

Selective buckling via states of self-stress in topological metamaterials

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States of self-stress—tensions and compressions of structural elements that result in zero net forces—play an important role in determining the load-bearing ability of structures ranging from bridges to metamaterials with tunable mechanical properties. We exploit a class of recently introduced states of self-stress analogous to topological quantum states to sculpt localized buckling regions in the interior of periodic cellular metamaterials. Although the topological states of self-stress arise in the linear response of an idealized mechanical frame of harmonic springs connected by freely hinged joints, they leave a distinct signature in the nonlinear buckling behavior of a cellular material built out of elastic beams with rigid joints. The salient feature of these localized buckling regions is that they are indistinguishable from their surroundings as far as material parameters or connectivity of their constituent elements are concerned. Furthermore, they are robust against a wide range of structural perturbations. We demonstrate the effectiveness of this topological design through analytical and numerical calculations as well as buckling experiments performed on two- and three-dimensional metamaterials built out of stacked kagome lattices.

isostatic lattices | topological modes | topological mechanics | tunable failure

Mechanical metamaterials are artificial structures with unusual properties that originate in the geometry of their constituents, rather than the specific material they are made of. Such structures can be designed to achieve a specific linear elastic response, like auxetic (negative Poisson ratio) (1) or pentamode (zero-shear modulus) (2) elasticity. However, it is often their nonlinear behavior that is exploited to engineer highly responsive materials, the properties of which change drastically under applied stress or confinement (3–7). Coordinated buckling of the building blocks of a metamaterial is a classic example of nonlinear behavior that can be used to drive the auxetic response (4, 8), modify the phononic properties (5), or generate 3D micro/nanomaterials from 2D templates (9).

Buckling-like shape transitions in porous and cellular metamaterials involve large deformations from the initial shape, typically studied through finite element simulations. However, many aspects of the buckling behavior can be successfully captured in an approximate description of the structure that is easier to analyze (5, 6, 10, 11). Here, we connect the mechanics of a cellular metamaterial, a foamlike structure made out of slender flexible elements (12, 13), to that of a frame—a simpler, idealized assembly of rigid beams connected by free hinges—with the same beam geometry. We exploit the linear response of a recently introduced class of periodic frames (14), inspired by topologically protected quantum materials, to induce a robust nonlinear buckling response in selected regions of two- and three-dimensional cellular metamaterials.

Frames, also known as trusses, are ubiquitous minimal models of mechanical structures in civil engineering and materials science. Their static response to an external load is obtained by balancing the forces exerted on the freely hinged nodes against internal stresses (tensions or compressions) of the beams. However, a unique set of equilibrium stresses may not always be found for any load (15). First, the structure may have loads that cannot be carried because they excite hinge motions called zero modes that leave all

beams unstressed. Second, the structure may support states of self-stress—combinations of tensions and compressions on the beams that result in zero net forces on each hinge. An arbitrary linear combination of states of self-stress can be added to an internal stress configuration without disrupting static equilibrium, implying that degenerate stress solutions exist for any load that can be carried. The respective counts N_m and N_{ss} of zero modes and states of self-stress in a d -dimensional frame with n_n nodes and n_b beams are related to each other by the generalized Maxwell relation (16, 17):

$$dn_n - n_b = N_m - N_{ss}. \quad [1]$$

As Eq. 1 shows, states of self-stress count the excess constraints imposed by the beams on the dn_n degrees of freedom provided by the hinge positions.

States of self-stress play a special role in the mechanical response of repetitive frames analyzed under periodic boundary conditions (18). Macroscopic stresses in such systems correspond to boundary loads, which can only be balanced by states of self-stress involving beams that cross the boundaries. If these states span the entire system, boundary loads are borne by tensions and compressions of beams throughout the structure. Conversely, by localizing states of self-stress to a small portion of a frame, load-bearing ability is conferred only to that region. Our approach consists of piling up states of self-stress in a specific region of a repetitive frame so that the beams participating in these states of self-stress are singled out to be compressed under a uniform load at the boundary. In a cellular material with the same beam geometry, these beams buckle when the compression exceeds their buckling threshold.

Significance

An ongoing challenge in modern materials science is to create artificial structures, termed metamaterials, with an unconventional response that can be programmed by suitable design of their geometry or topology. We demonstrate a novel metamaterial in which small structural variations single out regions that buckle selectively under external stresses. Surprisingly, these regions are at first glance indistinguishable from the rest of the structure. Our samples are examples of topological metamaterials that are the mechanical analogues of topologically protected electronic systems. As a result, the buckling response is robust against structural perturbations yet tunable without affecting secondary physical properties in the buckling region. This feature is desirable for optomechanical or thermomechanical metamaterials in which multiple characteristics need to be tuned simultaneously.

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Although our strategy is of general applicability, it is particularly suited to isostatic lattices (19, 20), with $dn_n = n_b$ and no zero modes other than the d rigid body motions under periodic boundary conditions. According to Eq. 1, these lattices only have d states of self-stress, insufficient to bear the $d(d+1)/2$ possible independent macroscopic stresses (18, 21). Additional states of self-stress can significantly modify the load-bearing ability of these structures at the cusp of elastic stability. Trivial states of self-stress can be created by locally adding extra beams between hinges. However, some isostatic periodic lattices can harbor topological states of self-stress that owe their existence to the global structure of the frame. These frames, introduced by Kane and Lubensky (14), are characterized by a polarization \mathbf{R}_T and its origin can be traced to topological invariants calculated from the geometry of the unit cell. Domain walls between different topological polarizations, as in Fig. 1A (14, 22), and lattice defects (23) can harbor localized states of self-stress, which can be used to drive localized buckling. Unlike their trivial counterparts, the existence of these topological states of self-stress cannot be discerned from a local count of the degrees of freedom or constraints in the region. An attractive feature for potential applications is their topological protection from perturbations of the lattice or changes in material parameters that do not close the acoustic gap of the structure (14, 24–27).

In the remainder of this article, we use robust states of self-stress to design buckling regions in topological metamaterials composed of flexible beams rigidly connected to each other at junctions. As shown in Fig. 1, the states of self-stress are localized to a quasi-2D domain wall obtained by stacking multiple layers of a pattern based on a deformed kagome lattice (14). The domain wall separates regions with different orientations of the same repeating unit (and hence of the topological polarization \mathbf{R}_T of the underlying frame). When the material is compressed uniaxially, beams participating in the topological states of self-stress will buckle out of their layers in the portion of the lattice highlighted in red, see [Movie S1](#). The region primed for buckling is indistinguishable from the remainder of the structure in terms of the density of beams per site and the characteristic beam slenderness. We verify using numerical calculations that the states of self-stress in the idealized frame influence the beam compressions of the cellular structure in response to in-plane compressions along the edges. The phenomenon survives even when the patterns on either side of the domain wall are nearly identical, reflecting the robustness of the topological design.

Linear Response of the Frame

The distinguishing feature of the isostatic periodic frame used in our design is the existence of a topological characterization of the underlying phonon band structure (14). If the only zero modes available to the structure are the d rigid-body translations, its phonon spectrum has an acoustic gap; i.e., all phonon modes have nonzero frequencies except for the translational modes at zero wavevector. A gapped isostatic spectrum is characterized by d topological indices n_i , one for each primitive lattice vector \mathbf{a}_i ($i \in \{1, 2\}$ for the 2D lattice used in our design). Although the phonon spectrum can be changed by smoothly deforming the unit cell geometry, the indices themselves are integer valued and thus invariant to smooth perturbations. Their value can only be changed by a deformation that closes the acoustic gap. Analogous to topological electronic systems such as quantum Hall layers and topological insulators that harbor protected edge states (28), the existence of these invariants guarantees the presence of zero modes or states of self-stress localized to lattice edges or domain walls where the n_i change value.

The count of topological mechanical states at an edge or domain wall is obtained via an electrostatic analogy. The lattice vector $\mathbf{R}_T = \sum_i n_i \mathbf{a}_i$ (Fig. 1A) can be interpreted as a polarization of net degrees of freedom in the unit cell. Just as Gauss's law

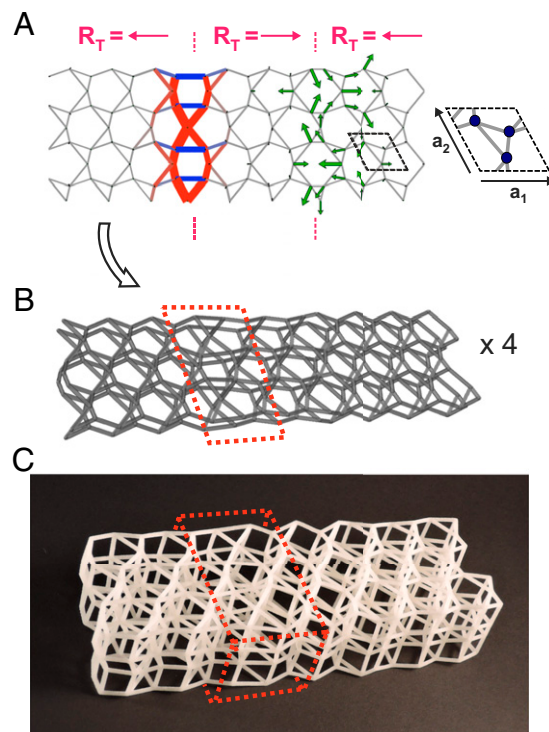


Fig. 1. Topological buckling zone in a cellular metamaterial. (A) Isostatic frame containing two domain walls that separate regions built out of opposite orientations of the same repeating unit (boxed; the zoom shows the three unique hinges in the unit cell as disks). This unit cell carries a polarization $\mathbf{R}_T = \mathbf{a}_1$ (solid arrow), and the periodic frame displays topological edge modes (14). The left domain wall harbors topological states of self-stress, one of which is visualized by thickened beams identifying equilibrium-maintaining tensions (red) and compressions (blue) with magnitudes proportional to the thickness. The right domain wall harbors zero modes, one of which is visualized by green arrows showing relative displacements that do not stretch or compress the beams. Modes were calculated using periodic boundary conditions. (B) A 3D cellular metamaterial is obtained by stacking four copies of the beam geometry in A, and connecting equivalent points with vertical beams to obtain a structure with each interior point connecting six beams. The beams are rigidly connected to each other at the nodes and have a finite thickness. Points are perturbed by random amounts in the transverse direction and a small offset is applied to each layer to break up straight lines of beams. A 3D-printed realization of the design made of flexible plastic (*Materials and Methods*) is shown in C. The sample has a unit cell size of 25 mm and beams with circular cross-section of 2 mm diameter. The stacking creates a pile-up of states of self-stress in a quasi-2D region, highlighted by dotted lines.

yields the net charge enclosed in a region from the flux of the electric polarization through its boundary, the net number of states of self-stress (minus the zero modes) in an arbitrary portion of an isostatic lattice is given by the flux of the topological polarization through its boundary (14). In Fig. 1A, the left domain wall has a net outflux of polarization and harbors topological states of self-stress (the right domain wall, with a polarization influx, harbors zero modes). Although only one of each mechanical state is shown, the number of states of self-stress and zero modes is proportional to the length of the domain wall. Similar results can be obtained by using other isostatic lattices with an acoustic gap and a topological polarization, such as the deformed square lattice in ref. 23.

The linear response of a frame can be calculated from its equilibrium matrix \mathbf{A} , a linear operator that relates the beam tensions \mathbf{t} (a vector with as many elements as the number of beams n_b) to the resultant forces on the nodes \mathbf{p} (a vector with one element for each of the $2n_n$ degrees of freedom of the n_n nodes) via $\mathbf{A}\mathbf{t} = \mathbf{p}$. States of self-stress are vectors \mathbf{t}^q that satisfy

$\mathbf{A}\tilde{\mathbf{t}}^q = \mathbf{0}$, i.e., they are beam stresses that do not result in net forces on any nodes (the index q identifies independent normalized states of self-stress that span the null space of \mathbf{A}). The same null vectors are also an orthogonal set of incompatible strains of the structure, i.e., beam extensions that cannot be realized through any set of point displacements (15).

Because we are interested in triggering buckling through uniform loads that do not pick out any specific region of the lattice, we focus on the response to affine strains, where affine beam extensions \mathbf{e}_a are imposed geometrically by some uniform strain ε_{ij} across the sample via

$$e_{a\alpha} = \tilde{r}_i^\alpha \varepsilon_{ij} r_j^\alpha. \quad [2]$$

Here, α indexes the beams and \mathbf{r}^α is the end-to-end vector of beam α . To attain equilibrium, the beams take on additional nonaffine extensions \mathbf{e}_{na} . Under periodic boundary conditions, affine strains are balanced by loads across the system boundary rather than loads on specific nodes, which implies that the resultant beam tensions $\mathbf{t}_a = k(\mathbf{e}_a + \mathbf{e}_{na})$ must be constructed solely out of states of self-stress (18, 29) (we assume for simplicity that all beams have identical spring constant k). Therefore, $\mathbf{t}_a = \sum_q x_q \tilde{\mathbf{t}}^q \equiv \tilde{\mathbf{T}}\mathbf{x}$ where $\tilde{\mathbf{T}} = [\tilde{\mathbf{t}}^1, \tilde{\mathbf{t}}^2, \dots, \tilde{\mathbf{t}}^{N_{ss}}]$ and the x_q are the weights of the N_{ss} states of self-stress. These weights are determined by requiring that the nonaffine strains have zero overlap with the set of incompatible strains (15) (the affine strains are automatically compatible with the affine node displacements):

$$\tilde{\mathbf{T}}^T \mathbf{e}_{na} = \tilde{\mathbf{T}}^T \left(\frac{1}{k} \tilde{\mathbf{T}}\mathbf{x} - \mathbf{e}_a \right) = 0, \quad [3]$$

which gives the solution

$$\mathbf{x} = k\tilde{\mathbf{T}}^T \mathbf{e}_a \quad [4]$$

$$\Rightarrow \mathbf{t}_a = k\tilde{\mathbf{T}}\tilde{\mathbf{T}}^T \mathbf{e}_a = k \sum_q (\tilde{\mathbf{t}}^q \cdot \mathbf{e}_a) \tilde{\mathbf{t}}^q. \quad [5]$$

Therefore, the beam tensions under an affine deformation are obtained by projecting the affine strains onto the space of states of self-stress.

Eq. 5 shows that the loading of beams under affine strains is completely determined by the states of self-stress. In a frame consisting of a single repeating unit cell, loads are borne uniformly across the structure. However, if the structure also has additional states of self-stress with nonzero entries in $\tilde{\mathbf{t}}^q$ confined to a small region of the frame, it would locally enhance tensions and compressions in response to affine strains, provided the $\tilde{\mathbf{t}}^q$ have a nonzero overlap with the affine bond extensions imposed by the strain. Fig. 2 shows the approximate states of self-stress for the domain wall geometry of Fig. 1A, which has a nonzero overlap with affine extensions \mathbf{e}_x and \mathbf{e}_y due to uniaxial strains $\varepsilon_{ij} = \delta_{ix}\delta_{jx}$ and $\varepsilon_{ij} = \delta_{iy}\delta_{jy}$, respectively. (These approximate states of self-stress become exact when the separation between the two domain walls becomes large.) The system-spanning states of self-stress $\tilde{\mathbf{t}}^a$ and $\tilde{\mathbf{t}}^b$ in Fig. 2A and B, respectively, do not single out any particular region. Although they have a nonzero overlap with both \mathbf{e}_x and \mathbf{e}_y , they do not provide significant stiffness to a uniaxial loading because a combined affine strain $\varepsilon_{ij} = \delta_{ix}\delta_{jx} + \beta\delta_{iy}\delta_{jy}$ exists with $\beta \approx -1.2$ such that the overlaps of the corresponding affine extensions with $\tilde{\mathbf{t}}^a$ and $\tilde{\mathbf{t}}^b$ are small (less than 10^{-5}). Upon compression along one direction, say y , the frame can expand in the perpendicular direction to keep the tensions and compressions low in the majority of the sample. In contrast, the localized states of self-stress shown in Fig. 2C and D have a significant overlap with \mathbf{e}_y but not \mathbf{e}_x . Therefore, according to Eq. 5, a uniform compression applied to the lattice along the

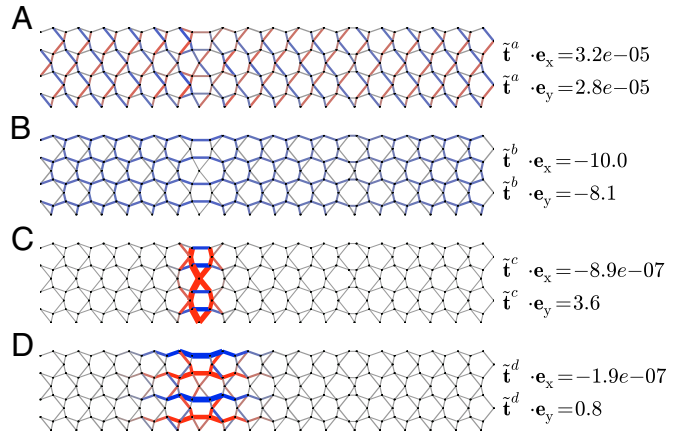


Fig. 2. Extended and localized states of self-stress of the frame. (A–D) States of self-stress in the infinite periodic frame obtained by tiling the design of Fig. 1A in both directions. The states A and B are largely uniform over the structure, whereas C and D are localized to the left domain wall. The overlaps of each state of self-stress with the affine strains \mathbf{e}_x and \mathbf{e}_y (main text) are also shown. Only states of self-stress with significant overlaps are shown; all other states of self-stress in the structure have $\tilde{\mathbf{t}}^i \cdot \mathbf{e}_{(x,y)} < 10^{-5}$.

vertical direction, with the horizontal direction free to respond by expanding, will significantly stretch or compress only the beams participating in the localized states of self-stress.

Whereas the topological polarization guarantees the presence of states of self-stress localized to the left domain wall, their overlap with one of the three independent affine strains is determined by the specific geometry of the hinges and bars. For the frame in Fig. 2, the states of self-stress visualized in Fig. 2C and D are crucial in triggering buckling response, which may be predicted to occur under compression along the y direction (or extension along the x direction, which would also lead to y compression because the lattice has a positive Poisson ratio set by β). In other domain wall geometries, or other orientations of the domain wall relative to the lattice, the localized states of self-stress could have small overlap with affine strains, which would make them inconsequential to the buckling behavior. Alternatively, states of self-stress that overlap significantly with shear deformations could also be realized that would enable buckling to be triggered via shear.

Buckling in Topological Cellular Metamaterials

The cellular metamaterial differs from the ideal frame in two important ways. First, real structures terminate at a boundary, and loads on the boundary are no longer equilibrated by states of self-stress, but rather by tension configurations that are in equilibrium with forces on the boundary nodes (15). However, these tension states are closely related to system-traversing states of self-stress in the periodic case, with the forces on the boundary nodes in the finite system playing the role of the tensions exerted by the boundary-crossing beams in the periodic system. Therefore, the states of self-stress also provide information about the load-bearing regions in the finite structure away from the boundary.

In addition to having boundaries, the cellular block probed in Fig. 1 also departs from the limit of an ideal frame, as it is made of flexible beams rigidly connected at the nodes and can support external loads through shear and bending of the beams in addition to axial stretching or compression. Nevertheless, the states of self-stress and tension states of the corresponding frame (with the same beam geometry) determine the relative importance of bending to stretching in the load-bearing ability of the cellular structure (21). To verify that the localized states of self-stress in the underlying frame influence the response of the finite cellular

structure, we numerically calculate the in-plane response of each layer treated as an independent 2D cellular structure with loading confined to the 2D plane. Each beam provides not just axial tension/compression resistance but also resistance to shear and bending. A beam of length L , cross-sectional area A , and area moment of inertia I resists (i) axial extensions e with a tension $t = (EA/L)e$, (ii) transverse deformations e_s with a restoring shear force $s = 12(EI/L^3)e_s$, and (iii) angular deflections of the end nodes θ with restoring moment $m = (EI/L)\theta$ (Fig. 3). Because $A \sim w^2$ and $I \sim w^4$ for a beam of width w , the relative contribution of the bending, shear and torsional components to the total stiffness is set by the aspect ratio w/L of each beam; the frame limit with no bending or shear stiffness is recovered when $w/L \rightarrow 0$. The physical sample has beams with aspect ratios in the range $0.1 \lesssim w/L \lesssim 0.2$, indicating a small but appreciable contribution of shear and torsion to the response.

The 2D linear response of such a structure is calculated by augmenting the equilibrium matrix \mathbf{A} to include an additional degree of freedom (a rigid rotation angle) at each node, and two additional restoring forces (shear and torsion) for each beam, see *Materials and Methods*. The resulting equilibrium matrix is of size $3n_n \times 3n_b$, which implies that cellular solids based on isostatic frames with $n_b \approx dn_n$ are severely overconstrained for $d=2,3$, and structurally stable even with free boundaries. In particular, they can support any loads exerted on the boundary nodes as long as the net forces and torques about the center of mass are zero. Once the equilibrium matrix is constructed, its singular value decomposition can be used to completely determine all stresses and torques experienced by the beams in response to forces

specified on the boundary nodes (15). Fig. 3 shows the linear response of each plane of the cellular pattern corresponding to the domain wall geometry of Fig. 2 under uniform compressive force applied to the nodes on the top and bottom edges. The beams participating in the states of self-stress at the left domain wall are singled out by their high compression; in contrast, the rest of the structure primarily supports the boundary load through shear rather than compression or bending. Remarkably, the unique compression-dominated response of the left domain wall (originating from the topological invariant in the idealized isostatic frame with freely hinged joints) survives in cellular structures away from the isostatic condition.

We expect a similar localized compression-dominated response in each layer of the stacked structure (Fig. 1 *B* and *C*). The enhanced compressions along the left domain wall trigger buckling when the compression exceeds the Euler beam buckling threshold, $t_i < -cEI/L_i^2$, where c is a positive numerical factor determined by the clamping conditions as well as cooperative buckling effects. Buckling is signified by a loss of ability to bear axial loads, as the beam releases its compression by bending out of plane. Upon compressing the 3D sample between two plates as shown in Fig. 4*A*, we see a significant out-of-plane deflection for beams along the left domain wall (Fig. 4 *B* and *C*), consistent with buckling of the maximally stressed beams in Fig. 3*A*. The deflection in different layers is coordinated by the vertical beams connecting equivalent points, so that beams within the same column buckle either upwards (column 1) or downward (column 2) to produce a distinctive visual signature when viewed along the compression axis. Beams connecting different planes in the stacked pattern create additional states of self-stress that traverse the sample vertically, but these do not single out any region of the material, and do not couple to the specific in-plane loading of Fig. 4*A*.

We emphasize that having as many states of self-stress as there are unit cells along the domain wall is crucial for the buckling to occur throughout the domain wall. When a beam buckles, its contribution to the constraints of distances between nodes essentially disappears, reducing the number of load-bearing configurations by one. If there were only a single localized state of self-stress in the system, the buckling of a single beam would eliminate this state, and the compressions on the other beams would be relaxed, preventing further buckling events. However, the presence of multiple states of self-stress, guaranteed in this case by the topological origin of the states, allows many buckling events along the domain wall. In the *SI Text*, we use an adaptive simulation that sequentially removes highly compressed beams to show that each repeating unit along the y direction experiences a unique buckling event even when the loss of constraints due to other buckling events is taken into account (Fig. S1).

Robustness of the Buckling Region

Finally, we show that the robustness of the topological states of self-stress predicted within linear elastic theory carries over to the buckling response in the nonlinear regime. There is a wide range of distortions of the deformed kagome unit cell which do not close the acoustic gap, leaving the topological invariants n_i and hence the polarization \mathbf{R}_T unchanged (14). These distortions may bring the unit cell arbitrarily close to the regular kagome lattice with equilateral triangles of beams (for which the gap is closed and the polarization is undefined). The unit cell of the 2D lattice shown in Fig. 5*A* is minimally distorted away from the regular kagome lattice, but this barely noticeable distortion (Fig. 5*A*, *Inset*) is sufficient to induce the same topological polarization \mathbf{R}_T in the underlying frame as before. As a result, the domain wall on the left (characterized by a net outflux of polarization) localizes states of self-stress (shown in Fig. S2), even though the unit cells are nearly identical on either side.

For ease of visualization, we tested this design in a 2D prototype cellular metamaterial, obtained by laser-cutting voids in a

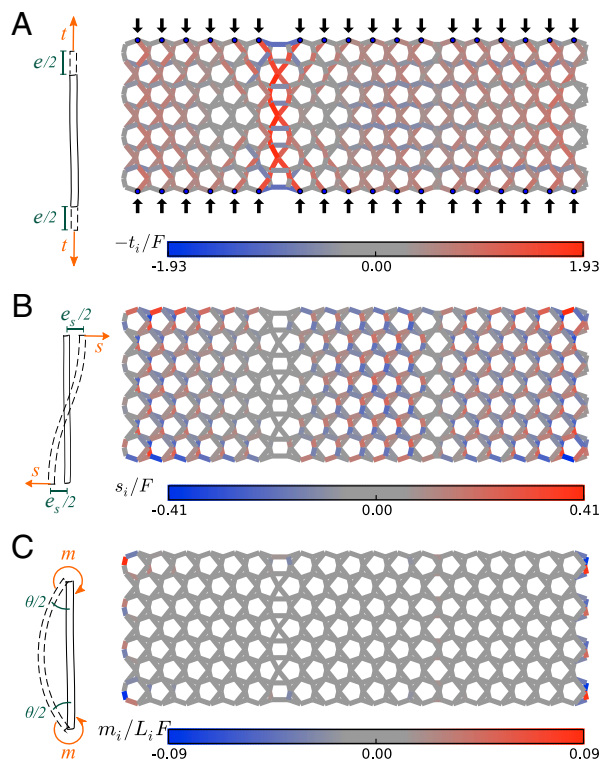


Fig. 3. Stretching, shear, and bending contributions to the linear in-plane response of the cellular metamaterial. Response of a planar cellular structure, related to the lattice in Fig. 2 but with free edges, subject to a vertical compressive force F (solid arrows) at each point highlighted along the top and bottom edges. The structure is modeled as a network of flexible beams connected by rigid joints at the nodes, and with each beam providing torsional stiffness in addition to axial stiffness. The beams are colored according to (A) axial compression; (B) shear load; and (C) bending moment.

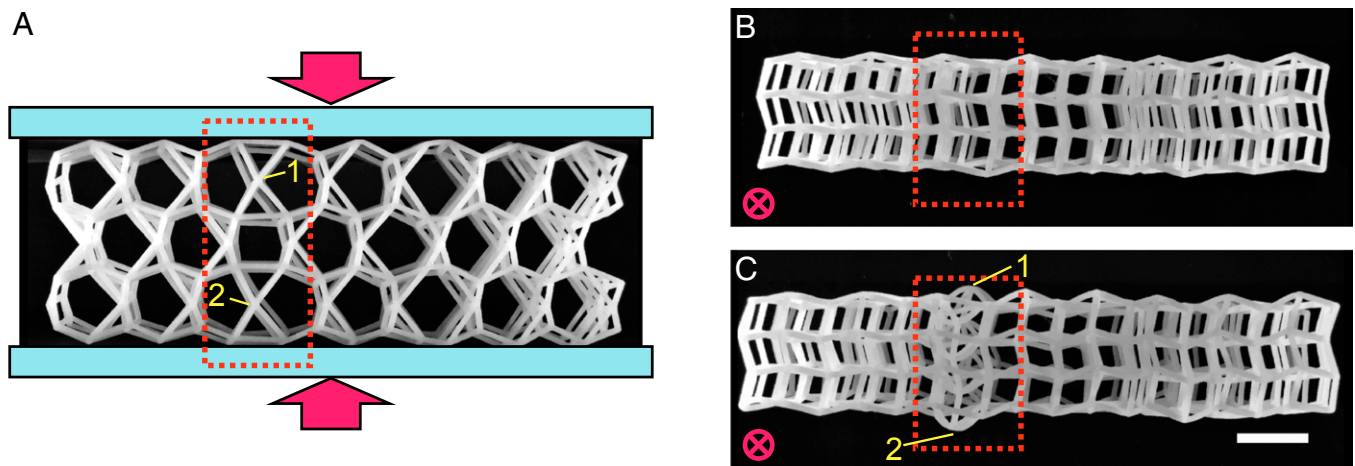


Fig. 4. Buckling in the 3D topological cellular metamaterial. (A) Top view of the 3D sample (constructed as outlined in Fig. 1) showing the compression applied by confining the sample between transparent plates in contact with the front and back surfaces. The buckling zone is highlighted in red as in Fig. 1. Two vertical columns within this zone are labeled in yellow. Magenta arrows show the compression direction. (B and C) View along the compression axis at compressions of 0 and 20%, respectively. The beams in the region with states of self-stress have buckled in the vertical direction, whereas other beams have largely deformed within their stacking planes. (Scale bar, 25 mm.)

1.5-cm-thick slab of polyethylene foam (*Materials and Methods*), leaving behind beams 1–2 mm wide and 10–12 mm long (Fig. 5A). The aspect ratio of the beams is comparable to that of the 3D sample. Because the slab thickness is much larger than the beam width, deformations are essentially planar and uniform through the sample thickness, and can be captured by an overhead image of the selectively illuminated top surface. The sample was confined between rigid acrylic plates in free contact with the top

and bottom edges, and subjected to a uniaxial in-plane compression along the vertical direction by reducing the distance between the plates. Using image analysis (*Materials and Methods*) we computed the beams' tortuosity, defined as the ratio of the contour length of each beam to its end-to-end distance. Buckled beams have a tortuosity significantly above 1. Under a vertical compression of 4%, only beams along the left domain wall show a tortuosity above 1.05, consistent with a localized buckling response (Fig. 5B). However, under higher strains, the response of the lattice qualitatively changes. When beams have buckled along the entire domain wall, its response to further loading is no longer compression dominated (Fig. S1). Future buckling events, which are triggered by coupling between torsional and compressional forces on beams due to the stiff hinges, no longer single out the left domain wall and happen uniformly throughout the sample (Fig. 5C and *Movie S2*).

We have demonstrated that piling up localized states of self-stress in a small portion of an otherwise bending-dominated cellular metamaterial can induce a local propensity for buckling. Whereas this principle is of general applicability, our buckling regions exploit topological states of self-stress (14), which provide two advantages. First, they are indistinguishable from the rest of the structure in terms of node connectivity and material parameters, allowing mechanical response to be locally modified without changing the thermal, electromagnetic, or optical properties. This feature could be useful for optomechanical (30) or thermo-mechanical (31) metamaterial design. Second, the buckling regions are robust against structural perturbations, as long as the acoustic bulk gap of the underlying frame is maintained. This gap is a property of the unit cell geometry. Large deformations that close the gap could be induced through external actuation (32) or confinement, potentially allowing a tunable response from localized to extended buckling in the same sample, reminiscent of electrically tunable band gaps in topological insulators (33).

Materials and Methods

Deformed Kagome Lattice Unit Cell. The deformed kagome lattices are obtained by decorating a regular hexagonal lattice, built from the primitive lattice vectors $\{\mathbf{a}_1 = a\hat{x}, \mathbf{a}_2 = -(a/2)\hat{x} + (\sqrt{3}a/2)\hat{y}\}$, where a is the lattice constant, with a three-point unit cell that results in triangles with equal sizes. The unit cells are described by a parameterization introduced in ref. 14, which uses three numbers (x_1, x_2, x_3) to quantify the distortion of lines of bonds away from a regular kagome lattice ($x_i = 0$). The unit cell in Fig. 1A, which forms the basis for the 3D cellular structure, is reproduced by $(x_1, x_2, x_3) = (-0.1, 0.06, 0.06)$

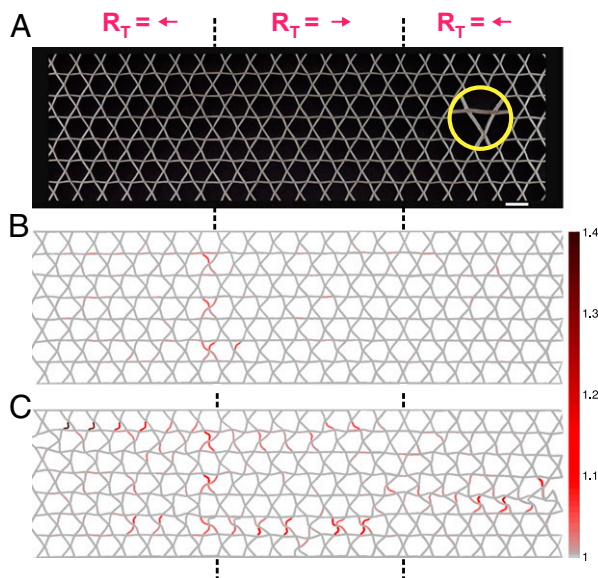


Fig. 5. Buckling is robust under polarization-preserving changes of the unit cell. (A) A 2D foam cellular prototype, whose unit cell maintains the topological polarization R_T even though its distortion away from the regular kagome lattice is small (a zoom of the constituent triangles is shown within the yellow circle). The domain wall geometry is identical to that of the 3D sample, with the left domain wall localizing states of self-stress. (Scale bar, 2 cm.) (B) Response of the structure under a vertical compression of 4% with free left and right edges. The beams are colored by the tortuosity, the ratio of the initial length of the beam to the end-to-end distance of the deformed segment (color bar). (C) Response of the structure under 7% compression, with beams colored by tortuosity using the same color scale as in B.

for which the topological polarization is $\mathbf{R}_T = -\mathbf{a}_i$ (14). The unit cell for the design in Fig. 5A parameterized by $(-0.025, 0.025, 0.025)$ has the same polarization. In all cases we describe the unit cell and polarization in the outer region; the inner region between the domain walls rotates this unit cell by π , which flips the polarization direction.

Construction of the Equilibrium/Compatibility Matrix. Analysis of the linear response of a frame or a cellular material begins with the construction of the equilibrium matrix \mathbf{A} . Because it is more natural to relate node displacements to beam length changes, we describe how to build the compatibility matrix $\mathbf{C} = \mathbf{A}^T$, relating point displacements \mathbf{u} to extensions \mathbf{e} via $\mathbf{Cu} = \mathbf{e}$. We construct the compatibility matrix from the contributions of individual beams. Consider a single beam aligned to the x axis connecting hinge 1 at $(0, 0)$ to hinge 2 at $(L, 0)$. There are six degrees of freedom potentially constrained by the beam: the positions (x_i, y_i) at each node i and the torsion angle θ_i . Within Euler–Bernoulli beam theory, these values are sufficient to determine the shape of the beam along its length as well as the forces and torques at each end needed to maintain equilibrium (34). Three independent combinations of forces and torques can be identified and which are proportional to generalized strains experienced by the beam: pure tension $t \propto x_2 - x_1$, pure shear $s \propto y_2 - y_1 - L(\theta_1 + \theta_2)/2$, and pure bending torque $m \propto \theta_2 - \theta_1$, as illustrated schematically in Fig. 3. In matrix form, this gives

$$\mathbf{e} = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & -L/2 & 0 & 1 & -L/2 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ y_1 \\ \theta_1 \\ x_2 \\ y_2 \\ \theta_2 \end{pmatrix} \equiv \mathbf{Cu}. \quad [6]$$

The forces and torque (generalized stresses) are obtained from the generalized strains \mathbf{e} through the stiffness matrix \mathbf{K} , which depends on the Young's modulus E , the cross-sectional area A , the area moment of inertia I , and the beam length L :

$$\boldsymbol{\sigma} \equiv \begin{pmatrix} t \\ s \\ m \end{pmatrix} = \begin{pmatrix} EA/L & 0 & 0 \\ 0 & 12EI/L^3 & 0 \\ 0 & 0 & EI/L \end{pmatrix} \cdot \mathbf{e} \equiv \mathbf{Ke}. \quad [7]$$

The compatibility matrix for a beam with arbitrary orientation is obtained by projecting the displacement vectors at the end of each beam onto the axial and

transverse directions using the appropriate rotation matrix, which depends on the angle made by the beam with the x axis. Each beam in an assembly provides three rows to the compatibility matrix, with additional columns set to zero for the degrees of freedom unassociated with that beam. In the simpler frame limit, each beam only resists axial extensions, and contributes one row (the first row of the \mathbf{C} matrix in Eq. 6) to the compatibility matrix.

Once the equilibrium matrix is constructed, the approximate states of self-stress of the periodic frame (Fig. 2) as well as the linear response of the cellular material under loads (Fig. 3) are obtained from its singular value decomposition following the methods of ref. 15. More details of the computation are provided in the *SI Text*.

Construction and Characterization of 2D and 3D Prototypes. The 3D structures were printed by Materialise N.V. through laser sintering of their proprietary thermoplastic polyurethane TPU 92A-1 (tensile strength 27 MPa, density 1.2 g/cm³, Young's modulus 27 MPa).

The 2D structures were cut using a VersaLaser 3.5 laser cutter (Laser & Sign Technology) from 1.5 cm thick sheets of closed-cell cross-linked polyethylene foam EKI-1306 (EKI B.V.; tensile strength 176 kPa, density 0.03 g/cm³, Young's modulus 1.7 MPa). A characteristic load-compression curve of a 2D sample is shown in Fig. S3.

Image Analysis of 2D Experiment. Images of the 2D cellular prototype (Fig. 5) were obtained using a Nikon CoolPix P340 camera and stored as 3,000 × 4,000 px 24-bit JPEG images. To quantitatively identify the buckled beams in the 2D prototype under confinement, we extracted the tortuosity τ of each beam, defined as the ratio of the length of the beam to the distance between its endpoints. Tortuosity was estimated from the sample images through a series of morphological operations, as detailed in *SI Text* and Fig. S4. Straight beams have $\tau = 1$, whereas beams that buckle under axial compression have $\tau > 1$. Unlike other measures of curvature, tortuosity distinguishes buckled beams from sheared beams (schematic, Fig. 3B), which have $\tau \geq 1$.

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