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Title: Some case studies of random walks in dynamic random environments
Date: 2012-12-19

Some Case Studies of Random Walks in Dynamic Random Environments

Proefschrift

ter verkrijging van

de graad van Doctor aan de Universiteit Leiden,

op gezag van Rector Magnificus prof. mr. P. F. van der Heijden,

volgens besluit van het College voor Promoties

te verdedigen op woensdag 19 december 2012

klokke 13:45 uur

door

Renato Soares dos Santos

geboren te Nova Lima, Brazilië

in 1974

Samenstelling van de promotiecommissie:

About the cover: the picture was obtained via computer simulations of systems of many random walks in dynamic random environments. The black-gray colors in the background represent the number (on an increasing scale) of random walks on a simple symmetric exclusion process and the blue-white colors represent the number of random walks on a supercritical contact process, which is in turn represented by the green colors.

Contents

Contents

1 Introduction

1.1 Background and outline of the introduction

1.1.1 Background

Random walks in random environments have been an active subject of research in the mathematics and physics literature for over 40 years. They are random walks whose transition kernels or rates are themselves random and may evolve in time in a stochastic manner. They model the motion of a "tracer particle" in a disordered medium, called a *random environment*, which can be *static* if it stays constant in time, or *dynamic* if it evolves in time. These cases reflect, respectively, situations where changes in the medium take place at a much larger time scale than displacements of the tracer (static case), and situations where these time scales are comparable (dynamic case). Examples are a photon in an amorphous solid, or a contaminant particle in a turbulent fluid. We could ask whether, depending on the statistical properties of the medium, the tracer particle behaves ballistically, and whether its fluctuations are diffusive. The subject of this thesis is the analysis of the asymptotic behaviour of random walks in dynamic random environments from a mathematical standpoint.

1.1.2 Outline of the introduction

In Section 1.2, we define the class of models and state the main questions that will be discussed in this thesis. Section 1.3 contains a brief historical overview of random walk in static and dynamic random environment, with emphasis on results that are relevant for our discussion. A description of the remaining chapters is given in Section 1.4.

1.2 Model and questions

In this section, we setup our notation, define our model and state the main questions that we will discuss.

1.2.1 Random walk in dynamic random environment

The role of the random environment will be taken by a Feller process $\xi = (\xi_t)_{t \geq 0}$ on the product space $\Omega := E^{\mathbb{Z}^d}$, where $d \in \mathbb{N}$ and E is a Polish space. We call the random environment *static*

if $\xi_t = \xi_0 \ \forall \ t \geq 0$, and we call it *dynamic* otherwise. Endow Ω with the space-shift operators $(\theta_x)_{x \in \mathbb{Z}^d}$ defined by

$$(\theta_x \eta)(y) := \eta(x + y), \quad \eta \in \Omega, \ x, y \in \mathbb{Z}^d. \tag{1.2.1}$$

Conditionally on ξ, the random walk in random environment, denoted by $W = (W_t)_{t \geq 0}$, is a Markov process on $\mathbb{Z}^d$ with rates as follows. Let

$$\pi : \Omega \times \mathbb{Z}^d \to [0, \infty) \tag{1.2.2}$$

be a given measurable function. Then the rate at time t for the random walk to jump to $x + z$ given that $W_t = x$ is equal to $\pi(\theta_x \xi_t, z)$. In a dynamic random environment, this gives rise to a time-inhomogeneous Markov process.

Discrete-time models can be defined analogously by taking π to be transition probabilities (i.e. $\pi(\eta, z) \in [0, 1]$ and $\sum_{z \in \mathbb{Z}^d} \pi(\eta, z) = 1$ for μ-a.e. η), and letting W jump at integer times. In this case, the random environment ξ may evolve in continuous or discrete time.

We will denote by $\mathbb{P}_\eta^z$ the joint law of ξ and W when $\xi_0 = \eta \in \Omega$ and $W_0 = z \in \mathbb{Z}^d$, omitting the superscript when $z = 0$. We assume that ξ is *translation-invariant*, i.e.,

$$\mathbb{P}_\eta (\theta_x \xi_t \in \cdot) = \mathbb{P}_{\theta_x \eta} (\xi_t \in \cdot) \ \ \forall \ \eta \in \Omega, x \in \mathbb{Z}^d, \tag{1.2.3}$$

and that there exists a translation-invariant probability measure μ on Ω that is an equilibrium for ξ. Then, under the probability measure $\mathbb{P}_\mu$ given by

$$\mathbb{P}_\mu(\cdot) := \int_\Omega \mathbb{P}_\eta(\cdot) \mu(\mathrm{d}\eta), \tag{1.2.4}$$

the distribution of ξ is invariant w.r.t. space-time shifts.

The random walk W is said to have *bounded jumps* if there exists a deterministic finite set $\mathcal{J} \subset \mathbb{Z}^d$ such that

$$\pi(\eta, x) = 0 \ \forall \ x \notin \mathcal{J} \ \text{ for } \mu\text{-a.e. } \eta. \tag{1.2.5}$$

The majority of the models studied in the literature are of this type. Unless explicitly stated otherwise, in the following we will always assume that W has bounded jumps. A particular case is that of *nearest-neighbour* jumps, i.e., $\mathcal{J} = \{x \in \mathbb{Z}^d : \|x\| = 1\}$.

Another important notion is *ellipticity*: the model is called *elliptic* if there exists a basis $(e_i)_{i=1}^d$ of $\mathbb{Z}^d$ such that

$$\pi(\eta, \pm e_i) > 0 \text{ for } i \in \{1, \ldots, d\}, \text{ for } \mu\text{-a.e. } \eta. \tag{1.2.6}$$

It is called *uniformly* elliptic if there exists a $\delta \in (0, 1)$ such that, for μ-a.e. η,

$$\begin{aligned} \pi(\eta, x) &\leq \delta^{-1} & \forall \ x \in \mathbb{Z}^d, \\ \pi(\eta, \pm e_i) &\geq \delta & \forall \ i \in \{1, \ldots, d\}. \end{aligned} \tag{1.2.7}$$

A particular case is when E is finite (e.g. $E = \{0, 1\}$) and $\pi(\eta, x) > 0$ if and only if $x \in \{\pm e_i : i = 1, \ldots, d\}$.

An important tool that appears recurrently in the literature is the process $\bar{\xi} = (\bar{\xi}_t)_{t \geq 0}$ of the *environment as seen from the random walk* (ESRW), defined by

$$\bar{\xi} := \theta_{W_t} \xi_t. \tag{1.2.8}$$

When ξ is translation-invariant, $\bar{\xi}$ is a Markov process. Many questions about the model can be formulated in terms of the ESRW.

In the following, we will write RWRE to abbreviate "random walk in static random environment" and RWDRE to abbreviate "random walk in dynamic random environment".

1.2.2 Questions

The main objective is to understand the asymptotic behaviour of W_t as $t \to \infty$. For example, we seek criteria for recurrence/transience, laws of large numbers, large deviation principles and estimates, as well as central limit theorems or other scaling limits. These questions come in two flavours: *quenched* and *annealed*. The difference is the law w.r.t. which W satisfies a given property. The annealed law is the law of W under $\mathbb{P}_\mu$, while the quenched law is the law of W under $\mathbb{P}_\mu(\cdot \mid \xi)$, i.e., conditional on ξ. An annealed property is one that holds for W under $\mathbb{P}_\mu$, while a quenched property is one that holds under $\mathbb{P}_\mu(\cdot \mid \xi)$ for $\mathbb{P}_\mu$-a.e. ξ.

The random walk W is called *transient* if

$$\lim_{t \to \infty} \|W_t\| = \infty \quad \mathbb{P}_\mu\text{-a.s.} \tag{1.2.9}$$

and is called *recurrent* if, for every $x \in \mathbb{Z}^d$,

$$\mathbb{P}_\mu \left(\exists\, t > 0 \colon\, W_t = x \right) = 1. \tag{1.2.10}$$

It is not difficult to see that (1.2.10) implies that each site of $\mathbb{Z}^d$ is visited infinitely often by W; this follows from the fact that the law of $\theta_{W_t} \xi_t$ under $\mathbb{P}_\mu$ is, for each $t \geq 0$, absolutely continuous w.r.t. μ.

We say that W satisfies a *strong law of large numbers* (SLLN) if there exists a $w \in \mathbb{R}^d$, called the *velocity* of the random walk (or, alternatively, *speed* in one dimension), such that

$$\lim_{t \to \infty} t^{-1} W_t = w \quad \mathbb{P}_\mu\text{-a.s.} \tag{1.2.11}$$

Note that transience, recurrence and the SLLN require no distinction between annealed and quenched laws. If large deviation bounds of the type

$$\limsup_{t \to \infty} t^{-1} \log \mathbb{P}_\mu \left(t^{-1} W_t \notin (w - \varepsilon, w + \varepsilon) \right) < 0 \quad \forall\, \varepsilon > 0 \tag{1.2.12}$$

are available, then (1.2.11) holds by the Borel-Cantelli lemma.

We say that W satisfies the annealed *large deviation principle* (LDP) in $\mathbb{R}^d$, with rate function $H_a \colon \mathbb{R}^d \to [0, \infty]$ if

$$
\begin{aligned}
\limsup_{t \to \infty} \ t^{-1} \log \mathbb{P}_\mu \left(t^{-1} W_t \in F \right) &\leq \ -\inf_{x \in F} H_a(x) \quad \forall \ \text{closed } F \subset \mathbb{R}^d, \\
\liminf_{t \to \infty} \ t^{-1} \log \mathbb{P}_\mu \left(t^{-1} W_t \in G \right) &\geq \ -\inf_{x \in G} H_a(x) \quad \forall \ \text{open } G \subset \mathbb{R}^d.
\end{aligned}
\tag{1.2.13}
$$

Analogously, W satisfies the quenched LDP if (1.2.13) holds with $\mathbb{P}_\mu(\cdot \mid \xi)$ in place of $\mathbb{P}_\mu$, and H_a replaced by a (in general, different) rate function H_q that is *deterministic*, i.e., does not depend on ξ. Usual requirements for rate functions are lower semi-continuity and compact level sets. In many cases they are also finite and convex.

Finally, we say that W satisfies the *functional central limit theorem* (FCLT) if there exists a constant $\sigma^2 \in (0, \infty)$ (the limiting variance) such that

$$
\left(\frac{W_{nt} - ntw}{n^{1/2}\sigma} \right)_{t \geq 0} \Longrightarrow B \quad \text{as } n \to \infty,
\tag{1.2.14}
$$

where B is standard Brownian motion and $\Longrightarrow$ denotes weak convergence in Skorohod space. The FCLT is called quenched or annealed depending on which law is considered for W. In contrast to the LDP (where annealed and quenched rate functions may differ), a quenched FCLT with variance σ^2 implies the annealed FCLT with the same variance.

Models admitting limits other than Gaussian and with different spatial and temporal scalings will be mentioned in Section 1.3.

1.3 History and discussion

In this section we give a brief overview of the literature on random walks in static and dynamic random environments. The exposition will be far from complete, as we focus on results relevant for our subsequent discussion. For more information on RWRE we refer the reader to the lecture notes of Sznitman [77, 78] and Zeitouni [83, 84], and for RWDRE to the PhD thesis of Avena [3].

1.3.1 Static random environment

In this section, all models are in discrete time and have bounded or nearest-neighbour jumps.

One dimension

Models of RWRE have been around since the late 1960's, when they appeared as toy models for DNA replication (see e.g. Chernov [26]). A mathematical treatment of the one-dimensional model in the i.i.d. setting (i.e., when μ is a product measure) was given by Solomon [73], who proved, among other results, a criterion for transience/recurrence and a SLLN. The model has some surprising features: for example, depending on the parameters, the RWRE can have speed

zero but still be transient, which is impossible for random processes with stationary increments (see Kesten [48]). Furthermore, according to the formula provided by Solomon, the speed of the RWRE is smaller than the speed of a homogeneous random walk whose transition probabilities are given by the average of π over μ. These *slow-down phenomena* are related to the presence of *traps*, i.e., regions in the lattice where the random walk spends a long time because the random environment gives the random walk a drift towards the center of the region.

The slow-down can be noticed also at the level of large deviations. For example, it was shown by Greven and den Hollander [40] for the i.i.d. setting that, when the speed w is positive, the quenched rate function has a flat piece (i.e., is equal to zero) in the interval $[0, w]$. This means that deviations to travel at speeds slower than typical are not exponentially costly.

Another interesting feature of RWRE is the possibility to have both diffusive and non-diffusive scaling limits, as was shown by Kesten, Kozlov and Spitzer [53] in the transient case, and Sinai [72] in the recurrent case, both under the annealed law. The latter case is particularly dramatic: the random walk is so slow that its position after n steps scales as $(\log n)^2$ as $n \to \infty$.

One-dimensional RWRE is by now very well understood. As examples of additional results, we mention:

- extensions to ergodic random environments, i.e., when μ is invariant and ergodic w.r.t. space-shifts (recurrence vs. transience criteria and SLLNs by Alili [2]; quenched, annealed and functional LDPs by Comets, Gantert and Zeitouni [28]);
- precise identification of the law of annealed scaling limits in the i.i.d. setting (recurrent case by Golosov [39] and Kesten [49]; transient case by Enriquez, Sabot and Zindy [36]).

Two and more dimensions

Much is known also in dimensions two and higher, but the picture is far less complete. For example, even in the i.i.d. uniformly elliptic setting there is still no complete characterization of transience vs. recurrence (see, however, Peres, Popov and Sousi [64] for a quite general transience criterion), and the SLLN has only been proven under additional restrictions, except for $d = 2$ (see Sznitman and Zerner [79] and Zerner [85]).

One such class of restrictions are *ballisticity* conditions, which ensure that the random walk has a strong drift in some direction. For example, one may require the model to be *non-nestling*, meaning that the random walk has a minimum drift in a certain direction independently of the random environment. The nestling case is more challenging, because the random walk can become "trapped", as in the one-dimensional case, in large regions of the random environment where the random walk is pushed towards the center.

Two ballisticity conditions formulated for the nestling case are Kalikow's condition [47] and Sznitman's (T') condition [76]. The first of these concerns the drift of a certain "coarse-grained" auxiliary chain, while the second requires stretched-exponential decay of certain slab exit probabilities under the annealed measure. Recently, Berger, Drewitz and Ramirez [13] proved a polynomial criterion for ballisticity that is equivalent to condition (T'). In the i.i.d. uniformly elliptic setting, Kalikow's condition implies condition (T'), which in turn implies the SLLN, the

annealed FCLT, and large deviation estimates (see Sznitman [74, 75]).

The requirement of i.i.d. random environment can be substituted by suitable mixing conditions. For example, Rassoul-Agha [65] proved a SLLN assuming Kalikow's condition combined with a strong mixing condition named after Dobrushin and Shlosman. His proof uses the ESRW process, in this case given by $(\theta_{W_n}\xi_0)_{n\in\mathbb{N}_0}$. A similar SLLN was proven by Comets and Zeitouni [29], again requiring Kalikow's condition but introducing the weaker condition of "ϕ-mixing on ℓ-cones", which amounts, roughly speaking, to the requirement that the states of the random environment inside a cone aligned with a vector ℓ are asymptotically independent of the states across a distant hyperplane that is perpendicular to ℓ. Their proof relies on the construction of *regeneration times.* In [30], the same authors prove a CLT under a modified mixing condition involving multiple cones.

Among the host of additional results available, we mention only a few:

- quenched FCLT for i.i.d. random environments under transience and moment assumptions by Rassoul-Agha and Seppäläinen [67];

- quenched FCLT for balanced ergodic random environments by Guo and Zeitouni [42] and Lawler [55];

- quenched LPD in the ergodic setting and annealed LPD in the i.i.d. setting by Varadhan [80].

1.3.2 Dynamic random environment

In this section, most of the random walk models are again in discrete time, while the evolution of the random environment can be in discrete or continuous time.

Comparison with static random environment

As a model for phenomena with comparable time scales for the random environment and the random walk, RWDRE has a motivation of its own. Nevertheless, the comparison with RWRE is natural and interesting. For example, by considering time as an additional dimension, one can view RWDRE in dimension d as a special case of RWRE in dimension $d + 1$, with the random walk transient and the random environment Markovian in the time direction. Therefore, for a fixed dimension, one could hope to obtain more detailed information for RWDRE, at least in some cases.

Another motivation for comparison comes from the "slow-down phenomena" mentioned in Section 1.3.1. As noted there, the "anomalous behaviour" of RWRE in one dimension is related to the occurrence of *traps* that, in a dynamic random environment, would not stay fixed in time but would eventually disappear. Is this enough to restore the "ordinary behaviour" (i.e., ballistic or diffusive) of the random walk? One would expect that when the dynamics of the environment is fast enough this will indeed be the case, while when it is slow enough some "anomaly" will survive.

Currently, these questions can be only partially answered, mostly by identifying situations where regular diffusive and/or ballistic behaviour occurs. This includes, as we will see, dynamic random environments with fast enough and uniform mixing, as well as a few examples outside this class. In contrast, a random walk considered by Avena, den Hollander and Redig [5] on the simple symmetric exclusion process was shown to exhibit slow-down at the level of annealed large deviations, and simulations of the same model by Avena and Thomann [8] suggest non-diffusive behaviour depending on parameters. Other examples of RWDRE with anomalous behaviour can be constructed (Völlering [81]). However, no general criteria to decide whether this happens or not have been found yet. In fact, even for some classical and well-studied dynamic random environments (e.g. the exclusion process) the behaviour is still wide open, so there is a lot to be done.

Brief history of the model

One of the first models in dynamic random environment was introduced by Madras [61] in 1986. In this model, the random environment is i.i.d. in space and Markovian in time, and the random walk is a deterministic functional of the random environment. In particular, the model is *non-elliptic*. He obtained recurrence/transience criteria, a SLLN and an annealed FCLT.

In 1992, Boldrighini, Ignatyuk, Malyshev and Pellegrinotti [16] introduced a model of RW-DRE, for which they proved a CLT under different assumptions. Their model has *mutual interactions*, i.e., the evolution of the random environment is influenced by the random walk. Since then, RWDRE models have been intensively studied in different settings, falling mostly into one of the following two categories:

1. *Independent in time,* where the random environment is resampled after each time unit. Under the annealed measure, the random walk is homogeneous, so the focus is on quenched results. The quenched FCLT was studied e.g. by Bérard [11], Boldrighini, Minlos and Pellegrinotti [19, 21, 23], Joseph and Rassoul-Agha [46], and Rassoul-Agha and Seppäläinen [66]. The quenched LDP was treated by Yilmaz [82].

2. *Independent in space and Markovian in time,* where the random environment consists of identically distributed Markov processes, evolving independently at each site of $\mathbb{Z}^d$. The SLLN and both quenched and annealed FCLTs were considered e.g. by Boldrighini, Minlos and Pellegrinotti [20, 22], Bandyophadyay and Zeitouni [9], and Dolgopyat and Liverani [33, 34]. Boldrighini, Minlos, Nardi and Pellegrinotti [17, 18] studied the decay of correlations for the ESRW process.

Cases where the random environment has correlations in both space and time have been considered only recently. Dolgopyat, Keller and Liverani [32] obtained a SLLN and a quenched CLT for random environments with strong space-time mixing under absolute-continuity assumptions on the equilibrium of the ESRW process. A quenched FCLT was also obtained by Bricmont and Kupiainen [24] for random walks that depend weakly on the random environment, under an assumption that implies exponential mixing of the random environment. The latter is not assumed to be Markovian.

1 Introduction

Avena, den Hollander and Redig [6] considered random environments consisting of *interacting particle systems* (IPS) with state space $E = \{0, 1\}$ (see Liggett [57]). The transition rates π for this model are defined as follows:

$$\pi(\eta, x) = \begin{cases} \alpha\eta(0) + \beta\left[1 - \eta(0)\right] & \text{if } x = 1, \\ \beta\eta(0) + \alpha\left[1 - \eta(0)\right] & \text{if } x = -1, \\ 0 & \text{otherwise,} \end{cases} \qquad (1.3.1)$$

where $\alpha, \beta \in (0, \infty)$. Hereafter we will refer to this model as the (α, β)-*model*. In [6], the authors prove, among other results, a SLLN under a milder mixing condition called *cone-mixing*. This condition, and the ensuing proof strategy using regeneration times, is an adaptation of the "ϕ-mixing on cones" condition used by Comets and Zeitouni [29] in the static setting. It requires, roughly speaking, that the states of the random environment inside a space-time cone opening upwards in time depend weakly on the initial configuration when the time coordinate of the tip of the cone is very large.
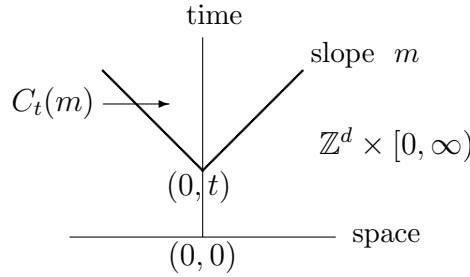
Figure 1.1: The cone defined in (1.3.2).

To define this condition more precisely (in one dimension), for $t \geq 0$ and $m > 0$, let

$$C_t(m) := \{(x, s) \in \mathbb{Z} \times [t, \infty) \colon |x| \leq m(s - t)\} \qquad (1.3.2)$$

be the cone with tip at $(0, t)$ and slope m (see Figure 1.1), and put

$$\mathscr{C}_t(m) := \sigma\left\{\xi_s(x) \colon (x, s) \in C_t(m)\right\}. \qquad (1.3.3)$$

Then the cone-mixing condition holds if, for any slope m, there exists a function $\Phi(t)$ satisfying $\lim_{t \to \infty} \Phi(t) = 0$ such that, for all non-negative f measurable in $\mathscr{C}_t(m)$,

$$|\mathbb{E}_\eta[f] - \mathbb{E}_\mu[f]| \leq \Phi(t)\|f\|_\infty \quad \text{for } \mu\text{-a.e. } \eta. \qquad (1.3.4)$$

No assumptions on the speed of decay of $\Phi(t)$ are required. Note, however, that cone-mixing is a *uniform* mixing condition; this is crucial for the implementation of the regeneration strategy. An annealed FCLT was later obtained in [3] in the same setting, under a stricter condition involving multiple cones.

The SLLN and annealed FCLT were obtained also by Redig and Völlering [70] in a fairly general setup, not requiring e.g. E to be finite or W to have bounded jumps and assuming instead moment and regularity conditions on π. Also this work requires conditions that ensure uniform and fast enough mixing of the random environment (polynomial with a high enough degree). These conditions are formulated in terms of *coupling* between different realizations of the random environment. For example, a mixing condition that implies the SLLN is the existence, for any $\eta, \zeta \in \Omega$, of a strong Markovian coupling $\widehat{\mathbb{P}}_{\eta,\zeta}$ of two copies ξ^1 and ξ^2 of the random environment, starting respectively from η and ζ, such that

$$\int_0^\infty t^d \sup_{\eta,\zeta \in \Omega} \widehat{\mathbb{E}}_{\eta,\zeta} \left[\mathrm{dist} \left(\xi_t^1(0), \xi_t^2(0) \right) \right] dt < \infty, \tag{1.3.5}$$

where $\mathrm{dist}(\cdot, \cdot)$ is the distance in the Polish space E. Extra conditions are required to obtain the FCLT. The approach in [70] is to study the ESRW process, for which unique ergodicity is proven with control of the speed of relaxation towards equilibrium. This also yields information about how the limiting velocity and the limiting variance depend on π.

Apart from general results such as the quenched LDP (see Avena, den Hollander and Redig [5], Campos, Drewitz, Ramirez, Rassoul-Agha and Seppäläinen [25], and Rassoul-Agha, Seppäläinen and Yilmaz [69]), the annealed LDP for one-dimensional attractive spin systems (Avena, den Hollander and Redig [5]), and a recent transience criterion (Peres, Popov and Sousi [64]), very little is known for random environments that are not uniformly mixing. Some results are available in specific cases. One example is [43], where an approximate law of large numbers is obtained when the random environment is a Poisson system of independent simple random walks. Another example is [15], where a SLLN and both quenched and annealed FCLTs are obtained for a random walk in the "backbone" of supercritical oriented percolation. Three of the four remaining chapters of this thesis (Chapters 3–5) are dedicated to RWDREs without uniform mixing. Their contents are described in Section 1.4.

1.4 Overview of the thesis

We give here an overview of Chapters 2–5 of the thesis from the perspective of the previous discussion.

1.4.1 Chapter 2: Law of large numbers for non-elliptic random walks in dynamic random environment

The setup in this chapter is slightly different from the one described in Section 1.2. The reason is that its motivation comes from a non-elliptic model with rates that can be infinite. More precisely, we are interested in analysing the (α, β)-model of [6] in the limit as $\alpha \uparrow \infty$ and $\beta \downarrow 0$, which we call the $(\infty, 0)$-*model*. In this limit, the random walk is almost a deterministic functional of the random environment, being in particular non-elliptic. Using a new uniform

mixing condition called *conditional cone-mixing*, we are able to generalize the regeneration-time argument of [6] and obtain a SLLN in a general setup that includes the $(\infty, 0)$-model, the (α, β)-model, as well as various other models in one or more dimensions. Convergence in L^p, $p \geq 1$, and a ballisticity criterion are obtained for the $(\infty, 0)$-model in two classes of examples, namely: spin-flip systems with bounded flip rates that are either in the $M < \epsilon$ regime (see Liggett [57]) or have small enough ratio of maximal/minimum flip rates.

This chapter is based on a paper with Frank den Hollander and Vladas Sidoravicius [45], which has been accepted for publication in Stochastic Processes and their Applications.

1.4.2 Chapter 3: Transient random walk in symmetric exclusion: limit theorems and an Einstein relation

In this chapter, the random environment is given by a one-dimensional simple symmetric exclusion process (SSEP). This is the IPS on $\mathbb{Z}$ with state space $E = \{0, 1\}$ whose generator L acts on real bounded cylinder functions f as

$$Lf(\eta) := \sum_{\substack{x,y \in \mathbb{Z} \\ x \sim y}} [f(\eta^{x,y}) - f(\eta)], \quad \eta \in \{0, 1\}^{\mathbb{Z}}, \tag{1.4.1}$$

where $x \sim y$ means that x and y are nearest-neighbours and $\eta^{x,y}$ is the configuration obtained from η by interchanging the states at x and y. The interpretation is that particles move independently on the lattice, except for a hard-core repulsion (the exclusion interaction), which prevents them to jump to a site that is already occupied. We take the initial configuration to be distributed as a Bernoulli product measure with density $\rho \in [0, 1]$, denoted by ν_ρ. The measures $\{\nu_\rho \colon \rho \in [0, 1]\}$ are known to be the only extremal invariant measures for ξ. Due to conservation of particles, the SSEP has (relatively) *slow* and *non-uniform* mixing, and as such it does not fall into any of the categories treated previously.

We consider a nearest-neighbour model with transition rates given by

$$\pi(\eta, x) = \begin{cases} \alpha_1 \eta(0) + \alpha_0 [1 - \eta(0)] & \text{if } x = 1, \\ \beta_1 \eta(0) + \beta_0 [1 - \eta(0)] & \text{if } x = -1, \\ 0 & \text{otherwise}, \end{cases} \tag{1.4.2}$$

where $\alpha_i, \beta_i \in (0, \infty)$, $i \in \{0, 1\}$. Under the strong drift assumption

$$\alpha_1 \wedge \alpha_0 - \beta_1 \vee \beta_0 > 1, \tag{1.4.3}$$

we prove a SLLN, an annealed FCLT, as well as large deviation bounds. The method uses regeneration times, obtained and controlled with the help of (1.4.3). The basic idea is that the random walk is moving so fast that it very quickly "leaves behind" the information accumulated in its past.

Regarding the RWDRE as a perturbation of a homogeneous random walk, we also obtain a so-called *Einstein relation* between the linear response of the speed to the perturbation and the

unperturbed variance, in the limit when the perturbation is very weak. This is an example of a *fluctuation-dissipation theorem* in statistical physics.

This chapter is based on a paper with Luca Avena and Florian Völlering [7], which has been submitted.

1.4.3 Chapter 4: Scaling of a random walk on a supercritical contact process

In this chapter we study another example of an IPS on $\{0,1\}^{\mathbb{Z}}$ exhibiting non-uniform mixing, namely, the *contact process* in the supercritical phase. This is the translation-invariant Feller process whose local transition rates at a site $x \in \mathbb{Z}$, given that the current configuration is $\eta \in \{0,1\}^{\mathbb{Z}}$, are given by

$$
\begin{aligned}
1 &\to 0 \quad \text{at rate} \quad 1, \\
0 &\to 1 \quad \text{at rate} \quad \lambda\left[\eta(x+1) + \eta(x-1)\right],
\end{aligned}
\tag{1.4.4}
$$

where $\lambda \in (0,\infty)$ is called the *infection parameter*. We interpret the states of ξ by saying that a site x is *infected* at time t if $\xi_t(x) = 1$, and is *healthy* otherwise. In words, (1.4.4) means that infected sites heal spontaneously at rate 1, while healthy sites get infected at a rate proportional to the number of infected neighbours; the proportionality constant is the infection parameter λ.

This random environment exhibits a *phase transition*: there exists a critical infection parameter $\lambda_c \in (0,\infty)$ such that if $\lambda \le \lambda_c$, then the only equilibrium for ξ is the measure concentrated on the configuration with all sites healthy, while if $\lambda > \lambda_c$, then there exists a unique non-trivial equilibrium ν_λ that is invariant and ergodic under space-shifts, and assigns positive density to the set of infected sites. In this regime, there is also a characteristic *infection propagation speed* $\iota(\lambda) > 0$, which is the asymptotic speed of the rightmost infected site when ξ is started from a configuration that is all infected to the left of the origin and all healthy to the right of the origin.

Noting that long stretches of healthy sites have positive probability under ν_λ, it is not hard to see that the contact process is *not* uniformly mixing when $\lambda > \lambda_c$, as the typical time required for an infection to reach the center of such a stretch is linear in the length of the stretch. Therefore, also the supercritical contact process is not covered by previous results, even though it mixes *exponentially* fast when starting from configurations with enough infections.

We consider jump rates as in (1.4.2) under the following restrictions:

$$
\alpha_0 + \beta_0 = \alpha_1 + \beta_1 =: \gamma > 0
\tag{1.4.5}
$$

and

$$
v_0 \le v_1 \quad \text{where } v_i := \alpha_i - \beta_i, \ i \in \{0,1\},
\tag{1.4.6}
$$

i.e., the total jump rate is constant and equal to γ everywhere, while the local drift is larger on infections than on healthy sites. The latter assumption is made w.l.o.g. as the contact process is invariant under reflection through the origin.

Under (1.4.5)–(1.4.6), we obtain a SLLN and some properties of the speed. The proof consists of two parts: a subadditivity argument when ξ starts from all sites infected, and a coupling between the latter law and $\mathbb{P}_{\nu_\lambda}$. Under the additional restriction that

$$\iota(\lambda) > |v_0| \vee |v_1|, \tag{1.4.7}$$

i.e., the infection propagates faster than the maximum absolute speed at which the random walk can move, we construct regeneration times that allow us to prove an annealed FCLT and an annealed LDP, as well as continuity of the speed and the limiting variance as functions of λ.

This chapter is based on a paper with Frank den Hollander [44], which has been submitted.

1.4.4 Chapter 5: Non-trivial linear bounds for a random walk driven by a simple symmetric exclusion process

In this chapter, we revisit the random walk on the one-dimensional SSEP. We take the jump rates to be as in (1.4.2) but drop the strong drift assumption (1.4.3), assuming instead (1.4.5)–(1.4.6) as in Chapter 4. The proof of the SLLN in this setting is still an open challenge.

We propose instead a much simpler question: Can the random walk travel, even along a subsequence of times, at one of the extremal speeds v_0 or v_1? In other words, is it possible that $\liminf_{t\to\infty} t^{-1}W_t = v_0$ or $\limsup_{t\to\infty} t^{-1}W_t = v_1$? This is equivalent to the question of whether the random walk spends a negligible amount of time on top of particles or holes. Using a renormalization technique due to Kesten and Sidoravicius [50], we perform a multiscale analysis of the SSEP that allows us to answer this question in the negative.

2 Law of large numbers for non-elliptic random walks in dynamic random environments

This chapter is based on a paper with Frank den Hollander and Vladas Sidoravicius.

Abstract

We prove a law of large numbers for a class of $\mathbb{Z}^d$-valued random walks in dynamic random environments, including non-elliptic examples. We assume for the random environment a mixing property called *conditional cone-mixing* and that the random walk tends to stay inside wide enough space-time cones. The proof is based on a generalization of a regeneration scheme developed by Comets and Zeitouni [29] for static random environments and adapted by Avena, den Hollander and Redig [6] to dynamic random environments. A number of one-dimensional examples are given. In some cases, the sign of the speed can be determined.

2.1 Introduction

2.1.1 Background

Random walk in random environment (RWRE) has been an active area of research for more than three decades. Informally, RWRE's are random walks in discrete or continuous space-time whose transition kernels or transition rates are not fixed but are random themselves, constituting a random environment. Typically, the law of the random environment is taken to be translation invariant. Once a realization of the random environment is fixed, we say that the law of the random walk is *quenched*. Under the quenched law, the random walk is Markovian but not translation invariant. It is also interesting to consider the quenched law averaged over the law of

the random environment, which is called the *annealed law*. Under the annealed law, the random walk is not Markovian but translation invariant. For an overview on RWRE, we refer the reader to Zeitouni [83, 84], Sznitman [77, 78], and references therein.

In the past decade, several models have been considered in which the random environment itself evolves in time. These are referred to as *random walk in dynamic random environment* (RWDRE). By viewing time as an additional spatial dimension, RWDRE can be seen as a special case of RWRE, and as such it inherits the difficulties present in RWRE in dimensions two or higher. However, RWDRE can be harder than RWRE because it is an interpolation between RWRE and homogeneous random walk, which arise as limits when the dynamics is slow, respectively, fast. For a list of mathematical papers dealing with RWDRE, we refer the reader to Avena, den Hollander and Redig [5]. Most of the literature on RWDRE is restricted to situations in which the space-time correlations of the random environment are either absent or rapidly decaying.

One paper in which a milder space-time mixing property is considered is Avena, den Hollander and Redig [6], where a law of large numbers (LLN) is derived for a class of one-dimensional RWDRE's in which the role of the random environment is taken by an *interacting particle system* (IPS) with configuration space

$$\Omega := \{0,1\}^{\mathbb{Z}}. \tag{2.1.1}$$

Figure 2.1: Jump rates of the (α, β)-walk on top of a hole $(= 0)$, respectively, a particle $(= 1)$.

In their paper, the random walk starts at 0 and has transition rates as in Fig. 2.1: on a *hole* (i.e., on a 0) the random walk has rate α to jump one unit to the left and rate β to jump one unit to the right, while on a *particle* (i.e., on a 1) the rates are reversed (w.l.o.g. it may be assumed that $0 < \beta < \alpha < \infty$, so that the random walk has a drift to the left on holes and a drift to the right on particles). Hereafter, we will refer to this model as the (α, β)-model. The LLN is proved under the assumption that the IPS satisfies a space-time mixing property called *cone-mixing* (see Fig. 2.2), which means that the states inside a space-time cone are almost independent of the states in a space plane far below this cone. The proof uses a regeneration scheme originally developed by Comets and Zeitouni [29] for RWRE and adapted to deal with RWDRE. This proof can be easily extended to $\mathbb{Z}^d$, $d \geq 2$, with the appropriate corresponding notion of cone-mixing.
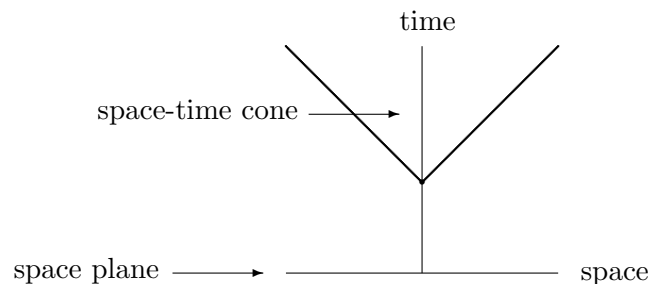
Figure 2.2: Cone-mixing property: asymptotic independence of states inside a space-time cone from states inside a space plane.

2.1.2 Elliptic vs. non-elliptic

The original motivation for the present paper was to study the (α, β)-model in the limit as $\alpha \to \infty$ and $\beta \downarrow 0$. In this limit, which we will refer to as the $(\infty, 0)$-model, the walk is almost a deterministic functional of the IPS; in particular, it is non-elliptic. The challenge was to find a way to deal with the *lack of ellipticity*. As we will see in Section 2.3, our set-up will be rather general and will include the (α, β)-model, the $(\infty, 0)$-model, as well as various other models. Examples of papers that deal with non-elliptic (actually, deterministic) RW(D)RE's are Madras [61] and Matic [63], where a recurrence vs. transience criterion, respectively, a large deviation principle are derived.

In the RW(D)RE literature, ellipticity assumptions play an important role. In the static case, RWRE in $\mathbb{Z}^d$, $d \geq 1$, is called *elliptic* when, almost surely w.r.t. the random environment, all the rates are *finite* and there is a basis $\{e_i\}_{1 \leq i \leq d}$ of $\mathbb{Z}^d$ such that the rate to go from x to $x + e_i$ is *positive* for $1 \leq i \leq d$. It is called *uniformly elliptic* when these rates are *bounded away from infinity*, respectively, *bounded away from zero*. In [29], in order to take advantage of the mixing property assumed on the random environment, it is important to have uniform ellipticity not necessarily in all directions, but in at least one direction in which the random walk is transient. One way to state this "uniform directional ellipticity" in a way that encompasses also the dynamic setting is to require the existence of a deterministic time $T > 0$ and a vector $e \in \mathbb{Z}^d$ such that the quenched probability for the random walk to displace itself along e during time T is uniformly positive for almost every realization of the random environment. This is satisfied by the (α, β)-model for $e = 0$ and any $T > 0$. This model is also transient (indeed, non-nestling) in the time direction, which enables the use of the cone-mixing property of [6]. In the case of the $(\infty, 0)$-model, however, there are in general no such T and e. For example, when the random environment is a spin-flip system with bounded flip rates, any fixed space-time position has positive probability of being unreachable by the random walk. For all such models, the approach in [6] fails.

In the present paper, in order to deal with the possible lack of ellipticity we require a different space-time mixing property for the dynamic random environment, which we call *conditional cone-mixing*. Moreover, as in [29] and [6], we must require the random walk to have a tendency to stay

inside space-time cones. Under these assumptions, we are able to set up a regeneration scheme and prove a LLN. Our result includes the LLN for the (α, β)-model in [6], the $(\infty, 0)$-model for at least two subclasses of IPS's that we will exhibit, as well as models that are intermediate, in the sense that they are neither uniformly elliptic in any direction, nor deterministic as the $(\infty, 0)$-model.

2.1.3 Outline

The rest of the paper is organized as follows. In Section 2.2 we discuss, still informally, the $(\infty, 0)$-model and the regeneration strategy. This section serves as a motivation for the formal definition in Section 2.3 of the class of models we are after, which is based on three *structural assumptions*. Section 2.4 contains the statement of our LLN under four *hypotheses*, and a description of two classes of one-dimensional IPS's that satisfy these hypotheses for the $(\infty, 0)$-model, namely, spin-flip systems with bounded flip rates that either are in Liggett's $M < \epsilon$ regime, or have finite range and a small enough ratio of maximal/minimal flip rates. Section 2.5 contains preparation material, given in a general context, that is used in the proof of the LLN given in Section 2.6. In Section 2.7 we verify our hypotheses for the two classes of IPS's described in Section 2.4. We also obtain a criterion to determine the sign of the speed in the LLN, via a comparison with independent spin-flip systems. Finally, in Section 2.8, we discuss how to adapt the proofs in Section 2.7 to other models, namely, generalizations of the (α, β)-model and the $(\infty, 0)$-model, and mixtures thereof. We also give an example where our hypotheses fail. The examples in our paper are all one-dimensional, even though our LLN is valid in $\mathbb{Z}^d$, $d \geq 1$.

2.2 Motivation

2.2.1 The $(\infty, 0)$-model

Let

$$\xi := (\xi_t)_{t \geq 0} \quad \text{with} \quad \xi_t := \big(\xi_t(x)\big)_{x \in \mathbb{Z}} \tag{2.2.1}$$

be a càdlàg Markov process on Ω. We will interpret ξ by saying that at time t site x contains either a *hole* ($\xi_t(x) = 0$) or a *particle* ($\xi_t(x) = 1$). Typical examples are interacting particle systems on Ω, such as independent spin-flips and simple exclusion.

Suppose that we run the (α, β)-model on ξ with $0 < \beta \ll 1 \ll \alpha < \infty$. Then the behavior of the random walk is as follows. Suppose that $\xi_0(0) = 1$ and that the walk starts at 0. The walk rapidly moves to the first hole on its right, typically before any of the particles it encounters manages to flip to a hole. When it arrives at the hole, the walk starts to rapidly jump back and forth between the hole and the particle to the left of the hole: we say that it sits in a *trap*. If $\xi_0(0) = 0$ instead, then the walk rapidly moves to the first particle on its left, where it starts to rapidly jump back and forth in a trap. In both cases, before moving away from the trap, the walk typically waits until one or both of the sites in the trap flip. If only one site flips, then the walk typically moves in the direction of the flip until it hits a next trap, etc. If both sites flip

simultaneously, then the probability for the walk to sit at either of these sites is close to $\frac{1}{2}$, and hence it leaves the trap in a direction that is close to being determined by an independent fair coin.

The limiting dynamics when $\alpha \to \infty$ and $\beta \downarrow 0$ can be obtained from the above description by removing the words "rapidly, "typically" and "close to". Except for the extra Bernoulli($\frac{1}{2}$) random variables needed to decide in which direction to go to when both sites in a trap flip simultaneously, the walk up to time t is a deterministic functional of $(\xi_s)_{0 \le s \le t}$. In particular, if ξ changes only by single-site flips, then apart from the first jump the walk is completely deterministic. Since the walk spends all of its time in traps where it jumps back and forth between a hole and a particle, we may imagine that it lives on the edges of $\mathbb{Z}$. We implement this observation by associating with each edge its left-most site, i.e., we say that the walk is at x when we actually mean that it is jumping back and forth between x and $x+1$. See Figure 2.3.
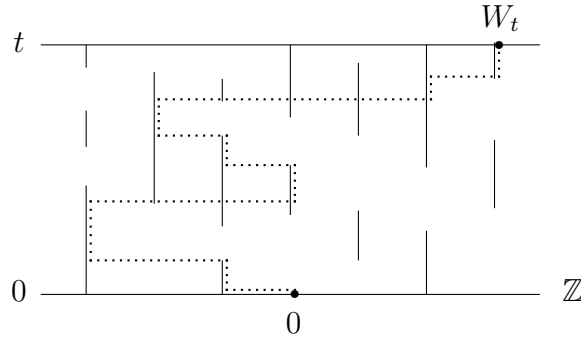
Figure 2.3: The vertical lines represent the presence of particles. The dotted line is the path of the $(\infty, 0)$-walk.

Let

$$W := (W_t)_{t \ge 0} \tag{2.2.2}$$

denote the random walk path. By the description above, W is càdlàg and

$$W_t \text{ is a function of } \big((\xi_s)_{0 \le s \le t}, Y\big), \tag{2.2.3}$$

where Y is a sequence of i.i.d. Bernoulli($\frac{1}{2}$) random variables independent of ξ. Note that W also has the following three properties:

(1) For any fixed time s, the increment $W_{s+t} - W_s$ is found by applying the same function in (2.2.3) to the environment shifted in space and time by (W_s, s) and an independent copy of Y; in particular, the pair (W_t, ξ_t) is Markovian.

(2) Given that W stays inside a space-time cone until time t, $(W_s)_{0 \le s \le t}$ is a functional only of Y and of the states in ξ up to time t inside a slightly larger cone, obtained by by adding all neighboring sites to the right.

(3) Each jump of the path follows the same mechanism as the first jump, i.e., $W_t - W_{t-}$ is computed using the same rules as those for W_0 but applied to the environment shifted in space and time by (W_{t-}, t).

The reason for emphasizing these properties will become clearer in Section 2.2.2.

Figure 2.4: Regeneration at time τ.

2.2.2 Regeneration

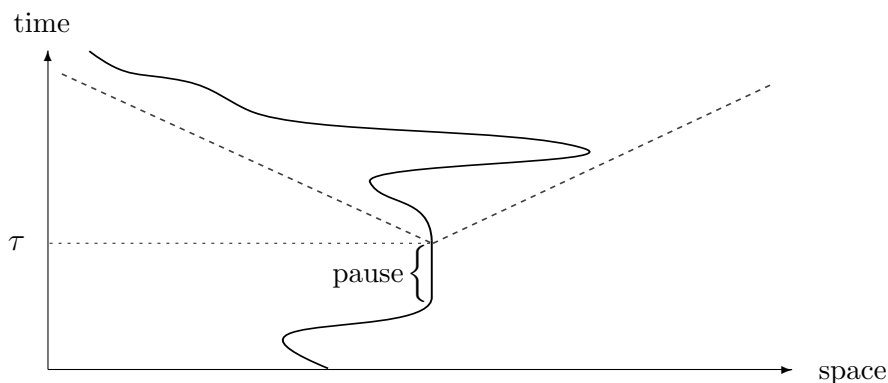The cone-mixing property that is assumed in [6] to prove the LLN for the (α, β)-model can be loosely described as the requirement that all the states of the IPS inside a space-time cone opening upwards depend weakly on the states inside a space plane far below the tip (recall Fig. 2.2). Let us give a rough idea of how this property can lead to *regeneration*. Consider the event that the walk stands still for a long time. Since the jump times of the walk are independent of the IPS, so is this event. During this pause, the environment around the walk is allowed to mix, which by the cone-mixing property means that by the end of the pause all the states inside a cone with a tip at the space-time position of the walk are almost independent of the past of the walk. If thereafter the walk stays confined to the cone, then its future increments will be almost independent of its past, and so we get an approximate regeneration. Since in the (α, β)-model there is a uniformly positive probability for the walk to stay inside a space-time cone with a large enough inclination, we see that this regeneration strategy can indeed be made to work. See Figure 2.4.

For the actual proof of the LLN in [6], cone-mixing must be more carefully defined. For technical reasons, there must be some uniformity in the decay of correlations between events in the space-time cone and in the space plane. This uniformity holds, for instance, for any spin-flip system in the $M < \epsilon$ regime (Liggett [57], Section I.3), but not for the exclusion process or the supercritical contact process. Therefore the approach outlined above works for the first IPS, but not for the other two.

There are three properties of the (α, β)-model that make the above heuristics plausible. First, to be able to apply the cone-mixing property relative to the space-time position of the walk, it is important that the pair (IPS,walk) is Markovian and that the law of the environment as seen from the walk at any time is comparable to the initial law. Second, there is a uniformly positive probability for the walk to stand still for a long time and afterwards stay inside a space-time cone. Third, once the walk stays inside a space-time cone, its increments depend on the IPS only through the states inside that cone. Let us compare these observations with what happens in the $(\infty, 0)$-model. Property (1) from Section 2.2.1 gives us the Markov property, while property (2) gives us the measurability inside cones. As we will see, when the environment is translation-

invariant, property (3) implies absolute continuity of the law of the environment as seen from the walk at any positive time with respect to its counterpart at time zero. Therefore, as long as we can make sure that the walk has a tendency to stay inside space-time cones (which is reasonable when we are looking for a LLN), the main difference is that the event of standing still for a long time is not independent of the environment, but rather is a *deterministic* functional of the environment. Consequently, it is not at all clear whether cone-mixing is enough to allow for regeneration. On the other hand, the event of standing still is local, since it only depends on the states of the two neighboring sites of the trap where the walk is pausing. For many IPS's, the observation of a local event will not affect the weak dependence between states that are far away in space-time. Hence, if such IPS's are cone-mixing, then states inside a space-time cone remain almost independent of the initial configuration even when we condition on seeing a trap for a long time.

Thus, under suitable assumptions, the event "standing still for a long time" is a candidate to induce regeneration. In the (α, β)-model this event does not depend on the environment whereas in the $(\infty, 0)$-model it is a deterministic functional of the environment. If we put the (α, β)-model in the form (2.2.3) by taking for Y two independent Poisson processes with rates α and β, then we can restate the previous sentence by saying that in the (α, β)-model the regeneration-inducing event depends only on Y, while in the $(\infty, 0)$-model it depends only on ξ. We may therefore imagine that, also for other models of the type (2.2.3) and that share properties (1)–(3), it will be possible to find more general regeneration-inducing events that depend on both ξ and Y in a non-trivial manner. This motivates our setup in Section 2.3.

2.3 Model setting

So far we have mostly been discussing RWDRE driven by an IPS. However, there are convenient constructions of IPS's on richer state spaces (such as graphical representations) that can facilitate the construction of the regeneration-inducing events mentioned in Section 2.2.2. We will therefore allow for more general Markov processes to represent the dynamic random environment ξ. Notation is set up in Section 2.3.1. Section 2.3.2 contains the three structural assumptions that define the class of models we will consider.

2.3.1 Notation and setup

Let $\mathbb{N} = \{1, 2, \ldots\}$ be the set of natural numbers, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let E be a Polish space and $\xi := (\xi_t)_{t \geq 0}$ a Markov process with state space $E^{\mathbb{Z}^d}$ where $d \in \mathbb{N}$. Let $Y := (Y_n)_{n \in \mathbb{N}}$ be an i.i.d. sequence of random elements independent of ξ. For $I \subset [0, \infty)$, abbreviate $\xi_I := (\xi_u)_{u \in I}$, and analogously for Y. The joint law of ξ and Y when $\xi_0 = \eta \in E^{\mathbb{Z}^d}$ will be denoted by $\mathbb{P}_\eta$. For $n \in \mathbb{N}$, put $\mathscr{Y}_n := \sigma(Y_{[1,n]})$. Let $\mathscr{F}_0 := \sigma(\xi_0)$ and, for $t > 0$, $\mathscr{F}_t := \sigma(\xi_{[0,t]}) \vee \mathscr{Y}_{\lceil t \rceil}$.

For $t \geq 0$ and $x \in \mathbb{Z}^d$, let θ_t and θ_x be the time-shift and space-shift operators given by

$$\theta_t(\xi, Y) := \big((\xi_{t+s})_{s \geq 0}, (Y_{\lfloor t \rfloor + n})_{n \in \mathbb{N}}\big), \qquad \theta_x(\xi, Y) := \big((\theta_x \xi_t)_{t \geq 0}, (Y_n)_{n \in \mathbb{N}}\big), \qquad (2.3.1)$$

where $\theta_x \xi_t(y) = \xi_t(x + y)$. In the sequel, whether θ is a time-shift or a space-shift operator will always be clear from the index.

We assume that ξ is translation-invariant, i.e., $\theta_x \xi$ has under $\mathbb{P}_\eta$ the same distribution as ξ under $\mathbb{P}_{\theta_x \eta}$. We also assume the existence of a (not necessarily unique) translation-invariant equilibrium distribution μ for ξ, and write $\mathbb{P}_\mu(\cdot) := \int \mu(d\eta)\, \mathbb{P}_\eta(\cdot)$ to denote the joint law of ξ and Y when ξ_0 is drawn from μ.

The random walk will be denoted by $W = (W_t)_{t \geq 0}$, and we will write $\bar\xi := (\bar\xi_t)_{t \geq 0}$ to denote the *environment process as seen from W*, i.e., $\bar\xi_t := \theta_{W_t} \xi_t$. Let $\bar\mu_t$ denote the law of $\bar\xi_t$ under $\mathbb{P}_\mu$. We abbreviate $\bar\mu := \bar\mu_0$. Note that $\bar\mu = \mu$ when $\mathbb{P}_\mu(W_0 = 0) = 1$.

For $m > 0$ and $R \in \mathbb{N}_0$, define the *R-enlarged m-cone* by

$$C_R(m) := \big\{ (x, t) \in \mathbb{Z}^d \times [0, \infty) \colon \|x\| \leq mt + R \big\}, \tag{2.3.2}$$

where $\|\cdot\|$ is the L^1 norm. Let $\mathscr{C}_{R,t}(m)$ be the σ-algebras generated by the states of ξ up to time t inside $C_R(m)$.

2.3.2 Structural assumptions

We will assume that W is random translation of a random walk starting at 0. More precisely, we assume that $Z = (Z_t)_{t \geq 0}$ is a càdlàg $\mathscr{F}$-adapted $\mathbb{Z}^d$-valued process with $Z_0 = 0$ $\mathbb{P}_{\bar\mu}$-a.s. such that

$$W_t = W_0 + \theta_{W_0} Z_t \quad \forall\, t \geq 0. \tag{2.3.3}$$

We also assume that $W_0 \in \mathbb{Z}^d$ and depends on ξ and Y only through ξ_0, i.e.,

$$\mathbb{P}_\mu(W_0 = x \mid \mathscr{F}_\infty) = \mathbb{P}_\mu(W_0 = x \mid \xi_0) \text{ a.s. } \forall\, x \in \mathbb{Z}^d. \tag{2.3.4}$$

Under these assumptions, $(W_t - W_0)_{t \geq 0}$ has under $\mathbb{P}_\mu$ the same distribution as Z under $\mathbb{P}_{\bar\mu}$. In what follows we make *three structural assumptions* on Z:

(A1) **(Additivity)**
For all $n \in \mathbb{N}$,
$$(Z_{t+n} - Z_n)_{t \geq 0} = \theta_{Z_n} \theta_n Z \quad \mathbb{P}_{\bar\mu}\text{-a.s.} \tag{2.3.5}$$

(A2) **(Locality)**
For $m > 0$, let $\mathcal{D}_m := \{\|Z_t\| \leq mt \ \forall\, t \geq 0\}$. Then there exists $R \in \mathbb{N}_0$ such that, $\forall\, m > 0$, both $\mathcal{D}_m$ and $(1_{\mathcal{D}_m} Z_t)_{t \geq 0}$ are measurable w.r.t. $\mathscr{C}_{R,\infty}(m) \vee \mathscr{Y}_\infty$.

(A3) **(Homogeneity of jumps)**
For all $n \in \mathbb{N}$ and $x \in \mathbb{Z}^d$,
$$\mathbb{P}_{\bar\mu}\big(Z_n - Z_{n-} = x \mid \xi_{[0,n]}, Z_{[0,n)}\big) = \mathbb{P}_{\theta_{Z_{n-}} \xi_n}\big(W_0 = x\big) \qquad \mathbb{P}_{\bar\mu}\text{-a.s.} \tag{2.3.6}$$

These properties are analogues of properties (1)–(3) of the $(\infty, 0)$-model mentioned in Section 2.2.1, with the difference that we only require them to hold at integer times; this will be

enough as our proof relies on integer-valued regeneration times. We also assume the 'extra randomness' Y to be split independently among time intervals of length 1; for example, in the case of the $(\infty, 0)$-model, each Y_n would *not* be a Bernoulli$(\frac{1}{2})$ random variable but a whole *sequence* of such variables instead. This is discussed in detail in Section 2.7.1.

Another remark: assumption (A3) might seem strange since many random walk models have no deterministic jumps, which is indeed the case for the examples described in Section 2.4. Note however that, in this case, (A3) severely restricts W_0, implying $W_0 = 0$ a.s. when ξ is started from $\theta_{Z_{n-}}\xi_n$. Furthermore, our main theorem (Theorem 2.4.1 below) is not restricted to this situation and includes also cases with deterministic jumps. For example, one could modify the $(\infty, 0)$-walk to jump exactly at integer times. Additional examples with deterministic jumps are described in item 4 of Section 2.8. The relevance of assumption (A3) is in showing that the law of the environment as seen by the RW after any jump is absolutely continuous w.r.t. the law after the first jump; this is done in Lemma 2.6.1 below.

2.4 Main results

Theorems 2.4.1 and 2.4.2 below are the main results of our paper. Theorem 2.4.1 in Section 2.4.1 is our LLN. Theorem 2.4.2 in Section 2.4.2 verifies the hypotheses in this LLN for the $(\infty, 0)$-model in two classes of one-dimensional IPS's. For these classes some more information is available, namely, convergence in L^p, $p \geq 1$, and a criterion to determine the sign of the speed.

2.4.1 Law of large numbers

In order to develop a regeneration scheme for a random walk subject to assumptions (A1)–(A3) based on the heuristics discussed in Section 2.2.2, we need suitable regeneration inducing events. In the *four hypotheses* stated below, these events appear as a sequence $(\Gamma_L)_{L \in \mathbb{N}}$ such that, for a certain fixed $m \in (0, \infty)$ and R as in (A2), $\Gamma_L \in \mathscr{C}_{R,L}(m) \vee \mathscr{Y}_L$ for all $L \in \mathbb{N}$.

(H1) **(Determinacy)**
   On Γ_L, $Z_t = 0$ for all $t \in [0, L]$ $\mathbb{P}_{\bar{\mu}}$-a.s.

(H2) **(Non-degeneracy)**
   For L large enough, there exists a $\gamma_L > 0$ such that $\mathbb{P}_\eta(\Gamma_L) \geq \gamma_L$ for $\bar{\mu}$-a.e. η.

(H3) **(Cone constraints)**
   Let $\mathcal{S} := \inf\{t > 0 \colon \|Z_t\| > mt\}$. Then there exist $a \in (1, \infty)$, $\kappa_L \in (0, 1]$ and $\psi_L \in [0, \infty)$ such that, for L large enough and $\bar{\mu}$-a.e. η,

$$
\begin{aligned}
&(1) \quad \mathbb{P}_\eta(\theta_L \mathcal{S} = \infty \mid \Gamma_L) \geq \kappa_L, \\
&(2) \quad \mathbb{E}_\eta\left[1_{\{\theta_L \mathcal{S} < \infty\}}(\theta_L \mathcal{S})^a \mid \Gamma_L\right] \leq \psi_L^a.
\end{aligned}
\tag{2.4.1}
$$

(H4) **(Conditional cone-mixing)**
   There exists a sequence of non-negative numbers $(\Phi_L)_{L \in \mathbb{N}}$ satisfying

$\lim_{L\to\infty} \kappa_L^{-1} \Phi_L = 0$ such that, for L large enough and for $\bar{\mu}$-a.e. η,

$$|\mathbb{E}_\eta (\theta_L f \mid \Gamma_L) - \mathbb{E}_{\bar{\mu}}(\theta_L f \mid \Gamma_L)| \le \Phi_L \, \|f\|_\infty \qquad \forall\, f \in \mathscr{C}_{R,\infty}(m), f \ge 0. \tag{2.4.2}$$

We are now ready to state our LLN.

Theorem 2.4.1. *Under assumptions* (A1)–(A3) *and hypotheses* (H1)–(H4)*, there exists a* $w \in \mathbb{R}^d$ *such that*

$$\lim_{t\to\infty} t^{-1} W_t = w \qquad \mathbb{P}_\mu - a.s. \tag{2.4.3}$$

Remark 1: Hypothesis (H4) above without the conditioning on Γ_L in (2.4.2) and with constant κ_L is the same as the cone-mixing condition used by Avena, den Hollander and Redig [6]. There, $W_0 = 0$ $\mathbb{P}_\mu$-a.s., so that $\bar{\mu} = \mu$.

Remark 2: Theorem 2.4.1 provides no information about the value of w, not even its sign when $d = 1$. Understanding the dependence of w on model parameters is in general a highly non-trivial problem.

2.4.2 Examples

We next describe two classes of one-dimensional IPS's for which the $(\infty, 0)$-model satisfies hypotheses (H1)–(H4). Further details will be given in Section 2.7. In both classes, ξ is a spin-flip system in $\Omega = \{0,1\}^{\mathbb{Z}}$ with bounded and translation-invariant single-site flip rates. We may assume that the flip rates at the origin are of the form

$$c(\eta) = \begin{cases} c_0 + \lambda_0 p_0(\eta) & \text{if } \eta(0) = 1, \\ c_1 + \lambda_1 p_1(\eta) & \text{if } \eta(0) = 0, \end{cases} \qquad \eta \in \Omega, \tag{2.4.4}$$

for some $c_i, \lambda_i \ge 0$ and $p_i \colon \Omega \to [0,1]$, $i = 0, 1$.

Example 1: $c(\cdot)$ is in the $M < \epsilon$ regime (see Liggett [57], Section I.3).

Example 2: $p(\cdot)$ has finite range and $(\lambda_0 + \lambda_1)/(c_0 + c_1) < \lambda_c$, where λ_c is the critical infection rate of the one-dimensional contact process with the same range.

Theorem 2.4.2. *Consider the* $(\infty, 0)$*-model. Suppose that* ξ *is a spin-flip system with flip rates given by* (2.4.4)*. Then for Examples* 1 *and* 2 *there exist a version of* ξ *and events* $\Gamma_L \in \mathscr{C}_{R,L}(m) \vee \mathscr{Y}_L$*,* $L \in \mathbb{N}$*, satisfying hypotheses* (H1)–(H4)*. Furthermore, the convergence in Theorem* 2.4.1 *holds also in* L^p *for all* $p \ge 1$*, and*

$$\begin{aligned} w &\ge \tfrac{c_0+\lambda_0}{c_1+c_0+\lambda_0}(c_1 - c_0 - \lambda_0) & \text{if } c_1 \ge c_0 + \lambda_0, \\ w &\le -\tfrac{c_1+\lambda_1}{c_0+c_1+\lambda_1}(c_0 - c_1 - \lambda_1) & \text{if } c_0 \ge c_1 + \lambda_1. \end{aligned} \tag{2.4.5}$$

For independent spin-flip systems (i.e., when $\lambda_0 = \lambda_1 = 0$), (2.4.5) shows that w is positive, zero or negative when the density $c_1/(c_0 + c_1)$ is, respectively, larger than, equal to or smaller than $\frac{1}{2}$. The criterion for other ξ is obtained by comparison with independent spin-flip systems.

We expect hypotheses (H1)–(H4) to hold for a very large class of IPS's and walks. For each choice of IPS and walk, the *verification* of hypotheses (H1)–(H4) constitutes a *separate problem*. Typically, (H1)–(H2) are immediate, (H3) requires some work, while (H4) is hard.

Additional models will be discussed in Section 2.8. We will consider generalizations of the (α, β)-model and the $(\infty, 0)$-model, namely, *internal noise* models and *pattern* models, as well as mixtures of them. The verification of (H1)–(H4) will be analogous to the two examples discussed above and will not be carried out in detail.

This concludes the motivation and the statement of our main results. The remainder of the paper will be devoted to the proofs of Theorems 2.4.1 and 2.4.2, with the exception of Section 2.8, which contains additional examples and remarks.

2.5 Preparation

The aim of this section is to prove two propositions (Propositions 2.5.2 and 2.5.4 below) that will be needed in Section 2.6 to prove the LLN. In Section 2.5.1 we deal with approximate laws of large numbers for general discrete- or continuous-time random walks in $\mathbb{R}^d$. In Section 2.5.2 we specialize to additive functionals of a Markov chain whose transition kernel satisfies a certain absolute-continuity property.

2.5.1 Approximate law of large numbers

This section contains two fundamental facts that are the basis of our proof of the LLN. They deal with the notion of an approximate law of large numbers.

Definition 2.5.1. *Let $W = (W_t)_{t \geq 0}$ be a random process in $\mathbb{R}^d$ with $t \in \mathbb{N}_0$ or $t \in [0, \infty)$. For $\varepsilon \geq 0$ and $v \in \mathbb{R}^d$, we say that W has an ε-approximate asymptotic velocity v, written $W \in AV(\varepsilon, v)$, if*

$$\limsup_{t \to \infty} \left\| \frac{W_t}{t} - v \right\| \leq \varepsilon \quad a.s. \tag{2.5.1}$$

We take $\| \cdot \|$ to be the L_1-norm. A simple observation is that W a.s. has an asymptotic velocity if and only if for every $\varepsilon > 0$ there exists a $v_\varepsilon \in \mathbb{R}^d$ such that $W \in AV(\varepsilon, v_\varepsilon)$. In this case $\lim_{\varepsilon \downarrow 0} v_\varepsilon$ exists and is equal to the asymptotic velocity.

First key proposition: skeleton approximate velocity

The following proposition gives conditions under which an approximate velocity for the process observed along a random sequence of times implies an approximate velocity for the full process.

Proposition 2.5.2. *Let W be as in Definition 2.5.1. Set $\tau_0 := 0$, let $(\tau_k)_{k \in \mathbb{N}}$ be an increasing sequence of random times in $(0, \infty)$ (or $\mathbb{N}$) with $\lim_{k \to \infty} \tau_k = \infty$ a.s. and put $X_k := (W_{\tau_k}, \tau_k) \in \mathbb{R}^{d+1}$, $k \in \mathbb{N}_0$. Suppose that the following hold:*
(i) There exists an $m > 0$ such that

$$\limsup_{k \to \infty} \sup_{s \in (\tau_k, \tau_{k+1}]} \left\| \frac{W_s - W_{\tau_k}}{s - \tau_k} \right\| \leq m \quad a.s. \tag{2.5.2}$$

(ii) There exist $v \in \mathbb{R}^d$, $u > 0$ and $\varepsilon \geq 0$ such that $X \in AV(\varepsilon, (v, u))$.
Then $W \in AV((3m+1)\varepsilon/u, v/u)$.

Proof. First, let us check that *(i)* implies

$$\limsup_{t \to \infty} \frac{\|W_t\|}{t} \leq m \quad a.s. \tag{2.5.3}$$

Suppose that

$$\limsup_{k \to \infty} \sup_{s > \tau_k} \left\| \frac{W_s - W_{\tau_k}}{s - \tau_k} \right\| \leq m \quad a.s. \tag{2.5.4}$$

Since, for every k and $t > \tau_k$,

$$\left\| \frac{W_t}{t} \right\| \leq \frac{\|W_{\tau_k}\|}{t} + \left\| \frac{W_t - W_{\tau_k}}{t - \tau_k} \right\| \left| 1 - \frac{\tau_k}{t} \right| \leq \frac{\|W_{\tau_k}\|}{t} + \sup_{s > \tau_k} \left\| \frac{W_s - W_{\tau_k}}{s - \tau_k} \right\| \left| 1 - \frac{\tau_k}{t} \right|, \tag{2.5.5}$$

(2.5.3) follows from (2.5.4) by letting $t \to \infty$ followed by $k \to \infty$.

To check (2.5.4), define, for $k \in \mathbb{N}_0$ and $l \in \mathbb{N}$,

$$m(k, l) := \sup_{s \in (\tau_k, \tau_{k+l}]} \left\| \frac{W_s - W_{\tau_k}}{s - \tau_k} \right\| \quad \text{and} \quad m(k, \infty) := \sup_{s > \tau_k} \left\| \frac{W_s - W_{\tau_k}}{s - \tau_k} \right\| = \lim_{l \to \infty} m(k, l). \tag{2.5.6}$$

Using the fact that $(x_1 + x_2)/(y_1 + y_2) \leq (x_1/y_1) \vee (x_2/y_2)$ for all $x_1, x_2 \in \mathbb{R}$ and $y_1, y_2 > 0$, we can prove by induction that

$$m(k, l) \leq \max\{m(k, 1), \ldots, m(k+l-1, 1)\}, \qquad l \in \mathbb{N}. \tag{2.5.7}$$

Fix $\varepsilon > 0$. By *(i)*, a.s. there exists a k_ε such that $m(k, 1) \leq m + \varepsilon$ for $k > k_\varepsilon$. By (2.5.7), the same is true for $m(k, l)$ for all $l \in \mathbb{N}$, and therefore also for $m(k, \infty)$. Since ε is arbitrary, (2.5.4) follows.

Let us now proceed with the proof of the proposition. Assumption *(ii)* implies that, a.s.,

$$\limsup_{k \to \infty} \left\| \frac{W_{\tau_k}}{k} - v \right\| \leq \varepsilon \quad \text{and} \quad \limsup_{k \to \infty} \left| \frac{\tau_k}{k} - u \right| \leq \varepsilon. \tag{2.5.8}$$

For $t \geq 0$, let k_t be the (random) non-negative integer such that

$$\tau_{k_t} \leq t < \tau_{k_t + 1}. \tag{2.5.9}$$

Since $\tau_1 < \infty$ a.s., $k_t > 0$ for large enough t. From (2.5.8) and (2.5.9) we deduce that

$$\limsup_{t\to\infty}\left|\frac{t}{k_t}-u\right|\le\varepsilon\quad\text{and so}\quad\limsup_{t\to\infty}\left|\frac{t}{k_t}-\frac{\tau_{k_t}}{k_t}\right|\le 2\varepsilon. \tag{2.5.10}$$

For t large enough we may write

$$\left\|\frac{uW_t}{t}-v\right\|\le\frac{\|W_t\|}{t}\left|u-\frac{t}{k_t}\right|+\left\|\frac{W_t-W_{\tau_{k_t}}}{k_t}\right\|+\left\|\frac{W_{\tau_{k_t}}}{k_t}-v\right\|$$

$$\le\frac{\|W_t\|}{t}\left|u-\frac{t}{k_t}\right|+\sup_{s\in(\tau_{k_t},\tau_{k_t+1}]}\left\|\frac{W_s-W_{\tau_{k_t}}}{s-\tau_{k_t}}\right\|\left|\frac{t-\tau_{k_t}}{k_t}\right|+\left\|\frac{W_{\tau_{k_t}}}{k_t}-v\right\|, \tag{2.5.11}$$

from which we obtain the conclusion by taking the limsup as $t\to\infty$ in (2.5.11), using (i), (2.5.3), (2.5.8) and (2.5.10), and then dividing by u. ∎

Conditions for the skeleton to have an approximate velocity

The following lemma states sufficient conditions for a discrete-time process to have an approximate velocity. It will be used in the proof of Proposition 2.5.4 below.

Lemma 2.5.3. *Let $X = (X_k)_{k\in\mathbb{N}_0}$ be a sequence of random vectors in $\mathbb{R}^d$ with joint law P such that $P(X_0 = 0) = 1$. Suppose that there exist a probability measure Q on $\mathbb{R}^d$ and numbers $\phi\in[0,1)$, $a > 1$, $K > 0$ with $\int_{\mathbb{R}^d}\|x\|^a\,Q(dx)\le K^a$, such that, P-a.s. for all $k\in\mathbb{N}_0$,*
(i) $|P(X_{k+1}-X_k\in A\mid X_0,\dots,X_k)-Q(A)|\le\phi$ for all A measurable;
(ii) $E[\|X_{k+1}-X_k\|^a|X_0,\dots,X_k]\le K^a$.
Then

$$\limsup_{n\to\infty}\left\|\frac{X_n}{n}-v\right\|\le 2K\phi^{(a-1)/a}\qquad P\text{-a.s.,} \tag{2.5.12}$$

where $v = \int_{\mathbb{R}^d}x\,Q(dx)$. In other words, $X\in AV(2K\phi^{(a-1)/a},v)$.

Proof. The proof is an adaptation of the proof of Lemma 3.13 in [29]; we include it here for completeness. With regular conditional probabilities, we can, using (i), couple P and $Q^{\otimes\mathbb{N}_0}$ according to a standard splitting representation (see e.g. Berbee [10]). More precisely, on an enlarged probability space we can construct random variables

$$(\Delta_k,V_k,R_k)_{k\in\mathbb{N}} \tag{2.5.13}$$

such that

(1) $(\Delta_k)_{k\in\mathbb{N}}$ is an i.i.d. sequence of Bernoulli(ϕ) random variables.

(2) $(V_k)_{k\in\mathbb{N}}$ is an i.i.d. sequence of random vectors with law Q.

(3) $(\Delta_l)_{l\ge k}$ is independent of $(\Delta_l,V_l,R_l)_{0\le l<k}$, R_k.

(4) Setting $\hat{X}_0 := 0$ and, for $k\in\mathbb{N}_0$, $\hat{X}_{k+1}-\hat{X}_k := (1-\Delta_k)V_k+\Delta_k R_k$, then $\hat{X}$ is equal in distribution to X.

(5) Setting $\mathcal{G}_k := \sigma(\Delta_l, V_l, R_l \colon 0 \le l \le k)$, then $E[f(V_k) \mid \mathcal{G}_{k-1}]$ is measurable w.r.t. $\sigma(\hat{X}_l \colon 0 \le l \le k-1)$ for any Borel nonnegative function f.

Using (4), we may write

$$\frac{X_n}{n} \stackrel{d}{=} \frac{\hat{X}_n}{n} = \frac{1}{n}\sum_{k=1}^n V_k - \frac{1}{n}\sum_{k=1}^n \Delta_k V_k + \frac{1}{n}\sum_{k=1}^n \Delta_k R_k. \qquad (2.5.14)$$

As $n \to \infty$, the first term on the r.h.s. converges a.s. to v by the LLN for i.i.d. random variables. By Hölder's inequality, the norm of the second term is at most

$$\left(\frac{1}{n}\sum_{k=1}^n \Delta_k\right)^{(a-1)/a} \left(\frac{1}{n}\sum_{k=1}^n \|V_k\|^a\right)^{1/a}, \qquad (2.5.15)$$

which, by (1) and (2), converges a.s. as $n \to \infty$ to

$$\phi^{(a-1)/a} \left(\int_{\mathbb{R}^d} \|x\|^a \, Q(dx)\right)^{1/a} \le K\phi^{(a-1)/a}. \qquad (2.5.16)$$

To control the third term, put $R_k^* := E[R_k \mid \mathcal{G}_{k-1}]$. Since $\|\Delta_k R_k\| \le \|\hat{X}_{k+1} - \hat{X}_k\|$, using (1), (3), (4), (5) and (ii), we get

$$\phi E[\|R_k\|^a \mid \mathcal{G}_{k-1}] = E[\Delta_k \|R_k\|^a \mid \mathcal{G}_{k-1}] \le E[\|\hat{X}_{k+1} - \hat{X}_k\|^a \mid \mathcal{G}_{k-1}] \le K^a. \qquad (2.5.17)$$

Combining (2.5.17) with Jensen's inequality, we obtain

$$\|R_k^*\| \le E\big[\|R_k\|^a \mid \mathcal{G}_{k-1}\big]^{1/a} \le \frac{K}{\phi^{1/a}}, \qquad (2.5.18)$$

so that

$$\left\|\frac{1}{n}\sum_{k=1}^n \Delta_k R_k^*\right\| \le \frac{K}{\phi^{1/a}}\left(\frac{1}{n}\sum_{k=1}^n \Delta_k\right) \xrightarrow[n\to\infty]{} K\phi^{(a-1)/a}. \qquad (2.5.19)$$

Now fix $y \in \mathbb{R}^d$ and put

$$M_n^y := \sum_{k=1}^n \frac{\Delta_k}{k}\langle R_k - R_k^*, y\rangle. \qquad (2.5.20)$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product. Then $(M_n^y)_{n\in\mathbb{N}_0}$ is a $(\mathcal{G}_n)_{n\in\mathbb{N}_0}$-martingale whose quadratic variation is

$$\langle M^y\rangle_n = \sum_{k=1}^n \frac{\Delta_k}{k^2}\langle R_k - R_k^*, y\rangle^2. \qquad (2.5.21)$$

By the Burkholder-Gundy inequality and (2.5.17–2.5.18), we have

$$\begin{aligned}
E\left[\sup_{n\in\mathbb{N}} |M_n^y|^{a\wedge 2}\right] &\le C\,E\left[\langle M^y\rangle_\infty^{(a\wedge 2)/2}\right] \\
&\le C\,E\left[\sum_{k=1}^\infty \frac{\Delta_k}{k^{a\wedge 2}}|\langle R_k - R_k^*, y\rangle|^{a\wedge 2}\right] \le C\,\|y\|^{a\wedge 2}K^{a\wedge 2},
\end{aligned} \qquad (2.5.22)$$

where C is a positive constant that may change after each inequality. This implies that M_n^y is uniformly integrable and therefore converges a.s. as $n \to \infty$. Kronecker's lemma then gives

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \Delta_k \langle R_k - R_k^*, y \rangle = 0 \ \text{ a.s.} \tag{2.5.23}$$

Since y is arbitrary, this in turn implies that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \Delta_k (R_k - R_k^*) = 0 \ \text{ a.s.} \tag{2.5.24}$$

Therefore, by (2.5.19) and (2.5.24), the limsup of the norm of the last term in the r.h.s. of (2.5.14) is also bounded by $K\phi^{(a-1)/a}$, which finishes the proof. ∎

2.5.2 Additive functionals of a discrete-time Markov chain

Notation

Let $\mathcal{X} = (\mathcal{X}_n)_{n\in\mathbb{N}_0}$ be a time-homogeneous Markov chain in the canonical space equipped with the time-shift operators $(\theta_n)_{n\in\mathbb{N}_0}$. For $n \geq 1$, put $\mathcal{F}_n := \sigma(\mathcal{X}_{[1,n]})$ (note that $\mathcal{X}_0 \notin \mathcal{F}_\infty$) and let P_χ denote the law of $(\mathcal{X}_n)_{n\in\mathbb{N}_0}$ when $\mathcal{X}_0 = \chi$. Fix an initial measure ν and suppose that, for any nonnegative $f \in \mathcal{F}_\infty$,

$$P_\nu(E_{\mathcal{X}_n}[f] \in \cdot) \ll P_\nu(E_{\mathcal{X}_0}[f] \in \cdot), \tag{2.5.25}$$

where $P_\nu := \int \nu(d\chi) P_\chi$.

Let $Z = (Z_n)_{n\in\mathbb{N}_0}$ be a $\mathbb{Z}^d$-valued $\mathcal{F}$-adapted process that is an *additive functional* of $(\mathcal{X}_n)_{n\in\mathbb{N}}$, i.e., $Z_0 = 0$ and, for any $k \in \mathbb{N}_0$,

$$(Z_{k+n} - Z_k)_{n\in\mathbb{N}_0} = \theta_k Z \qquad P_\nu\text{-a.s.} \tag{2.5.26}$$

We are interested in finding random times $(\tau_k)_{k\in\mathbb{N}_0}$ such that $(Z_{\tau_k}, \tau_k)_{k\in\mathbb{N}_0}$ satisfies the hypotheses of Lemma 2.5.3. In the Markovian setting it makes sense to look for τ_k of the form

$$\tau_0 = 0, \qquad \tau_{k+1} = \tau_k + \theta_{\tau_k}\tau, \quad k \in \mathbb{N}_0, \tag{2.5.27}$$

where τ is a random time.

Condition (i) of Lemma 2.5.3 is a "decoupling condition". It states that the law of an increment of the process depends weakly on the previous increments. Such a condition can be enforced by the occurrence of a "decoupling event" under which the increments of $(Z_{\tau_k}, \tau_k)_{k\in\mathbb{N}_0}$ lose dependence. In this setting, τ is a time at which the decoupling event is observed.

Second key proposition: approximate regeneration times

Proposition 2.5.4 below is a consequence of Lemma 2.5.3 and is the main result of this section. It will be used together with Proposition 2.5.2 to prove the LLN in Section 2.6. It gives a way to construct τ when the decoupling event can be detected by "probing the future" with a stopping time.

For a random variable $\mathcal{T}$ taking values in $\mathbb{N}_0 \cup \{\infty\}$, we define the *image* of $\mathcal{T}$ by $\mathcal{I}_{\mathcal{T}} := \{n \in \mathbb{N}\colon P_\nu(\mathcal{T} = n) > 0\}$, and its closure under addition by $\bar{\mathcal{I}}_{\mathcal{T}} := \{n \in \mathbb{N}\colon \exists\, l \in \mathbb{N},\, i_1, \ldots, i_l \in \mathcal{I}_{\mathcal{T}}\colon n = i_1 + \cdots + i_l\}$. Note that $\mathcal{I}_{\mathcal{T}} = \emptyset$ if and only if $\mathcal{T} \in \{0, \infty\}$ a.s.

Proposition 2.5.4. *Let $\mathcal{T}$ be a stopping time for the filtration $\mathcal{F}$ taking values in $\mathbb{N} \cup \{\infty\}$. Put $D := \{\mathcal{T} = \infty\}$ and suppose that the following properties hold:*
(i) For every $n \in \bar{\mathcal{I}}_{\mathcal{T}}$ there exists a $D_n \in \mathcal{F}_n$ such that

$$D \cap \theta_n D = D_n \cap \theta_n D \quad P_\nu\text{-a.s.}$$

(ii) There exist numbers $\rho \in (0,1]$, $a > 1$, $C > 0$, $m > 0$ and $\phi \in [0,1)$ such that, P_ν-a.s.,

> *(a)* $P_{\mathcal{X}_0}(D) \geq \rho$,
> *(b)* $E_{\mathcal{X}_0}\left[1_{\{\mathcal{T}<\infty\}}\mathcal{T}^a\right] \leq C^a$,
> *(c)* On D, $\|Z_t\| \leq mt$ for all $t \in \mathbb{N}_0$,
> *(d)* $\forall\, f \geq 0$ measurable,
> $\left|E_{\mathcal{X}_0}\left[f(Z, (\theta_n\mathcal{T})_{n\in\bar{\mathcal{I}}_{\mathcal{T}}}) \mid D\right] - E_\nu\left[f(Z, (\theta_n\mathcal{T})_{n\in\bar{\mathcal{I}}_{\mathcal{T}}}) \mid D\right]\right| \leq \phi\|f\|_\infty$.

Then there exists a random time $\tau \in \mathcal{F}_\infty$ taking values in $\mathbb{N}$ such that, setting τ_k as in (2.5.27) and $X_k := (Z_{\tau_k}, \tau_k)$, then $X \in AV(\varepsilon, (v, u))$ where $(v, u) = E_\nu[(Z_\tau, \tau) \mid D]$, $u > 0$ and $\varepsilon = 12(m+1)u\phi^{(a-1)/a}$.

Two further propositions

In order to prove Proposition 2.5.4, we will need two further propositions (Propositions 2.5.5 and 2.5.6 below).

Proposition 2.5.5. *Let τ be a random time measurable w.r.t. $\mathcal{F}_\infty$ taking values in $\mathbb{N}$. Put τ_k as in (2.5.27) and $X_k := (Z_{\tau_k}, \tau_k)$. Suppose that there exists an event $D \in \mathcal{F}_\infty$ such that the following hold P_ν-a.s.:*
(i) For $n \in \mathcal{I}_\tau$, there exist events H_n and $D_n \in \mathcal{F}_n$ such that

$$
\begin{aligned}
&\text{(a)}\quad \{\tau = n\} = H_n \cap \theta_n D,\\
&\text{(b)}\quad D \cap \theta_n D = D_n \cap \theta_n D.
\end{aligned}
\tag{2.5.28}
$$

(ii) There exist $\phi \in [0,1)$, $K > 0$ and $a > 1$ such that, on $\{P_{\mathcal{X}_0}(D) > 0\}$,

$$
\begin{aligned}
&\text{(a)}\quad E_{\mathcal{X}_0}[\|X_1\|^a|D] \leq K^a,\\
&\text{(b)}\quad |P_{\mathcal{X}_0}(X_1 \in A|D) - P_\nu(X_1 \in A|D)| \leq \phi \quad \forall\, A \text{ measurable.}
\end{aligned}
\tag{2.5.29}
$$

Then $X \in AV\big(\varepsilon, (v, u)\big)$, where $\varepsilon = 2K\phi^{(a-1)/a}$ and $(v, u) := E_\nu[X_1|D]$.

Proof. Since $\tau < \infty$, by (i)(a) and (2.5.25) we must have $P_\nu(D) > 0$. Let $\mathcal{F}_{\tau_k}$ be the σ-algebra of the events $B \in \mathcal{F}_\infty$ such that, for all $n \in \mathbb{N}$, there exists $B_n \in \mathcal{F}_n$ with $B \cap \{\tau_k = n\} = B_n \cap \{\tau_k = n\}$. We will show that, P_ν-a.s., for all $k \in \mathbb{N}$,

$$E_\nu \left[\|\theta_{\tau_k} X_1\|^a | \mathcal{F}_{\tau_k} \right] \leq K^a \tag{2.5.30}$$

and

$$|P_\nu \left(\theta_{\tau_k} X_1 \in A | \mathcal{F}_{\tau_k} \right) - P_\nu(X_1 \in A | D)| \leq \phi \quad \forall A \text{ measurable}. \tag{2.5.31}$$

Then, setting $Q(\cdot) := P_\nu(X_1 \in \cdot | D)$ and noting that $\theta_{\tau_k} X_1 = X_{k+1} - X_k$ and $X_j \in \mathcal{F}_{\tau_k}$ for all $0 \leq j \leq k$, we will be able to conclude since (2.5.30–2.5.31) and (ii)(a) imply that the conditions of Lemma 2.5.3 are all satisfied.

To prove (2.5.30–2.5.31), first note that, using (i), one can verify by induction that (i)(a) holds also for τ_k, i.e., for every $n \in \mathcal{I}_{\tau_k}$ there exists $H_{k,n} \in \mathcal{F}_n$ such that

$$\{\tau_k = n\} = H_{k,n} \cap \theta_n D \qquad P_\nu\text{-a.s.} \tag{2.5.32}$$

Take $B \in \mathcal{F}_{\tau_k}$ and a measurable nonnegative function f, and write

$$E_\nu \left[1_B \theta_{\tau_k} f(X_1) \right] = \sum_{n \in \mathcal{I}_{\tau_k}} E_\nu \left[1_{B \cap \{\tau_k = n\}} \theta_n f(X_1) \right] = \sum_{n \in \mathcal{I}_{\tau_k}} E_\nu \left[1_{B_n \cap H_{k,n}} \theta_n \big(1_D f(X_1) \big) \right]$$

$$= \sum_{n \in \mathcal{I}_{\tau_k}} E_\nu \left[1_{B_n \cap H_{k,n}} P_{\mathcal{X}_n}(D) E_{\mathcal{X}_n} \left[f(X_1) | D \right] \right]. \tag{2.5.33}$$

Noting that $P_\nu(B) = \sum_{n \in \mathcal{I}_{\tau_k}} E_\nu \left[1_{B_n \cap H_{k,n}} P_{\mathcal{X}_n}(D) \right]$, obtain (2.5.30) by taking $f(x) = \|x\|^a$ and using (ii)(a) together with (2.5.25). For (2.5.31), choose $f = 1_A$, subtract $P_\nu(B) E_\nu \left[f(X_1) | D \right]$ from (2.5.33) and use (ii)(b). ∎

Proposition 2.5.6. *Let $\mathcal{T}$ be a stopping time as in Proposition 2.5.4 and suppose that conditions (ii)(a) and (ii)(b) of that proposition are satisfied. Define a sequence of stopping times $(T_k)_{k \in \mathbb{N}_0}$ as follows. Put $T_0 = 0$ and, for $k \in \mathbb{N}_0$,*

$$T_{k+1} := \begin{cases} \infty & \text{if } T_k = \infty \\ T_k + \theta_{T_k} \mathcal{T} & \text{otherwise.} \end{cases} \tag{2.5.34}$$

Put

$$N := \inf\{k \in \mathbb{N}_0 : T_k < \infty \text{ and } T_{k+1} = \infty\}. \tag{2.5.35}$$

Then $N < \infty$ a.s. and there exists a constant $\varkappa = \varkappa(a, \rho) \in (0, \infty)$ such that, P_ν-a.s.,

$$E_{\mathcal{X}_0} \left[T_N^a \right] \leq (\varkappa C)^a. \tag{2.5.36}$$

Furthermore, $\mathcal{I}_{T_N} \subset \bar{\mathcal{I}}_{\mathcal{T}}$.

Proof. First, let us check that

$$P_{\mathcal{X}_0}(N \geq n) \leq (1 - \rho)^n. \tag{2.5.37}$$

Indeed, $N \geq n$ if and only if $T_n < \infty$, so that, for $k \in \mathbb{N}_0$,

$$P_{\mathcal{X}_0}(T_{k+1} < \infty) = E_{\mathcal{X}_0}\left[1_{\{T_k < \infty\}} P_{\mathcal{X}_{T_k}}(\mathcal{T} < \infty)\right] \leq (1 - \rho) P_{\mathcal{X}_0}(T_k < \infty), \tag{2.5.38}$$

where we use (ii)(a) and the fact that (2.5.25) holds also with a stopping time in place of n. Clearly, (2.5.37) follows from (2.5.38) by induction. In particular, $N < \infty$ a.s.

From (2.5.34) we see that, for $0 \leq k \leq n$,

$$T_n = T_k + \theta_{T_k} T_{n-k} \quad \text{on} \quad \{T_k < \infty\}. \tag{2.5.39}$$

Using (ii)(a) and (b), with the help of (2.5.25) again, we can a.s. estimate, for $0 \leq k < n$,

$$
\begin{aligned}
E_{\mathcal{X}_0}\left[1_{\{T_n < \infty\}} |T_{k+1} - T_k|^a\right] &= E_{\mathcal{X}_0}\left[1_{\{T_{k+1} < \infty\}} |T_{k+1} - T_k|^a P_{\mathcal{X}_{T_{k+1}}}(T_{n-k-1} < \infty)\right] \\
&\leq (1 - \rho)^{n-k-1} E_{\mathcal{X}_0}\left[1_{\{T_k < \infty, \theta_{T_k}\mathcal{T} < \infty\}} \theta_{T_k}\mathcal{T}^a\right] \\
&= (1 - \rho)^{n-k-1} E_{\mathcal{X}_0}\left[1_{\{T_k < \infty\}} E_{\mathcal{X}_{T_k}}\left[1_{\{\mathcal{T} < \infty\}} \mathcal{T}^a\right]\right] \\
&\leq (1 - \rho)^{n-k-1} C^a P_{\mathcal{X}_0}(T_k < \infty) \\
&\leq (1 - \rho)^{n-1} C^a.
\end{aligned}
\tag{2.5.40}
$$

Now write

$$T_N = \sum_{k=0}^{N-1} T_{k+1} - T_k. \tag{2.5.41}$$

By Jensen's inequality,

$$T_N^a \leq N^{a-1} \sum_{k=0}^{N-1} |T_{k+1} - T_k|^a \tag{2.5.42}$$

so that, by (2.5.40),

$$E_{\mathcal{X}_0}[T_N^a] \leq \sum_{n=1}^{\infty} n^{a-1} \sum_{k=0}^{n-1} E_{\mathcal{X}_0}\left[1_{\{N=n\}} |T_{k+1} - T_k|^a\right] \leq C^a \sum_{n=1}^{\infty} n^a (1 - \rho)^{n-1} \quad \text{a.s.} \tag{2.5.43}$$

and (2.5.36) follows by taking $\varkappa = (\sum_{n=1}^{\infty} n^a (1 - \rho)^{n-1})^{1/a}$.

As for the claim that $\mathcal{I}_{T_N} \subset \bar{\mathcal{I}}_{\mathcal{T}}$, write, for $n \in \mathbb{N}$,

$$\{T_N = n\} = \sum_{k=1}^{\infty} \{T_k = n, N = k\} \tag{2.5.44}$$

to see that $\mathcal{I}_{T_N} \subset \bigcup_{k=1}^{\infty} \mathcal{I}_{T_k}$. Using (2.5.34), we can verify by induction that, for each $k \in \mathbb{N}$, $\mathcal{I}_{T_k} \subset \{n \in \mathbb{N} : \exists i_1, \ldots, i_k \in \mathcal{I}_{\mathcal{T}} : n = i_1 + \cdots + i_k\} \subset \bar{\mathcal{I}}_{\mathcal{T}}$, and the claim follows. ∎

Proof of Proposition 2.5.4

We can now combine Propositions 2.5.5 and 2.5.6 to prove Proposition 2.5.4.

Proof. In the following we will refer to the hypotheses of Proposition 2.5.5 with the prefix P. For example, P(i)(a) denotes hypothesis (i)(a) in that proposition. The hypotheses in Proposition 2.5.4 will be referred to without a prefix. Since the hypotheses of Proposition 2.5.6 are a subset of those of Proposition 2.5.4, the conclusions of the former are valid.

We will show that, if $\tau := t_0 + \theta_{t_0} T_N$ for a suitable $t_0 \in \mathbb{N}$, then τ satisfies the hypotheses of Proposition 2.5.5 for a suitable K. There are two cases. If $\mathcal{I}_{\mathcal{T}} = \emptyset$, then $T_N \equiv 0$. Choosing $t_0 = 1$, we basically fall in the context of Lemma 2.5.3. P(i)(a) and P(i)(b) are trivial, (ii)(c) implies that P(ii)(a) holds with $K = (m+1)$, while P(ii)(b) follows immediately from (ii)(d). Therefore, we may suppose that $\mathcal{I}_{\mathcal{T}} \neq \emptyset$ and put $\iota := \min \mathcal{I}_{\mathcal{T}} \in \mathbb{N}$. Let $\hat{C} := 1 \vee (\varkappa C)$ and $t_0 := \iota \lceil \hat{C} \rho^{-1/a} \rceil$. We will show that τ satisfies the hypotheses of Proposition 2.5.5 with $K = 6\iota(m+1)\hat{C}\rho^{-1/a}$.

P(i)(a): First we show that this property is true for T_N. Indeed,

$$\{T_N = n\} = \sum_{k \in \mathbb{N}_0} \{N = k, T_k = n\} = \sum_{k \in \mathbb{N}_0} \{T_k = n, \theta_n \mathcal{T} = \infty\} \qquad (2.5.45)$$

$$= \theta_n D \cap \left(\bigcup_{k \in \mathbb{N}_0} \{T_k = n\} \right), \qquad (2.5.46)$$

and $\hat{H}_n := \bigcup_{k \in \mathbb{N}_0} \{T_k = n\} \in \mathcal{F}_n$ since the T_k's are all stopping times. Now we observe that $\{\tau = n\} = \theta_{t_0}\{T_N = n - t_0\}$, so we can take $H_n := \emptyset$ if $n < t_0$ and $H_n := \theta_{t_0} \hat{H}_{n-t_0}$ otherwise.

P(i)(b): By (i), it suffices to show that $\mathcal{I}_\tau \subset \bar{\mathcal{I}}_{\mathcal{T}}$. Since $t_0 \in \bar{\mathcal{I}}_{\mathcal{T}}$ (as an integer multiple of ι), this follows from the definition of τ and the last conclusion of Proposition 2.5.6.

P(ii)(a): By (ii)(c), $\|X_1\|^a = (\|Z_\tau\| + \tau)^a \leq ((m+1)\tau)^a$ on D. Therefore, we just need to show that

$$E_{\mathcal{X}_0}[\tau^a | D] \leq (6\iota\hat{C})^a/\rho. \qquad (2.5.47)$$

Now, $\tau^a \leq 2^{a-1}(t_0^a + \theta_{t_0} T_N^a)$ and, by Proposition 2.5.6 and (2.5.25),

$$E_{\mathcal{X}_0}[\theta_{t_0} T_N^a] = E_{\mathcal{X}_0}\left[E_{\mathcal{X}_{t_0}}[T_N^a]\right] \leq \hat{C}^a. \qquad (2.5.48)$$

Using (ii)(a), we obtain

$$E_{\mathcal{X}_0}[\theta_{t_0} T_N^a | D] \leq \hat{C}^a/\rho. \qquad (2.5.49)$$

Since $t_0 \leq 2\iota\hat{C}\rho^{-1/a}$ and $\iota \geq 1$, (2.5.47) follows.

P(ii)(b): Let $S = (S_n)_{n \in \bar{\mathcal{I}}_{\mathcal{T}}}$ with $S_n := \theta_n \mathcal{T}$. By (ii)(d), it is enough to show that $X_1 = (Z_\tau, \tau) \in \sigma(Z, S)$ a.s.. Since $Z_\tau = \sum_{n=0}^{\infty} 1_{\{\tau=n\}} Z_n \in \sigma(Z, \tau)$, it suffices to show that $\tau \in \sigma(S)$ a.s. Using the definition of the T_k's, we verify by induction that each T_k is a.s. measurable in $\sigma(S)$. Since $N \in \sigma((T_k)_{k \in \mathbb{N}_0})$, both N and T_N are also a.s. in $\sigma(S)$. Therefore, a.s. $\tau \in \sigma(\theta_{t_0} S) \subset \sigma(S)$.

With all hypotheses verified, Proposition 2.5.5 implies that $X \in AV(\hat{\varepsilon}, (v, u))$, where $(v, u) = E_\nu[X_1|D]$ and $\hat{\varepsilon} = 2K\phi^{(a-1)/a}$. To conclude, observe that $u = E_\nu[\tau|D] \geq t_0 \geq \iota\hat{C}\rho^{-1/a} > 0$, so that $K = 6(m+1)\iota\hat{C}\rho^{-1/a} \leq 6(m+1)u$. Therefore, $\hat{\varepsilon} \leq \varepsilon$ and the proposition follows. In the case $\mathcal{I}_\mathcal{T} = \emptyset$, we conclude similarly since $u = 1$ and $K = (m+1)$. ∎

2.6 Proof of the law of large numbers

In this section we show how to put the model defined in Section 2.3 in the context of Section 2.5, and we prove Theorem 2.4.1 using Propositions 2.5.2 and 2.5.4.

2.6.1 Two further lemmas

Before we start, we first derive two lemmas (Lemmas 2.6.1 and 2.6.2 below) that will be needed in Section 2.6.2. The first lemma relates the laws of the environment as seen from W_n and from W_0. The second lemma is an extension of the conditional cone-mixing property for functions that depend also on Y.

Lemma 2.6.1. $\bar{\mu}_n \ll \bar{\mu}$ *for all* $n \in \mathbb{N}$.

Proof. For $t \geq 0$, let $\bar{\mu}_{t-}$ denote the law of $\theta_{W_{t-}}\xi_t$ under $\mathbb{P}_\mu$. First we will show that $\bar{\mu}_{t-} \ll \mu$. This is a consequence of the fact that μ is translation-invariant equilibrium, and remains true if we replace W_{t-} by any random variable taking values in $\mathbb{Z}^d$. Indeed, if $\mu(A) = 0$ then $\mathbb{P}_\mu(\theta_x\xi_t \in A) = 0$ for every $x \in \mathbb{Z}^d$, so

$$\bar{\mu}_{t-}(A) = \mathbb{P}_\mu(\theta_{W_{t-}}\xi_t \in A) = \sum_{x \in \mathbb{Z}^d} \mathbb{P}_\mu(W_{t-} = x, \theta_x\xi_t \in A) = 0. \tag{2.6.1}$$

Now take $n \in \mathbb{N}$ and let $g_n := \frac{d\bar{\mu}_{n-}}{d\mu}$. For any measurable $f \geq 0$,

$$\begin{aligned}
\mathbb{E}_\mu\left[f(\theta_{W_n}\xi_n)\right] = \mathbb{E}_{\bar{\mu}}\left[f(\theta_{Z_n}\xi_n)\right] &= \sum_{x \in \mathbb{Z}^d} \mathbb{E}_{\bar{\mu}}\left[1_{\{Z_n - Z_{n-}=x\}}f(\theta_x\theta_{Z_{n-}}\xi_n)\right] \\
&= \sum_{x \in \mathbb{Z}^d} \mathbb{E}_{\bar{\mu}}\left[\mathbb{P}_{\theta_{Z_{n-}}\xi_n}(W_0 = x)f(\theta_x\theta_{Z_{n-}}\xi_n)\right] \\
&= \sum_{x \in \mathbb{Z}^d} \mathbb{E}_\mu\left[\mathbb{P}_{\theta_{W_{n-}}\xi_n}(W_0 = x)f(\theta_x\theta_{W_{n-}}\xi_n)\right] \\
&= \sum_{x \in \mathbb{Z}^d} \mathbb{E}_\mu\left[g_n(\xi_0)\mathbb{P}_{\xi_0}(W_0 = x)f(\theta_x\xi_0)\right] \\
&= \sum_{x \in \mathbb{Z}^d} \mathbb{E}_\mu\left[g_n(\xi_0)1_{\{W_0=x\}}f(\theta_x\xi_0)\right] \\
&= \mathbb{E}_\mu\left[g_n(\xi_0)f(\theta_{W_0}\xi_0)\right] \tag{2.6.2}
\end{aligned}$$

where, for the second equality, we use (A3). ∎

Lemma 2.6.2. *For L large enough and for all nonnegative $f \in \mathscr{C}_{R,\infty}(m) \vee \mathscr{Y}_\infty$,*

$$|\mathbb{E}_\eta\left[\theta_L f \mid \Gamma_L\right] - \mathbb{E}_{\bar{\mu}}[\theta_L f \mid \Gamma_L]| \leq \Phi_L \|f\|_\infty \quad \textit{for } \bar{\mu}\textit{-a.e. } \eta. \tag{2.6.3}$$

Proof. Put $f_y(\eta) = f(\eta, y)$ and abbreviate $Y^{(L)} = (Y_k)_{k>L}$. Then $\theta_L f = \theta_L f_{Y^{(L)}}$. Since Γ_L depends on Y only through $(Y_k)_{k \leq L}$, we have

$$\mathbb{E}_\eta[\theta_L f \, 1_{\Gamma_L} \mid Y^{(L)}] = \mathbb{E}_\eta\left[\theta_L f_{(\cdot)} \, 1_{\Gamma_L}\right] \circ (Y^{(L)}), \tag{2.6.4}$$

and (2.6.3) follows from (H4) applied to f_y. ∎

2.6.2 Proof of Theorem 2.4.1

Proof. Extend ξ and Z for times $t \in [-1, 0]$ by taking them constant in this interval, and let Y_0 be a copy of Y_1 independent of $\mathscr{F}_\infty$. Put

$$\begin{aligned}
\mathcal{X}_0 &:= \left(\xi_{[-1,0]}, Z_{[-1,0]}, Y_0\right), \\
\mathcal{X}_{n+1} &:= \left(\theta_{Z_n}\xi_{[n,n+1]}, (Z_{t+n} - Z_n)_{0 \leq t \leq 1}, Y_{n+1}\right), \quad n \in \mathbb{N}_0.
\end{aligned} \tag{2.6.5}$$

Then $(\mathcal{X}_n)_{n \in \mathbb{N}_0}$ is a time-homogeneous Markov chain; to avoid confusion, we will denote its time-shift operator by $\bar{\theta}_n$. Note that $\mathcal{F}_n = \mathscr{F}_n \, \forall \, n \in \mathbb{N} \cup \{\infty\}$ and that, for functions $f \in \mathcal{F}_\infty$, $\bar{\theta}_n f = \theta_{Z_n} \theta_n f \, \forall \, n \in \mathbb{N}_0$.

Fix $L \in \mathbb{N}$ large enough and put

$$\mathcal{T}_L := L + 1_{\Gamma_L}\lceil \theta_L \mathcal{S}\rceil. \tag{2.6.6}$$

By (2.3.5) and since $\Gamma_L \in \mathscr{F}_L$ and Z is $\mathscr{F}$-adapted, $\mathcal{T}_L$ is an $\mathcal{F}$-stopping time and $(Z_n)_{n \in \mathbb{N}_0}$ is an additive functional of $(\mathcal{X}_n)_{n \in \mathbb{N}}$ as in Section 2.5.2.

Next, we will verify (2.5.25) for $\mathcal{X}$ and the hypotheses of Proposition 2.5.4 for Z and $\mathcal{T}_L$ under $\mathbb{P}_{\bar{\mu}}$. These hypotheses will be referred to with the prefix P. The notation here is consistent in the sense that parameters in Section 2.3 are named according to their role in Section 2.5; the presence/absence of a subscript L indicates whether the parameter depends on L or not.

(2.5.25): Noting that, for nonnegative $f \in \mathcal{F}_\infty$ and $n \in \mathbb{N}_0$,

$$E_{\mathcal{X}_n}\left[f\right] = \mathbb{E}_{\theta_{Z_n}\xi_n}\left[f\right] \qquad \mathbb{P}_{\bar{\mu}}\text{-a.s.}, \tag{2.6.7}$$

this follows from Lemma 2.6.1 and (2.3.3–2.3.4).

P(i): We will find D_n for $n \geq L$. This is enough, since both $\mathcal{I}_{\mathcal{T}_L}$ and $\bar{\mathcal{I}}_{\mathcal{T}_L}$ are subsets of $[L, \infty) \cap \mathbb{N}$. Using (A1) and (H1), we may write

$$\begin{aligned}
D &= \Gamma_L \cap \{\|Z_{t+L}\| \leq mt \, \forall \, t \geq 0\}, \\
\bar{\theta}_n D &= \bar{\theta}_n \Gamma_L \cap \{\|Z_{t+n+L} - Z_n\| \leq mt \, \forall \, t \geq 0\}.
\end{aligned} \tag{2.6.8}$$

Intersecting the two above events, we get

$$D \cap \bar{\theta}_n D = \Gamma_L \cap \{\|Z_t\| \le mt \ \forall \ t \in [0, n]\} \cap \bar{\theta}_n D, \tag{2.6.9}$$

i.e., P(i) holds with $D_n := \Gamma_L \cap \{\|Z_t\| \le mt \ \forall \ t \in [0, n]\} \in \mathcal{F}_n$ for $n \ge L$.

For the remaining items, note that, by (2.6.7), the distribution of $(Z, \mathcal{T}_L)$ under $P_{\mathcal{X}_0}$ is $\mathbb{P}_{\bar{\mu}}$-a.s. the same as under $\mathbb{P}_{\xi_0}$.

P(ii)(a): Since $\{\mathcal{T}_L = \infty\} = \{\theta_L \mathcal{S} = \infty\} \cap \Gamma_L$, we get from (H2) and (H3)(1) that, $\mathbb{P}_{\bar{\mu}}$-a.s.,

$$\mathbb{P}_{\xi_0}(\mathcal{T}_L = \infty) = \mathbb{P}_{\xi_0}(\theta_L \mathcal{S} = \infty \mid \Gamma_L) \, \mathbb{P}_{\xi_0}(\Gamma_L) \ge \kappa_L \gamma_L > 0, \tag{2.6.10}$$

so that we can take $\rho_L := \kappa_L \gamma_L$.

P(ii)(b): By the definition of $\mathcal{T}_L$, we have

$$\begin{aligned} \mathcal{T}_L^a 1_{\{\mathcal{T}_L < \infty\}} &= L^a 1_{\Gamma_L^c} + (L + \lceil \theta_L \mathcal{S} \rceil)^a \, 1_{\Gamma_L \cap \{\theta_L \mathcal{S} < \infty\}} \\ &\le L^a 1_{\Gamma_L^c} + (L + 1 + \theta_L \mathcal{S})^a \, 1_{\Gamma_L \cap \{\theta_L \mathcal{S} < \infty\}} \\ &\le 2^{a-1}(L+1)^a + 2^{a-1} \left((\theta_L \mathcal{S})^a 1_{\{\theta_L \mathcal{S} < \infty\}} \right) 1_{\Gamma_L}. \end{aligned} \tag{2.6.11}$$

Therefore, by (H3)(2), we get

$$\mathbb{E}_{\xi_0}\left[\mathcal{T}_L^a 1_{\{\mathcal{T}_L < \infty\}} \right] \le 2^a((L+1)^a + (1 \vee \psi_L)^a) \le [2(L+1+1 \vee \psi_L)]^a \quad \mathbb{P}_{\bar{\mu}}\text{-a.s.}, \tag{2.6.12}$$

so that we can take $C_L := 2(L + 1 + 1 \vee \psi_L)$.

P(ii)(c): This follows from (H1) and the definition of $\mathcal{S}$.

P(ii)(d): First note that, for any $n \in \bar{\mathcal{I}}_{\mathcal{T}_L}$, $\bar{\theta}_n \mathcal{T}_L \in \sigma(Z, \bar{\theta}_n \Gamma_L)$. Since $n \ge L$, on $\{\mathcal{T}_L = \infty\} = \overline{\Gamma_L \cap \{\theta_L \mathcal{S} = \infty\}}$, $Z, \bar{\theta}_n \Gamma_L$ and $\{\theta_L \mathcal{S} = \infty\}$ are all measurable in $\theta_L(\mathscr{C}_{R,\infty}(m) \vee \mathscr{Y}_\infty)$; this follows from (A2), (H1) and the assumptions on Γ_L. Noting that, for any two probability measures ν_1, ν_2 and an event A,

$$\|\nu_1(\cdot \mid A) - \nu_2(\cdot \mid A)\|_{TV} \le 2 \frac{\|\nu_1 - \nu_2\|_{TV}}{\nu_1(A) \vee \nu_2(A)} \tag{2.6.13}$$

where $\|\cdot\|_{TV}$ stands for total variation distance, we see that P(ii)(d) follows from Lemma 2.6.2 and (H3)(1) with $\phi_L := 2\Phi_L/\kappa_L \to 0$ as $L \to \infty$ by (H4).

Thus, for large enough L, we can conclude by Proposition 2.5.4 that there exists a sequence of times $(\tau_k)_{k \in \mathbb{N}_0}$ with $\lim_{k \to \infty} \tau_k = \infty$ a.s. such that

$$(Z_{\tau_k}, \tau_k)_{k \in \mathbb{N}_0} \in AV(\varepsilon_L, (v_L, u_L)),$$

where

$$\begin{aligned} v_L &= \mathbb{E}_{\bar{\mu}}[Z_{\tau_1} | D], \\ u_L &= \mathbb{E}_{\bar{\mu}}[\tau_1 | D] > 0, \\ \varepsilon_L &= 12(m+1)u_L \phi_L^{(a-1)/a}. \end{aligned} \tag{2.6.14}$$

From (2.6.14) and P(ii)(c), Proposition 2.5.2 implies that $Z \in AV(\delta_L, w_L)$, where

$$
\begin{aligned}
w_L &= v_L/u_L, \\
\delta_L &= (3m+1)12(m+1)\phi_L^{(a-1)/a}.
\end{aligned}
\tag{2.6.15}
$$

By (H4), $\lim_{L\to\infty} \delta_L = 0$. As was observed after Definition 2.5.1, this implies that $w := \lim_{L\to\infty} w_L$ exists and that $\lim_{t\to\infty} t^{-1} Z_t = w$ $\mathbb{P}_{\bar{\mu}}$-a.s., which, by (2.3.3–2.3.4), implies the same for W, $\mathbb{P}_\mu$-a.s. ∎

We have at this point finished the proof of our LLN. In the following sections, we will look at examples that satisfy (H1)–(H4). Section 2.7 is devoted to the $(\infty, 0)$-model for two classes of one-dimensional spin-flip systems. In Section 2.8 we discuss three additional models where the hypotheses are satisfied, and one where they are not.

2.7 Verification of the examples for the $(\infty, 0)$-model

In this Section we give the proof of Theorem 2.4.2. We begin with a proper definition of the $(\infty, 0)$-model in Section 2.7.1, where we identify Z and W_0 of Section 2.3.2. In Section 2.7.2, we first define suitable versions of spin-flip systems with bounded rates. After checking assumptions (A1)–(A3), we define events Γ_L satisfying (H1) and (H2) for which we then verify (H3). We also derive uniform integrability properties of $t^{-1} W_t$ which are the key for convergence in L^p once we have the LLN. In Sections 2.7.3 and 2.7.4, we specialize to particular constructions in order to prove (H4), which is the hardest of the four hypotheses. Section 2.7.5 is devoted to proving a criterion for positive or negative speed.

2.7.1 Definition of the model

Assume that ξ is a càdlàg process with state space $E := \{0, 1\}^{\mathbb{Z}}$. We will define the walk W in several steps, and a monotonicity property will follow.

Identification of Z and W_0

First, let $Tr^+ = Tr^+(\eta)$ and $Tr^- = Tr^-(\eta)$ denote the locations of the closest traps to the right and to the left of the origin in the configuration $\eta \in E$, i.e.,

$$
\begin{aligned}
Tr^+(\eta) &:= \inf\{x \in \mathbb{N}_0 \colon \eta(x) = 1, \eta(x+1) = 0\}, \\
Tr^-(\eta) &:= \sup\{x \in -\mathbb{N}_0 \colon \eta(x) = 1, \eta(x+1) = 0\},
\end{aligned}
\tag{2.7.1}
$$

with the convention that $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$. For $i, j \in \{0, 1\}$, abbreviate $\langle i, j \rangle := \{\eta \in E \colon \eta(0) = i, \eta(1) = j\}$. Let $\bar{E} := \langle 1, 0 \rangle$, i.e., the set of all the configurations with a trap at the origin.

Next, we define the functional J that gives the jumps in W. For $b \in \{0,1\}$ and $\eta \in E$, let

$$J(\eta, b) := Tr^+ \left(1_{\langle 1,1 \rangle} + b 1_{\langle 0,1 \rangle}\right) + Tr^- \left(1_{\langle 0,0 \rangle} + (1-b) 1_{\langle 0,1 \rangle}\right), \tag{2.7.2}$$

i.e., J is equal to either the left or the right trap, depending on the configuration around the origin. In the case of an inverted trap ($\langle 0,1 \rangle$), the direction of the jump is decided by the value of b. Observe that $J = Tr^+ = Tr^- = 0$ when $\eta \in \bar{E}$, independently of the value of b.

Let b_0 be a Bernoulli($\frac{1}{2}$) random variable independent of ξ and set

$$W_0 = X_0 := J(\xi_0, b_0). \tag{2.7.3}$$

Now let $(b_{n,k})_{n,k \in \mathbb{N}}$ be a double-indexed i.i.d. sequence of Bernoulli($\frac{1}{2}$) r.v.'s independent of (ξ, b_0). Put $\tau_0 := 0$ and, for $k \geq 0$,

$$
\begin{aligned}
\tau_{k+1} &:= \begin{cases} \infty & \text{if } |X_k| = \infty, \\ \inf\{t > \tau_k: \ (\xi_t(X_k), \xi_t(X_k + 1)) \neq (1,0)\} & \text{otherwise}, \end{cases} \\
X_{k+1} &:= \begin{cases} X_k & \text{if } \tau_{k+1} = \infty, \\ X_k + J\left(\theta_{X_k} \xi_{\tau_k}, b_{\lceil \tau_{k+1} \rceil, k+1}\right) & \text{otherwise}. \end{cases}
\end{aligned}
\tag{2.7.4}
$$

Since ξ is càdlàg, for any $k \in \mathbb{N}_0$ we either have $\tau_k = \infty$ or $\tau_{k+1} > \tau_k$. We define $(W_t)_{t \geq 0}$ as the path that jumps $X_{k+1} - X_k$ at time τ_{k+1} and is constant between jumps, i.e.,

$$W_t := \sum_{k=0}^{\infty} 1_{\{\tau_k \leq t < \tau_{k+1}\}} X_k. \tag{2.7.5}$$

With this definition, it is clear that W_t is càdlàg and, by (2.7.3–2.7.4),

$$W_{n+t} - W_n = \theta_{W_n} \theta_n W_t \ \text{ on } \{W_n < \infty\} \ \forall \, n \in \mathbb{N}_0, t \geq 0. \tag{2.7.6}$$

Therefore, defining Z by

$$Z_t := 1_{\{\xi_0 \in \bar{E}\}} W_t, \qquad t \geq 0, \tag{2.7.7}$$

we get $W_t = W_0 + \theta_{W_0} Z_t$ on $\{W_0 < \infty\}$ since, in this case, $\theta_{W_0} \xi_0 \in \bar{E}$, and $W_0 = 0$ on $\bar{E}$.

Monotonicity

The following monotonicity property will be helpful in checking (H3). In order to state it, we first endow both E and $D([0,\infty), E)$ with the usual partial ordering, i.e., for $\eta_1, \eta_2 \in E$, $\eta_1 \leq \eta_2$ means that $\eta_1(x) \leq \eta_2(x)$ for all $x \in \mathbb{Z}$, while, for $\xi^{(1)}, \xi^{(2)} \in D([0,\infty), E)$, $\xi^{(1)} \leq \xi^{(2)}$ means that $\xi_t^{(1)} \leq \xi_t^{(2)}$ for all $t \geq 0$.

Lemma 2.7.1. *Fix a realization of b_0 and $(b_{n,k})_{n,k \in \mathbb{N}}$. If $\xi^{(1)} \leq \xi^{(2)}$, then*

$$W_t\left(\xi^{(1)}, b_0, (b_{n,k})_{n,k \in \mathbb{N}}\right) \leq W_t\left(\xi^{(2)}, b_0, (b_{n,k})_{n,k \in \mathbb{N}}\right) \tag{2.7.8}$$

for all $t \geq 0$.

Proof. This is a straightforward consequence of the definition. We need only to understand what happens when the two walks separate and, at such moments, the second walk is always to the right of the first. ∎

2.7.2 Spin-flip systems with bounded flip rates

Dynamic random environment

From now on we will take ξ to be a single-site spin-flip system with translation-invariant and bounded flip rates. We may assume that the rates at the origin are of the form

$$c(\eta) = \begin{cases} c_0 + \lambda_0 p_0(\eta) & \text{when} \quad \eta(0) = 1, \\ c_1 + \lambda_1 p_1(\eta) & \text{when} \quad \eta(0) = 0, \end{cases} \tag{2.7.9}$$

where $c_i, \lambda_i > 0$ and $p_i \in [0, 1]$. We assume the existence conditions of Liggett [57], Chapter I, which in our setting amounts to the additional requirement that $c(\cdot)$ has finite triple norm. This is automatically satisfied in the $M < \epsilon$ regime or when $c(\cdot)$ has finite range.

From (2.7.9), we see that the IPS is stochastically dominated by the system ξ^+ with rates

$$c^+(\eta) = \begin{cases} c_0 & \text{when} \quad \eta(0) = 1, \\ c_1 + \lambda_1 & \text{when} \quad \eta(0) = 0, \end{cases} \tag{2.7.10}$$

while it stochastically dominates the system ξ^- with rates

$$c^-(\eta) = \begin{cases} c_0 + \lambda_0 & \text{when} \quad \eta(0) = 1, \\ c_1 & \text{when} \quad \eta(0) = 0. \end{cases} \tag{2.7.11}$$

These are the rates of two independent spin-flip systems with respective densities $\rho^+ := (c_1 + \lambda_1)/\lambda^+$ and $\rho^- := c_1/\lambda^-$ where $\lambda^+ := c_0 + c_1 + \lambda_1$ and $\lambda^- := c_0 + \lambda_0 + c_1$. Consequently, any equilibrium for ξ is stochastically dominated by ν_{ρ^+} and dominates ν_{ρ^-}, where ν_ρ is a Bernoulli product measure with density ρ.

We will take as the dynamic random environment the triple $\Xi := (\xi^-, \xi, \xi^+)$ starting from the same initial configuration and coupled together via the basic (or Vasershtein) coupling, which implements the stochastic ordering as an a.s. partial ordering. More precisely, Ξ is the IPS with state space E^3 whose rates are translation invariant and at the origin are given schematically by (the configuration of the middle coordinate is η),

$$\begin{array}{ll}
(000) \rightarrow & \begin{cases} (111) & c_1, \\ (011) & c(\eta) - c_1, \\ (001) & c_1 + \lambda_1 - c(\eta), \end{cases} \\
(001) \rightarrow & \begin{cases} (111) & c_1, \\ (011) & c(\eta) - c_1, \\ (000) & c_0, \end{cases} \\
(011) \rightarrow & \begin{cases} (111) & c_1, \\ (000) & c_0, \\ (001) & c(\eta) - c_0, \end{cases} \\
(111) \rightarrow & \begin{cases} (000) & c_0, \\ (001) & c(\eta) - c_0, \\ (011) & c_0 + \lambda_0 - c(\eta). \end{cases}
\end{array} \tag{2.7.12}$$

Verification of (A1)–(A3)

Under our assumptions, $\lim_{k\to\infty} \tau_k = \infty$ and $X_0 < \infty$ $\mathbb{P}_\mu$-a.s., as ξ has bounded flip rates per site and μ dominates and is dominated by non-trivial product measures. By induction, $X_k < \infty$ a.s. for every $k \in \mathbb{N}$ as well, since the law of $\theta_{X_{k-1}} \xi_{\tau_k}$ is absolutely continuous w.r.t. μ, which can be verified by approximating τ_k from above by times taking values in a countable set. Therefore, W_t is finite for all $t \geq 0$.

Set $Y_n := (b_{n,k})_{k\in\mathbb{N}}$. Then Z is $\mathscr{F}$-adapted as it is independent of b_0. (A1) follows by (2.7.6) and (2.7.7), and (A3) follows either from the recursive construction (2.7.4) or by noting that Z has no deterministic jumps and $\theta_{Z_n} \theta_n W_0 = 0$. To verify (A2), note that $\{J = x\}$ depends on η only through $(\eta(y))_{y\in\{0\wedge x,\ldots,0\vee x+1\}}$ so we may take $R = 1$.

Definition of Γ_L and verification of (H1)–(H3)

Using Ξ, we can define the events Γ_L by

$$\Gamma_L := \left\{ \xi_t^\pm(x) = \xi_0^\pm(x) \ \forall \ t \in [0, L], \ x = 0, 1 \right\}. \tag{2.7.13}$$

Then $\Gamma_L \in \mathscr{C}_{1,L}(m)$ for any $m > 0$. When $\xi_0^\pm \in \bar{E}$, Γ_L implies that there is a trap at the origin between times 0 and L; since $\bar{\mu}(\bar{E}) = 1$, (H1) holds. The probability of Γ_L is positive and depends on Ξ_0 only through the states at 0 and 1, so (H2) is also satisfied.

In order to verify (H3), we will take advantage of Lemma 2.7.1 and the stochastic domination in Ξ to define two auxiliary processes $H^\pm = (H_t^\pm)_{t\geq 0}$ which we can control and which will bound Z. This will also allow us to deduce uniform integrability properties.

In the following we will suppose that $\xi_0^\pm \in \bar{E}$. Let $G_0 = U_0 := 0$ and, for $k \geq 0$,

$$\begin{aligned}
U_{k+1} &:= \inf\left\{ t > U_k \colon \xi_t^+(G_k + 1) = 1 \right\}, \\
G_{k+1} &:= G_k + Tr^+\!\left(\theta_{G_k} \xi_{U_{k+1}}^+ \right)
\end{aligned} \tag{2.7.14}$$

and put

$$H_t^+ := \sum_{k=0}^{\infty} 1_{\{U_k \leq t < U_{k+1}\}} G_{k+1}. \tag{2.7.15}$$

Define H^- analogously, using Tr^- and ξ^- instead and switching 1's to 0's in (2.7.14). Then H^+ (H^-) is the process that, observing ξ^+ (ξ^-), waits to the left of a hole (on a particle) until it flips to a particle (hole), and then jumps to the right (left) to the next trap. Therefore, by Lemma 2.7.1 and the definition of Z, $H_t^- \leq Z_t \leq H_t^+ \ \forall \ t \geq 0$. Note that H^+ depends only on $(\xi^+(x))_{x\geq 1}$, and analogously for H^-.

In the following, we will write $\mathbb{Z}_{\leq x} := \mathbb{Z} \cap (-\infty, x]$ and analogously for $\mathbb{Z}_{\geq x}$.

Lemma 2.7.2. *Fix $\rho_* \in (0, \rho^-]$ and $\rho^* \in [\rho^+, 1)$. There exist $m, a, \psi_* \in (0, \infty)$ and $\kappa_* \in (0, 1)$, depending on ρ_*, ρ^* and $\lambda^\pm$, such that, for any probability measure $\bar{\nu}$ on $\bar{E}$ that stochastically dominates ν_{ρ_*} on $\mathbb{Z}_{\leq -1}$ and is dominated by ν_{ρ^*} on $\mathbb{Z}_{\geq 2}$,*

$$(a) \ \sup_{t\geq 1} \mathbb{E}_{\bar{\nu}}\left[e^{a\left(t^{-1}|H_t^\pm|\right)} \right] \leq \psi_* \tag{2.7.16}$$

and, setting

$$\mathcal{S}^\pm := \inf\{t > 0 \colon |H_t^\pm| > mt\}, \qquad \widehat{\mathcal{S}}^\pm := \sup\{t > 0 \colon |H_t^\pm| > mt\}, \qquad (2.7.17)$$

then

$$\begin{array}{ll} (b) & \mathbb{P}_{\bar{\nu}}\left(\mathcal{S}^\pm = \infty\right) \geq \kappa_*, \\ (c) & \mathbb{E}_{\bar{\nu}}\left[e^{a\widehat{\mathcal{S}}^\pm}\right] \leq \psi_*. \end{array} \qquad (2.7.18)$$

Before proving this lemma, let us see how it leads to (H3). We will show that there exist $m, a, \psi \in (0, \infty)$ and $\kappa \in (0, 1)$ such that, for all $L \geq 1$ and $\eta \in \bar{E}$,

$$\mathbb{P}_\eta\left(\theta_L \mathcal{S} = \infty \mid \Gamma_L\right) \geq \kappa \qquad (2.7.19)$$

and

$$\mathbb{E}_\eta\left[e^{a(\theta_L \mathcal{S})} 1_{\{\theta_L \mathcal{S} < \infty\}} \mid \Gamma_L\right] \leq \psi, \qquad (2.7.20)$$

which clearly imply (H3).

Let us verify (2.7.19). First note that $\theta_L \mathcal{S} \geq \theta_L(\mathcal{S}^+ \wedge \mathcal{S}^-)$, and that the latter is nonincreasing in $(\eta(x))_{x \geq 2}$ and nondecreasing in $(\eta(x))_{x \leq -1}$. Therefore we may assume that $\eta = \eta_{01}$ which is the configuration in $\bar{E}$ with all 0's on $\mathbb{Z}_{\leq -1}$ and all 1's on $\mathbb{Z}_{\geq 2}$. In this case, ξ_L^- is distributed as $\nu_{\rho_0^L}$ on $\mathbb{Z}_{\leq -1}$ and ξ_L^+ as $\nu_{\rho_1^L}$ on $\mathbb{Z}_{\geq 2}$, where $\rho_0^L = \rho^-(1 - e^{-\lambda^- L})$ and $\rho_1^L = \rho^+ + e^{-\lambda^+ L}(1 - \rho^+)$. Furthermore, on Γ_L, $\xi_L^\pm \in \bar{E}$.

Let now m, a, ψ_* and κ^* as in Lemma 2.7.2 for $\rho_* := \rho_0^1$ and $\rho^* := \rho_1^1$, and let $\bar{\nu}_L$ be the distribution of $\bar{\eta}_L \in \bar{E}$ given by ξ_L^- on $\mathbb{Z}_{\leq -1}$ and ξ_L^+ on $\mathbb{Z}_{\geq 2}$. Noting that $\bar{\eta}_L$ is independent of Γ_L and that $\mathcal{S}^+$ and $\mathcal{S}^-$ are independent, we use the previous observations, the Markov property and Lemma 2.7.2(b) to write

$$\begin{aligned} \mathbb{P}_\eta\left(\theta_L \mathcal{S} = \infty \mid \Gamma_L\right) &\geq \mathbb{P}_{\eta_{01}}\left(\theta_L(\mathcal{S}^+ \wedge \mathcal{S}^-) = \infty \mid \Gamma_L\right) \\ &= \mathbb{E}_{\eta_{01}}\left[1_{\Gamma_L}\mathbb{P}_{\bar{\eta}_L}\left(\mathcal{S}^+ \wedge \mathcal{S}^- = \infty\right)\right]\mathbb{P}_{\eta_{01}}\left(\Gamma_L\right)^{-1} \\ &= \mathbb{P}_{\bar{\nu}_L}\left(\mathcal{S}^+ = \infty\right)\mathbb{P}_{\bar{\nu}_L}\left(\mathcal{S}^- = \infty\right) \geq \kappa_*^2 \in (0, 1), \end{aligned} \qquad (2.7.21)$$

and we may take $\kappa := \kappa_*^2$. For (2.7.20), note now that, when finite, $\theta_L \mathcal{S} < \theta_L(\widehat{\mathcal{S}}^+ \vee \widehat{\mathcal{S}}^-)$ and the latter is nondecreasing in $(\eta(x))_{x \geq 2}$ and nonincreasing in $(\eta(x))_{x \leq -1}$. Therefore we may again assume $\eta = \eta_{01}$ and write, using Lemma 2.7.2(c),

$$\begin{aligned} \mathbb{E}_\eta\left[\theta_L\left(e^{a\mathcal{S}} 1_{\{\mathcal{S} = \infty\}}\right) \mid \Gamma_L\right] &\leq \mathbb{E}_{\eta_{01}}\left[\theta_L e^{a\left(\widehat{\mathcal{S}}^+ + \widehat{\mathcal{S}}^-\right)} \mid \Gamma_L\right] \\ &= \mathbb{E}_{\bar{\nu}_L}\left[e^{a\widehat{\mathcal{S}}^+}\right]\mathbb{E}_{\bar{\nu}_L}\left[e^{a\widehat{\mathcal{S}}^-}\right] \leq \psi_*^2 \in (0, \infty), \end{aligned} \qquad (2.7.22)$$

and we can take $\psi := \psi_*^2$. All that is left to do is to prove Lemma 2.7.2.

Proof of Lemma 2.7.2. By symmetry, it is enough to prove (a)–(c) for H^+. Since H^+, $\mathcal{S}^+$ and $\widehat{\mathcal{S}}^+$ are monotone, we may assume that ξ^+ has rates $\lambda^+ \rho^*$ to flip from holes to particles and

$\lambda^+(1-\rho^*)$ from particles to holes and starts from ν_{ρ^*}, which is the equilibrium measure. In this case, the increments $G_{k+1} - G_k$ are i.i.d. Geom$(1-\rho^*)$, and $U_{k+1} - U_k$ are i.i.d. Exp$(\lambda^+ \rho^*)$, independent from $(G_k)_{k \in \mathbb{N}_0}$. Therefore, H^+ is a càdlàg Lévy process and H_1^+ has an exponential moment, so (a) promptly follows. Moreover, H^+ satisfies a large deviation estimate of the type

$$\mathbb{P}_{\nu_{\rho^*}} \left(\exists\, s > t \text{ such that } H_s^+ > ms \right) \leq K_1 e^{-K_2 t} \ \text{ for all } t > 0, \qquad (2.7.23)$$

where m, K_1 and K_2 are functions of (ρ^*, λ^+), which proves (c). In particular, $\widehat{S}^+ < \infty$ a.s., which implies that $\mathbb{P}_{\nu_{\rho^*}}(H_s^+ \leq m(s + n^*) \ \forall\, s \geq 0) \geq \frac{1}{2}$ for some n^* large enough; then

$$\begin{aligned}
\mathbb{P}_{\nu_{\rho^*}} \left(S^+ = \infty \right) &\geq \mathbb{P}_{\nu_{\rho^*}} \left(H_{n^*}^+ = 0, H_{n^*+s}^+ - H_{n^*}^+ \leq m(s + n^*) \ \forall\, s \geq 0 \right) \\
&= \mathbb{P}_{\nu_{\rho^*}} \left(H_{n^*}^+ = 0 \right) \mathbb{P}_{\nu_{\rho^*}} \left(H_s^+ \leq m(s + n^*) \ \forall\, s \geq 0 \right) =: \kappa_* > 0, \qquad (2.7.24)
\end{aligned}$$

proving (b). ∎

Uniform integrability

The following corollary implies that, for systems given by (2.7.9), $(t^{-1}|W_t|^p)_{t \geq 1}$ is uniformly integrable for any $p \geq 1$, so that, whenever we have a LLN, the convergence holds also in L^p.

Corollary 2.7.3. *Let ξ be a spin-flip system with rates as in* (2.7.9), *starting from equilibrium. Then $(t^{-1}W_t)_{t \geq 1}$ is bounded in L^p for all $p \geq 1$.*

Proof. The claim for Z under $\mathbb{P}_{\bar{\mu}}$ follows from Lemma 2.7.2(a) by noting that $\bar{\mu}$ stochastically dominates ν_{ρ^-} on $\mathbb{Z}_{\leq -1}$ and is dominated by ν_{ρ^+} on $\mathbb{Z}_{\geq 2}$; this can be verified noting that $W_0 \geq 0$ corresponds to finding particles to the left of W_0, and $W_0 \leq 0$ to holes to its right. The same for W follows from (2.3.3–2.3.4) since W_0 has exponential moments under $\mathbb{P}_\mu$. ∎

We still need to verify (H4). This will be done in Sections 2.7.3 and 2.7.4 below. As κ in (2.7.19) could be taken independently of L for (H3), we only need $\lim_{L \to \infty} \Phi_L = 0$ in (H4).

2.7.3 Example 1: $M < \epsilon$

We recall the definition of M and ϵ for a translation-invariant spin-flip system:

$$\begin{aligned}
M &:= \sum_{x \neq 0} \sup_\eta |c(\eta^x) - c(\eta)|, \qquad &(2.7.25) \\
\epsilon &:= \inf_\eta \left\{ c(\eta) + c(\eta^0) \right\}, \qquad &(2.7.26)
\end{aligned}$$

where η^x is the configuration obtained from η by flipping the x-coordinate.

Mixing for ξ

If ξ is in the $M < \epsilon$ regime, then there is exponential decay of space-time correlations (see Liggett [57], Section I.3). In fact, if ξ, ξ' are two copies starting from initial configurations η, η' and coupled according to the Vasershtein coupling, then, as was shown in Maes and Shlosman [62], the following estimate holds uniformly in $x \in \mathbb{Z}$ and in the initial configurations:

$$\mathbb{P}_{\eta, \eta'} \left(\xi_t(x) \neq \xi'_t(x) \right) \leq e^{-(\epsilon - M)t}. \tag{2.7.27}$$

Since the system has uniformly bounded flip rates, it follows that there exist constants $K_1, K_2 \in (0, \infty)$, independent of $x \in \mathbb{Z}$ and of the initial configurations, such that

$$\mathbb{P}_{\eta, \eta'} \left(\exists \, s > t \text{ s.t. } \xi_s(x) \neq \xi'_s(x) \right) \leq K_1 e^{-K_2 t}. \tag{2.7.28}$$

For $A \subset \mathbb{Z} \times \mathbb{R}_+$ measurable, let $\text{Discr}(A)$ be the event in which there is a discrepancy between ξ and ξ' in A, i.e., $\text{Discr}(A) := \{\exists \, (x, t) \in A \colon \xi_t(x) \neq \xi'_t(x)\}$. Recall the definition of $C_R(m)$ in Section 2.3.1, and let $C_{R,t}(m) := C_R(m) \cap \mathbb{Z} \times [0, t]$. From (2.7.28) we deduce that, for any fixed $m > 0$ and $R \in \mathbb{N}_0$, there exist (possibly different) constants $K_1, K_2 \in (0, \infty)$ such that

$$\mathbb{P}_{\eta, \eta'} \left(\text{Discr}\left(C_R(m) \setminus C_{R,t}(m) \right) \right) \leq K_1 e^{-K_2 t}. \tag{2.7.29}$$

Mixing for Ξ

Bounds of the same type as (2.7.27)–(2.7.29) hold for $\xi^{\pm}$, since $M = 0$ and $\epsilon > 0$ for independent spin-flips. Therefore, in order to have such bounds for the triple Ξ, we need only couple a pair Ξ, Ξ' in such a way that each coordinate is coupled with its primed counterpart by the Vasershtein coupling. A set of coupling rates for Ξ, Ξ' that accomplishes this goal is given in (2.9.1), in Appendix 2.9. Redefining $\text{Discr}(A) := \{\exists \, (x, t) \in A \colon \Xi_t(x) \neq \Xi'_t(x)\}$, by the previous results we see that (2.7.29) still holds for this coupling, with possibly different constants. As a consequence, we get the following lemma.

Lemma 2.7.4. *Define* $d(\eta, \eta') := \sum_{x \in \mathbb{Z}} 1_{\{\eta(x) \neq \eta'(x)\}} 2^{-|x|-1}$*. For any* $m > 0$ *and* $R \in \mathbb{N}_0$,

$$\lim_{d(\Xi_0, \Xi'_0) \to 0} \mathbb{P}_{\Xi_0, \Xi'_0} \left(\text{Discr}(C_R(m)) \right) = 0. \tag{2.7.30}$$

Proof. For any $t > 0$, we may split $\text{Discr}(C_R(m)) = \text{Discr}(C_{R,t}(m)) \cup \text{Discr}(C_R(m) \setminus C_{R,t}(m))$, so that

$$\mathbb{P}_{\eta, \eta'} \left(\text{Discr}(C_R(m)) \right) \leq \mathbb{P}_{\eta, \eta'} \left(\text{Discr}(C_{R,t}(m)) \right) + \mathbb{P}_{\eta, \eta'} \left(\text{Discr}(C_R(m) \setminus C_{R,t}(m)) \right). \tag{2.7.31}$$

Fix $\varepsilon > 0$. By (2.7.29), for t large enough the second term in (2.7.31) is smaller than ε uniformly in η, η'. For this fixed t, the first term goes to zero as $d(\eta, \eta') \to 0$, since $C_{R,t}(m)$ is contained in a finite space-time box and the coupling in (2.9.1) is Feller with uniformly bounded total flip rates per site. (Note that the metric d generates the product topology, under which the configuration space is compact.) Therefore $\limsup_{d(\eta, \eta') \to 0} \mathbb{P}_{\eta, \eta'} \left(\text{Discr}(C_R(m)) \right) \leq \varepsilon$. Since ε is arbitrary, (2.7.30) follows. ∎

Conditional mixing

Next, we define an auxiliary process $\bar{\bar{\Xi}}$ that, for each L, has the law of Ξ conditioned on Γ_L up to time L. We restrict to initial configurations $\eta \in \bar{E}$. In this case, $\bar{\bar{\Xi}}$ is a process on $\left(\{0,1\}^{\mathbb{Z}\backslash\{0,1\}}\right)^3$ with rates that are equal to those of Ξ, evaluated with a trap at the origin. More precisely, for $\bar{\eta} \in \{0,1\}^{\mathbb{Z}\backslash\{0,1\}}$, denote by $(\bar{\eta})_{1,0}$ the configuration in $\{0,1\}^{\mathbb{Z}}$ that is equal to $\bar{\eta}$ in $\mathbb{Z}\backslash\{0,1\}$ and has a trap at the origin. Then set $\bar{C}_x(\bar{\eta}) := C_x((\bar{\eta})_{1,0})$, where $\bar{C}_x$ are the rates of $\bar{\bar{\Xi}}$ and C_x the rates of Ξ at a site $x \in \mathbb{Z}$. Observe that the latter depend only on the middle configuration η, and not on $\eta^{\pm}$. These rates give the correct law for $\bar{\bar{\Xi}}$ because Ξ conditioned on Γ_L is Markovian up to time L. Indeed, the probability of Γ_L does not depend on η (for $\eta \in \bar{E}$) and, for $s < L$, $\Gamma_L = \Gamma_s \cap \theta_s \Gamma_{L-s}$. Thus, the rates follow by uniqueness. Observe that they are no longer translation-invariant.

Two copies of the process $\bar{\bar{\Xi}}$ can be coupled analogously to Ξ by restricting the rates in (2.9.1) to $\bar{E}$. Since each coordinate of $\bar{\bar{\Xi}}$ has similar properties as the corresponding coordinate in Ξ (i.e., $\bar{\xi}^{\pm}$ are independent spin-flip systems and $\bar{\xi}$ is in the $M < \epsilon$ regime), it satisfies an estimate of the type

$$\bar{\bar{\mathbb{P}}}_{\eta,\eta'}\left(\mathrm{Discr}([-t,t]\times\{t\})\right) \leq K_1 e^{-K_2 t} \quad \forall \ \eta,\eta' \in \bar{E}, \tag{2.7.32}$$

for appropriate constants $K_1, K_2 \in (0,\infty)$. From this estimate we see that $d(\bar{\bar{\Xi}}_t, \bar{\bar{\Xi}}'_t) \to 0$ in probability as $t \to \infty$, uniformly in the initial configurations. By Lemma 2.7.4, this is also true for $\mathbb{P}_{(\bar{\bar{\Xi}}_t)_{1,0},(\bar{\bar{\Xi}}'_t)_{1,0}}(\mathrm{Discr}(C_R(m)))$. Since the latter is bounded, the convergence holds in L_1 as well, uniformly in η, η'.

Proof of (H4)

Let f be a bounded function measurable in $\mathscr{C}_{R,\infty}(m)$ and estimate

$$|\mathbb{E}_\eta[\theta_L f \mid \Gamma_L] - \mathbb{E}_{\eta'}[\theta_L f \mid \Gamma_L]| \leq 2\|f\|_\infty \mathbb{P}_{\eta,\eta'}\left(\theta_L \mathrm{Discr}(C_R(m)) \mid \Gamma_L\right)$$
$$\leq 2\|f\|_\infty \sup_{\eta,\eta'} \bar{\bar{\mathbb{E}}}_{\eta,\eta'}\left[\mathbb{P}_{(\bar{\bar{\Xi}}_L)_{1,0},(\bar{\bar{\Xi}}'_L)_{1,0}}\left(\mathrm{Discr}(C_R(m))\right)\right], \tag{2.7.33}$$

where $\bar{\bar{\mathbb{E}}}$ denotes expectation under the (coupled) law of $\bar{\bar{\Xi}}$. Therefore (H4) follows with

$$\Phi_L := 2\sup_{\eta,\eta'} \bar{\bar{\mathbb{E}}}_{\eta,\eta'}\left[\mathbb{P}_{(\bar{\bar{\Xi}}_L)_{1,0},(\bar{\bar{\Xi}}'_L)_{1,0}}\left(\mathrm{Discr}(C_R(m))\right)\right], \tag{2.7.34}$$

which converges to zero as $L \to \infty$ by the previous discussion. This is enough since κ_L could be taken constant in the verification of (H3)(1), as we saw in (2.7.19).

2.7.4 Example 2: subcritical dependence spread

In this section, we suppose that the rates $c(\eta)$ have a finite range of dependence $r \in \mathbb{N}_0$. In this case, the system can be constructed via a graphical representation as follows.

Graphical representation

For each $x \in \mathbb{Z}$, let $I_t^j(x)$ and $\Lambda_t^j(x)$ be independent Poisson processes with rates c_j and λ_j respectively, where $j = 0, 1$. At each event of $I_t^j(x)$, put a j-cross on the corresponding space-time point. At each event of $\Lambda^j(x)$, put two j-arrows pointing at x, one from each side, extending over the whole range of dependence. Start with an arbitrary initial configuration $\xi_0 \in \{0, 1\}^{\mathbb{Z}}$. Then obtain the subsequent states $\xi_t(x)$ from ξ_0 and the Poisson processes by, at each j-cross, choosing the next state at site x to be j and, at each j-arrow pair, choosing the next state to be j if an independent Bernoulli$(p_j(\theta_x \xi_s))$ trial succeeds, where s is the time of the j-arrow event. This algorithm is well defined since, because of the finite range, up to each fixed positive time it can a.s. be performed locally.

Any collection of processes with the same range and with rates of the form (2.7.9) with c_i, λ_i fixed ($i = 0, 1$) can be coupled together via this representation by fixing additionally for each site x a sequence $(U_n(x))_{n \in \mathbb{N}}$ of independent Uniform$[0, 1]$ random variables to evaluate the Bernoulli trials at j-arrow events. In particular, $\xi^{\pm}$ can be coupled together with ξ in the graphical representation by noting that, for ξ^-, $p_0 \equiv 1$ and $p_1 \equiv 0$ and the opposite is true for ξ^+. For example, ξ^- is the process obtained by ignoring all 1-arrows and using all 0-arrows. This gives the same coupling as the one given by the rates (2.7.12). In particular, we see that in this setting the events Γ_L are given by (when $\xi_0 \in \bar{E}$)

$$\Gamma_L := \left\{ I_L^0(0) = \Lambda_L^0(0) = I_L^1(1) = \Lambda_L^1(1) = 0 \right\}. \tag{2.7.35}$$

Coupling with a contact process

We will couple Ξ with a contact process $\zeta = (\zeta_t)_{t \geq 0}$ in the following way. We keep all Poisson events and start with a configuration $\zeta_0 \in \{i, h\}^{\mathbb{Z}}$, where i stands for "infected" and h for "healthy". We then interpret every cross as a recovery, and every arrow pair as infection transmission from any infected site within a neighborhood of radius r to the site the arrows point to. This gives rise to a 'threshold contact process' (TCP), i.e., a process with transitions at a site x given by

$$
\begin{array}{lll}
i \to h & \text{with rate} & c_0 + c_1, \\
h \to i & \text{with rate} & (\lambda_0 + \lambda_1) 1_{\{\exists \text{ infected site within range } r \text{ of } x\}}.
\end{array} \tag{2.7.36}
$$

In the graphical representation for ξ, we can interpret crosses as moments of memory loss and arrows as propagation of influence from the neighbors. Therefore, looking at the pair $(\Xi_t(x), \zeta_t(x))$, we can interpret the second coordinate being healthy as the first coordinate being independent of the initial configuration.

Proposition 2.7.5. *Let $\underline{i}$ represent the configuration with all sites infected, and let $\Xi_0, \Xi_0' \in E^3$. Couple Ξ, Ξ' and ζ by fixing a realization of all crosses, arrows and uniform random variables, where Ξ and Ξ' are obtained from the respective initial configurations and ζ is started from $\underline{i}$. Then a.s. $\Xi_t(x) = \Xi_t'(x)$ for all $t > 0$ and $x \in \mathbb{Z}$ such that $\zeta_t(x) = h$.*

Proof. Fix $t > 0$ and $x \in \mathbb{Z}$. With all Poisson and Uniform random variables fixed, an algorithm to find the state at (x, t), simultaneously for any collection of systems of type (2.7.9) with fixed

c_i, λ_i and finite range r from their respective initial configurations runs as follows. Find the first Poisson event before t at site x. If it is a j-cross, then the state is j. If it is a j-arrow, then to decide the state we must evaluate p_j and, therefore, we must first take note of the states at this time at each site within range r of x, including x itself. In order to do so, we restart the algorithm for each of these sites. This process ends when time 0 or a cross is reached along every possible path from (x, t) to $\mathbb{Z} \times \{0\}$ that uses arrows (transversed in the direction opposite to which they point) and vertical lines. In particular, if along each of these paths time 0 is never reached, then the state at (x, t) does not change when we change the initial configuration. On the other hand, time 0 is not reached if and only if every path ends in a cross, which is exactly the description of the event $\{\zeta_t(x) = h\}$.	∎

Cone-mixing in the subcritical regime

The process $(\zeta_t)_{t \geq 0}$ is stochastically dominated by a standard (linear) contact process (LCP) with the same range and rates. Therefore, if the LCP is subcritical, i.e., if $\lambda := (\lambda_0 + \lambda_1)/(c_0 + c_1) < \lambda_c$ where λ_c is the critical parameter for the corresponding LCP, then the TCP will die out as well. Moreover, we have the following lemma:

Lemma 2.7.6. *Let A_t be the set of infected sites at time t. If $\lambda < \lambda_c$, then there exist positive constants K_1, K_2, K_3, K_4 such that*

$$\mathbb{P}_{\underline{i}}\big(\exists\, s > t \colon A_s \cap [-K_1 e^{K_2 s}, K_1 e^{K_2 s}] \neq \emptyset\big) \leq K_3 e^{-K_4 t}. \tag{2.7.37}$$

Proof. This is a straightforward consequence of Proposition 1.1 in Aizenman-Jung [1].	∎

According to Lemma 2.7.6, the infection disappears exponentially fast around the origin. For $r = 1$, a proof can be found in Liggett [57], Chapter VI, but it relies strongly on the nearest-neighbor nature of the interaction.

Let us now prove cone-mixing for ξ when the rates are subcritical. Pick a cone C_t with any inclination and tip at time t, and let $\mathcal{H}_t := \{\text{all sites inside } C_t \text{ are healthy}\}$. This event is independent of ξ_0 and, because of Lemma 2.7.6, has large probability if t is large. Furthermore, by Proposition 2.7.5, on $\mathcal{H}_t$ the states of ξ in C_t are equal to a random variable that is independent ξ_0, which implies the cone-mixing property.

Proof of (H4)

In order to prove the *conditional* cone-mixing property, we couple the conditioned process to a conditioned contact process as follows. First, let

$$\tilde{\Gamma}_L := \big\{ I_L^j(i) = \Lambda_L^j(i) = 0 \colon j, i \in \{0, 1\} \big\}. \tag{2.7.38}$$

Proposition 2.7.7. *Let $\hat{i}$ represent the configuration with all sites infected except for $\{0, 1\}$, which are healthy. Let $\Xi_0, \Xi'_0 \in \bar{E}^3$. Couple Ξ, Ξ' conditioned on Γ_L and ζ conditioned on $\tilde{\Gamma}_L$ by fixing a realization of all crosses, arrows and uniform random variables as in Proposition 2.7.5 and starting, respectively, from Ξ_0, Ξ'_0 and $\hat{i}$, but, for Ξ and Ξ', remove the Poisson events that characterize Γ_L and, for ζ, remove all Poisson events up to time L at sites 0 and 1, which characterizes $\tilde{\Gamma}_L$. Then a.s. $\Xi_t(x) = \Xi'_t(x)$ for all $t > 0$ and $x \in \mathbb{Z}$ such that $\zeta_t(x) = h$.*

Proof. On Γ_L, the states at sites 0 and 1 are fixed for time $[0, L]$. Therefore, in order to determine the state at (x, t), we need not extend paths that touch $\{0, 1\} \times [0, L]$: when every path from (x, t) either ends in a cross or touches $\{0, 1\} \times [0, L]$, the state at (x, t) does not change when the initial configuration is changed in $\mathbb{Z} \setminus \{0, 1\}$. But this is precisely the characterization of $\{\eta_t(x) = h\}$ on $\tilde{\Gamma}_L$ when started from $\hat{i}$. $\blacksquare$

The proof of (H4) is finished by noting that $(\eta_t)_{t \geq 0}$ starting from $\hat{i}$ and conditioned on $\tilde{\Gamma}_L$ is stochastically dominated by $(\eta_t)_{t \geq 0}$ starting from $\underline{i}$. Therefore, by Lemma 2.7.6, the "dependence infection" still dies out exponentially fast, and we conclude as for the unconditioned cone-mixing.

2.7.5 The sign of the speed

For independent spin flips, we are able to characterize with the help of a coupling argument the regimes in which the speed is positive, zero or negative. By the stochastic domination described in Section 2.7.2, this gives us a criterion for positive (or negative) speed in the two classes addressed in Sections 2.7.3 and 2.7.4 above.

Lipschitz property of the speed for independent spin-flip systems

Let ξ be an independent spin-flip system with rates d_0 and d_1 to flip to holes and particles, respectively. Since it fits both classes of IPS considered in Sections 2.7.3 and 2.7.4, by Theorem 2.4.1 there exists a $w(d_0, d_1) \in \mathbb{R}$ that is the a.s. speed of the $(\infty, 0)$-walk in this environment. This speed has the following local Lipschitz property.

Lemma 2.7.8. *Let $d_0, d_1, \delta > 0$. Then*

$$w(d_0, d_1 + \delta) - w(d_0, d_1) \geq \frac{d_0}{d_0 + d_1} \delta. \tag{2.7.39}$$

Proof. Our proof strategy is based on the proof of Theorem 2.24, Chapter VI in [57]. Construct ξ from a graphical representation by taking, for each site $x \in \mathbb{Z}$, two independent Poisson processes $N^i(x)$ with rates d_i, $i = 0, 1$, with each event of N^i representing a flip to state i. For a fixed $\delta > 0$, a second system ξ^δ with rates d_0 and $d_1 + \delta$ can be coupled to ξ by starting from a configuration $\xi_0^\delta \geq \xi_0$ and adding to each site x an independent Poisson process $N^\delta(x)$ with rate δ, whose events also represent flips to particles, but only for ξ^δ. Let us denote by W and W^δ

the walks in these respective environments. Under this coupling, $\xi \leq \xi^\delta$, so, by monotonicity, $W_t \leq W_t^\delta$ for all $t \geq 0$ as well. We aim to prove that

$$\mathbb{E}_{\mu^\delta}\left[W_t^\delta\right] - \mathbb{E}_\mu\left[W_t\right] \geq \frac{d_0}{d_0 + d_1}\delta t, \tag{2.7.40}$$

where μ and μ^δ are the equilibria of the respective systems. From this the conclusion will follow after dividing by t and letting $t \to \infty$.

Define a third walk W^* that is allowed to use one and only one event of N^δ. More precisely, let S be the first time when there is an event of N^δ at $W_S + 1$. Take W^* equal to W on $[0, S)$ and, for times $\geq S$, let W^* evolve by the same rules as W but adding a particle at $W_S + 1$ at time S, and using no more N^δ events. By construction, we have $W_t \leq W_t^* \leq W_t^\delta \, \forall \, t \geq 0$.

Let $\eta_1 := \theta_{W_S}\xi_S \in \bar{E}$ and $\eta_2 := (\eta_1)^1$ be the configurations around W_S and W_{S-}^*, respectively. Then

$$\mathbb{E}_{\mu^\delta}\left[W_t^\delta\right] - \mathbb{E}_\mu\left[W_t\right] \geq \mathbb{E}_\mu\left[W_t^* - W_t, S \leq t\right] \geq \mathbb{E}_\mu\left[W_t^* - W_t, S \leq t, \eta_1(2) = 0\right]$$
$$= \mathbb{E}_\mu\left[\mathbb{E}_{\eta_1,\eta_2}\left[W_{t-S}^2 - W_{t-S}^1\right], \eta_1(2) = 0, S \leq t\right], \tag{2.7.41}$$

where W^i, $i = 1, 2$ are copies of W starting from η_i and coupled via the graphical representation. We claim that, if $\eta_1(2) = 0$,

$$\mathbb{E}_{\eta_1,\eta_2}\left[W_s^2 - W_s^1\right] \geq 1 \quad \forall \, s \geq 0. \tag{2.7.42}$$

Indeed, we will argue that the difference $W_s^2 - W_s^1$ can only decrease when we flip all states of η_1, η_2 on $\mathbb{Z}_{\leq -1}$ to particles and on $\mathbb{Z}_{\geq 2}$ to holes; but after doing these operations, we find that W^2 has the same distribution as $W^1 + 1$, which gives (2.7.42). It is enough to consider a single $x > 2$. Let $\tau := \inf\{t > 0 \colon N_t^0(x) + N_t^1(x) > 0\} \wedge s$, and put $T := \inf\{t > 0 \colon W_t^1 = x - 1\}$. There are two cases: either $T > \tau$ or not. In the first case, W_s^1 remains constant if we set $\eta_{1,2}(x) = 0$, while W_s^2 does not increase. In the second case, if $\eta_{1,2}(x) = 0$, then $W_T^1 = W_T^2$; but then they must remain equal thereafter since, for them to meet, the state at site 1 must have flipped, and therefore they see the same configuration in the environment at time T. Hence, in this case, $W_s^2 - W_s^1 = 0$ which is the minimum value, and our claim follows.

From (2.7.41) and (2.7.42) we get

$$\mathbb{E}_{\mu^\delta}\left[W_t^\delta\right] - \mathbb{E}_\mu\left[W_t\right] \geq \mathbb{P}_\mu\left(\eta_1(2) = 0, S \leq t\right). \tag{2.7.43}$$

Consider the event $\{\eta_1(2) = 0\}$. There are two possible situations: either at time S the site $W_S + 2$ was not yet visited by W, in which case $\eta_1(2)$ is still in equilibrium, or it was. In the latter case, let s be the time of the last visit to this site before S. By geometrical constraints, at time s only a hole could have been observed at this site, so the probability that its state at time S is a hole is larger than at equilibrium, which is $d_0/(d_0 + d_1)$. In other words,

$$\mathbb{P}_\mu\left(\eta_1(2) = 0 \mid S, W_{[0,S]}\right) \geq \frac{d_0}{d_0 + d_1}, \tag{2.7.44}$$

which, together with (2.7.43) and the fact that S has distribution $\text{Exp}(\delta)$, gives us

$$\mathbb{E}_{\mu^\delta}\left[W_t^\delta\right] - \mathbb{E}_\mu\left[W_t\right] \geq \frac{d_0}{d_0 + d_1}\left(1 - e^{\delta t}\right). \tag{2.7.45}$$

Since δ is arbitrary, we may repeat the argument for systems with rates $d_1 + (k/n)\delta$, $n \in \mathbb{N}$ and $k = 0, 1, \ldots, n$, to obtain

$$\mathbb{E}_{\mu^\delta}\left[W_t^\delta\right] - \mathbb{E}_\mu\left[W_t\right] \geq \frac{d_0}{d_0 + d_1}n\left(1 - e^{\delta t/n}\right), \tag{2.7.46}$$

and we get (2.7.40) by letting $n \to \infty$. ∎

Sign of the speed

If $d_0 = d_1$, then $w = 0$, since by symmetry $W_t = -W_t$ in distribution. Hence we can summarize as follows.

Corollary 2.7.9. *For an independent spin-flip system with rates d_0 and d_1,*

$$\begin{aligned}
w &\geq \tfrac{d_0}{d_0+d_1}\left(d_1 - d_0\right) &&\text{if } d_1 > d_0, \\
w &= 0 &&\text{if } d_1 = d_0, \\
w &\leq -\tfrac{d_1}{d_0+d_1}\left(d_0 - d_1\right) &&\text{if } d_1 < d_0.
\end{aligned} \tag{2.7.47}$$

Applying this result to the systems $\xi^\pm$ of Section 2.7.2, we obtain the following.

Proposition 2.7.10. *Let W be the random walk for the $(\infty, 0)$-model in a spin-flip system with rates given by (2.7.9). Then, $\mathbb{P}_\mu$-a.s.,*

$$\begin{aligned}
\liminf_{t\to\infty} t^{-1}W_t &\geq \tfrac{c_0+\lambda_0}{c_1+c_0+\lambda_0}\left(c_1 - c_0 - \lambda_0\right) &&\text{if } c_1 \geq c_0 + \lambda_0, \\
\limsup_{t\to\infty} t^{-1}W_t &\leq -\tfrac{c_1+\lambda_1}{c_0+c_1+\lambda_1}\left(c_0 - c_1 - \lambda_1\right) &&\text{if } c_0 \geq c_1 + \lambda_1.
\end{aligned} \tag{2.7.48}$$

This concludes the proof of Theorem 2.4.2 and the discussion of our two classes of IPS's for the $(\infty, 0)$-model. In Section 2.8 we give additional examples and discuss some limitations of our setting.

2.8 Other examples

We describe here three types of examples that satisfy our hypotheses: generalizations of the (α, β)-model and of the $(\infty, 0)$-model, and mixed models. We also discuss an example that is beyond the reach of our setting.

1. Internal noise models. For $x \in \mathbb{Z} \setminus \{0\}$ and $\eta \in E$, let $\pi_x(\eta)$ be functions with a finite range of dependence R. These are the rates to jump x from the position of the walk. Let $\pi_x := \sup_\eta \pi_x(\eta)$ and suppose that, for some $u > 0$,

$$\sum_{x \in \mathbb{Z} \setminus \{0\}} e^{u|x|} \pi_x < \infty. \tag{2.8.1}$$

This implies that also

$$\Pi := \sum_{x \in \mathbb{Z} \setminus \{0\}} \pi_x < \infty. \tag{2.8.2}$$

The walk starts at the origin, and waits an independent Exponential(Π) time τ until it jumps to x with probability $\pi_x(\xi_\tau)/\Pi$. These probabilities do not necessarily sum up to one, so the walk may well stay at the origin. The subsequent jumps are obtained analogously, with ξ_τ substituted by the environment around the walk at the time of the attempted jump. It is clear that (A1)–(A3) hold. The walk has a bounded probability of standing still independently of the environment, and its jumps have an exponential tail. We take

$$\Gamma_L := \{\tau > L\}. \tag{2.8.3}$$

By defining an auxiliary walk $(H_t)_{t \geq 0}$ that also tries to jump at time τ, but only to sites $x > 0$ with probability π_x/Π, we see that $W_t \leq H_t$ and that H_t has properties analogous to the process defined in the proof of Lemma 2.7.2. Therefore, (H1)–(H3) are always satisfied for this model. Since Γ_L is independent of ξ, (H4) is the (unconditional) cone-mixing property. Observe that $W_0 = 0$, so that $\bar{\mu} = \mu$. Therefore the LLN for this model holds in both examples discussed in Section 2.7, and also for the IPS's for which cone-mixing was shown in Avena, den Hollander and Redig [6]. The (α, β)-model is an internal noise model with $R = 0$ (the rates depend only on the state of the site where the walker is) and $\pi_x(\eta) = 0$, except for $x = \pm 1$, for which $\pi_1(1) = \alpha = \pi_{-1}(0)$ and $\pi_1(0) = \beta = \pi_{-1}(1)$.

2. Pattern models. Take $\aleph$ to be a finite sequence of 0's and 1's, which we call a *pattern*, and let R be the length of this sequence. Take the environment ξ to be of the same type used to define the $(\infty, 0)$-walk. Let $q : \{0, 1\}^R \setminus \{\aleph\} \to [0, 1]$. The pattern walk is defined similarly as the $(\infty, 0)$-walk, with the trap being substituted by the pattern, and a Bernoulli(q) random variable being used to decide whether the walk jumps to the right or to the left. More precisely, let $\vartheta = (\xi_0(0), \ldots, \xi_0(R - 1))$. If $\vartheta = \aleph$, then we set $W_0 = 0$, otherwise we sample b_0 as an independent Bernoulli($q(\vartheta)$) trial. If $b_0 = 1$, then W_0 is set to be the starting position of the first occurrence of $\aleph$ in ξ_0 to the right of the origin, while if $b_0 = 0$, then the first occurrence of $\aleph$ to the left of the origin is taken instead. Then the walk waits at this position until the configuration of one of the R states to its right changes, at which time the procedure to find the jump is repeated with the environment as seen from W_0. Subsequent jumps are obtained analogously. The $(\infty, 0)$-model is a pattern model with $\aleph := (1, 0)$, $q(1, 1) := 1$, $q(0, 0) := 0$ and $q(0, 1) := 1/2$.

For spin-flip systems given by (2.7.9), the pattern walk is defined and finite for all times, no matter what $\aleph$ is, the reasoning being exactly the same as for the $(\infty, 0)$-walk. Also, it may be

analogously defined so as to satisfy assumptions (A1)–(A3). Defining the events Γ_L as

$$\Gamma_L := \left\{\xi_s^\pm(j) = \xi_0^\pm(j) \;\forall\; s \in [0, L] \text{ and } j \in \{0, \ldots, R-1\}\right\}, \tag{2.8.4}$$

we may indeed, by completely analogous arguments, reobtain all the results of Section 2.7, so that hypotheses (H1)–(H4) hold and, therefore, the LLN as well.

3. Pattern models with extra jumps. Examples of models that fall into our setting and for which the events Γ_L depend non-trivially both on ξ and Y can be constructed by taking a pattern model and adding noise in the form of non-zero jump rates while sitting on the pattern. More precisely, add to Y an independent Poisson process N with positive rate and let W jump also at events of N but with the same jump mechanism, i.e., choosing the sign of the jump according to the result of a Bernoulli(q) random variable, and the displacement using the pattern. Taking $\Gamma_L := \Gamma_L^\aleph \cap \{N_L = 0\}$, where $\Gamma_L^\aleph$ is the corresponding event for the pattern model, we may check that, for the two examples of dynamic random environments considered in Theorem 2.4.2, (A1)–(A3) and (H1)–(H4) are all verified.

4. Mixtures of pattern and internal noise models. Another class of models with nontrivial dependence structure for the regeneration-inducing events can be constructed as follows. Let W^0 be an internal noise model and W^1 a pattern model (with or without extra jumps) on the same random environment ξ and let Y^i, $i \in \{0, 1\}$, be the corresponding random elements associated to each model. Let $X = (X)_{n \in \mathbb{N}}$ be a sequence of i.i.d. Bernoulli(p) random variables independent of all the rest, where $p \in (0, 1)$. Then the mixture is the model for which the dynamics associated to $i \in \{0, 1\}$ are applied in the time interval $[n-1, n)$ when $X_n = i$. Note that this model will have deterministic jumps.

Letting $Y := (Y^0, Y^1, X)$ where Y^i is the corresponding random element associated to the model i, it is easily checked that this model indeed falls into our setting.

Choosing
$$\Gamma_L := \Gamma_L^1 \cap \{X_k = 1, k = 1, \ldots, L\} \tag{2.8.5}$$

where Γ_L^1 is the corresponding event for the pattern model, it is not hard to verify, using the results of Section 2.7, that, for the two classes of random environments considered in Theorem 2.4.2, the mixed model satisfies (A1)–(A3) and (H1)–(H4).

An open example. We will close with an example of a model that does not satisfy the hypotheses of our LLN (in dynamic random environments given by spin-flip systems). When $\xi(0) = j$, let C^j be the cluster of j's around the origin. Define jump rates for the walk as follows:

$$\begin{aligned}
\pi_1(\eta) &= \begin{cases} |C^1| & \text{if} & \eta(0) = 1, \\ |C^0|^{-1} & \text{if} & \eta(0) = 0, \end{cases} \\
\pi_{-1}(\eta) &= \begin{cases} |C^0| & \text{if} & \eta(0) = 0, \\ |C^1|^{-1} & \text{if} & \eta(0) = 1. \end{cases}
\end{aligned} \tag{2.8.6}$$

Even though this looks like a fairly natural model, it does not satisfy (A2). It also will not satisfy (H1) and (H2) together for any reasonable random environment, which is actually the

hardest obstacle. The problem is that, while we are able to transport a.s. properties of the equilibrium measure to the measure of the environment as seen from the walk, we cannot control the distortion in events of positive measure. Thus, even if Γ_L has positive probability at time zero, there is no a priori guarantee that it will have an appreciable probability from the point of view of the walk at later times. Because of this, we cannot implement our regeneration strategy, and our proof of the LLN breaks down.

2.9 Appendix: coupling rates

Here we give the rates for a coupling between Ξ and Ξ', mentioned in Section 2.7.3, such that corresponding pairs of coordinates are distributed according to the Vasershtein coupling. Let η, η' be the state of the middle coordinates ξ and ξ'; the states outside the origin of the other coordinates play no role. Then the flip rates at the origin are given schematically by

$$
\begin{aligned}
(000)(000) \;\to\; &\begin{cases}
(111)(111) & c_1, \\
(011)(011) & c(\eta) \wedge c(\eta') - c_1, \\
(011)(001) & c(\eta) - c(\eta) \wedge c(\eta'), \\
(001)(011) & c(\eta') - c(\eta) \wedge c(\eta'), \\
(001)(001) & c_1 + \lambda_1 - c(\eta) \vee c(\eta'),
\end{cases} \\[4pt]
(001)(001) \;\to\; &\begin{cases}
(111)(111) & c_1, \\
(011)(011) & c(\eta) \wedge c(\eta') - c_1, \\
(011)(001) & c(\eta) - c(\eta) \wedge c(\eta'), \\
(001)(011) & c(\eta') - c(\eta) \wedge c(\eta'), \\
(000)(000) & c_0,
\end{cases} \\[4pt]
(001)(011) \;\to\; &\begin{cases}
(111)(111) & c_1, \\
(011)(011) & c(\eta) - c_1, \\
(001)(001) & c(\eta') - c_0, \\
(000)(000) & c_0,
\end{cases} \\[4pt]
(000)(001) \;\to\; &\begin{cases}
(111)(111) & c_1, \\
(011)(011) & c(\eta) \wedge c(\eta') - c_1, \\
(011)(001) & c(\eta) - c(\eta) \wedge c(\eta'), \\
(001)(011) & c(\eta') - c(\eta) \wedge c(\eta'), \\
(001)(001) & c_1 + \lambda_1 - c(\eta) \vee c(\eta'), \\
(000)(000) & c_0,
\end{cases} \\[4pt]
(000)(011) \;\to\; &\begin{cases}
(111)(111) & c_1, \\
(011)(011) & c(\eta) - c_1, \\
(001)(011) & c_1 + \lambda_1 - c(\eta), \\
(000)(000) & c_0, \\
(000)(001) & c(\eta') - c_0,
\end{cases} \\[4pt]
(000)(111) \;\to\; &\begin{cases}
(111)(111) & c_1, \\
(011)(111) & c(\eta) - c_1, \\
(001)(111) & c_1 + \lambda_1 - c(\eta), \\
(000)(000) & c_0, \\
(000)(001) & c(\eta') - c_0, \\
(000)(011) & c_0 + \lambda_0 - c(\eta').
\end{cases}
\end{aligned}
\tag{2.9.1}
$$

The other transitions, starting from

$$
(111)(111), \quad (011)(011), \quad (011)(001), \quad (111)(011), \quad (111)(001) \text{ and } (111)(000), \tag{2.9.2}
$$

can be obtained from the ones in (2.9.1) by symmetry, by exchanging the roles of η/η' or of particles/holes.

3 Transient random walk in symmetric exclusion: limit theorems and an Einstein relation

This chapter is based on a paper with Luca Avena and Florian Völlering.

Abstract

We consider a one-dimensional simple symmetric exclusion process in equilibrium as a dynamic random environment for a nearest-neighbor random walk that on occupied/vacant sites has two different local drifts to the right. We obtain a LLN, a functional CLT and large deviation bounds for the random walk under the annealed measure by means of a renewal argument. We also obtain an Einstein relation under a suitable perturbation. A brief discussion on the topic of random walks in slowly mixing dynamic random environments is presented.

3.1 Introduction: model, results and motivation

3.1.1 The model

Let
$$\xi = (\xi_t)_{t \geq 0} \quad \text{with} \quad \xi_t = \big(\xi_t(x)\big)_{x \in \mathbb{Z}} \tag{3.1.1}$$
be a càdlàg Markov process with state space $\Omega = \{0,1\}^{\mathbb{Z}}$. We say that at time t the site x is *occupied by a particle* if $\xi_t(x) = 1$ and is *vacant* or, alternatively, *occupied by a hole*, if $\xi_t(x) = 0$. For $\eta \in \Omega$, we write P^η to denote the law of ξ starting from $\xi_0 = \eta$, and denote by
$$P^\mu(\cdot) = \int_\Omega P^\eta(\cdot)\, \mu(\mathrm{d}\eta) \tag{3.1.2}$$
the law of ξ when ξ_0 is drawn from a probability measure μ on Ω.

Having fixed a realization of ξ, let
$$W = (W_t)_{t \geq 0} \tag{3.1.3}$$
be the Random Walk (RW) that starts from 0 and has local transition rates

$$
\begin{aligned}
x \to x+1 &\quad \text{at rate} \quad \alpha_1\, \xi_t(x) + \alpha_0\,[1 - \xi_t(x)], \\
x \to x-1 &\quad \text{at rate} \quad \beta_1\, \xi_t(x) + \beta_0\,[1 - \xi_t(x)],
\end{aligned} \tag{3.1.4}
$$

where
$$\alpha_0, \alpha_1, \beta_0, \beta_1 \in (0, \infty), \tag{3.1.5}$$
i.e., on occupied (resp. vacant) sites the random walk jumps to the right at rate α_1 and to the left at rate β_1 (resp. α_0 and β_0). We write P_W^ξ to denote the law of W when ξ is fixed and, for an initial measure μ,

$$\mathbb{P}_\mu(\cdot) = \int P_W^\xi(\cdot)\, P^\mu(\mathrm{d}\xi) \tag{3.1.6}$$

to denote the law of W averaged over ξ. We refer to P_W^ξ as the *quenched* law and to $\mathbb{P}_\mu$ as the *annealed* law.

We are interested in studying the RW W when ξ is a one-dimensional *Simple Symmetric Exclusion Process* (SSEP), i.e., an Interacting Particle System (IPS) (see [57]) whose generator L acts on a real cylinder function f as

$$(Lf)(\eta) = \sum_{\substack{x,y \in \mathbb{Z} \\ x \sim y}} [f(\eta^{xy}) - f(\eta)], \qquad \eta \in \Omega, \tag{3.1.7}$$

where the sum runs over unordered pairs of neighboring sites in $\mathbb{Z}$, and η^{xy} is the configuration obtained from η by interchanging the states at sites x and y. For any $\rho \in (0,1)$, the Bernoulli product measure with density ρ, which we denote by ν_ρ, is an ergodic measure for the SSEP ([57], Theorem VIII.1.44).

We will assume that
$$\alpha_0 \wedge \alpha_1 - \beta_0 \vee \beta_1 > 1. \tag{3.1.8}$$
Condition (3.1.8) implies that the local drifts on occupied and vacant sites, $\alpha_1 - \beta_1$ and $\alpha_0 - \beta_0$ respectively, are both bigger than 1. Thus the RW is not only transient, but travels faster than local information can spread in the SSEP. This is a strong property which is key to our argument; it allows us, roughly speaking, to overcome the slow mixing in time of the SSEP with the good mixing in space of ν_ρ, giving rise to a regenerative structure for the random walk.

3.1.2 Results

In the three theorems below we fix $\rho \in [0,1]$ and assume (3.1.5), (3.1.8).

Theorem 3.1.1. (Law of large numbers)
There exists $v \geq \alpha_0 \wedge \alpha_1 - \beta_0 \vee \beta_1 > 1$ such that

$$\lim_{t \to \infty} \frac{W_t}{t} = v \qquad \mathbb{P}_{\nu_\rho}\text{-a.s. and in } L^p \ \forall \ p \geq 1. \tag{3.1.9}$$

Theorem 3.1.2. (Annealed large deviations)

For any $\epsilon > 0$,

$$\limsup_{t \to \infty} t^{-1} \log \mathbb{P}_{\nu_\rho}(|W_t - tv| \geq t\epsilon) < 0. \tag{3.1.10}$$

Theorem 3.1.3. (Annealed functional central limit theorem)

There exists $\sigma \in (0, \infty)$ such that, under $\mathbb{P}_{\nu_\rho}$,

$$\left(\frac{W_{nt} - ntv}{\sqrt{n}} \right)_{t \geq 0} \Rightarrow \sigma B \tag{3.1.11}$$

where B is a standard Brownian motion.

For the next result, we interpret the model of Section 3.1.1 as a perturbation of a homogeneous RW. We regard the exclusion process as an oscillating random field which interacts weakly with the RW, affecting its asymptotic speed. The Einstein relation then says that the rate of change of the speed when the interaction is very weak is given by the diffusion coefficient of the unperturbed walk. This is a form of the fluctuation-dissipation theorem from statistical physics, which concerns the response of thermodynamical systems to small external perturbations, connecting it with spontaneous fluctuations of the system. As references we mention [31, 37, 54].

Theorem 3.1.4. (Einstein Relation)
Fix $\alpha, \beta > 0$ with $\alpha - \beta > 1$. Let $\lambda \in (0, \infty)$ be the perturbation strength, and fix interaction constants $F_0, F_1 \in \mathbb{R}$ with $F_0 + F_1 = 1$. Let the perturbed rates be given by:

$$\alpha_0 = \alpha \exp\left\{ F_0 \frac{\lambda}{1 - \rho} + o(\lambda) \right\}, \quad \beta_0 = \beta \exp\left\{ -F_0 \frac{\lambda}{1 - \rho} + o(\lambda) \right\},$$

$$\alpha_1 = \alpha \exp\left\{ F_1 \frac{\lambda}{\rho} + o(\lambda) \right\}, \qquad \beta_1 = \beta \exp\left\{ -F_1 \frac{\lambda}{\rho} + o(\lambda) \right\}. \tag{3.1.12}$$

When λ is small enough, (3.1.8) is satisfied. For such λ, let $v(\lambda)$ be the speed as in (3.1.9). Then

$$\lim_{\lambda \downarrow 0} \frac{v(\lambda) - v(0)}{\lambda} = \alpha + \beta. \tag{3.1.13}$$

The rest of the paper is organized as follows. In Section 3.1.3, we present a brief introduction to RW in static and dynamic Random Environment (RE), and discuss slowly mixing dynamic REs. In Section 3.2, we construct a particular version of our model. Section 3.3 is the core of the paper; there we develop a regeneration scheme that is used in Section 3.4 to prove Theorems 3.1.1–3.1.4.

3.1.3 Motivation

Random Walks in Random Environments (RWRE) on $\mathbb{Z}^d$ are RWs whose transition probabilities or rates depend on a random field (*static* case) or on a random process (*dynamic* case) which

is called a *random environment.* They model the motion of a particle in an inhomogeneous medium.

RWs in *static* REs have been an intensive research area since the 1970's (see e.g. [73]). One-dimensional models are well understood. In particular, recurrence vs. transience criteria, LLNs and CLTs have been derived, as well as quenched and annealed LDPs. In higher dimensions the picture is much less complete, but several results are available for RWs that are *transient* in some direction. In particular, LLNs and CLTs for i.i.d. REs ([79, 74, 75, 67]) and for uniformly (fast) mixing REs ([29, 30, 65]) have been obtained under ballisticity conditions. See [77, 83, 84] for an overview.

By considering time as an additional dimension, one can view RWs in *dynamic* REs in dimension d as RWs in *static* REs in dimension $d + 1$ which are transient in the time direction (see e.g. [6]). Thus there are results analogous to the static, transient case. In particular, LLNs and CLTs have been obtained when the dynamic RE has either no correlations in space and/or time, or has uniform and fast mixing, where 'fast' means either exponential or (more recently) polynomial with a high enough degree. A few references are: [6, 9, 11, 21, 23, 24, 45, 32, 46, 70]. Further references can be found in [3, 5].

Very little is known for dynamics with slow and/or non-uniform mixing (e.g. exclusion, supercritical contact, and zero-range processes), apart from recent LLNs for specific cases ([44], [43]). A special interest in studying RW in slowly mixing dynamic REs comes from the static, one-dimensional case, where unusual asymptotic behavior can be observed. More specifically, there are regimes exhibiting transience with zero speed ([73]), non-diffusivity ([53, 72]) and subexponential decay of the probability of travelling at speeds slower than typical ([28, 40]). Such phenomena do not occur in dynamic RE with fast mixing (as discussed in the previous paragraph), but one would expect them to persist when the dynamics are slow enough. Indeed, for a RW in the SSEP with symmetric drifts on holes/particles (i.e., dropping (3.1.8) and taking $\alpha_0 = \beta_1$, $\beta_0 = \alpha_1$), it was shown in [5] that the cost for travelling with zero speed is subexponential; furthermore, simulation results ([8]) suggest the existence of non-diffusive regimes. Thus the SSEP, being a natural example where mixing is both slow and non-uniform due to particle conservation, is an interesting and challenging choice of dynamic RE.

In the present paper, we study the RW in the SSEP under the additional assumption of a strong spatial drift (3.1.8), which significantly facilitates the analysis. We believe that the regeneration strategy developed in Section 3.3 could be adapted to other dynamic REs (for instance, asymmetric exclusion processes or a Poissonian field of independent RWs) under similar drift assumptions.

3.2 Construction of the model

In this section we construct particular versions of the random walk and of the exclusion process, and introduce the notion of *marked agents.* The resulting Lemma 3.2.1 plays a key role throughout the paper.

3.2.1 Coupling with the minimal walker

We will construct the RW W defined in (3.1.3) from four independent Poisson processes and the RE. This is valid in any dynamic RE given by a two-state IPS.

Let the following set of Poissonian clocks be given, each independent of all the other variables:

$$
\begin{array}{lll}
N^+ = (N^+_t)_{t\geq 0} & \text{with rate} & \alpha_0 \wedge \alpha_1, \\
N^- = (N^-_t)_{t\geq 0} & \text{with rate} & \beta_0 \wedge \beta_1, \\
\widehat{N}^+ = (\widehat{N}^+_t)_{t\geq 0} & \text{with rate} & \alpha_0 \vee \alpha_1 - \alpha_0 \wedge \alpha_1, \\
\widehat{N}^- = (\widehat{N}^-_t)_{t\geq 0} & \text{with rate} & \beta_0 \vee \beta_1 - \beta_0 \wedge \beta_1.
\end{array}
\tag{3.2.1}
$$

Now define W by the following rules:

1. W jumps only when one of the Poisson clocks ring;
2. When N^+ rings, W jumps to the right; when N^- rings, W jumps to the left;
3. When $\widehat{N}^+$ rings, W jumps to the right if the state j at its position is such that $\alpha_j = \alpha_0 \vee \alpha_1$. When $\widehat{N}^-$ rings, W jumps to the left if $\beta_j = \beta_0 \vee \beta_1$. Otherwise, W stays still.

In this construction, W is a function of $(N^\pm, \widehat{N}^\pm, \xi)$ and depends on the environment only through the states it sees when $\widehat{N}^+$ or $\widehat{N}^-$ ring.

Let $M = (M_t)_{t\geq 0}$ be defined by

$$
M_t := N^+_t - N^-_t - \widehat{N}^-_t.
\tag{3.2.2}
$$

By construction, for any $t \geq s \geq 0$,

$$
M_t - M_s \leq W_t - W_s,
\tag{3.2.3}
$$

and we are thus justified to call M the *minimal walker*.

Let

$$
N_t := N^+_t + N^-_t + \widehat{N}^+_t + \widehat{N}^-_t
\tag{3.2.4}
$$

be the number of attempted jumps before time t and

$$
\widehat{N}_t := \widehat{N}^+_t + \widehat{N}^-_t
\tag{3.2.5}
$$

the number of times before time t when the random walk observes the environment. Note that, by construction,

$$
|W_t - W_s| \leq N_t - N_s \quad \forall \ t \geq s \geq 0.
\tag{3.2.6}
$$

As a consequence, for all $p \geq 1$, there is a $C(p) \in (0, \infty)$ such that

$$
\sup_{\eta \in \Omega} \mathbb{E}_\eta[|W_t|^p] \leq C(p) t^p.
\tag{3.2.7}
$$

Therefore, by uniform integrability, as soon as a LLN holds, convergence in L^p, $p \geq 1$, will follow as well.

3.2.2 Graphical representation

The SSEP can be constructed from a graphical representation as follows. Let

$$I = (I(x))_{x \in \mathbb{Z}} \tag{3.2.8}$$

be a collection of i.i.d. Poisson processes with rate 1. Draw the events of $I(x)$ on $\mathbb{Z} \times [0, \infty)$ as arrows between the points x and $x + 1$. Then, for each $t > 0$ and $x \in \mathbb{Z}$, there exists (a.s.) a unique path in $\mathbb{Z} \times [0, \infty)$ starting at (x, t) and ending in $\mathbb{Z} \times \{0\}$ going downwards in time but forced to cross any arrows it encounters; see Figure 3.1. Denote by $\gamma_t(x) \in \mathbb{Z}$ the end position of this path. The process $\gamma = (\gamma_t)_{t \geq 0}$ is called the *interchange process*. On the other hand, for
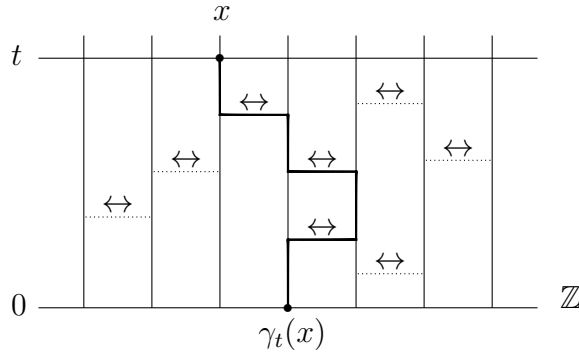

Figure 3.1: Graphical representation. The dotted lines represent events of I. The thick lines mark the path of the agent $\gamma_t(x)$.

each $t \geq 0$ and $x \in \mathbb{Z}$, there is a unique y in $\mathbb{Z}$ such that $\gamma_t(y) = x$; denote by

$$\gamma^{-1} = (\gamma_t^{-1})_{t \geq 0} \tag{3.2.9}$$

the process such that $\gamma_t^{-1}(x) = y$.

We interpret these processes by saying that there are *agents* on the lattice, named after their initial positions, who move around by exchanging places with their neighbors at events of I. Then $\gamma_t^{-1}(x)$ is the position at time t of agent x and $\gamma_t(x)$ is the agent who at time t is at position x.

The SSEP $\xi = (\xi_t)_{t \geq 0}$ starting from a configuration $\eta \in \Omega = \{0, 1\}^{\mathbb{Z}}$ is obtained from γ by putting

$$\xi_t(x) := \eta(\gamma_t(x)), \quad x \in \mathbb{Z}. \tag{3.2.10}$$

The description under the 'agent interpretation' is that we assign at time 0 to each agent x a state $\eta(x)$ and declare the state of the exclusion process at a space time position (x, t) to be the state of the agent who is there.

We will call $\widetilde{P}$ the joint law of $(N^+, N^-, \widehat{N}^+, \widehat{N}^-, I)$. For simplicity of notation, we redefine $\mathbb{P}_\mu$ as the joint law of $(N^+, N^-, \widehat{N}^+, \widehat{N}^-, I)$ and η when the latter is distributed as μ, i.e., $\mathbb{P}_\mu = \mu \times \widetilde{P}$. Then ξ as defined in (3.2.10) is under $\mathbb{P}_\mu$ indeed distributed as a SSEP started from μ.

3.2.3 Marked agents

In our proof, regeneration comes as a consequence of the fact that, even though the environment is slowly mixing, the environment *perceived* by the walker is fast mixing in some sense. The idea is that, since W has a strong drift and the information spread is limited, the dependence on the observed environment is left behind very fast. In the exclusion process, this dependence is carried by the agents of the interchange process whom the RW meets; we will therefore keep track of them via the following time-increasing set of *marked agents*:

$$A_t := \bigcup_{\substack{0 < s \leq t \\ \widehat{N}_{s-} \neq \widehat{N}_s}} \{\gamma_s(W_{s-})\}. \tag{3.2.11}$$

In words, A_t consists of all the agents $x \in \mathbb{Z}$ whose states the walker observes up to time t. Set also

$$R_t := \sup_{x \in A_t} \gamma_t^{-1}(x), \tag{3.2.12}$$

i.e., R_t is the position of the rightmost marked agent at time t. As usual we take $\sup \emptyset = -\infty$.

An important observation is that the walker depends on the initial configuration only through the states of the agents in A_t. More precisely, W is adapted to the filtration $\mathcal{G} = (\mathcal{G}_t)_{t \geq 0}$ given by

$$\mathcal{G}_t := \sigma((N_s^\pm, \widehat{N}_s^\pm, I_s)_{0 \leq s \leq t}, A_t, (\eta(x))_{x \in A_t}). \tag{3.2.13}$$

Moreover, as the next lemma shows, a consequence of the i.i.d. structure and exchangeability of ν_ρ is that the states of the agents who are not in A_t are still, given $\mathcal{G}_t$, distributed as under ν_ρ.

Lemma 3.2.1. *For any $t \geq 0$ and $x_1, \ldots x_n \in \mathbb{Z}$,*

$$\mathbb{E}_{\nu_\rho}\left[\prod_{i=1}^n \xi_t(x_i) \,\middle|\, \mathcal{G}_t\right] = \rho^n \quad a.s. \ on \ \{\gamma_t(x_1), ..., \gamma_t(x_n) \notin A_t\}. \tag{3.2.14}$$

Moreover, (3.2.14) is still valid when t is replaced with a finite $\mathcal{G}$-stopping time.

Proof. From the definition of A_t it follows that, for $A \subset \mathbb{Z}$,

$$\{A_t = A\} \in \sigma((N_s^\pm, \widehat{N}_s^\pm, I_s)_{0 \leq s \leq t}, (\eta(x))_{x \in A}). \tag{3.2.15}$$

With (3.2.15) we can verify by summing over A that, for any $x_1, ..., x_n \in \mathbb{Z}$,

$$\mathbb{E}_{\nu_\rho}\left[\prod_{i=1}^n \eta(x_i) \,\middle|\, \mathcal{G}_t\right] = \rho^n \quad \text{a.s. on the set } \{x_1, ..., x_n \notin A_t\}. \tag{3.2.16}$$

The summation is justified because A_t is a finite set. Since γ is $\mathcal{G}$-adapted and $\xi_t(x) = \eta(\gamma_t(x))$, (3.2.14) follows. The extension to a $\mathcal{G}$-stopping time is done by approximating it from above by stopping times taking values in a countable set (to which (3.2.14) easily extends) and then using the right-continuity of A_t and ξ_t. ∎

3.3 Regeneration

In this section we will develop a regenerative structure for the path of the RW W. Let us first give an informal description of the regeneration strategy. Since W is travelling fast to the right, there will be moments, called *trial times*, when the RW has left behind all agents previously met. At these times, it may 'try to regenerate', and we say that it succeeds if afterwards it never meets those agents again. In case it does not succeed, we wait for the moment when it meets an agent from the past, which we call a *failure time*, and repeat the procedure by waiting for the next trial time. Summarizing, the regeneration strategy consists of two steps: waiting for a trial time when there is a chance for the walker to forget its past, and then checking whether it succeeds or fails in its regeneration attempt. These steps are repeated until the walker succeeds, which will eventually happen by the strong drift assumption (3.1.8).

We proceed to formalize the regeneration scheme, beginning with the trial times. Let $(T_t)_{t\geq 0}$ be the family of $\mathcal{G}$-stopping times defined by:

$$T_t := \inf\left\{s \geq J_t \colon W_s > R_s\right\}. \tag{3.3.1}$$

where $J_t := \inf\{s \geq t : N_t \neq N_s\}$ is the time of the next possible jump after time t. The previous discussion justifies calling T_t the first *trial time* after time t. From the definition it is clear that they are indeed $\mathcal{G}$-stopping times. Note that, a.s., $T_t > t$.

In order to define the failure times, first let, for $t \geq 0, x \in \mathbb{Z}$,

$$Y^t(x) = (Y^t_s(x))_{s\geq t} \tag{3.3.2}$$

be the path starting at time t from x and jumping to the right across the arrows of the process I in (3.2.8); see Figure 3.2. Then $(Y^t_{t+u}(x) - x)_{u\geq 0}$ is a Poisson process with rate 1.
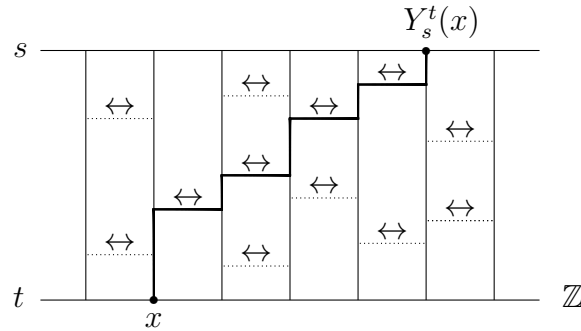

Figure 3.2: As in Figure 3.1, the dotted lines are events of I. The path $Y^t(x)$ starts at x and goes upwards in time and to the right across the arrows.

Now let $(F_t)_{t\geq 0}$ be the family of $\mathcal{G}$-stopping times defined by

$$F_t := \inf\{s > t \colon W_s \leq Y^t_s(W_t - 1)\}. \tag{3.3.3}$$

As usual we take $\inf \emptyset = \infty$. We call F_t the first *failure time* after time t. The F_t's are smaller than the failure times informally discussed in the beginning of the section. Indeed, agents to the left of W_t at time t can never cross $Y^t(W_t - 1)$, as can be seen on the graphical representation. In particular, if $F_t = \infty$, then W will after time t never meet such agents again.

In the following lemma we obtain exponential moment bounds for the trial times T_t, showing in particular that they are a.s. finite.

Lemma 3.3.1. *For every $a > 0$, there exists $b_1 \in (0, \infty)$ such that, for all $t \geq 0$,*

$$\mathbb{E}_{\nu_\rho}[e^{b_1(T_t - t)}|\mathcal{G}_t] \leq (1 + a) \, e^{a(R_t - W_t)^+} \quad \mathbb{P}_{\nu_\rho}\text{-}a.s. \tag{3.3.4}$$

Proof. Let

$$\widetilde{Y}^t = Y^t(R_t \vee W_t) \tag{3.3.5}$$

be the Poisson path starting at time t from the position $R_t \vee W_t$.

Define $H_t := \inf\{s > t : \ M_s - M_t + W_t > \widetilde{Y}_s^t\}$. Let us check that

$$T_t \leq H_t \vee J_t. \tag{3.3.6}$$

Indeed, if $W_{J_t} > \widetilde{Y}_{J_t}^t$ (which can happen only if $R_t \leq W_t$), then $T_t = J_t$. Suppose now that $W_{J_t} \leq \widetilde{Y}_{J_t}^t$. Recall the definition of γ^{-1} in (3.2.9). By geometrical constraints, if $\gamma_s^{-1}(x) \leq \widetilde{Y}_s^t$ for some $s \geq t$, then this will also hold for all future times. In particular, agents marked by W before it crosses $\widetilde{Y}^t$ will never be able to cross $\widetilde{Y}^t$ themselves. This implies that T_t is smaller than the first time after t when W is to the right of $\widetilde{Y}^t$, which is in turn smaller than H_t by (3.2.3).

Since M is independent of I, $(M_{t+u} - M_t - (\widetilde{Y}_{t+u}^t - R_t \vee W_t))_{u \geq 0}$ is under $\mathbb{P}_{\nu_\rho}(\cdot|\mathcal{G}_t)$ a continuous-time RW starting from 0 that has a positive drift by (3.1.8). Furthermore, $H_t - t$ is the first time when it hits $(R_t - W_t)^+ + 1$. Now, if $\mathcal{T}_x$ is the first time when a continuous-time RW with drift $d > 0$ hits a site $x > 0$, then $\sup_{x \geq 1} (\mathcal{T}_x - 2x/d)^+$ has an exponential moment, which can be taken arbitrarily close to 1. Therefore, by (3.3.6), (3.3.4) holds for b_1 sufficiently small. ∎

For $t \geq 0$, denote by $W^{(t)}$ the increments of the walk after time t, that is,

$$W_u^{(t)} := W_{t+u} - W_t. \tag{3.3.7}$$

The next lemma shows that the second step of the regeneration strategy indeed works.

Lemma 3.3.2. *For each $t \geq 0$,*

$$\mathbb{P}_{\nu_\rho}\left(F_t = \infty, W^{(t)} \in \cdot \mid \mathcal{G}_t\right) = \mathbb{P}_{\nu_\rho}\left(\Gamma, W \in \cdot\right) \ a.s. \ on \ \{R_t < W_t\}, \tag{3.3.8}$$

where $\Gamma := \{F_0 = \infty\}$.

Proof. First note that

$$\eta \mapsto \mathbb{P}_\eta \left(\Gamma, W \in \cdot \right) \text{ does not depend on } (\eta(x))_{x<0}. \tag{3.3.9}$$

This can be verified using the graphical representation. Indeed, the agents $x < 0$ can never cross $Y^0(-1)$. Therefore, on Γ, none of them ever meets W, i.e., $A_t \cap (\mathbb{Z} \setminus \mathbb{N}_0) = \emptyset$ for all t. On the other hand, Γ is itself measurable in $\sigma(W, I)$; since W is adapted to $\mathcal{G}$, (3.3.9) follows.

Now, letting $\bar{\xi}_t(\cdot) := \xi_t(W_t + \cdot)$, we can write

$$
\begin{aligned}
\mathbb{P}_{\nu_\rho} \left(R_t < W_t, F_t = \infty, W^{(t)} \in \cdot \mid \mathcal{G}_t \right) &= \mathbb{E}_{\nu_\rho} \left[\mathbb{1}_{\{R_t < W_t\}} \mathbb{P}_{\bar{\xi}_t} \left(\Gamma, W \in \cdot \right) \mid \mathcal{G}_t \right] \\
&= \mathbb{1}_{\{R_t < W_t\}} \mathbb{P}_{\nu_\rho} \left(\Gamma, W \in \cdot \right),
\end{aligned}
\tag{3.3.10}
$$

where the first equality holds by the Markov property and translation-invariance of the graphical representation and the second is justified since, by (3.3.9), $\mathbb{P}_{\bar{\xi}_t} \left(\Gamma, W \in \cdot \right)$ is a function only of $(\bar{\xi}_t(x))_{x \geq 0}$, whose distribution under $\mathbb{P}_{\nu_\rho}(\cdot|\mathcal{G}_t)$ is, by Lemma 3.2.1, a.s. equal to ν_ρ when $R_t < W_t$. ∎

Before proceeding we make a simple but nonetheless important remark:

Remark 3.3.3. Replacing t in T_t and F_t with a finite $\mathcal{G}$-stopping time still yields a stopping time, and Lemmas 3.3.1–3.3.2 (as well as Lemmas 3.3.5 and 3.3.6 below) remain true with a finite stopping time in place of t.

Remark 3.3.3 is justified by right-continuity as in the proof of Lemma 3.2.1. Recall also that a stopping time multiplied by the indicator function of the set where it is finite is again a stopping time.

We are now in shape to prove our main result.

Theorem 3.3.4. *There exists a $\mathbb{P}_{\nu_\rho}$-a.s. positive and finite random time τ such that, $\mathbb{P}_{\nu_\rho}$-a.s.,*

$$\mathbb{P}_{\nu_\rho} \left((W_{\tau+s} - W_\tau)_{s \geq 0} \in \cdot \mid \tau, (W_s)_{s \leq \tau} \right) = \mathbb{P}_{\nu_\rho} \left(W \in \cdot \mid \Gamma \right); \tag{3.3.11}$$

$$\mathbb{P}_{\nu_\rho} \left((W_{\tau+s} - W_\tau)_{s \geq 0} \in \cdot \mid \Gamma, \tau, (W_s)_{s \leq \tau} \right) = \mathbb{P}_{\nu_\rho} \left(W \in \cdot \mid \Gamma \right). \tag{3.3.12}$$

Proof. We will obtain the regeneration time τ with the help of an increasing sequence $(U_n)_{n \in \mathbb{N}_0}$ of $\mathcal{G}$-stopping times in $[0, \infty]$, which will be defined using T_t and F_t. We will throughout the proof tacitly use Remark 3.3.3.

Set $U_0 := 0$. Supposing that for some $n \geq 0$, $(U_k)_{k \leq 2n}$ are all defined, let

$$
\begin{aligned}
U_{2n+1} &:= \begin{cases} \infty & \text{if } U_{2n} = \infty \\ T_{U_{2n}} & \text{otherwise,} \end{cases} \\
U_{2(n+1)} &:= \begin{cases} \infty & \text{if } U_{2n+1} = \infty \\ F_{U_{2n+1}} & \text{otherwise.} \end{cases}
\end{aligned}
\tag{3.3.13}
$$

Then $(U_n)_{n \in \mathbb{N}_0}$ is an increasing sequence of $\mathcal{G}$-stopping times. Now define

$$K = \inf\{n \in \mathbb{N}_0 \colon U_{2n+1} < \infty, F_{U_{2n+1}} = \infty\} \in [0, \infty], \tag{3.3.14}$$

i.e., $2K + 1$ is the first index before the sequence U hits infinity.

Set $\kappa := \mathbb{P}_{\nu_\rho}(\Gamma)$. Then $\kappa > 0$ since W dominates M and $M - Y^0(-1)$ has a positive drift. By Lemma 3.3.2,

$$\mathbb{P}_{\nu_\rho}(K \geq n) = (1 - \kappa)^n \ \ \forall \, n \in \mathbb{N}_0. \tag{3.3.15}$$

In particular, $K < \infty$ $\mathbb{P}_{\nu_\rho}$-a.s. and we can define

$$\tau := U_{2K+1} < \infty \quad \mathbb{P}_{\nu_\rho}\text{-a.s.} \tag{3.3.16}$$

Since $\mathbb{P}_{\nu_\rho}(\cdot|\Gamma) \ll \mathbb{P}_{\nu_\rho}$, τ is a.s. well-defined and finite also under $\mathbb{P}_{\nu_\rho}(\cdot|\Gamma)$.

We will now proceed to verify (3.3.11). Define $\mathcal{G}_\tau$ as the sigma-algebra of the events B such that, for all $n \in \mathbb{N}_0$, there exist $B_n \in \mathcal{G}_{U_{2n+1}}$ such that $B \cap \{K = n\} = B_n \cap \{K = n\}$. Note that τ and $(W_s)_{s \leq \tau}$ are measurable in $\mathcal{G}_\tau$.

Take $f \geq 0$ measurable, $B \in \mathcal{G}_\tau$, and write

$$\mathbb{E}_{\nu_\rho}\left[\mathbb{1}_B f(W^{(\tau)})\right] = \sum_{n=0}^{\infty} \mathbb{E}_{\nu_\rho}\left[\mathbb{1}_{B_n} \mathbb{1}_{\{K=n\}} f(W^{(U_{2n+1})})\right]$$

$$= \sum_{n=0}^{\infty} \mathbb{E}_{\nu_\rho}\left[\mathbb{1}_{B_n} \mathbb{1}_{\{U_{2n+1} < \infty, F_{U_{2n+1}} = \infty\}} f(W^{(U_{2n+1})})\right]$$

$$= \sum_{n=0}^{\infty} \mathbb{E}_{\nu_\rho}\left[\mathbb{1}_{B_n} \mathbb{1}_{\{U_{2n+1} < \infty\}} \mathbb{E}_{\nu_\rho}\left[\mathbb{1}_{\{F_{U_{2n+1}} = \infty\}} f(W^{(U_{2n+1})}) \,\Big|\, \mathcal{G}_{U_{2n+1}}\right]\right].$$

When $U_{2n+1} < \infty$, $R_{U_{2n+1}} < W_{U_{2n+1}}$ so, by Lemma 3.3.2, the last line equals

$$\mathbb{E}_{\nu_\rho}[f(W)\mathbb{1}_\Gamma] \sum_{n=0}^{\infty} \mathbb{E}_{\nu_\rho}\left[\mathbb{1}_{B_n} \mathbb{1}_{\{U_{2n+1} < \infty\}}\right]$$

$$= \mathbb{E}_{\nu_\rho}[f(W) \mid \Gamma] \sum_{n=0}^{\infty} \mathbb{E}_{\nu_\rho}\left[\mathbb{1}_{B_n} \mathbb{1}_{\{U_{2n+1} < \infty\}}\right] \mathbb{P}_{\nu_\rho}(\Gamma)$$

which, by Lemma 3.3.2 again, is equal to

$$\mathbb{E}_{\nu_\rho}[f(W) \mid \Gamma] \sum_{n=0}^{\infty} \mathbb{E}_{\nu_\rho}\left[\mathbb{1}_{B_n} \mathbb{1}_{\{U_{2n+1} < \infty\}} \mathbb{P}_{\nu_\rho}\left(F_{U_{2n+1}} = \infty \,\Big|\, \mathcal{G}_{U_{2n+1}}\right)\right]$$

$$= \mathbb{E}_{\nu_\rho}[f(W) \mid \Gamma] \sum_{n=0}^{\infty} \mathbb{P}_{\nu_\rho}(B_n, K = n)$$

$$= \mathbb{E}_{\nu_\rho}[f(W) \mid \Gamma] \mathbb{P}_{\nu_\rho}(B). \tag{3.3.17}$$

3 Transient random walk in symmetric exclusion: limit theorems and an Einstein relation

This proves (3.3.11). To finish the proof, note that $\Gamma \in \mathcal{G}_\tau$ since, for any $t \geq 0$,

$$\Gamma \cap \{F_t = \infty\} = \{W_s > Y_s^0(-1) \; \forall \, s \leq t\} \cap \{F_t = \infty\}. \tag{3.3.18}$$

So (3.3.12) follows by applying (3.3.17) to $B \cap \Gamma$ in place of B. ∎

In Proposition 3.3.7 below, we will show that τ and W_τ have exponential moments. For its proof, we will need the following two lemmas.

Lemma 3.3.5. *For all $\epsilon > 0$, there exists $a_1 \in (0, \infty)$ such that, for all $t \geq 0$,*

$$\mathbb{E}_{\nu_\rho}\left[\mathbb{1}_{\{F_t < \infty\}} e^{a_1(F_t - t)} \mid \mathcal{G}_t\right] \leq 1 + \epsilon \qquad \mathbb{P}_{\nu_\rho}\text{-a.s.} \tag{3.3.19}$$

Proof. Let

$$D_t := \sup\{s > t; M_s - M_t + W_t \leq Y_s^t(W_t - 1)\}. \tag{3.3.20}$$

If $F_t < \infty$, then $F_t \leq D_t$ because, when finite, F_t is smaller than the last time $s > t$ when $W_s \leq Y_s^t(W_t - 1)$, which is in turn smaller than D_t by (3.2.3). On the other hand, $(M_{t+u} - M_t + W_t - Y_{t+u}^t(W_t - 1))_{u \geq 0}$ is under $\mathbb{P}_{\nu_\rho}(\cdot | \mathcal{G}_t)$ a continuous-time RW with positive drift starting at 1. Since $D_t - t$ is the last time when this random walk is less or equal to 0, (3.3.19) follows. ∎

Lemma 3.3.6. *For all $\epsilon > 0$, there exists $a_2 \in (0, \infty)$ such that, for all $t \geq 0$,*

$$\mathbb{E}_{\nu_\rho}\left[\mathbb{1}_{\{F_t < \infty\}} e^{a_2(R_{F_t} - W_{F_t})^+} \mid \mathcal{G}_t\right] \leq 1 + \epsilon \quad \mathbb{P}_{\nu_\rho}\text{-a.s. on } \{R_t < W_t\}. \tag{3.3.21}$$

Proof. Take D_t as in (3.3.20) and recall that, when finite, $F_t \leq D_t$. Let $\chi_t := W_t + N_{D_t} - N_t$ and consider $Y^t(\chi_t)$ (see (3.3.2)). If $R_t < W_t$, then $R_{F_t} \leq Y_{F_t}^t(\chi_t)$ and so

$$R_{F_t} - W_{F_t} \leq Y_{D_t}^t(\chi_t) - \chi_t + N_{D_t} - N_t + 1. \tag{3.3.22}$$

Now (3.3.21) follows by noting that, even though χ_t is not in $\mathcal{G}_t$, it is independent of $(Y_{t+u}^t(\chi_t) - \chi_t)_{u \geq 0}$ (as they depend on disjoint regions of the graphical representation), so that the latter is still a Poisson process under $\mathbb{P}_{\nu_\rho}(\cdot | \mathcal{G}_t)$. ∎

Proposition 3.3.7. *There exists $b \in (0, \infty)$ such that*

$$\mathbb{E}_{\nu_\rho}[e^{b\tau}], \; \mathbb{E}_{\nu_\rho}[e^{bN_\tau}] < \infty, \tag{3.3.23}$$

the same being true under $\mathbb{P}_{\nu_\rho}(\cdot | \Gamma)$.

Proof. The last sentence follows from (3.3.23) and $\kappa = \mathbb{P}_{\nu_\rho}(\Gamma) > 0$. Since N is a Poisson process, it is enough prove to that τ has exponential moments under $\mathbb{P}_{\nu_\rho}$. To this end, let $\epsilon > 0$ such that $(1 + \epsilon)^2(1 - \kappa) < 1$. Take $a \in (0, \epsilon)$ such that, for all $t \geq 0$,

$$\mathbb{E}_{\nu_\rho}\left[\mathbb{1}_{\{F_t < \infty\}} e^{a(F_t - t) + a(R_{F_t} - W_{F_t})^+} \mid \mathcal{G}_t\right] \leq 1 + \epsilon \quad \mathbb{P}_{\nu_\rho}\text{-a.s. on } \{R_t < W_t\}. \tag{3.3.24}$$

Such a exists by Lemmas 3.3.5 and 3.3.6 and an application of Hölder's inequality. For this a, take b_1 as in Lemma 3.3.1 and let $b := (a \wedge b_1)/2$. Now fix $n \geq 1$ and estimate, recalling that $R_{U_{2n-1}} < W_{U_{2n-1}}$ when $U_{2n-1} < \infty$,

$$
\begin{aligned}
\mathbb{E}_{\nu_\rho} \left[\mathbb{1}_{\{U_{2n} < \infty\}} e^{2bU_{2n+1}} \right] &= \mathbb{E}_{\nu_\rho} \left[\mathbb{1}_{\{U_{2n} < \infty\}} e^{2bU_{2n}} \mathbb{E}_{\nu_\rho} \left[e^{2b(T_{U_{2n}} - U_{2n})} \mid \mathcal{G}_{U_{2n}} \right] \right] \\
&\leq (1+a) \mathbb{E}_{\nu_\rho} \left[\mathbb{1}_{\{U_{2n} < \infty\}} e^{2bU_{2n} + a(R_{U_{2n}} - W_{U_{2n}})^+} \right] \\
&= (1+a) \mathbb{E}_{\nu_\rho} \Bigg\{ \mathbb{1}_{\{U_{2n-2} < \infty\}} e^{2bU_{2n-1}} \\
&\quad \times \mathbb{E}_{\nu_\rho} \left[\mathbb{1}_{\{F_{U_{2n-1}} < \infty\}} e^{2b(F_{U_{2n-1}} - U_{2n-1}) + a\left(R_{F_{U_{2n-1}}} - W_{F_{U_{2n-1}}} \right)^+} \mid \mathcal{G}_{U_{2n-1}} \right] \Bigg\} \\
&\leq (1+\epsilon)^2 \mathbb{E}_{\nu_\rho} \left[\mathbb{1}_{\{U_{2(n-1)} < \infty\}} e^{2bU_{2(n-1)+1}} \right].
\end{aligned}
$$

By induction, we get

$$
\mathbb{E}_{\nu_\rho} \left[\mathbb{1}_{\{U_{2n} < \infty\}} e^{2bU_{2n+1}} \right] \leq (1+\epsilon)^{2n+1}. \tag{3.3.25}
$$

To conclude, use Hölder's inequality and (3.3.15) to write:

$$
\begin{aligned}
\mathbb{E}_{\nu_\rho} \left[e^{b\tau} \right] &= \sum_{n=0}^\infty \mathbb{E}_{\nu_\rho} \left[\mathbb{1}_{\{K=n\}} e^{bU_{2n+1}} \right] = \sum_{n=0}^\infty \mathbb{E}_{\nu_\rho} \left[\mathbb{1}_{\{K=n\}} \mathbb{1}_{\{U_{2n} < \infty\}} e^{bU_{2n+1}} \right] \\
&\leq \sum_{n=0}^\infty \mathbb{P}_{\nu_\rho} (K=n)^{\frac{1}{2}} \mathbb{E}_{\nu_\rho} \left[\mathbb{1}_{\{U_{2n} < \infty\}} e^{2bU_{2n+1}} \right]^{\frac{1}{2}} \\
&\leq \sqrt{1+\epsilon} \sum_{n=0}^\infty \left(\sqrt{(1-\kappa)(1+\epsilon)^2} \right)^n < \infty.
\end{aligned}
$$

∎

Finally, due to Theorem 3.3.4, we can construct a sequence of i.i.d. regeneration times.

Theorem 3.3.8. *By enlarging the probability space, one can assume the existence of a sequence* $(\tau_n)_{n \in \mathbb{N}}$ *of random times with* $\tau_1 := \tau$ *and such that, setting* $S_n := \sum_{i=1}^n \tau_i$,

$$
\left(\tau_{n+1}, \left(W_s^{(S_n)} \right)_{0 \leq s \leq \tau_{n+1}} \right)_{n \in \mathbb{N}} \tag{3.3.26}
$$

is under $\mathbb{P}_{\nu_\rho}$ *an i.i.d. sequence which is independent from* $(\tau, (W_s)_{0 \leq s \leq \tau})$, *each of its terms being distributed as* $(\tau, (W_s)_{0 \leq s \leq \tau})$ *under* $\mathbb{P}_{\nu_\rho}(\cdot | \Gamma)$.

Proof. A version of W with the claimed properties can be constructed on a product space using Theorem 3.3.4, as is standard for "delayed regenerative processes" (see e.g. [71]). This version can be assumed to be the one constructed in Section 3.2.1 again by a standard coupling argument. ∎

3.4 Limit theorems

As a fruit of the regenerative structure constructed in Section 3.3, we now obtain the asymptotic results stated in Section 3.1.2.

3.4.1 Proofs of Theorems 3.1.1 — 3.1.3

Let us collect some useful facts. First of all, by Theorem 3.3.8, Proposition 3.3.7 and (3.2.6),

$$\left(\sup_{s \in [0, \tau_{n+1}]} \left| W_s^{(S_n)} \right| \right)_{n \in \mathbb{N}_0} \quad \text{have a uniform exponential moment.} \tag{3.4.1}$$

Furthermore, again by Theorem 3.3.8 and Proposition 3.3.7,

$$\lim_{n \to \infty} \frac{S_n}{n} = \mathbb{E}_{\nu_\rho}[\tau | \Gamma] \quad \text{and} \quad \lim_{n \to \infty} \frac{W_{S_n}}{n} = \mathbb{E}_{\nu_\rho}[W_\tau | \Gamma] \quad \mathbb{P}_{\nu_\rho}\text{-a.s.} \tag{3.4.2}$$

For $t \geq 0$, let k_t be the random integer such that

$$S_{k_t} \leq t < S_{k_t+1}. \tag{3.4.3}$$

Then a.s. $\lim_{t \to \infty} t^{-1} k_t = \mathbb{E}_{\nu_\rho}[\tau | \Gamma]^{-1}$. Thus the candidate velocity for W is

$$v := \frac{\mathbb{E}_{\nu_\rho}[W_\tau | \Gamma]}{\mathbb{E}_{\nu_\rho}[\tau | \Gamma]}. \tag{3.4.4}$$

Proof of Theorems 3.1.1 and 3.1.2. We first prove (3.1.10). From Theorem 3.3.8 and Proposition 3.3.7 we obtain LDP's for both S_n and W_{S_n} with rate functions which are only zero at $\mathbb{E}_{\nu_\rho}[\tau | \Gamma]$ and $\mathbb{E}_{\nu_\rho}[W_\tau | \Gamma]$, respectively. Since k_t is the inverse of S_n, it also satisfies a LDP with a rate function which is zero only at $\mathbb{E}_{\nu_\rho}[\tau | \Gamma]^{-1}$ (see [38]). Fix $\epsilon > 0$. From the LDP's for W_{S_n} and k_t, we get exponential decay of $\mathbb{P}_{\nu_\rho}\left(|t^{-1} W_{S_{k_t}} - v| \geq \epsilon \right)$, while the same is obtained for $\mathbb{P}_{\nu_\rho}\left(|W_t - W_{S_{k_t}}| \geq \epsilon t \right)$ from (3.4.1) and the LDP for k_t. From this, (3.1.10) is readily obtained, and the LLN follows by the Borel-Cantelli lemma. By (3.2.3), $v \geq \alpha_0 \wedge \alpha_1 - \beta_0 \vee \beta_1 > 1$. Convergence in L^p follows from (3.2.7). ∎

Proof of Theorem 3.1.3. Let $\hat{\sigma}^2$ be the variance of W_τ under $\mathbb{P}_{\nu_\rho}(\cdot | \Gamma)$ which is finite due to (3.3.23) and positive since W_τ is not a.s. constant. For the process $(W_{S_k})_{k \in \mathbb{N}}$, a functional CLT with variance $\hat{\sigma}^2$ holds since, by Theorem 3.3.8 and (3.3.23), the assumptions of the Donsker-Prohorov invariance principle are satisfied. With a random time change argument as in Section 17 of [14], we obtain for $(W_{S_{k_t}})_{t \geq 0}$ a functional CLT with variance $\sigma^2 = \hat{\sigma}^2 \mathbb{E}_{\nu_\rho}[\tau | \Gamma]^{-1}$. To extend it to W, note that

$$\lim_{n \to \infty} n^{-1/2} \sup_{t \leq T} \left| W_{nt} - W_{S_{k_{nt}}} \right| = 0 \quad \mathbb{P}_{\nu_\rho}\text{-a.s. for any } T > 0. \tag{3.4.5}$$

This follows from Theorem 3.3.8, (3.4.1) and the LDP for k_t (mentioned in the previous proof), and implies that the Skorohod distance between diffusive rescalings of W and $(W_{S_{k_t}})_{t \geq 0}$ goes to zero almost surely as $n \to \infty$. ∎

3.4.2 Einstein relation: proof of Theorem 3.1.4

We first show how the speed v is related to the observed density of particles, and that the latter approaches the density of the environment as $\lambda \downarrow 0$.

Proposition 3.4.1. *The limit*

$$\hat{\rho}(\lambda) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{E}_{\nu_\rho} \left[\xi_s(W_s) \right] \mathrm{d}s \tag{3.4.6}$$

exists and satisfies

$$v(\lambda) = \left[\alpha_1(\lambda) - \beta_1(\lambda) \right] \hat{\rho}(\lambda) + \left[\alpha_0(\lambda) - \beta_0(\lambda) \right] \left[1 - \hat{\rho}(\lambda) \right], \tag{3.4.7}$$

$$\lim_{\gamma \downarrow 0} \hat{\rho}(\lambda) = \rho. \tag{3.4.8}$$

Proof. Since W is Markovian under the quenched measure,

$$W_t - \int_0^t (\alpha_1 - \beta_1)\xi_s(W_s) + (\alpha_0 - \beta_0)(1 - \xi_s(W_s))\mathrm{d}s \tag{3.4.9}$$

is a martingale under P_W^ξ for a.e. ξ. Hence by Theorem 3.1.1 the limit in (3.4.6) exists and satisfies (3.4.7). We proceed to prove (3.4.8). Write

$$\int_0^t \mathbb{E}_{\nu_\rho} \left[\xi_s(W_s) \right] \mathrm{d}s = \int_0^t \mathbb{P}_{\nu_\rho} \left(\gamma_s(W_s) \in A_s, \xi_s(W_s) = 1 \right) \mathrm{d}s$$

$$+ \int_0^t \mathbb{P}_{\nu_\rho} \left(\gamma_s(W_s) \notin A_s, \xi_s(W_s) = 1 \right) \mathrm{d}s.$$

The first term is bounded by

$$L_t := \mathbb{E}_{\nu_\rho} \left[\int_0^t \mathbb{1}_{\{\gamma_s(W_s)\in A_s\}}\mathrm{d}s \right], \tag{3.4.10}$$

the expected time spent by the walker on marked agents up to time t. For the second term, we use Lemma 3.2.1:

$$\int_0^t \mathbb{P}_{\nu_\rho} \left(\gamma_s(W_s) \notin A_s, \xi_s(W_s) = 1 \right) \mathrm{d}s = \int_0^t \mathbb{E}_{\nu_\rho} \left[\mathbb{1}_{\{\gamma_s(W_s)\notin A_s\}}\mathbb{E}_{\nu_\rho} \left[\xi_s(W_s) \,|\, \mathcal{G}_s \right] \right] \mathrm{d}s$$

$$= \rho \int_0^t \mathbb{P}_{\nu_\rho} \left(\gamma_s(W_s) \notin A_s \right) \mathrm{d}s = \rho \left(t - L_t \right).$$

Hence

$$\left| \int_0^t \mathbb{E}_{\nu_\rho} \left[\xi_s(W_s) \right] \mathrm{d}s - \rho t \right| \leq L_t. \tag{3.4.11}$$

In order to bound L_t, consider the total time that the walker spends on top of a single marked agent x. If t is the time when this agent is marked, the agent will never cross to the right of

$Y^t(\gamma_t^{-1}(x))$. On the other hand, after time t, W will never be to the left of $M - M_t + \gamma_t^{-1}(x) - 1$. Hence the time spent on the marked agent x is bounded by the total time during which $Y^t(\gamma_t^{-1}(x))$ is to the right of $M - M_t + \gamma_t^{-1}(x)$. Writing $t_x = \inf\{t \geq 0 : x \in A_t\}$, we get

$$
\begin{aligned}
L_t &\leq \sum_{x \in \mathbb{Z}} \mathbb{E}_{\nu_\rho} \left[\mathbb{1}_{\{t_x < t\}} \int_{t_x}^\infty \mathbb{1}_{\{Y_s^{t_x}(\gamma_{t_x}^{-1}(x)) > M_s - M_{t_x} + \gamma_{t_x}^{-1}(x)\}} \mathrm{d}s \right] \\
&= \mathbb{E}_{\nu_\rho} \left[|A_t| \right] \mathbb{E}_{\nu_\rho} \left[\int_0^\infty \mathbb{1}_{\{Y_s^0(0) > M_s\}} \mathrm{d}s \right].
\end{aligned}
\tag{3.4.12}
$$

When λ is small enough, (3.1.8) is satisfied, and the term with the integral in (3.4.12) is uniformly bounded by some constant $C \in (0, \infty)$. On the other hand, the number of marked agents $|A_t|$ is bounded by $\widehat{N}_t$, so finally we have

$$
\left| \int_0^t \mathbb{E}_{\nu_\rho} \left[\xi_s(W_s) \right] \mathrm{d}s - \rho t \right| \leq L_t \leq tC \Big(|\alpha_1(\lambda) - \alpha_0(\lambda)| + |\beta_1(\lambda) - \beta_0(\lambda)| \Big),
$$

proving (3.4.8). ∎

Proof of Theorem 3.1.4. Write

$$
\begin{aligned}
\frac{v(\lambda) - v(0)}{\lambda} &= \frac{(\alpha_1(\lambda) - \beta_1(\lambda)) - (\alpha_1(0) - \beta_1(0))}{\lambda} \hat{\rho}(\lambda) \\
&\quad + (\alpha_1(0) - \beta_1(0)) \frac{\hat{\rho}(\lambda) - \hat{\rho}(0)}{\lambda} \\
&\quad + \frac{(\alpha_0(\lambda) - \beta_0(\lambda)) - (\alpha_0(0) - \beta_0(0))}{\lambda} (1 - \hat{\rho}(\lambda)) \\
&\quad + (\alpha_0(0) - \beta_0(0)) \frac{(1 - \hat{\rho}(\lambda)) - (1 - \hat{\rho}(0))}{\lambda} \\
&= \frac{(\alpha_1(\lambda) - \beta_1(\lambda)) - (\alpha_1(0) - \beta_1(0))}{\lambda} \hat{\rho}(\lambda) \\
&\quad + \frac{(\alpha_0(\lambda) - \beta_0(\lambda)) - (\alpha_0(0) - \beta_0(0))}{\lambda} (1 - \hat{\rho}(\lambda)).
\end{aligned}
$$

Now take the limit as $\lambda \downarrow 0$ and use (3.4.8) to get

$$
\begin{aligned}
v'(0) &= \left(\alpha \frac{F_1}{\rho} + \beta \frac{F_1}{\rho} \right) \rho + \left(\alpha \frac{F_0}{1 - \rho} + \beta \frac{F_0}{1 - \rho} \right) (1 - \rho) \\
&= (\alpha + \beta)(F_1 + F_0) = \alpha + \beta.
\end{aligned}
$$

∎

4 Scaling of a random walk on a supercritical contact process

This chapter is based on a paper with Frank den Hollander.

Abstract

A proof is provided of a strong law of large numbers for a one-dimensional random walk in a dynamic random environment given by a supercritical contact process in equilibrium. The proof is based on a coupling argument that traces the space-time cones containing the infection clusters generated by single infections and uses that the random walk eventually gets trapped inside the union of these cones. For the case where the local drifts of the random walk are smaller than the speed at which infection clusters grow, the random walk eventually gets trapped inside a single cone. This in turn leads to the existence of regeneration times at which the random walk forgets its past. The latter are used to prove a functional central limit theorem and a large deviation principle.

The qualitative dependence of the speed, the volatility and the rate function on the infection parameter is investigated, and some open problems are mentioned.

4.1 Introduction

4.1.1 Background, motivation and outline

Background. *A random walk in a dynamic random environment* on $\mathbb{Z}^d$, $d \geq 1$, is a random process where a "particle" makes random jumps with transition rates that depend on its location and themselves evolve with time. A typical example is when the dynamic random environment is given by an interacting particle system

$$\xi = (\xi_t)_{t \geq 0} \text{ with } \xi_t = \{\xi_t(x) \colon x \in \mathbb{Z}^d\} \in \Omega, \tag{4.1.1}$$

where Ω is the configuration space, and ξ_0 is typically drawn from equilibrium. In the case where $\Omega = \{0, 1\}^{\mathbb{Z}^d}$, the configurations can be thought of as consisting of "particles" and "holes". Given ξ, run a random walk $W = (W_t)_{t \geq 0}$ on $\mathbb{Z}^d$ that jumps at a fixed rate, but uses different transition kernels on a particle and on a hole. The key question is: What are the scaling properties of W and how do these properties depend on the law of ξ?

The literature on random walks in dynamic random environments is still modest (for a recent overview, see Avena [3], Chapter 1). In Avena, den Hollander and Redig [6] a strong law of large numbers (SLLN) was proved for a class of interacting particle systems satisfying a mild space-time mixing condition, called *cone-mixing*. Roughly speaking, this is the requirement that for every $m > 0$ all states inside the space-time cone (see Fig. 4.1)

$$\text{CONE}_t := \big\{ (x, s) \in \mathbb{Z}^d \times [t, \infty) \colon \|x\| \leq m(s - t) \big\}, \tag{4.1.2}$$

are conditionally independent of the states at time zero in the limit as $t \to \infty$. The proof of the SLLN uses a *regeneration-time* argument. Under a cone-mixing condition involving multiple cones, a functional central limit theorem (FCLT) can be derived as well, and under monotonicity conditions also a large deviation principle (LDP).
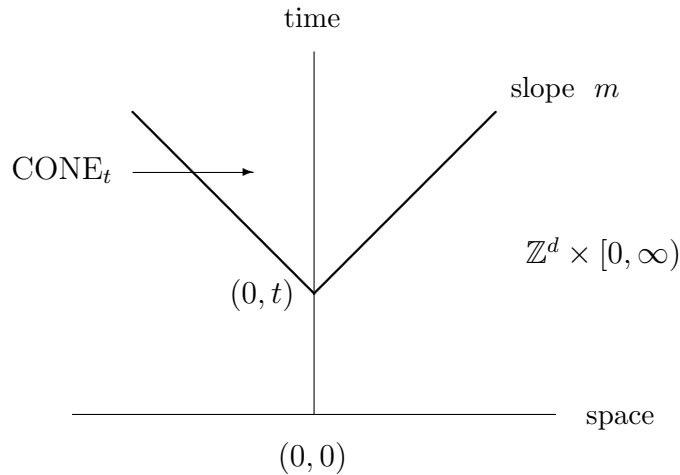

Figure 4.1: The cone defined in (4.1.2).

Many interacting particle systems are cone-mixing, including spin-flip systems with spin-flip rates that are weakly dependent on the configuration, e.g. the stochastic Ising model above the critical temperature. However, also many interacting particle systems are not cone-mixing, including independent simple random walks, the exclusion process, the contact process and the voter model. Indeed, these systems have *slowly decaying space-time correlations*. For instance, in the exclusion process particles are conserved and cannot sit on top of each other. Therefore, if at time zero there are particles everywhere in the box $[-t^2, t^2] \cap \mathbb{Z}^d$, then these particles form a "large traffic jam around the origin". This traffic jam will survive up to time t with a probability tending to 1 as $t \to \infty$, and will therefore affect the states near the tip of CONE_t. Similarly,

in the contact process, if at time zero there are no infections in the box $[-t^2, t^2] \cap \mathbb{Z}^d$, then no infections will be seen near the tip of CONE_t as well.

Motivation. Several attempts have been made to extend the SLLN to interacting particle systems that are not cone-mixing, with partial success. Examples include: independent simple random walks (den Hollander, Kesten and Sidoravicius [43]) and the exclusion process (Avena, dos Santos and Völlering [7], Avena [4]). The present paper considers the *supercritical contact process*. We exploit the graphical representation, which allows us to simultaneously couple all realizations of the contact process starting from different initial configurations. This coupling in turn allows us to first prove the SLLN when the initial configuration is "all infected" (with the help of a subadditivity argument), and then show that the same result holds when the initial configuration is drawn from equilibrium. The main idea is to use the coupling to show that configurations agree in large space-time cones containing the infection clusters generated by single infections and that the random walk eventually gets trapped inside the *union* of these cones.

Under the assumption that the local drifts of the random walk are smaller than the speed at which infection clusters grow, the random walk eventually gets trapped inside a *single* cone. We show that this implies the existence of *regeneration times* at which the random walk "forgets its past". The latter in turn allow us to prove the FCLT and the LDP.

It is typically difficult to obtain information about the speed in the SLLN, the volatility in the FCLT and the rate function in the LDP. In general, these are non-trivial functions of the parameters in the model, a situation that is well known from the literature on random walks in static random environments (for overviews, see Sznitman [77] and Zeitouni [83]). The reason is that these quantities depend on the *environment process* (i.e., the process of environments as seen from the location of the walk), which is typically hard to analyze. For the supercritical contact process we are able to derive a few qualitative properties as a function of the infection parameter, but it remains a challenge to obtain a full quantitative description.

A model of a random walk on the infinite cluster of supercritical oriented percolation (the discrete-time analogue of the contact process) is treated in Birkner, Černý, Depperschmidt and Gantert [15], where a SLLN and a quenched and annealed CLT are obtained. This model can be viewed as a random walk in a dynamic random environment, but it has non-elliptic transition probabilities different from the ones we consider here, because the random walk is confined to the infinite cluster.

Outline. In Section 4.1.2 we define the model. In Section 4.1.3 we state our main results: two theorems claiming the SLLN, the FCLT and the LDP under appropriate conditions on the model parameters. In Section 4.1.4 we mention some open problems. The proofs of the theorems are given in Sections 4.3 and 4.5, respectively, Section 4.6. Sections 4.2 and 4.4 contain preparatory work.

4.1.2 Model

In this paper we consider the case where the dynamic random environment is the one-dimensional linear *contact process* $\xi = (\xi_t)_{t \geq 0}$, i.e., the spin-flip system on $\Omega := \{0,1\}^{\mathbb{Z}}$ with local transition rates given by

$$\eta \to \eta^x \text{ with rate } \begin{cases} 1 & \text{if } \eta(x) = 1, \\ \lambda\left\{\eta(x-1) + \eta(x+1)\right\} & \text{if } \eta(x) = 0, \end{cases} \tag{4.1.3}$$

where $\lambda \in (0, \infty)$ and η^x is defined by $\eta^x(y) := \eta(y)$ for $y \neq x$, $\eta^x(x) := 1 - \eta(x)$. We call a site *infected* when its state is 1, and *healthy* when its state is 0. See Liggett [57], Chapter VI, for proper definitions.

The empty configuration $\mathbf{0} \in \Omega$, given by $\mathbf{0}(x) = 0$ for all $x \in \mathbb{Z}$, is an absorbing state for ξ, while the full configuration $\mathbf{1} \in \Omega$, given by $\mathbf{1}(x) = 1$ for all $x \in \mathbb{Z}$, evolves towards an equilibrium measure ν_λ that is stationary and ergodic under space-shifts. There is a critical threshold $\lambda_c \in (0, \infty)$ such that: (1) for $\lambda \in (0, \lambda_c]$, $\nu_\lambda = \delta_{\mathbf{0}}$; (2) for $\lambda \in (\lambda_c, \infty)$, $\rho_\lambda := \nu_\lambda(\eta(0) = 1) > 0$. In the latter case, $\delta_{\mathbf{0}}$ and ν_λ are the only equilibrium measures. It is known that ν_λ has exponentially decaying correlations, and that $\lambda \mapsto \rho_\lambda$ is continuous and non-decreasing with $\lim_{\lambda \to \infty} \rho_\lambda = 1$.

Fixed a realization of ξ, we define the random walk $W := (W_t)_{t \geq 0}$ as the time-inhomogeneous Markov process on $\mathbb{Z}$ that, given $W_t = x$, jumps to

$$\begin{array}{lll} x+1 & \text{at rate} & \alpha_1 \xi_t(x) + \alpha_0\left[1 - \xi_t(x)\right], \\ x-1 & \text{at rate} & \beta_1 \xi_t(x) + \beta_0\left[1 - \xi_t(x)\right], \end{array} \tag{4.1.4}$$

where $\alpha_i, \beta_i \in (0, \infty)$, $i = 0, 1$. We assume that

$$\alpha_0 + \beta_0 = \alpha_1 + \beta_1 =: \gamma, \tag{4.1.5}$$

and that

$$v_1 > v_0 \text{ with } v_1 := \alpha_1 - \beta_1 \text{ and } v_0 := \alpha_0 - \beta_0, \tag{4.1.6}$$

i.e., the jump rate is constant and equal to γ everywhere, while the drift to the right is larger on infected sites than on healthy sites. Observe that the assumption in (4.1.6) is made without loss of generality: since the contact process is invariant under reflection in the origin, $-W$ has the same law as W with inverted jump rates.

4.1.3 Theorems

Let $\mathbb{P}_{\nu_\lambda}$ denote the joint law of W and ξ when the latter is started from ν_λ. Our SLLN reads as follows.

Theorem 4.1.1. *Suppose that (4.1.5–4.1.6) hold.*
(a) For every $\lambda \in (\lambda_c, \infty)$ there exists a $v(\lambda) \in [v_0, v_1]$ such that

$$\lim_{t \to \infty} t^{-1} W_t = v(\lambda) \quad \mathbb{P}_{\nu_\lambda}\text{-a.s. and in } L^p, \ p \geq 1. \tag{4.1.7}$$

(b) The function $\lambda \mapsto v(\lambda)$ is non-decreasing and right-continuous on (λ_c, ∞), with $v(\lambda) \in (v_0, v_1)$ for all $\lambda \in (\lambda_c, \infty)$ and $\lim_{\lambda \to \infty} v(\lambda) = v_1$.

We note in passing that if $\lambda \in (0, \lambda_c)$, then ξ_t agrees with $\mathbf{0}$ on an interval that grows exponentially fast in t (Liggett [57], Chapter VI), and so it is trivial to deduce that W satisfies the SLLN with $v(\lambda) = v_0$.

A FCLT and an LDP hold under an additional restriction, namely, $\lambda \in (\lambda_W, \infty)$ with

$$\lambda_W := \inf \left\{ \lambda \in (\lambda_c, \infty) \colon |v_0| \vee |v_1| < \iota(\lambda) \right\}. \qquad (4.1.8)$$

Here, $\lambda \mapsto \iota(\lambda)$ is the infection propagation speed (see (4.2.4) in Section 4.2.1), which is known to be continuous, strictly positive and strictly increasing on (λ_c, ∞), with $\lim_{\lambda \downarrow \lambda_c} \iota(\lambda) = 0$ and $\lim_{\lambda \to \infty} \iota(\lambda) = \infty$.

Theorem 4.1.2. *Suppose that* (4.1.5–4.1.6) *hold.*
(a) For every $\lambda \in (\lambda_W, \infty)$ there exists a $\sigma(\lambda) \in (0, \infty)$ such that, under $\mathbb{P}_{\nu_\lambda}$,

$$\left(\frac{W_{nt} - v(\lambda)nt}{\sigma(\lambda)\sqrt{n}} \right)_{t \geq 0} \implies (B_t)_{t \geq 0} \qquad as \ n \to \infty, \qquad (4.1.9)$$

where B is standard Brownian motion and $\implies$ denotes weak convergence in path space.
(b) The functions $\lambda \mapsto v(\lambda)$ and $\lambda \mapsto \sigma(\lambda)$ are continuous on (λ_W, ∞).
(c) For every $\lambda \in (\lambda_W, \infty)$, $(t^{-1}W_t)_{t>0}$ under $\mathbb{P}_{\nu_\lambda}$ satisfies the large deviation principle on $\mathbb{R}$ with a finite and convex rate function that has a unique zero at $v(\lambda)$.

The intuitive reason why the rate function has a unique zero is that deviations of the empirical speed in the: (i) upward direction require a density of infected sites larger than ρ_λ, which is costly because infections become healthy independently of the states at the other sites; (ii) downward direction require a density of infected sites smaller than ρ_λ, which is costly because infection clusters grow at a linear speed and rapidly fill up healthy intervals everywhere.

4.1.4 Discussion

1. It is natural to expect that $\lambda \mapsto v(\lambda)$ is continuous and strictly increasing on (λ_c, ∞) with $\lim_{\lambda \downarrow \lambda_c} v(\lambda) = v_0$. Fig. 4.2 shows a qualitative plot of the speed in that setting. If $0 \in (v_0, v_1)$, then there is a critical threshold $\lambda^* \in (\lambda_c, \infty)$ at which the speed changes sign. It is natural to ask whether $\lambda \mapsto v(\lambda)$ is concave on (λ_c, ∞) and Lipshitz at λ_c.

2. We know that W is transient when $v(\lambda) \neq 0$. Is W recurrent when $v(\lambda) = 0$?

3. We expect (4.1.8) to be redundant. Moreover, we expect that for every $\lambda \in (\lambda_c, \infty)$ the *environment process* (i.e., the process of environments as seen from the location of the random walk) has a unique and non-trivial equilibrium measure that is absolutely continuous with respect to ν_λ.

4. Theorems 4.1.1–4.1.2 can presumably be extended to $\mathbb{Z}^d$ with $d \geq 2$. Also in higher dimensions single infections create infection clusters that grow at a linear speed (i.e., asymptotically form
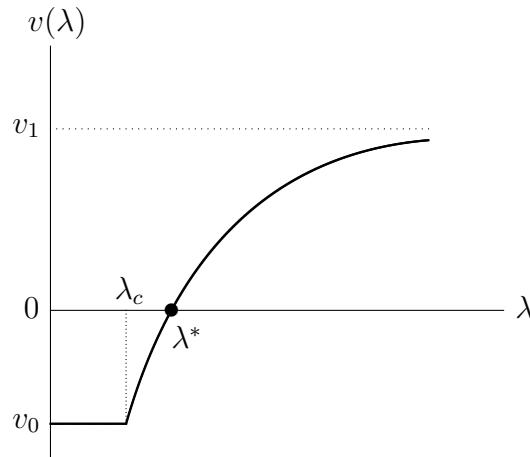
Figure 4.2: Qualitative plot of $\lambda \mapsto v(\lambda)$ when $0 \in (v_0, v_1)$.

a ball with a linearly growing radius). The construction of the regeneration times when $\lambda \in (\lambda_W, \infty)$, with λ_W the analogue of (4.1.8), is straightforward.

5. It would be interesting to extend Theorems 4.1.1–4.1.2 to *multi-type contact processes*. On each type i the random walk has transition rates α_i, β_i such that $\alpha_i + \beta_i = \gamma$ for all i. As long as the dynamics is monotone and $i \mapsto v_i$ is non-decreasing, many of the arguments in the present paper carry over.

4.2 Construction

In Section 4.2.1 we construct the contact process, in Section 4.2.2 the random walk on top of the contact process.

4.2.1 Contact process

A càdlàg version of the contact process can be constructed from a graphical representation in the following standard fashion. Let $:= (H(x))_{x \in \mathbb{Z}}$ and $I := (I(x))_{x \in \mathbb{Z}}$ be two independent collections of i.i.d. Poisson processes with rates 1 and λ, respectively. On $\mathbb{Z} \times [0, \infty)$, draw the events of $H(x)$ as crosses over x and the events of $I(x)$ as two-sided arrows between x and $x + 1$ (see Fig. 4.3).

(The standard graphical representation uses Poisson processes of one-sided arrows to the right and to the left on every time line, each with rate λ. This gives the same dynamics.)

For $x, y \in \mathbb{Z}$ and $0 \le s \le t$, we say that (x, s) and (y, t) are *connected*, written $(x, s) \leftrightarrow (y, t)$, if and only if there exists a nearest-neighbor path in $\mathbb{Z} \times [0, \infty)$ starting at (x, s) and ending at (y, t), going either upwards in time or sideways in space across arrows without hitting crosses.
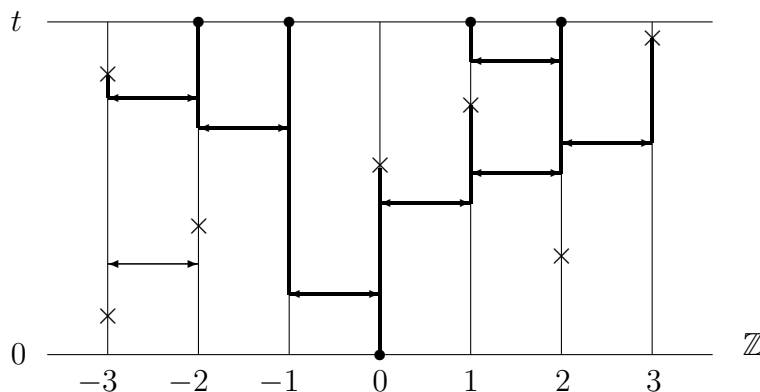
Figure 4.3: Graphical representation. The crosses are events of H and the arrows are events of I. The thick lines cover the region that is infected when the initial configuration has a single infection at the origin.

For $x \in \mathbb{Z}$, we define the cluster of x at time t by

$$C_t(x) := \big\{y \in \mathbb{Z} \colon (x, 0) \leftrightarrow (y, t)\big\}. \tag{4.2.1}$$

For example, in Fig. 4.3, $C_t(0) = \{-2, -1, 1, 2\}$ and $C_t(2) = \emptyset$. Note that $C_t(x)$ is a function of H and I.

For a fixed initial configuration η, we declare $\xi_t(y) = 1$ if there exists an x such that $y \in C_t(x)$ and $\eta(x) = 1$, and we declare $\xi_t(y) = 0$ otherwise. Then ξ is adapted to the filtration

$$\mathcal{F}_t := \sigma\big(\xi_0, (H_s, I_s)_{s \in [0,t]}\big). \tag{4.2.2}$$

This construction allows us to *simultaneously couple* copies of the contact process starting from *all* configurations $\eta \in \Omega$. In the following we will write $\xi(\eta)$ and $\xi_t(\eta)(x)$ when we want to exhibit that the initial configuration is η.

We note two consequences of the graphical construction, stated in Lemmas 4.2.1–4.2.3 below. The first is the monotonicity of $\eta \mapsto \xi(\eta)$, the second concerns the state of the sites surrounded by the cluster of an infected site. The notation $\eta \leq \eta'$ stands for $\eta(x) \leq \eta'(x)$ for all $x \in \mathbb{Z}$.

Lemma 4.2.1. *If $\eta \leq \eta'$, then $\xi_t(\eta) \leq \xi_t(\eta')$ for all $t \geq 0$.*

Proof. Immediate from the definition of ξ_t in terms of η and $(C_t(x))_{x \in \mathbb{Z}}$. ■

For $x \in \mathbb{Z}$, define the left-most and the right-most site influenced by site x at time t as

$$\begin{aligned} L_t(x) &:= \inf C_t(x), \\ R_t(x) &:= \sup C_t(x), \end{aligned} \tag{4.2.3}$$

where $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$. By symmetry, for any $t \geq 0$, $R_t(x) - x$ and $x - L_t(x)$ have the same distribution, independently of x.

Lemma 4.2.2. *Fix $x \in \mathbb{Z}$ and $t \geq 0$. If $C_t(x) \neq \emptyset$ and $y \in [L_t(x), R_t(x)] \cap \mathbb{Z}$, then $\eta \mapsto \xi_t(\eta)(y)$ is constant on $\{\eta \in \Omega \colon \eta(x) = 1\}$.*

Proof. It suffices to show that, under the conditions stated, $\xi_t(\eta)(y) = 1$ if and only if $y \in C_t(x)$. The 'if' part is obvious. For the 'only if' part, note that if there is a $z \neq x$ such that $(z, 0) \leftrightarrow (y, t)$, then any path realizing the connection must cross a path connecting $(x, 0)$ to either $(R_t(x), t)$ or $(L_t(x), t)$, so that $(x, 0) \leftrightarrow (y, t)$ as well. ∎

If $\xi_0 = \mathbb{1}_x$, then $R_t(x)$ and $L_t(x)$ are, respectively, the right-most and the left-most infections present at time t. In particular, in this case the infection survives for all times if and only if $R_t(x) - L_t(x) \geq 0$ for all $t \geq 0$. For $\lambda \in (\lambda_c, \infty)$ it is well known that, given $\xi_0 = \mathbb{1}_0$, the infection survives with positive probability and there exists a constant $\iota = \iota(\lambda) > 0$ such that, conditionally on survival,

$$\lim_{t \to \infty} t^{-1} R_t(0) = \iota \quad \xi\text{-a.s.} \tag{4.2.4}$$

4.2.2 Random walk on top of contact process

Under assumptions (4.1.5–4.1.6), the random walk W can be constructed as follows. Let $N := (N_t)_{t \geq 0}$ be a Poisson process with rate γ. Denote by $J := (J_k)_{k \in \mathbb{N}_0}$ its generalized inverse, i.e., $J_0 = 0$ and $(J_{k+1} - J_k)_{k \in \mathbb{N}_0}$ are i.i.d. EXP(γ) random variables. Let $U := (U_k)_{k \in \mathbb{N}}$ be an i.i.d. sequence of UNIF([0, 1]) random variables, independent of N. Set $S_0 := 0$ and, recursively for $k \in \mathbb{N}_0$,

$$S_{k+1} := S_k + 2 \left(\mathbb{1}_{\{0 \leq U_{k+1} \leq \alpha_0/\gamma\}} + \xi_{J_{k+1}}(S_k) \mathbb{1}_{\{\alpha_0/\gamma < U_{k+1} \leq \alpha_1/\gamma\}} \right) - 1, \tag{4.2.5}$$

i.e., $S_{k+1} = S_k + 1$ with probability α_i/γ and $S_{k+1} = S_k - 1$ with probability $\beta_i/\gamma = 1 - \alpha_i/\gamma$ when $\xi_{J_{k+1}}(S_k) = i$, for $i = 0, 1$ (recall that $\alpha_0 < \alpha_1$ by (4.1.5–4.1.6)). Setting

$$W_t := S_{N_t}, \tag{4.2.6}$$

we can use the right-continuity of ξ to verify that W indeed is a Markov process with the correct jump rates.

A useful property of the above construction is that it is monotone in the environment, in the following sense. For two dynamic random environments ξ and ξ', we say that $\xi \leq \xi'$ when $\xi_t \leq \xi'_t$ for all $t \geq 0$. Writing $W = W(\xi)$ in the previous construction (i.e., exhibiting W as a function of ξ), it is clear from (4.2.5) that

$$\xi \leq \xi' \quad \Longrightarrow \quad W_t(\xi) \leq W_t(\xi') \quad \forall t \geq 0. \tag{4.2.7}$$

We denote by

$$\mathcal{G}_t := \mathcal{F}_t \vee \sigma\left((N_s)_{s \in [0,t]}, (U_k)_{1 \leq k \leq N_t} \right) \tag{4.2.8}$$

the filtration generated by all the random variables that are used to define the contact process ξ and the random walk W.

4.3 Proof of the law of large numbers

Theorem 4.1.1(a) is proved in two steps. In Section 4.3.1 we use subadditivity to prove the SLLN when ξ starts from $\delta_{\mathbf{1}}$. In Section 4.3.2 we couple two copies of ξ starting from ν_λ and $\delta_{\mathbf{1}}$, transfer the SLLN, and show that the speed is the same.

In the following, for a random process $X = (X_t)_{t \in \mathcal{I}}$ with $\mathcal{I} = \mathbb{R}$ or $\mathcal{I} = \mathbb{Z}$, we write

$$X_{[0,t]} := (X_s)_{s \in [0,t] \cap \mathcal{I}}. \tag{4.3.1}$$

4.3.1 Starting from the full configuration: subadditivity

Since $\eta \leq \mathbf{1}$ for all $\eta \in \Omega$, it follows from (4.2.7) and Lemma 4.2.1 that $W_t(\xi(\eta)) \leq W_t(\xi(\mathbf{1}))$ for all $t \geq 0$. Therefore, if in the graphical construction we replace ξ_s by $\mathbf{1}$ at any given time s, then the new increments after time s lie to the right of the old increments after time s, and are independent of the increments before time s. This leads us to a subadditivity argument, which we now formalize.

For $n \in \mathbb{N}_0$, let

$$
\begin{aligned}
H^{(n)} &= \left(H_t^{(n)}(x) \right)_{t \geq 0, x \in \mathbb{Z}} &:=& \left(H_{t+n}(x + W_n) - H_n(x + W_n) \right)_{t \geq 0, x \in \mathbb{Z}}, \\
I^{(n)} &= \left(I_t^{(n)}(x) \right)_{t \geq 0, x \in \mathbb{Z}} &:=& \left(I_{t+n}(x + W_n) - I_n(x + W_n) \right)_{t \geq 0, x \in \mathbb{Z}}, \\
N^{(n)} &= \left(N_t^{(n)} \right)_{t \geq 0} &:=& \left(N_{t+n} - N_n \right)_{t \geq 0}, \\
U^{(n)} &= \left(U_k^{(n)} \right)_{k \in \mathbb{N}} &:=& \left(U_{k+N_n} \right)_{k \in \mathbb{N}}.
\end{aligned}
\tag{4.3.2}
$$

Then, for any $n \in \mathbb{N}_0$, $(H^{(n)}, I^{(n)}, N^{(n)}, U^{(n)})$ has the same distribution as (H, I, N, U) and is independent of

$$H_{[0,n-j]}^{(j)}, I_{[0,n-j]}^{(j)}, N_{[0,n-j]}^{(j)}, U_{[1,N_{n-j}^{(j)}]}^{(j)}, \qquad 0 \leq j \leq n - 1. \tag{4.3.3}$$

Abbreviate $\xi = \xi(\eta, H, I)$ and $W = W(\xi, N, U)$. For $n \in \mathbb{N}_0$, let

$$
\begin{aligned}
\xi^{(n)} &:= \xi(\mathbf{1}, H^{(n)}, I^{(n)}), \\
W^{(n)} &:= W(\xi^{(n)}, N^{(n)}, U^{(n)}),
\end{aligned}
\tag{4.3.4}
$$

and define the double-indexed sequence

$$X_{m,n} := W_{n-m}^{(m)}, \qquad n, m \in \mathbb{N}_0,\ n \geq m. \tag{4.3.5}$$

Lemma 4.3.1. *The following properties hold:*
(i) For all $n, m \in \mathbb{N}_0$, $n \geq m$: $X_{0,n} \leq X_{0,m} + X_{m,n}$.
(ii) For all $n \in \mathbb{N}_0$: $(X_{n,n+k})_{k \in \mathbb{N}_0}$ has the same distribution as $(X_{0,k})_{k \in \mathbb{N}_0}$.
(iii) For all $k \in \mathbb{N}$: $(X_{nk,(n+1)k})_{n \in \mathbb{N}_0}$ is i.i.d.
(iv) $\sup_{n \in \mathbb{N}} \mathbb{E}_{\delta_{\mathbf{1}}} \left[n^{-1} |X_{0,n}| \right] < \infty$.

Proof. (i) Fix $n, m \in \mathbb{N}_0$, $n \geq m$ and define $\hat{\xi} := \xi(\hat{\eta}, H^{(m)}, I^{(m)})$, where $\hat{\eta}(x) = \xi_m(x + W_m)$. This is the contact process after time m as seen from W_m. Note that $X_{0,n} - X_{0,m} = W_n - W_m = W_{n-m}(\hat{\xi}, N^{(m)}, U^{(m)})$. Since $\hat{\eta} \leq \mathbf{1}$, it follows from (4.2.7) and Lemma 4.2.1 that the latter is $\leq W_{n-m}(\xi^{(m)}, N^{(m)}, U^{(m)}) = W_{n-m}^{(m)}$.

(ii) Immediate from the construction.

(iii) By definition, $X_{nk,(n+1)k} = W_k(\xi^{(nk)}, N^{(nk)}, U^{(nk)})$. For each $t \geq 0$, $W_t(\xi, N, U)$ is by construction a function of $N_{[0,t]}$, $U_{[1,N_t]}$ and $\xi_{[0,t]}$, which in turn is a function of $H_{[0,t]}$, $I_{[0,t]}$ and η. Therefore $X_{nk,(n+1)k}$ is equal to a (fixed) function of

$$H_{[0,k]}^{(nk)}, I_{[0,k]}^{(nk)}, N_{[0,k]}^{(nk)}, U_{[1,N_{(n+1)k}^{(nk)}]}^{(nk)}, \tag{4.3.6}$$

which are jointly i.i.d. in n (when k is fixed).

(iv) This follows from the fact that $|W_t| \leq N_t$. ∎

Lemma 4.3.1 allows us to prove the SLLN when ξ starts from $\delta_{\mathbf{1}}$.

Proposition 4.3.2. *Let*

$$v(\lambda) := \inf_{n \in \mathbb{N}} \mathbb{E}_{\delta_{\mathbf{1}}} \left[n^{-1} W_n \right]. \tag{4.3.7}$$

Then

$$\lim_{t \to \infty} t^{-1} W_t = v(\lambda) \quad \mathbb{P}_{\delta_{\mathbf{1}}}\text{-}a.s. \text{ and in } L^p, \ p \geq 1. \tag{4.3.8}$$

Proof. Conditions (i)–(iv) in Lemma 4.3.1 allow us to apply the subbaditive ergodic theorem of Liggett [58] (see also Liggett [57], Theorem VI.2.6) to the sequence $(X_{0,n})_{n \in \mathbb{N}_0} = (W_n)_{n \in \mathbb{N}_0}$, which gives $\lim_{n \to \infty} n^{-1} W_n = v$ $\mathbb{P}_{\delta_{\mathbf{1}}}$-a.s. Via a standard argument this can subsequently be extended to $(t^{-1} W_t)_{t \geq 0}$ by using that, for any $n \in \mathbb{N}_0$,

$$\sup_{s \in [0,1]} |W_{n+s} - W_n| \leq N_{n+1} - N_n, \tag{4.3.9}$$

which implies that $\lim_{t \to \infty} t^{-1} |W_t - W_{\lfloor t \rfloor}| = 0$ $\mathbb{P}_{\delta_{\mathbf{1}}}$-a.s. The convergence also holds in L^p, because $|W_t| \leq N_t$ and so $(t^{-p} |W_t|^p)_{t \geq 1}$ is uniformly integrable for any $p \geq 1$. ∎

4.3.2 Starting from equilibrium: coupling

In this section we show that two copies of the contact process starting from ν_λ and $\delta_{\mathbf{1}}$ and coupled via the graphical representation are with a large probability equal inside space-time cones with tips at large times. Since the random walk eventually gets trapped inside a dense union of such cones, this will be enough to transfer the result of Proposition 4.3.2 from $\mathbb{P}_{\delta_{\mathbf{1}}}$ to $\mathbb{P}_{\nu_\lambda}$, with the same velocity $v(\lambda)$, and will complete the proof of Theorem 4.1.1(a).

For $m, r > 0$ and $t \geq 0$, let

$$V_{m,r}(t) := \left\{ (x, s) \in \mathbb{Z} \times [t, \infty) \colon |x| \leq r \vee m(s - t) \right\}, \tag{4.3.10}$$

i.e., $V_{m,r}(t)$ is the union of the cylinder $[-r, r] \cap \mathbb{Z} \times [t, \infty)$ and the cone with tip at $(0, t)$ opening upwards in space-time with inclination m (recall (4.1.2)).

Let η be distributed according to ν_λ, and let $\xi^{(1)} := \xi(\eta)$, $\xi^{(2)} := \xi(\mathbf{1})$, i.e., take $\xi^{(1)}$ and $\xi^{(2)}$ to be copies of the contact process constructed from the same graphical representation and initial configurations η and $\mathbf{1}$, respectively. Denote by $\mathbb{P}$ the joint distribution of all random variables needed to define $\xi^{(1)}$, $\xi^{(2)}$ and W, i.e., $\mathbb{P}$ is the product of the distributions of η, H, I, N and U.

Lemma 4.3.3. *For any $m, r > 0$,*

$$\lim_{T \to \infty} \mathbb{P} \left(\exists\, (x, t) \in V_{m,r}(T) \colon\, \xi_t^{(1)}(x) \neq \xi_t^{(2)}(x) \right) = 0. \tag{4.3.11}$$

Before proving Lemma 4.3.3, we show how it leads to Theorem 4.1.1(a).

Proof of Theorem 4.1.1(a). Fix $\epsilon > 0$. Let $D_T^1(r) := \{N_{T+t} - N_T \leq r \vee 2\gamma t \; \forall\, t \geq 0\}$. Since $\lim_{t \to \infty} t^{-1} N_t = \gamma$ a.s. and $(N_{T+t} - N_T)_{t \geq 0}$ is equal in distribution to N, there exists an $r_0 > 0$ such that

$$\mathbb{P} \left(D_T^1(r_0) \right) \geq 1 - \tfrac{1}{2}\epsilon \qquad \forall\, T > 0. \tag{4.3.12}$$

Let $D_T^2 := \{\xi_t^{(1)}(x) = \xi_t^{(2)}(x) \; \forall\, (x, t) \in V_{2\gamma, r_0}(T)\}$ and $D_T := D_T^1(r_0) \cap D_T^2$. By (4.3.12) and Lemma 4.3.3, there exists a $T_0 > 0$ large enough such that

$$\mathbb{P}(D_{T_0}) > 1 - \epsilon. \tag{4.3.13}$$

Let $\Gamma_0 := \{N_{T_0} = 0\}$, which has positive probability and is independent of $\xi^{(i)}$, $i = 1, 2$. Let $W^{(i)} := W(\xi^{(i)})$, $i = 1, 2$. Note that $W^{(1)} = W^{(2)}$ on $\Gamma_0 \cap D_{T_0}$. Since $\lim_{t \to \infty} t^{-1} W_t^{(2)} = v(\lambda)$ $\mathbb{P}$-a.s., we therefore get

$$\mathbb{P} \left(\lim_{t \to \infty} t^{-1}(W_{t+T_0}^{(1)} - W_{T_0}^{(1)}) = v \;\Big|\; \Gamma_0 \right) \geq 1 - \epsilon. \tag{4.3.14}$$

However, because ν_λ is an equilibrium and $W_{T_0}^{(1)} = 0$ on Γ_0, $(W_{t+T_0}^{(1)} - W_{T_0}^{(1)})_{t \geq 0}$ has under $\mathbb{P}(\cdot \mid \Gamma_0)$ the same distribution as W under $\mathbb{P}_{\nu_\lambda}$, so the SLLN is obtained by letting $\epsilon \downarrow 0$. Convergence in L^p, $p \geq 1$, follows as in the proof of Proposition 4.3.2. ∎

Proof of Lemma 4.3.3. Denote by P the joint law of η, H and I. The law of $(\xi^{(1)}, \xi^{(2)})$ is the same under P or $\mathbb{P}$. We can regard P as a law on the product space

$$\left(\{0, 1\} \times D(\mathbb{N}_0, [0, \infty))^2 \right)^{\mathbb{Z}} = \{0, 1\}^{\mathbb{Z}} \times \left(D(\mathbb{N}_0, [0, \infty))^2 \right)^{\mathbb{Z}}. \tag{4.3.15}$$

P is shift-ergodic because it is the product of probability measures that are shift-ergodic, namely, ν_λ and the distributions of H and I. Let

$$\Lambda_x := \left\{ \eta(x) = 1, \, (x - L_t(x)) \wedge (R_t(x) - x) \geq \lfloor (\iota/2)t \rfloor \; \forall\, t \geq 0 \right\}, \tag{4.3.16}$$

i.e., the event that x generates a "wide-spread infection" (moving at speed at least half the typical asymptotic speed ι). Since Λ_x is a translation of Λ_0, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{x=1}^{n} \mathbb{1}_{\Lambda_x} = P(\Lambda_0) =: \varrho > 0 \qquad P\text{-a.s.,} \tag{4.3.17}$$

where the last inequality is justified by (4.2.4) and local modifications of the graphical representation.

Next, for $n \in \mathbb{N}$, define Z_n by the equation

$$\sum_{x=1}^{Z_n} \mathbb{1}_{\Lambda_x} = n. \tag{4.3.18}$$

Then we also have

$$\lim_{n\to\infty} \frac{Z_n}{n} = \varrho^{-1} \qquad P\text{-a.s.} \tag{4.3.19}$$

$(Z_n)_{n\in\mathbb{N}}$ marks the positions of wide-spread infections to the right of the origin, i.e., $x > 0$ such that Λ_x occurs. Equation (4.3.19) means that these wide-spread infections are not too far apart. Extending the definition of Z_n to the negative integers, we obtain analogously that $\lim_{n\to\infty} n^{-1}(-Z_{-n}) = \varrho^{-1}$ P-a.s. Let $\mathcal{Z} := \cup_{n\in\mathbb{N}}\{Z_n, Z_{-n}\}$ and

$$\mathcal{S} := \big\{(y,t) \in \mathbb{Z} \times [2/\iota, \infty) \colon \exists\, x \in \mathcal{Z} \text{ such that } |y - x| \le (\iota/2)t - 1\big\}. \tag{4.3.20}$$

Then $\mathcal{S}$ is the union of cones of inclination angle $\iota/2$ with tips at $(2/\iota, z)$ with $z \in \mathcal{Z}$ (see Fig. 4.4). We call $\mathcal{S}$ the *safe region*. This is justified by the following fact, whose proof is a direct consequence of Lemma 4.2.2.

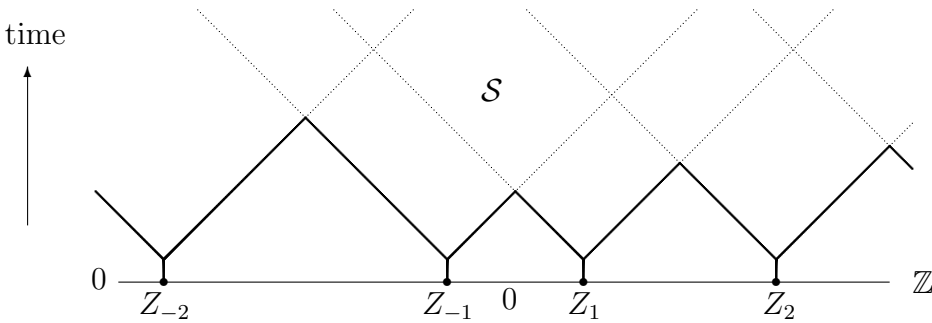**Lemma 4.3.4.** *If $(x,t) \in \mathcal{S}$, then $\xi_t^{(1)}(x) = \xi_t^{(2)}(x)$.*

Figure 4.4: Cones have inclination angle $\iota/2$. The safe region $\mathcal{S}$ lies above the thick lines.

By Lemma 4.3.4, it is enough to prove that $\mathcal{S}$ contains $V_{m,r}(t)$ with a large probability when t is large. Instead, we will prove that, for any $m > 0$,

$$V_{m,0}(0) \cap \mathcal{S}^c \text{ is a bounded subset of } \mathbb{Z} \times [0,\infty) \quad P\text{-a.s.} \tag{4.3.21}$$

This will also be enough, because it implies that $V_{m,r}(t) \subset \mathcal{S}$ for large t, P-a.s. for any $r > 0$,

Now, $\mathcal{S}^c$ is contained in the union of space-time "houses" (unions of triangles and rectangles) with base at time 0. The tips of the houses to the right of 0 form a sequence with spatial coordinates $\frac{1}{2}(Z_{n+1} + Z_n)$ and temporal coordinates $(Z_{n+1} - Z_n + 2)/\iota$, $n \in \mathbb{N}$. By (4.3.19), the ratio of temporal/spatial coordinates tends to 0 as $n \to \infty$, so that only finitely many tips can be inside $V_{m,0}(0)$. The same is true for the tips of the houses to the left of 0. Therefore $V_{m,0}(0)$ touches only finitely many houses, which proves (4.3.21). ∎

4.4 More on the contact process

In this section we collect some additional facts about the contact process on $\mathbb{Z}$ that will be needed in the remainder of the paper. The proofs rely on geometric observations that will also illuminate the proof strategies developed in Sections 4.5–4.6.

In the following we will use the notation

$$\mathbb{Z}_{\leq x} := \mathbb{Z} \cap (-\infty, x] \tag{4.4.1}$$

and analogously for $\mathbb{Z}_{\geq x}$.

Stochastic domination. We start with a useful alternative construction of the equilibrium ν_λ. Let $\eta(x) := \mathbb{1}_{\{C_t(x) \neq \emptyset \,\forall\, t \geq 0\}}$. Then, by the graphical representation, η has distribution ν_λ. This follows from duality (see Liggett [57], Chapter VI). We can also graphically construct the contact process starting from ν_λ: extend the graphical representation to negative times, and declare $\xi_t(x) = 1$ if and only if for all $0 \leq s \leq t$ there exists a y such that $(y, s) \leftrightarrow (x, t)$, i.e., if and only if there exists an infinite infection path going backwards in time from (x, t).

Let $\bar{\nu}_\lambda$ denote the restriction of ν_λ to $\mathbb{Z}_{\leq -1}$. Abusing notation, we will write the same symbol to denote the measure on Ω that is the product of $\bar{\nu}_\lambda$ with the measure concentrated on all sites healthy to the right of -1. Using the alternative construction above, we can prove that the restriction of $\nu_\lambda(\cdot \mid \eta(0) = 1)$ to $\mathbb{Z}_{\leq -1}$ is stochastically larger than $\bar{\nu}_\lambda$. In the following, we will focus on a similar result for the distribution of ξ_t to the left of certain infection paths.

For $\varpi_{[0,t]}$ a nearest-neighbor càdlàg path with values in $\mathbb{Z}$, let

$$\bar{\mathcal{R}}_t^\varpi := \sigma\left((\xi_0(x))_{x \geq \varpi_0}, (H_s(x), I_s(x))_{s \in [0,t], x \geq \varpi_s} \right). \tag{4.4.2}$$

Suppose that $\pi_{[0,t]}$ is a random path of the same type, with the following properties:

(p1) $\xi_0(\pi_0) = 1$ a.s. and $(\pi_s, s) \leftrightarrow (\pi_u, u)$ for all $s, u \in [0, t]$.

(p2) π is $\mathcal{F}$-adapted and $\{\pi_s \geq \varpi_s \,\forall\, s \in [0, t]\} \in \bar{\mathcal{R}}_t^\varpi$ for all deterministic paths ϖ.

We call π a *random infection path* (see Fig. 4.5), a name that is justified by (p1). Property (p2) means that π is causal and that, when we discover it, we leave the graphical representation to its left untouched. For such π, let

$$\mathcal{R}_t^\pi := \sigma\left(\pi, (\xi_0(x))_{x \geq \pi_0}, (H_s(x), I_s(x))_{s \in [0,t], x \geq \pi_s} \right). \tag{4.4.3}$$

Note that, since π is an infection path, also $(\xi_s(x))_{x \geq \pi_s} \in \mathcal{R}_t^\pi$ for each $s \in [0,t]$ (see the proof of Lemma 4.2.2). We have the following stochastic domination result.

Lemma 4.4.1. *For any random infection path $\pi_{[0,t]}$ as above, the law of $\xi_t(\cdot + \pi_t + 1)$ under $\mathbb{P}_{\bar{\nu}_\lambda}(\cdot \mid \mathcal{R}_t^\pi)$ is stochastically larger than $\bar{\nu}_\lambda$.*

Proof. Construct $\mathbb{P}_{\bar{\nu}_\lambda}$ from a graphical representation on $\mathbb{Z} \times \mathbb{R}$ as outlined above by adding healing events on $(x,0)$ for each $x \in \mathbb{Z}_{\geq 0}$. Extend π to negative times by making it equal to the right-most infinite infection path going backwards in time from $(\pi_0, 0)$. (Such a path exists because $\xi_0(\pi_0) = 1$.) We may check that the resulting path still has properties (p1) and (p2). Extend also $\mathcal{R}_t^\pi$ to include negative times.

Next, regard H and I as Poisson point processes on subsets of $\mathbb{Z} \times \mathbb{R}$. Let (see Fig. 4.5)

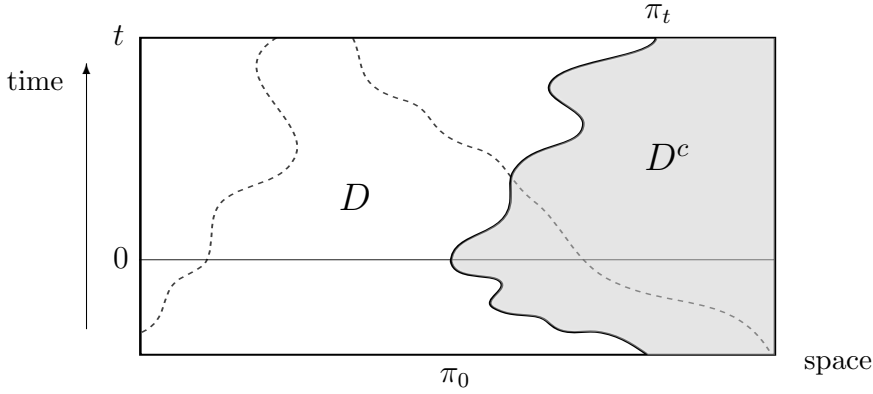$$D := \big\{ (x,s) \in \mathbb{Z} \times \mathbb{R} \colon \ s > t \text{ or } \pi_s > x \big\}. \tag{4.4.4}$$

Figure 4.5: The thick line represents the random infection path π. The dashed lines represent other infection paths.

Given $\mathcal{R}_t^\pi$, by (p2) H and I are still Poisson point processes with the same densities on D. This can be justified first for π taking values in a countable set and then for general π using right-continuity.

With this observation we can couple $\mathbb{P}_{\nu_\lambda}$ to $\mathbb{P}_{\bar{\nu}_\lambda}(\cdot \mid \mathcal{R}_t^\pi)$ in the following way. Draw independent Poisson point processes $\hat{H}$, $\hat{I}$ on D^c. Take $\hat{\xi}$ to be the contact process obtained by using H, I on D and $\hat{H}$, $\hat{I}$ on D^c. Then $\hat{\xi}$ is distributed as the contact process under $\mathbb{P}_{\nu_\lambda}$, and is independent of $\mathcal{R}_t^\pi$. Furthermore, $\xi_t(x) \geq \hat{\xi}_t(x)$ for all $x < \pi_t$. Indeed, if $\hat{\xi}_t(x) = 1$, then infinite infection paths going backwards in time must either stay inside D or cross π, so that, by (p1), $\xi_t(x) = 1$ as well. ∎

Remark 4.4.2. *In Lemma 4.4.1, we may replace t by a finite stopping time $\mathcal{T}$ w.r.t. the filtration $\mathcal{F}$, as long as the event in (p2) is replaced by $\{\mathcal{T} \leq t, \pi_s \geq \varpi_s \ \forall \ s \in [0, \mathcal{T}]\}$ and we add $\mathcal{T}$ to $\mathcal{R}_{\mathcal{T}}^\pi$. We may also enlarge all filtrations by adding information that is independent of ξ_0, H, I, in particular, $N_{[0,t]}$ and $U_{[1,N_t]}$ (recall Section 4.2.2).*

Infection range. Lemma 4.4.3 below concerns the positions of wide-spread infections. For $\delta \in (0, \iota)$ and $x \in \mathbb{Z}$, let $\mathcal{W}_x^\delta := \{(z, t) \in \mathbb{Z} \times [0, \infty) \colon (\iota - \delta)t - 1 < z - x \leq (\iota + \delta)t\}$ be a wedge between two lines of inclination $\iota - \delta$ and $\iota + \delta$. Set $C_t^\delta(x) := \{y \in \mathbb{Z} \colon (y, t) \leftrightarrow (x, 0) \text{ via a path contained in } \mathcal{W}_x^\delta\}$, and

$$Z_\delta(x) := \sup \left\{ z \in \mathbb{Z}_{<x} \colon \xi_0(z) = 1, C_t^\delta(z) \neq \emptyset \ \forall \, t \geq 0 \right\}, \tag{4.4.5}$$

i.e., the first infected site to the left of x that spreads its infection forever inside a wedge.

Lemma 4.4.3. *If $\lambda \in (\lambda_c, \infty)$ then $|Z_\delta(x) - x|$ has exponential moments under $\mathbb{P}_{\bar{\nu}_\lambda}$ for every $\delta \in (0, \iota)$, uniformly in $x \in \mathbb{Z}_{\leq 0}$.*

Proof. We will use the fact that, for any $\lambda \in (\lambda_c, \infty)$, ν_λ stochastically dominates a non-trivial Bernoulli product measure μ_λ. This follows from Liggett and Steif [60], Theorem 1.2, Durrett and Schonmann [35], Theorem 1, and van den Berg, Häggström and Kahn [12], Theorem 3.5. Since $Z_\delta(x)$ is monotone in ξ_0, it is therefore enough to prove the statement under $\mathbb{P}_{\mu_\lambda}$. We may also assume $x = 0$, as $Z_\delta(x)$ does not depend on $(\xi_0(z))_{z \geq x}$.

Construct a sequence of pairs $(Z_n, T_n)_{n \in \mathbb{N}_0}$ as follows. Set $Z_0 = T_0 := 0$ and, recursively for $n \in \mathbb{N}_0$,

$$
\begin{aligned}
Z_{n+1} &:= \begin{cases} Z_n & \text{if } T_n = \infty, \\ \sup\{z < Z_n - \lceil (\iota + \delta)T_n \rceil \colon \xi_0(z) = 1\} & \text{otherwise}, \end{cases} \\
T_{n+1} &:= \begin{cases} \infty & \text{if } T_n = \infty, \\ \inf\{t > 0 \colon C_t^\delta(Z_{n+1}) = \emptyset\} & \text{otherwise}. \end{cases}
\end{aligned}
\tag{4.4.6}
$$

Conditionally on $T_n < \infty$, $\Delta_{n+1} := Z_{n+1} - Z_n + \lceil (\iota + \delta)T_n \rceil$ and T_{n+1} are independent of $(Z_k, T_k)_{k=1}^n$ and distributed as (Z_1, T_1). This is because the region of the graphical representation plus initial configuration on which T_{n+1} and Δ_{n+1} depend is disjoint from the region on which the previous random variables depend. Since μ_λ is a non-trivial product measure, $|Z_1|$ has exponential moments. Noting that T_1 is independent of Z_1 we conclude, using standard facts about the contact process (see Liggett [57], Chapter VI, Theorem 2.2, Corollary 3.22 and Theorem 3.23), that $\mathbb{P}_{\mu_\lambda}(T_1 = \infty) > 0$ and that, conditionally on $T_1 < \infty$, T_1 has exponential moments. Defining the random index

$$K := \inf\{n \in \mathbb{N} \colon T_n = \infty\} \tag{4.4.7}$$

whose distribution is $\mathrm{GEO}(\mathbb{P}_{\mu_\lambda}(T_1 = \infty))$, we see that $|Z_\delta(0)| \leq |Z_K|$. Taking $a > 0$ such that $\mathbb{E}_{\mu_\lambda}[e^{a(|Z_1| + \lceil (\iota + \delta)T_1 \rceil)} \mid T_1 < \infty] < 1/\mathbb{P}_\mu(T_1 < \infty)$, we get after a short calculation that $\mathbb{E}_{\mu_\lambda}[\mathbb{1}_{\{K=n\}} e^{a|Z_n|}]$ decays exponentially in n. ∎

4.5 Properties of the speed

In this section we prove Theorem 4.1.1(b).

For each $n \in \mathbb{N}$, W_n depends on ξ in a finite space-time region. Therefore $\lambda \mapsto \mathbb{E}_{\delta_1}[n^{-1}W_n]$ is continuous (see Liggett [59], Part I). Since, by monotonicity, the latter is non-decreasing, it follows from (4.3.7) that $\lambda \mapsto v(\lambda)$ is right-continuous and non-decreasing.

It remains to show that $v(\lambda) \in (v_0, v_1)$ and $\lim_{\lambda \to \infty} v(\lambda) = v_1$. This will be done in Sections 4.5.1–4.5.2 below. These properties come from the fact that the random walk spends positive fractions of its time on top of infected sites and on top of healthy sites. To keep track of this, define $N_t^i := \#\{n \in \mathbb{N} \colon \xi_{J_n}(W_{J_{n-1}}) = i\}$, $i \in \{0, 1\}$. Recalling the construction of W in Section 4.2.2, we may write

$$W_t = S_{N_t^0}^0 + S_{N_t^1}^1, \tag{4.5.1}$$

where S_n^i, $i = 0, 1$, are discrete-time homogeneous random walks that jump to the right with probability α_i/γ and to the left with probability β_i/γ. From this representation we immediately get the following.

Lemma 4.5.1.
$$\begin{aligned}
\liminf_{t \to \infty} t^{-1}W_t &= v_0 + (v_1 - v_0)\liminf_{t \to \infty}(\gamma t)^{-1}N_t^1, \\
\limsup_{t \to \infty} t^{-1}W_t &= v_1 - (v_1 - v_0)\liminf_{t \to \infty}(\gamma t)^{-1}N_t^0.
\end{aligned} \tag{4.5.2}$$

Lemma 4.5.1 is valid for any dynamic random environment, even without a SLLN for W. But (4.5.2) shows that a SLLN for W holds with speed v if and only if a SLLN holds for N^1 with limit $\gamma\rho_{\text{eff}}$, where $\rho_{\text{eff}} := (v - v_0)/(v_1 - v_0)$ is the effective density of 1's seen by W. Thus, $v > v_0$ and $v < v_1$ are equivalent to, respectively, $\rho_{\text{eff}} > 0$ and $\rho_{\text{eff}} < 1$.

4.5.1 Proof of $v(\lambda) < v_1$

In the contact process, infected sites heal spontaneously. Therefore it is easier to find 0's than 1's. For this reason, it is easier to prove that W often jumps from healthy sites than from infected sites.

Proof. For $k \in \mathbb{N}$, let $Y_k := \xi_{J_k}(W_{J_{k-1}})$, and note that $\{Y_{k+1} = 0\}$ contains all configurations that between times J_k and J_{k+1} have a cross at site W_{J_k} and no arrows between W_{J_k} and its nearest-neighbors, i.e., such that the events $H_{J_{k+1}}(W_{J_k}) - H_{J_k}(W_{J_k}) \geq 1$ and $I_{J_{k+1}}(W_{J_k}) - I_{J_k}(W_{J_k}) = I_{J_{k+1}}(W_{J_k} - 1) - I_{J_k}(W_{J_k} - 1) = 0$ occur. The probability of the latter events given $\sigma\{(J_k, \xi_s, W_s)_{0 \leq s \leq J_k}\}$ is constant in k and equal to $p := \gamma/(\gamma + 2\lambda)(1 + \gamma + 2\lambda)$. Therefore the sequence $(Y_k)_{k \in \mathbb{N}}$ is stochastically dominated by a sequence of i.i.d. BERN$(1 - p)$ random variables, which implies that $\liminf_{t \to \infty} t^{-1}N_t^0 \geq \gamma p > 0$, so that $v(\lambda) < v_1$ by Lemma 4.5.1. ∎

4.5.2 Proof of $v(\lambda) > v_0$ and $\lim_{\lambda \to \infty} v(\lambda) = v_1$

This is the harder part of the proof. We will need results from Section 4.4. In the following we will assume that $v_0 \leq 0$. The case $v_0 > 0$ can be treated analogously.

Let us start with an informal description of the argument. The idea is that there are "waves of infection" coming from $\pm\infty$ from which the random walk cannot escape. When $v_0 \leq 0$, we can concentrate on the waves coming from the left, represented schematically in Fig. 4.6. Each time the random walk hits a new wave, there is an infection path starting from its current location and going backwards in time entirely to the left of the random walk path. By Lemma 4.4.1, at this time the law of ξ to the left of the random walk has an appreciable density, which means that there are new waves coming in from locations not very far to the left. On the other hand, any infections to the right of the random walk can be ignored, since they only push it to the right. But doing so makes the random walk behave as a homogeneous random walk with a non-positive drift, meaning that it does not take the random walk long to hit the next infection wave. Since at each collision there is a fixed probability for the random walk to jump while sitting on an infection, $v(\lambda) > v_0$ will follow from Lemma 4.5.1. With some care in the computations we also get the limit for large λ.
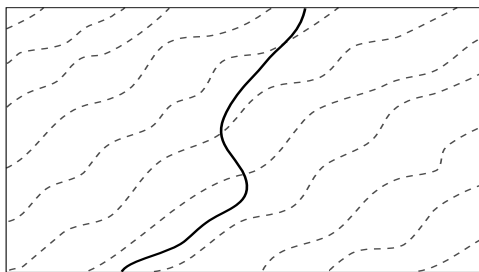

Figure 4.6: The dashed lines represent infection waves. The thick line represents the path of W.

Proof. Using the graphical representation, we will construct, on a larger probability space, a second random walk $\hat{W}$ coupled to W in such a way that $\hat{W}_t \leq W_t$ for all $t \geq 0$ and that $\hat{W}$ has a speed with the desired properties. Let

$$V_1 := \inf\{t > 0 \colon \xi_t(W_t) = 1\}. \tag{4.5.3}$$

Note that V_1 has exponential moments under $\mathbb{P}_{\bar{\nu}_\lambda}$ by Lemma 4.4.3 and the fact that $v_0 \leq 0$. Let

$$\tau_1 := \inf\big\{t > V_1 \colon W_t \neq W_{V_1} \text{ or } H_t(W_{V_1}) > H_{V_1}(W_{V_1})\big\}, \tag{4.5.4}$$

i.e., τ_1 is the first time after time V_1 at which either W jumps or there is a healing event at the position of the random walk. Note that τ_1 is a stopping time w.r.t. the filtration $\mathcal{G}$ and that, given $\mathcal{G}_{V_1}$, $\tau_1 - V_1$ has distribution $\mathrm{EXP}(1+\gamma)$.

We will construct a sequence $(W^{(n)}, \tau_n)_{n\in\mathbb{N}}$ with the following properties:

(A1) $W_t^{(n+1)} \leq W_{\tau_n+t}^{(n)} - W_{\tau_n}^{(n)}$ for all $t \geq 0$;

(A2) $(W^{(n)}, \tau_n)$ is distributed as (W, τ_1) under $\mathbb{P}_{\bar{\nu}_\lambda}$;

(A3) $(W^{(n)}_{[0,\tau_n]}, \tau_n)_{n\in\mathbb{N}}$ is i.i.d.;

(A4) If $\hat{v}(\lambda) := \mathbb{E}_{\bar{\nu}_\lambda}[W_{\tau_1}]/\mathbb{E}_{\bar{\nu}_\lambda}[\tau_1]$, then $\hat{v}(\lambda) > v_0$ and $\lim_{\lambda\to\infty} \hat{v}(\lambda) = v_1$.

Once we have this sequence, we can put $T_0 := 0$, $T_n := \sum_{k=1}^n \tau_k$ for $n \in \mathbb{N}$, and

$$\hat{W}_t := \sum_{k=1}^n W_{\tau_k}^{(k)} + W_{t-T_n}^{(n+1)} \quad \text{for} \quad T_n \leq t < T_{n+1}. \tag{4.5.5}$$

By (A1), $\hat{W}_t \leq W_t^{(1)}$ for all $t \geq 0$. By (A2), the latter is distributed as W under $\mathbb{P}_{\bar{\nu}_\lambda}$, which by monotonicity is stochastically smaller than W under $\mathbb{P}_{\nu_\lambda}$. By (A3), $\lim_{n\to\infty} T_n^{-1} \hat{W}_{T_n} = \hat{v}(\lambda)$, and so the claim follows from (A4). Thus, it remains to construct the sequence $(W^{(n)}, \tau_n)_{n\in\mathbb{N}}$ with properties (A1)–(A4).

To do so, we draw ξ_0 from $\bar{\nu}_\lambda$, let $\xi^{(1)} := \xi$, $W^{(1)} := W$, define τ_1 as above, and note the following.

Lemma 4.5.2. *Under $\mathbb{P}_{\bar{\nu}_\lambda}(\cdot \mid \tau_1, W_{[0,\tau_1]})$, the law of $\xi_{\tau_1}(\cdot + W_{\tau_1})$ is stochastically larger than $\bar{\nu}_\lambda$.*

Proof. Since $\xi_{V_1}(W_{V_1}) = 1$, there exists a right-most path $\pi_{[0,V_1]}$ connecting (W_{V_1}, V_1) to $\mathbb{Z}_{\leq -1} \times \{0\}$. Extend π to $[V_1, \tau_1]$ by making it constant and equal to W_{V_1} on this time interval. Since $\pi_s \leq W_s$ for all $0 \leq s < \tau_1$, we have $(\tau_1, W_{[0,\tau_1]}) \in \mathcal{R}_{\tau_1}^\pi \vee \sigma(N_{[0,\tau_1]}, U_{[1,N_{\tau_1}]})$. Note that π is not an infection path, but only because of a possible healing event at time τ_1, which does not affect $(\xi_{\tau_1}(x + W_{V_1}))_{x\leq -1}$. Therefore, by Lemma 4.4.1, the distribution of the latter given $(\tau_1, W_{[0,\tau_1]})$ is stochastically larger than $\bar{\nu}_\lambda$. Using this observation and noting that $W_{\tau_1} \neq W_{V_1}$ if and only if $\xi_{\tau_1}(W_{V_1}) = 1$, we can verify that the claim holds for each possible outcome of $W_{\tau_1} - W_{V_1} \in \{0, \pm 1\}$. ∎

By Lemma 4.5.2, there exists a configuration $\xi_0^{(2)}$ distributed as $\bar{\nu}_\lambda$, independent of $(\tau_1, W_{[0,\tau_1]})$ and stochastically smaller than $\xi_{\tau_1}^{(1)}(\cdot + W_{\tau_1})$. We may now define $\xi^{(2)}$ by using the events of the graphical representation that lie above time τ_1 with the origin shifted to W_{τ_1}, using $\xi_0^{(2)}$ as starting configuration. We may then define $W^{(2)}$ and τ_2 from $\xi^{(2)}$, $(N_{t+\tau_1} - N_{\tau_1})_{t\geq 0}$ and $(U_k)_{k>N_{\tau_1}}$. With this coupling, clearly $W_t^{(2)} \leq W_{\tau_1+t}^{(1)} - W_{\tau_1}^{(1)}$ for all $t \geq 0$. Furthermore, since $\xi_0^{(2)}$ is independent of $(\tau_1, W_{[0,\tau_1]})$, the distribution of $\xi_{\tau_2}^{(2)}(\cdot + W_{\tau_2}^{(2)})$ given $(W_{[0,\tau_i]}^{(i)}, \tau_i)_{i=1,2}$ depends only on the random variables with $i = 2$ and hence, by Lemma 4.5.2, is again stochastically larger than $\bar{\nu}_\lambda$.

We may therefore repeat the argument. More precisely, suppose by induction that we have defined $\xi^{(k)}$, $W^{(k)}$ and τ_k for $k = 1, \ldots, n$ and $n \geq 2$, in such a way that:

(B1) $W_t^{(k+1)} \leq W_{\tau_n+t}^{(k)} - W_{\tau_n}^{(n)}$ for all $t \geq 0$ and $k = 1 \ldots n - 1$;

(B2) $(W^{(k)}, \tau_k)$ is distributed as (W, τ_1) under $\mathbb{P}_{\bar{\nu}_\lambda}$ for all $k = 1, \ldots, n$;

(B3) $(W_{[0,\tau_k]}^{(k)}, \tau_k)_{k=1}^n$ is i.i.d.;

(B4) The law of $\xi^{(n)}(\cdot + W_{\tau_n}^{(n)})$ given $(W_{[0,\tau_k]}^{(k)}, \tau_k)_{k=1}^n$ is stochastically larger than $\bar{\nu}_\lambda$.

Then we proceed as before: there exists a configuration $\xi_0^{(n+1)}$ distributed as $\bar{\nu}_\lambda$, stochastically smaller than $\xi^{(n)}(\cdot + W_{\tau_n}^{(n)})$ and independent of $(W_{[0,\tau_k]}^{(k)}, \tau_k)_{k=1}^n$, from which we obtain $\xi^{(n+1)}$,

$W^{(n+1)}$ and τ_{n+1}, and we prove (B1)–(B4) like in the case $n = 2$. This settles the existence of the sequence $(W^{(n)}, \tau_n)_{n \in \mathbb{N}}$. All that is left to show is that $\hat{v}(\lambda) > v_0$ and $\lim_{\lambda \to \infty} \hat{v}(\lambda) = v_1$.

Note that Lemma 4.5.1 is valid also for $\hat{W}$, and write $\hat{N}_t^1$ to denote the number of jumps that $\hat{W}$ takes on infected sites. Then $\hat{N}_{T_n}^1$ has distribution $\mathrm{BINOM}(n, \gamma/(1 + \gamma))$, and by standard arguments we obtain

$$\lim_{t \to \infty} t^{-1} \hat{N}_t^1 = \frac{\gamma}{(1 + \gamma) \mathbb{E}_{\bar{\nu}_\lambda}[\tau_1]} > 0, \tag{4.5.6}$$

which proves $\hat{v}(\lambda) > v_0$. Furthermore, we claim that $\lim_{\lambda \to \infty} \mathbb{E}_{\bar{\nu}_\lambda}[V_1] = 0$. Indeed, V_1 is non-increasing in λ and, since $\lim_{\lambda \to \infty} \rho_\lambda = 1$ (recall Section 4.1.2), it is not hard to see that V_1 converges in probability to zero as $\lambda \to \infty$. Therefore $\lim_{\lambda \to \infty} \mathbb{E}_{\bar{\nu}_\lambda}[\tau_1] = 1/(1 + \gamma)$, and so $\lim_{\lambda \to \infty} \hat{v}(\lambda) = v_1$. ∎

4.6 Regeneration, central limit theorem and large deviations

The proof of Theorem 4.1.2 depends on the construction of *regeneration times*, i.e., times at which the random walk forgets its past. This construction will be carried out in Section 4.6.1 and is based on two propositions (Propositions 4.6.1–4.6.2 below), which are proved in Sections 4.6.2–4.6.3. At the end of Section 4.6.1 we will see that these propositions imply Theorem 4.1.2(a,c). The proof of Theorem 4.1.2(b) is deferred to Section 4.6.4.

4.6.1 Regeneration times

If the infection propagation speed $\iota = \iota(\lambda)$ is larger than $|v_0| \vee |v_1|$, the maximum absolute speed at which the random walk can move, then each time W finds itself on an infected site it can become "trapped" forever in an infection cluster generated by this site alone. In that case, by Lemma 4.2.2, the future increments of W become independent of its past. The issue is therefore to find enough moments when W sits on an infection. This can be dealt with in a way similar to what was done in the proof of $v(\lambda) > v_0$ in Section 4.5.2.

Hitting, failure and trial times. In order to build the regeneration structure, we first need to extend some definitions related to clusters and right-most infections. For $s \geq t$ and $x \in \mathbb{Z}$, let

$$C_{t,s}(x) := \big\{ y \in \mathbb{Z} \colon (x, t) \leftrightarrow (y, s) \big\} \tag{4.6.1}$$

and

$$R_{t,s}(x) := \sup C_{t,s}(x), \qquad L_{t,s}(x) := \inf C_{t,s}(x). \tag{4.6.2}$$

Furthermore, let

$$r_{t,s}(x) := \sup_{\substack{y < x \\ \xi_t(y) = 1}} R_{t,s}(y), \tag{4.6.3}$$

i.e., the right-most infection at time s that comes from $\mathbb{Z}_{\leq x-1} \times \{t\}$.

For $t \geq 0$ and $z \in \mathbb{Z}$, let

$$V_t(z) := \inf \big\{ s > t \colon W_s = r_{t,s}(z) \big\} \tag{4.6.4}$$

be the first time after time t at which W meets the right-most infection coming from $\mathbb{Z}_{\leq z-1}$. We will call this the *z-wave hitting time* after t. It is not hard to see that $V_t(z) < \infty$ $\mathbb{P}_{\nu_\lambda}$-a.s. for any t and $z \leq W_t$. Indeed, at any time t there is an infected site $x < z$ whose infection survives forever, and in this case $\lim_{s\to\infty} s^{-1} R_{t,s}(x) = \iota > |v_0| \vee |v_1|$. Therefore there must be an $s > t$ for which $R_{t,s}(x) = W_s$. By right-continuity, $\mathbb{P}_{\nu_\lambda}(V_t(z) < \infty \ \forall\, z \leq W_t, t \geq 0) = 1$ as well.

Now define the first *failure time* after time t by (see Fig. 4.8)

$$F_t := \inf \big\{ s > t \colon W_s \notin [L_{t,s}(W_t), R_{t,s}(W_t)] \big\}, \tag{4.6.5}$$

i.e., the first time after time t when W exits the region surrounded by the cluster of (W_t, t). To keep track of the space-time region on which the failure time depends, define, for $t \geq 0$ and $x \in \mathbb{Z}$,

$$(Y_{t,s}(x))_{s \geq t} \tag{4.6.6}$$

as the process with values in $\mathbb{Z}$ that starts at time t at site x and jumps down by following the infection arrows to the left in the graphical representation (see Fig. 4.7). Then, given $\mathcal{G}_t$, $(x - Y_{t,t+s}(x))_{s \geq 0}$ is a Poisson process with rate λ.
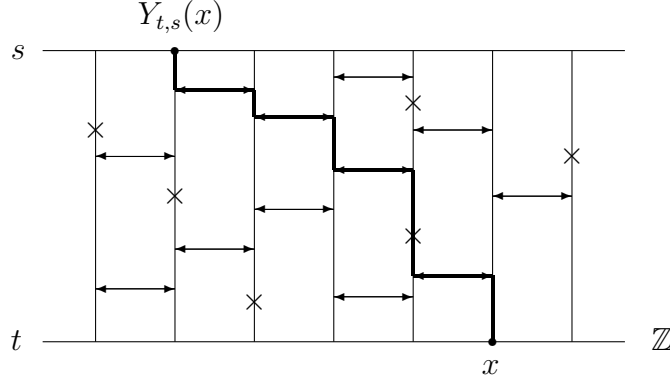
Figure 4.7: $Y_{t,s}(x)$ starts at x and goes upwards and to the left across the arrows of the graphical representation.

With the above observations we can define the *trial time* after a failure time (see Fig. 4.8):

$$T_t := \begin{cases} \infty & \text{if } F_t = \infty, \\ V_{F_t}(Y_{t,F_t}(W_t)) & \text{otherwise.} \end{cases} \tag{4.6.7}$$

i.e., T_t is the $Y_{t,F_t}(W_t)$-wave time after time F_t when the latter is finite. This wave ensures "good conditions" at the trial time, meaning an appreciable density of infections to the left of W.

Figure 4.8: A failure time F_t and a trial time T_t after time t. The dashed lines represent infection paths. The thick line represents the path of W.
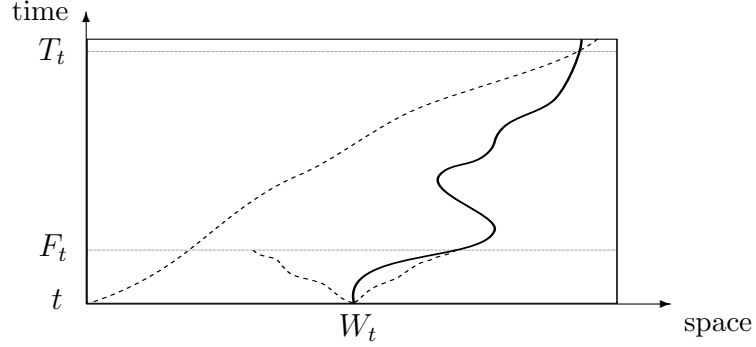
Regeneration times.   We can now define our regeneration time τ. First let

$$\mathcal{T}_1 := V_0(0) \tag{4.6.8}$$

and, under the assumption that $\mathcal{T}_1, \ldots, \mathcal{T}_k$, $k \in \mathbb{N}$, are all defined, let

$$\mathcal{T}_{k+1} := \begin{cases} \infty & \text{if } \mathcal{T}_k = \infty, \\ T_{\mathcal{T}_k} & \text{otherwise.} \end{cases} \tag{4.6.9}$$

Note that the $\mathcal{T}_k$'s are stopping times w.r.t. the filtration $\mathcal{G}$. Finally, put

$$K := \inf\left\{ k \in \mathbb{N} \colon \mathcal{T}_k < \infty, \mathcal{T}_{k+1} = \infty \right\}, \tag{4.6.10}$$

and let

$$\tau := \mathcal{T}_K. \tag{4.6.11}$$

Note that $K < \infty$ a.s. since, at any trial time, the probability for the next failure time to be infinite is uniformly bounded from below. We will prove in Sections 4.6.2–4.6.3 that τ is a regeneration time and has exponential moments. This is stated in the following two propositions.

Proposition 4.6.1. *The distribution of $(W_{t+\tau} - W_\tau)_{t \geq 0}$ under both $\mathbb{P}_{\nu_\lambda}(\cdot \mid \tau, W_{[0,\tau]})$ and $\mathbb{P}_{\nu_\lambda}(\cdot \mid \Gamma, \tau, W_{[0,\tau]})$ is the same as that of W under $\mathbb{P}_{\nu_\lambda}(\cdot \mid \Gamma)$, where*

$$\Gamma := \{\xi_0(0) = 1, F_0 = \infty\}. \tag{4.6.12}$$

Proposition 4.6.2. *τ and $|W_\tau|$ have exponential moments under both $\mathbb{P}_{\nu_\lambda}$ and $\mathbb{P}_{\nu_\lambda}(\cdot \mid \Gamma)$, uniformly in $\lambda \in [\lambda_-, \lambda_+]$ for any fixed $\lambda_-, \lambda_+ \in (\lambda_W, \infty)$.*

These two propositions imply the LLN and Theorem 4.1.2(a), with

$$v(\lambda) = \frac{\mathbb{E}_{\nu_\lambda}[W_\tau \mid \Gamma]}{\mathbb{E}_{\nu_\lambda}[\tau \mid \Gamma]}, \qquad \sigma(\lambda)^2 = \frac{\mathbb{E}_{\nu_\lambda}[(W_\tau)^2 \mid \Gamma] - \mathbb{E}_{\nu_\lambda}[W_\tau \mid \Gamma]^2}{\mathbb{E}_{\nu_\lambda}[\tau \mid \Gamma]}. \tag{4.6.13}$$

They also imply that

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}_{\nu_\lambda} \left(t^{-1} W_t \notin (v - \epsilon, v + \epsilon) \right) < 0 \quad \forall\, \epsilon > 0. \tag{4.6.14}$$

For a proof of these facts, the reader can follow word-by-word the arguments given in Avena, dos Santos and Völlering [7], Theorem 3.8 and Section 4.1 (which do not require (4.1.5)–(4.1.6)).

Theorem 4.1.2(c) follows from (4.6.14) and the partial LDP proven in Avena, den Hollander and Redig [5] for attractive spin-flip systems (including the contact process). Here, partial means that the LDP is shown to hold outside a possible interval where the rate function is zero. However, (4.6.14) precisely precludes the presence of such an interval. (See Glynn and Whitt [38], Theorem 3, for more details.)

The proof of Theorem 4.1.2(b) is deferred to Section 4.6.4.

4.6.2 Proof of Proposition 4.6.1

We first show that the regeneration strategy indeed makes sense.

Lemma 4.6.3. *For all $t \geq 0$,*

$$\mathbb{P}_{\nu_\lambda} \left(F_t = \infty, (W_{s+t} - W_t)_{s \geq 0} \in \cdot \mid \mathcal{G}_t \right) = \mathbb{P}_{\mathbb{1}_0} \left(\Gamma_0, W \in \cdot \right) \;\; a.s. \;\; on \; \{\xi_t(W_t) = 1\}, \tag{4.6.15}$$

where $\Gamma_0 := \{F_0 = \infty\}$. The same is true for a finite stopping time w.r.t. $\mathcal{G}$ replacing t.

Proof. First note that $\mathbb{P}_\eta(\Gamma_0, W \in \cdot) = \mathbb{P}_{\mathbb{1}_0}(\Gamma_0, W \in \cdot)$ for any η with $\eta(0) = 1$. This follows from Lemma 4.2.2 because, on Γ_0, W depends on ξ only through $\{\xi_t(x)\colon t \geq 0, x \in [L_t(0), R_t(0)]\}$, and Γ_0 does not depend on ξ_0. Now, letting $\hat{\xi}_t(\cdot) := \xi_t(\cdot + W_t)$, we can write (recall (4.6.5))

$$\mathbb{P}_{\nu_\lambda} \left(\xi_t(W_t) = 1, F_t = \infty, (W_{s+t} - W_t)_{s \geq 0} \in \cdot \mid \mathcal{G}_t \right)$$
$$= \mathbb{E}_{\nu_\lambda} \left[\xi_t(W_t) \mathbb{P}_{\hat{\xi}_t}(\Gamma_0, W \in \cdot) \mid \mathcal{G}_t \right] = \xi_t(W_t) \, \mathbb{P}_{\mathbb{1}_0}(\Gamma_0, W \in \cdot), \tag{4.6.16}$$

where the first equality is justified by the Markov property and the translation invariance of the graphical representation. To extend the result to stopping times we can use the strong Markov property of (ξ, W). ∎

With the help of Lemma 4.6.3 we are ready to prove Proposition 4.6.1.

Proof. We will closely follow the proof of Theorem 3.4 in [7]. Let $\mathcal{G}_\tau$ be the σ-algebra of all events B such that, for all $n \in \mathbb{N}_0$, there exists a $B_n \in \mathcal{G}_{\mathcal{T}_n}$ such that $B \cap \{K = n\} = B_n \cap \{K = n\}$. Note that τ and $W_{[0,\tau]}$ are in $\mathcal{G}_\tau$.

In the following, we abreviate $W^{(t)} := (W_{s+t} - W_t)_{s \geq 0}$. Pick f bounded and measurable, $B \in \mathcal{G}_\tau$, and write (recall (4.6.9))

$$
\begin{aligned}
\mathbb{E}_{\nu_\lambda}\left[\mathbb{1}_B f(W^{(\tau)})\right] &= \sum_{n \in \mathbb{N}_0} \mathbb{E}_{\nu_\lambda}\left[\mathbb{1}_{B_n} \mathbb{1}_{\{K=n\}} f(W^{(\mathcal{T}_n)})\right] \\
&= \sum_{n \in \mathbb{N}_0} \mathbb{E}_{\nu_\lambda}\left[\mathbb{1}_{B_n} \mathbb{1}_{\{\mathcal{T}_n < \infty\}} \mathbb{E}_{\nu_\lambda}\left[\mathbb{1}_{\{F_{\mathcal{T}_n} = \infty\}} f(W^{(\mathcal{T}_n)}) \,\big|\, \mathcal{G}_{\mathcal{T}_n}\right]\right].
\end{aligned}
\tag{4.6.17}
$$

Since $\xi_{\mathcal{T}_n}(W_{\mathcal{T}_n}) = 1$ on $\{\mathcal{T}_n < \infty\}$, by Lemma 4.6.3 the last line of (4.6.17) equals

$$
\begin{aligned}
\mathbb{E}_{\mathbb{1}_0}\left[f(W)\mathbb{1}_{\Gamma_0}\right] &\sum_{n \in \mathbb{N}_0} \mathbb{E}_{\nu_\lambda}\left[\mathbb{1}_{B_n} \mathbb{1}_{\{\mathcal{T}_n < \infty\}}\right] \\
&= \mathbb{E}_{\mathbb{1}_0}\left[f(W) \mid \Gamma_0\right] \sum_{n \in \mathbb{N}_0} \mathbb{E}_{\nu_\lambda}\left[\mathbb{1}_{B_n} \mathbb{1}_{\{\mathcal{T}_n < \infty\}}\right] \mathbb{P}_{\mathbb{1}_0}(\Gamma_0),
\end{aligned}
\tag{4.6.18}
$$

which, again by Lemma 4.6.3, equals

$$
\begin{aligned}
\mathbb{E}_{\mathbb{1}_0}\left[f(W) \mid \Gamma_0\right] &\sum_{n \in \mathbb{N}_0} \mathbb{E}_{\nu_\lambda}\left[\mathbb{1}_{B_n} \mathbb{1}_{\{\mathcal{T}_n < \infty\}} \mathbb{P}_{\nu_\lambda}\left(F_{\mathcal{T}_n} = \infty \mid \mathcal{G}_{\mathcal{T}_n}\right)\right] \\
&= \mathbb{E}_{\mathbb{1}_0}\left[f(W) \mid \Gamma_0\right] \sum_{n \in \mathbb{N}_0} \mathbb{P}_{\nu_\lambda}\left(B_n, K = n\right) \\
&= \mathbb{E}_{\mathbb{1}_0}\left[f(W) \mid \Gamma_0\right] \mathbb{P}_{\nu_\lambda}(B) \\
&= \mathbb{E}_{\nu_\lambda}\left[f(W) \mid \Gamma\right] \mathbb{P}_{\nu_\lambda}(B),
\end{aligned}
\tag{4.6.19}
$$

where the last equality is, one more time, justified by Lemma 4.6.3. This proves the claim under $\mathbb{P}_{\nu_\lambda}$.

To extend the claim to $\mathbb{P}_{\nu_\lambda}(\cdot \mid \Gamma)$, note that $\Gamma \in \mathcal{G}_\tau$ since

$$
\Gamma \cap \{K = n\} = \left\{\xi_0(0) = 1, W_s \in [L_s(0), R_s(0)] \;\forall\; s \in [0, \mathcal{T}_n]\right\} \cap \{K = n\},
\tag{4.6.20}
$$

and apply (4.6.19) to $B \cap \Gamma$ instead of B. ∎

4.6.3 Proof of Proposition 4.6.2

Exponential moments. We first show that T_0 has exponential moments when it is finite, uniformly for λ in compact sets. Fix $\lambda_-, \lambda_+ \in (\lambda_W, \infty)$.

Lemma 4.6.4. *For every $\lambda \in [\lambda_-, \lambda_+]$ and $\epsilon > 0$ there exists an $a = a(\lambda_-, \lambda_+, \epsilon) > 0$ such that, for any probability measure μ stochastically larger than $\bar{\nu}_\lambda$,*

$$
\begin{aligned}
&(a) \quad \mathbb{E}_\mu\left[\mathbb{1}_{\{T_0 < \infty\}} e^{aT_0}\right] \leq 1 + \epsilon. \\
&(b) \quad \mathbb{E}_\mu\left[e^{aV_0(0)}\right] \leq 1 + \epsilon.
\end{aligned}
\tag{4.6.21}
$$

Proof. We couple systems with infection rates λ_-, λ and λ_+ starting, respectively, from $\bar{\nu}_{\lambda_-}$, μ and $\mathbf{1}$, by coupling their initial configurations and their infection events monotonically. Denote their joint law by $\mathbb{P}$. In what follows, we will refer to these systems by their rates and we will use a superscript to indicate on which system a random variable depends.

We will bound $T_0 \mathbb{1}_{\{T_0 < \infty\}} = T_0 \mathbb{1}_{\{F_0 < \infty\}}$ by a time D_0 that depends only on systems $\lambda_\pm$ and has exponential moments under $\mathbb{P}$. We start by bounding $F_0 \mathbb{1}_{\{F_0 < \infty\}}$ by a variable D_1 depending only on system λ_-. Let

$$r_t := \sup_{x \in \mathbb{Z}_{\leq 0}} R_t(x), \qquad l_t := \inf_{x \in \mathbb{Z}_{\geq 0}} L_t(x). \tag{4.6.22}$$

Then r_t is the same as $r_{0,t}(0)$ in (4.6.3) when all sites in $\mathbb{Z}_{\leq 0}$ are infected, and analogously for l_t. Furthermore, $R_t(0), L_t(0)$ are equal to r_t, l_t while $C_t(0) \neq \emptyset$: this can be seen by using the graphical representation (see e.g. Liggett [57] Chapter VI, Theorem 2.2). Therefore

$$F_0 = \inf\{t \geq 0 \colon r_t < W_t \text{ or } l_t > W_t\}. \tag{4.6.23}$$

Let $m := \frac{1}{2}(\iota(\lambda_-) + |v_0| \vee |v_1|)$. Take homogeneous random walks X^i jumping at rates α_i, β_i, $i \in \{0, 1\}$, independent of ξ and coupled to W in such a way that $X_t^0 \leq W_t \leq X_t^1$ for all $t \geq 0$. Set

$$\begin{aligned} D_{1a} &:= \sup\{t \geq 0 \colon l_t^{\lambda_-} \geq -mt \text{ or } r_t^{\lambda_-} \leq mt\}, \\ D_{1b} &:= \sup\{t \geq 0 \colon |X_t^0| \vee |X_t^1| > mt\}. \end{aligned} \tag{4.6.24}$$

Then D_{1a} depends only on system λ_- and has exponential moments by known large deviation bounds for r_t (see Liggett [57] Chapter VI, Corollary 3.22), while D_{1b} is independent of ξ and has exponential moments by standard large deviation bounds for X^0 and X^1. Noting that r_t and l_t are monotone, we can take $D_1 := D_{1a} \vee D_{1b}$, which does not depend on the initial configuration.

Set $\delta := \frac{1}{2}(\iota(\lambda_-) - m)$, $x_0 := Y_{0,D_1}^{\lambda_+}(0) - \lceil (\iota(\lambda_+) + \delta) D_1 \rceil$ and note, using the graphical representation, that $\Delta_0 := x_0 - Z_\delta^{\lambda_-}(x_0)$ is independent of x_0, where $Z_\delta(x)$ is as in (4.4.5). Then

$$D_0 := \frac{\Delta_0 + |x_0| + 1}{\iota(\lambda_-) - \delta - m} = 4 \frac{\Delta_0 + |x_0| + 1}{\iota(\lambda_-) - |v_0| \vee |v_1|} \tag{4.6.25}$$

depends only on λ_-, λ_+ and has exponential moments under $\mathbb{P}$ by Lemma 4.4.3. It is easy to check that D_0 is the intersection time of the line of inclination $\iota(\lambda_-) - \delta$ passing through $(Z_\delta^{\lambda_-}(x_0) - 1, 0)$ and the line of inclination m passing through the origin. Since system λ has more infections than system λ_- and $D_0 \geq D_1$, we have $T_0 \mathbb{1}_{\{T_0 < \infty\}} \leq D_0$, which proves (a). For (b), we can bound $V_0(0)$ analogously, taking $x_0 = 0$ instead. $\blacksquare$

Infections at trial times. We next show that at trial times there are more infections to the left of the random walk than under $\bar{\nu}_\lambda$.

Lemma 4.6.5. *For all $n \in \mathbb{N}$, on the event $\{\mathcal{T}_n < \infty\}$ the law of $\xi_{\mathcal{T}_n}(\ \cdot + W_{\mathcal{T}_n})$ under $\mathbb{P}_{\nu_\lambda}(\cdot \mid \mathcal{T}_{[1,n]}, W_{[0,\mathcal{T}_n]})$ a.s. is stochastically larger than $\bar{\nu}_\lambda$.*

Proof. Suppose that $n \geq 2$ (the case $n = 1$ is simpler). Using the definition of $\mathcal{T}_n$, we can show by induction that, if $\mathcal{T}_n < \infty$, then there exist infection paths connecting $(W_{\mathcal{T}_n}, \mathcal{T}_n)$ to $\mathbb{Z}_{\leq -1} \times \{0\}$ and never touching the paths $Y^{\mathcal{T}_k}(W_{\mathcal{T}_k})$, $k = 1, \ldots, n-1$, or the region to the right of W. Take π to be the right-most of these infection paths. Then π is a random infection path with properties (p1) and (p2), and

$$(\mathcal{T}_{[1,n]}, W_{[0,\mathcal{T}_n]}) \in \mathcal{R}^{\pi}_{\mathcal{T}_n} \vee \sigma\big(N_{[0,\mathcal{T}_n]}, U_{[1,N_{\mathcal{T}_n}]}\big). \tag{4.6.26}$$

Therefore the result follows from Lemma 4.4.1. ∎

Conclusion. We are now ready to prove Proposition 4.6.2.

Proof. Let

$$\kappa := \mathbb{P}_{\mathbb{1}_0}(\Gamma_0). \tag{4.6.27}$$

By Lemma 4.6.3, $\mathbb{P}_{\nu_\lambda}(\Gamma) = \kappa \rho_\lambda \geq \kappa \rho_{\lambda_-}$ by monotonicity (recall the definition of ρ_λ from Section 4.1.2). Also, there exists a $\kappa_- > 0$ such that $\kappa \geq \kappa_-$ for any $\lambda \geq \lambda_-$: we can take κ_- to be the probability that X^0 and X^1 in the proof of Lemma 4.6.4 never cross $L(0)$ or $R(0)$ in system λ_-. Therefore it is enough to prove the claim for $\mathbb{P}_{\nu_\lambda}$. Since $|W|$ is dominated by N, which is Poisson process independent of ξ, we only need to worry about τ.

For $\epsilon > 0$ such that $(1+\epsilon)(1-\kappa_-) < 1$, take $a > 0$ as in Lemma 4.6.4. On the event $\{\mathcal{T}_n < \infty\}$, let $\hat{\xi}_n := \xi_{\mathcal{T}_n}(\cdot + W_{\mathcal{T}_n})$ and note that, given $\mathcal{G}_{\mathcal{T}_n}$, $\mathcal{T}_{n+1} - \mathcal{T}_n$ is distributed as T_0 under $\mathbb{P}_{\hat{\xi}_n}$. By Lemma 4.6.5, the law of $\hat{\xi}_n$ under $\mathbb{P}_{\nu_\lambda}(\cdot \mid \mathcal{T}_{[1,n]}, W_{[0,\mathcal{T}_n]})$ is stochastically larger than $\bar{\nu}_\lambda$, and we get from Lemma 4.6.4 that

$$\mathbb{E}_{\nu_\lambda}\Big[\mathbb{1}_{\{\mathcal{T}_{n+1} < \infty\}} e^{a(\mathcal{T}_{n+1} - \mathcal{T}_n)} \mid \mathcal{T}_{[1,n]}, W_{[0,\mathcal{T}_n]}\Big]$$
$$= \mathbb{E}_{\nu_\lambda}\Big[\mathbb{E}_{\hat{\xi}_n}\big[\mathbb{1}_{\{T_0 < \infty\}} e^{aT_0}\big] \mid \mathcal{T}_{[1,n]}, W_{[0,\mathcal{T}_n]}\Big] \leq 1 + \epsilon. \tag{4.6.28}$$

Using this bound, estimate

$$\mathbb{E}_{\nu_\lambda}\big[\mathbb{1}_{\{\mathcal{T}_{n+1} < \infty\}} e^{a\mathcal{T}_{n+1}}\big] = \mathbb{E}_{\nu_\lambda}\Big[\mathbb{1}_{\{\mathcal{T}_n < \infty\}} e^{a\mathcal{T}_n} \mathbb{E}_{\nu_\lambda}\big[\mathbb{1}_{\{\mathcal{T}_{n+1} < \infty\}} e^{a(\mathcal{T}_{n+1} - \mathcal{T}_n)} \mid \mathcal{T}_n\big]\Big]$$
$$\leq (1 + \epsilon)\mathbb{E}_{\nu_\lambda}\big[\mathbb{1}_{\{\mathcal{T}_n < \infty\}} e^{a\mathcal{T}_n}\big], \tag{4.6.29}$$

so that, by induction,

$$\mathbb{E}_{\nu_\lambda}\big[\mathbb{1}_{\{\mathcal{T}_n < \infty\}} e^{a\mathcal{T}_n}\big] \leq (1 + \epsilon)^n. \tag{4.6.30}$$

Using Lemma 4.6.3, write, for $n \in \mathbb{N}$,

$$\mathbb{P}_{\nu_\lambda}(K \geq n+1) = \mathbb{P}_{\nu_\lambda}(\mathcal{T}_n < \infty, F_{\mathcal{T}_n} < \infty) = (1 - \kappa)\mathbb{P}_{\nu_\lambda}(K \geq n) \tag{4.6.31}$$

to note that K has distribution $\text{GEO}(\kappa)$. To conclude, use (4.6.30)–(4.6.31) to write

$$\mathbb{E}_{\nu_\lambda}\big[e^{\frac{a}{2}\tau}\big] = \sum_{n \in \mathbb{N}} \mathbb{E}_{\nu_\lambda}\big[\mathbb{1}_{\{K=n\}} e^{\frac{a}{2}\mathcal{T}_n}\big] = \sum_{n \in \mathbb{N}} \mathbb{E}_{\nu_\lambda}\big[\mathbb{1}_{\{K=n\}}\mathbb{1}_{\{\mathcal{T}_n < \infty\}} e^{\frac{a}{2}\mathcal{T}_n}\big]$$
$$\leq \sum_{n \in \mathbb{N}} \mathbb{P}_{\nu_\lambda}(K = n)^{\frac{1}{2}} \mathbb{E}_{\nu_\lambda}\big[\mathbb{1}_{\{\mathcal{T}_n < \infty\}} e^{a\mathcal{T}_n}\big]^{\frac{1}{2}} \tag{4.6.32}$$
$$\leq (1 - \kappa_-)^{-\frac{1}{2}} \sum_{n \in \mathbb{N}} \Big(\sqrt{(1 - \kappa_-)(1 + \epsilon)}\Big)^n < \infty,$$

where in the second line we use the Cauchy-Schwarz inequality. ∎

4.6.4 Continuity of the speed and the volatility

Given $\lambda_- \leq \lambda_+$ in (λ_W, ∞) and $(\lambda_n)_{n \in \mathbb{N}}$, $\lambda_* \in [\lambda_-, \lambda_+]$ such that either $\lambda_n \uparrow \lambda_*$ or $\lambda_n \downarrow \lambda_*$ as $n \to \infty$, we can simultaneously construct systems with infection rates $(\lambda_n)_{n \in \mathbb{N}}$, λ_* and $\lambda_\pm$, starting from equilibrium, with a single graphical representation in the standard fashion, taking a monotone sequence of Poisson processes for infection events and coupling the initial configurations monotonically. For $n \in \mathbb{N} \cup \{*, +, -\}$, denote by $\Lambda^n := (\xi_0^n, H, I^n, N, U)$ the system with infection rate λ_n, and by $\mathbb{P}$ their joint law. In the following, we will use a superscript n to indicate functionals of Λ^n.

In view of (4.6.13) and Proposition 4.6.2, in order to prove convergence of $v(\lambda_n)$ and $\sigma(\lambda_n)$ it is enough to prove convergence in distribution of Γ^n and of $(W_{\tau^n}^n, \tau^n)\mathbb{1}_{\Gamma^n}$. The main step to achieve this will be to approximate relevant random variables with uniformly large probability by random variables depending on bounded regions of the graphical representation.

Note that, by monotonicity and continuity of $\lambda \mapsto \rho_\lambda$ (see Liggett [57] Chapter VI, Theorem 1.6),

$$\lim_{n \to \infty} \xi_0^n(x) = \xi_0^*(x) \quad \forall\, x \in \mathbb{Z} \quad \mathbb{P}\text{-a.s.} \tag{4.6.33}$$

Recall the definitions of F_0, $\mathcal{T}_k$ and K in (4.6.5), (4.6.8)–(4.6.9) and (4.6.10), respectively. For $n \in \mathbb{N} \cup \{*\}$ and $k \in \mathbb{N}$, let

$$\Gamma_k^n := \left\{ \xi_0^n(0) = 1, W_s^n \in [L_s^n(0), R_s^n(0)] \,\forall\, s \in [0, \mathcal{T}_k^n] \cap \mathbb{R} \right\}, \tag{4.6.34}$$

so that $\Gamma^n = \Gamma_k^n$ on $\{K^n = k\}$ as in (4.6.20).

Proposition 4.6.6. *For every $k \in \mathbb{N}$, $(W_{\mathcal{T}_k^n}^n, \mathcal{T}_k^n, \mathbb{1}_{\Gamma_k^n})\mathbb{1}_{\{\mathcal{T}_k^n < \infty\}}$, $\mathbb{1}_{\{\mathcal{T}_k^n < \infty\}}$ and $\mathbb{1}_{\{F_0^n < \infty\}}$ converge in probability as $n \to \infty$ to the corresponding functionals of Λ^*.*

Proof. We first show that, for every fixed $T \in (0, \infty)$,

$$(W_{\mathcal{T}_k^n}^n, \mathcal{T}_k^n, \mathbb{1}_{\Gamma_k^n})\mathbb{1}_{\{\mathcal{T}_k^n \leq T\}}, \quad \mathbb{1}_{\{\mathcal{T}_k^n \leq T\}}, \quad \mathbb{1}_{\{F_0^n \leq T\}}, \tag{4.6.35}$$

converge a.s. as $n \to \infty$ to the corresponding functionals of Λ^*. To that end, let $\bar{Y}_{t,s}(x)$ be the increasing analogue of $Y_{t,s}(x)$ in (4.6.6), starting from x but jumping across the arrows of I to the right. Let $\bar{Z}_\delta(x)$, analogously to $Z_\delta(x)$ in (4.4.5), be the first infected site to the right of x whose infection spreads inside a wedge between lines of inclination $-(\iota + \delta)$ and $-(\iota - \delta)$. Take $\delta := \iota(\lambda_-)/2$, set $\underline{y} := Y_{0,T}^+(-N_T)$ and $\underline{z} := Z_\delta^-(\underline{y} - \lceil(\iota(\lambda_-) + \delta)T\rceil)$. Analogously, put $\overline{y} := \bar{Y}_{0,T}^+(N_T)$ and $\overline{z} := \bar{\bar{Z}}_\delta^-(\overline{y} + \lceil(\iota(\lambda_-) + \delta)T\rceil)$.

Now observe that, for any $n \in \mathbb{N} \cup \{*\}$, all random variables in (4.6.35) depend on Λ^n only in the space-time box $\mathcal{B} := [\underline{z}, \overline{z}] \times [0, T]$. Indeed, for any $0 \leq t \leq s \leq T$, we have $L_{t,s}^n(W_t^n) \geq Y_{t,s}^n(W_t^n) \geq y^-$ and $R_{t,s}^n(W_t^n) \leq y^+$, so that $\{F_t^n \leq s\}$ depends on Λ^n only inside $[\underline{y}, \overline{y}] \times [0, T]$. Also, there are infection paths from time 0 to time T inside $[\underline{z}, \underline{y})$ and $(\overline{y}, \overline{z}]$. Therefore the states of ξ^n inside $[\underline{y}, \overline{y}] \times [0, T]$ depend on Λ^n only in $\mathcal{B}$ (see the proof of Lemma 4.2.2). The same is true for $\{T_t^n \leq s\}$, since any infection path needed to discover T_t^n can be taken inside $\mathcal{B}$.

Therefore, by (4.6.33) (and since the graphical representation is a.s. eventually constant inside bounded space-time regions), the claim after (4.6.35) follows.

To conclude note that, because $\mathcal{T}_k \mathbb{1}_{\{\mathcal{T}_k < \infty\}} \leq \tau$ and $F_0 \mathbb{1}_{\{F_0 < \infty\}} \leq T_0 \mathbb{1}_{\{T_0 < \infty\}}$,

$$\lim_{T \to \infty} \sup_{n \in \mathbb{N} \cup \{*\}} \mathbb{P}\big(T < \mathcal{T}_k^n < \infty \ \text{ or } \ T < F_0^n < \infty\big) = 0 \tag{4.6.36}$$

by Proposition 4.6.2 and Lemma 4.6.4, which implies that, for large T, the random variables in the statement are equal to the ones in (4.6.35) with uniformly large probability. ∎

Corollary 4.6.7. *Let κ^n be as in (4.6.27). Then $\lim_{n \to \infty} \kappa^n = \kappa^*$ and K^n converges in distribution to K^*.*

Proof. This follows directly from Proposition 4.6.6 and the definition of κ since, by (4.6.31), K^n is a geometric random variable with parameter κ^n. ∎

With these results we can conclude the proof of Theorem 4.1.2(c).

Proof. Let f be a bounded measurable function. For $k \in \mathbb{N}$, write

$$\begin{aligned}
\mathbb{E}\left[f(W_{\tau^n}^n, \tau^n)\mathbb{1}_{\Gamma^n}\mathbb{1}_{\{K^n = k\}}\right] &= \mathbb{E}\left[f\big(W_{\mathcal{T}_k^n}^n, \mathcal{T}_k^n\big)\mathbb{1}_{\Gamma_k^n}\mathbb{1}_{\left\{\mathcal{T}_k^n < \infty, F_{\mathcal{T}_k^n}^n = \infty\right\}}\right] \\
&= \kappa^n\, \mathbb{E}\left[f\big(W_{\mathcal{T}_k^n}^n, \mathcal{T}_k^n\big)\mathbb{1}_{\Gamma_k^n}\mathbb{1}_{\left\{\mathcal{T}_k^n < \infty\right\}}\right] \\
&\xrightarrow{n \to \infty} \kappa^*\mathbb{E}\left[f\big(W_{\mathcal{T}_k^*}^*, \mathcal{T}_k^*\big)\mathbb{1}_{\Gamma_k^*}\mathbb{1}_{\left\{\mathcal{T}_k^* < \infty\right\}}\right] \\
&= \mathbb{E}\left[f(W_{\tau^*}^*, \tau^*)\mathbb{1}_{\Gamma^*}\mathbb{1}_{\{K^* = k\}}\right],
\end{aligned} \tag{4.6.37}$$

where for the second and the third equality we use Lemma 4.6.3 and the strong Markov property, and for the convergence we use Proposition 4.6.6 and Corollary 4.6.7. Therefore

$$\begin{aligned}
\big|\mathbb{E}&\left[f(W_{\tau^n}^n, \tau^n)\mathbb{1}_{\Gamma^n}\right] - \mathbb{E}\left[f(W_{\tau^*}^*, \tau^*)\mathbb{1}_{\Gamma^*}\right]\big| \\
&\leq \|f\|_\infty\big\{\mathbb{P}(K^n > M) + \mathbb{P}(K^* > M)\big\} \\
&\quad + \sum_{k=1}^M \left|\mathbb{E}\left[f(W_{\tau^n}^n, \tau^n)\mathbb{1}_{\Gamma^n}\mathbb{1}_{\{K^n = k\}}\right] - \mathbb{E}\left[f(W_{\tau^*}^*, \tau^*)\mathbb{1}_{\Gamma^*}\mathbb{1}_{\{K^* = k\}}\right]\right|,
\end{aligned} \tag{4.6.38}$$

and we conclude by taking $n \to \infty$, using Corollary 4.6.7 and (4.6.37), and taking $M \to \infty$. ∎

5 Non-trivial linear bounds for a random walk driven by a simple symmetric exclusion process

Abstract

Non-trivial linear bounds are obtained for the displacement of a random walk in a dynamic random environment given by a one-dimensional simple symmetric exclusion process in equilibrium. The proof uses an adaptation of multiscale renormalization methods of Kesten and Sidoravicius [50].

5.1 Introduction, results and motivation

5.1.1 Setup

In this chapter, we discuss linear scaling properties of a random walk in a dynamic random environment (RWDRE), where the role of the random environment is taken by a one-dimensional simple symmetric exclusion process (SSEP). The latter is the càdlàg Markov process $\xi = (\xi_t)_{t \geq 0}$ with state space $E := \{0, 1\}^{\mathbb{Z}}$ whose infinitesimal generator $\mathcal{L}$ acts on bounded local functions f in the following manner:

$$(\mathcal{L}f)(\eta) := \sum_{x \in \mathbb{Z}} f(\eta^{x,x+1}) - f(\eta) \tag{5.1.1}$$

where $\eta \in \{0, 1\}^{\mathbb{Z}}$ and $\eta^{x,y}$ is defined by

$$\eta^{x,y}(z) = \begin{cases} \eta(x) & \text{if } z = y; \\ \eta(y) & \text{if } z = x; \\ \eta(z) & \text{otherwise.} \end{cases} \tag{5.1.2}$$

For a detailed description, we refer the reader to Liggett [57], Chapter VIII. We say that the site x is occupied by a *particle* at time t if $\xi_t(x) = 1$ and is *vacant* (alternatively, occupied by a *hole*) if $\xi_t(0) = 0$.

For a fixed realization of ξ, the random walk in dynamic random environment $W = (W_t)_{t \geq 0}$ is the time-inhomogeneous Markov process that starts at 0 and, given that $W_t = x$, jumps to

$$
\begin{array}{lll}
x + 1 & \text{with rate} & \alpha_1 \xi_t(x) + \alpha_0 \left[1 - \xi_t(x)\right], \\
x - 1 & \text{with rate} & \beta_1 \xi_t(x) + \beta_0 \left[1 - \xi_t(x)\right],
\end{array}
\tag{5.1.3}
$$

where $\alpha_i, \beta_i \in (0, \infty)$, $i \in \{0, 1\}$. We will assume that

$$
\alpha_0 + \beta_0 = \alpha_1 + \beta_1 =: \gamma
\tag{5.1.4}
$$

and

$$
v_1 > v_0 \text{ with } v_0 := \alpha_0 - \beta_0 \text{ and } v_1 := \alpha_1 - \beta_1,
\tag{5.1.5}
$$

i.e., the total jump rate is constant and equal to γ, and the local drift is larger on particles than on holes. The latter is made w.l.o.g., since the SSEP is invariant under reflection through the origin. We will denote by $\mathbb{P}_\eta$ the joint law of W and ξ when $\xi_0 = \eta$. We will draw ξ_0 from a Bernoulli product measure ν_ρ with $\rho \in (0, 1)$; these are known to be the only non-trivial extremal invariant measures for the SSEP.

While many results for RWDRE have been obtained in the past few years for random environments exhibiting uniform and fast enough mixing (see e.g. Avena [3], as well as the introduction to this thesis), very little is known when the random environment mixes in a non-uniform way, as happens in the SSEP. For example, there are still no general laws of large numbers available for such cases. In particular, for the model described here, the law of large numbers has only been proven under the restriction that $v_1 > v_0 > 1$ (see Avena, dos Santos and Völlering [7]). Another recent result is the paper by den Hollander, Kesten and Sidoravicius [43], where an approximate law of large numbers is proven when the random environment is a high-density Poisson field of independent random walks.

5.1.2 Main result

It is easy to see, with a coupling argument, that W lies between two homogeneous random walks with drifts v_0 and v_1. In particular, any subsequential limit of $t^{-1} W_t$ as $t \to \infty$ lies in the interval $[v_0, v_1]$. But would it be possible, even along a subsequence, for W to travel at one of the extremal speeds? For the case of the SSEP, the following theorem answers this question in the negative.

Theorem 5.1.1. *For any $\rho \in (0, 1)$, there exist $v_-, v_+ \in (v_0, v_1)$ such that*

$$
v_- \leq \liminf_{t \to \infty} t^{-1} W_t \leq \limsup_{t \to \infty} t^{-1} W_t \leq v_+ \quad \mathbb{P}_{\nu_\rho}\text{-}a.s.
\tag{5.1.6}
$$

While this result is "intuitively obvious", it does not seem a trivial fact to prove. For dynamic random environments consisting of single-site spin-flips with bounded flip rates, there is a simple proof strategy since particles and holes can be found locally "around" the random walk. For the supercritical contact process, the proof by den Hollander and dos Santos [44] that W cannot

travel with speed v_0 is already non-trivial and relies on model-specific features. The proof of Theorem 5.1.1 given here is based on the *multiscale analysis* scheme put forth by Kesten in Sidoravicius [50], and seems exceedingly heavy for such a simple fact. It has however the advantage of being easier to generalize; while several technical facts are verified here specifically for the SSEP, the overall proof strategy should work in much greater generality. For example, the analogous result for the supercritical contact process can be reobtained with this approach.

5.1.3 Essential enhancements

Our question can also be formulated in terms of *essential enhancements*, in analogy with percolation theory (see e.g. Grimmett [41], Chapter 3). From this point of view, W is seen as a perturbation of a homogeneous random walk with drift v_0, and ρ is the intensity of the perturbation. The question then becomes: is this perturbation, for any $\rho > 0$, an "essential enhancement" in the sense that it changes the linear scaling of W?

Let us look at what can happen for random walks in *static* one-dimensional random environments. For these models, there are criteria for recurrence/transience as well as laws of large numbers proven under very general assumptions (see e.g. Zeitouni [83]). If $v_0 = 0 < v_1$, then the random walk is always transient to the right in any ergodic random environment with a positive density of particles. But random walks in static random environments can exhibit *slow-down* phenomena; for example, there are regimes where the random walk can be transient to the right with zero speed. In the case of i.i.d. static random environments, the latter can only happen when $v_0 < 0 < v_1$. Therefore, if $v_0 = 0 < v_1$, then the perturbation given by a static i.i.d. random environment is always an essential enhancement, as long as the density of 1's is positive.

Consider, however, the following example of a stationary and ergodic static random environment with positive particle density that does *not* result in an essential enhancement. Let L be an $\mathbb{N}$-valued random variable with finite first moment but infinite second moment. Partition $\mathbb{Z}$ into intervals in a translation-invariant way such that the length of each interval is independent and distributed as L. Let η be obtained by coloring each interval with 1's or 0's according to independent fair coin tosses. On top of this static random environment, put a random walk with $\beta_0 = \alpha_0 = 1/2$, $\beta_1 = 0$ and $\alpha_1 = 1$. As discussed above, this random walk is transient to the right; therefore, it eventually reaches a point where there is an interval full of 1's to its left (into which it cannot backtrack) and an interval to its right whose law is still independent of the past. In other words, the times when W crosses the boundary between an interval full of 1's and the next interval are *regeneration times*. This observation allows us to calculate the speed of W as the ratio between the expectation of L and the expected time required by W to cross one interval, given that the interval to the left is occupied. The latter turns out to be infinite, so that W has speed $0 = v_0$. Therefore, in this example the random environment is not an essential enhancement, despite having particle density equal to $1/2$.

5.1.4 Outline

The rest of this chapter is organized as follows. In Section 5.2, we construct particular versions of the SSEP and of the random walk. In Section 5.3, we give the proof of Theorem 5.1.1 with the help of a proposition (Proposition 5.3.1 below) concerning *rarefied* and *turbulent* regions in the SSEP. In Section 5.4, we lay out the basic tools that will be used to prove Proposition 5.3.1 in Section 5.5, where all constructions and estimates specific to the SSEP are carried out.

5.2 Construction of the model

In Section 5.2.1 we construct the SSEP and, in Section 5.2.2, the random walk on top of the SSEP.

5.2.1 Graphical construction of the SSEP

It will be convenient to have a graphical construction of the SSEP including negative times. Let $\mathcal{E}$ be the set of edges of $\mathbb{Z}$, i.e., all unordered pairs of neighbouring sites, and let $\mathcal{A} = (\mathcal{A}_e)_{e \in \mathcal{E}}$ be a collection of independent Poisson point processes on $\mathbb{R}$ with intensity 1. Draw each event of $\mathcal{A}_e$ in space-time as an arrow between the two sites connected by e. This gives rise to a system of random paths in $\mathbb{Z} \times \mathbb{R}$ as follows. For each $(x,t) \in \mathbb{Z} \times \mathbb{R}$, there exists a.s. a unique doubly infinite right-continuous path that goes either vertically in time or (forcibly) across arrows of $\mathcal{A}$. For $s \in \mathbb{R}$, let $\zeta_s^t(x)$ denote the position of this path at time s.
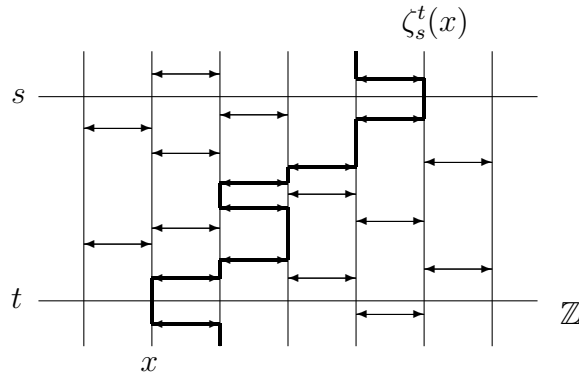
Figure 5.1: Graphical representation. The arrows represent events of $\mathcal{A}$. The thick lines mark the path $\zeta_s^t(x)$.

Given $\eta \in \{0,1\}^{\mathbb{Z}}$, we will define the SSEP $\xi = (\xi_t)_{t \in \mathbb{R}}$ by

$$\xi_t(x) := \eta(\zeta_0^t(x)), \tag{5.2.1}$$

i.e., a space-time point (x,t) is occupied if and only if the path going through it hits an occupied site at time 0. If we take η to be distributed as ν_ρ, $\rho \in (0,1)$, then we may check that this

construction indeed results in a stationary process with the correct distribution. To verify this, we only need to note that $\xi_t(x) = \xi_s(\zeta_s^t(x))$ for any $s, t \in \mathbb{R}$ and that, by the product structure and exchangeability of ν_ρ, ξ_s is independent of $\mathcal{A}$.

5.2.2 The random walk on top of the SSEP

We next give a particular construction of the random walk model described in the introduction. Take a Poisson process $N = (N_t)_{t \geq 0}$ with rate γ and two sequences $J^1 = (J_k^1)_{k \in \mathbb{N}}$ and $J^0 = (J_k^0)_{k \in \mathbb{N}}$ of i.i.d. $\{-1, +1\}$-valued random variables taking the value $+1$ with probability α_1/γ and α_0/γ , respectively. These random variables are taken such that ξ, N, J^1, J^0 are jointly independent.

The random walk W is a functional of (ξ, N, J^1, J^0) obtained as follows. We set $W_0 := 0$. At a time $t > 0$, W jumps if and only if N jumps, and the increment is given by $W_t - W_{t-} = J_{N_t}^i$, where $i = \xi_t(W_{t-})$ is the state of the exclusion process at the position of W just before the jump.

Setting
$$
\begin{aligned}
N_t^1 &:= \#\{t \in [0,t]: W_t \neq W_{t-} \text{ and } \xi_t(W_{t-}) = 1\}, \\
N_t^0 &:= \#\{t \in [0,t]: W_t \neq W_{t-} \text{ and } \xi_t(W_{t-}) = 0\},
\end{aligned}
\tag{5.2.2}
$$

then $N_t^0 + N_t^1 = N_t$ and we see that W has the following representation:

$$
W_t = S_{N_t^1}^1 + S_{N_t^0}^0
\tag{5.2.3}
$$

where $(S_n^i)_{n \in \mathbb{N}_0}$, $i \in \{0, 1\}$, are discrete-time simple random walks that jump to the right with probability α_i/γ. From this we immediately get

$$
\begin{aligned}
\liminf_{t \to \infty} t^{-1} W_t &= v_0 + (v_1 - v_0) \liminf_{t \to \infty} (\gamma t)^{-1} N_t^1, \\
\limsup_{t \to \infty} t^{-1} W_t &= v_1 - (v_1 - v_0) \liminf_{t \to \infty} (\gamma t)^{-1} N_t^0.
\end{aligned}
\tag{5.2.4}
$$

5.3 Proof of the main theorem

Since the *holes* of a SSEP under $\mathbb{P}_{\nu_\rho}$ have the same distribution as the *particles* of a SSEP under $\mathbb{P}_{\nu_{1-\rho}}$, we may w.l.o.g. restrict ourselves to proving the statement for the $\liminf$ in (5.1.6).

The main idea in the proof of Theorem 5.1.1 is that, because the jump rates are positive and bounded, the random walk can spend time on top of particles whenever it is in a region of the environment that is not too rough, namely, neither too rarefied nor too turbulent. A *rarefied* region is one where the density of the environment is atypically low. A *turbulent* region is one where the environment is moving atypically fast. It is of course not possible to control such deviations of the environment in all space and time simultaneously, but, as we will see in Proposition 5.3.1 below, it is possible to show that, in most of the regions *accessible* to the random walk, the environment cannot deviate too much from its typical behaviour.

In Section 5.3.1 we state Proposition 5.3.1. In Section 5.3.2, we use this proposition to prove Theorem 5.1.1. The proof of Proposition 5.3.1 is given in Section 5.5.

5.3.1 Rarefied and turbulent regions

For $r \in \mathbb{N}$, let $\omega_r \leq \Delta_r \in \mathbb{N}$ and $\rho_r, \epsilon_r \in (0,1)$ be given parameters. Let

$$B_r(k,s) := [k, k + \Delta_r) \times [s, s + \Delta_r), \quad k, s \in \Delta_r \mathbb{Z}, \tag{5.3.1}$$

be blocks in $\mathbb{R}^d$ with side length Δ_r, called r-blocks. For $x \in \mathbb{Z}$ and $t \in \mathbb{R}$, we write

$$\Sigma_r^x(\xi_t) := \sum_{y \in [x, x + \omega_r)} \xi_t(y) \tag{5.3.2}$$

to denote the number of particles present in $[x, x + \omega_r)$ at time t. We call a set $A \subset \mathbb{R}^2$ r-*rarefied* if there exists $(x,t) \in \mathbb{Z}^2$ with $[x, x + \omega_r) \times \{t\} \subset A$ and such that $\Sigma_r^x(\xi_t) < \rho_r \omega_r$. We call A r-*turbulent* if there exists $(x,t) \in A \cap \mathbb{Z}^2$ and $s \in (0, \epsilon_r)$ such that $\xi_{t+s}(x) \neq \xi_t(x)$.

For $\ell \in (0, \infty)$, let

$$\begin{aligned} \mathcal{W}_\ell := \{&\text{all paths in } \mathbb{R}^2 \text{ starting at } 0 \text{ which are continuous,} \\ &\text{piecewise } C^1, \text{ and have length at most } \ell\}, \end{aligned} \tag{5.3.3}$$

and put

$$\begin{aligned} \Phi_r^{\mathrm{r}}(\ell) &:= \sup_{w \in \mathcal{W}_\ell} \#\{r\text{-rarefied } r\text{-blocks intersected by } w\}, \\ \Phi_r^{\mathrm{t}}(\ell) &:= \sup_{w \in \mathcal{W}_\ell} \#\{r\text{-turbulent } r\text{-blocks intersected by } w\}. \end{aligned} \tag{5.3.4}$$

The key ingredient in the proof of Theorem 5.1.1 is the following proposition.

Proposition 5.3.1. *For any $\rho \in (0,1)$, there exist $(\Delta_r, \omega_r, \rho_r, \epsilon_r)_{r \in \mathbb{N}}$ as above such that, $\mathbb{P}_{\nu_\rho}$-a.s.,*

$$\begin{aligned} (a) &\quad \lim_{r \to \infty} \limsup_{\ell \to \infty} \ell^{-1} \Delta_r^2 \Phi_r^{\mathrm{r}}(\ell) = 0, \\ (b) &\quad \lim_{r \to \infty} \limsup_{\ell \to \infty} \ell^{-1} \Delta_r^2 \Phi_r^{\mathrm{t}}(\ell) = 0. \end{aligned} \tag{5.3.5}$$

Part (a) will be proved using a *multiscale renormalization scheme* developed by Kesten and Sidoravicius (see [50]; we also borrow some ideas from [51]). Very little adaptation will be necessary, and some simplifications are possible in our setting. For completeness, we include all the details. The proof of part (b) uses a similar strategy, but is much simpler.

To simplify the exposition, we present the proof in dimension one only. There are no technical issues to extend it to higher dimensions. Small complications arise in the proof of Lemma 5.5.2 below, but they can be dealt with straightforwardly.

5.3.2 Proof of Theorem 5.1.1

Proof. Fix $\rho \in (0,1)$ and recall the definition of N^1 in (5.2.2). By (5.2.4), it is enough to prove the existence of a $\delta_0 > 0$ such that

$$\liminf_{t \to \infty} t^{-1} N_t^1 \geq \delta_0 \quad \mathbb{P}_{\nu_\rho}\text{-a.s.} \tag{5.3.6}$$

Regard $(W_s)_{s \in [0,t]}$ as a path in $\mathbb{R}^2$ and denote its length by $\ell_t = t + N_t$. Recall that N is a Poisson process with rate $\gamma > 0$, independent of ξ. Using Proposition 5.3.1, fix $\ell_* \in (1, \infty)$ and $r_* \in \mathbb{N}$ such that

$$\Delta_{r_*}^2 \left\{ \Phi_{r_*}^{\mathrm{r}}(\ell) + \Phi_{r_*}^{\mathrm{t}}(\ell) \right\} \le \frac{\ell}{2(1+\gamma)} \quad \mathbb{P}_{\nu_\rho}\text{-a.s. } \forall \, \ell \ge \ell_*. \tag{5.3.7}$$

Let $B_t^*(W)$ be the unique r_*-block containing the spacetime point (W_t, t). We call $B_t^*(W)$ *rough* if it is either r_*-rarified or turbulent, and call it *smooth* otherwise. For $t \ge 0$, let

$$\Theta_t^*(W) := \sum_{s=0}^{\lfloor t \rfloor} \mathbb{1}_{\{B_s^*(W) \text{ is rough}\}} \tag{5.3.8}$$

denote the total number of integer times between 0 and t at which W is inside a rough block. Since W can spend at most Δ_{r_*} time units in each rough block, if $t \ge \ell_*$, then by (5.3.7) we have

$$\begin{aligned}
\Theta_t^*(W) &\le \Delta_{r_*} \left\{ \Phi_{r_*}^{\mathrm{r}}(\ell_t) + \Phi_{r_*}^{\mathrm{t}}(\ell_t) \right\} \\
&\le \frac{1}{\Delta_{r_*}} \frac{\ell_t}{2(1+\gamma)} \le \frac{\ell_t}{2(1+\gamma)} \quad \mathbb{P}_{\nu_\rho}\text{-a.s.}
\end{aligned} \tag{5.3.9}$$

For $s \in \mathbb{N}_0$, let

$$Y_{s+1} := \mathbb{1}_{\{N_{s+1}^1 > N_s^1\}}. \tag{5.3.10}$$

Note that $N_{s+1}^1 > N_s^1$ if and only if W jumps at least once from a particle in the time interval $(s, s+1]$. Since W has uniformly positive jump rates, for any $s \ge 0$, $r \in \mathbb{N}$, $\epsilon > 0$ and $j \in [W_s - r, W_s + r]$,

$$\mathbb{P}_{\nu_\rho} \left(W \text{ jumps once from } j \text{ in the time interval } (s, s+\epsilon) \mid (W_u)_{u \in [0,s]}, \xi \right) \ge \delta \tag{5.3.11}$$

for some $\delta = \delta(r, \epsilon) > 0$. Therefore, if $B_s^*(W)$ is smooth, then

$$\mathbb{P}_{\nu_\rho} \left(Y_{s+1} = 1 \mid (W_u)_{u \in [0,s]}, \xi \right) \ge \delta_* := \delta(r_*, \epsilon_{r_*}) \tag{5.3.12}$$

since there is at time s at least one particle in $[W_s - r_*, W_s + r_*]$ that does not move before time $s + \epsilon_{r_*}$. Therefore we can couple Y with an i.i.d. sequence $(\tilde{Y}_s)_{s \in \mathbb{N}}$ of Bernoulli(δ_*) random variables such that $Y_{s+1} \ge \tilde{Y}_{s+1}$ if $B_s^*(W)$ is smooth.

Using these observations, we can write, for $t \ge \ell_*$,

$$\begin{aligned}
t^{-1} N_t^1 &\ge t^{-1} \sum_{s=1}^{\lfloor t \rfloor} Y_s \ge t^{-1} \sum_{\substack{s \in [1,t] \cap \mathbb{N}: \\ B_{s-1}^*(W) \text{ is smooth}}} \tilde{Y}_s \\
&\ge \left(\frac{\lfloor t \rfloor - \Theta_t^*(W)}{t} \right) \# \left\{ \begin{aligned} &\scriptstyle s \in [1,t] \cap \mathbb{N}: \\ &\scriptstyle B_{s-1}^*(W) \text{is smooth} \end{aligned} \right\}^{-1} \sum_{\substack{s \in [1,t] \cap \mathbb{N}: \\ B_{s-1}^*(W) \text{ is smooth}}} \tilde{Y}_s.
\end{aligned} \tag{5.3.13}$$

By (5.3.9), the $\liminf$ as $t \to \infty$ of the term in parentheses in the r.h.s. of (5.3.13) is at least $1/2$. The remaining term converges to δ_*, since the number of integer times s in $[1,t]$ for which $B_{s-1}^*(W)$ is smooth is unbounded. Thus (5.3.6) holds with $\delta_0 = \delta_*/2$. ∎

5.4 Block percolation and partitioned systems

In this section we present a percolation result, Proposition 5.4.3 below, which will play an important role in the proof of Proposition 5.3.1 in Section 5.5.

5.4.1 Percolative systems

Fix $d \in \mathbb{N} \setminus \{1\}$ and $\Delta \in (0, \infty)$. For $k = (k_1, \ldots, k_d) \in \Delta\mathbb{Z}^d$, let

$$B_\Delta(k) := \prod_{i=1}^{d} [k_i, k_i + \Delta) \tag{5.4.1}$$

be the block in $\mathbb{R}^d$ of side length Δ with lower-left corner at k. A collection of random variables

$$\Upsilon = (\Upsilon(k))_{k \in \Delta\mathbb{Z}^d}, \quad \Upsilon(k) \in \{0, 1\} \text{ for each } k \in \Delta\mathbb{Z}^d, \tag{5.4.2}$$

is called call a *percolative system* (PS) with scale Δ. We interpret Υ by saying that a block $B_\Delta(k)$ is *open* if $\Upsilon(k) = 1$, and *closed* otherwise. See Figure 5.2.

We aim to bound the number of open blocks that intersect paths of a certain fixed length in $\mathbb{R}^d$. For $\ell \in (0, \infty)$, let, analogously to (5.3.3),

$$\mathcal{W}_\ell := \{\text{all paths in } \mathbb{R}^d \text{ starting at } 0 \text{ which are continuous,}$$
$$\text{piecewise } C^1, \text{ and have length at most } \ell\}. \tag{5.4.3}$$
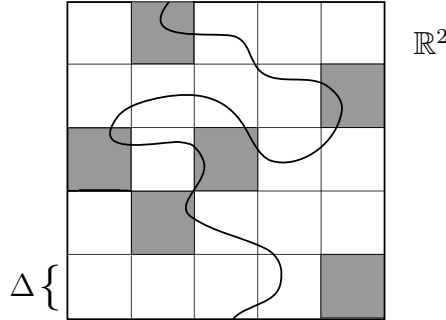

Figure 5.2: Block percolation in $\mathbb{R}^2$. Gray blocks are open. The curve represents a path in $\mathcal{W}_\ell$.

For $w \in \mathcal{W}_\ell$, put

$$\psi(w) := \#\{k \in \Delta\mathbb{Z}^d : w \text{ intersects } B_\Delta(k) \text{ and } \Upsilon(k) = 1\} \tag{5.4.4}$$

and let

$$\Psi(\ell) := \sup_{w \in \mathcal{W}_\ell} \psi(w). \tag{5.4.5}$$

In order to control $\Psi(\ell)$, we need to restrict the class of allowed percolative systems. We will call a PS Υ *homogeneous* with parameter $p \in (0,1)$ if $\Upsilon(k)$ has distribution Bernoulli(p) for each $k \in \Delta\mathbb{Z}^d$. We call it (finitely) *partitioned* if there exists a finite partition $\mathcal{P}$ of $\Delta\mathbb{Z}^d$ such that, for each $I \in \mathcal{P}$,

$$(\Upsilon(k))_{k \in I} \text{ are jointly independent.} \tag{5.4.6}$$

In other words, Υ is partitioned if its dependence graph has a finite chromatic number. In that case, we let $|\mathcal{P}| := \#\mathcal{P}$. In the following, we use the abbreviation p-PPS for "homogeneous partitioned percolative system with parameter p".

5.4.2 Key lemma

The following lemma is the key to the proof of Proposition 5.4.3 below.

Lemma 5.4.1. *There exist constants $c_1, c_2 \in (0, \infty)$ depending on d only such that, for any percolative system Υ with scale Δ that is stochastically dominated by a p-PPS with partition $\mathcal{P}$,*

$$P\left(\Psi(\ell) > |\mathcal{P}|c_1\frac{\theta\ell}{\Delta}\right) \leq |\mathcal{P}|e^{-c_2\left(\frac{\theta\ell}{\Delta}-1\right)} \quad \text{for any } \theta \in [p^{\frac{1}{d}}, 1]. \tag{5.4.7}$$

Our proof of Lemma 5.4.1 is an adaptation of the proof of Lemma 8 in [50]. It is based on geometric constraints of $\mathbb{R}^d$ and an application of Bernstein's inequality, which we recall for the case of i.i.d. bounded random variables.

Lemma 5.4.2. *(Bernstein's inequality) Suppose that $(X_i)_{i \in \mathbb{N}}$ is an i.i.d. sequence of a.s. bounded random variables with joint law P. Then*

$$P\left(\sum_{i=1}^{n} X_i - EX_i > x\right) \leq e^{-\frac{x}{2}\left(\|X_1\|_\infty + \frac{nVar(X_1)}{x}\right)^{-1}}. \tag{5.4.8}$$

For a proof of Lemma 5.4.2, see e.g. Chow and Teicher [27], Exercise 4.3.14.

Proof of Lemma 5.4.1. There exist $K_1, K_2 \in \mathbb{N}$, depending on d only, with the following properties. For any ℓ and Δ, the total number of Δ-blocks intersecting any path in $\mathcal{W}_\ell$ is at most $K_1\lceil \ell/\Delta\rceil$ and, for any n and Δ, the number of connected subsets of $\mathbb{R}^d$ that are unions of exactly n Δ-blocks and contain the origin is at most $e^{K_2 n}$. We will show that (5.4.7) holds with $c_2 := 2^d K_1 K_2$ and $c_1 := 16c_2$.

Since $\Psi(\ell)$ does not decrease if additional Δ-blocks are opened, we may suppose that Υ is a p-PPS with partition $\mathcal{P}$. Let

$$L := \lceil \theta^{-1}\rceil, \quad N := K_1\lceil \ell/(L\Delta)\rceil. \tag{5.4.9}$$

As discussed in the first paragraph, N is an upper bound for the number of $L\Delta$-blocks intersected by any path in $\mathcal{W}_\ell$. If $\ell/(L\Delta) < \frac{1}{2}$, then $\theta\ell/\Delta < 1$ and (5.4.7) holds trivially. Therefore, we may assume that $\ell/(L\Delta) \geq \frac{1}{2}$, in which case $N \leq 3K_1\ell/(L\Delta)$. Letting

$$\mathscr{C}_L^N := \{\text{connected subsets of } \mathbb{R}^d \text{ containing the origin} \\ \text{that are the union of } N \text{ distinct } L\Delta\text{-blocks}\}, \tag{5.4.10}$$

we can estimate, for $x > 0$,

$$
P\left(\exists\, w \in \mathcal{W}_\ell \colon \psi(w) > x\right) \leq \sum_{C \in \mathscr{C}_L^N} P\left(\exists\, w \in \mathcal{W}_\ell, w \subset C \colon \psi(w) > x\right)
$$
$$
\leq \sum_{C \in \mathscr{C}_L^N} P\left(\#\{\text{open } \Delta\text{-blocks in } C\} > x\right). \tag{5.4.11}
$$

To estimate for a fixed $C \in \mathscr{C}_L^N$ the corresponding term in (5.4.11), we use the partition.

$$
P\left(\#\{\text{open } \Delta\text{-blocks in } C\} > x\right) \leq \sum_{I \in \mathcal{P}} P\left(\#\{\text{open } \Delta\text{-blocks in } C \cap I\} > \frac{x}{|\mathcal{P}|}\right)
$$
$$
\leq |\mathcal{P}| P\left(\mathrm{Bin}(NL^d, p) > \frac{x}{|\mathcal{P}|}\right), \tag{5.4.12}
$$

where $\mathrm{Bin}(NL^d, p)$ is a Binomial random variable and (5.4.12) is justified by (5.4.6) and the fact that each $C \in \mathscr{C}_L^N$ is the union of exactly NL^d Δ-blocks. By the definition of L and our choice of c_1, we can check that $pNL^d < \frac{1}{2}c_1\theta\ell/\Delta$. Therefore, substituting x in (5.4.12) by $|\mathcal{P}|c_1\theta\ell/\Delta$ and applying Bernstein's inequality (5.4.8), we obtain

$$
P\left(\#\{\text{open } \Delta\text{-blocks in } C\} > |\mathcal{P}|c_1\frac{\theta\ell}{\Delta}\right) \leq |\mathcal{P}| \exp\left(-\frac{c_1\theta\ell}{8\Delta}\right). \tag{5.4.13}
$$

Since $N \leq 3K_1\ell/(L\Delta) \leq 3K_1\ell\theta/\Delta$, we have $K_2N < c_2\theta\ell/\Delta$. Hence, combining (5.4.11) and (5.4.13), we get

$$
P\left(\Psi(\ell) > |\mathcal{P}|c_1\frac{\theta\ell}{\Delta}\right) \leq |\mathcal{P}|e^{K_2N - 2c_2\frac{\theta\ell}{\Delta}} \leq |\mathcal{P}|e^{-c_2\frac{\theta\ell}{\Delta}}. \tag{5.4.14}
$$

∎

5.4.3 Sequences of percolative systems

The following proposition concerns sequences of percolative systems, and will be used in Section 5.5 in the proof of Proposition 5.3.1.

Proposition 5.4.3. *Let $(\Upsilon_r)_{r \in \mathbb{N}}$ be a sequence of percolative systems in $\mathbb{R}^d$ with scales Δ_r, defined jointly in the same probability space through an arbitrary coupling. Suppose that, for each $r \in \mathbb{N}$, Υ_r is stochastically dominated by a p_r-PPS with partition $\mathcal{P}_r$ such that the following hold:*

$$
\begin{aligned}
&(i) && \limsup_{r \to \infty} |\mathcal{P}_r| < \infty. \\
&(ii) && m := \limsup_{r \to \infty} r^{-1} \log(\Delta_r) < \infty. \\
&(iii) && M := -\limsup_{r \to \infty} r^{-1} \log(p_r) > md.
\end{aligned} \tag{5.4.15}
$$

Then, for any $\kappa \in (0, (md)^{-1})$,

$$
\lim_{n \to \infty} \limsup_{\ell \to \infty} \frac{1}{\ell} \sum_{r=n}^{\lfloor \kappa \log(\ell) \rfloor} \Delta_r^d \Psi_r(\ell) = 0 \quad a.s. \tag{5.4.16}
$$

where $\Psi_r(\ell)$ is defined for Υ_r as in (5.4.4)–(5.4.5).

Proof. Let $0 < \varepsilon < \frac{1}{2}(1/\kappa - md)$ and put $\theta_r := \sqrt[d]{p_r} \vee e^{-br}$ with $b = m(d-1) + \varepsilon$. By (i), there exists $K_1 \in (0, \infty)$ such that $\sup_r |\mathcal{P}_r| \leq K_1$ and, by (ii), there exist $K_2 \in (0, \infty)$ and $r_0 \in \mathbb{N}$ such that $\Delta_r \leq K_2 e^{(m+\varepsilon)r}$ whenever $r \geq r_0$. Hence, by Lemma 5.4.1,

$$P\left(\exists\, r_0 \leq r \leq \lfloor \kappa \log(\ell) \rfloor \colon \Psi_r(\ell) > K_1 c_1 \ell \frac{\theta_r}{\Delta_r}\right) \leq K_1 \kappa \log(\ell) e^{c_2} \exp\left(-c_2 \ell \frac{\ell^{-\kappa b}}{K_2 \ell^{\kappa(m+\varepsilon)}}\right)$$

$$= K_1 \kappa \log(\ell) e^{c_2} \exp\left(-\frac{c_2 \ell^a}{K_2}\right), \qquad (5.4.17)$$

where $a := 1 - \kappa(m + \varepsilon + b) > 0$ by our choice of ε and b. Thus, (5.4.17) is summable in ℓ. By the Borel-Cantelli lemma, a.s. for $n \geq r_0$ and ℓ large enough we may estimate

$$\frac{1}{\ell} \sum_{r=n}^{\lfloor \kappa \log(\ell) \rfloor} \Delta_r^d \Psi_r(\ell) \leq K_1 c_1 \sum_{r=n}^{\lfloor \kappa \log(\ell) \rfloor} \Delta_r^{d-1} \theta_r. \qquad (5.4.18)$$

By (ii)-(iii) and the definition of θ_r, $\Delta_r^{d-1}\theta_r$ is summable in r. Therefore (5.4.16) follows by first letting $\ell \to \infty$ and then $n \to \infty$. ∎

5.5 Multiscale analysis

Section 5.5.1 contains the proof of Proposition 5.3.1(a), Section 5.5.2 the proof of Proposition 5.3.1(b).

Most of the work is concentrated in Section 5.5.1, where the renormalization scheme for rarefied blocks is defined and analyzed using the results from Section 5.4. Central to this work are estimates for systems of independent simple random walks, stated in Lemma 5.5.3 below, which are used for comparison with the system of *holes* of the SSEP via a result due to Liggett. These estimates are used to control a *recursive formula* that, roughly speaking, transfers properties from larger to smaller scales, allowing us to deduce microscopic properties from mesoscopic and macroscopic properties.

In Section 5.5.2, a similar approach is used to analyze turbulent blocks from the point of view of Section 5.4. There the construction and estimates are much simpler.

5.5.1 Proof of Proposition 5.3.1(a)

Bad blocks

Fix $\rho_- \in (0, \rho)$, let $N_0 \in \mathbb{N}$ be large enough such that

$$\bar{\rho}_\infty := \prod_{r=1}^{\infty}(1 - N_0^{-r/4}) \geq 1 - \rho_- =: \bar{\rho}_+ \qquad (5.5.1)$$

and put

$$\bar{\rho}_r := \prod_{k=1}^{r}(1 - N_0^{-k/4}).\qquad(5.5.2)$$

Set

$$\omega_r := N_0^r, \ \ \Delta_r := N_0^{6r} \ \text{ and } \ \rho_r := 1 - \bar{\rho}_r.\qquad(5.5.3)$$

The parameters ϵ_r will be defined in Section 5.5.2. Set also $\bar{\rho} := 1 - \rho$ and, for $\eta \in \{0,1\}^{\mathbb{Z}}$, define $\bar{\eta}$ by

$$\bar{\eta}(x) := 1 - \eta(x).\qquad(5.5.4)$$

In the following, we will also need r-superblocks, defined as

$$\mathbf{B}_r(k,s) := [k - 5\Delta_r, k + 6\Delta_r) \times [s - 2\Delta_r, s + \Delta_r), \quad k,s \in \Delta_r\mathbb{Z}.\qquad(5.5.5)$$

We call the r-block $B_r(k,s)$ *bad* if $\mathbf{B}_r(k,s)$ is r-rarefied. Thus, any r-rarefied r-block is bad. We call r-*dense* any set in $\mathbb{R}^2$ that is not r-rarefied.

Lemma 5.5.1. *For any $\kappa > 0$, $\mathbb{P}_{\nu_\rho}$-a.s. there exists a (random) $\ell_0 \in (0,\infty)$ such that, if $\ell \geq \ell_0$, no bad r-blocks with $r \geq \lfloor \kappa \log(\ell) \rfloor$ intersect $[-\ell, \ell]^2$.*

Proof. Since the product Bernoulli measure ν_ρ is a translation-invariant equilibrium, for any $r \in \mathbb{N}$, $x \in \mathbb{Z}$ and $t \in \mathbb{R}$, we have

$$\mathbb{P}_{\nu_\rho}\left(\Sigma_r^x(\xi_t) < \rho_r\omega_r\right) \leq \mathbb{P}_{\nu_\rho}\left(\Sigma_r^x(\xi_t) < \rho_-\omega_r\right)$$
$$= P\left(\text{Bin}(\omega_r, \rho) - \rho\omega_r < -(\rho - \rho_-)\omega_r\right) \leq e^{-\varepsilon\omega_r},\qquad(5.5.6)$$

where $\text{Bin}(\omega_r, \rho)$ is a Binomial random variable and $\varepsilon > 0$. The last step can be justified e.g. by using Bernstein's inequality (5.4.2). Therefore, for any $(k,s) \in \Delta_r\mathbb{Z}^2$,

$$\mathbb{P}_{\nu_\rho}\left(B_r(k,s) \text{ is bad}\right) \leq \sum_{(x,t)\in\mathbf{B}_r(k,s)\cap\mathbb{Z}^2} \mathbb{P}_{\nu_\rho}\left(\Sigma_r^x(\xi_t) < \rho_r\omega_r\right)$$
$$\leq 33\Delta_r^2 e^{-\varepsilon\omega_r} \leq Ce^{-\frac{\varepsilon}{2}N_0^r}\qquad(5.5.7)$$

for some $C \in (0,\infty)$. Since at most $(2\ell + 1)^2$ r-blocks intersect $[-\ell, \ell]^2$, we can estimate

$$\mathbb{P}_{\nu_\rho}\left(\exists\, r > \kappa \log(\ell) \text{ and a bad } r\text{-block intersecting } [-\ell, \ell]^2\right)$$
$$\leq C(2\ell + 1)^2 \sum_{r=\lfloor k \log(\ell) \rfloor}^{\infty} e^{-\frac{\varepsilon}{2}N_0^r}\qquad(5.5.8)$$

which is summable in ℓ, and so the claim follows by the Borel-Cantelli lemma. $\blacksquare$

Locally spoiled blocks

For $(k, s) \in \Delta_r \mathbb{Z}^2$, let

$$\mathcal{B}_r(k, s) := [k - \Delta_r, k + 2\Delta_r) \times [s - \Delta_r, s + \Delta_r) \qquad (5.5.9)$$

be the *neighbourhood* of the r-block $B_r(k, s)$, and let

$$\mathbf{\Lambda}_r(k, s) := [k - 5\Delta_r, k + 6\Delta_r) \times \{s - 2\Delta_r\} \qquad (5.5.10)$$

be the *base* of the r-superblock $\mathbf{B}_r(k, s)$. Define also $\mathbf{V}_r^k := [k - 5\Delta_r, k + 6\Delta_r) \subset \mathbb{Z}$, so that $\mathbf{\Lambda}_r(k, s) = \mathbf{V}_r^k \times \{s - 2\Delta_r\}$. See Figure 5.3. We also need the *interior* of $\mathbf{B}_r(k, s)$,

$$\mathring{\mathbf{B}}_r(k, s) := [k - 5\Delta_r + 1, k + 6\Delta_r - 1) \times [s - 2\Delta_r, s + \Delta_r), \qquad (5.5.11)$$

and, for $(x, t) \in \mathcal{B}_{r+1}(k, s)$,

$$\widehat{\Sigma}_r^{k,s}(x, t) := \#\{\text{all particles of the SSEP in } [x, x + \omega_r) \times \{t\} \text{ that}$$
$$\text{stayed in } \mathring{\mathbf{B}}_{r+1}(k, s) \text{ during the time interval } [s - 2\Delta_{r+1}, t]\}. \qquad (5.5.12)$$
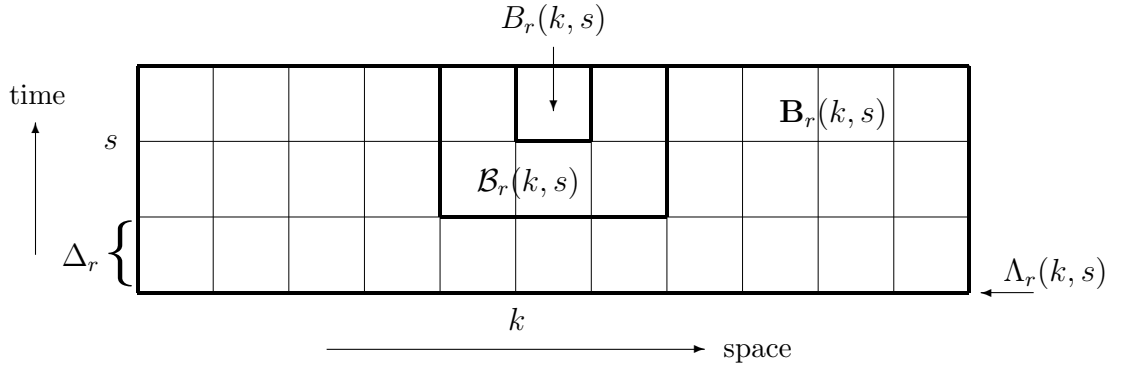

Figure 5.3: Relative position of $B_r(k, s)$, $\mathcal{B}_r(k, s)$, $\mathbf{B}_r(k, s)$ and $\Lambda_r(k, s)$.

A block $B_{r+1}(k, s)$ is called *locally spoiled* if $\mathbf{\Lambda}_{r+1}(k, s)$ is $(r + 1)$-dense but there is a point (x, t) such that $[x, x + \omega_r) \times \{t\} \subset \mathcal{B}_r(k, s)$ and $\widehat{\Sigma}_r^{k,s}(x, t) < \rho_r \omega_r$. Being locally spoiled means that, in the scale Δ_{r+1}, the $(r + 1)$-block "has good conditions", meaning that the base of its $(r + 1)$-superblock is $(r + 1)$-dense, but nonetheless there are not enough particles transfered locally (i.e., inside $\mathbf{B}_{r+1}(k, s)$) to ensure that in the finer scale Δ_r the neighbourhood $\mathcal{B}_{r+1}(k, s)$ is r-dense (which would in turn guarantee that $B_{r+1}(k, s)$ contains no bad r-blocks). We will see below that, with our choice of parameters, being locally spoiled is an extremely unlikely event.

Define a percolative system Υ_r with scale Δ_r by

$$\Upsilon_r(k, s) := \mathbb{1}_{\{B_r(k,s) \text{ is locally spoiled}\}}, \qquad (5.5.13)$$

and, for each $a = (a_1, a_2) \in \mathbf{B}_r(0, 2\Delta_r) \cap \Delta_r \mathbb{Z}^2$, let

$$I_a := \{(z_1, z_2) \in \Delta \mathbb{Z}^2 \colon z_1 \equiv a_1 \pmod{11} \text{ and } z_2 \equiv a_2 \pmod 3\}. \tag{5.5.14}$$

Then

$$\mathcal{P}_r := \{I_a \colon a \in \mathbf{B}_r(0, 2\Delta_r) \cap \Delta_r \mathbb{Z}^2\} \tag{5.5.15}$$

is a partition of $\Delta_r \mathbb{Z}^2$ with $|\mathcal{P}_r| = 33$.

Lemma 5.5.2. *For all large enough $r \in \mathbb{N}$, Υ_r is stochastically dominated by a p_r-PPS with partition $\mathcal{P}_r$, where p_r tends to 0 super-exponentially fast as $r \to \infty$.*

The proof of Lemma 5.5.2 requires quite a bit of work, including estimates for systems of simple random walks for comparison with the SSEP. Therefore, we postpone it to Section 5.5.1, and show first how it is used to prove Proposition 5.3.1(a).

Proof of Proposition 5.3.1(a)

Proof. Let

$$\begin{aligned}
\Phi_r^{\mathrm{b}}(\ell) &:= \sup_{w \in \mathcal{W}_\ell} \#\{\text{bad } r\text{-blocks that intersect } w\}, \\
\Psi_r^{\mathrm{ls}}(\ell) &:= \sup_{w \in \mathcal{W}_\ell} \#\{\text{locally spoiled } r\text{-blocks that intersect } w\}.
\end{aligned} \tag{5.5.16}$$

Since $\Phi_r^{\mathrm{r}}(\ell) \leq \Phi_r^{\mathrm{b}}(\ell)$, it is enough to prove that

$$\lim_{r \to \infty} \limsup_{\ell \to \infty} \ell^{-1} \Delta_r^2 \Phi_r^{\mathrm{b}}(\ell) = 0. \tag{5.5.17}$$

We claim that, for all $r \in \mathbb{N}$,

$$\Phi_r^{\mathrm{b}}(\ell) \leq N_0^{12} \Phi_{r+1}^{\mathrm{b}}(\ell) + N_0^{12} \Psi_{r+1}^{\mathrm{ls}}(\ell). \tag{5.5.18}$$

Indeed, if an r-block is bad, then the unique $(r+1)$-block containing it is either bad or locally spoiled, and the number of r-blocks inside any given $(r+1)$-block is equal to N_0^{12}. By induction we get, for $R \geq r+1$,

$$\Delta_r^2 \Phi_r^{\mathrm{b}}(\ell) \leq \Delta_R^2 \Phi_R^{\mathrm{b}}(\ell) + \sum_{n=r+1}^{R} \Delta_n^2 \Psi_n^{\mathrm{ls}}(\ell). \tag{5.5.19}$$

For $\kappa \in (0, (12 \log(N_0))^{-1})$, take ℓ_0 as in Lemma 5.5.1 and $R = \lfloor \kappa \log(\ell) \rfloor$. Then, for $\ell \geq \ell_0$, we may estimate

$$\frac{1}{\ell} \Delta_r^2 \Phi_r^{\mathrm{b}}(\ell) \leq \frac{1}{\ell} \sum_{n=r+1}^{\lfloor \kappa \log(\ell) \rfloor} \Delta_n^2 \Psi_n^{\mathrm{ls}}(\ell), \tag{5.5.20}$$

and so (5.5.17) follows from Lemma 5.5.2 and Proposition 5.4.3. ∎

The rest of this section is dedicated to the proof of Lemma 5.5.2. In Section 5.5.1 we derive some estimates for systems of independent simple random walks. These are used in Lemma 5.5.4 below for comparison with the system of holes of the SSEP. The latter lemma is used in Section 5.5.1 to prove Lemma 5.5.2.

Estimates for systems of independent random walks

It will be useful to compare the system $\bar{\xi}$ of the *holes* of the exclusion process with a system of independent simple random walks, which we define next.

Let $(S^z)_{z \in \mathbb{Z}}$ be a collection of independent simple random walks on $\mathbb{Z}$, with $S_0^z = z$ for each $z \in \mathbb{Z}$. For $\eta \in \{0,1\}^{\mathbb{Z}}$, define the process $\xi^\circ = (\xi_t^\circ)_{t \geq 0}$ by

$$\xi_t^\circ(x) := \sum_{z \in \mathbb{Z}} \eta(z) \mathbb{1}_{\{S_t^z = x\}}, \quad (x,t) \in \mathbb{Z} \times [0,\infty). \tag{5.5.21}$$

The interpretation is that, if we launch from each site z with $\eta(z) = 1$ an independent simple random walk, then $\xi_t^\circ(x)$ is the number of random walks present at the site x at time t.

The following lemma states two estimates for $\Sigma_r^x(\xi_t^\circ)$, where $[x, x + \omega_r) \times \{t\} \subset \mathcal{B}_{r+1}(0, 2\Delta_{r+1})$. The first gives a bound on its exponential moments in terms of its first moment, while the second gives a bound on the first moment in terms of density properties of the initial configuration in the $(r+1)$-scale.

Lemma 5.5.3. *Let $\eta \in \{0,1\}^{\mathbb{Z}}$ and $\xi^\circ = (\xi_t^\circ)_{t \geq 0}$ be a system of independent SRWs, as discussed above, starting from $\bar{\eta}$ (recall (5.5.4)). Then the following hold:*

(i) For any $\lambda > 0$, $x \in \mathbb{Z}$ and $t \geq 0$,

$$\mathbb{E}_{\bar{\eta}} \left[\exp(\lambda \Sigma_r^x(\xi_t^\circ)) \right] \leq \exp \left\{ (e^\lambda - 1) \mathbb{E}_{\bar{\eta}} \left[\Sigma_r^x(\xi_t^\circ) \right] \right\}. \tag{5.5.22}$$

(ii) For large enough $r \in \mathbb{N}$ and any $(x,t) \in \mathcal{B}_{r+1}(0, 2\Delta_{r+1})$,

$$\mathbb{E}_{\bar{\eta}} \left[\Sigma_r^x(\xi_t^\circ) \right] \leq 1 + \bar{\rho}_{r+1} \omega_r \quad \text{if } \mathbf{\Lambda}_{r+1}(0, 2\Delta_{r+1}) \text{ is } (r+1)\text{-dense}, \tag{5.5.23}$$

i.e., $\sum_{y \in [x, x+\omega_{r+1})} \eta(y) \geq \rho_{r+1} \omega_{r+1} \ \forall \ x \in \mathbb{Z}$ s.t. $[x, x + \omega_{r+1}) \times \{0\} \subset \mathbf{\Lambda}_{r+1}(0, 2\Delta_{r+1})$.

Proof. (i) Using (5.5.21), we may write

$$\begin{aligned}
\mathbb{E}_{\bar{\eta}} \left[\exp(\lambda \Sigma_r^x(\xi_t^\circ)) \right] &= \prod_{z \in \mathbb{Z}} E \left[e^{\lambda \bar{\eta}(z) \mathbb{1}_{\{S_t^z \in [x, x+\omega_r)\}}} \right] \\
&= \prod_{z \in \mathbb{Z}} \left\{ \bar{\eta}(z)(e^\lambda - 1) P(S_t^z \in [x, x + \omega_r)) + 1 \right\} \\
&\leq \prod_{z \in \mathbb{Z}} \exp \left(\bar{\eta}(z)(e^\lambda - 1) P(S_t^z \in [x, x + \omega_r)) \right) \\
&= \exp \left\{ (e^\lambda - 1) \mathbb{E}_{\bar{\eta}} \left[\Sigma_r^x(\xi_t^\circ) \right] \right\}. \tag{5.5.24}
\end{aligned}$$

(ii) We recall two basic results for one-dimensional simple random walk: there exist $K_1, K_2 \in (0, \infty)$ such that

$$P \left(|S_t^0| > 2\sqrt{t} \log t \right) \leq K_1 e^{-K_2 (\log t)^2}, \quad t \geq 1, \tag{5.5.25}$$

and

$$|P(S_t^y = z_1) - P(S_t^y = z_2)| \le K_1 \frac{|z_1 - z_2|}{t}, \quad y, z_1, z_2 \in \mathbb{Z}, \ t \ge 1. \tag{5.5.26}$$

The first of these can be verified e.g. with the help of Bernstein's inequality (5.4.2); for the second, see e.g. Lawler and Limic [56], Theorem 2.3.5.

To simplify the notation, in the following we omit the coordinates $(0, 2\Delta_{r+1})$ of the sets involved. Let $k_t := \lceil 2\sqrt{t}\log(t)/\omega_{r+1}\rceil$ and put $A_t^x := [x - k_t\omega_{r+1}, x + (k_t + 1)\omega_{r+1})$. Since $(x, t) \in \mathcal{B}_{r+1}$, we have $A_t^x \times \{0\} \subset \mathbf{\Lambda}_{r+1}$. Write

$$\mathbb{E}_{\bar{\eta}}[\Sigma_r^x(\xi_t^\circ)] = \sum_{z \in \mathbb{Z}} \bar{\eta}(z)P(S_t^z \in [x, x + \omega_r))$$

$$\le \sum_{z \notin A_t^x} P(S_t^z \in [x, x + \omega_r)) + \sum_{z \in A_t^x} \bar{\eta}(z)P(S_t^z \in [x, x + \omega_r)). \tag{5.5.27}$$

The first term in the r.h.s. of (5.5.27) can be estimated by

$$\sum_{y \in [x, x+\omega_r)} P(S_t^y \notin A_t^x) \le \omega_r P\left(|S_t| > 2\sqrt{t}\log t\right) \le K_1 \omega_r e^{-K_2(\log \Delta_{r+1})^2} \le \frac{1}{2} \tag{5.5.28}$$

for r large enough, where we use (5.5.25) and the fact that $t \ge \Delta_{r+1}$. Decompose A_t^x into disjoint intervals $I_1, \ldots, I_n$ with length exactly ω_{r+1}, and let $z_i \in I_i$ be the maximizer of $z \mapsto P(S_t^z \in [x, x + \omega_r))$ in I_i. Then the second term in (5.5.27) is at most

$$\sum_{i=1}^n \sum_{z \in I_i} \bar{\eta}(z)P(S_t^{z_i} \in [x, x + \omega_r)) \le \bar{\rho}_{r+1}\omega_{r+1} \sum_{i=1}^n P(S_t^{z_i} \in [x, x + \omega_r))$$

$$= \bar{\rho}_{r+1} \sum_{i=1}^n \sum_{z \in I_i} P(S_t^{z_i} \in [x, x + \omega_r)). \tag{5.5.29}$$

The last double sum in the r.h.s. of (5.5.29) is bounded by

$$\sum_{z \in A_t^x} P(S_t^z \in [x, x + \omega_r)) + \sum_{y \in [x, x+\omega_r)} \sum_{i=1}^n \sum_{z \in I_i} |P(S_t^{z_i} = y) - P(S_t^z = y)|. \tag{5.5.30}$$

The first term in (5.5.30) can be estimated by

$$\sum_{y \in [x, x+\omega_r)} P(S_t^y \in A_t^x) \le \omega_r, \tag{5.5.31}$$

and, via (5.5.26), the second term in (5.5.30) by

$$\omega_r|A_t^x|K_1\frac{\omega_{r+1}}{t} \le 4K_1\frac{\omega_r\omega_{r+1}\log(t)}{\sqrt{t}} + 3K_1\frac{\omega_r\omega_{r+1}^2}{t}$$

$$\le \frac{4K_1}{N_0^{r-1}}\left\{\log(3N_0^{6(r+1)}) + \frac{1}{N_0^{2r-1}}\right\} \le \frac{1}{2} \tag{5.5.32}$$

for large enough r, where for the second inequality we use that $\Delta_{r+1} \le t \le 3\Delta_{r+1}$. Now (5.5.23) follows by combining (5.5.27)–(5.5.32) since $\bar{\rho}_{r+1} \le 1$. ∎

Proof of Lemma 5.5.2

In this section we give the proof of Lemma 5.5.2. The first step is to compare $\bar{\xi}$ with a system of independent simple random walks and use the estimates of Lemma 5.5.3 to show that, if $\Lambda_{r+1}(k,s)$ is $(r+1)$-dense, then it is extremely unlikely for $\mathcal{B}_{r+1}(k,s)$ to be r-rarefied. This will also imply that the probability to have a locally spoiled $B_{r+1}(k,s)$ is extremely low, since particles in the SSEP, with large probability, do not travel very large distances in a short time. This is the content of Lemma 5.5.4 below.

We will need the following σ-algebras:

$$\mathcal{F}_r^s := \sigma\big(\xi_t\colon\, t \in (-\infty, s - 2\Delta_r]\big), \quad r \in \mathbb{N},\ s \in \Delta_r\mathbb{Z}. \tag{5.5.33}$$

Lemma 5.5.4. *There exist $C_1, C_2 \in (0,\infty)$ such that, for all $r \in \mathbb{N}$ large enough, $(k,s) \in \Delta_{r+1}\mathbb{Z}^2$ and $(x,t) \in \mathcal{B}_{r+1}(k,s) \cap \mathbb{Z}^2$, if $\Lambda_{r+1}(k,s)$ is $(r+1)$-dense, then*

$$\mathbb{P}_{\nu_\rho}\left(\widehat{\Sigma}_r^{k,s}(s,t) < \rho_r\omega_r \mid \mathcal{F}_{r+1}^s\right) \le C_1 e^{-C_2\sqrt{\omega_r}}. \tag{5.5.34}$$

Proof. By translation invariance and the Markov property, it is enough to prove (5.5.34) for $(k,s) = (0, 2\Delta_{r+1})$ and under $\mathbb{P}_\eta$ for an arbitrary $\eta \in \{0,1\}^\mathbb{Z}$, under the assumption that $\Lambda_{r+1}(0, 2\Delta_{r+1})$ is $(r+1)$-dense. We will first do this for $\Sigma_r^x(\xi_t)$.

We claim that, for any $\eta \in \{0,1\}^\mathbb{Z}$ and $\lambda > 0$,

$$\mathbb{E}_\eta\left[\exp(\lambda\Sigma_r^x(\xi_t))\right] \le \mathbb{E}_\eta\left[\exp(\lambda\Sigma_r^x(\xi_t^\circ))\right] \tag{5.5.35}$$

where ξ° is a system of independent simple random walks as in Lemma 5.5.3. This can be justified using a result due to Liggett [57], Chapter VIII, Proposition 1.7, by noting that, for any $n \in \mathbb{N}$, the function $(y_1, \ldots, y_n) \mapsto \exp\lambda\sum_{i=1}^n \mathbb{1}_{[x,x+\omega_r]}(y_i)$ is symmetric and positive definite. Liggett's result only applies to initial configurations with finitely many particles, but, since $\Sigma_r^x(\xi_t)$ is monotone in η, (5.5.35) follows by the monotone convergence theorem.

Since $\bar{\xi}$ under $\mathbb{P}_\eta$ has the same distribution as ξ under $\mathbb{P}_{\bar{\eta}}$, we have, by Markov's inequality, (5.5.35) and Lemma 5.5.3, that, for any $\lambda > 0$ and r large enough,

$$\mathbb{P}_\eta\left(\Sigma_r^x(\xi_t) < \rho_r\omega_r\right) = \mathbb{P}_{\bar{\eta}}\left(\Sigma_r^x(\xi_t) > \bar{\rho}_r\omega_r\right) \tag{5.5.36}$$

$$\le \exp\left\{\left(e^\lambda - 1\right)\left(1 + \bar{\rho}_{r+1}\omega_r\right) - \lambda\bar{\rho}_r\omega_r\right\}$$

$$= e^{e^\lambda - 1}\exp\bar{\rho}_r\omega_r\left\{\left(e^\lambda - 1\right)\frac{\bar{\rho}_{r+1}}{\bar{\rho}_r} - \lambda\right\}. \tag{5.5.37}$$

Using $e^\lambda - 1 \le \lambda e^\lambda$ and the definition of $\bar{\rho}_r$, we see that the term in brackets in the r.h.s. of (5.5.36) is at most $\lambda e^\lambda(\lambda - \omega_r^{-1/4})$. Choosing $\lambda = \frac{1}{2}\omega_r^{-1/4}$, we obtain

$$\mathbb{P}_\eta\left(\Sigma_r^x(\xi_t) < \rho_r\omega_r\right) \le e^{\sqrt{e}-1}\exp-\frac{\bar{\rho}_+\sqrt{e}\sqrt{\omega_r}}{4} = \tilde{C}_1 e^{-\tilde{C}_2\sqrt{\omega_r}}. \tag{5.5.38}$$

To obtain (5.5.38) for $\widehat{\Sigma}_r(x,t)$ in place of $\Sigma_r^x(\xi_t)$ (with possibly different constants), we will argue that the two are equal with a uniformly large probability.

To that effect, let $(y_t^-)_{t \geq 0}$ denote the path starting at time 0 from the point $y_0^- := -5\Delta_{r+1}$ that goes upwards in time and (forcibly) jumps across any arrows of the graphical representation to the right. Likewise, let $(y_t^+)_{t \geq 0}$ denote the path that starts at $y_0^+ := 6\Delta_{r+1} - 1$ and follows the arrows of the graphical representation to the left. We see that, on the event $A := \{y^-, y^+ \text{ do not hit } \mathcal{B}_{r+1}\}$, no particles can move from outside $\mathbf{B}_{r+1}$ into $\mathcal{B}_{r+1}$. In particular, $\widehat{\Sigma}_r(x,t) = \Sigma_r^x(\xi_t)$ on A if $[x, x + \omega_r] \times \{t\} \subset \mathcal{B}_{r+1}$. On the other hand, $y_t^- - y_0^-$ and $y_0^+ - y_t^+$ are both distributed as a rate 1 Poisson process, and are independent of ξ_0; therefore, because of the shape chosen for $\mathbf{B}_{r+1}$, we have

$$\mathbb{P}_\eta(A) \geq 1 - Ce^{-\varepsilon \Delta_{r+1}} \tag{5.5.39}$$

for some $C, \varepsilon \in (0, \infty)$, which completes the proof. ∎

Proof of Lemma 5.5.2. If $\mathbf{\Lambda}_{r+1}(k,s)$ is $(r+1)$-dense, then we may use (5.5.34) to estimate

$$\mathbb{P}_{\nu_\rho}\left(\Upsilon_{r+1}(k,s) = 1 \mid \mathcal{F}_{r+1}^s\right) \leq \sum_{(x,t) \in \mathcal{B}_{r+1}(k,s) \cap \mathbb{Z}^2} \mathbb{P}_{\nu_\rho}\left(\widehat{\Sigma}_r^{k,s}(s,t) < \rho_r \omega_r \mid \mathcal{F}_{r+1}^{k,s}\right)$$
$$\leq C_1 6\Delta_{r+1}^2 e^{-C_2 \sqrt{\omega_r}} =: p_{r+1}, \tag{5.5.40}$$

which decays super-exponentially fast in r; in particular, $p_{r+1} < 1$ for large enough r. Since $\Upsilon_{r+1}(k,s) = 0$ if $\mathbf{\Lambda}_{r+1}(k,s)$ is $(r+1)$-rarefied, (5.5.40) holds $\mathbb{P}_{\nu_\rho}$-a.s..

To conclude, fix $a \in \mathbf{B}_{r+1}(0, 2\Delta_{r+1})$ and note that, by the definition of being locally spoiled, $\Upsilon_{r+1}(k,s)$ only depends on $\xi_{s-2\Delta_{r+1}}$ and on the graphical representation inside $\mathbf{B}_{r+1}(k,s)$. Therefore, for fixed s, the collection

$$\left(\Upsilon_{r+1}(k,s)\right)_{k \colon (k,s) \in I_a} \tag{5.5.41}$$

is jointly independent under $\mathbb{P}_{\nu_\rho}(\cdot \mid \mathcal{F}_{r+1}^s)$. Thus, by ordering any sequence $(k_i, s_i) \in I_a$, $i = 1, \ldots, n$, such that $s_i \leq s_j$ if $i \leq j$, we see that, by (5.5.40), $(\Upsilon_{r+1}(k_i, s_i))_{i=1}^n$ can be progressively coupled in a monotone way to n independent Bernoulli(p_{r+1}) random variables. ∎

5.5.2 Proof of Proposition 5.3.1(b)

In this section, we use the same proof strategy as in Section 5.5.1, but the arguments will be technically much simpler.

Set $\epsilon_r := e^{-\Delta_r}$. We call a point $(x,t) \in \mathbb{Z} \times \mathbb{R}$ r-*stuck* if both Poissonian clocks in the graphical representation that lie to the right and to the left of x fail to ring between times t and $t + \epsilon_r$. A subset of $\mathbb{Z} \times \mathbb{R}$ is called r-stuck if all its points with integer coordinates are r-stuck. Note that r-turbulent blocks are not r-stuck.

Let $\Upsilon_r^{ns}(k,s) := \mathbb{1}_{\{B_r(k,s) \text{ is not } r\text{-stuck}\}}$. Set $I_{odd} := \{(x,t) \in \Delta_r \mathbb{Z}^2 \colon x\Delta_r^{-1} \text{ is odd}\}$, $I_{even} := \{(x,t) \in \Delta_r \mathbb{Z}^2 \colon x\Delta_r^{-1} \text{ is even}\}$ and $\mathcal{P}_r^{ns} := \{I_{odd}, I_{even}\}$.

Lemma 5.5.5. Υ_r^{ns} *is stochastically dominated by a* $\tilde{p}_r$-*PPS with partition* $\mathcal{P}_r^{ns}$, *where* $\tilde{p}_r$ *decays super-exponentially fast in* r.

Proof. By the definition of being r-stuck, we have

$$\mathbb{P}_{\nu_\rho}\left((x,t) \text{ is } r\text{-stuck}\right) = e^{-2\epsilon_r}. \tag{5.5.42}$$

Therefore, for any $(k,s) \in \Delta_r \mathbb{Z}^2$,

$$\mathbb{P}_{\nu_\rho}\left(B_r(k,s) \text{ is not } r\text{-stuck}\right) \le \Delta_r^2(1 - e^{-2\epsilon_r}) \le 2\Delta_r^2 e^{-\Delta_r}, \tag{5.5.43}$$

i.e., for each (k,s), $\Upsilon_r^{\mathrm{ns}}(k,s)$ is stochastically dominated by a Bernoulli($\tilde{p}_r$) random variable, where $\tilde{p}_r := 2\Delta_r^2 e^{-\Delta_r}$ decays super-exponentially fast in r. Note that $\Upsilon_r^{\mathrm{ns}}(k,s)$ only depends on the graphical representation inside $B_r(k,s) \cup \{y_1, y_2\} \times [s - 2\Delta_r, s + \Delta_r]$, where $y_1 := k - 5\Delta_r - 1$ and $y_2 := k + 6\Delta_r$ are the sites on the spatial boundary of $B_r(k,s)$. Therefore $(\Upsilon_r^{\mathrm{ns}}(k,s))_{(k,s)\in I}$ are jointly independent if $I \in \{I_{odd}, I_{even}\}$, which finishes the proof. ∎

Proof of Proposition 5.3.1(b). Let

$$\Phi_r^{\mathrm{ns}}(\ell) := \sup_{w \in \mathcal{W}_\ell} \#\{r\text{-blocks which intersect } w \text{ and are not } r\text{-stuck}\}. \tag{5.5.44}$$

Since $\Phi_r^{\mathrm{t}}(\ell) \le \Phi_r^{\mathrm{ns}}(\ell)$, it is enough to prove that

$$\lim_{r \to \infty} \limsup_{\ell \to \infty} \ell^{-1} \Delta_r^2 \Phi_r^{\mathrm{ns}}(\ell) = 0. \tag{5.5.45}$$

But, for $\kappa \in (0, (6\log(N_0))^{-1})$ and $r \le \lfloor \kappa \log(\ell) \rfloor$,

$$\frac{1}{\ell}\Delta_r^2 \Phi_r^{\mathrm{ns}}(\ell) \le \frac{1}{\ell} \sum_{k=r}^{\lfloor \kappa \log(\ell) \rfloor} \Delta_k^2 \Phi_k^{\mathrm{ns}}(\ell), \tag{5.5.46}$$

so (5.5.45) follows from Lemma 5.5.5 and Proposition 5.4.3. ∎

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Samenvatting

Stochastische wandelingen in stochastische omgevingen in $\mathbb{Z}^d$ fungeren als modellen voor de beweging van een deeltje in een wanordelijk materiaal, dat een *stochastische omgeving* wordt genoemd. De stochastische wandeling is een Markovproces met overgangsintensiteiten die afhankelijk zijn van de stochastische omgeving. Het model wordt *statisch* genoemd als de stochastische omgeving constant in de tijd is, en *dynamisch* als die in de tijd evolueert. Gewoonlijk is de stochastische omgeving translatie-invariant en in evenwicht. Zulke modellen behoren tot het grotere onderzoeksgebied van *wanordelijke systemen*, en zijn sinds begin jaren zeventig uitgebreid bestudeerd in de natuurkunde en wiskunde literatuur. Het doel is om de schalingseigenschappen van de stochastische wandeling te begrijpen. Dit proefschrift concentreert zich op de analyse van stochastische wandelingen in dynamische stochastische omgevingen.

Het statische een-dimensionale model is goed begrepen. Criteria voor terugkerend of voorbijgaand gedrag, wetten van grote aantallen, grote afwijkingen principes en schalingslimieten zijn beschikbaar. Er is ook veel bekend in meer dimensies, hoewel daar het beeld veel minder volledig is. Een belangrijk open probleem is de karakterisering van ballistisch gedrag van de stochastische wandeling (d.w.z., niet-nul asymptotische snelheid) voor algemene klassen van stochastische omgevingen.

Reeds in één dimensie vertoont het statische model een rijk gedrag, met kenmerken die heel anders zijn dan die van homogene stochastische wandelingen. Bijvoorbeeld, de stochastische wandeling kan voorbijgaand zijn met snelheid nul, en kan niet-diffusieve schalingslimieten hebben. Zulke eigenschappen hangen nauw samen met de aanwezigheid van zogeheten *vallen* in de stochastische omgeving, d.w.z., gebieden waar de stochastische wandeling een zeer lange tijd doorbrengt. Dit verschaft een natuurlijke motivatie voor het dynamische model in één dimensie: aangezien vallen in een dynamische omgeving uiteindelijk verdwijnen, rijst de vraag wat er gebeurt met de kenmerken hierboven beschreven. Zullen die de dynamica overleven of zullen ze verdwijnen? Hoe zal de conclusie afhangen van de snelheid and van het type gekozen dynamica? Deze vragen zijn tot nu toe slechts gedeeltelijk beantwoord. Inderdaad heeft het dynamische model nog slechts een korte geschiedenis. De meeste resultaten in de literatuur veronderstellen dat de dynamische stochastische omgeving Markov in de tijd is, met uniforme en voldoend snelle mengingseigenschappen, en tonen standaardgedrag aan voor de stochastische wandeling.

De hoofdstukken van dit proefschrift zijn als volgt georganiseerd. Hoofdstuk 1 bevat een gedetailleerde inleiding. Hoofdstuk 2 beschrijft de wet van grote aantallen voor bepaalde klassen van stochastische wandelingen in dynamische stochastische omgevingen in $\mathbb{Z}^d$, $d \in \mathbb{N}$, inclusief enkele niet-elliptische gevallen (d.w.z, waar de overgangsintensiteiten van de stochastische wandeling willekeurig dicht bij nul of oneindig kunnen zijn). De hypothesen bevatten een uniforme mengingsconditie genaamd *conditional cone-mixing*, en andere condities noodzakelijk om de tijd

te controleren, die de stochastische wandeling in grote ruimte-tijdkegels doorbrengt. Het hoofd-model dat behandeld wordt is het $(\infty, 0)$-*model*, dat een bijna deterministische functie van de stochastische omgeving is.

In Hoofdstuk 3 nemen we als dynamische stochastische omgeving het naaste-buren symmetrische exclusieproces in één dimensie. Deze dynamica is uitdagend omdat hij geen snelle of uniforme menging heeft. Voor onze analyse veronderstellen we een sterke drift-conditie, wat betekent dat de stochastische wandeling altijd naar rechts beweegt met een voldoend grote snel-heid. Hierdoor kunnen we de langzame en niet-uniforme menging van de stochastische omgeving overwinnen en een wet van grote aantallen, een invariantie-principe, grote-afwijking grenzen, en ook een zogenoemde *Einstein relatie* voor de stochastische wandeling bewijzen door middel van een vernieuwingsargument.

In Hoofdstuk 4 bestuderen we nog een voorbeeld van dynamische stochastische omgeving zonder uniforme menging, namelijk, het superkritische contactproces in één dimensie. Mono-toniciteitseigenschappen, plus snelle convergentie van dit proces naar zijn grootste invariante kansverdeling als het vanuit een volle configuratie begint, laten het toe een wet van grote aan-talen voor de stochastische wandeling te bewijzen in de gehele superkritische fase. Als bovendien de stochastische wandeling langzaam genoeg is ten opzichte van de infectiesnelheid, dan kun-nen we een vernieuwingstructuur opbouwen om een invariantie-principe voor de stochastische wandeling te bewijzen, evenals de continuïteit van de asymptotische snelheid en variantie met betrekking tot de infectiesnelheid.

In Hoofdstuk 5 beschouwen we opnieuw het (naaste-buren en symmetrische) exclusieproces, dit keer zonder de sterke drift veronderstelling. In dit geval is de wet van grote aantallen voor de stochastische wandeling nog niet bewezen. We richten ons op een eenvoudiger vraag, namelijk, of de minimale/maximale lineaire (in de tijd) verplaatsingen van de stochastische wandeling gelijk kunnen zijn aan de overeenkomstige snelheden van de samenstellende homogene stochastische wandelingen die worden gebruikt om het model te definiëren. Met behulp van een multi-schaalanalyse van het exclusieproces kunnen we een negatief antwoord op deze vraag geven.

Acknowledgements

It would have been impossible to complete or even start a PhD, were it not for the help of several people. First of all, I am deeply grateful to my advisor Frank den Hollander. His constant support, encouragement and guidance throughout these four years were fundamental for my development as a scientist, and his dedication to research and education has been an inspiration for me.

I am just as indebted to my co-advisor Vladas Sidoravicius, who introduced me to the Dutch mathematical community and to the wonders of talking about probability on an intuitive level. I have benefited a lot from his knowledge and insight, and his passion for sharing beautiful problems and ideas might have infected me beyond possible cure.

I am also very grateful to Sônia de Carvalho and Bernardo de Lima, who introduced me to dynamical systems and to probability when I was studying in Belo Horizonte, and to Claudio Landim, who greatly contributed to my education when I was at IMPA. Their influence was pivotal in my decision to pursue a PhD in mathematics.

I would like to heartily thank Nina Gantert, Frank Redig and Christophe Sabot, who were very kind to take time to read the thesis manuscript, even with a very tight schedule. Thank you very much for your help and comments.

I would further like to thank all the staff of the Mathematical Institute and the Science Faculty at Leiden University for providing a good working environment and for their support with all the administrative tasks.

Of course, many thanks go also to my colleagues in the Mathematical Institute. It was a pleasure and a privilege to work with you, discuss with you, break coffee with you, and share so many joys and hardships during all this time. I hope to see you all again in the future!

Last but not least, thanks to my family and friends for making my life richer and supporting me to go after my dreams.

Curriculum Vitae

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