymmetric versions of the SIP and the BEP, for simplicity focusing on a one-dimensional context with N sites and closed boundary conditions. In this case both models have spatially inhomogeneous product measures as reversible measures (to be compared with the blocking measure of the asymmetric exclusion process). Conditioning on K particles in the system (resp. total energy E), we prove that that in the limit $K \to \infty$ "almost all" the particles (resp. all the energy) are concentrated on a single site, where the marginal of the reversible measure has the heaviest tail.

The other sites contain a finite number of particles (resp. finite amount of energy).

We further study condensation in inclusion processes with spatially homogeneous stationary measures, with the SIP as the main example. The strength of the diffusive part of the dynamics in comparison to the attraction is controlled by a system parameter $m > 0$. For fixed particle density ρ we study the limit $m \to 0$ where attraction dominates, and show that the single-site marginals converge to Dirac measures concentrated on zero mass. This corresponds to the fact that a typical configuration consists of rare piles of typical size $2\rho/m$ separated by empty sites. The distribution of pile sizes approaches a power law with exponent -1 and becomes degenerate in the limit $m \to 0$. This leads to a breakdown of the usual law of large numbers which we illustrate in detail.

Our results for the asymmetric case also cover condensation phenomena in zero-range processes, which have attracted a lot of recent research interest [5, 6]. For inhomogeneous systems, these have been studied before mainly in the context of a quenched disorder in the jump rates, which have to be non-decreasing functions of the number of particles [7, 8, 9, 10]. For such systems, the use of coupling techniques allowed in special cases to also obtain results on the dynamics of condensation. In contrast, our results cover only the stationary behaviour but apply to a much larger class of jump rates with essentially no restriction. The widely studied condensation in spatially homogeneous zero-range processes [11, 12, 13, 14, 15] has a somewhat different origin than our homogeneous results for the SIP. This is discussed in detail at the end of Section 4.2.

In the next section we describe the inclusion process and its stationary measures. In Sections 3 and 4 we study condensation in the asymmetric and spatially homogeneous case, and discuss extensions and relations to zero-range processes. In Section 5 we introduce the asymmetric Brownian energy process and discuss condensation in an example of a system with continuous state space.

3.2 Inclusion processes

The inclusion process on a general discrete set Λ has state space $\Omega = \mathbb{N}^\Lambda$ and we denote a configuration by $\eta = (\eta_i : i \in \Lambda)$ where η_i is interpreted as the number of particles at site $i \in \Lambda$. The dynamics is defined by the generator defined on the core of local functions $f : \Omega \to \mathbb{R}$:

$$Lf(\eta) = \sum_{i,j\in\Lambda} p(i,j)\eta_i \left(\frac{m}{2} + \eta_j\right) \left(f(\eta^{i,j}) - f(\eta)\right) , \tag{3.2.1}$$

where $\eta^{i,j}$ is the configuration obtained from η by removing a particle from site i and putting it to j. The $p(i,j) \geq 0$ are jump rates of an irreducible random walk on Λ with $p(i,i) = 0$, and the parameter $m > 0$ determines the rate of diffusion of the particles as compared to the aggregation part given by the product $\eta_i\eta_j$. We also assume the $p(i,j)$ to be uniformly bounded and of finite range, i.e. there exist $C, R > 0$ such that

$$\sup_{i,j\in\Lambda} p(i,j) < C \quad \text{and} \quad \left|\left\{j \in \Lambda \,:\, p(i,j) > 0\right\}\right| < R \text{ for all } i \in \Lambda\ . \tag{3.2.2}$$

This ensures that the dynamics is well defined even on infinite lattices (for a large class of 'reasonable' initial conditions) and contains all generic examples we are interested in, such as nearest-neighbour hopping on regular lattices.
If the $p(i,j)$ are symmetric the inclusion process is also called symmetric (SIP), otherwise asymmetric (ASIP).

3.2.1 Stationary product measures

For $\varphi \geq 0$ and $\lambda_i > 0$, $i \in \Lambda$, define the product probability measure

$$\nu_\varphi(d\eta) = \otimes_{i\in\Lambda}\nu^i_\varphi(d\eta_i) , \tag{3.2.3}$$

where the marginals ν^i are probability measures on $\mathbb{N}$ given by

$$\nu^i_\varphi(n) = \left(z_i(\varphi)\right)^{-1} \lambda_i^n \varphi^n \frac{\Gamma\left(\frac{m}{2} + n\right)}{n!\Gamma\left(\frac{m}{2}\right)} \tag{3.2.4}$$

with the normalizing constant

$$z_i(\varphi) = \sum_{n=0}^{\infty} \lambda_i^n \varphi^n \frac{\Gamma\left(\frac{m}{2} + n\right)}{n!\Gamma\left(\frac{m}{2}\right)} = \left(1 - \lambda_i\varphi\right)^{-m/2} \ . \tag{3.2.5}$$

The parameter $\varphi \geq 0$ is called fugacity and controls the particle density, which is invariant under the time evolution.

Theorem 3.2.1. *For all $\varphi < \varphi_c := \left(\sup_{i\in\Lambda} \lambda_i\right)^{-1}$, ν_φ is a stationary measure for the inclusion process with generator (3.2.1), provided that one of the following conditions holds:*

a) The $p(i,j)$ are doubly stochastic modulo a constant, i.e.

$$\sum_{j\in\Lambda} \big(p(i,j) - p(j,k)\big) = 0 \quad \textit{for all } i,k \in \Lambda \ , \tag{3.2.6}$$

and $\lambda_i = 1$ for all $i \in \Lambda$.

b) The λ_i are reversible w.r.t. the $p(i,j)$, i.e.

$$\lambda_i p(i,j) = \lambda_j p(j,i) \quad \textit{for all } i,j \in \Lambda \ , \tag{3.2.7}$$

and in that case ν_φ is also a reversible measure.

This is in direct analogy with well-known results for stationary measures for exclusion processes (see e.g. [16], Thm VIII.2.1). In both cases, the λ_i are special harmonic functions solving

$$\sum_{j\in\Lambda} \big(\lambda_i p(i,j) - \lambda_j p(j,i)\big) = 0 \quad \text{for all } i \in \Lambda \ , \tag{3.2.8}$$

i.e. they provide a (not necessarily normalized) stationary distribution for the underlying random walk of a single particle. For the above product measures to be stationary, the $p(i,j)$ have to be such that they admit a constant solution (first case) or a detailed balance solution (second case). It is not clear at this point whether these conditions are really necessary for the existence of stationary product measures in general. Note also that on infinite lattices $\varphi_c = 0$ is possible. But for finite Λ (which we mainly focus on in this paper), Theorem 3.2.1 guarantees the existence of a family of stationary measures.

Proof. We have to show for expected values w.r.t. ν_φ that

$$\nu_\varphi(Lf) = \sum_{\eta\in\Omega} \sum_{i,j\in\Lambda} p(i,j)\eta_i \left(\frac{m}{2} + \eta_j\right) (f(\eta^{i,j}) - f(\eta))\nu_\varphi(\eta) = 0 \tag{3.2.9}$$

for all local functions f. For fixed i,j we get after a change of variable

$$\begin{aligned}\sum_{\eta\in\Omega} & p(i,j)\eta_i \left(\frac{m}{2} + \eta_j\right) f(\eta^{i,j})\nu_\varphi(\eta) \\ &= \sum_{\eta\in\Omega} p(i,j)(\eta_i + 1)\left(\frac{m}{2} + \eta_j - 1\right) f(\eta)\nu_\varphi(\eta^{j,i}) \ .\end{aligned}$$

The form (3.2.4) of the marginals implies that for all $i \in \Lambda$ and $k \geq 0$

$$\frac{\nu_\varphi^i(k+1)}{\nu_\varphi^i(k)} = \varphi\, \frac{m+2k}{2(k+1)}\, \lambda_i \ .$$

Thus we get for each fixed pair $i, j \in \Lambda$

$$\nu_\varphi^i(n{+}1)\,\nu_\varphi^j(k{-}1)\,(n+1)\left(\frac{m}{2}+k-1\right) = \nu_\varphi^i(n)\,\nu_\varphi^j(k)\,k\left(\frac{m}{2}+n\right)\frac{\lambda_i}{\lambda_j}$$

for all $n \geq 0$ and $k \geq 1$. It is easy to check that boundary terms in the sums vanish consistently, and we do not consider them in the following. Plugging this into (3.2.9) we get

$$\nu_\varphi(Lf) = \sum_{\eta\in\Omega} f(\eta)\nu_\varphi(\eta) \sum_{i,j\in\Lambda} p(i,j)\left(\eta_j\left(\frac{m}{2}{+}\eta_i\right)\frac{\lambda_i}{\lambda_j}{-}\eta_i\left(\frac{m}{2}{+}\eta_j\right)\right) , \qquad (3.2.10)$$

and exchanging the summation variables $i \leftrightarrow j$ in the first part of the sum leads to

$$\nu_\varphi(Lf) = \sum_{\eta\in\Omega} f(\eta)\nu_\varphi(\eta) \sum_{i,j\in\Lambda} \frac{\eta_i\,(m/2+\eta_j)}{\lambda_i}\big(p(j,i)\lambda_j - p(i,j)\lambda_i\big) . \qquad (3.2.11)$$

This clearly vanishes under the reversibility condition (3.2.7) which implies stationarity under assumption b). In analogy to (3.2.10) we can derive

$$\begin{aligned}\nu_\varphi(gLf) \quad &= \quad \sum_{\eta\in\Omega} f(\eta)\nu_\varphi(\eta) \\ &\qquad \sum_{i,j\in\Lambda} p(i,j)\left(\eta_j\left(\frac{m}{2}{+}\eta_i\right)\frac{\lambda_i}{\lambda_j}g(\eta^{j,i}) - \eta_i\left(\frac{m}{2}{+}\eta_j\right)g(\eta)\right) ,\end{aligned}$$

and after using (3.2.7) and the exchange of summation variables this implies

$$\begin{aligned}\nu_\varphi(gLf) \quad &= \quad \sum_{\eta\in\Omega} f(\eta)\nu_\varphi(\eta) \sum_{i,j\in\Lambda} p(i,j)\eta_i\left(\frac{m}{2}{+}\eta_j\right)\big(g(\eta^{i,j}) - g(\eta)\big) \\ &= \quad \nu_\varphi(fLg) ,\end{aligned}$$

so ν_φ is also reversible.
Assuming a), the λ_i and λ_j in (3.2.11) cancel, and we write the linear (diffusive) part as

$$\sum_{i\in\Lambda} \eta_i \frac{m}{2} \sum_{j\in\Lambda} \big(p(j,i) - p(i,j)\big) = 0 ,$$

which vanishes due to (3.2.6). For the quadratic aggregation part we get

$$\sum_{i,j\in\Lambda} \eta_i\eta_j\,\big(p(j,i) - p(i,j)\big) = \sum_{i,j\in\Lambda} \eta_i\eta_j\,\big(p(j,i) - p(j,i)\big) = 0 ,$$

by another exchange of the summation variables in the second part, using that $\eta_i\eta_j$ is symmetric under $i \leftrightarrow j$. $\square$

3.2.2 Canonical measures for finite systems

Consider a finite lattice Λ_N of size N with corresponding state space $\Omega_N = \mathbb{N}^{\Lambda_N}$. Starting with a fixed number of K particles, the inclusion process with generator L_N as given in (3.2.1) is an irreducible continuous-time Markov chain on the finite set

$$A_K = \left\{ \eta \in \Omega_N : \sum_{i=1}^{N} \eta_i = K \right\} , \tag{3.2.12}$$

and has a unique stationary measure, which we denote by μ^K.

By conservation of the number of particles, the conditional measure

$$\nu_\varphi \left(d\eta \middle| \sum_{i=1}^{N} \eta_i = K \right)$$

is also invariant. Indeed for $f : \Omega_N \to \mathbb{R}$ we have

$$\begin{aligned} \int L_N f(\eta) \nu_\varphi \left(d\eta \middle| A_K \right) &= \frac{\int L_N f(\eta) 1_{A_K} \nu_\varphi(d\eta)}{\nu_\varphi(A_K)} \\ &= \frac{\int f(\eta) \left(L_N^* \left(1_{A_K} \right) \right) (\eta) \nu_\varphi(d\eta)}{\nu_\varphi(A_K)} = 0 , \end{aligned} \tag{3.2.13}$$

since it is easy to see that with the generator L_N also its adjoint L_N^* conserves the number of particles. In the case of reversible measures ν_φ, L_N is self-adjoint and there is nothing to check. By uniqueness of the stationary measure, we thus have

$$\nu_\varphi(.\,|A_K) = \mu^K \tag{3.2.14}$$

for all $\varphi < \varphi_c$ and $K \in \mathbb{N}$. So the conditioned product measures are actually independent of φ, and this connection provides an explicit form for the canonical measures μ^K.

3.3 Condensation in the ASIP

A generic situation where Theorem 3.2.1 gives rise to spatially inhomogeneous reversible measures is a one-dimensional lattice $\Lambda_N = \{1, \ldots, N\}$ with an underlying asymmetric nearest-neighbour walk. We consider the ASIP with generator

$$\begin{aligned} L_N f(\eta) &= \sum_{i=1}^{N-1} p\eta_i \left(\frac{m}{2} + \eta_{i+1}\right)(f(\eta^{i,i+1}) - f(\eta)) \\ &\quad + \sum_{i=1}^{N-1} q\eta_{i+1} \left(\frac{m}{2} + \eta_i\right)(f(\eta^{i+1,i}) - f(\eta)) \end{aligned} \tag{3.3.1}$$

where $p > q > 0$. In this case $\lambda_i = (p/q)^i$ fulfills condition (3.2.7) in Theorem 3.2.1. We will now proceed towards showing that in the limit $K \to \infty$, under the canonical measure μ^K, the typical situation will be that all but a finite number of particles condenses at the right site $i = N$, whereas the other sites contain a number of particles distributed according to $\nu^i_{\varphi_c}$.

At first sight one could be tempted to think that this is just a consequence of the asymmetry: particles are pushed to the right. This is, however, not the case. If we consider independent random walkers, moving at rate p to the right and q to the left, then the reversible profile measures are Poissonian and given by $\otimes_{i=1}^N \nu^i_\varphi(d\eta_i)$ with

$$\nu^i_\varphi(n) = \frac{1}{z_i(\varphi)} \left(\frac{p}{q}\right)^{ni} \frac{\varphi^n}{n!}$$

with a normalizing constant $z_i(\varphi) = e^{\varphi(p/q)^i}$ which is now finite for all values of φ. As a consequence, no condensation happens: if we condition on having K particles, and let K tend to infinity, all sites will carry a diverging number of particles. The condensation phenomenon is thus a combination of the asymmetry, together with the attractive interaction between the particles in the inclusion process. Indeed, it is the interaction which is responsible for the existence of a finite critical φ_c.

3.3.1 Condensation

Before we formulate the main result of this Section, we recall the marginals ν^i_φ for the ASIP

$$\nu^i_\varphi(n) = \frac{1}{z_i(\varphi)} \varphi^n \lambda_i^n w_i(n) \; , \tag{3.3.2}$$

where we have now

$$\lambda_i = (p/q)^i \tag{3.3.3}$$

and write

$$w_i(n) = \frac{\Gamma\left(n + \frac{m}{2}\right)}{n!\Gamma(n)} \ . \tag{3.3.4}$$

In the present case w_i does not dependend on i, but in generalizations explained below we will allow explicit dependence on i. The weights $w_i(n)$ have the asymptotic behavior

$$w_i(n) \sim n^{\frac{m}{2}-1} \tag{3.3.5}$$

where $a_n \sim b_n$ means that a_n/b_n converges to a strictly positive constant.

We remind that the normalizing constants are

$$z_i(\varphi) = (1 - \lambda_i\varphi)^{-m/2} = \left(1 - \left(\frac{p}{q}\right)^i \varphi\right)^{-m/2} \ . \tag{3.3.6}$$

Therefore, in the context of Theorem 3.2.1 we have $\lambda_1 < \lambda_2 < \ldots < \lambda_N$, $\varphi_c = 1/\lambda_N$, $z_i(\varphi_c) < \infty$ for all $1 \le i \le N-1$, and $z_N(\varphi) < \infty$ for all $\varphi < \varphi_c$. We then have the following result.

Theorem 3.3.1. *a) In the limit $K \to \infty$, $\eta_1, \ldots, \eta_{N-1}$ are asymptotically independent and converge in distribution to the critical product measure, i.e. for all $n_1, \ldots, n_{N-1} \in \mathbb{N}$*

$$\mu^K\left(\eta_1 = n_1, \ldots, \eta_{N-1} = n_{n-1}\right) \to \nu^1_{\varphi_c}(n_1) \cdots \nu^{N-1}_{\varphi_c}(n_{N-1}) \tag{3.3.7}$$

where $\varphi_c = 1/\lambda_N = (q/p)^N$.

b) In the limit $K \to \infty$, the right edge contains "almost all" particles, i.e., for all $\delta \in (0,1)$

$$\mu^K\big(\eta_N \le (1-\delta)K\big) \to 0 \ , \tag{3.3.8}$$

and we have a strong law of large numbers, $\eta_N/K \to 1$ a.s. .

Proof. We use that $\mu^K = \nu_\varphi(.\,|A_K)$ and write for $\Lambda' \subseteq \Lambda_N$

$$Z(\Lambda', K) = \sum_{\{n_i, i\in\Lambda' : \sum_{i\in\Lambda'} n_i = K\}} \prod_{i\in\Lambda'} w_i(n_i)\lambda_i^{n_i} \ . \tag{3.3.9}$$

We then have

$$\mu^K\left(\eta_1 = n_1, \ldots, \eta_{N-1} = n_{n-1}\right) = \\ = \frac{1}{Z(\Lambda_N, K)}\, w_N\left(K - \sum_{i=1}^{N-1} n_i\right) \lambda_N^{K - \sum_{i=1}^{N-1} n_i} \prod_{i=1}^{N-1} w_i(n_i)\, \lambda_i^{n_i} \ . \tag{3.3.10}$$

We first prove that

$$\lim_{K\to\infty} \frac{Z(\Lambda_N, K)}{\lambda_N^K w_N(K)} = \prod_{i=1}^{N-1} z_i(\varphi_c) \ . \tag{3.3.11}$$

To see this, we choose an appropriate order of summation,

$$\begin{aligned} Z(\Lambda_N, K) &= \sum_{n_1=0}^{K} \sum_{n_2=0}^{K-n_1} \cdots \sum_{n_{N-1}=0}^{K-(n_1+\ldots n_{N-2})} \\ &\qquad \left(\prod_{j=1}^{N-1} w_j(n_j)\lambda_j^{n_j}\right) w_N\left(K - \sum_{j=1}^{N-1} n_j\right) \lambda_N^{K-\sum_{j=1}^{N-1} n_j} \\ &= \lambda_N^K w_N(K) \sum_{n_1=0}^{\infty} \cdots \sum_{n_{N-1}=0}^{\infty} \\ &\qquad \Psi_K(n_1, \ldots, n_{N-1}) \prod_{j=1}^{N-1} w_j(n_j) \left(\frac{\lambda_j}{\lambda_N}\right)^{n_j} \end{aligned} \tag{3.3.12}$$

with

$$\Psi_K(\ldots) = \frac{w_N\left(K - \sum_{j=1}^{N-1} n_j\right)}{w_N(K)} 1_{n_1 \le K} \cdots 1_{n_{N-1} \le K - n_1 - .. - n_{N-2}} \ . \tag{3.3.13}$$

We see from (3.3.4) that $\Psi_K \le 1$. Therefore, by dominated convergence, using that $\varphi_c = \lambda_N^{-1}$ and

$$z_i(\varphi_c) = \sum_{n=0}^{\infty} \varphi_c^n \, w_i(n) \, \lambda_i^n = \sum_{n=0}^{\infty} w_i(n) \left(\frac{\lambda_i}{\lambda_N}\right)^n < \infty \ ,$$

we obtain (3.3.11). Combining (3.3.10) and (3.3.11) with the fact that

$$\lim_{K\to\infty} \frac{w_N(K-n)}{w_N(K)} = 1 \quad \text{for all } n \in \mathbb{N} \tag{3.3.14}$$

(which follows immediately from (3.3.5)), yields item a) of Theorem 3.3.1.

To prove item b), we start with

$$\mu^K\big(\eta_N \le (1-\delta)K\big) = \frac{\sum_{n \le (1-\delta)K} w_N(n) \lambda_N^n Z(\Lambda_N \setminus \{N\}, K-n)}{Z(\Lambda_N, K)}$$

and estimate, for $n \le (1-\delta)K$ and a small enough $\epsilon' > 0$ to be chosen below:

$$
\begin{aligned}
Z(\Lambda_N \setminus \{N\}, K-n) \quad &\le \quad (\lambda_{N-1}(1+\epsilon'))^{K-n} \\
&\qquad \sum_{n_1=0}^{K-n} \cdots \sum_{n_{N-2}=0}^{K-n-(n_1+\ldots+n_{N-3})} w_{N-1}\left(K-n-\sum_{j=1}^{N-2} n_j\right) \\
&\qquad\qquad \left(\prod_{j=1}^{N-2} w_j(n_j)\left(\frac{\lambda_j}{\lambda_{N-1}(1+\epsilon')}\right)^{n_j}\right) \\
&\le \quad C\left(\lambda_{N-1}(1+\epsilon')\right)^{K-n}(1+\epsilon)^K \ . \qquad (3.3.15)
\end{aligned}
$$

Here we have used that (cf. (3.3.5))

$$
w_{N-1}\left(K-n-\sum_{j=1}^{N-2} n_j\right) \le C(1+\epsilon)^K \qquad (3.3.16)
$$

for some $\epsilon > 0$ to be chosen below, and the fact that the remaining sums in the RHS of (3.3.15) converge to a finite value as $K \to \infty$. By (3.3.11) $Z(\Lambda_N, K)$ is bounded below by $C'\lambda_N^K w_N(K)$ for K large enough. This then gives

$$
\mu^K\big(\eta_N \le (1-\delta)K\big) \le C''\left(\sum_{n\le(1-\delta)K}\frac{w_N(n)}{w_N(K)}\right)\left(\frac{(1+\epsilon)^{1/\delta}(1+\epsilon')\lambda_{N-1}}{\lambda_N}\right)^{\delta K} ,
$$

since for the summation indices $K-n \ge \delta K$. Choosing $\epsilon, \epsilon' > 0$ small enough such that

$$
0 < \frac{(1+\epsilon)^{1/\delta}(1+\epsilon')\lambda_{N-1}}{\lambda_N} < q < 1
$$

and using that $\frac{w_N(n)}{w_N(K)} \le 1$, we obtain

$$
\mu^K(\eta_N \le (1-\delta)K) \le C'' q^{\delta K} \ . \qquad (3.3.17)
$$

Choosing $\delta = \delta_K = 1/\sqrt{K} \to 0$, we get a summable bound on the right-hand side. Since by definition $\eta_N \le K$ *a.s.* under the measure μ^K, this implies almost sure convergence and the strong law $\eta_N/K \to 1$ by Borel-Cantelli. $\square$

3.3.2 Generalizations

Notice that in the proof of Theorem 3.3.1 we did not use the specific form of w_i and λ_i. Therefore, the same proof shows a condensation phenomenon for a general family of independent random variables $\eta_1, \ldots, \eta_N$ with

$$\mathbb{P}(\eta_i = n) = \frac{1}{z_i(\varphi)}\, w_i(n)\lambda_i^n \varphi^n$$

under the following hypotheses on the w_i, λ_i:

a) The λ_i satisfy

$$\lambda_N > \max_{i=1}^{N-1} \lambda_i \ , \tag{3.3.18}$$

b) the weights $w_i(n)$ are subexponential in the following sense

$$\lim_{n\to\infty} \frac{w_i(n+1)}{w_i(n)} = 1 \tag{3.3.19}$$

for all $1 \le i \le N$.

From (3.3.19), (3.3.14) follows directly, and it further implies that for all $\alpha > 0$ there exists $C_\alpha > 0$ such that for all $n \ge 0$

$$C_\alpha^{-1} e^{-\alpha n} \le w_i(n) \le C_\alpha e^{\alpha n} \ .$$

From this bound we conclude that for all $\beta > 0$, there exists $C_\beta > 0$ such that for all $n, l \ge 0$,

$$C_\beta^{-1} e^{-\beta(n+l)} \le \frac{w_i(n)}{w_i(l)} \le C_\beta e^{\beta(n+l)} \ .$$

This is all we need in the dominated convergence argument to bound Ψ_K of (3.3.13), and to conclude (3.3.16), (3.3.17). Therefore, under the assumptions a), b) we conclude the statement of Theorem 3.3.1 with

$$\mu^K = \mathbb{P}\left(\cdot \,\middle|\, \sum_{i=1}^N \eta_i = K \right) \ .$$

Example: Zero-range processes

Consider a general zero-range process on $\Omega_N = \mathbb{N}^{\{1,\ldots,N\}}$ with generator

$$L_N f(\eta) = \sum_{i,j} p(i,j)\, g_i(\eta_i) \left(f(\eta^{i,j}) - f(\eta) \right) \ , \tag{3.3.20}$$

where $p(i,j)$ are rates of an irreducible continuous-time random walk on $\{1,\dots,N\}$ and where $g_i : \mathbb{N} \to [0,\infty)$ with $g_i(n) = 0$ if and only if $n = 0$. Moreover, we assume for the moment that $g_i(n) \to \gamma_i \in (0,\infty)$ as $n \to \infty$ for all $i = 1,\dots,N$.

By irreducibility of $p(i,j)$, up to multiplicative constants there exists a unique function $\kappa : \{1,\dots,N\} \to (0,\infty)$ such that for all i,

$$\sum_{i=1}^{N} \big(\kappa_j p(j,i) - \kappa_i p(i,j)\big) = 0 \ . \tag{3.3.21}$$

Under these conditions it is well known [17] that the zero-range process has stationary product measures with marginals

$$\nu_\varphi^i(n) = \frac{1}{z_i(\varphi)} \frac{\varphi^n \kappa_i^n}{\prod_{k=1}^n g_i(k)} \ , \tag{3.3.22}$$

which are of the form (3.3.2) with

$$w_i(n) = \frac{\gamma_i^n}{\prod_{k=1}^n g_i(k)} \quad \text{and} \quad \lambda_i = \frac{\kappa_i}{\gamma_i} \ .$$

So in order to apply the general result, we need

$$\frac{\kappa_N}{\gamma_N} > \max_{i=1}^{N-1} \frac{\kappa_i}{\gamma_i} \ ,$$

and the subexponentiality condition on w_i follows since

$$\lim_{n\to\infty} \frac{w_i(n+1)}{w_i(n)} = \lim_{n\to\infty} \frac{\gamma_i}{g_i(n+1)} = 1 \ .$$

Remark 3.3.2. *1. The case $\gamma_i = \infty$ for some $i \neq N$ can be included as well. In that case, $z_i(\varphi) < \infty$ in (3.3.22) for all $\varphi > 0$, in particular for $\varphi = \varphi_c = 1/\lambda_N$. Therefore the result of Theorem 3.3.1 still holds.*

2. If there are more sites i such that $\lambda_i = \lambda_N$, then a) of Theorem 3.3.1 holds for all i where $\lambda_i < \lambda_N$. Item b) becomes that all but a finite amount of mass is concentrated on the sites where $\lambda_i = \lambda_N$.

Note that we make no assumptions on the jump rates of the zero-range process except a regular limiting behaviour, in particular there are no monotonicity assumptions. The latter have been in place in previous work on inhomogeneous zero-range condensation where the g_i are non-decreasing [7, 8, 10, 9], which made it possible to make much stronger statements including also the time evolution of the condensation. In that sense Theorem 3.3.1 is a generalization of previous results regarding only the stationary distribution.

3.4 Condensation in homogeneous inclusion processes

In this section we study condensation in spatially homogeneous systems. There are two natural situations where Theorem 3.2.1 leads to spatially homogeneous product measures ν_φ. If the $p(i,j)$ are symmetric, i.e. $p(i,j) = p(j,i)$ for all $i,j \in \Lambda$ then the reversibility condition (3.2.7) is fulfilled by taking a constant $\lambda_i = 1$ for all $i \in \Lambda$ independent of the geometry of the lattice. The same solution holds for translation invariant, asymmetric processes according to condition (3.2.6), where

$$p(i,j) = q(j-i) \quad \text{for some } q : \Lambda \to [0,\infty) \text{ with bounded support .}$$

In the second case the lattice also has to be translation invariant, such as $\Lambda = \mathbb{Z}^d$ or a finite subset with periodic boundary conditions. The measures ν_φ are then not reversible and the system can support a non-zero stationary current of the form

$$J(\rho) = \rho\left(\frac{m}{2} + \rho\right) \sum_{k\in\Lambda} k\, q(k) \ .$$

3.4.1 Stationary measures

In both cases discussed above the inclusion process has a family of homogeneous stationary product measures with marginals

$$\nu_\varphi^i(n) = \frac{1}{z(\varphi)}\, \varphi^n \frac{\Gamma\left(\frac{m}{2}+n\right)}{n!\Gamma\left(\frac{m}{2}\right)} \ , \tag{3.4.1}$$

and the normalizing constant

$$z(\varphi) = \sum_{n=0}^{\infty} \varphi^n \frac{\Gamma\left(\frac{m}{2}+n\right)}{n!\Gamma\left(\frac{m}{2}\right)} = (1-\varphi)^{-m/2} \ . \tag{3.4.2}$$

The measures are well defined for all positive $\varphi < \varphi_c = 1$, and the average number of particles per site is given by

$$\rho_m(\varphi) = \varphi\, \partial_\varphi \log z(\varphi) = \frac{m}{2} \frac{\varphi}{1-\varphi} \ . \tag{3.4.3}$$

Inverting this relation $\varphi_m(\rho) = \frac{\rho}{m/2+\rho}$ allows us – with a slight abuse of notation – to index the measures by the density,

$$\nu_\rho^{(m)}(n) = \frac{1}{z(\varphi_m(\rho))} \left(\frac{\rho}{m/2+\rho}\right)^n \frac{\Gamma\left(\frac{m}{2}+n\right)}{n!\Gamma\left(\frac{m}{2}\right)} \ . \tag{3.4.4}$$

We also replace the superscript since the marginals are site-independent and we want to stress the dependence on the parameter m. Since the density can take all values between 0 and ∞, we see that for fixed $m > 0$ the attraction between the particles is not strong enough and the inclusion process does not exhibit condensation. However, if we increase the relative strength of the attractive part in the generator (3.2.1) by taking m smaller and smaller at a fixed density ρ, a condensation phenomenon occurs in the limit $m \to 0$.

Theorem 3.4.1. *As $m \to 0$, we have for all $\rho > 0$*

$$\begin{aligned} \nu_\rho^{(m)}(0) &= \left(\frac{m}{2\rho}\right)^{m/2} \big(1 + o(1)\big) \to 1 \qquad \textit{and} \\ \nu_\rho^{(m)}(n) &= \frac{m}{2}\left(\frac{m}{2\rho}\right)^{m/2} \left(1 - \frac{m}{2\rho}\right)^{n} n^{m/2-1}\big(1 + o(1)\big) \to 0 \end{aligned} \tag{3.4.5}$$

for $n \geq 1$, which implies

$$\frac{2}{m}\,\nu_\rho^{(m)}(n) \to \frac{1}{n}\ . \tag{3.4.6}$$

Proof. By direct computation we get that

$$z\left(\varphi_m(\rho)\right) = \left(1 - \frac{\rho}{m/2 + \rho}\right)^{-m/2} = \left(\frac{2\rho}{m}\right)^{m/2} \big(1 + o(1)\big) \to 1 \tag{3.4.7}$$

as $m \to 0$, which directly implies the statement for $\nu_\rho^{(m)}(0) = 1/z\left(\varphi_m(\rho)\right)$. For every fixed $n \geq 1$ we have

$$\left(\frac{\rho}{m/2 + \rho}\right)^{n} = \left(1 - \frac{m}{2\rho}\right)^{n} \big(1 + o(1)\big) \to 1\ ,$$

and using $\Gamma(x) \sim \frac{1}{x}$ as $x \to 0$ and (3.3.5) we obtain

$$\frac{\Gamma\left(\frac{m}{2} + n\right)}{n!\,\Gamma\left(\frac{m}{2}\right)} = \frac{m}{2}\, n^{m/2-1}\big(1 + o(1)\big) \to 0\ ,$$

which implies the second statement. The limit in (3.4.6) follows immediately from the asymptotic behaviour. □

Therefore, for small diffusion rate m sites are either empty with very high probability, or contain a large number of particles to match the fixed expected value $\rho > 0$. From

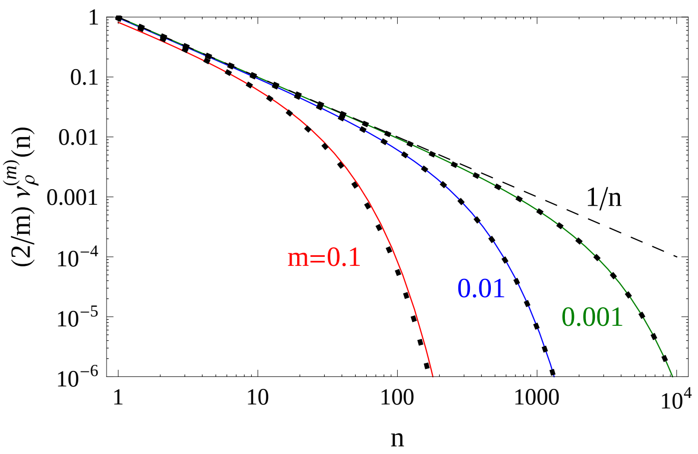

Figure 3.1: The scaled marginal $(2/m)\nu_\rho^{(m)}$ for $\rho = 1$ and for several values of m (full colored lines). The asymptotic behaviour as given in (3.4.8) is indicated by dotted lines.

theorem 3.4.1 we infer the following leading-order behaviour for small fixed m,

$$\nu_\rho^{(m)}(n) \simeq \frac{m}{2} \begin{cases} n^{-1} & , \ 1 \ll n \ll 2\rho/m \\ (1-\frac{m}{2\rho})^n n^{m/2-1} & , \ n \gg 2\rho/m \end{cases} , \tag{3.4.8}$$

where we have used

$$\left(1-\frac{m}{2\rho}\right)^n \simeq \left(1-\frac{mn}{2\rho}\right) \simeq 1 \quad \text{for } n \ll 2\rho/m ,$$

with the notation $a_m \simeq b_m$ if $a_m/b_m \to 1$ as $m \to 0$.

So the marginals show an approximate power law decay with exponent -1, until an exponential cut-off sets in at the scale $n \sim 2\rho/m$. This is illustrated in Figure 3.1, where we see that the asymptotic behaviour for large n fits very well also for smaller values of n. Despite the small prefactor $m/2$ the density $\rho > 0$ is realized by the asymptotic heavy-tail behaviour, and for each $m > 0$ the distribution is normalized due to the cut-off. Conditioned on a site being non-empty, its distribution is given by $\nu_\rho^{(m)}(n)/\big(1-\nu_\rho^{(m)}(0)\big)$. Using that to leading order

$$1-\nu_\rho^{(m)}(0) \simeq 1-\left(\frac{m}{2\rho}\right)^{m/2} \simeq -\frac{m}{2}\log m ,$$

we get with (3.4.8) for the conditional distributions

$$\frac{\nu_\rho^{(m)}(n)}{1-\nu_\rho^{(m)}(0)}\,|\log m| \to \frac{1}{n} \quad \text{as} \quad m\to 0\ . \tag{3.4.9}$$

Like in (3.4.6), convergence is clearly non-uniform due to the cut-off, and the limit is not a probability distribution. The interpretation of this result in terms of condensation depends on the geometry and is different for finite and infinite lattices Λ, as discussed below.

3.4.2 Finite systems

For finite lattices one can condition on the total number of particles in the system, defining the canonical measures as in Section 2.2. The basic features of this approach can already be understood on a system with two sites and $\Lambda = \{1,2\}$. Let η_1,η_2 be two random variables each distributed as $\nu_\rho^{(m)}$ and consider their joint distribution μ_m^K conditioned on their sum being equal to $K\in\mathbb{N}$, i.e.

$$\mu_m^K := \nu_\rho\big(.\,\big|\eta_1+\eta_2=K\big)\ . \tag{3.4.10}$$

For each $K\in\mathbb{N}$ and $m>0$ the inclusion process is irreducible and μ_m^K is the unique stationary measure (cf.(3.2.13)). A first observation is that, as before, μ_m^K does in fact not depend on ρ since due to cancellation

$$\begin{aligned}\mu_m^K(\eta_1=n_1,\eta_2=n_2) &= \frac{\delta_{n_1+n_2,K}\,\nu_\rho^{(m)}(n_1)\nu_\rho^{(m)}(n_2)}{\sum_{l=0}^K \nu_\rho^{(m)}(l)\nu_\rho^{(m)}(K-l)}\\ &= \frac{\delta_{n_1+n_2,K}\,\Gamma(m/2+n_1)\Gamma(m/2+n_2)/(n_1!n_2!)}{\sum_{l=0}^K \Gamma(m/2+l)\Gamma(m/2+K-l)/(l!(K-l)!)}\ .\end{aligned} \tag{3.4.11}$$

Proposition 3.4.2. *In the limit $m\to 0$ we have for all $K>0$*

$$\mu_m^K \to \frac{1}{2}\big(\delta_{(K,0)}+\delta_{(0,K)}\big)\ , \tag{3.4.12}$$

i.e. all particles concentrate on one of the sites with equal probability.

Proof. With $\eta_2 = K-\eta_1$ we have

$$\mu_m^K(\eta_1=n,\eta_2=K-n) = \frac{\Gamma(\frac{m}{2}+n)\Gamma(\frac{m}{2}+K-n)/(n!(K-n)!)}{\sum_{l=0}^K \Gamma(\frac{m}{2}+l)\Gamma(\frac{m}{2}+K-l)/(l!(K-l)!)}\ . \tag{3.4.13}$$

In the normalizing sum, as $m \to 0$, the two terms for $l = 0, K$ diverge like $\Gamma(m/2)/K$, whereas the rest of the sum converges. Also the term in the numerator of $\mu_m^K(\eta_1 = n)$ diverges like $\Gamma(m/2)/K$ if $n = 0$ or K and is finite otherwise. This implies the result. $\square$

The interpretation is that as $m \to 0$ aggregation dominates more and more over diffusion and the particles tend to cluster on one of the lattice sites. The onset of condensation for small m can be well illustrated in the limit of infinitely many particles.

Proposition 3.4.3. *In the limit $K \to \infty$ we have for all $m > 0$,*

$$\Big(\frac{\eta_1}{K}, \frac{\eta_2}{K}\Big) \stackrel{\mu^K}{\longrightarrow} (B, 1-B) \qquad \textit{in distribution} \ , \tag{3.4.14}$$

where $B \in [0,1]$ is a continuous random variable with Beta$\big(\frac{m}{2}, \frac{m}{2}\big)$ *distribution and PDF*

$$f_B(x) = \frac{\Gamma(m/2)^2}{\Gamma(m)}\, x^{m/2-1}(1-x)^{m/2-1} \ , \quad x \in [0,1] \ . \tag{3.4.15}$$

Proof. Using (3.3.5) we get as $K \to \infty$ and $n/K \to x \in [0,1]$ for the asymptotic form of the numerator of (3.4.13)

$$K^{m-2} x^{m/2-1}(1-x)^{m/2-1} \ .$$

For the denominator we get the integral

$$K^{m-2} \int_0^1 y^{m/2-1}(1-y)^{m/2-1} K\, dy = K^{m-1} \frac{\Gamma(m)}{\Gamma(m/2)^2} \ ,$$

using the representation $B(r,s) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}$ for the Beta function. Thus we have that $K\mu_m^K(\eta_1 = n) \to f_B(x)$ converges to the PDF of the Beta$\big(\frac{m}{2}, \frac{m}{2}\big)$ distribution. $\square$

We see that for $m < 2$ one site contains most of the particles while for $m > 2$ both sites are likely to have around $K/2$ particles. The boundary case is $m = 2$, where the particles are distributed uniformly among the two sites. This is a standard property of the symmetric Beta distribution and is illustrated in Figure 3.2. In the limit $m \to 0$ we recover the degenerate distribution (3.4.12).

Remark 3.4.4. *a) The result (3.4.12) can be immediately generalized to a finite set $\Lambda_N = \{1, \ldots, N\}$ of $N \geq 2$ sites. In the limit $m \to 0$ we have for all $K \in \mathbb{N}$*

$$\mu_m^K \to \frac{1}{N} \sum_{i=1}^{N} \delta_{K\, e_i} \ , \tag{3.4.16}$$

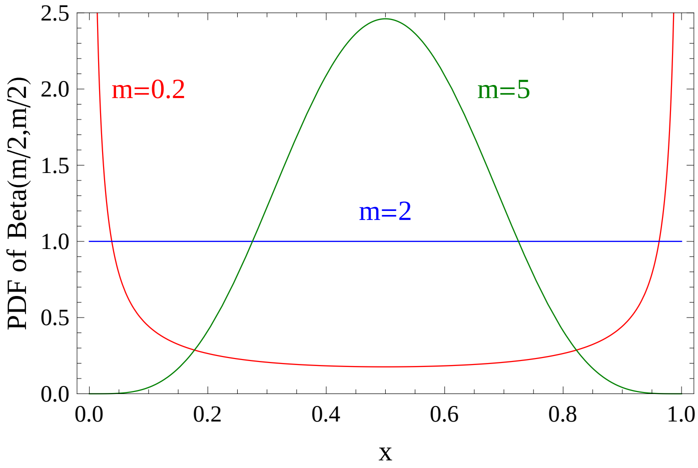

Figure 3.2: The limit distribution of η_1/K as $K \to \infty$, given by the PDF of $\mathrm{Beta}\left(\frac{m}{2}, \frac{m}{2}\right)$ (cf. 3.4.15) for several values of m.

where $e_i = (.., 0, 1, 0, ..) \in \mathbb{R}^N$ *is the standard unit vector in direction* i*.*

b) In the absence of diffusion for $m = 0$ *the inclusion process has in general many absorbing states which exhibit several isolated piles of particles. However, if the* $p(i, j) > 0$ *for all* $i, j \in \Lambda_N$*, then all absorbing states have exactly one pile containing all the particles. The stationary measures are then all possible mixtures*

$$\sum_{i=1}^{N} \alpha_i \delta_{K\,e_i} \quad \textit{with} \quad \alpha_i \in [0, 1] \quad \textit{and} \quad \sum_i \alpha_i = 1\ . \tag{3.4.17}$$

The limit result (3.4.16) leads only to the symmetric mixture, due to homogeneity and ergodicity of the process for $m > 0$*.*

Connection to zero-range processes.
This result is slightly different from most previous work on homogeneous zero-range condensation, which is mostly discussed in the limit of infinitely many particles [13] or the thermodynamic limit [11, 6]. In this case, above a certain density or particle number all sites have heavy-tailed distributions and condensation is a consequence of large deviation properties of such random variables, as discussed in detail in [14].

For the inclusion process we discuss the two extreme cases of a finite and an infinite lattice (see next section), in the limit of a vanishing system parameter $m \to 0$. The distributions of the occupation numbers always have exponential tails due to the cut-off (3.4.8), which disappears in the limit in a non-uniform way. This is very similar to results in [18], where a parameter was varied together with the system size in a joint limit. Analogous results are phrased here in terms of the law of large numbers in the next section. Size-dependent system parameters have also been studied in [19], which can lead to a cut-off similar to (3.4.8) and a typical maximal cluster size also in zero-range processes.

As a further difference to zero-range condensation, there is no non-trivial critical density ρ_c for the distribution of sites outside the maximum in the inclusion process. In fact, in the limit $m \to 0$ all N particles condense on a single site, which corresponds to $\rho_c = 0$ and is an absorbing state for the dynamics with $m = 0$. This is related to results on zero-range processes where the jump rates vanish in the limit of infinite occupation number, which has been studied in [20] and more recently also in [21].

3.4.3 The infinite-volume limit

For finite systems with a fixed number of particles the exponential part of the product measures that leads to a cut-off for large n (cf. (3.4.8)) did not play any role due to cancellation, but will be of importance for infinite systems. For simplicity we consider stationary configurations of the symmetric inclusion process (SIP) on the infinite lattice $\Lambda = \mathbb{N}$ which leads to a family of iid random variables $\eta_1, \eta_2, \ldots$ with distribution $\nu_\rho^{(m)}$ (3.4.4). In this context the condensation phenomenon for $m \to 0$ can be formulated as a breakdown of the usual law of large numbers.

For every $m > 0$ by definition $\mathbb{E}(\eta_i) = \rho$ and a usual law of large numbers holds, i.e.

$$S_K := \frac{1}{K} \sum_{i=1}^{K} \eta_i \to \rho \quad a.s. \quad \text{as} \quad K \to \infty \; . \tag{3.4.18}$$

On the other hand, $\eta_i \to 0$ as $m \to 0$ in distribution, and therefore we have for all $K \in \mathbb{N}$ even for the unnormalized sums

$$\sum_{i=1}^{K} \eta_i \to 0 \quad \text{in distr.} \quad \text{as} \quad m \to 0 \; .$$

This implies that the limiting behaviour of the empirical mean as $K \to \infty$ and $m \to 0$ depends on the order of limits. Thus we are interested in the joint limit $K_m \to \infty$ as $m \to 0$ to identify the scale on which the law of large numbers changes behaviour. It turns out that there are two interesting scales for K_m.

Proposition 3.4.5. *Let $\kappa_m = \frac{-1}{m \log m}$. Then as $m \to 0$ we have (in distr.)*

$$\Delta_{K_m} := \sum_{i=1}^{K_m} \big(1 - \delta_{0,\eta_i}\big) \longrightarrow \begin{cases} 0 & , \; K_m \ll \kappa_m \\ W_\delta & , \; K_m/\kappa_m \to \delta \in (0,\infty) \\ \infty & , \; K_m \gg \kappa_m \end{cases} , \tag{3.4.19}$$

where $W_\delta \sim Poi(\delta/2)$ is a Poisson random variable with mean $\delta/2$. In the last case, $\Delta_{K_m} = \frac{K_m}{2\kappa_m}\big(1 + o(1)\big)$.
Furthermore, on the larger scale $1/m \gg \kappa_m$ we have

$$S_{K_m} = \frac{1}{K_m} \sum_{i=1}^{K_m} \eta_i \longrightarrow \begin{cases} 0 & , \; K_m \ll 1/m \\ X_\gamma & , \; K_m m \to \gamma \in (0,\infty) \\ \rho & , \; K_m \gg 1/m \end{cases} , \tag{3.4.20}$$

where $X_\gamma \sim \mathrm{Gamma}\big(\frac{\gamma}{2}, \frac{2\rho}{\gamma}\big)$ is a Gamma random variable with mean ρ.

Proof. Denote the probability of $\eta_i > 0$ by

$$p_m := 1 - \nu_\rho^{(m)}(0) = -\frac{m}{2} \log m\big(1 + o(1)\big) = \frac{1}{2\kappa_m}\big(1 + o(1)\big) , \tag{3.4.21}$$

with asymptotics for $m \to 0$. Then $1 - \delta_{0,\eta_i} \sim \mathrm{Be}(p_m)$ are i.i.d. Bernoulli random variables and therefore $\Delta_{K_m} \sim \mathrm{Bi}(K_m, p_m)$ is a Binomial with

$$\mathbb{P}(\Delta_{K_m} = n) = \binom{K_m}{n} p_m^n (1 - p_m)^{K_m - n} ,$$

counting the non-zero contributions to the sum S_{K_m}. $p_m \to 0$ as $m \to 0$ with asymptotics given in (3.4.21), and (3.4.19) is a well-known scaling result for Binomial r.v.s. Since the rescaled random variables $(1 - \delta_{0,\eta_i})/p_m$ have mean 1, we have by the ususal law of large numbers

$$\frac{\Delta_{K_m}}{K_m p_m} = \frac{1}{K_m} \sum_{i=1}^{K_m} \frac{1}{p_m} (1 - \delta_{0,\eta_i}) \to 1 .$$

This holds whenever $K_m p_m \to \infty$ or, equivalently, $K_m \gg \kappa_m$ since the sum will have infinitely many non-zero contributions, and implies that $\Delta_{K_m} = \frac{K_m}{2\kappa_m}\big(1 + o(1)\big)$.

Analogous to (3.4.2) we get for the characteristic function of η_i

$$\chi_\eta(t) = \mathbb{E}\big(e^{it\eta_1}\big) = \left(\frac{1-\varphi}{1-e^{it}\varphi}\right)^{m/2} \quad .$$

For the rescaled sum S_{K_m} of K_m independent r.v.s we get

$$\chi_S(t) = \chi_\eta(t/K_m)^{K_m} = \left(1 + \frac{2\rho}{m}\left(1 - e^{it/K_m}\right)\right)^{-K_m m/2} \quad ,$$

where we used $\rho = \rho_m(\varphi) = \frac{m}{2}\frac{\varphi}{1-\varphi}$ as in (3.4.3) to fix the density. As $K \to \infty$ we have for all complex $z \neq 0$

$$1 - z^{1/K} = -\frac{1}{K}\log z\,\big(1 + o(1)\big) \ . \tag{3.4.22}$$

This leads to the asymptotics

$$\chi_S(t) = \left(1 - \frac{2\rho}{K_m m}\,it\right)^{-K_m m/2} \big(1 + o(1)\big) \ ,$$

since the correction terms from (3.4.22) are of order $1/K_m \ll 1/(mK_m)$. Therefore, as $m \to 0$

$$\chi_S(t) \longrightarrow \begin{cases} 1 & ,\ mK_m \to 0 \\ e^{it\rho} & ,\ mK_m \to \infty \end{cases} \quad ,$$

which implies the weak law of large numbers in the two extreme cases of (3.4.20). In the intermediate case $mK_m \to \gamma$ we have

$$\chi_S(t) \to \left(1 - \frac{2\rho}{\gamma}\,it\right)^{-\gamma/2} \quad ,$$

which is the characteristic function of a $\text{Gamma}\big(\frac{\gamma}{2}, \frac{2\rho}{\gamma}\big)$ random variable. □

This result leads to the following interpretation for the limiting behaviour of S_{K_m} as $m \to 0$.

a) $K_m \ll \kappa_m$: There are no non-zero contributions to S_{K_m} and even the unnormalized sum $K_m S_{K_m} \to 0$.

b) $K_m \sim \kappa_m$: There is a finite (Poisson distributed) number of non-zero contributions to S_{K_m}, but still $S_{K_m} \to 0$. Since the law of these contributions becomes degenerate as $m \to 0$ (cf. (3.4.9)) we have no scaling law for $K_m S_{K_m}$.

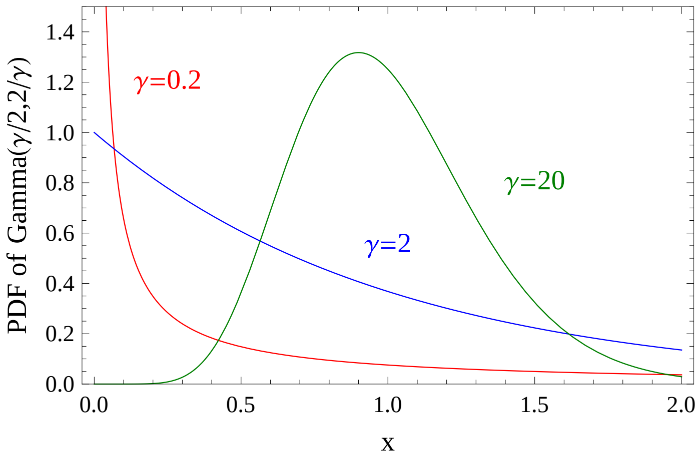

Figure 3.3: The limit distribution of S_{K_m} as $m \to 0$ with $mK_m \to \gamma$, given by the PDF of a Gamma$\left(\frac{\gamma}{2}, \frac{2\rho}{\gamma}\right)$ random variable (3.4.20). In all cases $\rho = 1$, and increasing γ (3 values shown) interpolates between the deterministic limits 0 and ρ.

c) $\kappa_m \ll K_m \ll 1/m$: S_{K_m} has an infinite number of non-zero contributions, but still vanishes as $m \to 0$.

d) $K_m \sim 1/m$: S_{K_m} has a random limiting value (Gamma distributed) with mean ρ, and infinitely many non-zero contributions. This interpolates between the deterministic limits 0 and ρ, as shown in Fig. 3.3.

e) $K_m \gg 1/m$: The usual weak law of large numbers holds, i.e. $S_{K_m} \to \rho$ as $m \to 0$.

If we interpret $\eta_1, \eta_2, \ldots$ as a configuration of the inclusion process, this result gives detailed information about the structure of such configurations as $m \to 0$. They are in direct analogy to results in [18] on a particular zero-range process, which have just been formulated in an inverted fashion corresponding to a parameter $m_K \to 0$ in the limit $K \to \infty$.

3.5 The Brownian energy process

In [1] we introduced the Brownian energy process with parameter $m > 0$ (abbreviation BEP(m)), and explained how, for integer values of m it is related to the Brownian momentum process with m momenta per site.

More precisely, the BEP(m) is an interacting diffusion process on $\Omega_N = [0,\infty)^{1,\dots,N}$ with generator

$$\begin{aligned} Lf(x) &= \sum_{i=1}^{N-1} 4x_i x_{i+1} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right)^2 \\ &\qquad -2m(x_i - x_{i+1}) \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right) , \end{aligned} \tag{3.5.1}$$

where for a configuration of "energies" $x \in \Omega_N$, x_i denotes the energy at site $i \in \{1,\dots,N\}$.

In [3] we introduced an asymmetric version of the Brownian momentum process. This model was later studied in [4]. Motivated by this asymmetric modification of the Brownian momentum process, we now introduce an asymmetric version of BEP(m) via its generator

$$\begin{aligned} Lf(x) &= \sum_{i=1}^{N-1} 4x_i x_{i+1} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right)^2 \\ &\qquad -2m(x_i - x_{i+1}) \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right) \\ &\qquad -2E x_i x_{i+1} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{i+1}}\right) . \end{aligned} \tag{3.5.2}$$

We focus on a one-dimensional nearest-neighbour lattice as for the ASIP in Section 3, but the definition could of course be generalized to arbitrary geometries. Obviously, the total energy $f(x) = \sum_{i=1}^{N} x_i$ is conserved, and for $E > 0$ the process has a drift to the left, which can most easily be seen from the stationary measures discussed in the next section.

3.5.1 Condensation in the ABEP

We first consider $m = 2$, $E > 0$, and two sites. This is the simplest case because the marginals of the stationary distribution are exponential, which makes explicit computations simple. The generalization to $m > 0$ and more sites is easy.

The generator, written in the variables $(x_1, x_2) =: (u, v)$, then reads:

$$\begin{aligned} L &= 4uv\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)^2 \\ &\quad -4(u-v)\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right) \\ &\quad +2Euv\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right) \end{aligned}$$

The adjoint (in $L^2(\mathbb{R}, dx)$) is given by (the closure of the operator)

$$\begin{aligned} L^* &= 4uv\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)^2 \\ &\quad -4(u-v)\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right) \\ &\quad +2E(u-v) - 2Euv\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right) \ . \end{aligned}$$

As an ansatz for the density of the stationary distribution we put

$$f(u, v) = abe^{-au}e^{-bv} \tag{3.5.3}$$

with $a, b > 0$. Plugging this in the equation for the stationary density $L^* f = 0$ gives

$$4uv(b-a)^2 + 4(v-u)(b-a) + 2E(u-v) - 2Euv(b-a) = 0 \ ,$$

which leads to

$$b = a + \frac{E}{2} \tag{3.5.4}$$

and

$$f(u, v) = a(a + E/2)e^{-au}e^{-av}e^{-Ev/2} \ . \tag{3.5.5}$$

In order to state our condensation result, denote by (U_K, V_K) the pair (U, V) with probability density (3.5.5) conditioned on $U + V = K$. We then have the following result, which should be thought of as the analogue of Theorem 3.3.1, but now in continuous state space setting.

Theorem 3.5.1. *a) As $K \to \infty$, V_K converges in distribution to a random variable with exponential distribution with parameter $E/2$, i.e., with probability density $(E/2)e^{-u(E/2)}$.*

b) As $K \to \infty$, $U_K/K \to 1$ almost surely.

Proof. The proof is a direct computation. Put $\lambda = a + E/2$, $\lambda' = a$, then $\lambda > \lambda'$, $\lambda - \lambda' = E/2$. First note that the distribution of $U + V$ has probabily density $\frac{\lambda\lambda'}{\lambda-\lambda'}(e^{-\lambda' x} - e^{-\lambda x})$.

Next, the conditional density of V given $U + V = K$ is given by

$$\frac{\lambda\lambda'(\lambda-\lambda')e^{-\lambda u}e^{-\lambda'(K-u)}}{\lambda\lambda'(e^{-\lambda' K} - e^{-\lambda K})} = \frac{(\lambda-\lambda')e^{-(\lambda-\lambda')u}}{1 - e^{-(\lambda-\lambda')K}}$$

which converges, as $K \to \infty$ to $(\lambda-\lambda')e^{-(\lambda-\lambda')u}$, implying statement a) of the theorem. To prove statement b): choose $0 < \delta < 1$, then

$$\begin{aligned}
\mathbb{P}\big(U \leq (1-\delta)K \big| U + V = K\big) &= \frac{\int_0^{(1-\delta)K} \lambda\lambda'(\lambda-\lambda')e^{-x\lambda'}e^{-(K-x)\lambda}dx}{\lambda\lambda'(e^{-\lambda' K} - e^{-\lambda K})} \\
&= \frac{(\lambda-\lambda')\int_0^{(1-\delta)K} e^{-x(\lambda-\lambda')}dx}{e^{(\lambda-\lambda')K} - 1} = \frac{e^{(\lambda-\lambda')K(1-\delta)} - 1}{e^{(\lambda-\lambda')K} - 1} \\
&= \left(e^{-\delta(\lambda-\lambda')}\right)^K \frac{1 - e^{(\lambda'-\lambda)K}}{1 - e^{(\lambda'-\lambda)K}} \to 0
\end{aligned}$$

as $K \to \infty$. As in the proof of Theorem 3.3.1 the bound is summable in K if we choose $\delta = 1/\sqrt{K}$ and $U_K/K \leq 1$ by definition, which implies almost sure convergence. □

To generalize the previous computation to the case of N sites and general parameter $m > 0$, it is easy to check along the lines of the proof of Theorem 3.2.1 that the process with generator (3.5.2) has a stationary measure which is a product of Gamma distributions with identical shape parameter m and site-dependent location parameter. More precisely, the PDF is given by

$$f(x_1, \ldots, x_N) = \prod_{i=1}^{N} \frac{a_i^{m/2} x_i^{m/2-1} e^{-a_i x_i}}{\Gamma(m/2)} \tag{3.5.6}$$

with

$$a_i = a + \frac{(i-1)E}{2} \tag{3.5.7}$$

for $i \in \{1, \ldots, N\}$. After conditioning on the sum $X_1 + \ldots + X_N = K$ we find, again by simple explicit computation, in the limit $K \to \infty$ that X_1/K converges to 1 almost surely, and that for $i = 2, \ldots, N$, the law of X_i converges to a shifted Gamma distribution with density

$$\lim_{K\to\infty} f_{X_i|X_1+\ldots X_N=K}(x_i) = C_i x_i^{\frac{m}{2}-1} e^{-\frac{(i-1)Ex_i}{2}} \tag{3.5.8}$$

where $C_i = (\frac{i-1}{2}e)^{\frac{m}{2}}/\Gamma(\frac{m}{2})$ is a normalization constant.

The interpretation of this result is the same as in the discrete case for the ASIP. Here almost all energy concentrates on the lattice site with the heaviest tail in the stationary distribution.

3.5.2 Generalizations

Exactly as in the case of condensation in the ASIP (section 3.2), we can formulate a more general condensation result for independent random variables $X_1, \ldots, X_N$ with values in $[0, \infty)$ and marginal densities

$$f_{X_i}(x) = \frac{1}{z_i(\mu)}\, e^{-\lambda_i x} w_i(x)\, e^{\mu x}\ . \tag{3.5.9}$$

where $0 < \lambda_1 < \min_{j=2}^N \lambda_j$. Here a notation with so-called chemical potentials $\mu \in \mathbb{R}$ is more convenient than the fugacity variable $\varphi = e^\mu$ used for the SIP, and values $-\infty < \mu < \mu_c := \lambda_1$ are possible. The normalization

$$z_i(\mu) = \int_0^\infty e^{-\lambda_i x} w_i(x)\, e^{\mu x}\, dx$$

is finite for $\mu < \mu_c$, and for indices $i < N$ also $z_i(\mu_c) < \infty$. The $w_i : [0, \infty) \to [0, \infty)$ are subexponential in the sense that for all $y \in \mathbb{R}$

$$\lim_{x\to\infty} \frac{w(x+y)}{w(x)} = 1\ . \tag{3.5.10}$$

The proof of this result follows the same steps as the proof of the analogous discrete result, except that we have to replace sums by integrals. As this is a straightforward extension, we leave the proof to the reader.

Theorem 3.5.2. *Denote by $(Y_1^K, \ldots, Y_N^K)$ the random variables $(X_1, \ldots, X_N)$ conditioned on $X_1 + \ldots + X_N = K$. Then under the above conditions (3.5.9) and (3.5.10) we have as $K \to \infty$:*

a) Condensation on the site with the heaviest tail, i.e.

$$\frac{Y_1^K}{K} \to 1 \quad \textit{almost surely} \; ;$$

b) Convergence to the critical distribution with $\mu = \mu_c$ for other sites, i.e.

$$(Y_2^K, \dots, Y_N^K) \to (Y_2, \dots, Y_N) \quad \textit{in distribution} \; ,$$

where the Y_i are independent with densities

$$f_{Y_i}(y) = \frac{1}{z_i(\mu_c)} e^{-\lambda_i y} e^{\mu_c y} w_i(y) \; .$$

Remark 3.5.3. *In the limit $m \to 0$, also for spatially homogeneous Brownian energy processes there will be a condensation phenomenon as $m \to 0$ completely analogous to the results in Section 4 for the inclusion process. Indeed, for a fixed average energy $\rho > 0$ (taking $a_i = m/(2\rho)$ in (3.5.6)), the marginal densities of the stationary product measure are*

$$f_{X_i}(x_i) = \frac{1}{\Gamma(m/2)} \left(\frac{m}{2\rho}\right)^{m/2} x_i^{m/2-1} e^{-m x_i/(2\rho)} \; .$$

Analogous to Theorem 3.4.1 one can easily show that this implies

$$\mathbb{P}(X_i < \delta) = \int_0^\delta f_{X_i}(x_i)\, dx_i \to 1$$

for all $\delta > 0$ as $m \to 0$, so that $X_i \to 0$ in probability. Further, all statements following from Theorem 3.4.1 in Section 4 can be derived in an appropriate version for continuous variables.

3.6 Conclusion

We have studied condensation phenomena for random variables with exponential tails, which arise in the inclusion process and related particle systems. In general, condensation can be due to the presence of subexponential tails resulting from a strong particle attraction, which has been studied in detail in the context of zero-range processes [5, 11, 12, 13, 14, 15]. For exponential tails considered in this work, the attraction between particles alone is not strong enough and a second ingredient is needed for condensation.

One possibility are spatial inhomogeneities, which will lead to a non-zero fraction of the particles to cluster on the sites with the heaviest tails in the limit of infinitely many particles. Our result on this in Section 3 applies in great generality, extending also previous related work on zero-range process [7, 8, 9, 10]. For homogeneous systems, varying a system parameter can induce condensation for fixed total particle density as studied in Section 4 for the inclusion process. Previous results in that direction include [18, 19] for zero-range processes and also [22] for a continuous mass model. The Brownian energy process studied in Section 5 provides an interesting example where both versions of condensation can be studied in a system with continuous state space and dynamics. Condensation for continuous variables has been studied before in the random average process [22] and mass transport models [23, 24], all of which use a discontinuous redistribution of mass (or energy) following a jump process.

To summarize, inclusion processes and related systems such as the BEP provide a rich class of models that exhibit condensation phenomena of several kinds in the presence of exponential tails, the description of which applies also in more general situations. For inhomogeneous models we have focused on finite systems, and a further question would be to consider thermodynamic limits where, for example, inhomogeneity is due to random disorder as studied in [7, 8, 9, 10] for zero-range processes. In the homogeneous case it would be of great interest to exploit duality in the SIP and BEP to get results on *the dynamics* of condensation.

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4 Weak coupling limits in a stochastic model of heat conduction

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4.0 Abstract

We study the Brownian momentum process, a model of heat conduction, weakly coupled to heat baths. In two different settings of weak coupling to the heat baths, we study the non-equilibrium steady state and its proximity to the local equilibrium measure in terms of the strength of coupling. For three- and four-site systems, we obtain the two-point correlation function and show it is generically not multilinear.

4.1 Introduction

In the study of non-equilibrium systems, exactly solvable models can serve as test-cases with which general statements about non-equilibrium, such as in [3], [11] can be tested. Recently, in [6], [7], [8], we studied the Brownian momentum process (BMP) and showed that this model is exactly solvable via duality with a particle system, the symmetric inclusion process. In this paper, we look at the close-to-equilibrium states of the BMP. First, we consider a close-to-equilibrium scenario where the temperature of the right heat bath is close to the temperature of the left heat bath, and show that the distance between the local equilibrium measure and the true non-equilibrium steady state is of order at most the square of the temperature difference, in agreement with the theory of Mc Lennan ensembles, see [11]. Next, we consider a situation where the linear chain is coupled weakly to heat baths to left and right ends (with fixed and different temperatures), and study which equilibrium measure is selected in the limit where the coupling strength λ tends to zero[1], as well as how far the true non-equilibrium steady state is from the local equilibrium measure for small coupling strengths. The temperature profile can be computed for all values of λ and is only linear in the chain including the extra sites associated to the heat baths for $\lambda = 1$, and linear if these sites are not included for all values of $\lambda > 0$. Finally, we explicitly compute the two-point correlation for all $\lambda > 0$ for a three and four sites system and show that the multilinear ansatz of the two-point function introduced in [6], see also [3], [4] fails for a system of four sites, except when $\lambda = 1$.

[1]The limit $\lambda \to 0$ is what we call here the weak coupling limit. This should not be confused with the notion of weak coupling limit in quantum systems introduced by Davis-Van Hove.

4.2 The model

The Brownian momentum process on a linear chain $\{1,\ldots,N\}$ coupled at the left and right end to a heat bath is a Markov process $\{x(t) : t \geq 0\}$ on the state space $\Omega_N = \mathbb{R}^{\{1,\ldots,N\}}$. The configuration $x(t) = x_i(t) : i \in \{1,\ldots N\}$ is interpreted as a collection of momenta associated to the sites $i \in \{1,\ldots,N\}$. The process is defined via its generator working on the core of smooth functions $f : \Omega_N \to \mathbb{R}$ which is given by

$$L = \lambda B_1 + \lambda B_N + \sum_{i,j}^{N} p(i,j) L_{i,j} \tag{4.2.1}$$

with

$$L_{i,j} = (x_i \partial_j - x_j \partial_i)^2$$

and where ∂_j is shorthand for $\frac{\partial}{\partial x_j}$. The underlying random walk transition rate $p(i,j)$ is chosen to be symmetric and nearest neighbor, i.e., $p_{i,i+1} = p_{i+1,i} = 1, i \in \{1,\ldots,N-1\}$, $p(i,j) = 0$ otherwise. Since $L_{i,j} = L_{j,i}$ the symmetry of $p(i,j)$ is no loss of generality.

The boundary operators B_1, B_N model the contact with the heat baths, and are chosen to be Ornstein-Uhlenbeck generators corresponding to the temperatures of the left and right heat bath, i.e.,

$$B_1 = T_L \partial_1^2 - x_1 \partial_1$$

$$B_N = T_R \partial_N^2 - x_N \partial_N$$

Finally, $\lambda > 0$ measures the strength of the coupling to the heat baths. The process with generator (4.2.1) is abbreviated as BMP_λ.

If $T_L = T_R = T$, then, for all $\lambda > 0$, the unique stationary measure of the process $\{x(t) : t \geq 0\}$ is the product of Gaussian measures with mean zero and variance T. If $T_L \neq T_R$ there exists a unique stationary measure; the so-called non-equilibrium steady state denoted by $\mu^\lambda_{T_L,T_R}$. The existence and uniqueness of the measure $\mu^\lambda_{T_L,T_R}$ follows from duality (see next section).

We will look at two different close-to-equilibrium scenarios:

1. $\lambda = 1$, $T_R = T_L + \epsilon$ and $\epsilon \to 0$,
2. $T_L \neq T_R$, and $\lambda \to 0$.

In both cases we look at the behavior of the measure $\mu^{\lambda}_{T_L,T_R}$, in case two, as $\lambda \to 0$, and in case one as $\epsilon \to 0$. Since for $\lambda = 0$, the system has infinitely many equilibrium measures, in the second case it is of interest to find out which of these measure is selected in the limit $\lambda \to 0$. Both in the first and second case, we want to understand how close the true non-equilibrium steady state is to the local equilibrium measure.

4.3 Duality

The BMP_λ can be analyzed via duality. The dual process is an interacting particle system, the so-called symmetric inclusion process [8], where particles are jumping on the lattice $\{0, 1, \ldots, N, N+1\}$ and interacting by "inclusion" (i.e., particles at site i can attract particles at site j). The "extra sites" $0, N+1$ -associated to the heat baths- are absorbing. I.e., a dual particle configuration is a map

$$\xi : \{0, \ldots, N+1\} \to \mathbb{N}$$

specifying at each site the number of particles present at that site. The space of dual particle configurations is denoted by Ω^d_N. For $\xi \in \Omega^d_N$, $\xi^{i,j}$ denotes the configuration obtained from ξ by removing a particle from i and putting it at j.

The generator of the dual process then reads

$$\begin{aligned} & L_d\varphi(\xi) = 2\lambda\xi_1[\varphi(\xi^{1,0}) - \varphi(\xi)] + \\ + \quad & \sum_{i,j=1}^{N-1} p(i,j)\left(2\xi_j(2\xi_i+1)[\varphi(\xi^{j,i}) - \varphi(\xi)] + 2\xi_i(2\xi_j+1)[\varphi(\xi^{i,j}) - \varphi(\xi)]\right) \\ & +2\lambda\xi_N[\varphi(\xi^{N,N+1}) - \varphi(\xi)]. \end{aligned} \tag{4.3.1}$$

In words, this means particles at site i jump to j at rate $2p(i,j)(2\xi_j + 1)$. At the boundary site 1 (resp. N) particles can jump at rate 2λ to the site 0 (resp. $N+1$) where they are absorbed. Absorbed particles do not interact with non-absorbed ones. The dual process is abbreviated as SIP_λ. The duality functions for duality between BMP_λ and SIP_λ are independent of λ and given by

$$D(\xi, x) = T_L^{\xi_0} T_R^{\xi_{N+1}} \prod_{i=1}^{N} \frac{x_i^{2\xi_i}}{(2\xi_i - 1)!!}$$

for $\xi \in \Omega^d_N$ a dual particle configuration, and $x \in \Omega_N$.

The duality relation then reads

$$LD(\xi, x) = L_d D(\xi, x) \tag{4.3.2}$$

where L works on x and L_d on ξ. The derivation of (4.3.2) is the same as in [6]. By passing to the semigroup, from (4.3.2) we obtain the duality relation

$$\mathbb{E}_x D(\xi, x(t)) = \mathbb{E}^d_\xi D(\xi(t), x) \tag{4.3.3}$$

where $\mathbb{E}_x$ is expectation in BMP_λ starting from $x \in \Omega_N$, and $\mathbb{E}^d_\xi$ is expectation in SIP_λ starting from $\xi \in \Omega^d_N$.

For $\xi \in \Omega^d_N$ we denote $|\xi| = \sum_{i=0}^{N+1} \xi_i$ the total number of particles in ξ. Since eventually all particles in a particle configuration $\xi \in \Omega^d_N$ will be absorbed, we have a unique stationary distribution $\mu^\lambda_{T_L,T_R}$ with

$$\int D(\xi, x)\mu^\lambda_{T_L,T_R}(dx) = \sum_{k,l:k+l=|\xi|} T_L^k T_R^l \mathbb{P}^d_\xi (\xi(t=\infty) = k\delta_0 + l\delta_{N+1}) \tag{4.3.4}$$

where $\xi(t = \infty)$ denotes the final configuration when all particles are absorbed and $k\delta_0 + l\delta_{N+1}$ the configuration with k particles at 0 and l particles at $N + 1$.

4.4 Temperature profile

The local temperature at site $i \in \{1, \dots, N\}$ is defined as

$$T_i = \int x_i^2 \mu^\lambda_{T_L,T_R}(dx)$$

and by definition $T_0 = T_L, T_{N+1} = T_R$. We say that the temperature profile is linear in the lattice interval $[K, L]$ if there exist $a, b \in \mathbb{R}$ with $T_i = ai + b$, for all $i \in [K, L]$. For the computation of the temperature profile we only need a single dual walker, which performs a continuous-time random walk with rates $2p(i, j)$ and absorption at rate 2λ from the sites $1, N$.

Indeed, using (4.3.4) we have

$$T_i = T_L \mathbb{P}^d_{\delta_i} (\xi(\infty) = \delta_0) + T_R \left(1 - \mathbb{P}^d_{\delta_i} (\xi(\infty) = \delta_0)\right) \tag{4.4.1}$$

From this expression, one obtains the following equations for the temperature profile:

$$\begin{aligned}
\sum_{i=1}^{N} p(i,1)T_i &= T_1 - \lambda(T_L - T_1) \\
\sum_{i=1}^{N} p(i,k)T_i &= T_k \\
\sum_{i=1}^{N} p(i,N)T_i &= T_N - \lambda(T_R - T_N)
\end{aligned} \tag{4.4.2}$$

The second equation expresses that the temperature profile is a harmonic function of the transition probabilities, whereas the first and third equation are boundary conditions. In the case $\lambda = 1$ and p corresponding to the simple nearest neighbor random walk, the equation for $T_i, i = 0, \ldots, N$ is the discrete Laplace equation, which gives a linear temperature profile in $[0, N+1]$.

Remark 4.4.1. *In this paper we restrict to the symmetric nearest neighbor walk kernel $p(i,j)$. The equations (4.4.2) hold for general symmetric $p(i,j)$. However, in the cases where it is not translation-invariant and/or not nearest neighbor, the temperature profile will not be linear.*

We have the following theorem that follows immediately from the equations (4.4.2).

Theorem 4.4.2. *For all $\lambda > 0$, the temperature profile is linear in $[1, N]$ and is given by*

$$T_i = ai + b \tag{4.4.3}$$

$i = 1, \ldots, N$ *with*

$$a = \frac{\lambda(T_R - T_L)}{\lambda(N-1) + 2}$$

$$b = \frac{T_L + T_R + \lambda(NT_L - T_R)}{\lambda(N-1) + 2}$$

We can now look at different limiting cases:

1. In the case $\lambda = 1$ we recover the result from [6]:

$$T = T_L + \frac{T_R - T_L}{N+1} i$$

In this case (only) the temperature profile is linear in $[0, N+1]$.

2. In the limit $\lambda \to 0$ we obtain for all $i \in \{1, \ldots, N\}$

$$\lim_{\lambda \to 0} T_i^{(\lambda)} = \frac{T_L + T_R}{2}$$

3. In the limit $\lambda \to \infty$ we obtain $T_1 = T_L, T_N = T_R$ and the profile is linear in $[1, N]$, similar to a system with $\lambda = 1$ and $N - 2$ sites.
4. In the limit $N \to \infty$, such that $i/N \to r \in [0, 1]$ fixed,

$$\lim_{N \to \infty, \frac{i}{N} \to r} T_i = T_L + r(T_R - T_L)$$

This means that the macroscopic profile is linear and does not depend on λ.

Remark 4.4.3. *The expectation of the heat current in the steady state in the system is $J = T_i - T_{i-1}$. This can be seen from computing the effect of the generator on x_i^2 (the local energy) and writing it in the form of a discrete gradient of the quantity $J_i = x_i^2 - x_{i-1}^2$ which is then defined to be heat current at site i. Heat conductivity κ is defined via the equation $J = \kappa \Delta T$. From Theorem 4.4.2 it follows that $\kappa = \frac{\lambda}{\lambda(N-1)+2}$ which is independent of the temperature (i.e. the system obeys the Fourier's law for all values of $\lambda > 0$).*

4.5 The stationary measure for $\epsilon \to 0$

We consider the first weak coupling setting, i.e, $\lambda = 1$, $T_R = T_L + \epsilon$. We will prove that up to corrections of order ϵ^2, the stationary measure is given by a product of Gaussian measures corresponding to the temperature profile, i.e., the local equilibrium measure.

Let us denote this local equilibrium measure

$$d\nu_{T_L, T_R} = \otimes_{i=1}^N G_{T_i}(x_i) dx_i$$

with T_i given by (4.4.3),

$$G_T(x) = \frac{1}{\sqrt{2\pi T}} \exp(-x^2/2T)$$

and $\mu_{T_L, T_L + \epsilon}$ the true non-equilibrium steady state (with $\lambda = 1$). Then we have the following result.

Theorem 4.5.1. *The true non-equilibrium steady state $\mu_{T_L,T_L+\epsilon}$ and the local equilibrium measure $\nu_{T_L,T_L+\epsilon}$ are at most order ϵ^2 apart, i.e., there exists $\epsilon_0 > 0$ such that for all $\xi \in \Omega_N^d$ there exists a constant $C = C(\xi) < \infty$ such that for all $0 \leq \epsilon \leq \epsilon_0$ we have*

$$\left| \int D(\xi,x)\mu_{T_L,T_L+\epsilon}(dx) - \int D(\xi,x)\nu_{T_L,T_L+\epsilon}(dx) \right| \leq C(\xi)\epsilon^2 \tag{4.5.1}$$

Proof. For the local equilibrium measure we have

$$\int D(\xi,x)\nu_{T_L,T_L+\epsilon}(dx) = \prod_{i=1}^{N} T_i^{\xi_i} \tag{4.5.2}$$

expanding this up to order ϵ we find,

$$\prod_i T_i^{\xi_i} = \prod_i \left(T_L + \frac{\epsilon i}{N+1}\right)^{\xi_i} = T_L^{|\xi|}\left(1 + \frac{\epsilon}{T_L(N+1)}\sum_i i\xi_i\right) + O(\epsilon^2)$$

Start now from (4.3.4) and expand up to order ϵ:

$$\begin{aligned}
& \int D(\xi,x)\mu_{T_L T_L+\epsilon}(dx) \\
= \quad & T_L^{|\xi|}\left(1 + \frac{\epsilon}{T_L}\sum_{k,l:k+l=|\xi|} l\mathbb{P}_\xi^d(\xi(\infty) = k\delta_0 + l\delta_{N+1})\right) + O(\epsilon^2)
\end{aligned} \tag{4.5.3}$$

Upon identification of (4.5.2) and (4.5.3) we see that we have to prove

$$\begin{aligned}
\sum_{k,l:k+l=|\xi|} l\mathbb{P}_\xi^d(\xi(\infty) = k\delta_0 + l\delta_{N+1}) &= \mathbb{E}_\xi(\xi_\infty(N+1)) \\
&= \frac{1}{(N+1)}\sum_{i=0}^{N+1} i\xi_i =: \psi(\xi)
\end{aligned} \tag{4.5.4}$$

The function $\varphi(\xi) := \mathbb{E}_\xi(\xi_\infty(N+1))$ is the harmonic function for the dual process, i.e.,

$$L_d\varphi = 0$$

which satisfies the boundary conditions

$$\varphi\left(k\delta_0 + \sum_{i=1}^{N}\xi_i\delta_i + l\delta_{N+1}\right) = \varphi\left(\sum_{i=1}^{N}\xi_i\delta_i\right) + l \tag{4.5.5}$$

Therefore, it suffices to show that

$$\frac{1}{(N+1)} \sum_{i=0}^{N+1} i\xi_i =: \psi(\xi)$$

both satisfies

$$L_d \psi = 0$$

and the boundary conditions (4.5.5). That ψ satisfies the boundary conditions is immediately clear. The fact that ψ is harmonic follows from explicit computation:

$$\begin{aligned}
L_d\psi(\xi) &= 2\xi_1[\psi(\xi^{1,0}) - \psi(\xi)] \\
&+ \sum_{i=1}^{N-1} \big(2\xi_{i+1}(2\xi_i+1)[\psi(\xi^{i+1,i}) - \psi(\xi)] + 2\xi_i(2\xi_{i+1}+1)[\psi(\xi^{i,i+1}) - \psi(\xi)]\big) \\
&+ 2\xi_N[\psi(\xi^{N,N+1}) - \psi(\xi)] \\
&= \frac{1}{N+1}\Big(2\xi_1[-1] + \\
&+ \sum_{i=1}^{N-1} (2\xi_{i+1}(2\xi_i+1)[-1] + 2\xi_i(2\xi_{i+1}+1)[+1]) \\
&+ 2\xi_N[+1]\Big) \\
&= \frac{1}{N+1}\left(2\xi_1[-1] + +2\sum_{i=1}^{N-1}(\xi_i - \xi_{i+1}) + 2\xi_N[+1]\right)
\end{aligned}$$

and since $\sum_{i=1}^{N-1}(\xi_i - \xi_{i+1}) = \xi_1 - \xi_N$ we indeed have

$$L_d\psi(\xi) = 0$$

□

4.6 The case $\lambda \to 0$

Next, we consider the second weak coupling setting, i.e., we fix $T_L \neq T_R$ and study the behavior of the measure $\mu^{\lambda}_{T_L,T_R}$ as a function of λ.

In this case, the local equilibrium measure is the product of Gaussian measures corresponding to the temperature profile (4.4.3), i.e., we have to compare $\mu^{\lambda}_{T_L,T_R}$ with $\nu^{\lambda}_{T_L,T_R}$ where

$$d\nu^{\lambda}_{T_L,T_R} = \otimes_{i=1}^{N} G_{T_i^{\lambda}}(x_i)(dx_i)$$

where T_i^{λ} is given by (4.4.3). Denote

$$\varphi(\xi) = \int D(\xi, x)\ \mu^{\lambda}_{T_L,T_R}(dx) \tag{4.6.1}$$

then φ is the harmonic function of the dual generator satisfying the boundary conditions

$$\varphi(\xi^* = \xi + k\delta_0 + l\delta_{N+1}) = \psi(\xi).\psi(k\delta_0 + l\delta_{N+1}) = T_L^k T_R^l \psi(\xi)$$

On the other hand if we put

$$\psi(\xi) := \int D(\xi, x)\ \nu^{\lambda}_{T_L,T_R}(dx) = T_L^k T_R^l \prod_i (T_i^{(\lambda)})^{\xi_i} \tag{4.6.2}$$

then we see immediately that ψ satisfies the boundary conditions.

We will now first prove

Lemma 4.6.1. *There exists $\lambda_0 > 0$ such that for all $\xi \in \Omega_N^d$ there exists $A(\xi) > 0$ such that for all $0 < \lambda \le \lambda_0$ we have*

$$|\left(L_d\psi\right)(\xi)| \le \lambda^2 A(\xi)$$

In particular, since there is only a finite number of dual particle configurations with total number of particles equal to K, we have, for all $0 < \lambda \le \lambda_0$

$$\sup_{\xi:|\xi|=K} |\left(L_d\psi\right)(\xi)| \le C(K)\lambda^2$$

for some $C(K) > 0$

Proof. Compute

$$\begin{aligned} L_d\psi(\xi) &= 2\psi(\xi)\left(\lambda\xi_1\left(\frac{T_L}{T_1} - 1\right) + \lambda\xi_N\left(\frac{T_R}{T_N} - 1\right)\right) \\ &+ 2\psi(\xi)\left(\sum_{i=1}^{N-1}\left(\xi_{i+1}(2\xi_i + 1)\left(\frac{T_i}{T_{i+1}} - 1\right) + \xi_i(2\xi_{i+1} + 1)\left(\frac{T_{i+1}}{T_i} - 1\right)\right)\right) \end{aligned}$$

Put $T_R - T_N = T_1 - T_L =: \gamma$

$$\begin{aligned} L_d\psi(\xi) &= 2\psi(\xi)\left(\lambda\xi_1\left(\frac{-\gamma}{T_1}\right) + \lambda\xi_N\left(\frac{\gamma}{T_N}\right)\right) \\ + \quad & 2\psi(\xi)\left(\sum_{i=1}^{N-1}\left(2\xi_{i+1}\xi_i\frac{(T_i - T_{i+1})^2}{T_iT_{i+1}} + (T_i - T_{i+1})\left(\frac{\xi_{i+1}}{T_{i+1}} - \frac{\xi_i}{T_i}\right)\right)\right) \quad (4.6.3) \end{aligned}$$

Remember from Theorem 4.4.2 that $T_i = \lambda ai + b$, hence $T_i - T_{i+1} = -\lambda\alpha$, with

$$\lambda\alpha = \frac{\lambda(T_R - T_L)}{\lambda(N-1)+2}$$

$$b = \frac{T_L + T_R + \lambda(NT_L - T_R)}{\lambda(N-1)+2}$$

We find

$$\gamma = \frac{T_R - T_L}{\lambda(N-1)+2} = \alpha$$

and hence, from (4.6.3)

$$L_d\psi(\xi) = 2\psi(\xi)\left(\lambda\xi_1[\frac{-\alpha}{T_1}] + \lambda\xi_N[\frac{\alpha}{T_N}] + \sum_{i=1}^{N-1}\left(2\lambda^2\alpha^2\frac{\xi_i\xi_{i+1}}{T_iT_{i+1}} - \lambda\alpha\left(\frac{\xi_{i+1}}{T_{i+1}} - \frac{\xi_i}{T_i}\right)\right)\right)$$

We then see that the first order terms form a vanishing telescopic sum:

$$\sum_{i=1}^{N-1}\left(\frac{\xi_i}{T_i} - \frac{\xi_{i+1}}{T_{i+1}}\right) = \frac{\xi_1}{T_1} - \frac{\xi_N}{T_N}$$

and therefore;

$$L_d\psi(\xi) = 4\lambda^2 a^2\psi(\xi)\sum_{i=1}^{N-1}\left(\frac{\xi_{i+1}\xi_i}{T_iT_{i+1}}\right)$$

□

Given this result, we will prove that the measures $\nu^\lambda_{T_L,T_R}$ and $\mu^\lambda_{T_L,T_R}$ are at most order $O(\lambda\log(1/\lambda))$ apart as $\lambda \to 0$.

Theorem 4.6.2. *Let φ, ψ be the functions defined in* (4.6.1) *and* (4.6.2)*, then we have the following. There exists $\lambda_0 > 0$, such that for all $\xi \in \Omega_N^d$ there is $C(\xi) > 0$, such that for all $0 < \lambda \leq \lambda_0$*

$$|\varphi(\xi) - \psi(\xi)| \leq C(\xi)\lambda \log \frac{1}{\lambda} \tag{4.6.4}$$

as a consequence,

$$\lim_{\lambda \to 0} \mu^{\lambda}_{T_L,T_R} = \otimes_{i=1}^{N} G_{\frac{T_L+T_R}{2}}(x_i)\, dx_i$$

i.e., in the limit $\lambda \to 0$, the equibrium measure corresponding to temperature $(T_L + T_R)/2$ is selected.

Proof. We start with the following lemma

Lemma 4.6.3. *For all $\xi \in \Omega_N^d$ a (dual) particle configuration, there exists $c = c(\xi) > 0$, $a = a(\xi) > 0$ such that for all $\lambda > 0$, and for all $t > 0$*

$$\left| \int \mathbb{E}_x D(\xi, x_t)\ \nu^{\lambda}_{T_L,T_R}(dx) - \int D(\xi, x)\ \mu^{\lambda}_{T_L,T_R}(dx) \right| \leq ce^{-\lambda a t}$$

Proof. Using duality between BMP_λ and SIP_λ, and (4.3.4)

$$\begin{aligned}
& \left| \int \mathbb{E}_x D(\xi, x_t)\ \nu^{\lambda}_{T_L,T_R}(dx) - \int D(\xi, x)\ \mu^{\lambda}_{T_L,T_R}(dx) \right| \\
= & \left| \mathbb{E}_\xi \left(\prod_{i=1}^{N} (T_i^{(\lambda)})^{\xi_i(t)} T_L^{\xi_0(t)} T_R^{\xi_{N+1}(t)} \right) - \mathbb{E}_\xi \left(T_L^{\xi_0(\infty)} T_R^{\xi_{N+1}(\infty)} \right) \right| \\
\leq & \ C(\xi)\mathbb{P}^d_\xi(\xi(t) \neq \xi(\infty)) \\
\leq & \ C(\xi)\mathbb{P}^d_\xi(\text{ there exist particles that are not absorbed at time t}) \\
\leq & \ C(\xi)e^{-a\lambda t}
\end{aligned}$$

In order to see the last inequality, we remark that for a particle at positions $1, N$, the probability to be absorbed at the next step is of order λ, as the maximal rate to move to the other (non-absorbing) neighbor is at most $2(|\xi| + 1)$. □

Proof of Theorem 4.6.2: using Lemma 6.1, and duality between SIP_λ and BMP_λ, we have

$$\begin{aligned}
\left| \int \mathbb{E}_x D(\xi, x_t)\ \nu^{\lambda}_{T_L,T_R}(dx) - \int D(\xi, x)\ \nu^{\lambda}_{T_L,T_R}(dx) \right| &= \left| \int_0^t L_d \psi(\xi_s) ds \right| \\
&\leq C(|\xi|)\lambda^2 t
\end{aligned}$$

Combining with Lemma 6.2 we have

$$\int D(\xi, x_t)\ \nu^{\lambda}_{T_L,T_R}(dx) - \int D(\xi, x)\ \mu^{\lambda}_{T_L,T_R}(dx) \leq C(\xi)\left(\lambda^2 t + e^{-a\lambda t}\right) \qquad (4.6.5)$$

Now optimize w.r.t. t by choosing $t = (1/a\lambda)\log(a/\lambda)$ $\square$

4.7 The two point correlation functions in the limit $\lambda \to 0$

In this section we prove that for the two-point correlation function in the non-equilibrium steady state, the deviation from local equilibrium is of order λ, which strengthens (4.6.4) for $\xi = \delta_i + \delta_j$ (i.e., we get rid of the $\log(1/\lambda)$-factor). In the appendix we give explicit expressions for the two-point function of some finite systems, and show in particular that it is not multilinear for $\lambda \neq 1$.

Define for $i, j \in \{1, \ldots, N\}$

$$\mathbf{Y}_{ij} = \int (x_i^2 x_j^2) \mu^{(\lambda)}_{T_L T_R}(dx)$$

and additionally $\mathbf{Y}_{0i} = T_L T_i$, $\mathbf{Y}_{i,N+1} = T_i T_R$.

Denote by $\mathbf{T}$ the matrix with elements $\mathbf{T}_{ij} = T_i T_j$ if $i \neq j$ and $\mathbf{T}_{ij} = 3T_i^2$ if $i = j$ where T_i is the temperature profile of Theorem 4.4.2

Theorem 4.7.1. *There exists $C > 0$ such that for all $i, j \in \{1, \ldots, N\}$ we have*

$$|\mathbf{Y}_{ij} - \mathbf{T}_{ij}| \leq C\lambda \qquad (4.7.1)$$

Proof. From the stationarity of $\mu^{\lambda}_{T_L,T_R}$ we find that $\mathbf{Y}$ satisfies the following system of linear equations for $k, l \in \{1, \ldots, N\}$

$$\begin{aligned} 0 &= (-4Y_{kl} + Y_{k-1l} + Y_{k+1l} + Y_{kl-1} + Y_{kl+1}) \\ &+ 4Y_{kk+1}\delta_{kl} + 4Y_{k-1k}\delta_{kl} - 4Y_{k-1k}\delta_{k,l+1} - 4Y_{kk+1}\delta_{k,l-1} \\ &+ \lambda(T_L T_l - Y_{1l})\delta_{1k} + \lambda(T_L T_k - Y_{1k})\delta_{1l} \\ &+ \lambda(T_R T_l - Y_{Nl})\delta_{Nk} + \lambda(T_R T_k - Y_{Nk})\delta_{Nl} \end{aligned} \qquad (4.7.2)$$

which has the form

$$\mathbf{M}.\mathbf{Y} = \mathbf{D}$$

By explicit computation we obtain

$$\mathbf{X} := \mathbf{M}.\mathbf{T} - \mathbf{D} = O(\lambda^2) \tag{4.7.3}$$

From this we will now derive that

$$\mathbf{Y} = \mathbf{T} + O(\lambda). \tag{4.7.4}$$

Put

$$\|\mathbf{Y} - \mathbf{T}\| = \|\mathbf{M}^{-1}\mathbf{M}(\mathbf{Y} - \mathbf{T})\| = \|\mathbf{M}^{-1}\mathbf{X}\|$$

We will show that

$$\|\mathbf{M}^{-1}\mathbf{X}\|^2 \leq \frac{c}{\lambda^2}\|\mathbf{X}\|^2 \tag{4.7.5}$$

which combined with (4.7.3) gives the desired result (4.7.4).

To obtain (4.7.5) consider

$$\begin{aligned} < \mathbf{M}^{-1}\mathbf{X}, \mathbf{M}^{-1}\mathbf{X} > &= < \mathbf{X}, (\mathbf{M}^{-1})^T\mathbf{M}^{-1}\mathbf{X} > \\ &= < \mathbf{X}, \mathbf{A}^{-1}\mathbf{X} > \end{aligned}$$

with $\mathbf{A} := \mathbf{M}\mathbf{M}^T$ Using the spectral decomposition of $\mathbf{A}$, we get

$$\begin{aligned} < \mathbf{X}, \mathbf{A}^{-1}\mathbf{X} > &= \sum_i \frac{1}{\lambda_i^{(\mathbf{A})}} < \mathbf{X}, \mathbf{e}_i >< \mathbf{e}_i, \mathbf{X} > \\ &\leq \frac{1}{\min_i(\lambda_i^{(\mathbf{A})})}||\mathbf{X}||^2 \end{aligned}$$

where λ_i are the eigenvalues of $\mathbf{A}$ with the corresponding eigenvectors $\mathbf{e}_i$. So it suffices now to see that

$$\min_i(\lambda_i^{(\mathbf{A})}) \geq c\lambda^2$$

We have

$$\min_i(\lambda_i^{(\mathbf{A})}) = \inf_{||\mathbf{X}||=1} < \mathbf{X}, \mathbf{A}\mathbf{X} >$$

The matrix $\mathbf{M}$ has the form $\mathbf{M} = \mathbf{K} + \lambda\mathbf{S}$ and hence

$$< \mathbf{X}, \mathbf{A}\mathbf{X} >=< (\mathbf{K}^T + \lambda\mathbf{S}^T)\mathbf{X}, (\mathbf{K}^T + \lambda\mathbf{S}^T)\mathbf{X} >$$

Therefore

$$\frac{< \mathbf{X}, \mathbf{A}\mathbf{X} >}{\lambda^2} = \frac{\lambda^2 ||\mathbf{S}^T\mathbf{X}||^2 + 2\lambda < \mathbf{S}^T\mathbf{X}, \mathbf{K}^T\mathbf{X} > + ||\mathbf{K}^T\mathbf{X}||^2}{\lambda^2}$$

and so we obtain

$$\liminf_{\lambda \to 0} \frac{\min_i(\lambda_i^{(\mathbf{A})})}{\lambda^2} > 0$$

Indeed, since $\mathbf{M} \equiv \mathbf{K} + \lambda\mathbf{S}$ is not singular, either $\mathbf{K}$ or $\mathbf{S}$ must not be singular, therefore $||\mathbf{S}^T\mathbf{X}||^2$ and $||\mathbf{K}^T\mathbf{X}||^2$ cannot be both zero. $\square$

Remark 4.7.2. *It follows from the correlation inequalities derived in [8] that* $\mathbf{Y}_{ij} \geq \mathbf{T}_{ij}$. *Indeed,* $\mathbf{T}_{ij}$ *would be the correlation function if the dual walkers were walking independently, however, two dual walkers interact by inclusion (attraction), and this leads to a positive covariance.*

4.8 Acknowledgment

We would like to thank Christian Giardina for usefull discussions.

4.9 Appendix

Here we derive explicit expressions for the two point correlation function for systems with three and four sites. We start from the equations (4.7.2).

Since Y_{kl} is symmetric in k and l it suffices to consider $k \leq l$. the different cases are as follows;

1. $\mathbf{1 < k = l < N}$; $(-2Y_{kk} + 3Y_{kk-1} + 3Y_{kk+1}) = 0$
2. $\mathbf{1 < k = l - 1 < N - 1}$; $(-8Y_{kk+1} + Y_{k-1k+1} + Y_{k+1k+1} + Y_{kk} + Y_{kk+2}) = 0$
3. $\mathbf{1 < k < l + 1 < N + 1}$; $(-4Y_{kl} + Y_{k-1l} + Y_{k+1l} + Y_{kl-1} + Y_{kl+1}) = 0$
4. $\mathbf{k = l = 1}$; $(-2Y_{11} + 3Y_{10} + 3Y_{12}) + \lambda(T_L T_1 - Y_{11}) + \lambda(T_L T_1 - Y_{11}) = 0$
5. $\mathbf{k = l = N}$; $(-2Y_{NN} + 3Y_{NN-1} + 3Y_{NN+1}) + \lambda(T_R T_N - Y_{NN}) + \lambda(T_R T_N - Y_{NN}) = 0$
6. $\mathbf{1 = k = l - 1}$; $(-8Y_{12} + Y_{02} + Y_{22} + Y_{11} + Y_{13}) + \lambda(T_L T_2 - Y_{12}) = 0$
7. $\mathbf{k = l - 1 = N - 1}$; $(-8Y_{N-1N} + Y_{N-2N} + Y_{NN} + Y_{N-1N-1} + Y_{N-1N+1}) + \lambda(T_R T_{N-1} - Y_{NN-1}) = 0$

8. $\mathbf{1 = k < l + 1 < N + 1}$; $(-4Y_{1l} + Y_{0l} + Y_{2l} + Y_{1l-1} + Y_{1l+1}) + \lambda(T_L T_1 - Y_{1l}) = 0$
9. $\mathbf{1 = k < l + 1 = N + 1}$; $(-4Y_{1N} + Y_{0N} + Y_{2N} + Y_{1N-1} + Y_{1N+1}) + \lambda(T_L T_1 - Y_{1N}) + \lambda(T_R T_1 - Y_{1N}) = 0$
10. $\mathbf{1 < k < l + 1 = N + 1}$; $(-4Y_{kN} + Y_{k-1N} + Y_{k+1N} + Y_{kN-1} + Y_{kN+1}) + \lambda(T_R T_k - Y_{kN}) = 0$

4.9.1 3 Sites System

The equations for the two-point correlation function are of the form $\mathbf{M}.\mathbf{Y} = \mathbf{D}$ where

$$\mathbf{Y} = \begin{pmatrix} Y_{11} \\ Y_{12} \\ Y_{13} \\ Y_{22} \\ Y_{23} \\ Y_{33} \end{pmatrix} \text{ and } \mathbf{D} = \begin{pmatrix} -\lambda T_L T_3 - \lambda T_R T_1 \\ -3\lambda T_R T_3 \\ -\lambda T_R T_2 \\ -3\lambda T_L T_1 \\ -\lambda T_L T_2 \\ 0 \end{pmatrix}$$

and the matrix $\mathbf{M}$ can be read from the previous equations as;

$$\mathbf{M} = \begin{pmatrix} 0 & 1 & -2(1+\lambda) & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 & -(1+\lambda) \\ 0 & 0 & 1 & 1 & -(7+\lambda) & 1 \\ -(1+\lambda) & 3 & 0 & 0 & 0 & 0 \\ 1 & -(7+\lambda) & 1 & 1 & 0 & 0 \\ 0 & 3 & 0 & -2 & 3 & 0 \end{pmatrix}$$

The explicit solution is via inversion of $\mathbf{M}$. The result for $\mathbf{Y}$ and the correlation functions $\mathbf{C}_{ij} = \mathbf{Y}_{ij} - T_i T_j (1 + 2\delta_{ij})$ then reads as follows:

$$\mathbf{Y}_{11} = \frac{3\left(T_R^2(5+3\lambda) + 2T_L T_R\left(5 + 9\lambda + 2\lambda^2\right) + T_L^2\left(5 + 23\lambda + 24\lambda^2 + 4\lambda^3\right)\right)}{4(1+\lambda)^2(5+\lambda)}$$

$$\mathbf{Y}_{12} = \frac{T_R^2(5+3\lambda) + 2T_L T_R\left(5 + 4\lambda + \lambda^2\right) + T_L^2\left(5 + 13\lambda + 2\lambda^2\right)}{4\left(5 + 6\lambda + \lambda^2\right)}$$

$$\mathbf{Y}_{13} = \frac{T_L^2\left(5 + 13\lambda + 2\lambda^2\right) + T_R^2\left(5 + 13\lambda + 2\lambda^2\right) + 2T_L T_R\left(5 + 9\lambda + 12\lambda^2 + 2\lambda^3\right)}{4(1+\lambda)^2(5+\lambda)}$$

$$\mathbf{Y}_{22} = \frac{3\left(2T_LT_R\left(5+4\lambda+\lambda^2\right)+T_L^2\left(5+8\lambda+\lambda^2\right)+T_R^2\left(5+8\lambda+\lambda^2\right)\right)}{4\left(5+6\lambda+\lambda^2\right)}$$

$$\mathbf{Y}_{23} = \frac{T_L^2(5+3\lambda)+2T_LT_R\left(5+4\lambda+\lambda^2\right)+T_R^2\left(5+13\lambda+2\lambda^2\right)}{4\left(5+6\lambda+\lambda^2\right)}$$

$$\mathbf{Y}_{33} = \frac{3\left(T_L^2(5+3\lambda)+2T_LT_R\left(5+9\lambda+2\lambda^2\right)+T_R^2\left(5+23\lambda+24\lambda^2+4\lambda^3\right)\right)}{4(1+\lambda)^2(5+\lambda)}$$

and

$$\mathbf{C}_{11} = \frac{3(T_L-T_R)^2\lambda}{2(1+\lambda)^2(5+\lambda)}, \mathbf{C}_{12} = \frac{(T_L-T_R)^2\lambda}{2\left(5+6\lambda+\lambda^2\right)}$$

$$\mathbf{C}_{13} = \frac{(T_L-T_R)^2\lambda}{2(1+\lambda)^2(5+\lambda)}, \mathbf{C}_{22} = \frac{3(T_L-T_R)^2\lambda}{2\left(5+6\lambda+\lambda^2\right)}$$

$$\mathbf{C}_{23} = \frac{(T_L-T_R)^2\lambda}{2\left(5+6\lambda+\lambda^2\right)}, \mathbf{C}_{33} = \frac{3(T_L-T_R)^2\lambda}{2(1+\lambda)^2(5+\lambda)}$$

We see that for all k, l

$$\mathbf{C}_{kl} \propto \lambda(T_L-T_R)^2$$

and also $\mathbf{C}_{kl} \geq 0$.

One might be interested to see if the bi-linear ansatz introduced in [6] for the special case $\lambda = 1$ is also valid here, i.e.

$$\begin{aligned} \mathbf{Y}_{ij} &= a+bi+cj+dij \\ \mathbf{Y}_{ii} &= A+Bi+Di^2 \end{aligned} \qquad (4.9.1)$$

with the boundary conditions $\mathbf{Y}_{0i} = T_LT_i$, $\mathbf{Y}_{i,N+1} = T_iT_R$.

However, to check the validity of the ansatz we must calculate the correlation functions for a 4 sites system, since in 3 sites systems we have only 6 correlation functions which are less than the 7 constants of the ansatz.

4.9.2 4 Sites System

Similar to the calculation for the 3 site system, we have $\mathbf{M}.\mathbf{Y} = \mathbf{D}$ where

$$\mathbf{Y} = \begin{pmatrix} Y_{11} \\ Y_{12} \\ Y_{13} \\ Y_{14} \\ Y_{22} \\ Y_{23} \\ Y_{24} \\ Y_{33} \\ Y_{34} \\ Y_{44} \end{pmatrix} \text{ and } \mathbf{D} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\lambda T_L T_3 \\ -\lambda T_L T_2 \\ -3\lambda T_L T_1 \\ -\lambda T_2 T_R \\ -\lambda T_R T_3 \\ -3\lambda T_4 T_R \\ -\lambda T_L T_4 - \lambda T_1 T_R \end{pmatrix}$$

and where the matrix $\mathbf{M}$ is given by

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 & -8 & 1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & -2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & -2 & 3 & 0 \\ 0 & 1 & -3-\lambda & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -7-\lambda & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1-\lambda & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -3-\lambda & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -7-\lambda & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -1-\lambda \\ 0 & 0 & 1 & -2-2\lambda & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The solution for $\mathbf{Y}$ is

$$\mathbf{Y}_{11} = \frac{6T_R^2(12+\lambda(14+3\lambda))+6T_LT_R(24+\lambda(76+5\lambda(8+\lambda)))+T_L^2(72+3\lambda(172+3\lambda(106+\lambda(46+5\lambda))))}{(6+\lambda)(2+3\lambda)(8+\lambda(16+5\lambda))}$$
$$\mathbf{Y}_{12} = \frac{2T_R^2(1+\lambda)(12+\lambda(14+3\lambda))+T_LT_R(48+\lambda(152+\lambda(128+\lambda(44+5\lambda))))+2T_L^2(12+\lambda(74+\lambda(121+\lambda(49+5\lambda))))}{(6+\lambda)(2+3\lambda)(8+\lambda(16+5\lambda))}$$
$$\mathbf{Y}_{13} = \frac{T_L^2(1+\lambda)(24+5\lambda(4+\lambda)(5+\lambda))+T_R^2(24+\lambda(76+\lambda(41+4\lambda)))+2T_LT_R(24+\lambda(76+\lambda(109+\lambda(47+5\lambda))))}{(6+\lambda)(2+3\lambda)(8+\lambda(16+5\lambda))}$$
$$\mathbf{Y}_{14} = \frac{T_L^2(24+5\lambda(4+\lambda)(5+\lambda))+T_R^2(24+5\lambda(4+\lambda)(5+\lambda))+T_LT_R(48+\lambda(152+\lambda(314+3\lambda(46+5\lambda))))}{(6+\lambda)(2+3\lambda)(8+\lambda(16+5\lambda))}$$
$$\mathbf{Y}_{22} = \frac{3\big(2T_LT_R(24+\lambda(76+\lambda(73+3\lambda(9+\lambda))))+T_L^2(24+\lambda(124+\lambda(181+7\lambda(10+\lambda))))+T_R^2(24+\lambda(76+\lambda(77+2\lambda(12+\lambda))))\big)}{(6+\lambda)(2+3\lambda)(8+\lambda(16+5\lambda))}$$
$$\mathbf{Y}_{23} = \frac{2T_L^2(12+\lambda(5+2\lambda)(10+\lambda(8+\lambda)))+2T_R^2(12+\lambda(5+2\lambda)(10+\lambda(8+\lambda)))+T_LT_R(2+\lambda)(24+\lambda(64+\lambda(50+7\lambda)))}{(6+\lambda)(2+3\lambda)(8+\lambda(16+5\lambda))}$$
$$\mathbf{Y}_{24} = \frac{T_R^2(1+\lambda)(24+5\lambda(4+\lambda)(5+\lambda))+T_L^2(24+\lambda(76+\lambda(41+4\lambda)))+2T_LT_R(24+\lambda(76+\lambda(109+\lambda(47+5\lambda))))}{(6+\lambda)(2+3\lambda)(8+\lambda(16+5\lambda))}$$
$$\mathbf{Y}_{33} = \frac{3\big(2T_LT_R(24+\lambda(76+\lambda(73+3\lambda(9+\lambda))))+T_R^2(24+\lambda(124+\lambda(181+7\lambda(10+\lambda))))+T_L^2(24+\lambda(76+\lambda(77+2\lambda(12+\lambda))))\big)}{(6+\lambda)(2+3\lambda)(8+\lambda(16+5\lambda))}$$
$$\mathbf{Y}_{34} = \frac{2T_L^2(1+\lambda)(12+\lambda(14+3\lambda))+T_LT_R(48+\lambda(152+\lambda(128+\lambda(44+5\lambda))))+2T_R^2(12+\lambda(74+\lambda(121+\lambda(49+5\lambda))))}{(6+\lambda)(2+3\lambda)(8+\lambda(16+5\lambda))}$$
$$\mathbf{Y}_{44} = \frac{6T_L^2(12+\lambda(14+3\lambda))+6T_LT_R(24+\lambda(76+5\lambda(8+\lambda)))+3T_R^2(24+\lambda(172+3\lambda(106+\lambda(46+5\lambda))))}{(6+\lambda)(2+3\lambda)(8+\lambda(16+5\lambda))}$$

and the corresponding correlation functions are

$$\mathbf{C}_{11}=\frac{3(T_L-T_R)^2\lambda(24+\lambda(50+13\lambda))}{(6+\lambda)(2+3\lambda)^2(8+\lambda(16+5\lambda))},\mathbf{C}_{12}=\frac{(T_L-T_R)^2\lambda(1+\lambda)(24+\lambda(50+13\lambda))}{(6+\lambda)(2+3\lambda)^2(8+\lambda(16+5\lambda))}$$
$$\mathbf{C}_{13}=\frac{2(T_L-T_R)^2\lambda(1+\lambda)(12+\lambda(16+\lambda))}{(6+\lambda)(2+3\lambda)^2(8+\lambda(16+5\lambda))},\mathbf{C}_{14}=\frac{2(T_L-T_R)^2\lambda(12+\lambda(16+\lambda))}{(6+\lambda)(2+3\lambda)^2(8+\lambda(16+5\lambda))}$$
$$\mathbf{C}_{22}=\frac{3(T_L-T_R)^2\lambda(2+\lambda(4+\lambda))(12+\lambda(16+\lambda))}{(6+\lambda)(2+3\lambda)^2(8+\lambda(16+5\lambda))},\mathbf{C}_{23}=\frac{(T_L-T_R)^2\lambda(24+\lambda(86+\lambda(93+\lambda(27+2\lambda))))}{(6+\lambda)(2+3\lambda)^2(8+\lambda(16+5\lambda))}$$
$$\mathbf{C}_{24}=\frac{2(T_L-T_R)^2\lambda(1+\lambda)(12+\lambda(16+\lambda))}{(6+\lambda)(2+3\lambda)^2(8+\lambda(16+5\lambda))},\mathbf{C}_{33}=\frac{3(T_L-T_R)^2\lambda(2+\lambda(4+\lambda))(12+\lambda(16+\lambda))}{(6+\lambda)(2+3\lambda)^2(8+\lambda(16+5\lambda))}$$
$$\mathbf{C}_{34}=\frac{(T_L-T_R)^2\lambda(1+\lambda)(24+\lambda(50+13\lambda))}{(6+\lambda)(2+3\lambda)^2(8+\lambda(16+5\lambda))},\mathbf{C}_{44}=\frac{3(T_L-T_R)^2\lambda(24+\lambda(50+13\lambda))}{(6+\lambda)(2+3\lambda)^2(8+\lambda(16+5\lambda))}$$

We see once more that for all k, l

$$\mathbf{C}_{kl}\propto\lambda(T_L-T_R)^2$$

and $\mathbf{C}_{kl}\geq 0$.

Now we can directly check the validity of the bi-linear ansatz. Direct calculation shows that the diagonal part of the ansatz, i.e., $\mathbf{Y}_{ii}=A+Bi+Di^2$ is valid, but the non-diagonal part $\mathbf{Y}_{ij}=a+bi+cj+dij$ is not.

If we determine the coeficients a, b, c, d by fitting the bilinear ansatz to $\mathbf{Y}_{12}, \mathbf{Y}_{13}, \mathbf{Y}_{23}, \mathbf{Y}_{34}$, then we obtain

$$\mathbf{Y}_{14}-(a+b+4c+4d)=\frac{3(T_L-T_R)^2(-1+\lambda)\lambda^2}{(6+\lambda)(2+3\lambda)(8+\lambda(16+5\lambda))}.$$

which shows that the bilinear form can not hold for $\lambda\notin\{0,1\}$. Remark that also when $\lambda\to\infty$ the deviation from the multilinear form vanishes, which is consistent with the intuition that this limit is the same as having $\lambda=1$ in a smaller system obtained by removing the sites $1, N$.

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Nederlandse Samenvatting

In dit proefschrift bestuderen wij enkele evenwichts en niet-evenwichts stochastische modellen die exact oplosbaar zijn met de techniek van dualiteit en zelfdualiteit. Deze modellen bevatten een nieuwe klasse van deeltjessystemen die bosonisch zijn, dat wil zegen modellen met een attractieve wisselwerking tussen de deeltjes. Als gevolg van deze attractieve interactie kan in deze modellen condensatie optreden. Ons doel hierbij is de studie van modellen die een profiel hebben dat exact kan worden berekend, alsook correlatiefuncties, zoals de tweepunts correlatiefunctie van deeltjesaantallen of van de energie. De verkregen exacte uitdrukkingen kunnen dan gebruikt worden om algemene theorieën uit de niet-evenwichts statistische mechanica te testen. De modellen die in dit proefschrift zijn bestudeerd zijn van het type "interacting particle systems", alsook systemen van interagerende diffusieprocessen.

De basistechniek die we ontwikkelen om de modellen in dit proefschrift te bestuderen is dualiteit. Wij verbinden via dualiteit modellen van interagerende diffusieprocessen met interacting particle systems, zowel in de evenwichts als in de niet-evenwichts context. Omdat dualiteit een zeer krachtige methode is, is een gedeelte van het proefschrift gewijd aan de ontwikkeling van een algemeen formalisme van dualiteit, gebaseerd op symmetrieën van de generator. Dit formalisme kan gebruikt worden om duale processen en geassocieerde dualiteitsfuncties of zelfdualiteitsfuncties te vinden. Wij hebben het "Brownian Momentum Process" (BMP) en zijn duaal proces, het Symmetric Inclusion Process (SIP), bestudeerd. Met behulp van de dualiteit tussen BMP en SIP verkrijgen wij exacte analytische formules voor de correlatiefuncties van BMP. Het BMP is een model voor warmtegeleiding via stochastische diffusie van impuls, en het SIP is een interacting particle system waarbij deeltjes een random walk op het rooster $\mathbb{Z}^d$ uitvoeren en met elkaar een aantrekkende wisselwerking hebben. Wij behandelen ook verschillende andere modellen en verkrijgen resultaten die toepasbaar zijn voor een grotere klasse van modellen. De twee basismodellen (SIP en BMP) en hun veralgemeningen zijn echter een essentieel startpunt in ons werk en zullen vaak worden gebruikt als illustrerende voorbeelden.

In hoofdstuk 1 laten wij zien hoe zelfdualiteit in direct verband staat met de niet abelse symmetrieën van de generator van het Markov proces (wij zeggen dat een operator S een symmetrie van de generator L is indien hij commuteert met de generator, i.e., S.L = L.S). Met niet abels bedoelen wij dat de symmetrie niet noodzakelijk een vermenigvuldigingsoperator is, of in de taal van matrices, niet noodzakelijk een diagonale matrix. Wij laten zien dat er voor iedere symmetrie van de generator een corresponderende zelfdualteitsfunctie bestaat, en dat omgekeerd, er voor iedere zelfdualiteitsfunctie een corresponderende symmetrie van de generator is. In het geval van dualiteit tussen twee verschillende Markov processen, komt dualiteit neer op een conjugatie tussen de twee corresponderende generatoren. En dus kan dualiteit tussen twee verschillenden processen worden beschouwd als het kiezen van een nieuwe representatie van de generator. Dualiteit wordt op deze wijze direct verwant met verschillende representaties van dezelfde Lie-algebra.

Wij behandelen in hoofdstuk 1 de samenhang tussen zelfdualiteit en symmetrieën in veel grotere algemeenheid en geven verschillende nieuwe voorbeelden van dualiteiten. Voor interacting particle systems of interagerende diffusies gekoppeld aan de randen met deeltjesreservoirs of warmtebaden, zoals SIP of BMP, tonen wij op welke manier de dualiteitsfunctie moeten worden veranderd om het effect van de reservoirs aan de randen mee te nemen. Voor energie transport modellen ontdekken wij een verborgen SU(1,1) symmetrie in een grote klasse van modellen (waaronder BMP, KMP modellen) die hun dualiteit verklaren, net als de SU(2) symmetrie voor SEP. Wij bewijzen ook de SU(1,1) symmetrie van het SIP en de corresponderende zelfdualiteit.

De extra sprongen in SIP (i.e., andere dan de random walk sprongen), de zogenaamde inclusie-sprongen, veroorzaken een netto attractieve wisselwerking tussen deeltjes. Dit moeten we vergelijken met SEP waar de deeltjes een repulsieve interactie hebben (omdat ze niet op de zelfde roosterplaats mogen zijn). In meer fysische terminologie kan men zo SIP als een bosonisch tegenhanger van het fermionische SEP beschouwen.

In hoofdstuk 2 analyseren wij het SIP in detail en bewijzen het analogon van Liggett's "comparison inequality", die de verwachtingswaarde van positief definiete functies in SEP vergelijkt met deze in een systeem van onafhankelijke random walkers. Met deze comparison inequality leiden wij een aantal correlatieongelijkheden af. Zoals men intuïtief verwacht, veranderen de correlaties van negatief in SEP naar positief in SIP. Dit feit is vanuit een ander standpunt bekeken echter vrij opmerkelijk, want SIP is geen monotoon proces en positieve correlaties zijn dus niet gerelateerd aan de FKG eigenschap, zoals bijvoorbeeld in de ferromagnetische Glauber dynamica. Aangezien SIP het duale van het warmtegeleidingsmodel BMP is, kunnen de correlatieongelijkheden voor SIP direct vertaald worden naar BMP en het Brownian Energy Process. Wij bestuderen ook het algemenere niet-evenwichts geval waar het systeem in contact is met deeltjesreservoirs en waar wij de zelfdualiteit van de SIP gebruiken om een correlatie ongelijkheid voor de niet-evenwichts stationaire toestand te verkrijgen. Wij bewijzen ook in grotere algemeenheid de negatieve correlaties voor niet-evenwichts stationaire toestanden in modellen van SEP-type, waarvan sommige reeds eerder expliciet berekend werden met de matrix methode van Derrida.

In hoofdstuk 3 bestuderen wij de condensatie fenomenen en wij laten zien dat, omdat de stationaire maat van SIP exponentiële staarten heeft, de attractie tussen deeltjes alleen niet sterk genoeg is, en er een extra factor nodig is voor condensatie. Wij laten zien dat deze extra factor ruimtelijke inhomogeniteit of ook asymmetrie in een eindig of half-oneindig systeem kan zijn. Een andere mogelijkheid voor het verkrijgen van condensatie in SIP is het introduceren van een parameter m gedefinieerd als de rate van random walk jumps terwijl de intensiteit (rate) van inclusion jumps onveranderd (gelijk aan 1) blijft. Het geval $m = 0$ levert dan een zuiver inclusieproces (geen random walk, enkel attractie) op. Wij laten zien dat in de limiet $m \to 0$ in het SIP condensatie optreedt. Wij tonen ook gerelateerde condensatiefenomenen in het Brownian Energy Process (afgeleid van BMP en dus verwant met SIP) zien, wat een interessant nieuw voorbeeld geeft van condensatie in een model met continue variabelen.

In hoofdstuk 4 bestuderen wij BMP dicht bij evenwicht. Een mogelijkheid om dicht-bij-evenwicht condities te verkrijgen is het systeem in contact te brengen met twee warmtebaden aan de randen, met temperaturen die dicht bij elkaar liggen. In dit geval tonen wij aan dat de afstand tussen de lokale evenwichtsverdeling en de echte niet-evenwichts stationaire toestand hoogstens van de orde is van het kwadraat van het verschil tussen de temperaturen van de warmtebaden, in overeenstemming met de niet-rigoureuze theorie van McLennan ensembles. Een alternatieve mogelijkheid om condities dicht bij evenwicht te verkrijgen is om de verschillende temperaturen van de twee warmtebaden vast te houden maar de koppeling van het bulk systeem met de warmtebaden met een parameter λ te verzwakken. Wij bestuderen dan het gedrag van de niet-evenwichts stationaire toestand voor kleine waarden van deze kopplingsconstante λ. In het bijzonder laten wij zien welke evenwichtsmaat wordt geselecteerd als $\lambda \to 0$. Voor beide gevallen zijn de temperatuursprofielen lineair in de bulk. Wij geven ook exacte berekeningen voor de tweepunts correlatiefuncties voor finite size systemen en wij laten zien dat zij in het algemeen niet multilineair zijn. Hiermee laten we zien we dat de veelgebruikte multilineaire ansatz voor de correlatiefuncties alleen waar kan zijn als bijkomende symmetrieën aanwezig zijn, of in de macroscopische limiet.

Curriculum Vitae

Kiamars Vafayi was born in Iran in the year 1981. He completed the high-school education in 1998 and a one year science college afterwards. From 1999 to 2004 he studied the Bachelor of Science at Sharif University of Technology in Tehran majoring in Applied Physics and Solid State Physics. From 2004 to 2006 he completed the Master of Science in Physics program at Stuttgart University in the framework of International Max Planck Research School for Advanced Materials. From 2006 to 2008 he worked at the Max-Planck-Institut für Festkörperforschung as a scientific collaborator in the fields of Quantum Monte Carlo Methods in Condensed Matter Physics and later in Structure Prediction in Solid State Chemistry. From 2008 till early 2012 he worked as a PhD student in Probability Theory at the Mathematics Institute of Leiden University.

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