SPECIAL VALUES OF CANONICAL GREEN'S FUNCTIONS

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ABSTRACT. We give a precise formula for the value of the canonical Green's function at a pair of Weierstrass points on a hyperelliptic Riemann surface. Further we express the 'energy' of the Weierstrass points in terms of a spectral invariant recently introduced by N. Kawazumi and S. Zhang. It follows that the energy is strictly larger than $\log 2$. Our results generalize known formulas for elliptic curves.

1. INTRODUCTION

Let M be a compact and connected Riemann surface of genus $h \geq 1$. On M one has a canonical Kähler form μ given as follows. The space H of holomorphic differentials on M is a complex vector space of dimension h. It carries a natural hermitian inner product given by $\langle \alpha, \beta \rangle = \frac{\sqrt{-1}}{2} \int_M \alpha \wedge \overline{\beta}$ for elements α, β in H. Let $(\omega_1, \ldots, \omega_h)$ be an orthonormal basis of H. By the Riemann-Roch theorem, the ω_i do not simultaneously vanish at any point of M, and we put

$$\mu = \frac{\sqrt{-1}}{2h} \sum_{i=1}^{h} \omega_i \wedge \overline{\omega}_i \, .$$

The associated metric has been studied in many contexts, ranging from arithmetic geometry [3] [9] [10] [19] [21] to perturbative string theory [1] [2] [5] [6] [13]. The metric is known to have everywhere non-positive curvature, and the curvature vanishes at $x \in M$ if and only if x is a Weierstrass point on a hyperelliptic Riemann surface [20], Main Theorem.

In this paper we study the canonical Green's function g_μ associated to μ (see Section 2 for definitions), and in particular its special values $g_\mu(w_i, w_j)$ at pairs (w_i, w_j) of distinct Weierstrass points on a hyperelliptic Riemann surface. The starting point of our discussion is the case $h = 1$, with M given as a 2-sheeted cover of the Riemann sphere $\mathbb{CP}^1$ branched at four points, say $\alpha_1, \alpha_2, \alpha_3$ and ∞. Let w_1, w_2, w_3, o be the four critical points of M lying over $\alpha_1, \alpha_2, \alpha_3, \infty$. It is then known [7] [15] [22] that the formula

$$g_\mu(w_i, w_j) = \frac{1}{3} \log 2 + \frac{1}{12} \log \left(\frac{|\alpha_i - \alpha_j|^2}{|\alpha_i - \alpha_k| \cdot |\alpha_j - \alpha_k|} \right) \tag{1.1}$$

holds. By the translation invariance of g_μ we obtain the formula

$$\sum_{i=1}^{3} g_\mu(w_i, o) = \log 2 \, . \tag{1.2}$$

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We will return to both these formulas in the text below.

Our purpose is to generalize formulas (1.1) and (1.2) to the case of a hyperelliptic Riemann surface of genus $h \geq 2$. Let M be such a surface, given as the 2-sheeted cover of $\mathbb{CP}^1$ branched at the $2h+2$ points $\alpha_1, \ldots, \alpha_{2h+2}$ (these may or may not include ∞). We put, for each pair (i,j) of distinct indices,

$$\delta_{ij} = \frac{(\alpha_i - \alpha_j)^{2h(2h+1)} \prod_{r \neq s}(\alpha_r - \alpha_s)}{\prod_{r \neq i}(\alpha_i - \alpha_r)^{2h+1} \prod_{r \neq j}(\alpha_j - \alpha_r)^{2h+1}} \,. \tag{1.3}$$

In this formula, we disregard any difference of two roots when one of the roots is ∞. It is readily verified that δ_{ij} is invariant under the action of $\mathrm{Aut}(\mathbb{CP}^1) = \mathrm{PGL}_2(\mathbb{C})$ on the α_i. In particular δ_{ij} defines an analytic modular invariant of the pair (w_i, w_j), where w_i, w_j are the critical points of M lying over α_i, α_j. More intrinsically δ_{ij} is the discriminant of the branch set that one obtains by sending α_i, α_j to $0, \infty$ using an element of $\mathrm{Aut}(\mathbb{CP}^1)$ and by normalizing the other $2h$ roots such that their product is unity [12].

We note that there is a tight connection with the more familiar cross ratios on the α_i. Put $\eta_{ijk} = \delta_{ik}\delta_{jk}^{-1}$, and let

$$\mu_{ijkr} = (\alpha_i - \alpha_k)(\alpha_j - \alpha_r)(\alpha_j - \alpha_k)^{-1}(\alpha_i - \alpha_r)^{-1}$$

be the cross ratio of the 4-tuple $(\alpha_i, \alpha_j, \alpha_k, \alpha_r)$. Then the relation

$$\mu_{ijkr}^{2h(2h+1)} = \eta_{ijk}\eta_{ijr}^{-1} = \delta_{ik}\delta_{jk}^{-1}\delta_{jr}\delta_{ir}^{-1}$$

holds. It is straightforward to verify that for each index i, the product $\prod_{j \neq i} \delta_{ij}$ is equal to unity.

The *energy* of the Weierstrass points on M is defined to be the following invariant of M:

$$\psi(M) = \frac{1}{2h+2} \sum_{i \neq j} g_\mu(w_i, w_j) \,,$$

where the summation runs over all pairs i, j of distinct indices. Our first result is the following.

Theorem A. *Let w_i, w_j be two distinct Weierstrass points of M. Then the formula*

$$g_\mu(w_i, w_j) = \frac{1}{4h(2h+1)} \log|\delta_{ij}| + \frac{1}{2h+1}\psi(M)$$

holds.

In our second result we give an expression for $\psi(M)$. Consider for the moment an arbitrary compact and connected Riemann surface M of genus $h \geq 1$. Let Δ_μ be the Laplacian on $L^2(M,\mu)$ given by putting $\partial\overline{\partial} f = \pi\sqrt{-1}\,\Delta_\mu(f)\,\mu$ for C^∞ functions in $L^2(M,\mu)$. Let $(\phi_\ell)_{\ell=0}^\infty$ be an orthonormal basis of real eigenfunctions of Δ_μ, with corresponding eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$. We then let $\varphi(M)$ denote the invariant

$$\varphi(M) = \sum_{\ell > 0} \frac{2}{\lambda_\ell} \sum_{m,n=1}^{h} \left| \int_M \phi_\ell \, \omega_m \wedge \bar{\omega}_n \right|^2$$

of (M,μ). This fundamental invariant was introduced and studied recently by N. Kawazumi [18] and S. Zhang [24]. Note that $\varphi(M) \geq 0$, and that $\varphi(M) = 0$ if and only if M has genus one. In [17] the asymptotic behavior of the φ-invariant is determined for surfaces degenerating into a stable curve with a single node. The result shows in particular that φ can be viewed as a Weil function, with respect to the boundary, on the stable (Deligne-Mumford) compactification of the moduli space of surfaces of a fixed genus $h \geq 2$.

Theorem B. *Let M be a hyperelliptic Riemann surface of genus $h \geq 2$, and let $\varphi(M)$ be the Kawazumi-Zhang invariant of M. Then for the energy $\psi(M)$ of the set of Weierstrass points of M, the formula*

$$\psi(M) = \frac{1}{2h}\,\varphi(M) + \log 2$$

holds.

In particular we find that $\psi(M) > \log 2$. With $h = 1$, the formulas in Theorems A and B specialize to equations (1.1) and (1.2), respectively.

In order to prove Theorem B we proceed as follows. First we prove that, when viewed as functions on the moduli space of hyperelliptic Riemann surfaces, both ψ and $\frac{1}{2h}\varphi$ have the same image under $\partial\overline{\partial}$. Then we study the asymptotic behavior of the difference of ψ and $\frac{1}{2h}\varphi$ on a family of hyperelliptic Riemann surfaces degenerating into a stable curve of compact type. By using induction on h we will then deduce that the difference is a constant on the moduli space, equal to $\log 2$. For the degeneration argument we will rely heavily on results obtained by R. Wentworth [23].

2. Canonical Green's functions

In this section we collect some basic results on canonical Green's functions. The main sources are [3] [10]. Proposition 2.2, about a 'parallel' property of certain $(1,1)$-forms related to the universal Riemann surface over the moduli space of Riemann surfaces, seems not to be well known, and is possibly of independent interest.

As before, let M be a compact and connected Riemann surface of genus $h \geq 1$ and μ its canonical Kähler form. Let Δ be the diagonal on the complex manifold $M \times M$. The canonical Green's function $g_\mu = g_\mu(x,y)$ is the C^∞-function on $M \times M \setminus \Delta$ uniquely characterized by the following conditions:

(1) $\partial\overline{\partial} g_\mu(x,y) = \pi\sqrt{-1}(\mu(y) - \delta_x)$ for all $x \in M$;
(2) $\int_M g_\mu(x,y)\mu(y) = 0$ for all $x \in M$;
(3) $g_\mu(x,y) = g_\mu(y,x)$ for all $x \neq y \in M$;
(4) $g_\mu(x,y) - \log|z(x) - z(y)|$ is bounded for all $x \neq y$ in a coordinate chart $(U, z\colon U \xrightarrow{\sim} D)$ of M, where D is the open unit disk.

The canonical Green's function inverts the Laplacian Δ_μ in the sense that

$$f(x) = -\int_M g_\mu(x,y)\,\Delta_\mu(f)(y)\,\mu(y) + \int_M f\,\mu$$

holds for all $x \in M$ and all f in $C^\infty(M)$. As a consequence we have a formal development

$$g_\mu(x,y) = \sum_{\ell>0} \frac{\phi_\ell(x)\phi_\ell(y)}{\lambda_\ell}$$

for g_μ, where ϕ_ℓ and λ_ℓ are the eigenfunctions and eigenvalues of Δ_μ that we have introduced above.

In [10] Section 7 using standard harmonic analysis explicit formulas are derived for the ϕ_ℓ and λ_ℓ in the case $h = 1$. As a corollary of these formulas [10] Section 7 contains a derivation of equation (1.2) from the resulting formal expression for g_μ. We would like to present here an alternative argument.

Proposition 2.1. *Assume (M, o) is a complex torus, and let N be a positive integer. Let $x_1, \ldots, x_{N^2-1}$ be the non-trivial N-torsion points of M. Then the formula*

$$\sum_{i=1}^{N^2-1} g_\mu(x_i, o) = \log N$$

holds.

Proof. From properties (1) and (2) it follows that for every x, y on M with $Nx \neq y$ the equality

$$g_\mu(Nx, y) = \sum_{w \colon Nw = y} g_\mu(x, w)$$

holds. Now choose x in a standard euclidean coordinate chart $z \colon U \xrightarrow{\sim} D$ around o, where D is the open unit disk. Then we have

$$\begin{aligned} g_\mu(Nx, o) &= \sum_{w \colon Nw = o} g_\mu(x, w) \\ &= \sum_{i=1}^{N^2-1} g_\mu(x_i, x) + g_\mu(x, o) \\ &= \sum_{i=1}^{N^2-1} g_\mu(x_i, x) + \log |z(x)| + a + o(1) \end{aligned}$$

as $x \to o$, where a is some constant, by properties (3) and (4). On the other hand we have

$$\begin{aligned} g_\mu(Nx, o) &= \log |Nz(x)| + a + o(1) \\ &= \log N + \log |z(x)| + a + o(1) \end{aligned}$$

as $x \to o$, by property (4). The desired equality follows. □

One obtains (1.2) by taking $N = 2$ in the above proposition.

Let $\mathcal{M}$ denote the moduli space of compact and connected Riemann surfaces of genus $h \geq 2$, and denote by $\pi \colon \mathcal{C} \to \mathcal{M}$ the universal surface over $\mathcal{M}$. Both are viewed as orbifolds. The canonical Green's functions on the fibers of π determine a generalized function g on the fiber product $\mathcal{C} \times_{\mathcal{M}} \mathcal{C}$ with logarithmic singularities along the diagonal Δ. Let ν be the $(1,1)$-form on $\mathcal{C} \times_{\mathcal{M}} \mathcal{C}$ determined by the condition $\partial\overline{\partial} g = \pi\sqrt{-1}(\nu - \delta_\Delta)$. Let e^A denote the restriction of ν to Δ, which we view as another copy of $\mathcal{C}$. By [3] the form e^A restricts to $2-2h$ times the canonical $(1,1)$-form μ in each fiber of $\pi \colon \mathcal{C} \to \mathcal{M}$.

The $(1,1)$-forms e^A and ν are parallel over $\mathcal{M}$ in the following sense.

Proposition 2.2. *Let s, t be arbitrary holomorphic sections of π over an open subset U of $\mathcal{M}$. Then the equalities $s^* e^A = t^* e^A = (s,t)^* \nu$ hold on U.*

Proof. Without loss of generality we may replace $\mathcal{M}$ by a finite cover so that we may assume that $\mathcal{M}$ is equipped with a universal theta characteristic α, i.e. a consistent choice of a divisor class α of degree $h-1$ on each surface M such that 2α is the canonical class. Let $\mathcal{J} \to \mathcal{M}$ be the universal jacobian over $\mathcal{M}$ and let $\Phi \colon \mathcal{C} \times_{\mathcal{M}} \mathcal{C} \to \mathcal{J}$ be the map over $\mathcal{M}$ given by sending a triple (M, x, y) to the pair consisting of the jacobian $J(M)$ of M and the class of the divisor $h{\cdot}x{-}y{-}\alpha$ in $J(M)$. Let $\bar{\pi}\colon \mathcal{C}\times_{\mathcal{M}}\mathcal{C} \to \mathcal{C}$ be the projection on the first factor and let $\bar{s},\bar{t}\colon \pi^{-1}U \to \mathcal{C}\times_{\mathcal{M}}\mathcal{C}$ be the local sections of $\bar{\pi}$ obtained by pulling back the given local sections s,t of π. It follows from [14] Lemma 3.2 that there exists a $(1,1)$-form κ on $\mathcal{C}$ and a $(1,1)$-form w on $\mathcal{J}$ which is translation-invariant in the fibers of $\mathcal{J} \to \mathcal{M}$, such that $\Phi^* w = h\,\nu + \bar{\pi}^*\kappa$ holds. We deduce from this that

$$h\,\bar{s}^*\nu = (\Phi\bar{s})^* w - \kappa \quad \text{and} \quad h\,\bar{t}^*\nu = (\Phi\bar{t})^* w - \kappa\,.$$

Let $T_{x,y}$ denote translation by $[x-y]$ over $\mathcal{J}|_U$. Then $\Phi\bar{t} = T_{s,t}\Phi\bar{s}$ and since w is fiberwise translation-invariant we obtain $\bar{s}^*\nu = \bar{t}^*\nu$ from the above equalities. We derive $(s,t)^*\nu = s^*\bar{t}^*\nu = s^*\bar{s}^*\nu = (s,s)^*\nu = s^*e^A$. By symmetry property (3) we have $(s,t)^*\nu = (t,s)^*\nu = t^*e^A$ as well. □

3. Proof of Theorem A

In this section we give a proof of Theorem A. Let $x\colon M \to \mathbb{CP}^1$ be the 2-sheeted cover with branch points $\alpha_1,\ldots,\alpha_{2h+2}$. Let $w_1,\ldots,w_{2h+2}$ be the corresponding Weierstrass points on M. For the moment we fix two distinct indices i,j. Consider then the meromorphic function $f_{ij} = (x-\alpha_i)(x-\alpha_j)^{-1}$ on M. We note that $\mathrm{div}(f_{ij}) = 2(w_i - w_j)$. From properties (1) and (2) we therefore obtain an equality

$$g_\mu(w_i,z) - g_\mu(w_j,z) = \frac{1}{2}\log|f_{ij}(z)| - \frac{1}{2}\int_M \log|f_{ij}|\cdot\mu$$

of generalized functions on M. In particular we find for any pair of indices k,r

$$\begin{aligned} g_\mu(w_i,w_k) - g_\mu(w_i,w_r) - g_\mu(w_j,w_k) + g_\mu(w_j,w_r) &= \frac{1}{2}\log|f_{ij}(w_k)f_{ij}(w_r)^{-1}| \\ &= \frac{1}{2}\log|\mu_{ijkr}| \\ &= \frac{1}{4h(2h+1)}\log|\eta_{ijk}\eta_{ijr}^{-1}| \end{aligned}$$

where we recall that $\eta_{ijk} = \delta_{ik}\delta_{jk}^{-1}$ with δ_{ij} the expression from (1.3). This equality implies that there exists a constant c_{ij} such that

$$g_\mu(w_i,w_k) - g_\mu(w_j,w_k) = \frac{1}{4h(2h+1)}\log|\eta_{ijk}| + c_{ij}$$

for each index k. It follows from [16], Theorem 1.4 that

$$\sum_{k\neq i} g_\mu(w_i,w_k) = \sum_{k\neq j} g_\mu(w_j,w_k)\,.$$

From the fact that $\prod_{k\neq i,j}\delta_{ij}$ equals unity we obtain that $\sum_{k\neq i,j}\log|\eta_{ijk}| = 0$. It follows that

$$2h\,c_{ij} = \sum_{k\neq i,j}\left(g_\mu(w_i,w_k) - g_\mu(w_j,w_k)\right) = 0\,,$$

hence $c_{ij} = 0$ and

$$g_\mu(w_i,w_k) - g_\mu(w_j,w_k) = \frac{1}{4h(2h+1)}\log|\eta_{ijk}| = \frac{1}{4h(2h+1)}\log|\delta_{ik}\delta_{jk}^{-1}|\,.$$

Now fix the index k and vary the indices i, j. We find that there exists a constant c_k such that

$$g_\mu(w_i, w_k) = \frac{1}{4h(2h+1)} \log |\delta_{ik}| + c_k$$

for all indices $i \neq k$. We obtain $c_k = \frac{1}{2h+1}\psi(M)$ by summing over $i \neq k$. The formula in Theorem A follows.

Note that a similar argument in the case $h = 1$ allows one to deduce (1.1) from (1.2).

4. Proof of Theorem B

As was announced in the Introduction, we proceed in several steps. Let $\mathcal{H}$ be the moduli space of hyperelliptic Riemann surfaces of a fixed genus $h \geq 2$. Let $\pi \colon \mathcal{C} \to \mathcal{H}$ be the universal surface over $\mathcal{H}$. Both are viewed as orbifolds. A first result is the following.

Proposition 4.1. *The equality of* $(1,1)$*-forms*

$$\partial\overline{\partial}\,\psi = \frac{1}{2h}\partial\overline{\partial}\,\varphi$$

holds on $\mathcal{H}$.

Proof. Let $\mathcal{U}$ be an open cover of $\mathcal{H}$ such that for each U in $\mathcal{U}$ there exist $2h+2$ holomorphic sections $w_1, \ldots, w_{2h+2} \colon U \to \pi^{-1}U$ of $\pi^{-1}U \to U$ such that each w_i is a Weierstrass point in each fiber of $\pi^{-1}U \to U$. Let U be an element of $\mathcal{U}$. It suffices to prove the required equality on U. Theorem 3.1 of [18] gives an expression for $\partial\overline{\partial}\,\varphi$ over the moduli space of Riemann surfaces of genus h (note that [18] considers the invariant a with $a = \frac{1}{2\pi}\varphi$). Since the 'harmonic volume' of a Weierstrass point on a hyperelliptic Riemann surface is trivial, this expression immediately implies that on U one has the equality

$$\partial\overline{\partial}\,\varphi = 2h(2h+1)\pi\sqrt{-1}\,w_i^* e^A \,,$$

for any choice of the index i. On the other hand by Proposition 2.2 we have

$$\partial\overline{\partial}\,\psi = \pi\sqrt{-1}\sum_{j \neq i}(w_i, w_j)^*\nu = (2h+1)\pi\sqrt{-1}\,w_i^* e^A$$

for each index i. The proposition follows. □

We next determine the limit behavior of the difference $\psi - \frac{1}{2h}\varphi$ in a holomorphic family M_t of hyperelliptic Riemann surfaces of genus h degenerating into a stable curve M_0 which is the union of two hyperelliptic Riemann surfaces M_1, M_2 of genera $h_1, h_2 \geq 1$, respectively, with two points $p_1 \in M_1$ and $p_2 \in M_2$ identified, as in Chapter III of the book 'Theta functions on Riemann surfaces' by J. Fay [11]. Here t runs through a small punctured open disk around 0 in the complex plane. Note that $h = h_1 + h_2$. Let g_1, g_2 be the canonical Green's functions on M_1, M_2, respectively, and let z_1, z_2 be the local coordinates around p_1, p_2 on M_1, M_2 that come with the degeneration model. We put, following [23], Section 6:

$$\log k_1 = \lim_{x \to p_1} \left[g_1(x, p_1) - \log |z_1(x)| \right] \,, \quad \log k_2 = \lim_{x \to p_2} \left[g_2(x, p_2) - \log |z_2(x)| \right] \,,$$

and then make the reparametrization $\tau = k_1 k_2 t$.

Theorem 4.2. *For the surfaces M_t in Fay's degeneration model, the limit formula*

$$\lim_{t\to 0}\left[\frac{1}{2h}\,\varphi(M_t)+\frac{h_1h_2}{h^2}\log|\tau|\right]=\frac{1}{2h}\,\varphi(M_1)+\frac{1}{2h}\,\varphi(M_2)$$

holds.

Theorem 4.3. *For the energy $\psi(M_t)$ of the Weierstrass points of M_t, the limit formula*

$$\lim_{t\to 0}\left[\psi(M_t)+\frac{h_1h_2}{h^2}\log|\tau|\right]=\frac{h_1}{h}\,\psi(M_1)+\frac{h_2}{h}\,\psi(M_2)$$

holds.

We deduce Theorem B from Theorems 4.2 and 4.3 by induction on h. Note that the assertion in Theorem B is true for $h=1$ by (1.2) and the fact that φ vanishes for $h=1$. Now assume that the assertion in Theorem B is true for all genera smaller than h. Consider a holomorphic family M_t of hyperelliptic Riemann surfaces of genus h as above. We find

$$\begin{aligned}\lim_{t\to 0}\left[\psi(M_t)-\frac{1}{2h}\varphi(M_t)\right] &=\frac{h_1}{h}\psi(M_1)-\frac{1}{2h}\varphi(M_1)+\frac{h_2}{h}\psi(M_2)-\frac{1}{2h}\varphi(M_2)\\ &=\frac{h_1}{h}\left(\psi(M_1)-\frac{1}{2h_1}\varphi(M_1)\right)+\frac{h_2}{h}\left(\psi(M_2)-\frac{1}{2h_2}\varphi(M_2)\right)\\ &=\frac{h_1}{h}\log 2+\frac{h_2}{h}\log 2=\log 2\,.\end{aligned}$$

Let $\overline{\mathcal{H}}$ be the closure of $\mathcal{H}$ in the Deligne-Mumford compactification $\overline{\mathcal{M}}$ of the moduli space $\mathcal{M}$ of compact Riemann surfaces of genus h. Let $\widetilde{\mathcal{H}}\subset\overline{\mathcal{H}}$ be the partial compactification of $\mathcal{H}$ given by stable curves of compact type (i.e. whose jacobians are principally polarized abelian varieties). Write $\chi=\psi-\frac{1}{2h}\varphi$. The above calculation implies that for hyperelliptic Riemann surfaces in $\mathcal{H}$ tending to a point in the boundary of $\mathcal{H}$ in $\widetilde{\mathcal{H}}$, the function χ tends to $\log 2$. In particular the function χ extends as a continuous function over $\widetilde{\mathcal{H}}$. By Proposition 4.1 we know that χ is a pluriharmonic function on $\mathcal{H}$. By the Riemann extension theorem we find that χ extends as a pluriharmonic function on $\widetilde{\mathcal{H}}$, which is a constant function with value $\log 2$ on $\widetilde{\mathcal{H}}\setminus\mathcal{H}$.

Let $\mathcal{A}^*$ be the Satake compactification of the moduli space $\mathcal{A}$ of principally polarized abelian varieties of genus h. This is a normal projective variety [4]. There exists a holomorphic map $t\colon\overline{\mathcal{H}}\to\mathcal{A}^*$ given by assigning to a stable curve M the jacobian of its normalization. The map t is injective on $\mathcal{H}$, and the function χ descends as a pluriharmonic function on $t(\widetilde{\mathcal{H}})$, which is a constant function with value $\log 2$ on $t(\widetilde{\mathcal{H}}\setminus\mathcal{H})$.

The boundary of $\widetilde{\mathcal{H}}$ in $\overline{\mathcal{H}}$ is a finite union of divisors Ξ_i, see [8] for example. From the moduli theoretic description of the points in the divisors Ξ_i it follows that the map t restricted to the boundary has positive dimensional fibers. Hence the boundary of $t(\widetilde{\mathcal{H}})$ in the closure of $t(\widetilde{\mathcal{H}})$ in $\mathcal{A}^*$ has codimension at least two. Let M be a point in $t(\mathcal{H})$. Using that $\mathcal{A}^*$ is projective one sees from intersecting with suitable hyperplanes in a projective embedding that the variety $t(\widetilde{\mathcal{H}})$ contains a complete curve X passing through M. As $t(\mathcal{H})$ is affine, the curve X necessarily contains a point of $t(\widetilde{\mathcal{H}}\setminus\mathcal{H})$. Let $Y\to X$ be the normalization of X. The function

χ pulls back to a pluriharmonic function on Y, which is then necessarily constant. It follows that the value of χ at M is equal to $\log 2$, and Theorem B is proven.

For the proof of Theorem 4.2 we refer to [17]. It remains to prove Theorem 4.3. We need the next result, which follows from [23], Theorem 6.10.

Theorem 4.4. *Let g_t be the canonical Green's function on M_t. Denote by p both the point p_1 on M_1 and the point p_2 on M_2. Then for local distinct holomorphic sections x, y of the family M_t specializing onto M_1 we have*

$$\lim_{t \to 0} \left[g_t(x,y) - \left(\frac{h_2}{h} \right)^2 \log|\tau| \right] = g_1(x,y) - \frac{h_2}{h} \left(g_1(x,p) + g_1(y,p) \right) .$$

For local distinct holomorphic sections x, y with x specializing onto M_1 and y specializing onto M_2 we have

$$\lim_{t \to 0} \left[g_t(x,y) + \frac{h_1 h_2}{h^2} \log|\tau| \right] = \frac{h_1}{h} g_1(x,p) + \frac{h_2}{h} g_2(y,p) .$$

Now let $w_1, \ldots, w_{2h+2}$ be the Weierstrass points on M_t. As $t \to 0$ a portion of $2h_1 + 1$ of them degenerate onto M_1, and $2h_2 + 1$ of them degenerate onto M_2. The point p is a Weierstrass point of both M_1, M_2. Let I be the set of indices such that $i \in I$ if and only if w_i degenerates onto M_1.

Lemma 4.5. *The equality*

$$\sum_{\substack{i,j \in I \\ i \neq j}} g_1(w_i, w_j) = 2h_1 \, \psi(M_1)$$

holds.

Proof. We calculate

$$(2h_1 + 2) \, \psi(M_1) = \sum_{\substack{i,j \in I \cup \{p\} \\ i \neq j}} g_1(w_i, w_j) = \sum_{\substack{i,j \in I \\ i \neq j}} g_1(w_i, w_j) + 2 \, \psi(M_1) .$$

The required equality follows. □

Using Theorem 4.4 we find

$$\begin{aligned}
(2h+2) \, \psi(M_t) &= \sum_{i \neq j} g_t(w_i, w_j) \\
&= \sum_{\substack{i,j \in I \\ i \neq j}} g_t(w_i, w_j) + \sum_{\substack{i,j \in I^c \\ i \neq j}} g_t(w_i, w_j) + 2 \sum_{\substack{i \in I \\ j \in I^c}} g_t(w_i, w_j) \\
&= \sum_{\substack{i,j \in I \\ i \neq j}} \left(\left(\frac{h_2}{h} \right)^2 \log|\tau| + g_1(w_i, w_j) - \frac{h_2}{h} \left(g_1(w_i,p) + g_1(w_j,p) \right) \right) \\
&\quad + \sum_{\substack{i,j \in I^c \\ i \neq j}} \left(\left(\frac{h_1}{h} \right)^2 \log|\tau| + g_2(w_i, w_j) - \frac{h_1}{h} \left(g_2(w_i,p) + g_2(w_j,p) \right) \right) \\
&\quad + 2 \sum_{\substack{i \in I \\ j \in I^c}} \left(- \frac{h_1 h_2}{h^2} \log|\tau| + \frac{h_1}{h} g_1(w_i,p) + \frac{h_2}{h} g_2(w_j,p) \right) + o(1)
\end{aligned}$$

as $t \to 0$. With the help of Lemma 4.5 this can be rewritten as

$$\begin{aligned}(2h+2)\,\psi(M_t) = &\left((2h_1+1)\,2h_1\left(\frac{h_2}{h}\right)^2 + (2h_2+1)\,2h_2\left(\frac{h_1}{h}\right)^2\right)\log|\tau| \\ &- 2(2h_1+1)(2h_2+1)\frac{h_1h_2}{h^2}\,\log|\tau| \\ &+ \left(2h_1 - \frac{4h_1h_2}{h} + \frac{2h_1(2h_2+1)}{h}\right)\psi(M_1) \\ &+ \left(2h_2 - \frac{4h_1h_2}{h} + \frac{2h_2(2h_1+1)}{h}\right)\psi(M_2) + o(1) \\ =&(2h+2)\left(-\frac{h_1h_2}{h^2}\log|\tau| + \frac{h_1}{h}\psi(M_1) + \frac{h_2}{h}\psi(M_2)\right) + o(1)\end{aligned}$$

as $t \to 0$. Theorem 4.3 follows.

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