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Super Landau models on odd cosets

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We construct $d = 1$ sigma models of the Wess-Zumino type on the $SU(n|1)/U(n)$ fermionic cosets. Such models can be regarded as a specific supersymmetric generalization (with a target space supersymmetry) of the classical Landau model, when a charged particle possesses only fermionic coordinates. We consider both classical and quantum models, and we prove the unitarity of the quantum model by introducing the metric operator on the Hilbert space of the quantum states, such that all their norms become positive definite. It is remarkable that the quantum $n = 2$ model exhibits hidden $SU(2|2)$ symmetry. We also discuss the planar limit of these models. The Hilbert space in the planar $n = 2$ case is shown to carry $SU(2|2)$ symmetry which is different from that of the $SU(2|1)/U(1)$ model.

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I. INTRODUCTION

The renowned Landau model [1] describes a charged nonrelativistic particle moving on a two-dimensional Euclidean plane $\mathbb{R}^2 \sim (z, \bar{z})$ under the influence of a uniform magnetic field which is orthogonal to the plane. Its simplest generalization to a curved manifold is the Haldane model [2], describing a charged particle on the two-sphere $S^2 \sim SU(2)/U(1)$ in the field of a Dirac monopole located at the center. These models are exactly solvable, on both the classical and the quantum levels. They can be interpreted as one-dimensional nonlinear sigma models with the Wess-Zumino (WZ) terms. For instance, the Haldane model is described by the $d = 1$ $SU(2)/U(1)$ sigma model action with the $U(1)$ WZ term.

There are two different approaches to supersymmetrizing this bosonic system. One is based on a worldline supersymmetry (see, e.g., Ref. [3]), while the other deals with a target-space supersymmetry. The latter option corresponds to extending the bosonic manifolds to supermanifolds by adding extra fermionic target coordinates. These supermanifolds are identified with cosets of some supergroups, so the relevant invariant actions describe $d = 1$ WZ sigma models on supergroups. Since supergroups possess a wider set of cosets as compared to their bosonic subgroups, there are several nonequivalent super-Landau models associated with the same supergroup. A minimal superextension of $SU(2)$ is the supergroup $SU(2|1)$ involving the bosonic subgroup $U(2) = SU(2) \times U(1)$ and a doublet of fermionic generators.[1] It possesses a few different cosets, each giving rise to some super-Landau $d = 1$ sigma model. The $SU(2)/U(1)$ model can be promoted either to a model on the $(2|2)$-dimensional supersphere $SU(2|1)/U(1|1)$ [4,5] or to a model on the $(2|4)$-dimensional superflag manifold $SU(2|1)/[U(1) \times U(1)]$ [5,6]. Both models are exactly solvable, like their bosonic prototypes, and exhibit further interesting properties [5]. For instance, for some special relations between the coefficients of the corresponding WZ terms they prove to be quantum-mechanically equivalent to each other. Another surprising feature of these models is that, contrary to the standard lore, the presence of noncanonical fermionic kinetic terms bilinear in time derivative in their actions does not necessarily lead, upon quantization, to ghost states with negative norms; all norms can be made positive definite by modifying the inner product on the Hilbert space.

The supergroup $SU(2|1)$ also possesses the pure odd supercoset $SU(2|1)/U(2)$ of the dimension $(0|4)$. The Landau-type quantum sigma models on the odd cosets $SU(n|1)/U(n)$ of the dimension $(0|2n)$, with the pure WZ term as the action, were studied in Ref. [7] (see also Ref. [4]). The relevant Hilbert spaces studied there involve only single (vacuum) states associated with the lowest Landau levels (LLLs). These LLL states reveal interesting $SU(n|1)$ representation content.

There remained a problem of finding out the complete sigma model actions on the supercoset $SU(n|1)/U(n)$, such that they contain both the WZ term and the term bilinear in the coset Cartan forms [i.e., the one-dimensional pullback of the Killing form on $SU(n|1)/U(n)$], and of exploring the relevant quantum mechanics. The basic aim of the present paper is to fill this gap. It is worth noting that sigma models with the supergroup target spaces have attracted a lot of attention over the last few years, in particular because of their intimate relation to superbranes (see, e.g., Refs. [8–10]). This increasing interest is one of the main motivations of our study.

The paper is organized as follows. In Sec. II, we construct the $SU(n|1)$ invariant action for this model, using the $d = 1$ version of the universal gauge approach which can be traced back to the construction of the two-dimensional

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[1] This is a minimal possibility if one assumes the standard complex conjugation for the generators.

bosonic $\mathbb{CP}^n$ sigma model actions in Ref. [11]. The same general approach works nicely in some other cases elaborated as instructive examples in the Appendix [Landau-type models on the bosonic coset $SU(n+1)/U(n) \sim \mathbb{CP}^n$ and the supersphere $SU(2|1)/U(1|1) \sim \mathbb{CP}^{(1|1)}$]. In Sec. III we construct the corresponding Hamiltonian and $SU(n|1)$ Noether charges, in both the classical and the quantum cases. In Sec. IV we present the complete set of eigenfunctions of the Hamiltonian and compute its spectrum. The salient feature of the quantum case is that the number of Landau level (LL) states is finite and equal to $n+1$. We discuss the $SU(2|1)$ representation assignment of the full set of states in the $n=2$ case by computing, in particular, the eigenvalues of the $SU(2|1)$ Casimir operators. In Sec. V we compute the norms of the LL states and find that for some values of the WZ term strength κ, there are states with negative and/or zero norms, like in other super-Landau models. In Sec. VI we give a more detailed treatment of the $n=2$ case. We present the explicit form of the metric operator which allows one to make all norms positive definite. At each LL, the quantum states are found to form short multiplets of some hidden $SU(2|2)$ symmetry which is an extension of the original $SU(2|1)$ symmetry (the phenomenon of such an enhancement of $SU(2|1)$ to $SU(2|2)$ at the quantum level was earlier revealed in the superflag Landau model [5]). In Sec. VII we study the planar limit of the odd-coset Landau models. We show, in particular, that the Hilbert space for $n=2$ also carries some extended $SU(2|2)$ symmetry. We finish with conclusions and outlook in Sec. VIII.

II. $SU(n|1)/U(n)$ ACTION FROM $U(1)$ GAUGING

The supergroup $SU(n|1)$ can be defined as the set of linear transformations of the $n+1$-component multiplet (z, ξ^i) $(i = 1, 2, \ldots n)$, such that they preserve its norm

$$z\bar{z} - \zeta \cdot \bar{\zeta} = \text{inv.} \tag{2.1}$$

Here, the components z, $\bar{z}$ are Grassmann even, while ζ^i, $\bar{\zeta}_i$ are Grassmann odd. Hereafter, $\zeta \cdot \bar{\zeta} = -\bar{\zeta} \cdot \zeta = \zeta^i \bar{\zeta}_i$. The fermionic transformations are

$$\delta_\epsilon z = \zeta \cdot \bar{\epsilon}, \qquad \delta \zeta^i = \epsilon^i z, \tag{2.2}$$

where ϵ^i, $\bar{\epsilon}_i$ are Grassmann-odd parameters. The variables ζ_i transform in the fundamental representation of the group $U(n)$. These $U(n)$ transformations are contained in the closure of (2.2).

Now we are going to show how, starting from this linear $SU(n|1)$ multiplet, one can construct a nonlinear $d=1$ WZ sigma model action associated with the odd $(0|2n)$-dimensional coset $SU(n|1)/U(n)$.

We start with the following $SU(n|1)$ invariant Lagrangian:

$$L = \nabla z \bar{\nabla} \bar{z} + \bar{\nabla} \bar{\zeta} \cdot \nabla \zeta + 2\kappa \mathcal{A}, \tag{2.3}$$

where

$$\nabla = \partial_t - i\mathcal{A}, \qquad \bar{\nabla} = \partial_t + i\mathcal{A}. \tag{2.4}$$

The auxiliary gauge field $\mathcal{A}(t)$ ensures the invariance of the Lagrangian (2.3) under the $U(1)$ gauge transformations:

$$\delta z = i\lambda z, \qquad \delta \zeta^i = i\lambda \zeta_i, \qquad \delta \mathcal{A} = \dot{\lambda}. \tag{2.5}$$

The last term in (2.3) is the Fayet-Iliopoulos (FI) term; it is invariant under (2.5) up to a total time derivative. The gauge field $\mathcal{A}(t)$ is a $SU(n|1)$ singlet; gauge transformations (2.5) commute with the rigid $SU(n|1)$ ones.

As the next steps, we impose the $SU(n|1)$ invariant constraint on the variables z, ζ^i

$$z\bar{z} - \zeta \cdot \bar{\zeta} = 1, \tag{2.6}$$

choose the $U(1)$ gauge

$$z = \bar{z} \equiv \rho \tag{2.7}$$

and, using (2.6), express ρ in terms of the fermionic variables,

$$\rho = \sqrt{1 + \zeta \cdot \bar{\zeta}}. \tag{2.8}$$

Finally, the auxiliary field $\mathcal{A}$ can be eliminated by its algebraic equation of motion:

$$\mathcal{A} = -\frac{1}{2}[2\kappa + i(\bar{\zeta} \cdot \dot{\zeta} - \dot{\bar{\zeta}} \cdot \zeta)]. \tag{2.9}$$

Upon substituting all of this back into the Lagrangian (2.3), the latter takes the form

$$L = \dot{\bar{\zeta}} \cdot \dot{\zeta} + \frac{2 + \zeta \cdot \bar{\zeta}}{4(1 + \zeta \cdot \bar{\zeta})}[(\dot{\zeta} \cdot \bar{\zeta})^2 + (\zeta \cdot \dot{\bar{\zeta}})^2] - \frac{\zeta \cdot \bar{\zeta}}{2(1 + \zeta \cdot \bar{\zeta})}(\dot{\zeta} \cdot \bar{\zeta})(\zeta \cdot \dot{\bar{\zeta}}) + i\kappa(\dot{\zeta} \cdot \bar{\zeta} - \zeta \cdot \dot{\bar{\zeta}}). \tag{2.10}$$

This Lagrangian is invariant, up to a total time derivative, under the following purely fermionic nonlinear realization of the odd $SU(n|1)$ transformations:

$$\delta \zeta^i = \sqrt{1 + \zeta \cdot \bar{\zeta}}\,\epsilon^i + \frac{\zeta^i}{2\sqrt{1 + \zeta \cdot \bar{\zeta}}}(\bar{\epsilon} \cdot \zeta + \epsilon \cdot \bar{\zeta}). \tag{2.11}$$

It precisely coincides with the one considered in Ref. [7]. This realization follows from (2.2) by taking into account the gauge (2.7), the constraint (2.6), and the necessity to accompany the original $SU(n|1)$ transformations by the compensating gauge transformations (2.5) with

$$\lambda = -\frac{i}{2\sqrt{1 + \zeta \cdot \bar{\zeta}}}(\bar{\epsilon} \cdot \zeta + \epsilon \cdot \bar{\zeta})$$

in order to preserve the gauge (2.7). The Lagrangian (2.10) describes the $d=1$ WZ sigma model on the odd coset $SU(n|1)/U(n)$, with the $d=1$ fields ζ^i and $\bar{\zeta}_i$ ($2n$ real fermionic variables) being the coset parameters.

The number of independent variables was reduced from the $(n+1)$ complex ones z, ζ^i to n such variables ζ^i by imposing the constraint (2.6) and choosing the gauge (2.7). In this aspect, the method we applied is quite similar to the gauge approach to the construction of the bosonic $d=2$ $\mathbb{CP}^n$ sigma models in Ref. [11]. It is worth pointing out that our $d=1$ gauging procedure automatically yields not only the standard sigma model part of the Lagrangian but also the WZ term with the strength 2κ.[2] In the "parent" linear sigma model action (2.3), this constant appears as a strength of the FI term. In Ref. [7], only the WZ term in (2.10) was considered. Such truncated Lagrangian can be treated as the large κ limit of (2.10).

In what follows, it will be convenient to deal with the coset coordinates ξ^i related to ζ^i by

$$\xi^i = \frac{\zeta^i}{\sqrt{1+\zeta\cdot\bar{\zeta}}} \tag{2.12}$$

and possessing the simple holomorphic transformation law

$$\delta\xi^i = \epsilon^i + (\bar{\epsilon}\cdot\xi)\xi^i. \tag{2.13}$$

In terms of ξ^i, the Lagrangian (2.10) is rewritten as

$$L = \frac{\dot{\bar{\xi}}\cdot\dot{\xi}}{1-\xi\cdot\bar{\xi}} + \frac{(\dot{\bar{\xi}}\cdot\xi)(\dot{\xi}\cdot\bar{\xi})}{(1-\xi\cdot\bar{\xi})^2} - i\kappa\frac{\bar{\xi}\cdot\dot{\xi} - \dot{\bar{\xi}}\cdot\xi}{1-\xi\cdot\bar{\xi}}. \tag{2.14}$$

The odd $SU(n|1)$ transformations may be expressed in terms of the fermionic $SU(n|1)$ generators Q_i, $Q^{\dagger j}$:

$$\delta\xi = [(\epsilon Q) + (\bar{\epsilon}Q^\dagger)]\xi, \tag{2.15}$$

where Q_i and $Q^{\dagger i} = (Q_i)^\dagger$ satisfy the relations

$$\{Q_i, Q_j\} = 0, \qquad \{Q^{\dagger i}, Q^{\dagger j}\} = 0, \tag{2.16}$$

$$\{Q_i, Q^{\dagger j}\} = \left(J_i^j - \delta_i^j\frac{1}{n}J_k^k\right) + \delta_i^j B. \tag{2.17}$$

Here $J_i^j - \frac{1}{n}\delta_i^j J_k^k \equiv \tilde{J}_j^i$ are the $SU(n)$ generators, and B is the generator of the $U(1)$ transformation. They constitute the bosonic "body" $U(n)$ of the supergroup $SU(n|1)$. In the realization on the variables ξ^i, $\bar{\xi}_k$, these generators are

$$J_i^j = \bar{\xi}_i\frac{\partial}{\partial\bar{\xi}_j} - \xi^j\frac{\partial}{\partial\xi^i}, \qquad B = \left(\frac{1}{n}-1\right)J_k^k. \tag{2.18}$$

The explicit expressions for the conserved Noether supercharges corresponding to the odd $SU(n|1)$ transformations are

$$Q_i = -\pi_i + \bar{\xi}_i(\bar{\xi}\cdot\bar{\pi}) - i\kappa\bar{\xi}_i, \tag{2.19}$$

$$Q^{\dagger i} = \bar{\pi}^i + \xi^i(\xi\cdot\pi) + i\kappa\xi^i. \tag{2.20}$$

[2] See the Appendix for further examples.

Here, $\pi_i = \partial L/\partial\dot{\xi}^i$ and $\bar{\pi}^i = \partial L/\partial\dot{\bar{\xi}}_i$ are the momenta canonically conjugate to the fields ξ^i and $\bar{\xi}_i$. The corresponding conserved $U(n)$ generators J_j^i and B are expressed as

$$\begin{aligned} J_i^j &= \bar{\xi}_i\bar{\pi}^j - \xi^j\pi_i, \\ B &= \left(\frac{1}{n}-1\right)(\bar{\xi}_i\bar{\pi}^i - \xi^i\pi_i) + \text{constant}, \end{aligned} \tag{2.21}$$

where a constant in B (the central charge) will be fixed in the quantum model by requiring the generators to close on the $su(2|1)$ algebra as in (2.16) and (2.17).

III. QUANTIZATION

A. Hamiltonian formulation

To quantize the classical $SU(n|1)/U(n)$ coset model constructed in the previous section, we have to build its Hamiltonian formulation. First of all, it is convenient to rewrite the Lagrangian (2.14) in a geometric way, in terms of the metric on the coset space $SU(n|1)/U(n)$ and the external gauge potentials given on this manifold. The corresponding Hamiltonian will have the form convenient for quantization, and it will be easy to find its spectrum.

We can write down the metric on the $SU(n|1)/U(n)$ coset parametrized by ξ^i coordinates, using the Kähler potential

$$K = \ln(1-\xi\cdot\bar{\xi}). \tag{3.1}$$

The metric is given by

$$g_j^i = \partial_j\partial^{\bar{i}}K = \frac{\delta_j^i}{1-\xi\cdot\bar{\xi}} - \frac{\xi^i\bar{\xi}_j}{(1-\xi\cdot\bar{\xi})^2}. \tag{3.2}$$

Also, we define the gauge connections

$$A_i = -i\partial_i K = i\frac{\bar{\xi}_i}{1-\xi\cdot\bar{\xi}}; \quad \bar{A}^i = i\partial^{\bar{i}}\quad K = i\frac{\xi^i}{1-\xi\cdot\bar{\xi}}. \tag{3.3}$$

Note that $\bar{A}^i = -\overline{(A_i)}$. The inverse metric is given by

$$(g^{-1})_j^i = (1-\xi\cdot\bar{\xi})(\delta_j^i + \xi^i\bar{\xi}_j). \tag{3.4}$$

In terms of these quantities, the Lagrangian (2.14) can be written as

$$L = g_j^i\dot{\bar{\xi}}_i\dot{\xi}^j + \kappa(\dot{\xi}^iA_i + \dot{\bar{\xi}}_i\bar{A}^i). \tag{3.5}$$

The momenta canonically conjugate to the variables ξ^i and $\bar{\xi}_i$ are

$$\pi_i = \frac{\partial L}{\partial\dot{\xi}^i} = -g_i^k\dot{\bar{\xi}}_k + \kappa A_i, \quad \bar{\pi}^i = \frac{\partial L}{\partial\dot{\bar{\xi}}_i} = g_k^i\dot{\xi}^k + \kappa\bar{A}^i. \tag{3.6}$$

Then the Hamiltonian is given by

$$H = (g^{-1})_j^i(\bar{\pi}^j - \kappa\bar{A}^j)(\pi_i - \kappa A_i). \tag{3.7}$$

Using the anticommutativity of the Grassmann variables, we rewrite the Hamiltonian in the form

$$H = \frac{1}{2}(g^{-1})^i_j[\bar{\pi}^j - \kappa\bar{A}^j, \pi_i - \kappa A_i], \tag{3.8}$$

and perform canonical quantization in the coordinate representation by replacing

$$\pi_i \to -i\partial_i, \qquad \bar{\pi}^i \to -i\partial^{\bar{i}}. \tag{3.9}$$

As a result, we obtain the quantum Hamiltonian

$$H = \frac{1}{2}(g^{-1})^i_j[\nabla^{(\kappa)}_i, \nabla^{(\kappa)\bar{j}}], \tag{3.10}$$

where we have introduced

$$\nabla^{(\kappa)}_i = \partial_i + \frac{\kappa\bar{\xi}_i}{1-\xi\cdot\bar{\xi}}, \quad \nabla^{(\kappa)\bar{j}} = \partial^{\bar{j}} + \frac{\kappa\xi^j}{1-\xi\cdot\bar{\xi}}. \tag{3.11}$$

These "semicovariant" derivatives satisfy the following anticommutation relations:[3]

$$\begin{aligned} &\{\nabla^{(\kappa)}_i, \nabla^{(\kappa)}_j\} = 0,\\ &\{\nabla^{(\kappa)\bar{i}}, \nabla^{(\kappa)\bar{j}}\} = 0,\\ &\{\nabla^{(\kappa)}_i, \nabla^{(\kappa')\bar{j}}\} = (\kappa + \kappa')g^j_i. \end{aligned} \tag{3.12}$$

Using them, we can rewrite the Hamiltonian (3.10) in the convenient equivalent form:

$$H = (g^{-1})^i_j\nabla^{(\kappa)}_i\nabla^{(\kappa)\bar{j}} - \kappa n \equiv H' - \kappa n. \tag{3.13}$$

For what follows, it will be useful to write H' explicitly:

$$\begin{aligned} H' = {}&(1-\xi\cdot\bar{\xi})(\partial_i\partial^{\bar{i}} + \xi^i\bar{\xi}_{\bar{j}}\partial_i\partial^{\bar{j}}) + \kappa(1-\xi\cdot\bar{\xi})(\bar{\xi}_i\partial^{\bar{i}} - \xi^j\partial_j)\\ &- \kappa^2\xi\cdot\bar{\xi} + \kappa n. \end{aligned} \tag{3.14}$$

The Hamiltonian is Hermitian with respect to the appropriate inner product (see below).

B. Quantum $SU(n|1)$ generators

The quantum $SU(n|1)$ generators can be obtained from the classical expressions (2.19), (2.20), and (2.21):

$$\begin{aligned} Q_i &= -\partial_i + \bar{\xi}_i\bar{\xi}_j\partial^{\bar{j}} + \kappa\bar{\xi}_i,\\ Q^{\dagger i} &= -\partial^{\bar{i}} - \xi^i\xi^j\partial_j + \kappa\xi^i, \end{aligned} \tag{3.15}$$

$$\begin{aligned} J^j_i &= \bar{\xi}_i\frac{\partial}{\partial\bar{\xi}_j} - \xi^j\frac{\partial}{\partial\xi^i} \equiv \tilde{J}^j_i + \frac{1}{n}\delta^j_i J^k_k,\\ F &= \left(\frac{1}{n} - 1\right)J^k_k - 2\kappa, \end{aligned} \tag{3.16}$$

where we properly fixed the ordering ambiguities, based on the same reasonings as in Ref. [7]. A constant in the expression for the Noether charge B [see (2.21)] was fixed to be -2κ and the resulting operator was denoted F in order to have the $su(n|1)$ algebra (2.16) and (2.17) in the quantum case:

$$\{Q_i, Q_j\} = 0, \qquad \{Q^{\dagger i}, Q^{\dagger j}\} = 0, \tag{3.17}$$

$$\{Q_i, Q^{\dagger j}\} = \tilde{J}^j_i + \delta^j_i F. \tag{3.18}$$

Using the explicit form (3.14) of H', it is straightforward to check that the $SU(n|1)$ generators (3.15) and (3.16) indeed commute with the Hamiltonian.

For further use, we explicitly present the generators in the $n = 2$ case. The corresponding Landau model possesses the eight-parameter symmetry supergroup $SU(2|1)$. We define

$$\Pi = Q_1, \qquad Q = Q_2. \tag{3.19}$$

Then the full set of the $SU(2|1)$ generators as differential operators acting on the manifold with the odd coordinates $(\xi^i, \bar{\xi}_i)$, $i = 1, 2$, is given by the expressions

$$Q = -\partial_2 + \bar{\xi}_2\bar{\xi}_1\partial^{\bar{1}} + \kappa\bar{\xi}_2, \tag{3.20}$$

$$\Pi = -\partial_1 + \bar{\xi}_1\bar{\xi}_2\partial^{\bar{2}} + \kappa\bar{\xi}_1, \tag{3.21}$$

$$J_+ = i\bar{\xi}_2\partial^{\bar{1}} - i\xi^1\partial_2, \tag{3.22}$$

$$J_- = i\xi^2\partial_1 - i\bar{\xi}_1\partial^{\bar{2}}, \tag{3.23}$$

$$J_3 = \frac{1}{2}(\xi^1\partial_1 - \xi^2\partial_2 - \bar{\xi}_1\partial^{\bar{1}} + \bar{\xi}_2\partial^{\bar{2}}), \tag{3.24}$$

$$F = \frac{1}{2}(\xi^1\partial_1 + \xi^2\partial_2 - \bar{\xi}_1\partial^{\bar{1}} - \bar{\xi}_2\partial^{\bar{2}}) - 2\kappa. \tag{3.25}$$

The corresponding nonvanishing (anti)commutation relations read

$$[J_+, J_-] = 2J_3, \qquad [J_3, J_\pm] = \pm J_\pm, \tag{3.26}$$

$$\begin{aligned} &[J_+, \Pi] = iQ, \qquad [J_-, Q] = -i\Pi,\\ &[J_3, \Pi] = -\frac{1}{2}\Pi, \qquad [J_3, Q] = \frac{1}{2}Q,\\ &[F, \Pi] = -\frac{1}{2}\Pi, \qquad [F, Q] = -\frac{1}{2}Q, \end{aligned} \tag{3.27}$$

$$\begin{aligned} &[J_-, \Pi^\dagger] = iQ^\dagger, \qquad [J_+, Q^\dagger] = -i\Pi^\dagger,\\ &[J_3, \Pi^\dagger] = \frac{1}{2}\Pi^\dagger, \qquad [J_3, Q^\dagger] = -\frac{1}{2}Q^\dagger,\\ &[F, \Pi^\dagger] = \frac{1}{2}\Pi^\dagger, \qquad [F, Q^\dagger] = \frac{1}{2}Q^\dagger, \end{aligned} \tag{3.28}$$

[3]The semicovariant derivatives are essentially complex, in accordance with the fact that the wave superfunctions are complex. Under the complex conjugation, they are transformed as $(\nabla^{(\kappa)}_i, \nabla^{(\kappa)\bar{j}}) \Rightarrow \pm(\nabla^{(-\kappa)\bar{i}}, \nabla^{(-\kappa)}_j)$, when acting, respectively, on Grassmann-odd or Grassmann-even superfunctions.

$$\{\Pi, \Pi^\dagger\} = -J_3 + F, \qquad \{Q, Q^\dagger\} = J_3 + F,$$
$$\{\Pi, Q^\dagger\} = iJ_-, \qquad \{\Pi^\dagger, Q\} = -iJ_+. \tag{3.29}$$

The quadratic Casimir operator of the superalgebra $su(2|1)$,

$$C_2 = -\frac{1}{2}\{J_+, J_-\} - J_3^2 + F^2 + \frac{1}{2}[Q, Q^\dagger] + \frac{1}{2}[\Pi, \Pi^\dagger], \tag{3.30}$$

is related to the Hamiltonian of the $n = 2$ model as

$$C_2 = H + 4\kappa^2 = H' - 2\kappa + 4\kappa^2. \tag{3.31}$$

One can also define a third-order Casimir operator:

$$C_3 = \frac{i}{2}J_+[Q^\dagger, \Pi] - \frac{i}{2}[\Pi^\dagger, Q]J_- + \frac{1}{2}J_3([Q, Q^\dagger] - [\Pi, \Pi^\dagger]) - \frac{1}{2}F([\Pi, \Pi^\dagger] + [Q, Q^\dagger]) + 2C_2F - \Pi^\dagger\Pi - QQ^\dagger. \tag{3.32}$$

In the $n = 2$ model it can be represented as

$$C_3 = 6\kappa H' + 2\kappa(2\kappa - 1)(4\kappa - 1). \tag{3.33}$$

IV. THE ENERGY SPECTRUM AND WAVE FUNCTIONS

In this section we turn to the study of the quantum $SU(n|1)/U(n)$ model. We construct the complete set of the wave superfunctions and find the corresponding energy levels. We also obtain the realization of the $SU(n|1)$ symmetry group on the wave superfunctions.

A. The spectrum

In this subsection we construct the complete set of wave superfunctions for the $SU(n)$ singlet sector of the full space of quantum states. The corresponding superfunctions carry no external $SU(n)$ indices, but possess the fixed B charge $-2\kappa \neq 0$, in accord with the explicit structure of the quantum generators (3.15) and (3.16). Possible wave functions with nonzero external $SU(n)$ spins form a subspace orthogonal to the $SU(n)$ singlet one with respect to the inner product to be defined below. A similar situation occurs in the case of super-Landau models associated with the supersphere $SU(2|1)/U(1|1)$ [5], where one can consider only those wave functions which are singlets of the semisimple part $SU(1|1) \subset U(1|1)$.

To proceed, we will need two important properties:

$$(g^{-1})^i_j[\nabla^{(\kappa)}_i, g^j_k] = \frac{(1-n)\bar{\xi}_k}{1 - \xi \cdot \bar{\xi}} \tag{4.1}$$

and

$$\nabla^{(\kappa)}_{[i}(g^j_{k]}\Phi) = g^j_{[k}\nabla^{(k+2)}_{i]}\Phi, \tag{4.2}$$

where Φ is an arbitrary superfunction, $\Phi = \Phi(\xi, \bar{\xi})$, and square brackets denote antisymmetrization of indices (with the factor $1/n!$). These relations can be proved using the definitions (3.2) and (3.11).

We will deal with the shifted Hamiltonian H' defined in (3.13). The lowest (vacuum) Landau level wave function Ψ_0 corresponds to the zero eigenvalue of H' and is defined by the same chirality condition as in Ref. [7]:

$$\nabla^{(\kappa)\bar{j}}\Psi_0 = 0 \rightarrow \Psi_0 = \frac{1}{(1 - \xi \cdot \bar{\xi})^\kappa}\Omega_0(\xi), \tag{4.3}$$

where the analytic wave function is defined by the expansion

$$\Omega_0(\xi) = c^0 + \xi^i c^0_i + \cdots + \xi^{i_1} \cdots \xi^{i_n} c^0_{i_1 \cdots i_n}. \tag{4.4}$$

The wave superfunctions corresponding to the excited Landau levels are constructed by the covariant derivatives $\nabla^{(\kappa')}$ acting on the chiral superfunctions. The latter should carry the appropriate external $SU(n)$ indices in order to ensure the full wave functions to be $SU(n)$ singlets.

The first LL wave superfunction is defined as

$$\Psi_1 = \nabla^{(\kappa+1-n)}_k \Phi^k, \tag{4.5}$$

where Φ^k is the chiral superfunction in the fundamental representation of $U(n)$:

$$\nabla^{(\kappa)\bar{j}}\Phi^k = 0 \rightarrow \Phi^k = \frac{1}{(1 - \xi \cdot \bar{\xi})^\kappa}\Omega^k(\xi). \tag{4.6}$$

Using (4.1), it is easy to check that Ψ_1 is the eigenfunction of H',

$$H'\Psi_1 = (2\kappa + 1 - n)\Psi_1. \tag{4.7}$$

The second LL wave superfunction is defined by

$$\Psi_2 = \nabla^{(\kappa+1-n)}_i \nabla^{(\kappa+3-n)}_k \Phi^{[ik]}, \tag{4.8}$$

where $\Phi^{[ik]}$ is a chiral superfield [it is expressed through the holomorphic reduced wave superfunction $\Omega^{[ik]}(\xi)$ in the same way as in (4.3) and (4.6)]. The reason why the chiral superfunctions should belong to the irreducible $U(n)$ representations constructed by antisymmetrizing the indices in the fundamental representation will be explained in the next subsection.

Using (4.1) and (4.2), one may verify that

$$H'\Psi_2 = 2(2\kappa + 2 - n)\Psi_2. \tag{4.9}$$

In the $n = 2$ case, when the indices take values 1 and 2, it is the last level in the spectrum, because no nonzero higher-rank antisymmetric tensors can be defined.

For the same reason, in the general case of an n-dimensional model, the spectrum terminates at the level $\ell = n$, so we are left with the finite set of n excited states. The wave superfunction for the ℓth LL is given by the expression

$$\Psi_\ell = \nabla^{(\kappa+1-n)}_{m_1}\nabla^{(\kappa+3-n)}_{m_2}\cdots\nabla^{(\kappa+2\ell-1-n)}_{m_\ell}\Phi^{[m_1 m_2\cdots m_\ell]}, \tag{4.10}$$

where the reduced wave superfunction $\Phi^{[m_1 m_2\cdots m_\ell]}$ is chiral:

$$\Phi^{[m_1 m_2\cdots m_\ell]} = \frac{1}{(1-\xi\cdot\bar\xi)^\kappa}\Omega^{[m_1 m_2\cdots m_\ell]}(\xi). \tag{4.11}$$

The corresponding energy eigenvalue is

$$E_\ell = \ell(\ell - n + 2\kappa). \tag{4.12}$$

Sometimes it is convenient to use the equivalent representation for Ψ_ℓ:

$$\Psi_\ell := \frac{1}{(1-\xi\cdot\bar\xi)^\kappa}\hat\Psi_\ell,$$
$$\hat\Psi_\ell = \nabla^{(2\kappa+1-n)}_{m_1}\nabla^{(2\kappa+3-n)}_{m_2}\cdots\nabla^{(2\kappa+2\ell-1-n)}_{m_\ell}\Omega^{[m_1 m_2\cdots m_\ell]}. \tag{4.13}$$

It is natural to require that the energies of the excited Landau levels are not negative and exceed (or at least are not less than) the energy of LLL. Therefore, in what follows we will consider only the options when the WZ term strength is restricted to the values

$$\kappa \geq (n-1)/2. \tag{4.14}$$

The obtained set of $n+1$ chiral superfunctions captures the whole spectrum of the model in the $SU(n)$ singlet sector. To prove this, we should check that any $SU(n)$ singlet superfunction can be expressed as a linear superposition of the wave superfunctions of $n+1$ Landau levels. The total number of independent functions of n complex Grassmann variables is equal to 2^{2n}. The total number of independent coefficients in the ξ^i expansion of a holomorphic superfunction corresponding to the level ℓ is $2^n\binom{n}{l}$. Different levels possess independent wave superfunctions. So the total number of independent coefficients is

$$2^n\sum_{\ell=0}^{n}\binom{n}{l} = 2^{2n}. \tag{4.15}$$

Thus, the constructed set of wave superfunctions indeed spans the full $SU(n)$ singlet Hilbert space.

In the remainder of this subsection we will briefly discuss the case with $\kappa < 0$. Consider, instead of the H' eigenvalues (4.12), those of the full Hamiltonian $H = H' - \kappa n$:

$$E_\ell \Rightarrow E_\ell = \ell(\ell - n + 2\kappa) - \kappa n. \tag{4.16}$$

Now we assume that $\kappa = -|\kappa|$ and redefine the level number ℓ as

$$\ell = -\ell' + n. \tag{4.17}$$

In terms of ℓ', the spectrum (4.16) for $\kappa < 0$ becomes

$$E^{(\kappa<0)}_{\ell'} = \ell'(\ell' - n + 2|\kappa|) - |\kappa| n. \tag{4.18}$$

This formula coincides with that for $\kappa > 0$,

$$E^{(\kappa>0)}_{\ell} = \ell(\ell - n + 2|\kappa|) - |\kappa| n, \tag{4.19}$$

modulo the substitution $\ell \to \ell'$. Thus, for $\kappa < 0$, the tower of the LL states is reversed: the highest LL with $\ell = n$ becomes the LLL with $\ell' = 0$, while the LLL with $\ell = 0$ becomes the highest LL with $\ell' = n$. In order to have, in both cases, the excited LL energies not less than the LLL energy, one needs to impose the following general condition:

$$|\kappa| \geq (n-1)/2. \tag{4.20}$$

The $\kappa < 0$ LLL wave superfunction $\tilde\Psi_{\ell'=0} := \Psi_{\ell=n}$ can be checked to satisfy the antichirality condition[4]

$$\nabla^{(\kappa)}_j\tilde\Psi_{\ell'=0} = 0 \to \tilde\Psi_{\ell'=0} = (1-\xi\cdot\bar\xi)^{-|\kappa|}\Omega_0(\bar\xi), \tag{4.21}$$

whereas all other ones, up to $\tilde\Psi_{\ell'=n} := \Psi_{\ell=0}$, are obtained through the successive action of the proper antiholomorphic covariant derivatives on the antichiral superfunctions $(1-\xi\cdot\bar\xi)^{-|\kappa|}\ \Omega^{[i_1\ldots i_{\ell'}]}(\bar\xi)$. Passing to the complex-conjugate set of the wave superfunctions, $\tilde\Psi_{\ell'} \to \tilde\Psi^\star_{\ell'}$ takes us back to the holomorphic representation, i.e., to the same picture as in the $\kappa > 0$ case (with replacing $\kappa \to |\kappa|$ everywhere). Thus, without loss of generality, we can basically limit our study to the $\kappa > 0$ option.

B. Transformation properties

The $SU(n|1)$ transformation law of the wave function for any LL,

$$\delta\Psi(\xi,\bar\xi) = (\epsilon\cdot Q + \bar\epsilon\cdot Q^\dagger)\Psi(\xi,\bar\xi), \tag{4.22}$$

is fully specified by the form of the quantum supercharges:

$$Q_i = -\partial_i + \bar\xi_i\bar\xi_j\partial^{\bar j} + \kappa\bar\xi_i, \tag{4.23}$$

$$Q^{\dagger i} = -\partial^{\bar i} - \xi^i\xi^j\partial_j + \kappa\xi^i. \tag{4.24}$$

The bosonic transformations are contained in the closure of these odd ones.

Sometimes it is more convenient to deal with the equivalent *passive* form of the same $SU(n|1)$ transformation:

$$\delta^*\Psi(\xi,\bar\xi) \simeq \Psi'(\xi',\bar\xi') - \Psi(\xi,\bar\xi)$$
$$= \kappa(\epsilon\cdot\bar\xi + \bar\epsilon\cdot\xi)\Psi(\xi,\bar\xi). \tag{4.25}$$

[4]The proof of this property is rather tricky. One rewrites $\nabla^{(\kappa)}_i$ as $\nabla^{(\kappa)}_i = \partial_i + \kappa B_i$, $B_i = \bar\xi_i/(1-\xi\cdot\bar\xi)$, and repeatedly uses the identity $\partial_k B_i = B_k B_i$ to represent the product of n covariant derivatives in $\hat\Psi_{\ell=n}$ [Eq. (4.13)], as a differential operator in ∂_i of the nth order, with the coefficients being monomials in B_k. To show that $\nabla^{(2\kappa)}_i\hat\Psi_{\ell=n} = 0$, one has to take into account the total antisymmetry in the indices $m_1\ldots m_n$ and to make use of the proper cyclic identities.

Note that this transformation law indicates that the wave superfunctions cannot be real unless $\kappa = 0$.

Given the transformation law of the full wave superfunction for the level ℓ, one can restore the transformation rules of the reduced chiral superfunctions defined in the previous subsection. To this end, one should take into account the "passive" transformation properties of the semicovariant derivatives

$$\begin{aligned}\delta^*\nabla_j^{(\kappa')} &= -(\bar\epsilon\cdot\xi)\nabla_j^{(\kappa')} + \bar\epsilon_j\xi^i\nabla_i^{(\kappa')} + \kappa'\bar\epsilon_j,\\ \delta^*\nabla^{(\kappa')\bar j} &= (\epsilon\cdot\bar\xi)\nabla^{(\kappa')\bar j} - \epsilon^j\bar\xi_i\nabla^{(\kappa')\bar i} + \kappa'\epsilon^j.\end{aligned} \tag{4.26}$$

Then we consider the transformation law of some chiral wave function $\Phi^{i_1\cdots i_\ell}$ with $\ell \geq 2$ and require that the corresponding transformation of Ψ_ℓ is given by the passive form of (4.22), i.e., by (4.25). We find, first, that $\Phi^{i_1\cdots i_\ell}$ should necessarily be fully antisymmetric in its $SU(n)$ indices, $\Phi^{i_1\cdots i_\ell} = \Phi^{[i_1\cdots i_\ell]}$, and second, that the transformation law of $\Phi^{[i_1\cdots i_\ell]}$ should be

$$\begin{aligned}\delta^*\Phi^{[i_1\cdots i_\ell]} &= \kappa(\epsilon\cdot\bar\xi)\Phi^{[i_1\cdots i_\ell]} + (\kappa+\ell)(\bar\epsilon\cdot\xi)\Phi^{[i_1\cdots i_\ell]}\\ &\quad + \xi^{i_1}\bar\epsilon_j\Phi^{[j\cdots i_\ell]} + \cdots + \xi^{i_\ell}\bar\epsilon_j\Phi^{[i_1\cdots i_{\ell-1}j]}.\end{aligned} \tag{4.27}$$

The chirality conditions are automatically covariant. For the first LL function Φ^i, we have the same transformation law; for the $SU(n)$ singlet LLL function Ψ_0, the $U(n)$ rotation terms are obviously absent and the transformation law coincides with (4.25). For the holomorphic wave superfunction

$$\Omega^{[i_1\cdots i_\ell]}(\xi) = (1-\xi\cdot\bar\xi)^\kappa\Phi^{[i_1\cdots i_\ell]} \tag{4.28}$$

the weight factors are properly combined in such a way that the holomorphy property is preserved:

$$\delta^*\Omega^{[i_1\cdots i_\ell]}(\xi) = (2\kappa+\ell)(\bar\epsilon\cdot\xi)\Omega^{[i_1\cdots i_\ell]}(\xi) + \cdots, \tag{4.29}$$

where "dots" stand for the holomorphic $U(n)$ rotations, which are the same as in (4.27). This transformation law is valid for any $\ell \geq 0$.

C. $n = 2$

Here we consider in more detail the $n = 2$ case, which corresponds to the $SU(2|1)/U(2)$ model. In this case we can lower and raise the $SU(2)$ indices with the help of skew-symmetric symbols,

$$\varepsilon_{ik},\ \varepsilon^{ik},\ \varepsilon^{ik}\varepsilon_{kj} = \delta^i_j, \qquad \varepsilon_{12} = \varepsilon^{21} = 1. \tag{4.30}$$

The holomorphic LLL superfunction $\Omega_0(\xi)$ has the following ξ expansion:

$$\Omega_0(\xi) = c^0 + \xi^i c_i^0 + \frac{1}{2}\xi^{i_1}\xi^{i_2}\varepsilon_{i_1i_2}c_1, \tag{4.31}$$

where c_0 and c_1 are $SU(2)$ singlets. From the transformation law

$$\delta\Omega_0 = 2\kappa(\bar\epsilon\cdot\xi)\Omega_0 - [\epsilon^i + (\bar\epsilon\cdot\xi)\xi^i]\partial_i\Omega_0, \tag{4.32}$$

we derive the following transformations of the component wave functions:

$$\begin{aligned}\delta c^0 = -\epsilon^i c_i^0, \qquad \delta c_i^0 &= -2\kappa\bar\epsilon_i c^0 - \epsilon_i c_1,\\ \delta c_1 &= (2\kappa-1)\bar\epsilon^k c_k^0.\end{aligned} \tag{4.33}$$

Here, the complex conjugation rule for ϵ_i is

$$(\epsilon^i)^* = \bar\epsilon_i, \qquad (\epsilon_i)^* = -\bar\epsilon^i. \tag{4.34}$$

Next, we consider the first-level wave superfunction:

$$\Omega^{[k]}(\xi) = c^{[k]} + \xi^j c_j^{[k]} + \frac{1}{2}\xi^{j_1}\xi^{j_2}\varepsilon_{j_1j_2}c_2^{[k]}. \tag{4.35}$$

It transforms according to the rule

$$\begin{aligned}\delta\Omega^{[k]} &= (2\kappa+1)(\bar\epsilon\cdot\xi)\Omega^{[k]} + \xi^k\bar\epsilon_i\Omega^{[i]}\\ &\quad - [\epsilon^i + (\bar\epsilon\cdot\xi)\xi^i]\partial_i\Omega^{[k]},\end{aligned} \tag{4.36}$$

which implies the following component transformation laws:

$$\begin{aligned}\delta c^{[k]} &= -\epsilon^i c_i^{[k]},\\ \delta c_j^{[k]} &= -\epsilon_j c_2^{[k]} - (2\kappa+1)\bar\epsilon_j c^{[k]} + \delta_j^k\bar\epsilon_i c^{[i]},\\ \delta c_2^{[k]} &= \varepsilon^{kj}\bar\epsilon_i c_j^{[i]} + 2\kappa\bar\epsilon^j c_j^{[k]}.\end{aligned} \tag{4.37}$$

Finally, consider the second-level wave superfunction:

$$\Omega^{[k_1k_2]}(\xi) = c^{[k_1k_2]} + \xi^j c_j^{[k_1k_2]} + \xi^{j_1}\xi^{j_2}c_{j_1j_2}^{[k_1k_2]}. \tag{4.38}$$

Introducing

$$A = \varepsilon_{i_1i_2}c^{[i_1i_2]}, \qquad B = \varepsilon^{k_1k_2}\varepsilon_{i_1i_2}c_{k_1k_2}^{[i_1i_2]}, \qquad F_k = \varepsilon_{i_1i_2}c_k^{[i_1i_2]}, \tag{4.39}$$

we find

$$\begin{aligned}\delta A = -\epsilon^k F_k, \qquad \delta B &= -2\kappa\bar\epsilon^k F_k,\\ \delta F_k &= \epsilon_k B - (1+2\kappa)\bar\epsilon_k A.\end{aligned} \tag{4.40}$$

Note that these transformations can be brought precisely into the form (4.33) after redefinition $B \to -B$, $2\kappa \to 2\kappa - 1$. This means that the $\ell = 0$ and $\ell = 2$ wave superfunctions constitute isomorphic $SU(2|1)$ multiplets.

It is appropriate here to give the relevant values of the $su(2|1)$ Casimirs C_2 and C_3 defined in (3.31) and (3.33).

The eigenvalues of the Casimir operators on the LLL state are

$$C_2 = 2\kappa(2\kappa-1), \qquad C_3 = 2\kappa(2\kappa-1)(4\kappa-1). \tag{4.41}$$

On the first LL, the Casimir operators become

$$C_2 = (2\kappa+1)(2\kappa-1), \qquad C_3 = 4\kappa(2\kappa-1)(2\kappa+1). \tag{4.42}$$

Finally, their eigenvalues on the second level are

$$C_2 = 2\kappa(2\kappa+1), \quad C_3 = 2\kappa(2\kappa+1)(4\kappa+1). \tag{4.43}$$

Note that the replacement $2\kappa \to 2\kappa - 1$ in (4.43) yields just (4.41), in accord with the remark after Eqs. (4.40).

The spectrum of Casimir operators for the finite-dimensional representations of $SU(2|1)$ was studied in Refs. [12,13]. These representations are characterized by some positive number λ ("highest weight") which can be a half-integer or an integer and an arbitrary additional real number β which is related to the eigenvalues of the generator F ("baryon charge"). The values (4.41), (4.42), and (4.43) can be uniformly written in the generic form given in Ref. [12] as

$$C_2 = (\beta^2 - \lambda^2), \quad C_3 = 2\beta(\beta^2 - \lambda^2) = 2\beta C_2, \tag{4.44}$$

with

$$\underline{\text{LLL}}: \qquad \lambda = \frac{1}{2}, \quad \beta = \frac{4\kappa - 1}{2}, \tag{4.45}$$

$$\underline{\text{LLL}}: \qquad \lambda = 1, \quad \beta = 2\kappa, \tag{4.46}$$

$$\underline{\text{2nd LL}}: \qquad \lambda = \frac{1}{2}, \quad \beta = \frac{4\kappa + 1}{2}. \tag{4.47}$$

Note that our C_2 and C_3 were defined to have the opposite sign of those in Refs. [12,13] and C_3 also differs by the factor 2. We also took into account that $\kappa \geq 1/2$ in our case. The isospins and F charges of the component wave functions are expressed through the appropriate quantum numbers λ and β in full agreement with the general formulas of Ref. [12].

At $\kappa > 1/2$, we deal with what is called "typical" $SU(2|1)$ representations (both Casimirs are nonzero); at the special value $\kappa = 1/2$, both Casimirs are zero for the LLL and first LL multiplets. So in this case the latter belong to the so-called "atypical" $SU(2|1)$ representations. In accord with the consideration in Ref. [13], they are not completely irreducible: they contain invariant subspaces, the quotients over which, in turn, yield some further irreducible representations. As is seen from the transformation properties (4.33) and (4.37), at $2\kappa = 1$ the component c_1 of Ω_0 is $SU(2|1)$ singlet and the subset $(c_k^{[k]}, c_2^{[k]})$ in Ω_1 also forms a closed $SU(2|1)$ multiplet.

In the alternative $\kappa < 0$ case, the eigenvalues of the quadratic Casimir are

$$\begin{aligned} C_2 &= 2|\kappa|(2|\kappa|+1), \qquad \ell = 0\ (\ell' = 2), \\ C_2 &= (2|\kappa|-1)(2|\kappa|+1), \qquad \ell = 1\ (\ell' = 1), \\ C_2 &= 2|\kappa|(2|\kappa|-1), \qquad \ell = 2\ (\ell' = 0). \end{aligned} \tag{4.48}$$

These values are not negative for $|\kappa| \geq 1/2$, in accord with the general condition (4.20).

V. $SU(n|1)$ INVARIANT NORMS

A. General case

The $SU(n|1)$ invariant Berezin integral is defined as [7]

$$\int d\mu = \int d\mu_0 [1 - (\xi \cdot \bar{\xi})]^{n-1}, \tag{5.1}$$

where

$$\int d\mu_0 = \prod \partial_i \partial^{\bar{i}}. \tag{5.2}$$

Using this integration measure, we can define the inner product on the Hilbert space of wave superfunctions:

$$\langle \Psi | \Omega \rangle = \int d\mu \Psi^\star \Omega. \tag{5.3}$$

It is manifestly $SU(n|1)$ invariant, taking into account the invariance of the measure $d\mu$ and the fact that the weight factor in the general transformation law (4.25) is imaginary.

To express the norms in terms of the reduced chiral superfunctions $\Phi^{[k_1 \cdots k_n]}$, defined in (4.10), we will need the following rule of integration by parts for two superfunctions, Θ and Φ:

$$\int d\mu \Theta (\nabla_k^{(\kappa)} \Phi)^\star = (-1)^{P(\Theta)+P(\Phi)} \int d\mu (\nabla^{(\kappa+n-1)\bar{k}} \Theta) \Phi^\star, \tag{5.4}$$

where $P(\Theta)$ and $P(\Phi)$ are Grassmann parities of the superfunctions. We will also employ the identity

$$\nabla^{(\kappa+2)[\bar{i}} g_j^{k]} = g_j^{[k} \nabla^{(\kappa)\bar{j}]}. \tag{5.5}$$

Using these rules, together with the anticommutation relations (3.12), and the antichirality condition

$$\nabla_j^{-(\kappa)} (\Phi^{[k_1 \cdots k_n]})^* = 0,$$

one can express the norm of the full wave superfunction Ψ_ℓ for the ℓth LL in terms of the chiral wave functions as

$$||\Psi_\ell||^2 = \frac{\ell!(2\kappa - n + 2\ell - 1)!}{(2\kappa - n + \ell - 1)!} \times \int d\mu g_{i_1}^{m_1} \cdots g_{i_\ell}^{m_\ell} (\Phi^{[m_1 \cdots m_\ell]})^\star \Phi^{[i_1 \cdots i_\ell]}. \tag{5.6}$$

It is also straightforward to show that the wave superfunctions associated with different Landau levels are mutually orthogonal.

Expressing $\Phi^{[k_1 \cdots k_n]}$ through holomorphic wave functions by Eq. (4.28),

$$\Phi^{[k_1 \cdots k_n]} = (1 - \xi \cdot \bar{\xi})^{-\kappa} \Omega^{[k_1 \cdots k_n]}(\xi),$$

one can perform the ξ integration in (5.6) and obtain the norms written in terms of the coefficients in the ξ

expansion of the $\Omega^{[k_1\cdots k_n]}(\xi)$. As an illustration, we present this final form of the norm of the LLL wave superfunction, with the ξ expansion defined in (4.4):

$$||\Psi_0||^2 = (-1)^n \sum_{k=0}^{n} \binom{n-2\kappa-1}{k} k!(n-k)!\bar{c}^{0i_1\cdots i_{n-k}} c^0_{i_1\cdots i_{n-k}}. \tag{5.7}$$

In the next subsection, as an instructive example, we will give the explicit expressions for norms of all three LLL states of the $n = 2$ model.

As should be clear from the expression (5.7), there are values of κ for which the squared norms are negative; the same is true for the norms of higher LLs. This situation is typical for quantum-mechanical systems with Grassmann-odd target-space coordinates [7]. In the next section, we analyze this issue in some detail on the $n = 2$ example. The way to make all norms positive definite is to modify the inner product by introducing some metric operator on the Hilbert space (like in all other known examples of super-Landau models). This operator proves to be especially simple in the planar limit (Sec. VII).

B. Norms for the $SU(2|1)/U(2)$ model

Here we specialize to the $n = 2$ case and present the explicit expressions for the "naive" norms, using the general formulas of the previous subsection.

The norm of the vacuum wave superfunction is given by

$$||\Psi_0||^2 = \bar{c}_1 c_1 + (1-2\kappa)\bar{c}^{0i} c^0_i + 2\kappa(2\kappa-1)\bar{c}^0 c^0, \tag{5.8}$$

where the component wave functions were defined in (4.31). The first excited level is described by the wave superfunction with the norm

$$||\Psi_1||^2 = (2\kappa-1)[(2\kappa+1)(2\kappa-1)\bar{c}_{[k]}c^{[k]} + (1-2\kappa)\bar{c}^i_{[k]}c^{[k]}_i - \bar{c}^k_{[k]}c^{[i]}_i + \bar{c}_{2[k]}c^{[k]}_2]. \tag{5.9}$$

The second-level wave superfunction has the norm

$$||\Psi_2||^2 = 2\kappa(2\kappa+1)[2\kappa(2\kappa+1)\bar{A}A + \bar{B}B - 2\kappa\bar{F}^k F_k]. \tag{5.10}$$

It is straightforward to check that these norms are invariant under the transformations (4.33), (4.34), (4.35), (4.36), (4.37), (4.38), (4.39), and (4.40). Also, we observe that these norms are not positive definite. In the next section, we will see in detail how this unwanted property can be cured. At $2\kappa = 1$ there are zero norms for the LLL and the first LL wave functions. In this case, it is natural to define the physical Hilbert space as a quotient over the subspace of zero-norm states, so it is spanned by the LLL $SU(2|1)$ singlet "wave function" c_1 and four wave functions (A, B, F^k) of the second LL.

VI. UNITARY NORMS FOR $n = 2$ AND HIDDEN $SU(2|2)$ SYMMETRY

A. Redefining the inner product

All norms for the case $n = 2$ can be made positive by modifying the inner product in the Hilbert space, like in the cases worked out in [5]:

$$\langle\langle\Psi|\Phi\rangle\rangle = \int d\mu \Psi^\star G\Phi, \tag{6.1}$$

where G is a metric operator on Hilbert space. As was already mentioned in Sec. VI A, we will consider only the case $|\kappa| \geq 1/2$, because only in this case are the energies of the excited Landau levels non-negative. For the case $\kappa \geq 1/2$, we choose the metric operator to be

$$G = 1 - 4(2F + 4\kappa + \ell) + 2(2F + 4\kappa + \ell)^2. \tag{6.2}$$

It satisfies the conditions

$$G^2 = 1, \qquad [H, G] = 0, \tag{6.3}$$

which mean that the metric operator just alters the sign in front of all negative terms in the expressions for the norms of the wave superfunctions. Accordingly, the positive norms of the wave superfunctions are given by

$$\begin{aligned} ||\Psi_0||^2 &= \bar{c}_1 c_1 + (2\kappa-1)\bar{c}^{0i}c^0_i + 2\kappa(2\kappa-1)\bar{c}^0c^0, \\ ||\Psi_1||^2 &= (2\kappa-1)[(2\kappa+1)(2\kappa-1)\bar{c}_{[k]}c^{[k]} \\ &\quad + (2\kappa-1)\bar{c}^i_{[k]}c^{[k]}_i + \bar{c}^k_{[k]}c^{[i]}_i + \bar{c}_{2[k]}c^{[k]}_2], \\ ||\Psi_2||^2 &= 2\kappa(2\kappa+1)[2\kappa(2\kappa+1)\bar{A}A + \bar{B}B + 2\kappa\bar{F}^kF_k]. \end{aligned} \tag{6.4}$$

With $\kappa = 1/2$, the norms become

$$\begin{aligned} ||\Psi_0||^2 &= \bar{c}_1 c_1, \qquad ||\Psi_1||^2 = 0, \\ ||\Psi_2||^2 &= 2\kappa(2\kappa+1)[2\kappa(2\kappa+1)\bar{A}A + \bar{B}B + 2\kappa\bar{F}^kF_k]. \end{aligned} \tag{6.5}$$

As was already mentioned, the Hilbert space in this special case, obtained as a quotient over the zero-norm states, involves only two physical states, one corresponding to LLL with unbroken $SU(2|1)$ symmetry and another one corresponding to the second LL.

Now, let $\mathcal{O}$ be some operator that commutes with the Hamiltonian, and hence, generates some symmetry of the model, and let $\mathcal{O}^\dagger$ be its Hermitian conjugated operator with respect to the naive inner product (5.3). Then its Hermitian conjugate with respect to the "improved" product (6.1) is given by

$$\mathcal{O}^\ddagger \equiv G\mathcal{O}^\dagger G = \mathcal{O}^\dagger + G[\mathcal{O}^\dagger, G]. \tag{6.6}$$

To find a new conjugation for the $SU(2|1)$ generators, we need to know their realization on the analytic wave functions $\Omega(\xi)$:

$$
\begin{aligned}
\Pi &= -\partial_1,\\
\Pi^\dagger &= -\xi^1\xi^2\partial_2 + \xi^1\Big(2\kappa + \frac{\ell}{2} - \hat{B}_3\Big) - \xi^2\hat{B}_+,\\
Q &= -\partial_2,\\
Q^\dagger &= -\xi^2\xi^1\partial_1 + \xi^2\Big(2\kappa + \frac{\ell}{2} + \hat{B}_3\Big) - \xi^1\hat{B}_-,\\
J_+ &= -i\xi^1\partial_2 + i\hat{B}_+, \qquad J_- = i\xi^2\partial_1 - i\hat{B}_-,\\
J_3 &= \frac{1}{2}(\xi^1\partial_1 - \xi^2\partial_2) - \hat{B}_3,\\
F &= \frac{1}{2}(\xi^1\partial_1 + \xi^2\partial_2 - 4\kappa - \ell).
\end{aligned}
\tag{6.7}
$$

The matrix parts $\hat{B}$ of the $SU(2)$ generators satisfy, on their own, the $su(2)$ commutation relations,

$$
[\hat{B}_+, \hat{B}_-] = -2\hat{B}_3, \qquad [\hat{B}_3, \hat{B}_\pm] = \mp\hat{B}_\pm. \tag{6.8}
$$

They can take nonvanishing values only when applied to the first-level analytic wave functions $\Omega^{[k]}$ which form a doublet with respect to the external index:

$$
\begin{aligned}
&\hat{B}_+\Omega^{[2]} = \Omega^{[1]}, \qquad \hat{B}_+\Omega^{[1]} = 0, \qquad \hat{B}_-\Omega^{[1]} = \Omega^{[2]},\\
&\hat{B}_-\Omega^{[2]} = 0, \qquad \hat{B}_3\Omega^{[1]} = \frac{1}{2}\Omega^{[1]}, \qquad \hat{B}_3\Omega^{[2]} = -\frac{1}{2}\Omega^{[2]}.
\end{aligned}
\tag{6.9}
$$

In the holomorphic realization, the differential "metric" operator G defined in (6.2) takes the form

$$
\begin{aligned}
G_{\rm an} &= 1 - 2\xi^i\partial_i + 4\xi^1\xi^2\partial_2\partial_1\\
&= (1 - 2\xi^1\partial_1)(1 - 2\xi^2\partial_2).
\end{aligned}
\tag{6.10}
$$

One can check that $G_{\rm an}$ anticommutes with the supercharges and commutes with the bosonic $SU(2)\times U(1)$ generators. Owing to these properties, we find

$$
Q^\ddagger = -Q^\dagger, \qquad \Pi^\ddagger = -\Pi^\dagger. \tag{6.11}
$$

In Ref. [5], there was given a general definition of supercharge that commutes with G:

$$
\tilde{O} = O + \frac{1}{2}[G, O]G. \tag{6.12}
$$

However, in the model under consideration, it identically vanishes. So one cannot define a modified $su(2|1)$ superalgebra which would commute with the operator G. Therefore, the $su(2|1)$ transformation properties of the component wave functions are slightly changed after passing to the new Hermitian conjugation.

The odd $SU(2|1)$ transformations are now generated by

$$
\delta\Psi \equiv (\epsilon Q + \bar{\epsilon}Q^\ddagger)\Psi, \tag{6.13}
$$

giving rise to the following modified transformation properties of the component wave functions:

$$
\begin{aligned}
\delta c^0 &= -\epsilon^i c^0_i,\\
\delta c^0_i &= 2\kappa\bar{\epsilon}_i c^0 - \epsilon_i c_1,\\
\delta c_1 &= -(2\kappa - 1)\bar{\epsilon}^k c^0_k,
\end{aligned}
\tag{6.14}
$$

$$
\begin{aligned}
\delta c^{[k]} &= -\epsilon^i c^{[k]}_i,\\
\delta c^{[k]}_j &= -\epsilon_j c^{[k]}_2 + (2\kappa + 1)\bar{\epsilon}_j c^{[k]} - \delta^k_j\bar{\epsilon}_i c^{[i]},\\
\delta c^{[k]}_2 &= -\varepsilon^{kj}\bar{\epsilon}_i c^{[i]}_j - 2\kappa\bar{\epsilon}^j c^{[k]}_j,
\end{aligned}
\tag{6.15}
$$

$$
\begin{aligned}
&\delta A = -\epsilon^k F_k, \qquad \delta B = -2\kappa\bar{\epsilon}^k F_k,\\
&\delta F_k = \epsilon_k B - (1 + 2\kappa)\bar{\epsilon}_k A.
\end{aligned}
\tag{6.16}
$$

These transformations are similar to (4.33), (4.37), and (4.40), the difference being the opposite sign before the terms with $\bar{\epsilon}_i$. The modified norms (6.4) are invariant just under these transformations. The set of the group generators is now Q, Π, $Q^\ddagger$, $\Pi^\ddagger$, F, $J_\pm$, J_3. The quadratic Casimir operator for them is

$$
C_2 = -\frac{1}{2}\{J_+, J_-\} - J_3^2 + F^2 - \frac{1}{2}[Q, Q^\ddagger] - \frac{1}{2}[\Pi, \Pi^\ddagger]. \tag{6.17}
$$

It is related to the Hamiltonian by the same Eq. (3.31) and so takes the same values (4.41), (4.42), and (4.43) on the LL wave superfunctions [the same is true for the 3d order Casimir (3.33)]. In particular, for $\kappa = 1/2$, Casimir operators vanish for the LLL and the first LL superfunctions, implying that these levels carry atypical representations of $SU(2|1)$. It is worthwhile to note that, although the $su(2|1)$ algebra (3.26), (3.27), (3.28), and (3.29) changes its form after the replacement $Q^\dagger$, $\Pi^\dagger \to -Q^\ddagger$, $-\Pi^\ddagger$, the original form can be restored by passing [in the $SU(2)$ covariant notation (3.17) and (3.18)] to the new generators $\tilde{Q}_i = -\epsilon_{ik}Q^{\ddagger k}$, $\tilde{Q}^{\ddagger i} = \epsilon^{ik}Q_k$, $\tilde{F} = -F$.

B. $SU(2|2)$ symmetry

It was shown in Ref. [5] that the quantum Hilbert space of the superflag Landau model carries hidden $SU(2|2)$ symmetry which is, in a sense, an analog of the hidden worldline $\mathcal{N} = 2$ supersymmetry of the superplane Landau model [14,15]. It turns out that this phenomenon of enhancing the original $SU(2|1)$ symmetry to $SU(2|2)$ at the quantum level persists as well in the considered odd-coset super-Landau model. Below we assume that $\kappa > 1/2$, i.e., that the Casimir operators are nonvanishing for all three LLs.

Because of the anticommutation property $\{G, \Pi\} = \{G, \Pi^\dagger\} = \{G, Q\} = \{G, Q^\dagger\} = 0$, the method of defining hidden supersymmetries applied in Ref. [5] is not directly applicable to the present case. In particular, just due to this property, the supercharges commuting with G and defined by the general formula (6.12) are identically vanishing. Yet, we can achieve our goal, though in a distinct way.

We define

$$\Pi_G \equiv \frac{1}{2}[\Pi, G] = \Pi G, \qquad \Pi^\ddagger_G \equiv -\Pi^\ddagger G,$$
$$Q_G \equiv \frac{1}{2}[Q, G] = QG, \qquad Q^\ddagger_G \equiv -Q^\ddagger G. \tag{6.18}$$

These operators can be used to generate the second $SU(2)$ algebra[5] as

$$\mathcal{J}_- \equiv \frac{1}{2\sqrt{C_2}}\{Q, \Pi_G\} = -\frac{1}{2\sqrt{C_2}}\{\Pi, Q_G\} = \frac{1}{\sqrt{C_2}} Q\Pi G,$$
$$\mathcal{J}_+ \equiv \frac{1}{2\sqrt{C_2}}\{\Pi^\ddagger_G, Q^\ddagger\} = -\frac{1}{2\sqrt{C_2}}\{Q^\ddagger_G, \Pi^\ddagger\}$$
$$= \frac{1}{\sqrt{C_2}}\Pi^\ddagger Q^\ddagger G, \qquad \mathcal{J}_3 = \frac{1}{2}[\mathcal{J}_+, \mathcal{J}_-]. \tag{6.19}$$

The Casimir operator C_2 was defined in (3.31). Thus, we have two distinct $SU(2)$ algebras

$$[J_+, J_-] = 2J_3, \qquad [J_3, J_\pm] = \pm J_\pm,$$
$$[\mathcal{J}_+, \mathcal{J}_-] = 2\mathcal{J}_3, \qquad [\mathcal{J}_3, \mathcal{J}_\pm] = \pm\mathcal{J}_\pm, \tag{6.20}$$

which can be checked to commute with each other. Explicitly, in the holomorphic realization, these generators read

$$J_+ = J^1_2 = \xi^1\partial_2 - \hat{B}_+, \qquad J_- = J^2_1 = \xi^2\partial_1 - \hat{B}_-,$$
$$J_3 = J^1_1 = -J^2_2 = \frac{1}{2}(\xi^1\partial_1 - \xi^2\partial_2) - \hat{B}_3, \tag{6.21}$$

$$\mathcal{J}_+ = \mathcal{J}^2_1 = \sqrt{C_2}\xi^1\xi^2, \qquad \mathcal{J}_- = \mathcal{J}^1_2 = \frac{1}{\sqrt{C_2}}\partial_2\partial_1,$$
$$\mathcal{J}_3 = \mathcal{J}^1_1 = -\mathcal{J}^2_2 = \frac{1}{2}(\xi^1\partial_1 + \xi^2\partial_2 - 1). \tag{6.22}$$

Remember that the $SU(2)$ matrix operators $\hat{B}$ are nonvanishing only at the first LL ($\ell = 1$) and are zero for other levels. Then we can define the supercharges $S^a_i (i = 1, 2; a = 1, 2)$ which are doublets with respect to either of two $SU(2)$ groups:

$$S^1_1 = \Pi, \qquad S^2_1 \equiv [\Pi, \mathcal{J}_+],$$
$$S^1_2 = Q, \qquad S^2_2 \equiv [Q, \mathcal{J}_+], \tag{6.23}$$

$$\bar{S}^i_a := (S^a_i)^\ddagger. \tag{6.24}$$

Explicitly, these supercharges are

$$S^1_1 = -\partial_1, \qquad S^2_1 = -\sqrt{C_2}\xi^2,$$
$$S^1_2 = -\partial_2, \qquad S^2_2 = +\sqrt{C_2}\xi^1, \tag{6.25}$$

$$\bar{S}^1_1 = \xi^1\xi^2\partial_2 - \xi^1\left(2\kappa + \frac{\ell}{2} - \hat{B}_3\right) + \xi^2\hat{B}_+,$$
$$\bar{S}^2_1 = \xi^2\xi^1\partial_1 - \xi^2\left(2\kappa + \frac{\ell}{2} + \hat{B}_3\right) + \xi^1\hat{B}_-,$$
$$\sqrt{C_2}\bar{S}^1_2 = +(1 - \xi^1\partial_1)\partial_2 - \left(2\kappa + \frac{\ell}{2} - \hat{B}_3\right)\partial_2 - \hat{B}_+\partial_1,$$
$$\sqrt{C_2}\bar{S}^2_2 = -(1 - \xi^2\partial_2)\partial_1 + \left(2\kappa + \frac{\ell}{2} + \hat{B}_3\right)\partial_1 + \hat{B}_-\partial_2. \tag{6.26}$$

It is straightforward to check that they satisfy the (anti) commutation relations of the superalgebra $su(2|2)$ with three central charges:

$$\{S^a_i, \bar{S}^j_b\} = \delta^a_b J^j_i - \delta^j_i \mathcal{J}^a_b + \delta^a_b\delta^j_i\left(2\kappa + \frac{\ell}{2} - \frac{1}{2}\right),$$
$$\{S^a_i, S^b_j\} = \varepsilon_{ij}\varepsilon^{ab}\sqrt{C_2},$$
$$\{\bar{S}^i_a, \bar{S}^j_b\} = \varepsilon^{ij}\varepsilon_{ab}\sqrt{C_2}, \tag{6.27}$$
$$[S^a_i, \mathcal{J}^c_b] = \delta^a_b S^c_i - \frac{1}{2}\delta^c_b S^a_i,$$
$$[S^a_i, J^j_k] = \delta^j_i S^a_k - \frac{1}{2}\delta^j_k S^a_i.$$

As an example, let us give how this $SU(2|2)$ symmetry is realized on the LLL (i.e., $\ell = 0$) wave functions. We denote θ^i, $\bar{\theta}_i$ as the parameters associated with the extra pair of fermionic generators, i.e., with S^2_i, $\bar{S}^i_2$. Then the additional transformations are

$$\delta c^0 = \frac{\sqrt{2\kappa - 1}}{\sqrt{2\kappa}}\bar{\theta}^i c^0_i, \qquad \delta c_1 = -\sqrt{(2\kappa - 1)(2\kappa)}\theta^k c^0_k,$$
$$\delta c^0_i = -\frac{\sqrt{2\kappa}}{\sqrt{2\kappa - 1}}\bar{\theta}_i c_1 - \sqrt{(2\kappa - 1)(2\kappa)}\theta_i c^0. \tag{6.28}$$

It is easy to check that they leave invariant the norm $||\Psi_0||$ in (6.4). Their bracket with the $SU(2|1)$ transformations (6.14) produces the second $SU(2)$, with respect to which c^0 and c_1 form a doublet. The remaining components c^0_i are singlets of the second $SU(2)$. It is also easy to find the realization of the θ transformations on the $\ell = 1$ and $\ell = 2$ wave function multiplets.

According to Ref. [16] (see also Ref. [17]), the central charges can be combined into the three-vector $\vec{C}$ as

$$\vec{C}(\ell) = (2\kappa + \ell/2 - 1/2, \sqrt{C_2(\ell)}, \sqrt{C_2(\ell)}) \equiv (C, P, K). \tag{6.29}$$

[5]Due to the anticommutativity of the metric operator G with Π and Q (and its commutativity with bosonic generators) the operators Π_G, $\Pi^\ddagger_G$ and Q_G, $Q^\ddagger_G$ form the same $su(2|1)$ algebra as Π, $\Pi^\ddagger$ and Q, $Q^\ddagger$ themselves, but they do not produce any obvious closed structure together with the latter. This feature is in contrast with the construction in Ref. [5], where just the generators Π_G, $\Pi^\ddagger_G$ and Q_G, $Q^\ddagger_G$ extend $su(2|1)$ to $su(2|2)$.

The norm of $\vec{C}$ defined as $\vec{C}^2 = C^2 - PK$ is invariant under the $so(1,2)$ outer automorphisms of the $su(2|2)$ superalgebra (6.27) [16]. Exploiting this $so(1,2)$ freedom, one can cast $\vec{C}$ in the form

$$\vec{C} = (Z, 0, 0), \tag{6.30}$$

where

$$Z = \frac{1}{2} \quad \text{for } \ell = 0, 2, \quad \text{and} \quad Z = 1 \quad \text{for } \ell = 1. \tag{6.31}$$

Explicitly, the $so(1,2)$ rotated supercharges for all three levels are

$$\begin{aligned} \tilde{S}^a_i &= \sqrt{2\kappa} S^a_i - \sqrt{2\kappa - 1}\varepsilon^{ab}\varepsilon_{ij}\bar{S}^j_b, & \ell &= 0, \\ \tilde{S}^a_i &= \sqrt{2\kappa + 1} S^a_i - \sqrt{2\kappa}\varepsilon^{ab}\varepsilon_{ij}\bar{S}^j_b, & \ell &= 2, \\ \tilde{S}^a_i &= \frac{1}{\sqrt{2}}[\sqrt{2\kappa + 1} S^a_i - \sqrt{2\kappa - 1}\varepsilon^{ab}\varepsilon_{ij}\bar{S}^j_b], & \ell &= 1. \end{aligned} \tag{6.32}$$

In the new frame (6.30) and (6.32) the (anti)commutation relations of $su(2|2)$ become

$$\begin{aligned} &\{\tilde{S}^a_i, \tilde{\bar{S}}^j_b\} = \delta^a_b J^j_i - \delta^j_i \mathcal{J}^a_b + Z\delta^a_b\delta^j_i, \qquad \{\tilde{S}^a_i, \tilde{S}^b_j\} = 0, \\ &\{\tilde{\bar{S}}^i_a, \tilde{\bar{S}}^j_b\} = 0, \qquad [\tilde{S}^a_i, \mathcal{J}^c_b] = \delta^a_b \tilde{S}^c_i - \frac{1}{2}\delta^c_b \tilde{S}^a_i, \\ &[\tilde{S}^a_i, J^j_k] = \delta^j_i \tilde{S}^a_k - \frac{1}{2}\delta^j_k \tilde{S}^a_i. \end{aligned} \tag{6.33}$$

With respect to the redefined $su(2|2)$ generators, the transformations of the LLL supermultiplet take the form

$$\begin{aligned} \delta c^0 &= -\frac{\rho^i c^0_i}{\sqrt{2\kappa}}, \\ \delta c^0_i &= \sqrt{2\kappa}\bar{\rho}_i c^0 - \frac{\omega_i c_1}{\sqrt{2\kappa - 1}}, \\ \delta c_1 &= -\sqrt{2\kappa - 1}\bar{\omega}^k c^0_k, \end{aligned} \tag{6.34}$$

where $\bar{\omega}^i$, ω_i and ρ^i, $\bar{\rho}_i$ are the parameters associated with the supercharges $\tilde{S}^1_i$, $\tilde{\bar{S}}^i_1$ and $\tilde{S}^2_i$, $\tilde{\bar{S}}^i_2$.

As the final topic of this section, we indicate which $SU(2|2)$ multiplets the wave functions form for different values of ℓ.

In general, irreps of $SU(2|2)$ are characterized by the triple [16]

$$\langle m, n; \vec{C}\rangle, \tag{6.35}$$

where the non-negative integers m, n are Dynkin labels of subalgebra $su(2) \oplus su(2)$ (in the considered case, they are twice the external isospins of the wave superfunctions) and $\vec{C}$ represents three central charges. The special case is the "short" irreps, with

$$\langle m, n; \vec{C}\rangle, \qquad \vec{C}^2 = \frac{1}{4}(m + n + 1)^2. \tag{6.36}$$

It turns out that our wave superfunction multiplets belong just to this restricted class of the $su(2|2)$ representations.

At the levels $\ell = 0$ and $\ell = 2$, the $su(2|2)$ algebra has the central charge $Z = 1/2$. Hence, the relevant wave superfunctions encompass the $2|2$ multiplets characterized by the triple

$$\langle 0, 0; \vec{C}\rangle, \qquad \vec{C}^2 = Z^2 = \frac{1}{4}. \tag{6.37}$$

This option corresponds to the fundamental representations of $su(2|2)$.

At the level $\ell = 1$, the wave superfunction multiplet comprises four bosonic and four fermionic components, with the central charge $Z = 1$. It is characterized by the triple

$$\langle 1, 0; \vec{C}\rangle, \qquad \vec{C}^2 = Z^2 = 1. \tag{6.38}$$

With respect to the bosonic subalgebra $su(2) \oplus su(2)$, the bosonic components of the $\ell = 1$ supermultiplet are in $(0,0) \oplus (1,0)$, while the fermionic ones are in $(1/2, 1/2)$. This means that, with respect to the first $SU(2)$, the bosonic fields are split into the singlet c_+ and the triplet $c^{[2]}_1$, $c^{[1]}_2$ c_-, where

$$c_+ = \frac{1}{2}(c^{[1]}_1 + c^{[2]}_2), \qquad c_- = \frac{1}{2}(c^{[1]}_1 - c^{[2]}_2).$$

With respect to the second $SU(2)$, all of these components are just singlets. The fermionic components constitute doublets with respect to both $SU(2)$ groups. To be more precise, either of $c^{[k]}$ and $c^{[k]}_2$ is a doublet of the first $SU(2)$ acting on the index $[k]$, while another $SU(2)$ combines them both into its doublet (irrespective of the value of $[k]$).

Let us briefly address the case $\kappa < 0$. With the condition $\kappa < -1/2$ taken into account, the norms (5.8) and (5.10) are positive, while the first-level norm (5.9) is negative (we leave aside the degenerate case $|\kappa| = 1/2$). The metric operator chosen in the form

$$G = 1 + \frac{8\kappa^2 - 2H^2}{1 - 4\kappa^2} \tag{6.39}$$

has the eigenvalues

$$G = (-1)^\ell, \tag{6.40}$$

and so makes all norms positive definite. The implications of this metric operator radically differ from those of G for $\kappa > 1/2$. In particular, it commutes with all symmetry generators and so does not affect their Hermitian conjugation properties. Thus, as opposed to the $\kappa > 0$ case, it cannot be directly employed for construction of the $SU(2|2)$ generators acting in the relevant Hilbert space. Nevertheless, the unitary norms for $\kappa < 0$ are still invariant under the appropriately defined $SU(2|2)$ symmetry.

The corresponding generators are obtained by making the substitutions

$$\kappa = |\kappa|, \quad \ell = n - \ell', \quad S_i^a = \Pi_i^a, \quad \bar{S}_a^i = -\bar{\Pi}_a^i \tag{6.41}$$

in the definitions (6.25) and (6.26). The $su(2|2)$ superalgebra which these generators form is slightly different from the one defined by the relations (6.27) pertinent to the $\kappa > 0$ case. In the basic anticommutator, the two sets of the bosonic $SU(2)$ generators, J_i^J and $\mathcal{J}_b^a$, switch their places, and in the central charge one should replace $\kappa \to |\kappa|$, $\ell \to \ell'$. The three-vector $\vec{C}$ defined in (6.29) preserves its form modulo these substitutions.

A different treatment of the $\kappa < 0$ case is based on the consideration in the end of Sec. IVA. The LLL wave superfunction is defined as $\tilde{\Psi}_{\ell'=0} := \Psi_{\ell=2}$. It satisfies the antichirality condition

$$\nabla_j^{(\kappa)}\tilde{\Psi}_{\ell'=0} = \nabla_j^{(\kappa)}\nabla_i^{(\kappa-1)}\nabla_k^{(\kappa+1)}\Phi^{[ik]} = 0 \to \tilde{\Psi}_{\ell'=0} = (1 - \xi\cdot\bar{\xi})^{-|\kappa|}\bar{\tilde{\Omega}}_0(\bar{\xi}). \tag{6.42}$$

Passing to the complex conjugate set of the wave superfunctions, i.e., $\tilde{\Psi}_{\ell'} \to \tilde{\Psi}^\star_{\ell'}$, $\tilde{\Psi}^\star_{\ell'=0} = (1-\xi\cdot\bar{\xi})^{-|\kappa|}\tilde{\Omega}_0(\xi)$, reduces the $\kappa < 0$ case to the already studied $\kappa > 0$ one. The relevant $SU(2|1)$ generators are obtained just through the replacement $\kappa \to |\kappa|$ in the $\kappa > 0$ expressions, and the passive form of the supertransformation of the wave superfunctions mimics (4.25)

$$\delta^*\tilde{\Psi}^\star(\xi,\bar{\xi}) = |\kappa|(\epsilon\cdot\bar{\xi} + \bar{\epsilon}\cdot\xi)\tilde{\Psi}^\star(\xi,\bar{\xi}). \tag{6.43}$$

Thus, the structure of the quantum Hilbert space in the $\kappa < 0$ case is the same as for $\kappa > 0$. For the "physical" values $\kappa < -1/2$, the metric operator G making all norms positive definite is of the same form as in (6.2), up to the substitutions $\kappa \to |\kappa|$ and $\ell \to \ell'$. The hidden $SU(2|2)$ symmetry generators leaving invariant the unitary norms are obtained from (6.25) and (6.26) through the same substitutions.

VII. THE PLANAR LIMIT

In this section, we consider the planar limit of the $SU(n|1)/U(n)$ coset model.

We introduce a scale parameter r, rescale the odd variables and the time coordinate in the action corresponding to the Lagrangian (2.14) as

$$\xi \to \xi/r, \qquad t \to t/r^2, \qquad \kappa \to \kappa r^2, \tag{7.1}$$

and then send $r \to \infty$. The Lagrangian (2.14) goes over to

$$L = -\dot{\xi}^i\dot{\bar{\xi}}_i + i\kappa(\dot{\xi}^i\bar{\xi}_i + \dot{\bar{\xi}}_i\xi^i). \tag{7.2}$$

This is equivalent to the following redefinition of the Kähler superpotential:

$$K = r^2\log\left(1 - \frac{\xi\cdot\bar{\xi}}{r^2}\right). \tag{7.3}$$

In the planar limit K becomes $\xi\cdot\bar{\xi}$, and the target metric and gauge connections read

$$g_j^i = \partial_j\partial^{\bar{i}}K = \delta_j^i, \tag{7.4}$$

$$A_i = -i\partial_i K = i\bar{\xi}_i, \qquad \bar{A}^i = i\partial^{\bar{i}}K = i\xi^i. \tag{7.5}$$

The Hamiltonian is rescaled as

$$H(\xi,\bar{\xi},\kappa) \to 1/r^2 H(\xi/r, \bar{\xi}/r, \kappa r^2) \tag{7.6}$$

and in the planar limit becomes

$$H = \nabla_i^{(\kappa)}\nabla^{(\kappa)\bar{i}} - \kappa n := H' - \kappa n, \tag{7.7}$$

where now

$$\nabla_i^{(\kappa)} = \partial_i + \kappa\bar{\xi}_i, \qquad \nabla^{(\kappa)\bar{i}} = \partial^{\bar{i}} + \kappa\xi^i. \tag{7.8}$$

The covariant derivatives obey the anticommutation relation

$$\{\nabla_i^{(\kappa)}, \nabla^{(\kappa)\bar{j}}\} = 2\kappa\delta_i^j. \tag{7.9}$$

The Lagrangian (7.2) enjoys the symmetry under odd magnetic translations with the generators

$$Q_i = -\partial_i + \kappa\bar{\xi}_i, \qquad Q^{\dagger i} = -\partial^{\bar{i}} + \kappa\xi^i,$$
$$\{Q_i, Q^{\dagger k}\} = -2\kappa\delta_i^k, \qquad \{Q_i, Q_k\} = 0, \tag{7.10}$$

as well as a symmetry under $U(n)$ rotations of the coordinates ξ^i, which define "R symmetry" of the superalgebra (7.10). The full symmetry structure is an obvious contraction of the superalgebra $su(n|1)$. The spinor generators form just an extended $\mathcal{N} = n$, $d = 1$ Poincaré superalgebra, with the central charge -2κ in place of the Hamiltonian appearing in the standard extended supersymmetric mechanics models (see, e.g., Ref. [18]). The Hamiltonian (7.7) admits the Sugawara-type representation

$$H = Q_i Q^{\dagger i} - 2\kappa F + \kappa n, \tag{7.11}$$

where $F = \xi^i\partial_i - \bar{\xi}_i\partial^{\bar{i}}$ is the $U(1)$ generator. After putting $\xi^{\hat{i}} = 0$, $\hat{i} = 1,\ldots n-1$, and replacing $\kappa \to -\kappa$, for the remaining fermionic variable ξ^n, the Lagrangian (7.2) is reduced to that of the "toy" fermionic Landau model of Refs. [14,19]. Also note that (7.2), up to some redefinitions, in the $n = 2$ case, coincides with the pure fermionic truncation of the Lagrangian of $\mathcal{N} = 4$ super-Landau model recently constructed in Ref. [20].

Let us sketch some salient features of the quantum theory. For simplicity, we will mainly concentrate on the case with $\kappa > 0$.

The ground state wave superfunction is defined by the chirality condition $\nabla^{(\kappa)\bar{i}}\Psi_0 = 0$, which amounts to expressing this superfunction through the holomorphic function

$$\Psi_0 = e^{\kappa\xi\cdot\bar{\xi}}\Omega_0(\xi). \tag{7.12}$$

The higher LL wave superfunctions are obtained as

$$\Psi_\ell = \nabla^{(\kappa)}_{i_1} \cdots \nabla^{(\kappa)}_{i_\ell} \Phi^{[i_1\cdots i_\ell]}, \quad \Phi^{[i_1\cdots i_\ell]} = e^{\kappa\xi\cdot\bar\xi}\Omega^{[i_1\cdots i_\ell]}(\xi), \tag{7.13}$$

with the energy of the ℓth Landau level being $E_\ell = 2\kappa\ell$. This formula for E_ℓ can be directly reproduced from (4.12), making rescaling (7.6) in (4.12) and taking the planar limit $r \to \infty$. Note that the exponential prefactors in (7.12) and (7.13) can be regarded as the $r \to \infty$ limit of the prefactor in (4.3), (4.6), and (4.11): $(1 - r^{-2}\xi\cdot\bar\xi)^{-\kappa r^2} \to e^{\kappa\xi\cdot\bar\xi}$.

The invariant norm is defined by

$$||\Psi||^2 = \int d\mu_0 \Psi^\star \Psi, \tag{7.14}$$

where $\int d\mu_0 = \prod \partial_i \partial^{\bar i}$. Using the anticommutation relation (7.9), it is straightforward to compute the norm of the ℓth-level wave superfunction:

$$||\Psi_\ell||^2 = \ell!(2\kappa)^\ell \int d\mu_0 \Phi^\star_{[i_1\cdots i_\ell]}\Phi^{[i_1\cdots i_\ell]} = \ell!(2\kappa)^\ell \int d\mu_0 e^{2\kappa\xi\cdot\bar\xi}\Omega^\star_{[i_1\cdots i_\ell]}\Omega^{[i_1\cdots i_\ell]}. \tag{7.15}$$

This integral is obviously not positive definite, for both $\kappa > 0$ and $\kappa < 0$. To make all norms positive, we should redefine the inner product:

$$\langle\langle\Psi|\Phi\rangle\rangle = \int d\mu \Psi^\star G \Phi. \tag{7.16}$$

The metric operator G is defined as

$$(1)\ G = (1 - 2\hat m_1)\ldots(1 - 2\hat m_n), \quad \text{for } \kappa > 0, \tag{7.17}$$

$$(2)\ G = (1 - 2\hat n_1)\ldots(1 - 2\hat n_n), \quad \text{for } \kappa < 0. \tag{7.18}$$

Here

$$\hat m_i = \frac{\nabla^{(\kappa)}_i \nabla^{(\kappa)\bar i}}{2\kappa} + (\xi^i\partial_i - \bar\xi_i\partial^{\bar i}), \qquad \hat n_i = -\frac{\nabla^{(\kappa)}_i \nabla^{(\kappa)\bar i}}{2|\kappa|}, \tag{7.19}$$

and no summation over the index i is assumed.

The operator G anticommutes with the supercharges for $\kappa > 0$,

$$\{Q_i, G\} = 0, \qquad \{Q^{\dagger i}, G\} = 0, \tag{7.20}$$

and commutes with them for $\kappa < 0$. Note that, within our conventions, just the latter option corresponds to the toy $SU(1|1)$ invariant fermionic Landau model considered in Refs. [14,19]. The relevant metric operator is the $n = 1$ case of (7.18). The alternative choice of κ, with the metric operator being the $n = 1$ case of (7.17), was not addressed in these papers.

Though in the notation (7.17) and (7.18) covariance with respect to the $U(n)$ R symmetry is not manifest, one can check that the operators G commute with $U(n)$ generators. For instance, in the $n = 2$ case and for $\kappa < 0$, the metric operator G can be rewritten in the manifestly $U(2)$ covariant form as $G = 1 + \frac{1}{2}|\kappa|^{-2}H'^2 + 2|\kappa|^{-1}H'$. This operator can be obtained as the planar $r \to \infty$ limit of the metric operator (6.39), in which ξ_i and κ are rescaled according to (7.1). Upon truncation to $n = 1$, we find that $H'^2 \to -2|\kappa|H'$ and $G \to 1 + |\kappa|^{-1}H'$. This G is the metric operator of the fermionic model of Refs. [14,19].

Finally, let us illustrate the general consideration by the example of the positive norm for the LLL wave function in the $n = 2$ case with $\kappa > 0$:

$$||\Psi_0||^2 = \bar c_1 c_1 + 2\kappa\bar c^{0i}c^0_i + (2\kappa)^2\bar c^0 c^0. \tag{7.21}$$

The coefficients in the ξ^i expansion of the holomorphic LLL superfunction $\Omega_0(\xi)$ were defined precisely as in (4.31).

A. Symmetries of the quantum planar $n = 2$ model

The superplane Landau models obtained as a large radius limit of the $SU(2|1)/U(1|1)$ supersphere or the $SU(2|1)/[U(1) \times U(1)]$ superflag Landau models respect the worldline $\mathcal{N} = 2$ supersymmetry which emerges as a contraction of the appropriate extensions of $SU(2|1)$, realized in the Hilbert space of the quantum curved models [5].

In the planar limit, no worldline supersymmetry appears in the $n = 2$ odd-coset model. To see what kind of algebraic structure is recovered in this limit from the $su(2|2)$ generators defined in Sec. VI B, we should make, in all odd generators (6.25) and (6.26), the rescaling of ξ, $\bar\xi$, κ according to (7.1) and multiply them by the additional factor $1/r$, while no such factor is needed in the case of the bosonic generators (6.21) and (6.22) (it is enough to make the rescaling of the odd variables and κ in them). Keeping in mind that, to the leading order, $\sqrt{C_2} = 2\kappa r^2 + \cdots$ for any level ℓ, it is easy to check that in the limit $r \to \infty$, all odd $SU(2|2)$ generators become either ∂_i or $2\kappa\xi^i$ (modulo signs), which are just the generators Q_i or $Q^{\dagger i}$ in the holomorphic representation, with the only nonvanishing anticommutator $\{Q, Q^\dagger\}$, as is given in (7.10) (for $n = 2$). Thus, no new fermionic generators appear in this limit. The bosonic generators (6.21) retain their form and define the $SU(2)$ R-symmetry group of the flat $n = 2$ superalgebra (7.10). The extra $SU(2)$ generators (6.22) also retain their form, modulo the substitution $\sqrt{C_2} \to 2\kappa$. They generate the second R-symmetry $SU(2)$ group of the flat $n = 2$ superalgebra, such that the generators Q_i and $Q^{\dagger i}$ are mixed under this $SU(2)$.

Surprisingly, it is still possible, at least for the choice of $\kappa > 0$, to show that the space of quantum states of the $n = 2$ fermionic planar Landau model exhibits $SU(2|2)$ symmetry.

For $\kappa > 0$, the Hermitian conjugates of the supercharges Q_i are given by

$$Q^{\ddagger i} \equiv G Q^{\dagger i} G = -Q^{\dagger i}, \qquad i = 1, 2. \tag{7.22}$$

The metric operator can be used to construct another pair of the supercharges:

$$(Q_G)_i \equiv \frac{1}{2}[Q_i, G] = Q_i G = -G Q_i. \qquad (7.23)$$

These two pairs can be combined into the complex quartet supercharges $\hat{S}^a_i$ as

$$\begin{aligned} \hat{S}^1_1 &= \frac{1}{2}(2\kappa)^{-\frac{1}{2}}(1+G)Q_1, \\ \hat{S}^2_1 &= -\frac{1}{2}(2\kappa)^{-\frac{1}{2}}(1+G)Q^{\ddagger 2}, \\ \hat{S}^1_2 &= \frac{1}{2}(2\kappa)^{-\frac{1}{2}}(1+G)Q_2, \\ \hat{S}^2_2 &= \frac{1}{2}(2\kappa)^{-\frac{1}{2}}(1+G)Q^{\ddagger 1}. \end{aligned} \qquad (7.24)$$

These supercharges together with their conjugates $\hat{S}^{\ddagger i}_a$ satisfy just the (anti)commutation relations (6.33) of the superalgebra $su(2|2)$, with $Z = 1/2$ and the $SU(2)$ generators $(\hat{J}_\pm, \hat{J}_3)$, $(\hat{\mathcal{J}}_\pm, \hat{\mathcal{J}}_3)$ defined as

$$\begin{aligned} &\hat{J}_+ = (2\kappa)^{-1}Q^{\ddagger 1}Q_2, \qquad \hat{J}_- = (2\kappa)^{-1}Q^{\ddagger 2}Q_1, \\ &\hat{J}_3 = \frac{1}{2}(\hat{m}_1 - \hat{m}_2), \qquad \hat{\mathcal{J}}_+ = (2\kappa)^{-1}Q^{\ddagger 1}Q^{\ddagger 2}, \\ &\hat{\mathcal{J}}_- = (2\kappa)^{-1}Q_2 Q_1, \qquad \hat{\mathcal{J}}_3 = -\frac{1}{2}(1 - \hat{m}_1 - \hat{m}_2). \end{aligned} \qquad (7.25)$$

It is instructive to rewrite the first set of the $SU(2)$ generators in the covariant form

$$\hat{J}^i_j = \frac{1}{4\kappa}\left([Q^{\ddagger i}, Q_j] - \frac{1}{2}\delta^i_j[Q^{\ddagger k}, Q_k]\right), \qquad \hat{J}_+ = \hat{J}^1_2, \\ \hat{J}_- = \hat{J}^2_1, \qquad \hat{J}_3 = \hat{J}^1_1 = -\hat{J}^2_2. \qquad (7.26)$$

We see that they do not coincide with the generators of the R-symmetry $SU(2)$: in the holomorphic basis, with $Q_i = -\partial_i$, $Q^{\ddagger i} = -2\kappa\xi^i$, one has

$$\hat{J}^i_j = \xi^i\partial_j - \frac{1}{2}\delta^i_j\xi^k\partial_k, \qquad (7.27)$$

which should be compared with the R-symmetry $SU(2)$ generators (6.21) in the same basis (their form is preserved upon passing to the planar limit). The difference is the absence of the matrix parts $\hat{B}$ in (7.27) for any level ℓ as compared to the generators (6.21) which necessarily include the $\hat{B}$ parts for $\ell = 1$. Nevertheless, the positive norms are invariant under both of these $SU(2)$ symmetries separately. The R-symmetry $SU(2)$ can be interpreted as one of the outer automorphisms of the considered $su(2|2)$ superalgebra: it uniformly rotates the doublet indices i, j, k of the fermionic generators $\hat{S}^a_i$ and the bosonic generators $\hat{J}^i_j$.

As for the second set of $SU(2)$ generators in (7.25), they coincide with those of the second R-symmetry group $SU(2)$ and have, in the holomorphic basis, the same form as in (6.22), up to the substitution $\sqrt{C_2} \to 2\kappa$.

Note that the pair of generators $(Q_G)_i = Q_i G$, $(Q_G)^{\ddagger i} = -Q^{\ddagger i}G$ form the same flat algebra $\{Q_G, Q_G^\ddagger\} = 2\kappa$ as the pair Q_i, $Q^{\ddagger i}$. Their crossing anticommutators produce the $su(2)$ generators (7.25). The fermionic generators Q, Q_G and their conjugates just fix another basis in the odd sector (7.24) of the $su(2|2)$ superalgebra (this basis was employed, e.g., in Ref. [20]).

The LLL and the second LL wave superfunctions belong to the short representation of this $su(2|2)$

$$\langle 0, 0; \vec{C}\rangle, \qquad (7.28)$$

with $\vec{C} = (1/2, 0, 0)$. The first level is described by the direct sum of such representations:

$$\langle 0, 0; \vec{C}\rangle \oplus \langle 0, 0; \vec{C}\rangle. \qquad (7.29)$$

VIII. SUMMARY AND OUTLOOK

In this paper, we continued the study of super-Landau models associated with the supergroup $SU(n|1)$. This study was initiated in Refs. [4,7] for an arbitrary n and further performed in more detail for $n = 2$ in Ref. [5,6]. We constructed the model on the pure odd coset $SU(n|1)/U(n)$ as a generalization of the consideration of Ref. [4], which dealt only with the lowest Landau level sector of such a model. An important peculiarity of this model is the finite number $n + 1$ of Landau levels in the sector spanned by the wave superfunctions with the vanishing external $SU(n)$ "spin." We presented the action of the model, as well as its quantum Hamiltonian, for an arbitrary n, found the energy spectrum, and defined the relevant wave superfunctions. For the particular case of $n = 2$, we showed that the space of quantum states of the model reveals hidden $SU(2|2)$ symmetry: at each Landau level $\ell = 0, 1, 2$, the relevant wave superfunctions constitute "short" $SU(2|2)$ multiplets. Like in other super-Landau models, the Hilbert space for any n includes the states the norms of which are not positive definite; for $n = 2$, we gave the explicit form of the metric operator which modifies the inner product in such a way that the norms of all states become non-negative, thus demonstrating that the model is unitary. Such operator is expected to exist for any n. We also studied the planar limit of the $SU(n|1)/U(n)$ models. In this limit, the superalgebra $su(n|1)$ converts into a superalgebra which is isomorphic to n-extended $d = 1$ Poincaré superalgebra, with a central charge playing the role of the $d = 1$ Hamiltonian. We presented, for an arbitrary n, the explicit form of the metric operator ensuring the norms of all quantum states to be non-negative. For $n = 2$, we find out that the Hilbert space for all three levels carries a dynamical hidden $SU(2|2)$ symmetry, like in the $SU(2|1)$ model, though these two $SU(2|2)$ symmetries are realized in entirely different ways.

Our consideration shows that the appearance of the hidden $SU(2|2)$ symmetries in the superflag and supersphere Landau models based on the supercosets $SU(2|1)/[U(1) \times U(1)]$ and $SU(2|1)/U(1|1)$ [5] is not accidental: the same phenomenon persists in the Landau model on the pure odd supercoset $SU(2|1)/U(2)$. Moreover, the planar limit of this model also surprisingly exhibits $SU(2|2)$ symmetry on the quantum states (as a substitute of the worldline $\mathcal{N} = 2$ supersymmetry of the planar limits of the superflag and supersphere models [14,15]). It would be interesting to inquire whether the Hilbert spaces of the odd-coset Landau models with $n > 2$ and their planar limits admit any extended hidden symmetry.

The presence of hidden $SU(2|2)$ symmetry in the quantum $SU(2|1)$ Landau models [or some other symmetries of the similar kind in the case of Landau models based on further extensions of $SU(2|1)$] suggests the possible relation of this class of models to such integrable systems as the $su(2|2)$ or $su(3|2)$ spin chain models and, finally, to $\mathcal{N} = 4$, $d = 4$ super-Yang-Mills theory and string theory (see, e.g., Refs. [10,16,17,21]). It would be also interesting to retrieve the super-Landau models via dimensional reduction from some higher-dimensional sigma models with the supergroup target spaces.

Regarding possible physical applications of the odd-coset Landau models, we would like to express a hope that the latter (or their analogs associated with other supergroups) could be relevant to the description of the quantum Hall effect and its spin extensions [22–24]. In a sense, the quantum fermionic Landau model could be regarded as a sort of "parent model" for those associated with other $SU(n|1)$ supercosets. This is based on the following reasonings. The maximal linearly realized symmetry of the $SU(n|1)/U(n)$ model is $U(n)$, and we could couple the fermionic coset variables to some $U(n)$ "matter" multiplets while preserving the full nonlinear $SU(n|1)$ symmetry, following the general recipes of the nonlinear realizations theory. With adding the proper $U(n)$ [and $SU(n|1)$] invariant potentials to the action, we could hope to trigger the spontaneous breaking of $U(n)$ down to a smaller symmetry $H \subset U(n)$, so that the original fermionic variables, together with the bosonic $U(n)/H$ ones, parametrize some "mixed" fermionic-bosonic supercosets $SU(n|1)/H$. For instance, the superflag $SU(2|1)/[U(1) \times U(1)]$ model could be recovered through the spontaneous breaking pattern $U(2) \to H = U(1) \times U(1)$. The presence of additional bosonic "sigma fields" in such extended models as compared to the pure coset $SU(n|1)/H$ case, with nontrivial potential terms, could drastically change the quantum properties of these models. The relevant $d = 1$ supercoset models should follow from the extended models in the "long-wave" limit, in the same way as the higher-dimensional nonlinear sigma models follow from the linear ones.

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APPENDIX: ONE-DIMENSIONAL SIGMA MODELS FROM GAUGING

In this Appendix, we demonstrate how the invariant $d = 1$ actions of some other Landau-type models can be recovered by the gauge method of Sec. II.

1. Landau model on the supersphere $SU(2|1)/U(1|1)$

The superspherical Landau model describes a motion of a charged particle on the supersphere $SU(2|1)/U(1|1)$ parametrized by one complex even and one complex odd coordinates [5].

Consider the set consisting of two complex bosonic fields u^i, $i = 1, 2$, and of one complex fermionic field ξ which form a fundamental representation of $SU(2|1)$. The $SU(2|1)$ transformations are defined as a group of linear transformations leaving invariant the following bilinear form:

$$u^1\bar{u}_1 + u^2\bar{u}_2 + \xi\bar{\xi} = \text{inv}. \tag{A1}$$

As the first step, we impose the manifestly $SU(2|1)$ covariant constraint

$$u^1\bar{u}_1 + u^2\bar{u}_2 + \xi\bar{\xi} = 1, \tag{A2}$$

from which we eliminate $|u^1|$ as

$$|u^1| = \sqrt{1 - u^2\bar{u}_2 - \xi\bar{\xi}}. \tag{A3}$$

As the second step, we gauge the $U(1)$ symmetry which acts as a multiplication of the fields (u^i, ξ) by the common phase and commutes with $SU(2|1)$. The sigma-model-type action, which is invariant under both $SU(2|1)$ and gauge $U(1)$ symmetries, is given by

$$L = \nabla u^i \bar{\nabla}\bar{u}_i + \nabla\xi\bar{\nabla}\,\bar{\xi} + 2\kappa\mathcal{A}, \tag{A4}$$

where we have introduced the covariant derivatives $\nabla = \partial_t - i\mathcal{A}$, $\bar{\nabla} = \partial_t + i\mathcal{A}$ involving the nonpropagating $U(1)$ gauge field, $\mathcal{A}(t)$, and added the Fayet-Iliopoulos term for the gauge field $\sim\kappa$.

The local $U(1)$ invariance allows us to gauge away a phase of u^1 and thus to make u^1 real. In this gauge,

$$u^1 = |u^1| = \sqrt{1 - u^2\bar{u}_2 - \xi\bar{\xi}},$$

and we are left with the complex bosonic and fermionic fields u^2 and ξ as the only independent degrees of freedom. The field $\mathcal{A}(t)$ enters (A4) without derivatives, so we can eliminate it by its algebraic equation of motion:

$$\mathcal{A}(t) = \frac{i}{2}(u \cdot \dot{\bar{u}} - \dot{u} \cdot \bar{u} + \xi\dot{\bar{\xi}} - \dot{\xi}\,\bar{\xi} - 2\kappa). \tag{A5}$$

It is also convenient to pass to the new independent variables, Grassmann-even $z(t)$ and Grassmann-odd $\zeta(t)$,

$$(u^2, \xi) \to (z, \zeta), \qquad u^2 = \frac{z}{\sqrt{1 + z\bar{z} + \zeta\bar{\zeta}}},$$
$$\xi = \frac{\zeta}{\sqrt{1 + z\bar{z} + \zeta\bar{\zeta}}}. \tag{A6}$$

Then the Lagrangian (A4), with $A(t)$ being expressed in terms of z, ζ by (A5) and (A6), can be written as

$$L = g_{\bar{B}A}\dot{Z}^A\dot{\bar{Z}}^B + \kappa(\dot{Z}^A A_A + \dot{\bar{Z}}^B A_{\bar{B}}), \tag{A7}$$

where gauge connections are $A_A = -i\partial_A K$, $A_{\bar{A}} = i\partial_{\bar{A}}K$, $K = \ln(1 + z\bar{z} + \zeta\bar{\zeta})$, and the metric on the supersphere is

$$g_{\bar{z}z} = \frac{1 + \zeta\bar{\zeta}}{(1 + z\bar{z} + \zeta\bar{\zeta})^2}, \qquad g_{\bar{z}\zeta} = -\frac{z\bar{\zeta}}{(1 + z\bar{z})^2}, \tag{A8}$$

$$g_{\bar{\zeta}z} = \frac{\zeta\bar{z}}{(1 + z\bar{z})^2}, \qquad g_{\bar{\zeta}\zeta} = \frac{1}{1 + z\bar{z}}. \tag{A9}$$

The metric can be concisely written as $g_{\bar{B}A} = \partial_{\bar{B}}\partial_A K$, i.e., the function $K(z, \bar{z}, \zeta, \bar{\zeta})$ is the corresponding super Kähler potential. The Lagrangian (A7) coincides with the one found in Ref. [5] by a different method. It should be pointed out that the WZ term in (A7) originates from the FI term in the original gauge action (A4), like in the odd-coset sigma model Lagrangian (2.14) and (3.5), (3.5).

2. Landau model on $SU(n)/U(n-1)$

Our second example is the purely bosonic extended Landau-type model on the coset space $SU(n)/U(n-1)$, which is a generalization of the $S^2 \sim \mathbb{CP}^1$ Haldane model [2].

Consider bosonic multiplet $u^\alpha(t)$, $\alpha = 1, \ldots, n$ in the fundamental representation of the $SU(n)$. This group acts on these $d = 1$ fields as

$$\delta u^\alpha = \lambda^\alpha_\beta u^\beta, \qquad \overline{(\lambda^\alpha_\beta)} = -\lambda^\beta_\alpha, \qquad \lambda^\alpha_\alpha = 0. \tag{A10}$$

Impose the $SU(n)$ invariant constraint

$$\bar{u}_\alpha u^\alpha = 1 \tag{A11}$$

and define the $SU(n)$ invariant Lagrangian

$$L = \bar{\nabla}\bar{u}_\alpha \nabla u^\alpha + 2\kappa\mathcal{A}. \tag{A12}$$

Here the auxiliary gauge field $\mathcal{A}(t)$ ensures the local $U(1)$ invariance of (A12), the corresponding gauge-covariant derivatives being defined as $\nabla = \partial_t - i\mathcal{A}$ and $\bar{\nabla} = \partial_t + i\mathcal{A}$. As in other examples, we may eliminate $\mathcal{A}(t)$ by its algebraic equation of motion,

$$\mathcal{A} = \frac{i}{2}(\dot{\bar{u}}_\alpha u^\alpha - \bar{u}_\alpha \dot{u}^\alpha + 2i\kappa). \tag{A13}$$

We can also make use of the local $U(1)$ invariance to choose the u^1 field as real. Then, making use of the constraint (A11), this field can be expressed through the remaining ones as

$$u^1 = \sqrt{1 - \bar{u}_a u^a}.$$

The natural realization of the $SU(n)$ group as a group of left shifts on the coset space $SU(n)/U(n-1)$ is in terms of the complex coordinates z^a, $a = 1, \ldots, n-1$, with the holomorphic $SU(n)$ transformations:

$$\delta z^a = \lambda^a_1 + \lambda^a_b z^b - \lambda^1_1 z^a - \lambda^1_b z^a z^b. \tag{A14}$$

Coordinates with such a transformation law correspond to the realization of the coset space $SU(n)/U(n-1)$ as the complex projective space $\mathbb{CP}^n$. The connection between the new coordinates z^a and the old coordinates u^α is as follows:

$$u^a = \frac{z^a}{\sqrt{1 + z^a\bar{z}_a}}, \tag{A15}$$

and therefore

$$u^1 = \frac{1}{\sqrt{1 + z^a\bar{z}_a}}. \tag{A16}$$

In terms of the new coordinates, after substituting the expression (A13) for $\mathcal{A}(t)$ back to the Lagrangian (A12), the latter takes the form

$$L = \frac{\dot{z}^a\dot{\bar{z}}_a}{1 + z^b\bar{z}_b} - \frac{\dot{z}^a\bar{z}_a z^b\dot{\bar{z}}_b}{(1 + z^c\bar{z}_c)^2} - i\kappa\frac{\dot{z}^a\bar{z}_a - z^a\dot{\bar{z}}_a}{1 + z^b\bar{z}_b}. \tag{A17}$$

In the $n = 2$ case, it is reduced to the Lagrangian of the Haldane model, whereas for $n = 3$ it is the Lagrangian used in Ref. [25] for description of a variant of the four-dimensional quantum Hall effect. Note that the coefficient in front of the $U(1)$ WZ term in (A17) comes from the FI term in (A12), like in other examples.

[1] L. Landau, Z. Phys. **64**, 629 (1930).
[2] F. D. M. Haldane, Phys. Rev. Lett. **51**, 605 (1983).
[3] E. Witten, Nucl. Phys. **B188**, 513 (1981).
[4] E. Ivanov, L. Mezincescu, and P. K. Townsend, arXiv:hep-th/0311159.
[5] A. Beylin, T. L. Curtright, E. Ivanov, L. Mezincescu, and P. K. Townsend, J. High Energy Phys. 10 (2008) 069.
[6] E. Ivanov, L. Mezincescu, and P. K. Townsend, in *From Fields to Strings*, edited by M. Shifman *et al.* (World Scientific, Singapore, 2005), Vol. 3, pp. 2123–2146.

[7] E. Ivanov, L. Mezincescu, A. Pashnev, and P. K. Townsend, Phys. Lett. B **566**, 175 (2003).

[8] R. R. Metsaev and A. A. Tseytlin, Nucl. Phys. **B533**, 109 (1998).

[9] V. Mitev, T. Quella, and V. Schomerus, J. High Energy Phys. 11 (2008) 086.

[10] T. J. Hollowood and J. L. Miramontes, J. High Energy Phys. 05 (2011) 136.

[11] V. L. Golo and A. M. Perelomov, Phys. Lett. **79B**, 112 (1978); E. Witten, Nucl. Phys. **B149**, 285 (1979).

[12] M. Scheunert, W. Nahm, and V. Rittenberg, J. Math. Phys. (N.Y.) **18**, 155 (1977).

[13] V. Rittenberg, Rockefeller University, Report No. COO-2232B-141, 1977.

[14] T. Curtright, E. Ivanov, L. Mezincescu, and P. K. Townsend, J. High Energy Phys. 04 (2007) 020.

[15] E. Ivanov, Theor. Math. Phys. **154**, 349 (2008).

[16] N. Beisert, arXiv:nlin/0610017.

[17] G. Arutyunov and S. Frolov, Nucl. Phys. **B804**, 90 (2008).

[18] S. Fedoruk, E. Ivanov, and O. Lechtenfeld, J. Phys. A **45**, 173001 (2012).

[19] E. Ivanov, L. Mezincescu, and P. K. Townsend, J. High Energy Phys. 01 (2006) 143.

[20] V. Bychkov and E. Ivanov, Nucl. Phys. **B863**, 33 (2012).

[21] N. Beisert, Adv. Theor. Math. Phys. **12**, 945 (2008); B. Stefański and A. A. Tseytlin, Nucl. Phys. **B718**, 83 (2005).

[22] K. Hasebe, Phys. Rev. Lett. **94**, 206802 (2005).

[23] K. Hasebe, Phys. Rev. D **72**, 105017 (2005).

[24] Z. F. Ezawa, *Quantum Hall Effects: Field Theoretical Approach and Related Topics* (World Scientific, Singapore, 2000).

[25] D. Karabali and V. P. Nair, Nucl. Phys. **B641**, 533 (2002).