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Transient random walk in symmetric exclusion: limit theorems and an Einstein relation

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Abstract. We consider a one-dimensional simple symmetric exclusion process in equilibrium as a dynamic random environment for a nearest-neighbor random walk that on occupied/vacant sites has two different local drifts to the right. We obtain a LLN, a functional CLT and large deviation bounds for the random walk under the annealed measure by means of a renewal argument. We also obtain an Einstein relation under a suitable perturbation. A brief discussion on the topic of random walks in slowly mixing dynamic random environments is presented.

1. Introduction: model, results and motivation

1.1. *The model.* Let

$$\xi = (\xi_t)_{t\geq 0} \quad \text{with} \quad \xi_t = \big(\xi_t(x)\big)_{x\in\mathbb{Z}} \tag{1.1}$$

be a càdlàg Markov process with state space $\Omega = \{0,1\}^{\mathbb{Z}}$. We say that at time t the site x is *occupied by a particle* if $\xi_t(x) = 1$ and is *vacant* or, alternatively, *occupied by a hole*, if $\xi_t(x) = 0$. For $\eta \in \Omega$, we write P^η to denote the law of ξ starting from $\xi_0 = \eta$, and denote by

$$P^\mu(\cdot) = \int_\Omega P^\eta(\cdot)\,\mu(\mathrm{d}\eta) \tag{1.2}$$

the law of ξ when ξ_0 is drawn from a probability measure μ on Ω.

Having fixed a realization of ξ, let

$$X = (X_t)_{t\geq 0} \tag{1.3}$$

be the Random Walk (RW) that starts from 0 and has local transition rates

$$\begin{aligned} x \to x+1 \quad &\text{at rate} \quad \alpha_1\,\xi_t(x) + \alpha_0\,[1-\xi_t(x)],\\ x \to x-1 \quad &\text{at rate} \quad \beta_1\,\xi_t(x) + \beta_0\,[1-\xi_t(x)], \end{aligned} \tag{1.4}$$

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where

$$\alpha_0, \alpha_1, \beta_0, \beta_1 \in (0, \infty), \tag{1.5}$$

i.e., on occupied (resp. vacant) sites the random walk jumps to the right at rate α_1 and to the left at rate β_1 (resp. α_0 and β_0). We write P_X^ξ to denote the law of X when ξ is fixed and, for an initial measure μ,

$$\mathbb{P}_\mu(\cdot) = \int P_X^\xi(\cdot)\, P^\mu(\mathrm{d}\xi) \tag{1.6}$$

to denote the law of X averaged over ξ. We refer to P_X^ξ as the *quenched* law and to $\mathbb{P}_\mu$ as the *annealed* law.

We are interested in studying the RW X when ξ is a one-dimensional *Simple Symmetric Exclusion Process* (SSEP), i.e., an Interacting Particle System (IPS) (see Liggett (2005)) whose generator L acts on a real cylinder function f as

$$(Lf)(\eta) = \sum_{\substack{x,y\in\mathbb{Z}\\ x\sim y}} \left[f(\eta^{xy}) - f(\eta)\right], \qquad \eta \in \Omega, \tag{1.7}$$

where the sum runs over unordered pairs of neighboring sites in $\mathbb{Z}$, and η^{xy} is the configuration obtained from η by interchanging the states at sites x and y. For any $\rho \in (0,1)$, the Bernoulli product measure with density ρ, which we denote by ν_ρ, is an ergodic measure for the SSEP (Liggett (2005), Theorem VIII.1.44).

We will assume that

$$\alpha_0 \wedge \alpha_1 - \beta_0 \vee \beta_1 > 1. \tag{1.8}$$

Condition (1.8) implies that the local drifts on occupied and vacant sites, $\alpha_1 - \beta_1$ and $\alpha_0 - \beta_0$ respectively, are both bigger than 1. Thus the RW is not only transient, but travels faster than local information can spread in the SSEP. This is a strong property which is key to our argument; it allows us, roughly speaking, to overcome the slow mixing in time of the SSEP with the good mixing in space of ν_ρ, giving rise to a regenerative structure for the random walk.

1.2. *Results.* In the three theorems below we fix $\rho \in [0,1]$ and assume (1.5), (1.8).

Theorem 1.1. (Law of large numbers)
There exists $v \geq \alpha_0 \wedge \alpha_1 - \beta_0 \vee \beta_1 > 1$ such that

$$\lim_{t\to\infty} \frac{X_t}{t} = v \qquad \mathbb{P}_{\nu_\rho}\text{-a.s. and in } L^p \ \ \forall \ p \geq 1. \tag{1.9}$$

Theorem 1.2. (Annealed large deviations)
For any $\epsilon > 0$,

$$\limsup_{t\to\infty} t^{-1} \log \mathbb{P}_{\nu_\rho}(|X_t - tv| \geq t\epsilon) < 0. \tag{1.10}$$

Theorem 1.3. (Annealed functional central limit theorem)
There exists $\sigma \in (0,\infty)$ such that, under $\mathbb{P}_{\nu_\rho}$,

$$\left(\frac{X_{nt} - ntv}{\sqrt{n}}\right)_{t\geq 0} \Rightarrow \sigma B \tag{1.11}$$

where B is a standard Brownian motion and "⇒" denotes weak convergence in Skorohod space.

For the next result, we interpret the model of Section 1.1 as a perturbation of a homogeneous RW. We regard the exclusion process as an oscillating random field which interacts weakly with the RW, affecting its asymptotic speed. The Einstein relation then says that the rate of change of the speed when the interaction is very weak is given by the diffusion coefficient of the unperturbed walk. This is a form of the fluctuation-dissipation theorem from statistical physics, which concerns the response of thermodynamical systems to small external perturbations, connecting it with spontaneous fluctuations of the system. As references we mention Dembo and Deuschel (2010); Gantert et al. (2012); Komorowski and Olla (2005).

Theorem 1.4. (Einstein Relation)
Fix $\alpha, \beta > 0$ with $\alpha - \beta > 1$. Let $\lambda \in (0, \infty)$ be the perturbation strength, and fix interaction constants $F_0, F_1 \in \mathbb{R}$ with $F_0 + F_1 = 1$. Let the perturbed rates be given by:

$$\begin{aligned} \alpha_0 &= \alpha \exp\left\{F_0 \frac{\lambda}{1-\rho} + o(\lambda)\right\}, \quad & \beta_0 &= \beta \exp\left\{-F_0 \frac{\lambda}{1-\rho} + o(\lambda)\right\}, \\ \alpha_1 &= \alpha \exp\left\{F_1 \frac{\lambda}{\rho} + o(\lambda)\right\}, & \beta_1 &= \beta \exp\left\{-F_1 \frac{\lambda}{\rho} + o(\lambda)\right\}. \end{aligned} \tag{1.12}$$

When λ is small enough, (1.8) is satisfied. For such λ, let $v(\lambda)$ be the speed as in (1.9). Then

$$\lim_{\lambda \downarrow 0} \frac{v(\lambda) - v(0)}{\lambda} = \alpha + \beta. \tag{1.13}$$

The rest of the paper is organized as follows. In Section 1.3, we present a brief introduction to RW in static and dynamic Random Environment (RE), and discuss slowly mixing dynamic REs. In Section 2, we construct a particular version of our model. Section 3 is the core of the paper; there we develop a regeneration scheme that is used in Section 4 to prove Theorems 1.1–1.4.

1.3. *Motivation.* Random Walks in Random Environments (RWRE) on $\mathbb{Z}^d$ are RWs whose transition probabilities or rates depend on a random field (*static* case) or on a random process (*dynamic* case) which is called a *random environment*. They model the motion of a particle in an inhomogeneous medium.

RWs in *static* REs have been an intensive research area since the 1970's (see e.g. Solomon (1975)). One-dimensional models are well understood. In particular, recurrence vs. transience criteria, LLNs and CLTs have been derived, as well as quenched and annealed LDPs. In higher dimensions the picture is much less complete, but several results are available for RWs that are *transient* in some direction. In particular, LLNs and CLTs for i.i.d. REs (Sznitman and Zerner (1999); Sznitman (2000, 2001); Rassoul-Agha and Seppäläinen (2009)) and for uniformly (fast) mixing REs (Comets and Zeitouni (2004, 2005); Rassoul-Agha (2003)) have been obtained under ballisticity conditions. See Bolthausen and Sznitman (2002); Zeitouni (2004, 2006) for an overview.

By considering time as an additional dimension, one can view RWs in *dynamic* REs in dimension d as RWs in *static* REs in dimension $d+1$ which are transient in the time direction (see e.g. Avena et al. (2011)). Thus there are results analogous to the static, transient case. In particular, LLNs and CLTs have been obtained when the dynamic RE has either no correlations in space and/or time, or has uniform and fast mixing, where 'fast' means either exponential or (more recently) polynomial with a high enough degree. A few references are: Avena et al. (2011); Bandyopadhyay and Zeitouni (2006); Bérard (2004); Boldrighini et al. (2004, 2009); Bricmont and Kupiainen (2009); den Hollander et al. (2013b); Dolgopyat et al. (2008); Joseph and Rassoul-Agha (2011); Redig and Völlering (2013). Further references can be found in Avena (2010); Avena et al. (2010).

Very little is known for dynamics with slow and/or non-uniform mixing (e.g. exclusion, supercritical contact, and zero-range processes), apart from recent results for specific cases (den Hollander and dos Santos (2013), den Hollander et al. (2013a)). A special interest in studying RW in slowly mixing dynamic REs comes from the static, one-dimensional case, where unusual asymptotic behavior can be observed. More specifically, there are regimes exhibiting transience with zero speed (Solomon (1975)), non-diffusivity (Kesten et al. (1975); Sinaǐ (1982)) and subexponential decay of the probability of travelling at speeds slower than typical (Comets et al. (2000); Greven and den Hollander (1994)). Such phenomena do not occur in dynamic RE with fast mixing (as discussed in the previous paragraph), but one would expect them to persist when the dynamics are slow enough. Indeed, for a RW in the SSEP with symmetric drifts on holes/particles (i.e., dropping (1.8) and taking $\alpha_0 = \beta_1$, $\beta_0 = \alpha_1$), it was shown in Avena et al. (2010) that the cost for travelling with zero speed is subexponential; furthermore, simulation results (Avena and Thomann (2012)) suggest the existence of non-diffusive regimes. Thus the SSEP, being a natural example where mixing is both slow and non-uniform due to particle conservation, is an interesting and challenging choice of dynamic RE.

In the present paper, we study the RW in the SSEP under the additional assumption of a strong spatial drift (1.8), which significantly facilitates the analysis. We believe that the regeneration strategy developed in Section 3 could be adapted to other dynamic REs (for instance, asymmetric exclusion processes or a Poissonian field of independent RWs) under similar drift assumptions.

2. Construction of the model

In this section we construct particular versions of the random walk and of the exclusion process, and introduce the notion of *marked agents*. The resulting Lemma 2.1 plays a key role throughout the paper.

2.1. *Coupling with the minimal walker.* We will construct the RW X defined in (1.3) from four independent Poisson processes and the RE. This is valid in any dynamic RE given by a two-state IPS.

Let the following set of Poissonian clocks be given, each independent of all the other variables:

$$\begin{array}{lll} N^+ = (N^+_t)_{t\geq 0} & \text{with rate} & \alpha_0 \wedge \alpha_1, \\ N^- = (N^-_t)_{t\geq 0} & \text{with rate} & \beta_0 \wedge \beta_1, \\ \widehat{N}^+ = (\widehat{N}^+_t)_{t\geq 0} & \text{with rate} & \alpha_0 \vee \alpha_1 - \alpha_0 \wedge \alpha_1, \\ \widehat{N}^- = (\widehat{N}^-_t)_{t\geq 0} & \text{with rate} & \beta_0 \vee \beta_1 - \beta_0 \wedge \beta_1. \end{array} \tag{2.1}$$

Now define X by the following rules:

(1) X jumps only when one of the Poisson clocks ring;
(2) When N^+ rings, X jumps to the right; when N^- rings, X jumps to the left;

(3) When $\widehat{N}^+$ rings, X jumps to the right if the state j at its position is such that $\alpha_j = \alpha_0 \vee \alpha_1$. When $\widehat{N}^-$ rings, X jumps to the left if $\beta_j = \beta_0 \vee \beta_1$. Otherwise, X stays still.

In this construction, X is a function of $(N^\pm, \widehat{N}^\pm, \xi)$ and depends on the environment only through the states it sees when $\widehat{N}^+$ or $\widehat{N}^-$ ring.

Let $M = (M_t)_{t\geq 0}$ be defined by

$$M_t := N_t^+ - N_t^- - \widehat{N}_t^- . \tag{2.2}$$

By construction, for any $t \geq s \geq 0$,

$$M_t - M_s \leq X_t - X_s, \tag{2.3}$$

and we are thus justified to call M the *minimal walker*.

Let

$$N_t := N_t^+ + N_t^- + \widehat{N}_t^+ + \widehat{N}_t^- \tag{2.4}$$

be the number of attempted jumps before time t and

$$\widehat{N}_t := \widehat{N}_t^+ + \widehat{N}_t^- \tag{2.5}$$

the number of times before time t when the random walk observes the environment. Note that, by construction,

$$|X_t - X_s| \leq N_t - N_s \quad \forall \;\; t \geq s \geq 0. \tag{2.6}$$

As a consequence, for all $p \geq 1$, there is a $C(p) \in (0, \infty)$ such that

$$\sup_{\eta \in \Omega} \mathbb{E}_\eta[|X_t|^p] \leq C(p) t^p. \tag{2.7}$$

Therefore, by uniform integrability, as soon as a LLN holds, convergence in L^p, $p \geq 1$, will follow as well.

2.2. *Graphical representation.* The SSEP can be constructed from a graphical representation as follows. Let

$$I = (I(x))_{x\in\mathbb{Z}} \tag{2.8}$$

be a collection of i.i.d. Poisson processes with rate 1. Draw the events of $I(x)$ on $\mathbb{Z} \times [0, \infty)$ as arrows between the points x and $x+1$. Then, for each $t > 0$ and $x \in \mathbb{Z}$, there exists (a.s.) a unique path in $\mathbb{Z} \times [0, \infty)$ starting at (x, t) and ending in $\mathbb{Z} \times \{0\}$ going downwards in time and crossing any arrows it encounters; see Figure 2.1. Denote by $\gamma_t(x) \in \mathbb{Z}$ the end position of this path. The process $\gamma = (\gamma_t)_{t\geq 0}$ is called the *interchange process*. On the other hand, for each $t \geq 0$ and $x \in \mathbb{Z}$, there is a unique y in $\mathbb{Z}$ such that $\gamma_t(y) = x$; denote by

$$\gamma^{-1} = (\gamma_t^{-1})_{t\geq 0} \tag{2.9}$$

the process such that $\gamma_t^{-1}(x) = y$.

We interpret these processes by saying that there are *agents* on the lattice, named after their initial positions, who move around by exchanging places with their neighbors at events of I. Then $\gamma_t^{-1}(x)$ is the position at time t of agent x and $\gamma_t(x)$ is the agent who at time t is at position x.

The SSEP $\xi = (\xi_t)_{t\geq 0}$ starting from a configuration $\xi_0 \in \Omega = \{0,1\}^{\mathbb{Z}}$ is obtained from γ by putting

$$\xi_t(x) := \xi_0(\gamma_t(x)), \quad x \in \mathbb{Z}. \tag{2.10}$$

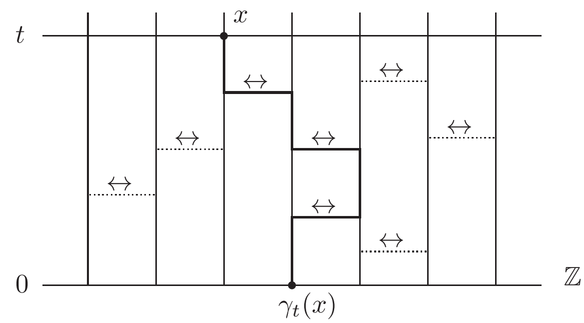

FIGURE 2.1. Graphical representation. The dotted lines represent events of I. The thick lines mark the path of the agent $\gamma_t(x)$.

The description under the 'agent interpretation' is that we assign at time 0 to each agent x a state $\xi_0(x)$ and declare the state of the exclusion process at a space time position (x,t) to be the state of the agent who is there.

We will call $\widetilde{P}$ the joint law of $(N^+, N^-, \widehat{N}^+, \widehat{N}^-, I)$. For simplicity of notation, we redefine $\mathbb{P}_\mu$ as the joint law of $(N^+, N^-, \widehat{N}^+, \widehat{N}^-, I)$ and ξ_0 when the latter is distributed as μ, i.e., $\mathbb{P}_\mu = \mu \times \widetilde{P}$. Then ξ as defined in (2.10) is under $\mathbb{P}_\mu$ indeed distributed as a SSEP started from μ.

2.3. *Marked agents.* In our proof, regeneration comes as a consequence of the fact that, even though the environment is slowly mixing, the environment *perceived* by the walker is fast mixing in some sense. The idea is that, since X has a strong drift and the information spread is limited, the dependence on the observed environment is left behind very fast. In the exclusion process, this dependence is carried by the agents of the interchange process whom the RW meets; we will therefore keep track of them via the following time-increasing set of *marked agents*:

$$A_t := \bigcup_{\substack{0<s\le t \\ \widehat{N}_{s-}\neq \widehat{N}_s}} \{\gamma_s(X_{s-})\}. \tag{2.11}$$

In words, A_t consists of all the agents $x \in \mathbb{Z}$ whose states the walker observes up to time t. Set also

$$R_t := \sup_{x\in A_t} \gamma_t^{-1}(x), \tag{2.12}$$

i.e., R_t is the position of the rightmost marked agent at time t. As usual we take $\sup \emptyset = -\infty$.

An important observation is that the walker depends on the initial configuration only through the states of the agents in A_t. More precisely, X is adapted to the filtration $\mathcal{G} = (\mathcal{G}_t)_{t\ge 0}$ given by

$$\mathcal{G}_t := \sigma((N_s^\pm, \widehat{N}_s^\pm, I_s)_{0\le s\le t}, A_t, (\xi_0(x))_{x\in A_t}). \tag{2.13}$$

Moreover, as the next lemma shows, a consequence of the i.i.d. structure and exchangeability of ν_ρ is that the states of the agents who are not in A_t are still, given $\mathcal{G}_t$, distributed as under ν_ρ.

Lemma 2.1. *For any $t \geq 0$ and $x_1, \dots x_n \in \mathbb{Z}$,*

$$\mathbb{E}_{\nu_\rho}\left[\prod_{i=1}^{n} \xi_t(x_i) \,\middle|\, \mathcal{G}_t\right] = \rho^n \quad a.s. \text{ on } \{\gamma_t(x_1), ..., \gamma_t(x_n) \notin A_t\}. \tag{2.14}$$

Moreover, (2.14) is still valid when t is replaced with a finite $\mathcal{G}$-stopping time.

Proof: From the definition of A_t it follows that, for $A \subset \mathbb{Z}$,

$$\{A_t = A\} \in \sigma((N_s^{\pm}, \widehat{N}_s^{\pm}, I_s)_{0 \leq s \leq t}, (\xi_0(x))_{x \in A}). \tag{2.15}$$

With (2.15) we can verify by summing over A that, for any $x_1, ..., x_n \in \mathbb{Z}$,

$$\mathbb{E}_{\nu_\rho}\left[\prod_{i=1}^{n} \xi_0(x_i) \,\middle|\, \mathcal{G}_t\right] = \rho^n \quad \text{a.s. on the set } \{x_1, ..., x_n \notin A_t\}. \tag{2.16}$$

The summation is justified because A_t is a finite set. Since γ is $\mathcal{G}$-adapted and $\xi_t(x) = \xi_0(\gamma_t(x))$, (2.14) follows. The extension to a $\mathcal{G}$-stopping time is done by approximating it from above by stopping times taking values in a countable set (to which (2.14) easily extends) and then using the right-continuity of A_t and ξ_t. ∎

3. Regeneration

In this section we will develop a regenerative structure for the path of the RW X. Let us first give an informal description of the regeneration strategy. Since X is travelling fast to the right, there will be moments, called *trial times*, when the RW has left behind all agents previously met. At these times, it may 'try to regenerate', and we say that it succeeds if afterwards it never meets those agents again. In case it does not succeed, we wait for the moment when it meets an agent from the past, which we call a *failure time*, and repeat the procedure by waiting for the next trial time. Summarizing, the regeneration strategy consists of two steps: waiting for a trial time when there is a chance for the walker to forget its past, and then checking whether it succeeds or fails in its regeneration attempt. These steps are repeated until the walker succeeds, which will eventually happen by the strong drift assumption (1.8).

We proceed to formalize the regeneration scheme, beginning with the trial times. Let $(T_t)_{t \geq 0}$ be the family of $\mathcal{G}$-stopping times defined by:

$$T_t := \inf\left\{s \geq J_t \colon X_s > R_s\right\}. \tag{3.1}$$

where $J_t := \inf\{s \geq t : N_t \neq N_s\}$ is the time of the next possible jump after time t. The previous discussion justifies calling T_t the first *trial time* after time t. From the definition it is clear that they are indeed $\mathcal{G}$-stopping times. Note that, a.s., $T_t > t$.

In order to define the failure times, first let, for $t \geq 0, x \in \mathbb{Z}$,

$$Y^t(x) = (Y_s^t(x))_{s \geq t} \tag{3.2}$$

be the path starting at time t from x and jumping to the right across the arrows of the process I in (2.8); see Figure 3.2. Then $(Y_{t+u}^t(x) - x)_{u \geq 0}$ is a Poisson process with rate 1.

Now let $(F_t)_{t \geq 0}$ be the family of $\mathcal{G}$-stopping times defined by

$$F_t := \inf\{s > t \colon X_s \leq Y_s^t(X_t - 1)\}. \tag{3.3}$$

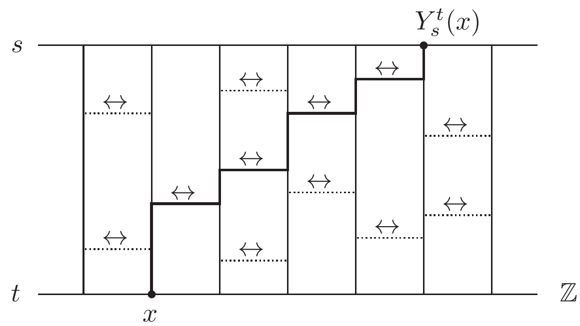

FIGURE 3.2. As in Figure 2.1, the dotted lines are events of I. The path $Y^t(x)$ starts at x and goes upwards in time and to the right across the arrows.

As usual we take $\inf\emptyset = \infty$. We call F_t the first *failure time* after time t. The F_t's are smaller than the failure times informally discussed in the beginning of the section. Indeed, agents to the left of X_t at time t can never cross $Y^t(X_t - 1)$, as can be seen on the graphical representation. In particular, if $F_t = \infty$, then X will after time t never meet such agents again.

In the following lemma we obtain exponential moment bounds for the trial times T_t, showing in particular that they are a.s. finite.

Lemma 3.1. *For every $a > 0$, there exists $b_1 \in (0,\infty)$ such that, for all $t \geq 0$,*

$$\mathbb{E}_{\nu_\rho}[e^{b_1(T_t - t)}|\mathcal{G}_t] \leq (1+a)\, e^{a(R_t - X_t)^+} \quad \mathbb{P}_{\nu_\rho}\text{-a.s.} \tag{3.4}$$

Proof: Let

$$\widetilde{Y}^t = Y^t(R_t \vee X_t) \tag{3.5}$$

be the Poisson path starting at time t from the position $R_t \vee X_t$.

Define $H_t := \inf\{s > t\colon\ M_s - M_t + X_t > \widetilde{Y}^t_s\}$. Let us check that

$$T_t \leq H_t \vee J_t \leq J_t + H_t. \tag{3.6}$$

Indeed, if $X_{J_t} > \widetilde{Y}^t_{J_t}$ (which can happen only if $R_t \leq X_t$), then $T_t = J_t$. Suppose now that $X_{J_t} \leq \widetilde{Y}^t_{J_t}$. Recall the definition of γ^{-1} in (2.9). By geometrical constraints, if $\gamma_s^{-1}(x) \leq \widetilde{Y}^t_s$ for some $s \geq t$, then this will also hold for all future times. In particular, agents marked by X before it crosses $\widetilde{Y}^t$ will never be able to cross $\widetilde{Y}^t$ themselves. This implies that T_t is smaller than the first time after t when X is to the right of $\widetilde{Y}^t$, which is in turn smaller than H_t by (2.3).

Next we show that, for any $a > 0$, we can find $b_1 > 0$ such that

$$\mathbb{E}_{\nu_\rho}[e^{2b_1(H_t - t)}|\mathcal{G}_t] \leq (1+a)\, e^{2a(R_t - X_t)^+}. \tag{3.7}$$

Indeed, since M is independent of I, $(Z_u)_{u\geq 0} := (M_{t+u} - M_t - (\widetilde{Y}^t_{t+u} - R_t \vee X_t))_{u \geq 0}$ is under $\mathbb{P}_{\nu\rho}(\cdot|\mathcal{G}_t)$ a continuous-time RW starting from 0 that has a positive drift d by (1.8). Let $\mathcal{T}_x$ be the first time when Z hits a site $x > 0$. Then $H_t - t = \mathcal{T}_x$ with $x = (R_t - X_t)^+ + 1$. Therewith,

$$\mathbb{E}_{\nu_\rho}[e^{2b_1(H_t - t)}|\mathcal{G}_t] = \mathbb{E}e^{2b_1\mathcal{T}_x} \leq \mathbb{E}\left(e^{2b_1 \sup_{y>0}(\mathcal{T}_y - \frac{2(y-1)}{d})}\right) e^{2b_1\frac{2(x-1)}{d}}.$$

Since the exponential moment is finite, we can choose b_1 sufficiently small so that the first factor is smaller than $1 + a$ and $4b_1/d \le 2a$, proving (3.7).

To prove (3.4), we use (3.6) and the Cauchy-Schwarz inequality to get

$$\mathbb{E}_{\nu_\rho}[e^{b_1(T_t-t)}|\mathcal{G}_t] \le \left(\mathbb{E}_{\nu_\rho}[e^{2b_1(H_t-t)}|\mathcal{G}_t]\right)^{\frac{1}{2}} \left(\mathbb{E}_{\nu_\rho}[e^{2b_1(J_t-t)}|\mathcal{G}_t]\right)^{\frac{1}{2}}.$$

Choosing b_1 so small that also $\mathbb{E}_{\nu_\rho}[e^{2b_1(J_t-t)}|\mathcal{G}_t] < 1 + a$ completes the proof. ∎

For $t \ge 0$, denote by $X^{(t)}$ the increments of the walk after time t, that is,

$$X^{(t)}_u := X_{t+u} - X_t. \tag{3.8}$$

The next lemma shows that the second step of the regeneration strategy indeed works.

Lemma 3.2. *For each* $t \ge 0$,

$$\mathbb{P}_{\nu_\rho}\left(F_t = \infty, X^{(t)} \in \cdot \,\middle|\, \mathcal{G}_t\right) = \mathbb{P}_{\nu_\rho}\left(\Gamma, X \in \cdot\right) \text{ a.s. on } \{R_t < X_t\}, \tag{3.9}$$

where $\Gamma := \{F_0 = \infty\}$.

Proof: First note that

$$\eta \mapsto \mathbb{P}_\eta\left(\Gamma, X \in \cdot\right) \text{ does not depend on } (\eta(x))_{x<0}. \tag{3.10}$$

This can be verified using the graphical representation. Indeed, the agents $x < 0$ can never cross $Y^0(-1)$. Therefore, on Γ, none of them ever meets X, i.e., $A_t \cap (\mathbb{Z} \setminus \mathbb{N}_0) = \emptyset$ for all t. On the other hand, Γ is itself measurable in $\sigma(X, I)$; since X is adapted to $\mathcal{G}$, (3.10) follows.

Now, letting $\bar{\xi}_t(\cdot) := \xi_t(X_t + \cdot)$, we can write

$$\begin{aligned}\mathbb{P}_{\nu_\rho}\left(R_t < X_t, F_t = \infty, X^{(t)} \in \cdot \,\middle|\, \mathcal{G}_t\right) &= \mathbb{E}_{\nu_\rho}\left[\mathbb{1}_{\{R_t<X_t\}}\mathbb{P}_{\bar{\xi}_t}\left(\Gamma, X \in \cdot\right) \,\middle|\, \mathcal{G}_t\right] \\ &= \mathbb{1}_{\{R_t<X_t\}}\mathbb{P}_{\nu_\rho}\left(\Gamma, X \in \cdot\right),\end{aligned} \tag{3.11}$$

where the first equality holds by the Markov property and translation-invariance of the graphical representation and the second is justified since, by (3.10), $\mathbb{P}_{\bar{\xi}_t}\left(\Gamma, X \in \cdot\right)$ is a function only of $(\bar{\xi}_t(x))_{x\ge0}$, whose distribution under $\mathbb{P}_{\nu_\rho}(\cdot|\mathcal{G}_t)$ is, by Lemma 2.1, a.s. equal to ν_ρ when $R_t < X_t$. ∎

Before proceeding we make a simple but nonetheless important remark:

Remark 3.3. *Replacing t in T_t and F_t with a finite $\mathcal{G}$-stopping time still yields a stopping time, and Lemmas 3.1–3.2 (as well as Lemmas 3.5 and 3.6 below) remain true with a finite stopping time in place of t.*

Remark 3.3 is justified by right-continuity as in the proof of Lemma 2.1. Recall also that a stopping time multiplied by the indicator function of the set where it is finite is again a stopping time.

We are now in shape to prove our main result.

Theorem 3.4. *There exists a $\mathbb{P}_{\nu_\rho}$-a.s. positive and finite random time τ such that, $\mathbb{P}_{\nu_\rho}$-a.s.,*

$$\mathbb{P}_{\nu_\rho}\left((X_{\tau+s} - X_\tau)_{s\ge0} \in \cdot \,\middle|\, \tau, (X_s)_{s\le\tau}\right) = \mathbb{P}_{\nu_\rho}\left(X \in \cdot \,\middle|\, \Gamma\right); \tag{3.12}$$

$$\mathbb{P}_{\nu_\rho}\left((X_{\tau+s} - X_\tau)_{s\ge0} \in \cdot \,\middle|\, \Gamma, \tau, (X_s)_{s\le\tau}\right) = \mathbb{P}_{\nu_\rho}\left(X \in \cdot \,\middle|\, \Gamma\right). \tag{3.13}$$

Proof: We will obtain the regeneration time τ with the help of an increasing sequence $(U_n)_{n\in\mathbb{N}_0}$ of $\mathcal{G}$-stopping times in $[0,\infty]$, which will be defined using T_t and F_t. We will throughout the proof tacitly use Remark 3.3.

Set $U_0 := 0$. Supposing that for some $n \geq 0$, $(U_k)_{k\leq 2n}$ are all defined, let

$$\begin{aligned} U_{2n+1} &:= \begin{cases} \infty & \text{if } U_{2n}=\infty \\ T_{U_{2n}} & \text{otherwise,} \end{cases} \\ U_{2(n+1)} &:= \begin{cases} \infty & \text{if } U_{2n+1}=\infty \\ F_{U_{2n+1}} & \text{otherwise.} \end{cases} \end{aligned} \tag{3.14}$$

Then $(U_n)_{n\in\mathbb{N}_0}$ is an increasing sequence of $\mathcal{G}$-stopping times. Now define

$$K = \inf\{n \in \mathbb{N}_0 \colon U_{2n+1} < \infty, F_{U_{2n+1}} = \infty\} \in [0,\infty], \tag{3.15}$$

i.e., $2K+1$ is the first index before the sequence U hits infinity.

Set $\kappa := \mathbb{P}_{\nu_\rho}(\Gamma)$. Then $\kappa > 0$ since X dominates M and $M - Y^0(-1)$ has a positive drift. By Lemma 3.2, for any $n \in \mathbb{N}$,

$$\begin{aligned} \mathbb{P}_{\nu_\rho}(K \geq n) &= \mathbb{P}_{\nu_\rho}(U_{2k+1} < \infty \;\forall\, k = 0,\dots,n) \\ &= \mathbb{E}_{\nu_\rho}\left[\mathbb{1}_{\{K\geq n-1\}}\mathbb{P}_{\nu_\rho}\left(F_{U_{2n-1}} < \infty \,\middle|\, \mathcal{G}_{U_{2n-1}}\right)\right] \\ &= (1-\kappa)\mathbb{P}_{\nu_\rho}(K \geq n-1), \end{aligned} \tag{3.16}$$

where for the last equality we used that $X_{U_{2n-1}} > R_{U_{2n-1}}$ if $U_{2n-1} < \infty$. Thus, by induction,

$$\mathbb{P}_{\nu_\rho}(K \geq n) = (1-\kappa)^n \;\;\forall\, n \in \mathbb{N}_0. \tag{3.17}$$

In particular, $K < \infty$ $\mathbb{P}_{\nu_\rho}$-a.s. and we can define

$$\tau := U_{2K+1} < \infty \quad \mathbb{P}_{\nu_\rho}\text{-a.s.} \tag{3.18}$$

Since $\mathbb{P}_{\nu_\rho}(\cdot|\Gamma) \ll \mathbb{P}_{\nu_\rho}$, τ is a.s. well-defined and finite also under $\mathbb{P}_{\nu_\rho}(\cdot|\Gamma)$.

We will now proceed to verify (3.12). Define $\mathcal{G}_\tau$ as the sigma-algebra of the events B such that, for all $n \in \mathbb{N}_0$, there exist $B_n \in \mathcal{G}_{U_{2n+1}}$ such that $B \cap \{K = n\} = B_n \cap \{K = n\}$. Note that τ and $(X_s)_{s\leq\tau}$ are measurable in $\mathcal{G}_\tau$.

Take $f \geq 0$ measurable, $B \in \mathcal{G}_\tau$, and write

$$\begin{aligned} \mathbb{E}_{\nu_\rho}\left[\mathbb{1}_B f(X^{(\tau)})\right] &= \sum_{n=0}^{\infty}\mathbb{E}_{\nu_\rho}\left[\mathbb{1}_{B_n}\mathbb{1}_{\{K=n\}} f(X^{(U_{2n+1})})\right] \\ &= \sum_{n=0}^{\infty}\mathbb{E}_{\nu_\rho}\left[\mathbb{1}_{B_n}\mathbb{1}_{\{U_{2n+1}<\infty, F_{U_{2n+1}}=\infty\}} f(X^{(U_{2n+1})})\right] \\ &= \sum_{n=0}^{\infty}\mathbb{E}_{\nu_\rho}\left[\mathbb{1}_{B_n}\mathbb{1}_{\{U_{2n+1}<\infty\}}\mathbb{E}_{\nu_\rho}\left[\mathbb{1}_{\{F_{U_{2n+1}}=\infty\}} f(X^{(U_{2n+1})}) \,\middle|\, \mathcal{G}_{U_{2n+1}}\right]\right]. \end{aligned}$$

When $U_{2n+1} < \infty$, $R_{U_{2n+1}} < X_{U_{2n+1}}$ so, by Lemma 3.2, the last line equals

$$\begin{aligned} &\mathbb{E}_{\nu_\rho}\left[f(X)\mathbb{1}_\Gamma\right]\sum_{n=0}^{\infty}\mathbb{E}_{\nu_\rho}\left[\mathbb{1}_{B_n}\mathbb{1}_{\{U_{2n+1}<\infty\}}\right] \\ &= \mathbb{E}_{\nu_\rho}\left[f(X)\,|\,\Gamma\right]\sum_{n=0}^{\infty}\mathbb{E}_{\nu_\rho}\left[\mathbb{1}_{B_n}\mathbb{1}_{\{U_{2n+1}<\infty\}}\right]\mathbb{P}_{\nu_\rho}(\Gamma) \end{aligned}$$

which, by Lemma 3.2 again, is equal to

$$\begin{aligned}
&\mathbb{E}_{\nu_\rho}\left[f(X)\,|\,\Gamma\right]\sum_{n=0}^{\infty}\mathbb{E}_{\nu_\rho}\left[\mathbb{1}_{B_n}\mathbb{1}_{\{U_{2n+1}<\infty\}}\mathbb{P}_{\nu_\rho}\left(F_{U_{2n+1}}=\infty\,\middle|\,\mathcal{G}_{U_{2n+1}}\right)\right]\\
&=\mathbb{E}_{\nu_\rho}\left[f(X)\,|\,\Gamma\right]\sum_{n=0}^{\infty}\mathbb{P}_{\nu_\rho}\left(B_n, K=n\right)\\
&=\mathbb{E}_{\nu_\rho}\left[f(X)\,|\,\Gamma\right]\mathbb{P}_{\nu_\rho}(B). \qquad (3.19)
\end{aligned}$$

This proves (3.12). To finish the proof, note that $\Gamma\in\mathcal{G}_\tau$ since, for any $t\geq 0$,

$$\Gamma\cap\{F_t=\infty\}=\{X_s>Y_s^0(-1)\ \forall\ s\leq t\}\cap\{F_t=\infty\}. \qquad (3.20)$$

So (3.13) follows by applying (3.19) to $B\cap\Gamma$ in place of B. ∎

In Proposition 3.7 below, we will show that τ and X_τ have exponential moments. For its proof, we will need the following two lemmas.

Lemma 3.5. *For all $\epsilon>0$, there exists $a_1\in(0,\infty)$ such that, for all $t\geq 0$,*

$$\mathbb{E}_{\nu_\rho}\left[\mathbb{1}_{\{F_t<\infty\}}e^{a_1(F_t-t)}\,\middle|\,\mathcal{G}_t\right]\leq 1+\epsilon \qquad \mathbb{P}_{\nu_\rho}\text{-a.s.} \qquad (3.21)$$

Proof: Let

$$D_t:=\sup\{s>t; M_s-M_t+X_t\leq Y_s^t(X_t-1)\}. \qquad (3.22)$$

If $F_t<\infty$, then $F_t\leq D_t$ because, when finite, F_t is smaller than the last time $s>t$ when $X_s\leq Y_s^t(X_t-1)$, which is in turn smaller than D_t by (2.3). On the other hand, $(M_{t+u}-M_t+X_t-Y_{t+u}^t(X_t-1))_{u\geq 0}$ is under $\mathbb{P}_{\nu_\rho}(\cdot|\mathcal{G}_t)$ a continuous-time RW with positive drift starting at 1. Since D_t-t is the last time when this random walk is less or equal to 0, (3.21) follows. ∎

Lemma 3.6. *For all $\epsilon>0$, there exists $a_2\in(0,\infty)$ such that, for all $t\geq 0$,*

$$\mathbb{E}_{\nu_\rho}\left[\mathbb{1}_{\{F_t<\infty\}}e^{a_2(R_{F_t}-X_{F_t})^+}\,\middle|\,\mathcal{G}_t\right]\leq 1+\epsilon \quad \mathbb{P}_{\nu_\rho}\text{-a.s. on }\{R_t<X_t\}. \qquad (3.23)$$

Proof: Take D_t as in (3.22) and recall that, when finite, $F_t\leq D_t$. Let $\chi_t:=X_t+N_{D_t}-N_t$ and consider $Y^t(\chi_t)$ (see (3.2)). If $R_t<X_t$, then $R_{F_t}\leq Y_{F_t}^t(\chi_t)$ and so

$$R_{F_t}-X_{F_t}\leq Y_{D_t}^t(\chi_t)-\chi_t+N_{D_t}-N_t+1. \qquad (3.24)$$

Now (3.23) follows by noting that, even though χ_t is not in $\mathcal{G}_t$, it is independent of $(Y_{t+u}^t(\chi_t)-\chi_t)_{u\geq 0}$ (as they depend on disjoint regions of the graphical representation), so that the latter is still a Poisson process under $\mathbb{P}_{\nu_\rho}(\cdot|\mathcal{G}_t)$. ∎

Proposition 3.7. *There exists $b\in(0,\infty)$ such that*

$$\mathbb{E}_{\nu_\rho}[e^{b\tau}],\ \mathbb{E}_{\nu_\rho}[e^{bN_\tau}]<\infty, \qquad (3.25)$$

the same being true under $\mathbb{P}_{\nu_\rho}(\cdot|\Gamma)$.

Proof: The last sentence follows from (3.25) and $\kappa=\mathbb{P}_{\nu_\rho}(\Gamma)>0$. Since N is a Poisson process, it is enough prove to that τ has exponential moments under $\mathbb{P}_{\nu_\rho}$. To this end, let $\epsilon>0$ such that $(1+\epsilon)^2(1-\kappa)<1$. Take $a\in(0,\epsilon)$ such that, for all $t\geq 0$,

$$\mathbb{E}_{\nu_\rho}\left[\mathbb{1}_{\{F_t<\infty\}}e^{a(F_t-t)+a(R_{F_t}-X_{F_t})^+}\,\middle|\,\mathcal{G}_t\right]\leq 1+\epsilon \quad \mathbb{P}_{\nu_\rho}\text{-a.s. on }\{R_t<X_t\}. \quad (3.26)$$

Such a exists by Lemmas 3.5 and 3.6 and an application of Hölder's inequality. For this a, take b_1 as in Lemma 3.1 and let $b := (a \wedge b_1)/2$. Now fix $n \geq 1$ and estimate, recalling that $R_{U_{2n-1}} < X_{U_{2n-1}}$ when $U_{2n-1} < \infty$,

$$\begin{aligned}
\mathbb{E}_{\nu_\rho}\left[\mathbb{1}_{\{U_{2n}<\infty\}} e^{2bU_{2n+1}}\right] &= \mathbb{E}_{\nu_\rho}\left[\mathbb{1}_{\{U_{2n}<\infty\}} e^{2bU_{2n}} \mathbb{E}_{\nu_\rho}\left[e^{2b(T_{U_{2n}}-U_{2n})} \,\middle|\, \mathcal{G}_{U_{2n}}\right]\right] \\
&\leq (1+a)\mathbb{E}_{\nu_\rho}\left[\mathbb{1}_{\{U_{2n}<\infty\}} e^{2bU_{2n}+a(R_{U_{2n}}-X_{U_{2n}})^+}\right] \\
&= (1+a)\mathbb{E}_{\nu_\rho}\Bigg\{\mathbb{1}_{\{U_{2n-2}<\infty\}} e^{2bU_{2n-1}} \\
&\quad \times \mathbb{E}_{\nu_\rho}\left[\mathbb{1}_{\{F_{U_{2n-1}}<\infty\}} e^{2b(F_{U_{2n-1}}-U_{2n-1})+a\left(R_{F_{U_{2n-1}}}-X_{F_{U_{2n-1}}}\right)^+} \,\middle|\, \mathcal{G}_{U_{2n-1}}\right]\Bigg\} \\
&\leq (1+\epsilon)^2 \mathbb{E}_{\nu_\rho}\left[\mathbb{1}_{\{U_{2(n-1)}<\infty\}} e^{2bU_{2(n-1)+1}}\right].
\end{aligned}$$

By induction, we get

$$\mathbb{E}_{\nu_\rho}\left[\mathbb{1}_{\{U_{2n}<\infty\}} e^{2bU_{2n+1}}\right] \leq (1+\epsilon)^{2n+1}. \tag{3.27}$$

To conclude, use Hölder's inequality and (3.17) to write:

$$\begin{aligned}
\mathbb{E}_{\nu_\rho}\left[e^{b\tau}\right] &= \sum_{n=0}^{\infty} \mathbb{E}_{\nu_\rho}\left[\mathbb{1}_{\{K=n\}} e^{bU_{2n+1}}\right] = \sum_{n=0}^{\infty} \mathbb{E}_{\nu_\rho}\left[\mathbb{1}_{\{K=n\}} \mathbb{1}_{\{U_{2n}<\infty\}} e^{bU_{2n+1}}\right] \\
&\leq \sum_{n=0}^{\infty} \mathbb{P}_{\nu_\rho}(K=n)^{\frac{1}{2}} \mathbb{E}_{\nu_\rho}\left[\mathbb{1}_{\{U_{2n}<\infty\}} e^{2bU_{2n+1}}\right]^{\frac{1}{2}} \\
&\leq \sqrt{1+\epsilon} \sum_{n=0}^{\infty} \left(\sqrt{(1-\kappa)(1+\epsilon)^2}\right)^n < \infty.
\end{aligned}$$

∎

Finally, due to Theorem 3.4, we can construct a sequence of i.i.d. regeneration times.

Theorem 3.8. *By enlarging the probability space, one can assume the existence of a sequence* $(\tau_n)_{n\in\mathbb{N}}$ *of random times with* $\tau_1 := \tau$ *and such that, setting* $S_n := \sum_{i=1}^n \tau_i$,

$$\left(\tau_{n+1}, \left(X_s^{(S_n)}\right)_{0\leq s\leq \tau_{n+1}}\right)_{n\in\mathbb{N}} \tag{3.28}$$

is under $\mathbb{P}_{\nu_\rho}$ *an i.i.d. sequence which is independent from* $(\tau, (X_s)_{0\leq s\leq\tau})$, *each of its terms being distributed as* $(\tau, (X_s)_{0\leq s\leq\tau})$ *under* $\mathbb{P}_{\nu_\rho}(\cdot|\Gamma)$.

Proof: Let $(\Omega, \mathcal{E})$ be the measurable space used to construct the random processes $N^\pm$, $\widehat{N}^\pm$, I and the random initial configuration ξ_0. For $n \in \mathbb{N}\cup\{\infty\}$, let $\mathbb{P}_n$ be the product probability measure on the product space $(\Omega^n, \mathcal{E}^n)$ whose marginals are $\mathbb{P}_{\nu_\rho}$ on the first coordinate and $\mathbb{P}_{\nu_\rho}(\cdot|\Gamma)$ on the remaining ones. For $i \in \mathbb{N}$, $i \leq n$, let $X(i,n) = (X_s(i,n))_{s\geq 0}$ be the random process obtained by evaluating X on the i-th coordinate. Define $\tau_{i,n}$ analogously by evaluating τ. Let $S_0(n) := 0$ and $S_k(n) := \sum_{i=1}^k \tau_{i,n}$ for $k < n$. Now define the process $\tilde{X}(n) = (\tilde{X}_s(n))_{s\geq 0}$ by

$$\tilde{X}_s(n) := \begin{cases} X_{s-S_i(n)}(i,n) & \text{for } s \in [S_{i-1}(n), S_i(n))\,, \quad i \in \mathbb{N}, i < n, \\ X_{s-S_{n-1}(n)}(n,n) & \text{for } s \geq S_{n-1}(n) \qquad\qquad\quad \text{if } n < \infty. \end{cases} \tag{3.29}$$

Using Theorem 3.4 we can verify by induction that, for any $n \in \mathbb{N}$, $\tilde{X}(n)$ has under $\mathbb{P}_n$ the same law as X under $\mathbb{P}$. Since the processes $\tilde{X}(n)$, $n \in \mathbb{N}\cup\{\infty\}$, can be coupled in such a way that, for any $L > 0$, there exists a (random) $n_0 \in \mathbb{N}$ such that

$$\left(\tilde{X}_s(n)\right)_{s\in[0,L]} = \left(\tilde{X}_s(\infty)\right)_{s\in[0,L]} \quad \text{if } n \geq n_0, \tag{3.30}$$

$\tilde{X}(\infty)$ also has the same law as X. Moreover, $\tilde{X}(\infty)$ has the properties claimed in the statement. These properties are passed to X using a coupling measure Q on $\Omega \times \Omega^{\mathbb{N}}$ such that $Q(X = \tilde{X}) = 1$. ∎

4. Limit Theorems

As a fruit of the regenerative structure constructed in Section 3, we now obtain the asymptotic results stated in Section 1.2.

4.1. *Proofs of Theorems 1.1 — 1.3.* Let us collect some useful facts. First of all, by Theorem 3.8, Proposition 3.7 and (2.6),

$$\left(\sup_{s\in[0,\tau_{n+1}]}\left|X_s^{(S_n)}\right|\right)_{n\in\mathbb{N}_0} \quad \text{have a uniform exponential moment.} \tag{4.1}$$

Furthermore, again by Theorem 3.8 and Proposition 3.7,

$$\lim_{n\to\infty}\frac{S_n}{n} = \mathbb{E}_{\nu_\rho}[\tau|\Gamma] \quad \text{and} \quad \lim_{n\to\infty}\frac{X_{S_n}}{n} = \mathbb{E}_{\nu_\rho}[X_\tau|\Gamma] \quad \mathbb{P}_{\nu_\rho}\text{-a.s.} \tag{4.2}$$

For $t \geq 0$, let k_t be the random integer such that

$$S_{k_t} \leq t < S_{k_t+1}. \tag{4.3}$$

Then a.s. $\lim_{t\to\infty} t^{-1}k_t = \mathbb{E}_{\nu_\rho}[\tau|\Gamma]^{-1}$. Thus the candidate velocity for X is

$$v := \frac{\mathbb{E}_{\nu_\rho}[X_\tau|\Gamma]}{\mathbb{E}_{\nu_\rho}[\tau|\Gamma]}. \tag{4.4}$$

Proof of Theorems 1.1 and 1.2: We first prove (1.10). From Theorem 3.8 and Proposition 3.7 we obtain LDP's for both S_n and X_{S_n} with rate functions which are only zero at $\mathbb{E}_{\nu_\rho}[\tau|\Gamma]$ and $\mathbb{E}_{\nu_\rho}[X_\tau|\Gamma]$, respectively. Since k_t is the inverse of S_n, it also satisfies a LDP with a rate function which is zero only at $\mathbb{E}_{\nu_\rho}[\tau|\Gamma]^{-1}$ (see Glynn and Whitt (1994)). Fix $\epsilon > 0$. From the LDP's for X_{S_n} and k_t, we get exponential decay of $\mathbb{P}_{\nu_\rho}\left(|t^{-1}X_{S_{k_t}} - v| \geq \epsilon\right)$, while the same is obtained for $\mathbb{P}_{\nu_\rho}\left(|X_t - X_{S_{k_t}}| \geq \epsilon t\right)$ from (4.1) and the LDP for k_t. From this, (1.10) is readily obtained, and the LLN follows by the Borel-Cantelli lemma. By (2.3), $v \geq \alpha_0 \wedge \alpha_1 - \beta_0 \vee \beta_1 > 1$. Convergence in L^p follows from (2.7). ∎

Proof of Theorem 1.3: Let $\hat{\sigma}^2$ be the variance of $X_\tau - \tau v$ under $\mathbb{P}_{\nu_\rho}(\cdot|\Gamma)$ which is finite due to (3.25) and positive since $X_\tau - \tau v$ is not a.s. constant. For the process $Y_k := X_{S_k} - S_k v$, $k \in \mathbb{N}$, a functional CLT with variance $\hat{\sigma}^2$ holds since, by Theorem 3.8 and (3.25), the assumptions of the Donsker-Prohorov invariance principle are satisfied.

Consider now the random time change $\varphi_n(t) := k_{nt}/n$. From the LLN and LDP for k_t it follows that

$$\lim_{n\to\infty}\sup_{t\in[0,L]}\left|\varphi_n(t) - \frac{t}{\mathbb{E}\left[\tau \mid \Gamma\right]}\right| = 0 \quad \mathbb{P}\text{-a.s. } \forall\, L > 0, \tag{4.5}$$

and hence φ_n converges a.s. in Skorohod space to the linear function $t \mapsto t/\mathbb{E}\left[\tau \mid \Gamma\right]$.

Let $Y_t^{(n)} := n^{-1/2} Y_{\lfloor nt \rfloor}$. With a time-change argument (see e.g. Billingsley (1968), (17.7)–(17.9) and Theorem 4.4), we see that $(Y^{(n)}_{\varphi_n(t)})_{t\geq 0}$ converges weakly to a Brownian motion with variance $\sigma^2 := \hat{\sigma}^2/\mathbb{E}_{\nu_\rho}[\tau|\Gamma]$. To extend this to X, note that, for any $T>0$,

$$\sup_{0\leq t\leq T}\left|\frac{X_{nt}-ntv}{\sqrt{n}}-Y^{(n)}_{\varphi_n(t)}\right| \leq \frac{1}{\sqrt{n}}\sup_{0\leq t\leq T}\left(\left|X_{nt}-X_{S_{k_{nt}}}\right|+v\left|S_{k_{nt}}-nt\right|\right) \tag{4.6}$$

which goes a.s. to 0 as $n\to\infty$ by Theorem 3.8, (4.1) and the LDP for k_t. ∎

4.2. *Einstein relation: proof of Theorem 1.4.* We first show how the speed v is related to the observed density of particles, and that the latter approaches the density of the environment as $\lambda \downarrow 0$.

Proposition 4.1. *The limit*

$$\hat{\rho}(\lambda) = \lim_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}_{\nu_\rho}\left[\xi_s(X_s)\right]\mathrm{d}s \tag{4.7}$$

exists and satisfies

$$v(\lambda) = \left[\alpha_1(\lambda)-\beta_1(\lambda)\right]\hat{\rho}(\lambda) + \left[\alpha_0(\lambda)-\beta_0(\lambda)\right]\left[1-\hat{\rho}(\lambda)\right], \tag{4.8}$$

$$\lim_{\lambda\downarrow 0}\hat{\rho}(\lambda) = \rho. \tag{4.9}$$

Proof: Since X is Markovian under the quenched measure,

$$X_t - \int_0^t (\alpha_1-\beta_1)\xi_s(X_s) + (\alpha_0-\beta_0)(1-\xi_s(X_s))\mathrm{d}s \tag{4.10}$$

is a martingale under P_X^ξ for a.e. ξ. Hence by Theorem 1.1 the limit in (4.7) exists and satisfies (4.8). We proceed to prove (4.9). Write

$$\int_0^t \mathbb{E}_{\nu_\rho}\left[\xi_s\left(X_s\right)\right]\mathrm{d}s = \int_0^t \mathbb{P}_{\nu_\rho}\left(\gamma_s(X_s)\in A_s, \xi_s(X_s)=1\right)\mathrm{d}s + \int_0^t \mathbb{P}_{\nu_\rho}\left(\gamma_s(X_s)\notin A_s, \xi_s(X_s)=1\right)\mathrm{d}s.$$

The first term is bounded by

$$L_t := \mathbb{E}_{\nu_\rho}\left[\int_0^t \mathbb{1}_{\{\gamma_s(X_s)\in A_s\}}\mathrm{d}s\right], \tag{4.11}$$

the expected time spent by the walker on marked agents up to time t. For the second term, we use Lemma 2.1:

$$\begin{aligned}\int_0^t \mathbb{P}_{\nu_\rho}\left(\gamma_s(X_s)\notin A_s, \xi_s(X_s)=1\right)\mathrm{d}s &= \int_0^t \mathbb{E}_{\nu_\rho}\left[\mathbb{1}_{\{\gamma_s(X_s)\notin A_s\}}\mathbb{E}_{\nu_\rho}\left[\xi_s\left(X_s\right)\mid\mathcal{G}_s\right]\right]\mathrm{d}s \\ &= \rho\int_0^t \mathbb{P}_{\nu_\rho}\left(\gamma_s(X_s)\notin A_s\right)\mathrm{d}s = \rho\left(t-L_t\right).\end{aligned}$$

Hence

$$\left|\int_0^t \mathbb{E}_{\nu_\rho}\left[\xi_s(X_s)\right]\mathrm{d}s - \rho t\right| \leq L_t. \tag{4.12}$$

In order to bound L_t, consider the total time that the walker spends on top of a single marked agent x. If t is the time when this agent is marked, the agent will never cross to the right of $Y^t(\gamma_t^{-1}(x))$. On the other hand, after time t, X will never be to the left of $M - M_t + \gamma_t^{-1}(x) - 1$. Hence the time spent on the marked agent x is bounded by the total time during which $Y^t(\gamma_t^{-1}(x))$ is to the right of $M - M_t + \gamma_t^{-1}(x)$. Writing $t_x = \inf\{t \geq 0 : x \in A_t\}$, we get

$$\begin{aligned} L_t &\leq \sum_{x\in\mathbb{Z}} \mathbb{E}_{\nu_\rho}\left[\mathbb{1}_{\{t_x<t\}} \int_{t_x}^{\infty} \mathbb{1}_{\{Y_s^{t_x}(\gamma_{t_x}^{-1}(x))>M_s-M_{t_x}+\gamma_{t_x}^{-1}(x)\}}\mathrm{d}s\right] \\ &= \mathbb{E}_{\nu_\rho}\left[|A_t|\right]\mathbb{E}_{\nu_\rho}\left[\int_0^\infty \mathbb{1}_{\{Y_s^0(0)>M_s\}}\mathrm{d}s\right]. \end{aligned} \tag{4.13}$$

When λ is small enough, (1.8) is satisfied, and the term with the integral in (4.13) is uniformly bounded by some constant $C \in (0,\infty)$. On the other hand, the number of marked agents $|A_t|$ is bounded by $\widehat{N}_t$, so finally we have

$$\left|\int_0^t \mathbb{E}_{\nu_\rho}\left[\xi_s(X_s)\right]\mathrm{d}s - \rho t\right| \leq L_t \leq tC\Big(|\alpha_1(\lambda)-\alpha_0(\lambda)| + |\beta_1(\lambda)-\beta_0(\lambda)|\Big),$$

proving (4.9). ∎

Proof of Theorem 1.4: Write

$$\begin{aligned} \frac{v(\lambda)-v(0)}{\lambda} &= \frac{(\alpha_1(\lambda)-\beta_1(\lambda))-(\alpha_1(0)-\beta_1(0))}{\lambda}\hat\rho(\lambda) \\ &\quad + (\alpha_1(0)-\beta_1(0))\frac{\hat\rho(\lambda)-\hat\rho(0)}{\lambda} \\ &\quad + \frac{(\alpha_0(\lambda)-\beta_0(\lambda))-(\alpha_0(0)-\beta_0(0))}{\lambda}(1-\hat\rho(\lambda)) \\ &\quad + (\alpha_0(0)-\beta_0(0))\frac{(1-\hat\rho(\lambda))-(1-\hat\rho(0))}{\lambda} \\ &= \frac{(\alpha_1(\lambda)-\beta_1(\lambda))-(\alpha_1(0)-\beta_1(0))}{\lambda}\hat\rho(\lambda) \\ &\quad + \frac{(\alpha_0(\lambda)-\beta_0(\lambda))-(\alpha_0(0)-\beta_0(0))}{\lambda}(1-\hat\rho(\lambda)). \end{aligned}$$

Now take the limit as $\lambda \downarrow 0$ and use (4.9) to get

$$\begin{aligned} v'(0) &= \left(\alpha\frac{F_1}{\rho} + \beta\frac{F_1}{\rho}\right)\rho + \left(\alpha\frac{F_0}{1-\rho} + \beta\frac{F_0}{1-\rho}\right)(1-\rho) \\ &= (\alpha+\beta)(F_1+F_0) = \alpha+\beta. \end{aligned}$$

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