Author: Erhard, Dirk
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The Parabolic Anderson Model and Long-Range Percolation

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Dirk Erhard

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Promotor:	Prof. Dr. W. Th. F. den Hollander (Universiteit Leiden)
Second promotor:	Dr. G. Maillard (Université d'Aix-Marseille)
Overige leden:	Prof. Dr. W. König (Technische Universität Berlin)
	Prof. Dr. P. Mörters (University of Bath)
	Prof. Dr. R. van der Hofstad (Technische Universiteit Eindhoven)

Contents

Preface

This thesis has two parts.

Part I deals with the parabolic Anderson model. This is the partial differential equation $\partial u(x,t)/\partial t = \kappa\Delta u(x,t) + \xi(x,t)u(x,t)$, $x \in \mathbb{Z}^d$, $t \geq 0$, where the u-field and the ξ-field are $\mathbb{R}$-valued, $\kappa \in [0,\infty)$ is the diffusion constant, and Δ is the discrete Laplacian. The ξ-field plays the role of a dynamic random environment that drives the equation. We take the initial condition $u(x,0) = u_0(x)$, $x \in \mathbb{Z}^d$, to be non-negative and bounded. The solution of the parabolic Anderson equation describes the evolution of a field of particles performing independent simple random walks with binary branching: particles jump at rate $2d\kappa$, split into two at rate $\xi \vee 0$, and die at rate $(-\xi) \vee 0$. The question of interest is how the exponential growth rate of u depends on the diffusion constant κ. This can be monitored via the annealed and quenched Lyapunov exponent. We focus on the latter.

Part II deals with two different percolation models. The occupied set of the first percolation model is obtained by taking the union of a collection of independent Brownian motions running up to time $t \geq 0$, whose initial positions are distributed according to a Poisson point process. The question we investigate is whether the occupied set undergoes a non-trivial percolation phase transition in t or not. We further investigate the uniqueness of the unbounded components in the supercritical regime. The occupied set of the second percolation model is given by the random interlacement set. This is a family of random subsets $\mathcal{I}^u$, $u \geq 0$, on $\mathbb{Z}^d$, $d \geq 3$, that locally describes the trace of a simple random walk on the torus $(\mathbb{Z}/N\mathbb{Z})^d$ running up to time uN^d. It has been shown that the vacant set $\mathcal{V}^u = \mathbb{Z}^d \setminus \mathcal{I}^u$ undergoes a non-trivial percolation phase transition in u. We describe the geometry of the vacant set $\mathcal{V}^u$ in the supercritical regime for intensities u that are close to the criticial percolation parameter.

Part I (Chapters $1-3$) deals with the parabolic Anderson model and is based on the articles [EdHM14a] and [EdHM14b]. **Part II** (Chapters $4-6$) deals with the two percolation models and is based on the articles [EMP14] and [DE14].

Part I The Parabolic Anderson Model

1 Introduction to Part I

1.1 The parabolic Anderson model

This section borrows parts of Section 1 in Erhard, den Hollander, Maillard [EdHM14a].

The Model

The parabolic Anderson model is the partial differential equation

$$\frac{\partial}{\partial t}u(x,t) = \kappa\Delta u(x,t) + \xi(x,t)u(x,t), \qquad x \in \mathbb{Z}^d,\ t \geq 0. \tag{1.1.1}$$

Here, the u-field is $\mathbb{R}$-valued, $\kappa \in [0,\infty)$ is the diffusion constant, Δ is the discrete Laplacian acting on u as

$$\Delta u(x,t) = \sum_{\substack{y\in\mathbb{Z}^d \\ \|y-x\|=1}} [u(y,t) - u(x,t)] \tag{1.1.2}$$

($\|\cdot\|$ is the l_1-norm), while

$$\xi = (\xi_t)_{t\geq 0} \quad \text{with} \quad \xi_t = \{\xi(x,t)\colon\ x \in \mathbb{Z}^d\} \tag{1.1.3}$$

is an $\mathbb{R}$-valued random field playing the role a of *dynamic random environment* that drives the equation. As initial condition for (1.1.1) we take

- ▶ $u(x,0) = u_0(x)$, $x \in \mathbb{Z}^d$, with u_0 non-negative and bounded. (1.1.4)

The Feynman-Kac formula

A formal solution of (1.1.1) and (1.1.4) is given by the Feynman-Kac formula

$$u(x,t) = E_x\left(\exp\left\{\int_0^t \xi(X^\kappa(s), t-s)\, ds\right\} u_0(X^\kappa(t))\right). \tag{1.1.5}$$

Here, $X^\kappa = (X^\kappa(t))_{t\geq 0}$ is the continuous-time simple random walk jumping at rate $2d\kappa$, with law P_x and expectation E_x when $X^\kappa(0) = x$. The representation in (1.1.5) may, on a heuristic level, be explained as follows. Consider the equations

$$\frac{\partial}{\partial t}v(x,t) = \kappa\Delta v(x,t) \quad \text{and} \quad \frac{\partial}{\partial t}w(x,t) = \xi(x,t)w(x,t) \tag{1.1.6}$$

with initial conditions v_0 and w_0, respectively. Note that $\kappa\Delta$ is the generator of X^κ. Therefore, the solution to the first equation is given by $v(x,t) = E_x(v_0(X^\kappa(t)))$. The solution to the second equation is given by $w(x,t) = e^{\int_0^t \xi(x,s)\,ds} w_0(x)$. Thus, (1.1.5) may be interpreted as a combination of these two solutions. The Laplacian in the first equation has the tendency to make the solution flat (e.g. $v_0 \equiv 1$ implies $v \equiv 1$). On the other hand, in the second equation there is no such smoothing and w is irregular. In (1.1.1), these two effects compete with each other, and it is this competition that makes the model more appealing but also more complicated to study.

Two interpretations of (1.1.1)

• One interpretation comes from *population dynamics.* Consider the special case where $\xi(x,t) = \gamma\bar{\xi}(x,t) - \delta$ with $\delta,\gamma \in (0,\infty)$ and $\bar{\xi}$ an $\mathbb{N}_0$-valued random field. Consider a system of two types of particles, A (catalyst) and B (reactant), subject to:

- A-particles evolve autonomously according to a prescribed dynamics with $\bar{\xi}(x,t)$ denoting the number of A-particles at site x at time t;
- B-particles perform independent simple random walks at rate $2d\kappa$ and split into two at a rate that is equal to γ times the number of A-particles present at the same location at the same time;
- B-particles die at rate δ;
- the average number of B-particles at site x at time 0 is $u_0(x)$.

Then

$$u(x,t) \quad = \quad \begin{array}{l}\text{the average number of } B\text{-particles at site } x \text{ at time } t \\ \text{conditioned on the evolution of the } A\text{-particles.}\end{array} \tag{1.1.7}$$

• Another interpretation comes from *random walk moving through a random field of sinks and sources.* Here, sites $(x,t) \in \mathbb{Z}^d \times [0,\infty)$ with $\xi(x,t) < 0$ are interpreted as sinks and sites $(x,t) \in \mathbb{Z}^d \times [0,\infty)$ with $\xi(x,t) > 0$ are interpreted as sources. The case in which ξ does not depend on time and is such that $\xi \in \{-\infty, 0\}$ has a particularly nice interpretation in terms of *survival probabilities,* and is sometimes referred to as *random walk among Bernoulli traps.* More precisely, assume that $u_0 \equiv 1$ and let

$$\mathcal{O} = \{z \in \mathbb{Z}^d : \xi(z) = -\infty\} \tag{1.1.8}$$

be the set of traps. Then the Feynman-Kac formula (1.1.5) reads

$$u(x,t) = P_x(X^\kappa(s) \subset \mathcal{O}^c \text{ for all } s \in [0,t]), \tag{1.1.9}$$

i.e., $u(x,t)$ equals the probability that the random walk X^κ starting at x does not get killed by any of the traps in $\mathcal{O}$ until time t.

Three problems related to (1.1.1)

• Burger's equation

Burger's equation is a fundamental equation in hydrodynamics (see [CM94]), and reads

$$\begin{cases} \frac{\partial}{\partial t} v(x,t) + v(x,t) \cdot \nabla v(x,t) = \kappa \Delta v(x,t) + f(x,t), \\ v(x,0) = v_0(x), \end{cases} \qquad x \in \mathbb{R}^d. \tag{1.1.10}$$

Here, Δ denotes the usual Laplacian in $\mathbb{R}^d$, ∇ denotes the gradient, the v-field is $\mathbb{R}^d$-valued and f is an external force. In the case where v_0 and f can be written as gradients, the solution of (1.1.10) may be obtained via the substitution

$$v(x,t) = -2\kappa \nabla \log \varphi(x,t). \tag{1.1.11}$$

Substituting (1.1.11) into (1.1.10), we get

$$\begin{cases} \frac{\partial}{\partial t} \varphi(x,t) = \kappa \Delta \varphi(x,t) + F(x,t)\varphi(x,t), \\ -2\kappa \nabla \varphi_0(x) = v_0(x), \\ \nabla F(x,t) = f(x,t). \end{cases} \tag{1.1.12}$$

Thus, (1.1.10) is transformed into the continuum version of the parabolic Anderson equation (1.1.1) and it is therefore enough to study the behaviour of φ.

• Advection-Convection equation for a temperature field

Consider the following equation for the scalar temperature field T:

$$\begin{cases} \frac{\partial}{\partial t} T(x,t) + v(x,t) \cdot \nabla T(x,t) = \kappa \Delta T(x,t), \\ T(x,0) = T_0(x), \quad x \in \mathbb{R}^d. \end{cases} \tag{1.1.13}$$

This equation is used for the analysis of turbulent diffusions. It is argued in [CM94] (see also [AM90]), that if the dimension is two and the velocity field v has the form

$$v(x_1,x_2) = \begin{bmatrix} 0 \\ v(x_1) \end{bmatrix}, \tag{1.1.14}$$

then, under some additional assumptions, the Laplace transform T_λ, $\lambda \in \mathbb{R}$, of T with respect to the second space coordinate, satisfies the equation

$$\frac{\partial T_\lambda(t,x_1)}{\partial t} = \kappa \frac{\partial^2 T_\lambda(t,x_1)}{\partial x_1^2} + (\lambda^2 - \lambda v(x_1)) T_\lambda(t,x_1) \tag{1.1.15}$$

with initial condition

$$T_\lambda(0,x_1) = \int_{-\infty}^{\infty} T(0,x_1,x_2) e^{\lambda x_2}\, dx_2. \tag{1.1.16}$$

This is a space continuum version of equation (1.1.1).

• **Random motion in random media**

The parabolic Anderson model is an example of a class of models for random motion in random media. Indeed, from the Feynman-Kac formula (1.1.5) we see that, in order to understand the solution of (1.1.1), we have to understand how X^κ sees ξ. This immediately links (1.1.1) to the model of *random walk in random scenery,* where the typical behaviour of $\int_0^t \xi(X^\kappa(s), s)\, ds$ is studied rather than the large deviation behaviour of $\int_0^t \xi(X^\kappa(s), t-s)\, ds$. Equation (1.1.1) is also deeply connected to random polymer models, since $\int_0^t \xi(X^\kappa(s), t-s)\, ds$ may be considered as a Hamiltonian and the Feynman-Kac formula is the corresponding partition function. *Random walks in random environment* are in the same spirit. Here, instead of the random branching rates as explained in the lines preceding (1.1.7), the transition probabilities of the random walk are random. All these models have in common that two types of randomness interact with each other and it is the goal to understand this interaction.

1.2 The parabolic Anderson model in a static random environment

In this section we give an overview of the state of the art of the parabolic Anderson equation when ξ does not depend on time, i.e., when (1.1.1) is of the form

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) = \kappa \Delta u(x,t) + \xi(x) u(x,t), \\ u(x,0) = u_0(x). \end{cases} \tag{1.2.1}$$

Here, $\xi = \{\xi(x), x \in \mathbb{Z}^d\}$ is a collection of i.i.d. random variables with law $\mathbb{P}$. We refer to (1.2.1) as the *static PAM.* What follows is based on the overview articles of Gärtner and König [GK05] and König and Wolff [KW13], and we refer the reader to these sources for a more detailed presentation.

The first rigorous mathematics on the static PAM can be traced back to the works of Gärtner and Molchanov [GM90, GM98] in 1990 and 1998, respectively. In these impressive papers a complete answer to the questions of existence and uniqueness of solutions to (1.2.1) is provided. Moreover, some geometric properties of the u-field are derived. In particular, it is shown that, under a mild assumption on the moments of ξ and under a weak condition on the negative tails of ξ, the unique non-negative solution of (1.2.1) is given by the Feynman-Kac formula

$$u(x,t) = E_x \left(\exp\left\{ \int_0^t \xi(X^\kappa(s))\, ds \right\} u_0(X^\kappa(t)) \right). \tag{1.2.2}$$

Three main questions have guided the research on the static PAM.

- What is the asymptotic behaviour of $u(\cdot,t)$ as $t \to \infty$?
- Where does the main mass of $u(\cdot,t)$ come from? Which regions contribute most to $u(\cdot,t)$? What determines these regions? How many are there and how far are they from each other?
- What do the typical shapes of the potential ξ and the solution $u(\cdot,t)$ look like?

These three questions may also be characterized by the behaviour of the typical paths $\{X^\kappa(s)\colon\ 0 \le s \le t\}$ that contribute most to the Feynman-Kac formula in (1.2.2). On the one hand, the typical paths should aim at finding spots where ξ obtains high values, to make the integral in (1.2.2) large. On the other hand, the probability for far away excursions is small, so that X^κ has to find a compromise between moving very quickly towards a region with exceptionally high values of ξ and performing typical excursions. The first order approximation to u comes from those paths that find such a good compromise. The second order approximation to u comes from the precise manner in which the paths move, i.e., from the geometry of the ξ-field, see [GM98].

The notion of *intermittency* has played a major role in the investigation of the above mentioned questions. Intermittency means that the main contribution to the solution comes from small islands that have large distances to each other and make the u-field look irregular. This irregularity can be quantified by looking at the moments of u in the following way.

Definition 1.2.1. *Fix $p \in \mathbb{N}$ and let*

$$\Lambda_p(\kappa,t) = \frac{1}{p}\log \mathbb{E}[u(0,t)^p]. \tag{1.2.3}$$

The solution of (1.2.1) *is called p-intermittent for $p \ge 2$, when* $\lim_{t\to\infty}[\Lambda_p(\kappa,t) - \Lambda_{p-1}(\kappa,t)] = \infty$.

In [GK05] the following argument was given to explain why Definition 1.2.1 indeed pertains to the geometric picture given above. Suppose that Definition 1.2.1 is fullfilled for some $p \in \mathbb{N}\setminus\{1\}$ and let l_p be such that $\Lambda_{p-1}(\kappa,\cdot) \ll l_p(\cdot) \ll \Lambda_p(\kappa,\cdot)$. Then

$$\begin{aligned}\mathbb{P}\Big(u(0,t) > e^{l_p(t)}\Big) &\le e^{-(p-1)l_p(t)}\mathbb{E}\Big(u(0,t)^{p-1}\Big)\\ &= \exp\Big\{(p-1)\Lambda_{p-1}(\kappa,t) - (p-1)l_p(t)\Big\} \to 0.\end{aligned} \tag{1.2.4}$$

Hence, by the stationarity of $u(\cdot,t)$, the density of the point process

$$\Gamma(t) = \{x \in \mathbb{Z}^d : \ u(x,t) > e^{l_p(t)}\} \tag{1.2.5}$$

vanishes as $t \to \infty$. However,

$$\mathbb{E}\Big(u(0,t)^p \mathbb{1}\{u(0,t) \le e^{l_p(t)}\}\Big) \le e^{pl_p(t)} = e^{pl_p(t)-p\Lambda_p(\kappa,t)}\mathbb{E}(u(0,t)^p) = o(\mathbb{E}(u(0,t)^p)), \tag{1.2.6}$$

so that

$$\mathbb{E}\Big(u(0,t)^p\Big) \sim \mathbb{E}\Big(u(0,t)^p \mathbb{1}\{u(0,t) > e^{l_p(t)}\}\Big). \tag{1.2.7}$$

Birkhoff's ergodic theorem yields for a large centered box $B \subseteq \mathbb{Z}^d$,

$$|B|^{-1} \sum_{x \in B} u(x,t)^p \approx |B|^{-1} \sum_{x \in B \cap \Gamma(t)} u(x,t)^p. \tag{1.2.8}$$

Hence Definition 1.2.1 means that the p-th moment of $u(0,t)$ is generated by the high values of $u(\cdot,t)$ on the thin set $\Gamma(t)$. The drawback however, is that the above approach does not yield any information about the geometric structure of $\Gamma(t)$. In what follows we refer to the connected components of $\Gamma(t)$ as relevant islands. Theorem 3.2 of [GM90] reads as follows.

Theorem 1.2.2. *Let ξ be an i.i.d. field of real-valued random variables. Assume that $\xi(0)$ has finite exponential moments of all positive orders. Then u is p-intermittent for all $p \in \mathbb{N} \setminus \{1\}$ when ess sup $[\xi(0)] = \infty$.*

Since [GM90, GM98] many more results were found. In particular, van der Hofstad, König and Mörters [HKM06] proved that exactly four qualitatively different types of asymptotic behaviour of u can occur. These four universality classes depend on the upper tail of the distribution of ξ. It turned out that the double-exponential distribution

$$\mathbb{P}(\xi(0) > r) = \exp\{-e^{r/\rho}\}, \quad r \in \mathbb{R}, \tag{1.2.9}$$

where $\rho \in (0,\infty)$ is a parameter, plays an important role. In terms of this distribution the four universality classes of the static PAM may be described as follows.

1. **The single peak case:** This is the boundary case with $\rho = \infty$, corresponding to cases beyond the double exponential distribution. Here the relevant islands shrink to single sites as $t \to \infty$.
2. **The double-exponential case:** This is the case with $\rho \in (0,\infty)$. Here the relevant islands stay bounded as $t \to \infty$.
3. **The almost bounded case:** Let ξ be bounded from above and in the vicinity of the distribution

$$\mathbb{P}(\xi(0) > -r) = \exp\{-Cr^{-\gamma/(1-\gamma)}\}, \quad \gamma \in (0,1), \tag{1.2.10}$$

   as $r \to 0$. The almost bounded case is an interpolation between the double-exponential distribution with $\rho = 0$ and the distribution (1.2.10) with $\gamma = 1$. Here the sizes of the relevant islands grow slower than any power of t to infinity.
4. **The bounded case:** Here the sizes of the relevant islands grow at least as fast as some power of t as $t \to \infty$.

In some cases more is known. König, Lacoin, Mörters and Sidorova [KLMS09] showed that if ξ is Pareto-distributed, then with high probability there is only one relevant island. A similar result was later obtained by Lacoin and Mörters [LM12] for the exponential distribution, and was extended by Sidorova and Twarovski [ST14] and Fiodorov and Muirhead [FM14] to the case of Weibull-distributed potentials. These potentials fall into the first universality class. For the double-exponential distribution, which falls into the second universality class, the concentration phenomenon on a single island has been explored by Biskup and König [BK14].

1.3 The parabolic Anderson model in a dynamic random environment

In this section we focus on equation (1.1.1) when ξ depends on time. This will be referred to as *dynamic PAM*. Unlike the static case, which was treated in the previous section, much less is known for the dynamic case. Most studies were concerned with the exponential growth rate, namely, the p-th *annealed* Lyapunov exponent

$$\lambda_p(\kappa) = \lim_{t\to\infty} \frac{1}{pt} \log \mathbb{E}[u^p(0,t)], \qquad p \in \mathbb{N} \tag{1.3.1}$$

and the *quenched* Lyapunov exponent, which is the almost sure limit

$$\lambda_0^{u_0}(\kappa) = \lim_{t\to\infty} \frac{1}{t} \log u(0,t). \tag{1.3.2}$$

These were investigated as a function of the diffusion constant κ.

In the next two sections we summarize the literature on the dynamic PAM.

1.3.1 White noise

This section is a copy of Section 1.3.1 in Gärtner, den Hollander, Maillard [GdHM12].

Carmona and Molchanov [CM94] obtained a qualitative description of both the *quenched* and the *annealed* Lyapunov exponents when ξ is white noise, i.e.,

$$\xi(x,t) = \frac{\partial}{\partial t} W(x,t), \tag{1.3.3}$$

where $W = (W_t)_{t\geq 0}$ with $W_t = \{W(x,t)\colon x \in \mathbb{Z}^d\}$ is a space-time field of independent Brownian motions. This choice is special because the increments of ξ are *independent in space and time.* They showed that if $u(\cdot,0)$ has compact support (e.g. $u(\cdot,0) = \delta_0(\cdot)$ as in (1.1.4)), then the quenched Lyapunov exponent $\lambda_0^{u_0}(\kappa)$ defined in (1.3.2) exists and is constant ξ-a.s., and is independent of $u(\cdot,0)$. Moreover, they found that the asymptotics of $\lambda_0^{u_0}(\kappa)$ as $\kappa \downarrow 0$ is singular, namely, there are constants $C_1, C_2 \in (0,\infty)$ and $\kappa_0 \in (0,\infty)$ such that

$$C_1 \frac{1}{\log(1/\kappa)} \leq \lambda_0^{u_0}(\kappa) \leq C_2 \frac{\log\log(1/\kappa)}{\log(1/\kappa)} \qquad \forall\, 0 < \kappa \leq \kappa_0. \tag{1.3.4}$$

Subsequently, Carmona, Molchanov and Viens [CMV96], Carmona, Koralov and Molchanov [CKM01], and Cranston, Mountford and Shiga [CMS02], proved the existence of $\lambda_0^{u_0}$ when $u(\cdot,0)$ has non-compact support (e.g. $u(\cdot,0) \equiv 1$), showed that there is a constant $C \in (0,\infty)$ such that

$$\lim_{\kappa\downarrow 0} \log(1/\kappa)\, \lambda_0^{u_0}(\kappa) = C, \tag{1.3.5}$$

and proved that

$$\lim_{p\downarrow 0} \lambda_p(\kappa) = \lambda_0^{u_0}(\kappa) \quad \forall\, \kappa \in [0,\infty). \tag{1.3.6}$$

(These results were later extended to Lévy white noise by Cranston, Mountford and Shiga [CMS05], and to colored noise by Kim, Viens and Vizcarra [KVV08].) Further refinements on the behavior of the Lyapunov exponents were proven in Carmona and Molchanov [CM94] and Greven and den Hollander [GdH07]. In particular, it was shown that $\lambda_1(\kappa) = \frac{1}{2}$ for all $\kappa \in [0,\infty)$, while for the other Lyapunov exponents the following dichotomy holds (see Figs. 1.1–1.2):

- $d = 1, 2$: $\lambda_0^{u_0}(\kappa) < \frac{1}{2}$, $\lambda_p(\kappa) > \frac{1}{2}$ for $p \in \mathbb{N}\backslash\{1\}$, for $\kappa \in [0,\infty)$;
- $d \geq 3$: there exist $0 < \kappa_0 \leq \kappa_2 \leq \kappa_3 \leq \ldots < \infty$ such that

$$\lambda_0^{u_0}(\kappa) - \tfrac{1}{2} \begin{cases} < 0, & \text{for } \kappa \in [0,\kappa_0), \\ = 0, & \text{for } \kappa \in [\kappa_0,\infty), \end{cases} \tag{1.3.7}$$

and

$$\lambda_p(\kappa) - \tfrac{1}{2} \begin{cases} > 0, & \text{for } \kappa \in [0,\kappa_p), \\ = 0, & \text{for } \kappa \in [\kappa_p,\infty), \end{cases} \qquad p \in \mathbb{N}\backslash\{1\}. \tag{1.3.8}$$

It was further shown in [CM94] that $\lambda_p(\kappa) > 1/2$ already implies the chain of inequalities $\lambda_p(\kappa) < \lambda_{p+1}(\kappa) < \cdots$, which yields p-intermittency, see (1.2.3). Moreover, variational formulas for κ_p were derived, which in turn led to upper and lower bounds on κ_p, and to the identification of the asymptotics of κ_p for $p \to \infty$ (κ_p grows linearly with p). In addition, it was shown that for every $p \in \mathbb{N}\backslash\{1\}$ there exists a $d(p) < \infty$ such that $\kappa_p < \kappa_{p+1}$ for $d \geq d(p)$. Moreover, it was shown that $\kappa_0 < \kappa_2$ in Birkner, Greven and den Hollander [BGdH08] ($d \geq 5$), Birkner and Sun [BS10] ($d = 4$), Berger and Toninelli [BT09], Birkner and Sun [BS11] ($d = 3$). Note that, by Hölder's inequality, all curves in Figs. 1.1–1.2 are distinct whenever they are different from $\frac{1}{2}$.

1.3.2 Interacting particle systems

This section is largely a copy of Sections 1.1 and 1.3.2 in Gärtner, den Hollander, Maillard [GdHM12].

Three examples for ξ which is dependent in *space and time* have received a special attention in recent years.

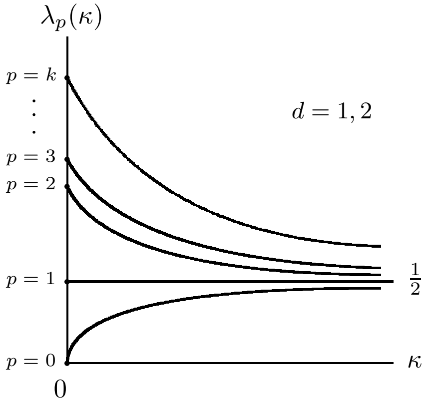

Figure 1.1: Quenched and annealed Lyapunov exponents when $d = 1, 2$ for white noise, with $\lambda_0(\kappa) = \lambda_0^{u_0}(\kappa)$.

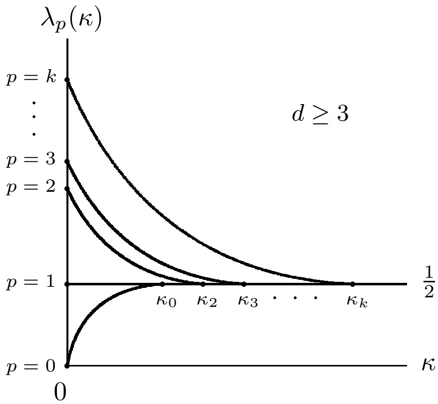

Figure 1.2: Quenched and annealed Lyapunov exponents when $d \geq 3$ for white noise, with $\lambda_0(\kappa) = \lambda_0^{u_0}(\kappa)$.

(1) *Independent Simple Random Walks* (ISRW) [Kipnis and Landim [KL99], Chapter 1]. Here, $\xi_t \in \Omega = (\mathbb{N} \cup \{0\})^{\mathbb{Z}^d}$ and $\xi(x,t)$ represents the number of particles at site x at time t. Under the ISRW-dynamics particles move around independently as simple random walks stepping at rate 1. ξ_0 is drawn according to the equilibrium ν_ρ with density $\rho \in (0,\infty)$, which is a Poisson product measure.

(2) *Symmetric Exclusion Process* (SEP) [Liggett [L85], Chapter VIII]. Here, $\xi_t \in \Omega = \{0,1\}^{\mathbb{Z}^d}$ and $\xi(x,t)$ represents the presence ($\xi(x,t) = 1$) or absence ($\xi(x,t) = 0$) of a particle at site x at time t. Under the SEP-dynamics particles move around independently according to an irreducible symmetric random walk transition kernel at rate 1, but subject to the restriction that no two particles can occupy the same site. ξ_0 is drawn according to the equilibrium ν_ρ with density $\rho \in (0,1)$, which is a Bernoulli product measure.

(3) *Symmetric Voter Model* (SVM) [Liggett [L85], Chapter V]. Here, $\xi_t \in \Omega = \{0,1\}^{\mathbb{Z}^d}$ and $\xi(x,t)$ represents the opinion of a voter at site x at time t. Under the SVM-dynamics each voter imposes its opinion on another voter according to an irreducible symmetric random walk transition kernel at rate 1. ξ_0 is either drawn according to the equilibrium distribution ν_ρ with density $\rho \in (0,1)$, which is not a product measure or according to a Bernoulli product measure.

Let $\gamma > 0$. Kesten and Sidoravicius [KS03], and Gärtner and den Hollander [GdH06], considered the case where ξ is γ times the number of particles in the ISRW dynamics. The survival versus extinction pattern [KS03] and the annealed Lyapunov exponents [GdH06] were analyzed, in particular, their dependence on d, κ, γ and ρ. The case where ξ is a single random walk was studied by Gärtner and Heydenreich [GH06]. Gärtner, den Hollander and Maillard [GdHM07], [GdHM09], [GdHM10] subsequently considered the cases where ξ is γ times an exclusion process, respectively, γ times a voter model. In each of these cases, a fairly complete picture of the behavior of the annealed Lyapunov exponents was obtained, including the presence or absence of *intermittency*, i.e., $\lambda_p(\kappa) > \lambda_{p-1}(\kappa)$ for some or all values of $p \in \mathbb{N}\backslash\{1\}$ and $\kappa \in [0,\infty)$. Several conjectures were formulated as well. In what follows we describe these results in some more detail. We refer the reader to Gärtner, den Hollander and Maillard [GdHM08] for an overview.

It was shown in Gärtner and den Hollander [GdH06], and Gärtner, den Hollander and Maillard [GdHM07], [GdHM09], [GdHM10] that for ISRW, SEP and SVM in equilibrium the function $\kappa \mapsto \lambda_p(\kappa)$ satisfies:

- If $d \geq 1$ and $p \in \mathbb{N}$, then the limit in (1.3.1) exists for all $\kappa \in [0,\infty)$. Moreover, if $\lambda_p(0) < \infty$, then $\kappa \mapsto \lambda_p(\kappa)$ is finite, continuous, strictly decreasing and convex on $[0,\infty)$.
- There are two regimes (we summarize results only for the case where the random walk transition kernel has finite second moment and we recall that ρ describes the density of particles):
  - *Strongly catalytic regime* (see Fig. 1.3):
    - ISRW: $d = 1,2$, $p \in \mathbb{N}$ or $d \geq 3$, $p \geq 1/\gamma G_d$: $\lambda_p \equiv \infty$ on $[0,\infty)$.
      (G_d is the Green function at the origin of simple random walk.)

    * SEP: $d = 1, 2$, $p \in \mathbb{N}$: $\lambda_p \equiv \gamma$ on $[0, \infty)$.
    * SVM: $d = 1, 2, 3, 4$, $p \in \mathbb{N}$: $\lambda_p \equiv \gamma$ on $[0, \infty)$.
  - *Weakly catalytic regime* (see Fig. 1.4–1.5):
    * ISRW: $d \geq 3$, $p < 1/\gamma G_d$: $\rho\gamma < \lambda_p < \infty$ on $[0, \infty)$.
    * SEP: $d \geq 3$, $p \in \mathbb{N}$: $\rho\gamma < \lambda_p < \gamma$ on $[0, \infty)$.
    * SVM: $d \geq 5$, $p \in \mathbb{N}$: $\rho\gamma < \lambda_p < \gamma$ on $[0, \infty)$.
- For all three dynamics, in the weakly catalytic regime $\lim_{\kappa\to\infty} \kappa[\lambda_p(\kappa) - \rho\gamma] = C_1 + C_2 p^2 1_{\{d=d_c\}}$ with $C_1, C_2 \in (0, \infty)$ and d_c a critical dimension: $d_c = 3$ for ISRW, SEP and $d_c = 5$ for SVM.
- Intermittent behavior:
  - In the strongly catalytic regime, there is no intermittency for all three dynamics.
  - In the weakly catalytic regime, there is full intermittency for:
    * all three dynamics when $0 \leq \kappa \ll 1$.
    * ISRW and SEP in $d = 3$ when $\kappa \gg 1$.
    * SVM in $d = 5$ when $\kappa \gg 1$.

Note: For SVM the convexity of $\kappa \mapsto \lambda_p(\kappa)$ and its scaling behavior for $\kappa \to \infty$ have not actually been proved, but have been argued on heuristic grounds.

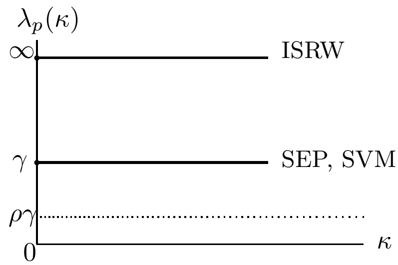

Figure 1.3: Triviality of the annealed Lyapunov exponents for ISRW, SEP, SVM in the strongly catalytic regime; i.e. below the critical dimension

Recently, there has been further progress for the case where ξ consists of 1 random walk (Schnitzler and Wolff [SW12]) or n independent random walks (Castell, Gün and Maillard [CGM12]), ξ is the SVM (Maillard, Mountford and Schöpfer [MMS12]), and for the trapping version of the PAM with $\gamma \in (-\infty, 0)$ (Drewitz, Gärtner, Ramírez and Sun [DGRS12]).

The first attempt to analyze the quenched Lyapunov exponent (1.3.2) for a dynamic ξ that has correlations in space and time was made in [GdHM12]. Several properties, such as (1) the existence of the quenched Lyapunov exponent for initial conditions u_0

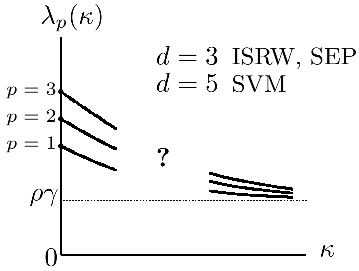

Figure 1.4: Non-triviality of the annealed Lyapunov exponents for ISRW, SEP and SVM in the weakly catalytic regime at the critical dimension.

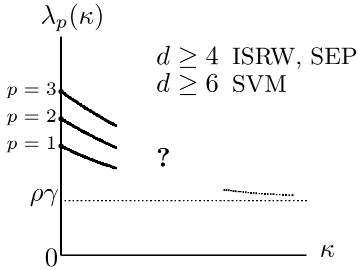

Figure 1.5: Non-triviality of the annealed Lyapunov exponents for ISRW, SEP and SVM in the weakly catalytic regime above the critical dimension.

with compact support; (2) the Lipschitz continuity of the map $\kappa \mapsto \lambda_0(\kappa)$ outside any neighborhood of zero; (3) the non-Lipschitz continuity of $\kappa \mapsto \lambda_0(\kappa)$ at $\kappa = 0$; (4) the strict lower bound $\lambda_0(\kappa) > \mathbb{E}(\xi(0,0))$, were derived under weak assumptions on ξ. Moreover assume that ξ satisfies a strong assumption on its occupation times, i.e., ξ is a Markov process such that uniformly in its initial configuration η there is a $c > 0$ such that for all $\mu, t > 0$

$$\mathbb{E}_\eta\left(\exp\left\{\mu\int_0^t (\xi(0,s) - \mathbb{E}(\xi(0,0)))\,ds\right\}\right) \leq \exp\{c\mu^2 t\}, \tag{1.3.9}$$

where $\mathbb{E}_\eta$ is the expectation of ξ when started at η, then even more can be said. Namely, the following asymptotic behaviour holds:

$$\limsup_{\kappa\to 0} \frac{\log(1/\kappa)}{\log\log(1/\kappa)}[\lambda_0(\kappa) - \mathbb{E}(\xi(0,0)] < \infty. \tag{1.3.10}$$

• *The goal of the work in this thesis is to broaden the understanding of the quenched Lyapunov exponent under assumptions on ξ that are as weak as possible.*

1.4 Overview of the results

In this section we give an overview over the main results, that will be presented in Chapter 2 and Chapter 3. Most of the results require space-time mixing conditions, which we call Gärtner-mixing conditions, whose precise definition can be found in the respective chapters.

1.4.1 Results of Chapter 2: basic properties of the quenched Lyapunov exponent

In Chapter 2 we derive basic properties of the solution of equation (1.1.1), such as existence and uniqueness, and of the quenched Lyapunov exponent (1.3.2) such as finiteness, independence on the initial condition and the non Lipschitz continuity in $\kappa = 0$.

Definition 1.4.1. *A field $q = \{q(x)\colon\ x \in \mathbb{Z}^d\}$ is said to be percolating from below if for every $\alpha \in \mathbb{R}$ the level set $\{x \in \mathbb{Z}^d\colon\ q(x) \le \alpha\}$ contains an infinite connected component. Otherwise q is said to be non-percolating from below.*

It was shown in [GM90] that if q is non-percolating from below, then (1.2.1) has at most one non-negative solution. We will show that a similar condition suffices for *dynamic* ξ, namely, (1.1.1) has at most one non-negative solution when there is a $T > 0$ such that

$$q^T = \{q^T(x)\colon\ x \in \mathbb{Z}^d\} \quad \text{with} \quad q^T(x) = \sup_{0 \le t \le T} q(x,t). \tag{1.4.1}$$

Theorem 1.4.2. ***[Uniqueness]*** *Consider a deterministic $q\colon\ \mathbb{Z}^d \times [0,\infty) \to \mathbb{R}$ such that:*
(1) There is a $T > 0$ such that q^T is non-percolating from below.
(2) $q^T(x) < \infty$ for all $T > 0$ and $x \in \mathbb{Z}^d$.
Then the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) = \kappa \Delta u(x,t) + q(x,t) u(x,t), \\ u(x,0) = u_0(x), \end{cases} \qquad x \in \mathbb{Z}^d,\ t \ge 0, \tag{1.4.2}$$

has at most one non-negative solution.

Theorem 1.4.3. ***[Existence]*** *Suppose that:*
(1) $s \mapsto \xi(x,s)$ is locally integrable for every x, ξ-a.s., i.e., for every compact subset $K \subset [0,\infty)$ and every x the map $s \mapsto \mathbb{1}\{s \in K\}\xi(x,s)$ is integrable ξ-a.s.
(2) $\mathbb{E}(e^{q\xi(0,0)}) < \infty$ for all $q \ge 0$.
Then the function defined by the Feynman-Kac formula

$$u(x,t) = E_x \left(\exp\left\{ \int_0^t \xi(X^\kappa(s), t-s)\, ds \right\} u_0(X^\kappa(t)) \right) \tag{1.4.3}$$

solves (1.1.1) with initial condition u_0.

From now on we assume that ξ satisfies the conditions of Theorems 1.4.2–1.4.3 (where q is replaced by ξ in Theorem 1.4.2).

The following two results concern the finiteness and existence of $\lambda_0^{u_0}(\kappa)$. See Corollary 1.4.8 for examples of ξ, that satisfy the assumptions of Theorems 1.4.4 and 1.4.5.

Theorem 1.4.4. ***[Finiteness]*** *If ξ is Gärtner-positive-hyper-mixing, then $\lambda_0^{\delta_0}(\kappa) < \infty$.*

From now on we also assume that ξ satisfies the conditions of Theorem 1.4.4. The following result extends Gärtner, den Hollander and Maillard [GdHM12], Theorem 1.1, in which it was shown that for the initial condition $u_0 = \delta_0$ the quenched Lyapunov exponent exists and is constant ξ-a.s.

Theorem 1.4.5. ***[Initial Condition]*** *If ξ is reversible in time or symmetric in space, type-II Gärtner-mixing and Gärtner-negative-hyper-mixing, then $\lambda_0^{u_0}(\kappa) = \lim_{t\to\infty} \frac{1}{t} \log u(0,t)$ exists ξ-a.s. and in $L^1(\mathbb{P})$, is constant ξ-a.s., and is independent of u_0.*

The next results concern the dependence on κ of $\lambda_0(\kappa)$ and are valid under certain conditions on the occupation times of ξ, which are similar to (1.3.9) but still weaker than (1.3.9).

Theorem 1.4.6. ***[Continuity at $\kappa = 0$]*** *If ξ is Gärtner-regular, then $\kappa \mapsto \lambda_0^{\delta_0}(\kappa)$ is continuous at zero.*

Theorem 1.4.7. ***[Not Lipschitz at $\kappa = 0$]*** *If ξ is Gärtner-volatile, then $\kappa \mapsto \lambda_0^{\delta_0}(\kappa)$ is not Lipschitz continuous in zero.*

1.4.1.1 Examples

Corollary 1.4.8. ***[Examples for Theorems 1.4.4–1.4.5]***
(1) Let $X = (X_t)_{t\geq 0}$ be a stationary and ergodic $\mathbb{R}$-valued Markov process. Let $(X_\cdot(x))_{x\in\mathbb{Z}^d}$ be independent copies of X. Define ξ by $\xi(x,t) = X_t(x)$. If

$$\mathbb{E}\left[e^{q \sup_{s\in[0,1]} X_s}\right] < \infty \qquad \forall\, q \geq 0, \tag{1.4.4}$$

then ξ fulfills the conditions of Theorem 1.4.4. If, moreover, the left-hand side of (1.4.4) is finite for all $q \leq 0$, then ξ satisfies the conditions of Theorem 1.4.5.
(2) Let ξ be the zero-range process with rate function $g\colon \mathbb{N}_0 \to (0,\infty)$, $g(k) = k^\beta$, $\beta \in (0,1]$, and transition probabilities given by a simple random walk on $\mathbb{Z}^d$. If ξ starts from the product measure π_ρ, $\rho \in (0,\infty)$, with marginals

$$\pi_\rho\big\{\eta \in \mathbb{N}_0^{\mathbb{Z}^d} \colon \eta(x) = k\big\} = \begin{cases} \gamma\,\frac{\rho^k}{g(1)\times\cdots\times g(k)}, & \text{if } k > 0,\\ \gamma, & \text{if } k = 0, \end{cases} \tag{1.4.5}$$

where $\gamma \in (0,\infty)$ is a normalization constant, then ξ satisfies the conditions of Theorems 1.4.4–1.4.5.

Corollary 1.4.9. ***[Examples for Theorem 1.4.6]***
(1) If ξ is a bounded interacting particle system in the so-called $M < \varepsilon$ regime (see Liggett [L85]), then the conditions of Theorem 1.4.6 are satisfied.
(2) If ξ is the exclusion process with an irreducible, symmetric and transient random walk transition kernel, then the conditions of Theorem 1.4.6 are satisfied.
(3) If ξ is the dynamics defined by

$$\xi(x,t) = \sum_{y\in\mathbb{Z}^d} \sum_{j=1}^{N_y} \delta_{Y_j^y(t)}(x), \tag{1.4.6}$$

where $\{Y_j^y\colon\, y \in \mathbb{Z}^d, 1 \le j \le N_y, Y_j^y(0) = y\}$ is a collection of independent continuous-time simple random walks jumping at rate one, and $(N_y)_{y\in\mathbb{Z}^d}$ is a Poisson random field with intensity ν for some $\nu \in (0,\infty)$. If $d \ge 3$, then the conditions of Theorem 1.4.6 are satisfied.

Remark 1.4.10. *Corollaries 1.4.8–1.4.9 list only a few examples that match the conditions. It is a separate problem to verify these conditions for as broad a class of interacting particle systems as possible.*

1.4.2 Result of Chapter 3: space-time ergodicity for the quenched Lyapunov exponent

In Chapter 3 we prove the following.

Theorem 1.4.11. *If $u_0 = \delta_0$ and ξ is Gärtner-hyper-mixing, then*

$$\lim_{\kappa\to\infty} \lambda_0^{\delta_0}(\kappa) = \mathbb{E}(\xi(0,0)). \tag{1.4.7}$$

• **Examples:** The two examples listed in Corollary 1.4.8 are examples of fields that are Gärtner-hyper-mixing.

Theorem 1.4.11 yields a partial answer to the question: Which random walk paths give the main contribution to the Feynman-Kac formula in (1.4.3)? Indeed, Theorem 1.4.11 shows that, for large κ and any *dynamic* ξ that is Gärtner-hyper-mixing, the main contribution comes from those paths that spend most of their time in regions where ξ looks typical. This is in sharp contrast with what is known for the parabolic Anderson model with a *static* i.i.d. random environment $\xi = \{\xi(x)\colon\, x \in \mathbb{Z}^d\}$. In this case the main contribution to the Feynman-Kac formula in (1.4.3) comes from those paths that are localized, in the sense that they spend almost all of their time in regions where ξ is large. The latter implies that for bounded ξ the quenched Lyapunov exponent equals $\operatorname{ess\,sup} \xi(0)$ instead of $\mathbb{E}(\xi(0))$.

Theorem 1.4.11, jointly with the results in Section 1.4.1, suggest the picture of $\kappa \mapsto \lambda_0^{\delta_0}(\kappa)$ in Figure 1.6.

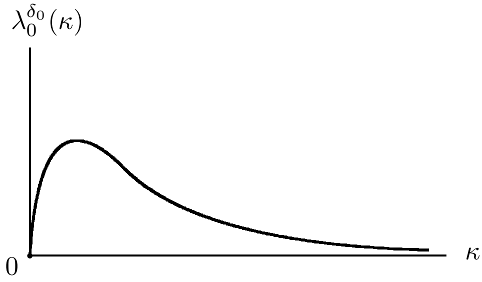

Figure 1.6: Qualitative behavior of $\kappa \mapsto \lambda_0^{\delta_0}(\kappa)$.

1.5 Open problems

We close this introduction by listing some open problems. We hope to address some of these in future works.

- Show that the graph of the quenched Lyapunov exponent indeed looks as indicated in Figure 1.6, i.e., show that $\kappa \mapsto \lambda_0(\kappa)$ has a unique maximum.
- Investigate (1.1.1) when the discrete Laplacian is replaced by a random discrete Laplacian. This amounts to replacing the simple random walk in the Feynman-Kac formula (1.1.5) by a random walk in a random environment.
- Determine the rate at which $\lambda_0(\kappa)$ converges to $\mathbb{E}(\xi(0,0))$ as $\kappa \to \infty$, i.e., determine a function $f : [0,\infty) \to [0,\infty)$ such that $\lim_{\kappa\to\infty} f(\kappa) = \infty$ and such that $f(\kappa)[\lambda_0(\kappa) - \mathbb{E}(\xi(0,0)]$ is bounded from above and below as $\kappa \to \infty$.
- Investigate the fluctuations of $\lambda_0(\kappa)$ as $\kappa \to \infty$.
- Investigate whether or not the parabolic Anderson model falls in the same universality class as the KPZ-equation.

2 Basic properties of the quenched Lyapunov exponent

This chapter is based on:
D. Erhard, F. den Hollander, G. Maillard. *The parabolic Anderson model in a dynamic random environment: basic properties of the quenched Lyapunov exponent.* Posted on *arXiv:1208.0330v2*, to appear in *Annales de l'Institut Henri Poincaré Probabilités et Statistiques 2014.*

Abstract

In this chapter we study the parabolic Anderson equation $\partial u(x,t)/\partial t = \kappa\Delta u(x,t) + \xi(x,t)u(x,t)$, $x \in \mathbb{Z}^d$, $t \geq 0$, where the u-field and the ξ-field are $\mathbb{R}$-valued, $\kappa \in [0,\infty)$ is the diffusion constant, and Δ is the discrete Laplacian. The ξ-field plays the role of a *dynamic random environment* that drives the equation. The initial condition $u(x,0) = u_0(x)$, $x \in \mathbb{Z}^d$, is taken to be non-negative and bounded. The solution of the parabolic Anderson equation describes the evolution of a field of particles performing independent simple random walks with binary branching: particles jump at rate $2d\kappa$, split into two at rate $\xi \vee 0$, and die at rate $(-\xi) \vee 0$. Our goal is to prove a number of *basic properties* of the solution u under assumptions on ξ that are as weak as possible. These properties will serve as a jump board for later refinements.

Throughout the chapter we assume that ξ is stationary and ergodic under translations in space and time, is not constant and satisfies $\mathbb{E}(|\xi(0,0)|) < \infty$, where $\mathbb{E}$ denotes expectation w.r.t. ξ. Under a mild assumption on the tails of the distribution of ξ, we show that the solution to the parabolic Anderson equation exists and is unique for all $\kappa \in [0,\infty)$. Our main object of interest is the *quenched Lyapunov exponent* $\lambda_0(\kappa) = \lim_{t\to\infty} \frac{1}{t} \log u(0,t)$. It was shown in Gärtner, den Hollander and Maillard [GdHM12] that this exponent exists and is constant ξ-a.s., satisfies $\lambda_0(0) = \mathbb{E}(\xi(0,0))$ and $\lambda_0(\kappa) > \mathbb{E}(\xi(0,0))$ for $\kappa \in (0,\infty)$, and is such that $\kappa \mapsto \lambda_0(\kappa)$ is globally Lipschitz on $(0,\infty)$ outside any neighborhood of 0 where it is finite. Under certain weak space-time mixing assumptions on ξ, we show the following properties: (1) $\lambda_0(\kappa)$ does not depend on the initial condition u_0; (2) $\lambda_0(\kappa) < \infty$ for all $\kappa \in [0,\infty)$; (3) $\kappa \mapsto \lambda_0(\kappa)$ is continuous on $[0,\infty)$ but not Lipschitz at 0. We further conjecture: (4) $\lim_{\kappa\to\infty}[\lambda_p(\kappa) - \lambda_0(\kappa)] = 0$ for all $p \in \mathbb{N}$, where $\lambda_p(\kappa) = \lim_{t\to\infty} \frac{1}{pt} \log \mathbb{E}([u(0,t)]^p)$ is the p-th *annealed Lyapunov exponent*. (In [GdHM12] properties (1), (2) and (4) were not addressed, while property (3) was shown under much more restrictive assumptions on ξ.) Finally, we prove that our weak space-time mixing conditions on ξ are satisfied for several classes of interacting particle systems.

MSC 2000. Primary 60H25, 82C44; Secondary 60F10, 35B40.
Key words and phrases. Parabolic Anderson equation, percolation, quenched Lyapunov exponent, large deviations, interacting particle systems.
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2.1 Introduction and main results

Section 2.1.1 defines the parabolic Anderson model and provides motivation, Section 2.1.2 describes our main targets and their relation to the literature, Section 2.1.3 contains our main results, while Section 2.1.4 discusses these results and state a conjecture.

2.1.1 The parabolic Anderson model (PAM)

The parabolic Anderson model is the partial differential equation

$$\frac{\partial}{\partial t}u(x,t) = \kappa\Delta u(x,t) + \xi(x,t)u(x,t), \qquad x \in \mathbb{Z}^d,\ t \geq 0. \tag{2.1.1}$$

Here, the u-field is $\mathbb{R}$-valued, $\kappa \in [0,\infty)$ is the diffusion constant, Δ is the discrete Laplacian acting on u as

$$\Delta u(x,t) = \sum_{\substack{y\in\mathbb{Z}^d \\ \|y-x\|=1}} [u(y,t) - u(x,t)] \tag{2.1.2}$$

($\|\cdot\|$ is the l_1-norm), while

$$\xi = (\xi_t)_{t\geq 0} \text{ with } \xi_t = \{\xi(x,t)\colon\ x \in \mathbb{Z}^d\} \tag{2.1.3}$$

is an $\mathbb{R}$-valued random field playing the role of a *dynamic random environment* that drives the equation. As initial condition for (2.1.1), we take

$$\blacktriangleright \quad u(x,0) = u_0(x),\ x \in \mathbb{Z}^d, \text{ with } u_0 \text{ non-negative and bounded.} \tag{2.1.4}$$

One interpretation of (2.1.1) and (2.1.4) comes from *population dynamics*. Consider the special case where $\xi(x,t) = \gamma\bar{\xi}(x,t) - \delta$ with $\delta, \gamma \in (0,\infty)$ and $\bar{\xi}$ an $\mathbb{N}_0$-valued random field. Consider a system of two types of particles, A (catalyst) and B (reactant), subject to:

- A-particles evolve autonomously according to a prescribed dynamics with $\bar{\xi}(x,t)$ denoting the number of A-particles at site x at time t;
- B-particles perform independent simple random walks at rate $2d\kappa$ and split into two at a rate that is equal to γ times the number of A-particles present at the same location at the same time;
- B-particles die at rate δ;
- the average number of B-particles at site x at time 0 is $u_0(x)$.

Then

$$u(x,t) \;=\; \begin{array}{l}\text{the average number of } B\text{-particles at site } x \text{ at time } t\\ \text{conditioned on the evolution of the } A\text{-particles.}\end{array} \tag{2.1.5}$$

The ξ-field is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Throughout the chapter we assume that

$$\begin{aligned} &\blacktriangleright \quad \xi \text{ is } \textit{stationary} \text{ and } \textit{ergodic} \text{ under translations in space and time.} \\ &\blacktriangleright \quad \xi \text{ is } \textit{not constant} \text{ and } \mathbb{E}(|\xi(0,0)|) < \infty. \end{aligned} \tag{2.1.6}$$

Without loss of generality we may assume that $\mathbb{E}(\xi(0,0)) = 0$.

2.1.2 Main targets and related literature

The goal of the present chapter is to prove a number of *basic properties* about the Cauchy problem in (2.1.1) with initial condition (2.1.4). In this section we describe these properties informally. Precise results will be stated in Section 2.1.3.

• **Existence and uniqueness of the solution.** For *static* ξ, i.e.,

$$\xi = \{\xi(x)\colon\ x \in \mathbb{Z}^d\}, \tag{2.1.7}$$

existence and uniqueness of the solution to (2.1.1) with initial condition (2.1.4) were addressed by Gärtner and Molchanov [GM90]. Namely, for arbitrary, deterministic $q\colon\ \mathbb{Z}^d \to \mathbb{R}$ and $u_0\colon\ \mathbb{Z}^d \to [0,\infty)$, they considered the equation

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) = \kappa \Delta u(x,t) + q(x) u(x,t), \\ u(x,0) = u_0(x), \end{cases} \qquad x \in \mathbb{Z}^d,\ t \geq 0, \tag{2.1.8}$$

with u_0 non-negative, and showed that there exists a non-negative solution if and only if the Feynman-Kac formula

$$v(x,t) = E_x \left(\exp\left\{ \int_0^t q(X^\kappa(s))\, ds \right\} u_0(X^\kappa(t)) \right) \tag{2.1.9}$$

is finite for all x and t. Here, $X^\kappa = (X^\kappa(t))_{t\geq 0}$ is the continuous-time simple random walk jumping at rate $2d\kappa$ (i.e., the Markov process with generator $\kappa\Delta$) starting in x under the law P_x. Moreover, they showed that v in (2.1.9) is the minimal non-negative solution to (2.1.8). From these considerations they deduced a criterion for the almost sure existence of a solution to equation (2.1.8) when $q = \xi$. This result was later extended to *dynamic* ξ by Carmona and Molchanov [CM94], who proved the following.

Proposition 2.1.1. *(Carmona and Molchanov [CM94]) Suppose that* $q\colon\ \mathbb{Z}^d \times [0,\infty) \to \mathbb{R}$ *is such that* $q(x,\cdot)$ *is locally integrable for every* x. *Then, for every non-negative initial condition* u_0, *the deterministic equation*

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) = \kappa \Delta u(x,t) + q(x,t) u(x,t), \\ u(x,0) = u_0(x), \end{cases} \qquad x \in \mathbb{Z}^d,\ t \geq 0, \tag{2.1.10}$$

has a non-negative solution if and only if the Feynman-Kac formula

$$v(x,t) = E_x \left(\exp\left\{ \int_0^t q(X^\kappa(s), t-s)\, ds \right\} u_0(X^\kappa(t)) \right) \tag{2.1.11}$$

is finite for all x and t. Moreover, v in (2.1.11) *is the minimal non-negative solution to (2.1.10).*

To complement Proposition 2.1.1, we need to find a condition on ξ that leads to uniqueness of (2.1.11). This will be the first of our targets. To answer the question of uniqueness for *static* ξ, Gärtner and Molchanov [GM90] introduced the following notion.

Definition 2.1.2. *A field $q = \{q(x)\colon\ x \in \mathbb{Z}^d\}$ is said to be percolating from below if for every $\alpha \in \mathbb{R}$ the level set $\{x \in \mathbb{Z}^d\colon\ q(x) \le \alpha\}$ contains an infinite connected component. Otherwise q is said to be non-percolating from below.*

It was shown in [GM90] that if q is non-percolating from below, then (2.1.8) has at most one non-negative solution. We will show that a similar condition suffices for *dynamic* ξ, namely, (2.1.10) has at most one non-negative solution when there is a $T > 0$ such that

$$q^T = \{q^T(x)\colon\ x \in \mathbb{Z}^d\} \quad \text{with} \quad q^T(x) = \sup_{0 \le t \le T} q(x,t) \tag{2.1.12}$$

is non-percolating from below (Theorem 2.1.12 below). This (surprisingly weak) condition is fulfilled ξ-a.s. for $q = \xi$ for most choices of ξ. Moreover, we show that this solution is given by the Feynman-Kac formula (Theorem 2.1.13 below).

• **Quenched Lyapunov exponent and initial condition.** The *quenched Lyapunov exponent* associated with (2.1.1) with initial condition u_0 is defined as

$$\lambda_0^{u_0}(\kappa) = \lim_{t\to\infty} \frac{1}{t} \log u(0,t). \tag{2.1.13}$$

Gärtner, den Hollander and Maillard [GdHM12] showed that if u_0 has finite support, then the limit exists ξ-a.s. and in $L^1(\mathbb{P})$, is ξ-a.s. constant, and does not depend on u_0. A natural question is whether the same is true for u_0 bounded with infinite support. This question was already addressed by Drewitz, Gärtner, Ramirez and Sun [DGRS12]. Define

$$\overline{\lambda}_0^{u_0}(\kappa) = \lim_{t\to\infty} \frac{1}{t} \log E_0\left(\exp\left\{\int_0^t \xi(X^\kappa(s), s)\, ds\right\} u_0(X^\kappa(t))\right) \tag{2.1.14}$$

and note that the expectation in (2.1.14) differs from the one in (2.1.11), since the time in the integral in (2.1.11) is reversed.

Proposition 2.1.3. *(Drewitz, Gärtner, Ramirez and Sun [DGRS12])*
(I) If ξ satisfies the first line of (2.1.6) *and is bounded, then $\overline{\lambda}_0^{\mathbb{1}}(\kappa)$ exists ξ-a.s. and in $L^1(\mathbb{P})$, and is ξ-a.s. constant.*
(II) If, in addition, ξ is reversible in time or symmetric in space, then, for all u_0 subject to (2.1.4), $\overline{\lambda}_0^{u_0}(\kappa)$ exists ξ-a.s. and in $L^1(\mathbb{P})$, and coincides with $\overline{\lambda}_0^{\mathbb{1}}(\kappa)$.

The time-reversal that distinguishes $\lambda_0^{\mathbb{1}}(\kappa)$ from $\overline{\lambda}_0^{\mathbb{1}}(\kappa)$ is non-trivial. Under appropriate space-time mixing conditions on ξ, we show how Proposition 2.1.3 can be used to settle the existence of $\lambda_0^{u_0}(\kappa)$ with the same limit for all u_0 subject to (2.1.4) (Theorem 2.1.15 below).

• **Finiteness of the quenched Lyapunov exponent.** On the one hand it follows by an application of Jensen's inequality that $\lambda_0^{u_0}(\kappa) \geq \mathbb{E}(\xi(0,0))$ for all κ (see Theorem 1.2(iii) in Gärtner, den Hollander and Maillard [GdHM12] for the details), while on the other hand if ξ is bounded from above, then also $\lambda_0^{u_0}(\kappa) < \infty$ for all κ. For unbounded ξ the same is expected to be true under a mild assumption on the positive tail of ξ. However, settling this issue seems far from easy (and it is even not true when ξ does not depend on time). The only two choices of ξ for which finiteness has been established in the literature are an i.i.d. field of Brownian motions (Carmona and Molchanov [CM94]) and a Poisson random field of independent simple random walks (Kesten and Sidoravicius [KS03]). We will show that finiteness holds under an appropriate mixing condition on ξ (Theorem 2.1.14 below).

• **Dependence on κ.** In [GdHM12] it was shown that $\lambda_0^{\delta_0}(0) = \mathbb{E}(\xi(0,0))$, $\lambda_0^{\delta_0}(\kappa) > \mathbb{E}(\xi(0,0))$ for $\kappa \in (0,\infty)$, and $\kappa \mapsto \lambda_0^{\delta_0}(\kappa)$ is globally Lipschitz outside any neighborhood of zero where it is finite. Under certain strong "noisiness" assumptions on ξ, it was further shown that continuity extends to zero while the Lipschitz property does not. It remained unclear, however, which characteristics of ξ are really necessary for the latter two properties to hold. We will show that if ξ is a Markov process, then in essence a weak condition on its Dirichlet form is enough to ensure continuity (Theorem 2.1.16 and Corollary 2.1.20 below), whereas the non Lipschitz property holds under a weak assumption on the fluctuations of ξ (Theorem 2.1.17). Finally, by the ergodicity of ξ in space, it is natural to expect (see Conjecture 2.1.21 below) that $\lim_{\kappa\to\infty}[\lambda_p^{\delta_0}(\kappa) - \lambda_0^{\delta_0}(\kappa)] = 0$ for all $p \in \mathbb{N}$, where

$$\lambda_p^{\delta_0}(\kappa) = \lim_{t\to\infty} \frac{1}{pt} \log \mathbb{E}([u(0,t)]^p) \tag{2.1.15}$$

is the p-th *annealed Lyapunov exponent* (provided this exists). It was proved for three special choices of ξ: (1) independent simple random walks; (2) the symmetric exclusion process; (3) the symmetric voter model, (for references, see [GdHM12]), that, when d is large enough, $\lim_{\kappa\to\infty} \lambda_p^{\delta_0}(\kappa) = \mathbb{E}(\xi(0,0))$, $p \in \mathbb{N}_0$. It is known from Carmona and Molchanov [CM94] that $\lim_{\kappa\to\infty} \lambda_p^{\delta_0}(\kappa) = \frac{1}{2} \neq \mathbb{E}(\xi(0,0))$ for all $p \in \mathbb{N}$ when ξ is an i.i.d. field of Brownian motions.

Remark 2.1.4. *We expect that one can define the p-th annealed Lyapunov exponent even for non-integer values of p and that in this case* $\lim_{p\to 0} \lambda_p^{\delta_0}(\kappa) = \lambda_0^{\delta_0}(\kappa)$. *This was indeed established by Cranston, Mountford and Shiga [CMS02] when ξ is an i.i.d. field of Brownian motions.*

2.1.3 Main results

This section contains five definitions of space-time mixing assumptions on ξ, six theorems subject to these assumptions, as well as examples of ξ for which these assumptions are satisfied. The material is organized as Sections 2.1.3.1–2.1.3.4. The first theorem refers to the deterministic PAM, the other four theorems to the random PAM. Recall that the initial condition u_0 is assumed to be non-negative and bounded. Further recall that ξ satisfies (2.1.6).

2.1.3.1 Definitions: Space-time blocks, Gärtner-mixing, Gärtner-regularity and Gärtner-volatility

• **Good and bad space-time blocks.** For $A \geq 1$, $R \in \mathbb{N}$, $x \in \mathbb{Z}^d$ and $k, b, c \in \mathbb{N}_0$, define the space-time blocks

$$\widetilde{B}_R^A(x,k;b,c) = \left(\prod_{j=1}^{d} \left[(x(j)-1-b)A^R, (x(j)+1+b)A^R \right) \cap \mathbb{Z}^d \right) \times [(k-c)A^R, (k+1)A^R), \tag{2.1.16}$$

abbreviate $B_R^A(x,k) = \widetilde{B}_R^A(x,k;0,0)$, and define the space-blocks

$$Q_R^A(x) = x + [0, A^R)^d \cap \mathbb{Z}^d. \tag{2.1.17}$$

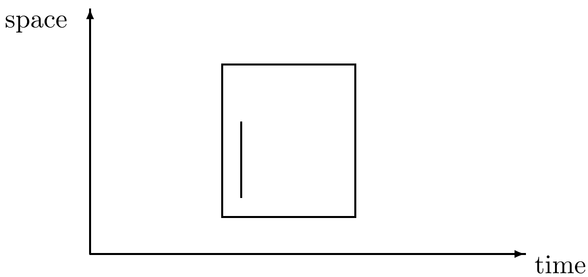

Figure 2.1: The box represents $B_R^A(x,k)$. The line is a possible realization of $Q_R^A(x)$.

It is convenient to extend the ξ-process to negative times, to obtain a two-sided process $\overline{\xi} = (\overline{\xi}_t)_{t\in\mathbb{R}}$. Abbreviate $M = \operatorname{ess\,sup}[\xi(0,0)]$.

Definition 2.1.5. *For $A \geq 1$, $R \in \mathbb{N}$, $x \in \mathbb{Z}^d$, $k \in \mathbb{N}$, $C \in [0,M]$ and $b, c \in \mathbb{N}_0$, the R-block $B_R^A(x,k)$ is called (C,b,c)-good when*

$$\sum_{z \in Q_R^A(y)} \xi(z,s) \leq CA^{Rd} \qquad \forall\, y \in \mathbb{Z}^d,\ s \geq 0\colon\ Q_R^A(y) \times \{s\} \subseteq \widetilde{B}_R^A(x,k;b,c). \tag{2.1.18}$$

Otherwise it is called (C,b,c)-bad.

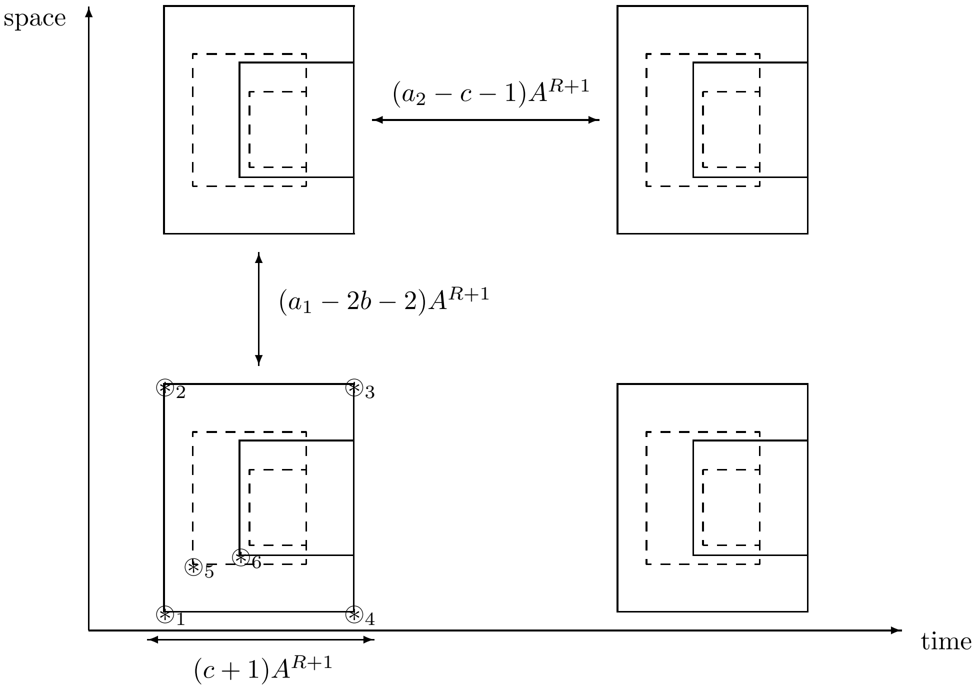

Figure 2.2: The dashed blocks are R-blocks, i.e., $B_R^A(x,k)$ (inner) and $\widetilde{B}_R^A(x,k;b,c)$ (outer) for some choice of A, x, k, b, c. The solid blocks are $(R+1)$-blocks, i.e., $B_{R+1}^A(y,l)$ (inner) and $\widetilde{B}_{R+1}^A(y,l;b,c)$ (outer) for a choice of A, y, l, b, c such that they contain the corresponding R-blocks. Furthermore, $\{\circledast_i\}_{i=1,2,3,4,5,6}$ represents the space-time coordinates $\circledast_1 = ((y-1-b)A^{R+1}, (l-c)A^{R+1})$,$\circledast_2 = ((y+1+b)A^{R+1}, (l-c)A^{R+1})$, $\circledast_3 = ((y+1+b)A^{R+1}, (l+1)A^{R+1})$, $\circledast_4 = ((y-1-b)A^{R+1}, (l+1)A^{R+1})$, $\circledast_5 = ((x-1-b)A^R, (k-c)A^R)$ and $\circledast_6 = ((y-1)A^{R+1}, lA^{R+1})$.

• **Gärtner-mixing.** For $A \geq 1$, $R \in \mathbb{N}$, $x \in \mathbb{Z}^d$, $k \in \mathbb{N}$, $C \in [0, M]$ and $b, c \in \mathbb{N}_0$, let

$$\begin{aligned} &\mathcal{A}_R^{A,C}(x,k;b,c) \\ &= \big\{ B_{R+1}^A(x,k) \text{ is } (C,b,c)\text{-good, but contains an } R\text{-block that is } (C,b,c)\text{-bad} \big\}. \end{aligned} \tag{2.1.19}$$

In terms of these events we define the following *space-time mixing conditions* (see Fig. 2.2). For $D \subset \mathbb{Z}^d \times \mathbb{R}$, let $\sigma(D)$ be the σ-field generated by $\{\overline{\xi}(x,t)\colon (x,t) \in D\}$.

Definition 2.1.6. ***[Gärtner-mixing]***
For $a_1, a_2 \in \mathbb{N}$, denote by $\Delta_n(a_1,a_2)$ the set of $\mathbb{Z}^d \times \mathbb{N}$-valued sequences $\{(x_i,k_i)\}_{i=0}^n$ that are increasing with respect to the lexicographic ordering of $\mathbb{Z}^d \times \mathbb{N}$ and are such that for all $0 \leq i < j \leq n$

$$x_j \equiv x_i \mod a_1 \quad \text{and} \quad k_j \equiv k_i \mod a_2. \tag{2.1.20}$$

(a) ξ is called (A,C,b,c)-type-I Gärtner-mixing when there are $a_1, a_2 \in \mathbb{N}$ and a constant $K > 0$ such that there is an $R_0 \in \mathbb{N}$ such that, for all $R \in \mathbb{N}$ with $R \geq R_0$ and all $n \in \mathbb{N}$,

$$\sup_{(x_i,k_i)_{i=0}^n \in \Delta_n(a_1,a_2)} \mathbb{P}\left(\bigcap_{i=0}^n \mathcal{A}_R^{A,C}(x_i,k_i;b,c) \right) \leq K \left(A^{(1+2d)} \right)^{-R(1+d)n}. \tag{2.1.21}$$

(b) ξ is called (A,C,b,c)-type-II Gärtner-mixing when for each family of events

$$\mathcal{A}_i^R \in \sigma(B_{R+1}^A(x_i,k_i)), \qquad (x_i,k_i)_{i=0}^n \in \Delta_n(a_1,a_2), \tag{2.1.22}$$

that are invariant under space-time shifts and satisfy

$$\lim_{R\to\infty} \mathbb{P}(\mathcal{A}_i^R) = 0, \tag{2.1.23}$$

there are $a_1, a_2 \in \mathbb{N}$ and a constant $K > 0$ such that for each $\delta > 0$ there is an $R_0 \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$,

$$\mathbb{P}\left(\bigcap_{i=0}^n \big\{ B_{R+1}^A(x_i,k_i) \textit{ is } (C,b,c)\textit{-good} \big\} \cap \mathcal{A}_i^R \right) \leq K\delta^n \qquad R \geq R_0, R \in \mathbb{N}. \tag{2.1.24}$$

(c) ξ is called type-I, respectively type-II, Gärtner-mixing, when there are $A \geq 1$, $C \in [0,M]$, $R \in \mathbb{N}$, $b, c \in \mathbb{N}$ such that ξ is (A,C,b,c)-type-I, respectively, (A,C,b,c)-type-II, Gärtner-mixing.

Definition 2.1.7. ***[Gärtner-hyper-mixing]***
(a) ξ is called Gärtner-positive-hyper-mixing when
(a1) $\mathbb{E}[e^{q \sup_{s\in[0,1]} \xi(0,s)}] < \infty$ for all $q \geq 0$.
(a2) There are $b, c \in \mathbb{N}$ and a constant C such that for each $A_0 > 1$ one can find $A \geq A_0$ such that $\xi \mathbb{1}\{\xi \geq 0\}$ is (A,C,b,c)-type-I Gärtner-mixing.
(a3) There are $R_0, C_0 \geq 1$ such that

$$\mathbb{P}\left(\sup_{s\in[0,1]} \frac{1}{|B_R|} \sum_{y\in B_R} \xi(y,s) \geq C \right) \leq |B_R|^{-\alpha} \quad \forall\, R \geq R_0, C \geq C_0, \tag{2.1.25}$$

for some $\alpha > (1+2d)(2+d)/d$, *where* $B_R = [-R,R]^d \cap \mathbb{Z}^d$.
(b) ξ *is called Gärtner-negative-hyper-mixing, when* $-\xi$ *is Gärtner-positive-hyper-mixing.*

Remark 2.1.8. *If* ξ *is bounded from above, then* ξ *is Gärtner-positive-hyper-mixing. For those examples where* $\xi(x,t)$ *represents "the number of particles at site* x *at time* t*", we may view Gärtner-mixing as a consequence of the fact that there are not enough particles in the blocks* $\widetilde{B}_R^A(x_i,k_i;b,c)$ *that manage to travel to the blocks* $\widetilde{B}_R^A(x_j,k_j;b,c)$. *Indeed, if there is a bad block on scale* R *that is contained in a good block on scale* $R+1$, *then in some neighborhood of this bad block the particle density cannot be too large. This also explains why we must work with the extended blocks* $\widetilde{B}_R^A(x,k;b,c)$ *instead of with the original blocks* $B_R^A(x,k;0,0)$. *Indeed, the surroundings of a bad block on scale* R *can be bad when it is located near the boundary of a good block on scale* $R+1$ *(see Fig. 2.2).*

• **Gärtner-regularity and Gärtner-volatility.** Recall that $||\cdot||$ denotes the lattice-norm, see the line following (2.1.2). We say that $\Phi\colon [0,t] \to \mathbb{Z}^d$ is a path when

$$\|\Phi(s) - \Phi(s-)\| \le 1 \qquad \forall\, s \in [0,t]. \tag{2.1.26}$$

We write $\Phi \in B_R$ when $\|\Phi(s)\| \le R$ for all $s \in [0,t]$ and denote by $N(\Phi,t)$ the number of jumps of Φ up to time t.

Definition 2.1.9. ***[Gärtner-regularity]***
ξ *is called Gärtner-regular when*
(a) ξ *is Gärtner-negative-hyper-mixing and Gärtner-positive-hyper-mixing.*
(b) There are $t_0 > 0$ *and* $n_0 \in \mathbb{N}$ *such that for every* $\delta_1 > 0$ *there is a* $\delta_2 = \delta_2(\delta_1) > 0$ *such that*

$$\mathbb{P}\left(\sum_{j=1}^{n} \int_{(j-1)t+1}^{jt} \xi\big(\Phi((j-1)t+1), s\big)\, ds \ge \delta_1 n t\right) \le e^{-\delta_2 n t} \qquad \forall\, t \ge t_0,\ n \ge n_0,\ \Phi \in B_{tn}. \tag{2.1.27}$$

Definition 2.1.10. ***[Gärtner-volatility]***
ξ *is called Gärtner-volatile when*
(a) ξ *is Gärtner-negative-hyper-mixing.*
(b)

$$\lim_{t\to\infty} \frac{1}{\log t} \mathbb{E}\left(\Big|\int_0^t [\xi(0,s) - \xi(e,s)]\, ds\Big|\right) = \infty \qquad \textit{for some } e \in \mathbb{Z}^d \textit{ with } \|e\| = 1, \tag{2.1.28}$$

Remark 2.1.11. *Corollary 2.1.20 below will show that condition (b) in Definition 2.1.9 is satisfied as soon as the Dirichlet form of* ξ *is non-degenerate, i.e., has a unique zero (see Section 2.7).*

2.1.3.2 Theorems: Uniqueness, existence, finiteness and initial condition

Recall the definition of q^T (see (2.1.12)), the condition on u_0 in (2.1.4) and the condition on ξ in (2.1.6).

Theorem 2.1.12. *[**Uniqueness**] Consider a deterministic $q\colon \mathbb{Z}^d \times [0,\infty) \to \mathbb{R}$ such that:*
(1) There is a $T>0$ such that q^T is non-percolating from below.
(2) $q^T(x)<\infty$ for all $T>0$ and $x \in \mathbb{Z}^d$.
Then the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) = \kappa\Delta u(x,t) + q(x,t)u(x,t), \\ u(x,0) = u_0(x), \end{cases} \qquad x \in \mathbb{Z}^d,\ t \geq 0, \tag{2.1.29}$$

has at most one non-negative solution.

Theorem 2.1.13. *[**Existence**] Suppose that:*
(1) $s \mapsto \xi(x,s)$ is locally integrable for every x, ξ-a.s.
(2) $\mathbb{E}(e^{q\xi(0,0)}) < \infty$ for all $q \geq 0$.
Then the function defined by the Feynman-Kac formula

$$u(x,t) = E_x\left(\exp\left\{\int_0^t \xi(X^\kappa(s), t-s)\,ds\right\} u_0(X^\kappa(t))\right) \tag{2.1.30}$$

solves (2.1.1) with initial condition u_0.

From now on we assume that ξ satisfies the conditions of Theorems 2.1.12–2.1.13 (where q is replaced by ξ in Theorem 2.1.12).

Theorem 2.1.14. *[**Finiteness**] If ξ is Gärtner-positive-hyper-mixing, then $\lambda_0^{\delta_0}(\kappa) < \infty$.*

From now on we also assume that ξ satisfies the conditions of Theorem 2.1.14. The following result extends Gärtner, den Hollander and Maillard [GdHM12], Theorem 1.1, in which it was shown that for the initial condition $u_0 = \delta_0$ the quenched Lyapunov exponent exists and is constant ξ-a.s.

Theorem 2.1.15. *[**Initial Condition**] If ξ is reversible in time or symmetric in space, type-II Gärtner-mixing and Gärtner-negative-hyper-mixing, then $\lambda_0^{u_0}(\kappa) = \lim_{t\to\infty} \frac{1}{t} \log u(0,t)$ exists ξ-a.s. and in $L^1(\mathbb{P})$, is constant ξ-a.s., and is independent of u_0.*

2.1.3.3 Theorems: Dependence on κ

Theorem 2.1.16. *[**Continuity at** $\kappa = 0$] If ξ is Gärtner-regular, then $\kappa \mapsto \lambda_0^{\delta_0}(\kappa)$ is continuous at zero.*

Theorem 2.1.17. *[**Not Lipschitz at** $\kappa = 0$] If ξ is Gärtner-volatile, then $\kappa \mapsto \lambda_0^{\delta_0}(\kappa)$ is not Lipschitz continuous in zero.*

Remark 2.1.18. *Theorem 2.1.17 was already shown in [GdHM12], under the additional assumption that ξ is bounded from below.*

2.1.3.4 Examples

We state two corollaries in which we give examples of classes of ξ for which the conditions in Theorems 2.1.14–2.1.16 are satisfied.

Corollary 2.1.19. ***[Examples for Theorems 2.1.14–2.1.15]***
(1) Let $X = (X_t)_{t\geq 0}$ be a stationary and ergodic $\mathbb{R}$-valued Markov process. Let $(X_\cdot(x))_{x\in\mathbb{Z}^d}$ be independent copies of X. Define ξ by $\xi(x,t) = X_t(x)$. If

$$\mathbb{E}\left[e^{q \sup_{s\in[0,1]} X_s}\right] < \infty \qquad \forall\, q \geq 0, \tag{2.1.31}$$

then ξ fulfills the conditions of Theorem 2.1.14. If, moreover, the left-hand side of (2.1.31) is finite for all $q \leq 0$, then ξ satisfies the conditions of Theorem 2.1.15.
(2) Let ξ be the zero-range process with rate function $g\colon \mathbb{N}_0 \to (0,\infty)$, $g(k) = k^\beta$, $\beta \in (0,1]$, and transition probabilities given by a simple random walk on $\mathbb{Z}^d$. If ξ starts from the product measure π_ρ, $\rho \in (0,\infty)$, with marginals

$$\pi_\rho\big\{\eta \in \mathbb{N}_0^{\mathbb{Z}^d}\colon\ \eta(x) = k\big\} = \begin{cases} \gamma\,\frac{\rho^k}{g(1)\times\cdots\times g(k)}, & \text{if } k > 0,\\ \gamma, & \text{if } k = 0, \end{cases} \tag{2.1.32}$$

where $\gamma \in (0,\infty)$ is a normalization constant, then ξ satisfies the conditions of Theorems 2.1.14–2.1.15.

Corollary 2.1.20. ***[Examples for Theorem 2.1.16]*** *(1) If ξ is a bounded interacting particle system in the so-called $M < \varepsilon$ regime (i.e., fast mixing, see Liggett [L85] for a more precise definition), then the conditions of Theorem 2.1.16 are satisfied.*
(2) If ξ is the exclusion process with an irreducible, symmetric and transient random walk transition kernel, then the conditions of Theorem 2.1.16 are satisfied.
(3) If ξ is the dynamics defined by

$$\xi(x,t) = \sum_{y\in\mathbb{Z}^d} \sum_{j=1}^{N_y} \delta_{Y_j^y(t)}(x), \tag{2.1.33}$$

where $\{Y_j^y\colon\ y \in \mathbb{Z}^d, 1 \leq j \leq N_y, Y_j^y(0) = y\}$ is a collection of independent continuous-time simple random walks jumping at rate one, and $(N_y)_{y\in\mathbb{Z}^d}$ is a Poisson random field with intensity ν for some $\nu \in (0,\infty)$. If $d \geq 3$, then the conditions of Theorem 2.1.16 are satisfied.

Corollaries 2.1.19–2.1.20 list only a few examples that match the conditions. It is a separate problem to verify these conditions for as broad a class of interacting particle systems as possible.

2.1.4 Discussion and a conjecture

The proofs of Theorems 2.1.12–2.1.17 and Corollaries 2.1.19–2.1.20 are given in Sections 2.2–2.7. The content of Theorems 2.1.12–2.1.17 is summarized in Fig. 2.3.

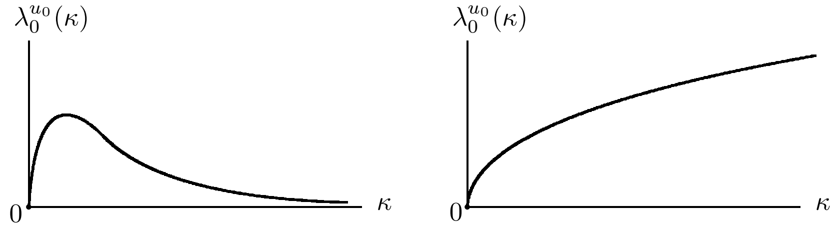

Figure 2.3: Qualitative picture of $\kappa \mapsto \lambda_0^{u_0}(\kappa)$ in the weakly, respectively, strongly catalytic regime.

The importance of $\lambda_0^{u_0}(\kappa)$ within the *population dynamics* interpretation of the parabolic Anderson model, as explained in Section 2.1.1, is the following. For $t > 0$, randomly draw a B-particle from the population of B-particles at the origin. Let L_t be the random time this B-particle and its ancestors have spent on top of A-particles. By appealing to the ergodic theorem, it may be shown that $\lim_{t\to\infty} L_t/t = \lambda_0^{u_0}(\kappa)$ a.s. Thus, $\lambda_0^{u_0}(\kappa)$ is the fraction of time the *best* B-particles spend on top of A-particles, where best means that they come from the fastest growing family ("survival of the fittest"). Fig. 2.3 shows that for all $\kappa \in (0,\infty)$ *clumping* occurs: the limiting fraction is strictly larger than the density of A-particles. In the limit as $\kappa \downarrow 0$ the clumping vanishes because the motion of the A-particles is ergodic in time. The clumping is hard to suppress for $\kappa \downarrow 0$: even a tiny bit of mobility allows the best B-particles and their ancestors to successfully "hunt down" the A-particles.

In the limit as $\kappa \to \infty$ we expect the quenched Lyapunov exponent to merge with the *annealed Lyapunov exponents* defined in (2.1.15), with δ_0 replaced by u_0.

Conjecture 2.1.21. $\lim_{\kappa\to\infty}[\lambda_p^{u_0}(\kappa) - \lambda_0^{u_0}(\kappa)] = 0$ *for all* $p \in \mathbb{N}$.

The reason is that for large κ the B-particles can easily find the largest clumps of A-particles and spend most of their time there, so that it does not matter much whether the largest clumps are close to the origin or not.

It remains to identify the scaling behaviour of $\lambda_0^{u_0}(\kappa)$ for $\kappa \downarrow 0$ and $\kappa \to \infty$. Under strong noisiness conditions on ξ, it was shown in Gärtner, den Hollander and Maillard [GdHM12] that $\lambda^{u_0}(\kappa)$ tends to zero like $1/\log(1/\kappa)$ (in a rough sense), while it tends to $\mathbb{E}(\xi(0,0))$ as $\kappa \to \infty$. For the *annealed* Lyapunov exponents $\lambda_p^{u_0}(\kappa)$, $p \in \mathbb{N}$, there is no singular behavior as $\kappa \downarrow 0$, in particular, they are Lipschitz continuous at $\kappa = 0$ with $\lambda_p^{u_0}(0) > \mathbb{E}(\xi(0,0))$. For three specific choices of ξ it was shown that $\lambda_p^{u_0}(\kappa)$ with $u_0 \equiv 1$ decays like $1/\kappa$ as $\kappa \to \infty$ (see [GdHM12] and references therein). A distinction is needed between the *strongly catalytic regime* for which $\lambda_p^{u_0}(\kappa) = \infty$ for all $\kappa \in [0,\infty)$, and the *weakly catalytic regime* for which $\lambda_p^{u_0}(\kappa) < \infty$ for all $\kappa \in [0,\infty)$. (These regimes were introduced by Gärtner and den Hollander [GdH06] for independent simple random walks.) We expect Conjecture 2.1.21 to be valid in both regimes.

2.2 Existence and uniqueness of the solution

In this section we prove Theorem 2.1.12 (uniqueness; Section 2.2.1) and Theorem 2.1.13 (existence; Section 2.2.2).

2.2.1 Uniqueness

The proof of Theorem 2.1.12 is based on the following lemma.

Lemma 2.2.1. *Let $q_i\colon \mathbb{Z}^d \times [0,\infty) \to \mathbb{R}$, $i \in \{1,2\}$, satisfy conditions (1)–(2) in Theorem 2.1.12 and be such that, for a given initial condition u_0, the two corresponding Cauchy problems*

$$\begin{cases} \frac{\partial}{\partial t}u_i(x,t) = \kappa\Delta u_i(x,t) + q_i(x,t)u_i(x,t), \\ u_i(x,0) = u_0(x), \end{cases} \qquad x \in \mathbb{Z}^d,\ t \geq 0,\ i \in \{1,2\}, \tag{2.2.1}$$

have a solution. If there exists a $T > 0$ such that $q_1(x,t) \geq q_2(x,t)$ for all $x \in \mathbb{Z}^d$ and $t \in [0,T]$, then $u_1(x,t) \geq u_2(x,t)$ for all $x \in \mathbb{Z}^d$ and $t \in [0,T]$, where u_1 and u_2 are any two solutions of (2.2.1).

We first prove Theorem 2.1.12 subject to Lemma 2.2.1.

Proof. Note from Definition 2.1.2 that whenever q^T is non-percolating from below for $T = T_0$ for some $T_0 > 0$, then the same is true for all $T \geq T_0$. Fix $T \geq T_0$, and let u be a non-negative solution of (2.1.29) with zero initial condition, i.e., $u_0(x) = 0$ for all $x \in \mathbb{Z}^d$. It is sufficient to prove that $u(x,t) = 0$ for all $x \in \mathbb{Z}^d$ and $t \in [0,T]$.

Let v be the solution of the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t}v(x,t) = \kappa\Delta v(x,t) + q^T(x)v(x,t), \\ v(x,0) = v_0(x) = 0, \end{cases} \qquad x \in \mathbb{Z}^d, t \in [0,T], \tag{2.2.2}$$

which exists because the corresponding Feynman-Kac representation is zero by Gärtner and Molchanov [GM90], Lemma 2.2. By Lemma 2.2.1 it follows that $0 \leq u \leq v$ on $\mathbb{Z}^d \times [0,T]$. Using that q^T is non-percolating from below, we may apply [GM90], Lemma 2.3, to conclude that (2.2.2) has at most one solution. Hence $u = v = 0$ on $\mathbb{Z}^d \times [0,T]$, which gives the claim. ∎

We next prove Lemma 2.2.1.

Proof. Fix $R \in \mathbb{N}$. Let $B_R = [-R,R]^d \cap \mathbb{Z}^d$, $\mathrm{int}(B_R) = (-R,R)^d \cap \mathbb{Z}^d$, and $\partial B_R = B_R \backslash \mathrm{int}(B_R)$. If u_1 and u_2 are solutions of (2.2.1) on $\mathbb{Z}^d \times [0,\infty)$, then they are also solutions on $B_R \times [0,T]$. More precisely, for $i \in \{1,2\}$, u_i is a solution of the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t}v(x,t) = \kappa\Delta v(x,t) + q_i(x,t)v(x,t), & (x,t) \in \mathrm{int}(B_R) \times [0,T], \\ v(x,0) = u_0(x), & x \in B_R, \\ v(x,t) = u_i(x,t), & (x,t) \in \partial B_R \times [0,T]. \end{cases} \tag{2.2.3}$$

Recall that $q_1 \geq q_2$ on $\mathbb{Z}^d \times [0,T]$. Choose c_R^T such that

$$c_R^T > \max_{x\in B_R,\, t\in[0,T]} q_1(x,t) \geq \max_{x\in B_R,\, t\in[0,T]} q_2(x,t), \tag{2.2.4}$$

and abbreviate

$$\begin{cases} v(x,t) = e^{-c_R^T t}\left[u_1(x,t) - u_2(x,t)\right], & (x,t) \in B_R \times [0,T], \\ \bar{Q}_i = q_i - c_R^T, & i \in \{1,2\}. \end{cases} \tag{2.2.5}$$

Then, by (2.2.3), v satisfies

$$\begin{cases} \frac{\partial}{\partial t} v(x,t) = \kappa\Delta v(x,t) + e^{-c_R^T t}\,\bar{Q}_1(x,t)u_1(x,t) & \\ \qquad\qquad -e^{-c_R^T t}\,\bar{Q}_2(x,t)u_2(x,t), & (x,t) \in \mathrm{int}(B_R) \times [0,T], \\ v(x,0) = 0, & x \in B_R, \\ v(x,t) = e^{-c_R^T t}\left[u_1(x,t) - u_2(x,t)\right], & (x,t) \in \partial B_R \times [0,T]. \end{cases} \tag{2.2.6}$$

Now, suppose that there exists a $(x_*,t_*) \in \mathrm{int}(B_R) \times [0,T]$ such that

$$v(x_*,t_*) = \min_{x\in\mathrm{int}(B_R),\, t\in[0,T]} v(x,t) < 0. \tag{2.2.7}$$

Then

$$\frac{\partial}{\partial t} v(x_*,t_*) \leq 0 \tag{2.2.8}$$

and

$$\Delta v(x_*,t_*) = \sum_{\substack{y\in\mathbb{Z}^d \\ \|y-x_*\|=1}} \left[v(y,t_*) - v(x_*,t_*)\right] \geq 0. \tag{2.2.9}$$

Moreover, by (2.2.4–2.2.5) and (2.2.7),

$$\begin{aligned} &e^{-c_R^T t_*}\,\bar{Q}_1(x_*,t_*)u_1(x_*,t_*) - e^{-c_R^T t_*}\,\bar{Q}_2(x_*,t_*)u_2(x_*,t_*) \\ &\quad = \left[q_1(x_*,t_*) - c_R^T\right] v(x_*,t_*) + \left[q_1(x_*,t_*) - q_2(x_*,t_*)\right] e^{-c_R^T t_*}\,u_2(x_*,t_*) > 0. \end{aligned} \tag{2.2.10}$$

But (2.2.8–2.2.10) contradict the first line of (2.2.6) at $(x,t) = (x_*,t_*)$. Hence (2.2.7) fails, and so it follows from (2.2.5) that $u_1(x,t) \geq u_2(x,t)$ for all $x \in \mathrm{int}(B_R)$ and $t \in [0,T]$. Since R can be chosen arbitrarily, the claim follows. ∎

2.2.2 Existence

In the sequel we use the abbreviations

$$\mathcal{I}^\kappa(a,b,c) = \int_a^b \xi(X^\kappa(s), c-s)\,ds, \qquad 0 \leq a \leq b \leq c, \tag{2.2.11}$$

$$\overline{\mathcal{I}}^\kappa(a,b,c) = \int_a^b \xi(X^\kappa(s), c+s)\,ds, \qquad 0 \leq a \leq b \leq c. \tag{2.2.12}$$

Proof. To prove Theorem 2.1.13, by Proposition 2.1.1 it is enough to show that

$$E_x\left(e^{\mathcal{I}^\kappa(0,t,t)}u_0(X^\kappa(t))\right) < \infty \qquad \forall x \in \mathbb{Z}^d,\, t \geq 0. \tag{2.2.13}$$

Since u_0 is assumed to be non-negative and bounded (recall (2.1.4)), without loss of generality we may take $u_0 \equiv 1$. We give the proof for $x = 0$, the extension to $x \in \mathbb{Z}^d$ being straightforward. Fix $q \in \mathbb{Q} \cap [0,\infty)$. Using Jensen's inequality and the stationarity of ξ, we have (recall (2.1.6))

$$\begin{aligned}
\mathbb{E}\left(E_0\left(e^{\mathcal{I}^\kappa(0,q,q)}\right)\right) &= E_0\left(\mathbb{E}\left(e^{\mathcal{I}^\kappa(0,q,q)}\right)\right)\\
&\leq E_0\left(\mathbb{E}\left(\frac{1}{q}\int_0^q \exp\left\{q\xi(X^\kappa(s),q-s)\right\}ds\right)\right)\\
&= E_0\left(\frac{1}{q}\int_0^q \mathbb{E}\Big(\exp\left\{q\xi(0,0)\right\}\Big)\,ds\right)\\
&= \mathbb{E}\big(e^{q\xi(0,0)}\big) < \infty,
\end{aligned} \tag{2.2.14}$$

where the finiteness follows by condition (2). Hence, for every $q \in \mathbb{Q} \cap [0,\infty)$ there exists a set A_q with $\mathbb{P}(A_q) = 1$ such that

$$E_0\left(e^{\mathcal{I}^\kappa(0,q,q)}\right) < \infty \qquad \forall\, \xi \in A_q. \tag{2.2.15}$$

To extend (2.2.15) to $t \in [0,\infty)$, note that, by the Markov property of X^κ applied at time $q-t$, $q > t$, we have

$$\begin{aligned}
E_0\left(e^{\mathcal{I}^\kappa(0,q,q)}\right) &\geq E_0\left(e^{\mathcal{I}^\kappa(0,q,q)}\mathbb{1}\Big\{X^\kappa(r) = 0 \;\forall\, r \in [0,q-t]\Big\}\right)\\
&= e^{\int_0^{q-t}\xi(0,q-s)\,ds}P_0\Big(X^\kappa(r) = 0\;\forall\, r\in[0,q-t]\Big)E_0\left(e^{\mathcal{I}^\kappa(0,t,t)}\right).
\end{aligned} \tag{2.2.16}$$

Because $s \mapsto \xi(0,s)$ is locally integrable ξ-a.s. by condition (1), we have $\int_0^{q-t}\xi(0,q-s)\,ds > -\infty$ ξ-a.s. The claim now follows from (2.2.15–2.2.16) by picking $q \in \mathbb{Q}\cap[0,\infty)$ and $t \in [0,\infty)$. ∎

2.3 Finiteness of the quenched Lyapunov exponent

In this section we prove Theorem 2.1.14. In Section 2.3.1 we sketch the strategy of the proof. In Sections 2.3.2–2.3.6 the details are worked out.

2.3.1 Strategy of the proof

The proof uses ideas from Kesten and Sidoravicius [KS03]. To simplify the notation, we assume that $\xi \geq 0$. Fix C, b, c according to our assumptions on ξ. For $j \in \mathbb{N}$ and $t > 0$, define the set of random walk paths

$$\Pi(j,t) = \left\{ \Phi\colon\ [0,t] \to \mathbb{Z}^d\colon\ \Phi \text{ makes } j \text{ jumps, } \Phi([0,t]) \subseteq [-C_1 t\log t, C_1 t\log t]^d \cap \mathbb{Z}^d \right\}, \tag{2.3.1}$$

where C_1 will be determined later on. Abbreviate $[C_1]_t = [-C_1 t\log t, C_1 t\log t]^d \cap \mathbb{Z}^d$. For $A \geq 1$, $R \in \mathbb{N}$ and $\Phi \in \Pi(j,t)$, define

$$\Psi_R^A(\Phi) = \begin{array}{c}\text{number of good } (R+1)\text{-blocks crossed}\\ \text{by } \Phi \text{ containing a bad } R\text{-block,}\end{array} \tag{2.3.2}$$

$$\Psi_R^{A,j} = \sup_{\Phi\in\Pi(j,t)} \Psi_R^A(\Phi), \tag{2.3.3}$$

$$\Xi_R^A(\Phi) = \text{number of bad } R\text{-blocks crossed by } \Phi, \tag{2.3.4}$$

$$\Xi_R^{A,j} = \sup_{\Phi\in\Pi(j,t)} \Xi_R^A(\Phi). \tag{2.3.5}$$

The proof comes in 5 steps, organized as Sections 2.3.2–2.3.6: (1) the Feynman-Kac formula may be restricted to paths contained in $[C_1]_t$; (2) there are no bad R-blocks for sufficiently large R; (3) the Feynman-Kac formula can be estimated in terms of bad R-blocks; (4) bounds can be derived on the number of bad R-blocks; (5) completion of the proof.

2.3.2 Step 1: Restriction to $[C_1]_t$

Lemma 2.3.1. *Fix $C_1 > 0$. Suppose that $\mathbb{E}\left(e^{q\sup_{s\in[0,1]}\xi(0,s)}\right) < \infty$ for all $q > 0$. Then:*
(a) ξ-a.s.

$$\limsup_{t\to\infty} \left[\frac{1}{\log t} \sup\{\xi(x,s)\colon\ x \in [C_1]_t, 0 \leq s \leq t\}\right] \leq 1. \tag{2.3.6}$$

(b) ξ-a.s. there exists a $t_0 \geq 0$ such that, for all $t \geq t_0$ and $x \notin [C_1]_t$,

$$\sup_{s\in[0,t]} \xi(x,s) \leq \log \|x\|. \tag{2.3.7}$$

Proof. (a) For any $\theta > 0$ and $t \geq 1$, we may estimate

$$\begin{aligned} &\mathbb{P}\left(\exists\, x \in [C_1]_t\colon\ \sup_{s\in[0,t]} \xi(x,s) \geq \log t\right) \\ &\leq \sum_{x\in[C_1]_t} \sum_{k=0}^{\lfloor t\rfloor} \mathbb{P}\left(\sup_{s\in[k,k+1]} \xi(x,s) \geq \log t\right) \\ &\leq (2C_1 t\log t + 1)^d (\lfloor t\rfloor + 1) \exp\{-\theta\log t\} \mathbb{E}\left(\exp\left\{\theta \sup_{s\in[0,1]} \xi(0,s)\right\}\right). \end{aligned} \tag{2.3.8}$$

Choosing $\theta > 2(d+1)+1$, we get that the right-hand side is summable over $t \in \mathbb{N}$. Hence, by the Borel-Cantelli Lemma, we get the claim.

(b) The proof is similar and is omitted. ∎

The main result of this section reads:

Lemma 2.3.2. *There exists a $C_0 > 0$ such that ξ-a.s. there exists a $t_0 > 0$ such that*

$$E_0\left(e^{\mathcal{I}^\kappa(0,t,t)} 1\!\!1\{X^\kappa([0,t]) \not\subseteq [C_1]_t\}\right) \le e^t \qquad \forall\, t \ge t_0, C_1 \ge C_0. \tag{2.3.9}$$

Proof. See Kesten and Sidoravicius [KS03], Eq. (2.38). We only sketch the main idea. Take a realization $\Phi\colon\ [0,t] \to \mathbb{Z}^d$ of a random walk path that leaves the box $[C_1]_t$. Then $\|\Phi\| = \max\{\|x\|\colon\ x \in \Phi([0,t])\} > C_1 t \log t$. By Lemma 2.3.1,

$$\sup_{s\in[0,t]} \sup_{\|x\|\le\|\Phi\|} \xi(x,s) \le \log\|\Phi\|, \tag{2.3.10}$$

and so we can estimate

$$\begin{aligned} &E_0\left(e^{\mathcal{I}^\kappa(0,t,t)} 1\!\!1\{X^\kappa([0,t]) \not\subseteq [C_1]_t\}\right) \\ &\le E_0\left(\exp\left\{t \sup_{s\in[0,t]} \log\|X^\kappa(s)\|\right\} 1\!\!1\{X^\kappa([0,t]) \not\subseteq [C_1]_t\}\right). \end{aligned} \tag{2.3.11}$$

The rest of the proof consists of balancing the exponential growth of the term with the supremum against the superexponential decay of $P_0(X^\kappa([0,t]) \not\subseteq [C_1]_t)$. See [KS03] for details. ∎

2.3.3 Step 2: No bad R-blocks for large R

Lemma 2.3.3. *Fix $C_0 > 0$ according to Lemma 2.3.2, and suppose that ξ satisfies condition (a3) in the Gärtner-positive-hyper-mixing definition. Then for every $C_1 \ge C_0$ and $\varepsilon > 0$ there exists an $A = A(\varepsilon) > 2$ such that*

$$\mathbb{P}\Big(\Xi_R^{A,j} > 0 \text{ for some } R \ge \varepsilon\log t \text{ and some } j \in \mathbb{N}_0\Big) \tag{2.3.12}$$

is summable over $t \in \mathbb{N}$. (It suffices to choose $A = \lfloor e^{1/a(1+2d)\varepsilon}\rfloor$ for some $a > 1$.)

Proof. Fix $C_1 \ge C_0$, $A > 2$ and assume that $\Xi_R^{A,j} > 0$ for some $j \in \mathbb{N}_0$. Then there is a bad R-block $B_R^A(x,k)$ that intersects $[C_1]_t \times [0,t]$. Hence there is a pair $(y,s) \in \mathbb{Z}^d \times [0,\infty)$ such that $Q_R^A(y) \times \{s\} \subseteq \widetilde{B}_R^A(x,k;b,c)$ and

$$\sum_{z\in Q_R^A(y)} \xi(z,s) > CA^{Rd}. \tag{2.3.13}$$

In particular, y and s satisfy $\mathrm{dist}(y,[C_1]_t)\le(b+2)A^R$ and $s\in[0,t+A^R]$. Hence, for $\varepsilon>0$,

$$
\begin{aligned}
&\mathbb{P}\Big(\Xi_R^{A,j}>0 \text{ for some } R\ge\varepsilon\log t, j\in\mathbb{N}_0\Big)\\
&\le \sum_{R\ge\varepsilon\log t}\mathbb{P}\Bigg(\sum_{z\in Q_R^A(y)}\xi(z,s)>CA^{Rd} \text{ for some } (y,s)\colon\\
&\qquad\qquad \mathrm{dist}(y,[C_1]_t)\le(b+2)A^R, s\in[0,t+A^R]\Bigg)\\
&\le \sum_{R\ge\varepsilon\log t}\ \sum_{y\colon \mathrm{dist}(y,[C_1]_t)\le(b+2)A^R}\ \sum_{k=0}^{\lfloor t+A^R\rfloor}\mathbb{P}\Big(\exists s\in[k,k+1)\colon \sum_{z\in Q_R^A(y)}\xi(z,s)>CA^{Rd}\Big).
\end{aligned}
\tag{2.3.14}
$$

By assumption (2.1.25), we may bound the two inner sums by

$$
(2C_1t\log t+1+(b+2)A^R)^d\times(\lfloor t+A^R\rfloor+1)\times(2A^R+1)^{-d\alpha}\stackrel{\mathrm{def}}{=}G(R,t). \tag{2.3.15}
$$

Recall the definition of α (see below 2.1.25)), to see that one can choose A as described in the formulation of Lemma 2.3.3 to get that

$$
\sum_{R\ge\varepsilon\log t}G(R,t) \tag{2.3.16}
$$

is summable over $t\in\mathbb{N}$. ∎

2.3.4 Step 3: Estimate of the Feynman-Kac formula in terms of bad blocks

Lemma 2.3.4. *Fix $\varepsilon>0$ and $A>2$. For all $C_1\ge C_0$ (where C_0 is determined by Lemma 2.3.2),*

$$
\begin{aligned}
&E_0\left(e^{\mathcal{I}^\kappa(0,t,t)}1\!\!1\{X^\kappa([0,t])\subseteq[C_1]_t\}\right)\\
&\le\sum_{j\in\mathbb{N}_0}\frac{(2dt\kappa)^j}{j!}\exp\left\{t(CA^d-2d\kappa)+\sum_{R=1}^{\infty}CA^{(R+1)d}A^R\Xi_R^{A,j}\right\}.
\end{aligned}
\tag{2.3.17}
$$

Proof. See [KS03], Lemma 9. We sketch the proof. Note that

$$
\begin{aligned}
E_0\left(e^{\mathcal{I}^\kappa(0,t,t)}1\!\!1\{X^\kappa([0,t])\subseteq[C_1]_t\}\right)&=\sum_{j\in\mathbb{N}_0}e^{-2dt\kappa}\frac{(2dt\kappa)^j}{j!}\\
\times\sum_{x_1,x_2,\dots,x_j\in\mathbb{Z}^d}\frac{1}{(2d)^j}&E_0\left(\exp\left\{\sum_{i=1}^j\int_{S_{i-1}}^{S_i}\xi(x_{i-1},t-u)\,du+\int_{S_j}^t\xi(x_j,t-u)\,du\right\}\right),
\end{aligned}
\tag{2.3.18}
$$

where j is the number of jumps, $0 = x_0, x_1, \ldots, x_j$, $x_i \in [C_1]_t, i \in \{0, 1, \cdots, j\}$, are the nearest-neighbor sites visited, and $0 = S_0 < S_1 < \cdots < S_j < t$ are the jump times. To analyze (2.3.18), fix $A > 2$, $R \in \mathbb{N}$ as well as $0 = s_0 < s_1 < \cdots < s_j$ and a path Φ with these jump times, and define

$$\begin{aligned}\Lambda_R(\Phi) = &\bigcup_{i=1}^{j} \left\{u \in [s_{i-1}, s_i)\colon\ CA^{Rd} < \xi(x_{i-1}, t-u) \leq CA^{(R+1)d}\right\} \\ &\bigcup \left\{u \in [s_j, t)\colon\ CA^{Rd} < \xi(x_j, t-u) \leq CA^{(R+1)d}\right\}.\end{aligned} \tag{2.3.19}$$

The contribution of Φ to the exponential in (2.3.18) may be bounded from above by

$$tCA^d + \sum_{R=1}^{\infty} CA^{(R+1)d} |\Lambda_R(\Phi)|, \tag{2.3.20}$$

where the first term comes from the space-time points $(x_{i-1}, t-u)$ with $\xi(x_{i-1}, t-u) \leq CA^d$. If $CA^{Rd} < \xi(x_{i-1}, t-u) \leq CA^{(R+1)d}$, then $(x_{i-1}, t-u)$ belongs to a bad R-block. There are at most $\Xi_R^{A,j}$ such blocks, and any path spends at most a time A^R in each R-block. Hence

$$|\Lambda_R(\Phi)| \leq A^R \Xi_R^{A,j}. \tag{2.3.21}$$

The claim now follows from (2.3.18), (2.3.20–2.3.21) and the fact that there are at most $(2d)^j$ nearest-neighbor paths $(0 = x_0, x_1, x_2, \ldots, x_j)$ that are contained in $[C_1]_t$. ∎

2.3.5 Step 4: Bound on the number of bad blocks

The goal of this section is to provide a bound on the number of bad blocks on all scales simultaneously (Lemma 2.3.5 below). In Section 2.3.6 we will combine Lemmas 2.3.2 and Lemmas 2.3.4–2.3.5 to prove Theorem 2.1.14.

Lemma 2.3.5. *Fix $\varepsilon > 0$, pick A according to Lemma 2.3.3 and assume that $\Xi_R^{A,j} = 0$ for all $R \geq \lceil \varepsilon \log t \rceil$. Then, for some $C_2 > 0$,*

$$\mathbb{P}\Big(\Psi_R^{A,j} \geq (t+j)(A^{(1+2d)})^{-R} \text{ for some } R \in \mathbb{N} \text{ and some } j \in \mathbb{N}_0\Big), \tag{2.3.22}$$

$$\mathbb{P}\Big(\Xi_R^{A,j} \geq C_2(t+j)(A^{(1+2d)})^{-R} \text{ for some } R \in \mathbb{N} \text{ and some } j \in \mathbb{N}_0\Big), \tag{2.3.23}$$

are summable on $t \in \mathbb{N}$.

The proof of Lemma 2.3.5 is based on Lemmas 2.3.6–2.3.7 below. The first estimates for fixed R the probability that there is a large number of good $(R+1)$-blocks containing a bad R-block, the second gives a recursion bound on the number of bad blocks in terms of $\Psi_R^{A,j}$.

Lemma 2.3.6. *Suppose that ξ satisfies condition (a2) in Definition 2.1.7. Then, for R large enough, $j \in \mathbb{N}_0$ and A chosen according to Lemma 2.3.3, for some constant $C_3 > 0$*

$$\mathbb{P}\left(\Psi_R^{A,j} \geq (t+j)(A^{(1+2d)})^{-R}\right) \leq \exp\left\{-C_3(t+j)(A^{(1+2d)})^{-R}\right\}. \tag{2.3.24}$$

Lemma 2.3.7. *Fix $\varepsilon > 0$, and pick A according to Lemma 2.3.3. Assume that $\Xi_R^{A,j} = 0$ for all $R \geq \lceil \varepsilon \log t \rceil$. Then, with $N = \lceil \varepsilon \log t \rceil$,*

$$\Xi_R^{A,j} \leq 2^d A^{(1+d)} \sum_{i=0}^{N-R-1} 2^{id} A^{i(1+d)} \Psi_{R+i}^{A,j}. \tag{2.3.25}$$

The proofs of Lemmas 2.3.6, 2.3.7 and 2.3.5 are given in Sections 2.3.5.1, 2.3.5.2 and 2.3.5.3, respectively.

2.3.5.1 Proof of Lemma 2.3.6

Proof. Throughout the proof, $x, x' \in \mathbb{Z}^d$ and $k, k' \in \mathbb{N}$. The idea of the proof is to divide space-time blocks into equivalence classes, such that in each equivalence class blocks are far enough away from each other so that they can be treated as being independent (see Fig. 2.2). The proof comes in four steps.

1. Fix $A > 2$ according to Lemma 2.3.3, fix $R \in \mathbb{N}$ and take $a_1, a_2, b, c \in \mathbb{N}_0$ according to condition (a2) in Definition 2.1.7. We say that (x,k) and (x',k') are equivalent if and only if

$$x \equiv x' \mod a_1 \quad \text{and} \quad k \equiv k' \mod a_2. \tag{2.3.26}$$

This equivalence relation divides $\mathbb{Z}^d \times \mathbb{N}$ into $a_1^d a_2$ equivalence classes. We write $\sum_{(x^*,k^*)}$ to denote the sum over all equivalence classes. Furthermore, we define

$$\chi^A(x,k) = \mathbb{1}\{B_{R+1}^A(x,k) \text{ is good, but contains a bad } R\text{-block}\}. \tag{2.3.27}$$

We tacitly assume that all blocks under consideration intersect $[C_1]_t \times [0,t]$. Then

$$\begin{aligned} &\mathbb{P}\left(\Psi_R^{A,j} \geq (t+j)(A^{(1+2d)})^{-R}\right) \\ &\leq \sum_{(x^*,k^*)} \mathbb{P}\left(\begin{array}{l} \exists \text{ a path with } j \text{ jumps that intersects at least} \\ (t+j)(A^{(1+2d)})^{-R}/a_1^d a_2 \text{ blocks } B_{R+1}^A(x,k) \\ \text{with } \chi^A(x,k)=1,\ (x,k) \equiv (x^*,k^*) \end{array}\right). \end{aligned} \tag{2.3.28}$$

2. For the rest of the proof we fix an equivalence class and define $\rho_R^{1/(1+d)} = (A^{(1+2d)})^{-R}$ (recall 2.1.21). To control the number of different ways to cross a prescribed number of R-blocks we consider enlarged blocks. To that end, as in [KS03], take $\nu = \lceil \rho_R^{-1/(1+d)} \rceil$ and define space-time blocks

$$\widetilde{B}_R^A(x,k) = \left(\prod_{j=1}^d [\nu(x(j)-1)A^R, \nu(x(j)+1)A^R) \cap \mathbb{Z}^d\right) \times [\nu k A^R, \nu(k+1)A^R). \tag{2.3.29}$$

By the same reasoning as in [KS03], we see that at most

$$\mu(j) \stackrel{\text{def}}{=} 3^d \left(\frac{t+j}{\nu A^{R+1}} + 2 \right) \tag{2.3.30}$$

blocks $\widetilde{B}^A_{R+1}(x,k)$ can be crossed by a path Φ with j jumps. We write

$$\bigcup_{(x_i,k_i)} \widetilde{B}^A_{R+1}(x_i,k_i) \quad \text{and} \quad \sum_{\widetilde{B}^A_{R+1}(x_i,k_i)} \tag{2.3.31}$$

to denote the union over at most $\mu(j)$ blocks $\widetilde{B}^A_{R+1}(x_i,k_i)$, $0 \leq i \leq \mu(j)-1$, and to denote the sum over all possible sequences of blocks $\widetilde{B}^A_{R+1}(x_i,k_i)$, that may be crossed by a path Φ with j jumps, respectively. As each block $B^A_{R+1}(x,k)$ that may be crossed by such a path is contained in the union in (2.3.31), we may estimate the probability in (2.3.28) from above by

$$\sum_{\widetilde{B}^A_{R+1}(x_i,k_i)} \mathbb{P}\left(\begin{array}{l} \text{the union in (2.3.31) contains at least } (t+j)\rho_R^{1/(1+d)}/a_1^d a_2 \\ \text{blocks } B^A_{R+1}(x,k) \text{ with } \chi^A(x,k)=1,\ (x,k)\equiv(x^*,k^*) \end{array} \right). \tag{2.3.32}$$

To estimate the probability in (2.3.32) write

$$\begin{aligned} &\mathcal{A}^n((x_0,k_0),\ldots,(x_{\mu(j)-1},k_{\mu(j)-1})) \\ &\qquad = \left\{ \begin{array}{l} \text{the union in (2.3.31) contains } n \text{ blocks } B^A_{R+1}(x,k) \\ \text{with } \chi^A(x,k)=1,\ (x,k)\equiv(x^*,k^*) \end{array} \right\}. \end{aligned} \tag{2.3.33}$$

Since the union (2.3.31) contains at most $L = \nu^{(1+d)}\mu(j)$ blocks $B^A_{R+1}(x,k)$, the probability in (2.3.32) is bounded from above by

$$\sum_{n=\frac{(t+j)\rho_R^{1/(1+d)}}{a_1^d a_2}}^{L} \mathbb{P}\Big(\mathcal{A}^n((x_0,k_0),\ldots,(x_{\mu(j)-1},k_{\mu(j)-1}))\Big). \tag{2.3.34}$$

Note that there are $\binom{L}{n}$ ways of choosing n blocks $B^A_{R+1}(x,k)$ with $\chi^A(x,k)=1$ out of L $(R+1)$-blocks. Hence, by condition (a2) in Definition 2.1.7, (2.3.34) is at most

$$\sum_{n=\frac{(t+j)\rho_R^{1/(1+d)}}{a_1^d a_2}}^{L} \binom{L}{n} \rho_R^n \leq \Big(1-\rho_R\Big)^{-L} K\mathbb{P}\left(T_L \geq \frac{(t+j)\rho_R^{1/(1+d)}}{a_1^d a_2} \right), \tag{2.3.35}$$

where $T_L = \mathrm{BIN}(L,\rho_R)$.

3. To estimate the binomial random variable, note that by Bernstein's inequality (compare with [KS03], Lemma 11), there is a constant C' such that, for all $\lambda \geq 2\mathbb{E}(T)$,

$$\mathbb{P}(T_L \geq \lambda) \leq \exp\{-C'\lambda\}. \tag{2.3.36}$$

We may assume that $\rho_R^{-1/(1+d)} \in \mathbb{N}$, so that

$$\mathbb{E}(T_L) = \nu^{(1+d)}\mu(j)\rho_R = 3^d \left(\frac{t+j}{A^{R+1}}\, \rho_R^{1/(1+d)} + 2 \right), \tag{2.3.37}$$

and hence, by Lemma 2.3.3 and the fact that $R \le \varepsilon \log t$,

$$\frac{(t+j)\rho_R^{1/(1+d)}}{a_1^d a_2} \ge 2\mathbb{E}(T_L). \tag{2.3.38}$$

Since a_1, a_2 are independent of R, we may estimate, using (2.3.36),

$$\begin{aligned} \mathbb{P}\left(T_L \ge \frac{(t+j)\rho_R^{1/(1+d)}}{a_1^d a_2} \right) &\le \exp\left\{ -C'(t+j)\rho_R^{1/(1+d)} \right\} \\ &= \exp\left\{ -C'(t+j)(A^{(1+2d)})^{-R} \right\}. \end{aligned} \tag{2.3.39}$$

It remains to show that the first term on the right hand side in (2.3.35) does not contribute. Note that $1/(1-\rho_R) = 1 + \rho_R/(1-\rho_R)$, so that

$$\log\left(\frac{1}{1-\rho_R} \right) \le \frac{\rho_R}{1-\rho_R}. \tag{2.3.40}$$

Thus, if we assume $\rho_r^{-1/(1+d)} \in \mathbb{N}$,

$$L \log\left(\frac{1}{1-\rho_R} \right) \le \frac{\mu(j)}{1-\rho_R}. \tag{2.3.41}$$

Inserting (2.3.41) into the second term on the right hand side of (2.3.35), comparing it with the right hand side of (2.3.39), and recalling the definition of $\mu(j)$ in (2.3.30), we see that the asymptotic of (2.3.35) is determined by the probability term in (2.3.39).

4. Finally, we estimate (2.3.28).

Claim 2.3.8. *There is $C > 0$ such that the number of summands in (2.3.32) is bounded from above by $e^{C\mu(j)}$.*

Before we proof the claim, we show how one deduces Lemma 2.3.6 from it. Insert (2.3.39) into (2.3.32) and use Claim 2.3.8 to obtain that

$$\mathbb{P}\left(\Psi_R^{A,j} \ge (t+j)(A^{(1+2d)})^{-R} \right) \le K a_1^d a_2 \exp\left\{ -C'(t+j)(A^{(1+2d)})^{-R} \right\}. \tag{2.3.42}$$

We now prove Claim 2.3.8.

Proof. Assume that $d = 1$. Divide time into intervals of length νA^{R+1} and fix an integer-valued sequence $(l_1, l_2, \ldots, l_{t/\nu A^{R+1}})$ such that

$$\sum_{i=1}^{t/\nu A^{R+1}} l_i \le \mu(j). \tag{2.3.43}$$

We first estimate the number of ways in which Φ can visit l_i R-blocks $\widetilde{B}^A_{R+1}(x,k)$ in the i-th time interval for $i=1,\dots,t/\nu A^{R+1}$. Let $\widetilde{B}^A_{R+1}(x_0,k_0),\widetilde{B}^A_{R+1}(x_1,k_1),\dots,$ $\widetilde{B}^A_{R+1}(x_{l_i-1},k_{l_i-1})$ be a possible sequence. Then $x_0=0$, $x_1=\pm 1$, $x_2\in(-1,x_1+1)$ if $x_1=1$ or $x_2\in(x_1-1,1)$ if $x_1=-1$, etc. For each j such that $l_i+1<j\le l_{i+1}$, x_j can take two values depending on x_{j-1}. Note that if $j=l_i+1$ for some j and i, then x_j is the space label of a block at the beginning of a new time interval. Consequently, $x_j\in\{x_{j-1}-1,x_{j-1},x_{j-1}+1\}$. Hence, for a fixed choice of $(l_1,l_2,\dots,l_{t/\nu A^{R+1}})$, there are at most $3^{l_1}\times 3^{l_2}\times\cdots\times 3^{l_{t/\nu A^{R+1}}}\le 3^{\mu(j)}$ possibilities to choose a sequence of blocks with a prescribed sequence $(l_1,l_2,\dots,l_{t/\nu A^{R+1}})$. Moreover, by Hardy and Ramanujan [HR18] and Erdös [E42] there are $a,b\in(0,\infty)$ such that there are no more than $(a/\mu(j))e^{b\sqrt{\mu(j)}}$ unordered integer-values sequences $(l_1,l_2,\dots,l_{t/\nu A^{R+1}})$ satisfying (2.3.43). From this it follows that there is a $c\in(0,\infty)$ such that the number of ordered integer-valued sequences $(l_1,l_2,\dots,l_{t/\nu A^{R+1}})$ satisfying (2.3.43) is bounded from above by $\mu(j)^{c\sqrt{\mu(j)}}(a/\mu(j))e^{b\sqrt{\mu(j)}}$. This in combination with the above arguments yields the claim. The extension to $d\ge 2$ is straightforward. ∎

This finishes the proof of Lemma 2.3.6. ∎

2.3.5.2 Proof of Lemma 2.3.7

Proof. We first show that

$$\Xi^{A,j}_R\le 2^dA^{(1+d)}\Xi^{A,j}_{R+1}+2^dA^{(1+d)}\Psi^{A,j}_R. \tag{2.3.44}$$

In order to see why (2.3.44) is true, take a bad R-block $B^A_R(x,k)$ that is crossed by a path with j jumps. Then there are two possibilities. Either $B^A_R(x,k)$ is contained in a bad $(R+1)$-block, or all $(R+1)$-blocks that contain $B^A_R(x,k)$ are good. Since an $(R+1)$-block contains $A^{(1+d)}$ R-blocks, and there are at most 2^d $(R+1)$-blocks, which may contain a given R-block, the first term in the above sum bounds the number of bad R-blocks contained in a bad $(R+1)$-block. In contrast, the second term bounds the number of bad R-blocks contained in a good $(R+1)$-block. Hence we obtain (2.3.44).

We can now prove the claim. Apply (2.3.44) iteratively to the terms in the sum, i.e., replace $\Xi^{A,j}_{R+i}$ by

$$2^dA^{(1+d)}\Xi^{A,j}_{R+i+1}+2^dA^{(1+d)}\Psi^{A,j}_{R+i}. \tag{2.3.45}$$

This yields

$$\Xi^{A,j}_R\le 2^dA^{(1+d)}\sum_{i=0}^{N-R-1}2^{id}A^{i(1+d)}\Psi^{A,j}_{R+i}, \tag{2.3.46}$$

from which the claim follows. ∎

2.3.5.3 Proof of Lemma 2.3.5

Proof. Fix $\varepsilon > 0$ and $0 < R \leq \varepsilon \log t$. Then, by Lemma 2.3.6,

$$\begin{aligned}
&\mathbb{P}\left(\Psi_R^{A,j} \geq (t+j)(A^{(1+2d)})^{-R} \text{ for some } j \in \mathbb{N}_0\right) \\
&\leq \sum_{j\in\mathbb{N}_0} \mathbb{P}\left(\Psi_R^{A,j} \geq (t+j)(A^{(1+2d)})^{-R}\right) \\
&\leq \sum_{j\in\mathbb{N}_0} \exp\left\{-tC_3(t+j)(A^{(1+2d)})^{-R}\right\} \\
&\leq \exp\left\{-C_3 t(A^{(1+2d)})^{-\varepsilon\log t}\right\} \sum_{j\in\mathbb{N}_0} \exp\left\{-C_3 j(A^{(1+2d)})^{-\varepsilon\log t}\right\}.
\end{aligned} \tag{2.3.47}$$

Recall Lemma 2.3.3, which implies that $\delta \overset{\text{def}}{=} \log(A)\varepsilon(1+2d) < 1$. Consequently, the right-hand side of (2.3.47) is at most

$$\begin{aligned}
&\exp\left\{-C_3 t^{(1-\delta)}\right\} \frac{1}{1-\exp\{-C_3 t^{-\delta}\}} \\
&\leq \frac{1}{C_3} \exp\left\{-C_3 t^{(1-\delta)}\right\} t^{\delta} \exp\left\{C_3 t^{-\delta}\right\}.
\end{aligned} \tag{2.3.48}$$

It therefore follows that

$$\begin{aligned}
&\mathbb{P}\left(\Psi_R^{A,j} \geq (t+j)(A^{(1+2d)})^{-R} \text{ for some } j \in \mathbb{N}_0,\ R \in \mathbb{N}\right) \\
&\leq \frac{1}{C_3} t^{\delta}\, \varepsilon \log t\, \exp\left\{-C_3 t^{(1-\delta)} + C_3 t^{-\delta}\right\},
\end{aligned} \tag{2.3.49}$$

which is summable over $t \in \mathbb{N}$. In order to prove the second statement, suppose that none of the events in (2.3.22) occurs. With Lemma 2.3.7 we may estimate

$$\begin{aligned}
\Xi_R^{A,j} &\leq 2^d A^{(1+d)} \sum_{i=0}^{N-R-1} 2^{id} A^{i(1+d)} \Psi_{R+i}^{A,j} \\
&\leq 2^d A^{(1+d)} \sum_{i=0}^{N-R-1} (t+j) 2^{id} A^{i(1+d)} (A^{(1+2d)})^{-i-R} \\
&\leq 2^d A^{(1+d)} (t+j) A^{-R(1+2d)} \sum_{i\in\mathbb{N}_0} 2^{id} A^{-id} \\
&\overset{\text{def}}{=} (t+j) A^{-R(1+2d)} C_2,
\end{aligned} \tag{2.3.50}$$

where we use that $A > 2$ (see Lemma 2.3.3). ∎

2.3.6 Step 5: Proof of the finiteness of the quenched Lyapunov exponent

Fix $\varepsilon > 0$ and A such that Lemma 2.3.5 applies. It follows from Lemma 2.3.3 that

$$\mathbb{P}\Big(\Xi_R^{A,j} > 0 \text{ for some } R \geq \varepsilon \log t, j \in \mathbb{N}_0\Big) \tag{2.3.51}$$

is summable over $t \in \mathbb{N}$. Hence, by the Borel-Cantelli Lemma, there is an $t_0 \in \mathbb{N}$ such that none of the events in the above probability occurs for integer $t \geq t_0$. Thus, by Lemma 2.3.4, for all integer $t \geq t_0$ we have, with $N = \lfloor \varepsilon \log t \rfloor$,

$$\begin{aligned} &E_0\left(e^{\mathcal{I}^\kappa(0,t,t)} 1\!\!1\{X^\kappa([0,t]) \subseteq [C_1]_t\}\right) \\ &\leq \sum_{j\in\mathbb{N}_0} \frac{(2dt\kappa)^j}{j!} \exp\left\{t(CA^d - 2d\kappa) + \sum_{R=1}^{N} CA^{(R+1)d} A^R \Xi_R^{A,j}\right\}. \end{aligned} \tag{2.3.52}$$

Using the bound of Lemma 2.3.5, we have

$$\begin{aligned} \sum_{R=1}^{N} CA^{(R+1)d} A^R \Xi_R^{A,j} &\leq \sum_{R=1}^{N} C(t+j) A^{(R+1)d} A^R A^{-R(1+2d)} C_2 \\ &\leq (t+j) A^d C' \sum_{R\in\mathbb{N}} A^{-Rd} \leq C_4(t+j). \end{aligned} \tag{2.3.53}$$

We can therefore estimate the last line of (2.3.52) by

$$\begin{aligned} &\sum_{j\in\mathbb{N}_0} \frac{(2dt\kappa)^j}{j!} \exp\left\{t(CA^d - 2d\kappa) + C_4(t+j)\right\} \\ &= \exp\left\{t(CA_2^d - 2d\kappa + C_4 + 2d\kappa e^{C_4})\right\}. \end{aligned} \tag{2.3.54}$$

From (2.3.54) and Lemma 2.3.2, we obtain

$$\limsup_{\substack{t\to\infty \\ t\in\mathbb{N}}} \frac{1}{t} \log E_0\left(e^{\mathcal{I}^\kappa(0,t,t)}\right) < \infty. \tag{2.3.55}$$

To extend this to sequences along $\mathbb{R}$ instead of $\mathbb{N}$, note that

$$u(0,t) \leq u(0,n+1)\, e^{-\int_t^{n+1} \xi(0,s)\, ds} e^{2d\kappa(n+1-t)}, \qquad t \in [n, n+1]. \tag{2.3.56}$$

Since ξ is ergodic in time, we have

$$\lim_{t\to\infty} \frac{1}{t} \int_t^{\lceil t \rceil} \xi(0,s)\, ds = 0. \tag{2.3.57}$$

Theorem 2.1.14 follows from (2.3.56–2.3.57).

2.4 Initial condition

In this section we prove Theorem 2.1.15. Section 2.4.1 contains some preparations. Section 2.4.2 states three lemmas (Lemmas 2.4.5–2.4.7 below) that are needed for the proof of Theorem 2.1.15, which is given in Section 2.4.3. Section 2.4.4 provides the proof of these three lemmas.

2.4.1 Preparations

In this section we first state and prove a lemma (Lemma 2.4.1 below) that will be needed for the proof of Theorem 2.1.15. After that we introduce some further notation (Definitions 2.4.2–2.4.4 below).

Fix $R_0 \in \mathbb{N}$ and take A, C according to our assumption (type-II Gärtner-mixing). Set $N = CA^{R_0 d}$ and abbreviate $\xi_N = (\xi \wedge N) \vee (-N)$. Let u_N be the solution of (2.1.1) with ξ replaced by ξ_N. Abbreviate (recall (2.2.11–2.2.12))

$$\begin{aligned} \mathcal{I}_N^\kappa(a,b,c) &= \int_a^b \xi_N(X^\kappa(s), c-s)\, ds, \qquad 0 \le a \le b \le c, &(2.4.1)\\ \overline{\mathcal{I}}_N^\kappa(a,b,c) &= \int_a^b \xi_N(X^\kappa(s), c+s)\, ds, \qquad 0 \le a \le b \le c. &(2.4.2)\end{aligned}$$

Lemma 2.4.1. *If, for all N of the form $N = CA^{R_0 d}$ and for all $\varepsilon > 0$ and some sequence $(t_r)_{r\in\mathbb{N}}$ of the form $t_r = rL$ with $L > 0$,*

$$\mathbb{P}\left(E_0\left(e^{\overline{\mathcal{I}}_N^\kappa(0,t_r,0)}\right) > e^{(\overline{\lambda}_0^{\mathbb{1}}(\kappa)+\varepsilon)t_r}\right) \tag{2.4.3}$$

is summable on r, then Theorem 2.1.15 holds.

Proof. Fix $\varepsilon > 0$. Note that $u_N(0,t)$ has the same distribution as $E_0(e^{\overline{\mathcal{I}}_N^\kappa(0,t,0)})$, so that we can replace the latter by $u_N(0,t)$ in (2.4.3) without violating the summability condition. Thus, by the Borel-Cantelli Lemma, we have

$$\limsup_{r\to\infty} \frac{1}{t_r} \log E_0\left(e^{\mathcal{I}_N^\kappa(0,t_r,t_r)}\right) \le \overline{\lambda}_0^{\mathbb{1}}(\kappa) + \varepsilon \qquad \xi\text{-a.s.} \tag{2.4.4}$$

The extension to sequences along $\mathbb{R}$ may be done as in the proof of Theorem 2.1.14 (recall (2.3.56)). Standard arguments yield

$$\limsup_{t\to\infty} \frac{1}{t} \log u_N(0,t) \le \overline{\lambda}_0^{\mathbb{1}}(\kappa) \qquad \xi\text{-a.s.} \tag{2.4.5}$$

To extend this to the solution of (2.1.1) with initial condition $u_0 \equiv 1$, we estimate

$$\begin{aligned}&\int_0^t \xi(X^\kappa(s), t-s) ds\\ &\le \int_0^t \xi(X^\kappa(s), t-s)\mathbb{1}\{\xi(X^\kappa(s), t-s) \ge N\}\, ds + \int_0^t \xi_N(X^\kappa(s), t-s)\, ds.\end{aligned} \tag{2.4.6}$$

Note that, by (2.3.53) and the arguments given in the proof of Lemma 2.3.4, we have, for $t \in \mathbb{N}$ sufficiently large,

$$\sup_{\Phi\in\Pi(j,t)} \int_0^t \xi(\Phi(s), t-s) 1\!\!1\{\xi(\Phi(s), t-s) \geq N\}\, ds \leq (t+j) A^d C' \sum_{R=R_0}^{\infty} A^{-Rd}. \qquad (2.4.7)$$

Next, choose $M > 1$ such that

$$\limsup_{t\to\infty} \frac{1}{t} \log E_0 \left(e^{\mathcal{I}^\kappa(0,t,t)} 1\!\!1\{N(X^\kappa, t) > Mt\}\right) < \overline{\lambda}_0^{1\!\!1}(\kappa), \qquad (2.4.8)$$

where $N(X^\kappa, t)$ denotes the number of jumps of the random walk X^κ up to time t (see (2.1.26)). Then, by (2.4.6–2.4.7), for $t \in \mathbb{N}$ sufficiently large,

$$\begin{aligned} &E_0 \left(e^{\mathcal{I}^\kappa(0,t,t)} 1\!\!1\{N(X^\kappa, t) \leq Mt\}\right) \\ &\leq \exp\left\{ (M+1) t A^d C' \sum_{R=R_0}^{\infty} A^{-Rd} \right\} E_0 \left(e^{\mathcal{I}_N^\kappa(0,t,t)} 1\!\!1\{N(X^\kappa, t) \leq Mt\}\right). \end{aligned} \qquad (2.4.9)$$

We infer from (2.4.5) and (2.4.8–2.4.9) that

$$\limsup_{\substack{t\to\infty \\ t\in\mathbb{N}}} \frac{1}{t} u(0,t) \leq (M+1) A^d C' \sum_{R=R_0}^{\infty} A^{-Rd} + \overline{\lambda}_0^{1\!\!1}(\kappa). \qquad (2.4.10)$$

Taking the limit $R_0 \to \infty$, $R_0 \in \mathbb{N}$, we obtain

$$\limsup_{\substack{t\to\infty \\ t\in\mathbb{N}}} \frac{1}{t} \log u(0,t) \leq \overline{\lambda}_0^{1\!\!1}(\kappa). \qquad (2.4.11)$$

The extension to sequences along $\mathbb{R}$ may again be done as in the proof of Theorem 2.1.14 (recall (2.3.56)). Furthermore, Proposition 2.1.3 gives $\lambda_0^{\delta_0}(\kappa) = \overline{\lambda}_0^{\delta_0}(\kappa) = \overline{\lambda}_0^{1\!\!1}(\kappa)$, so that

$$\limsup_{t\to\infty} \frac{1}{t} \log u(0,t) \leq \lambda_0^{\delta_0}(\kappa). \qquad (2.4.12)$$

By monotonicity, the reverse inequality holds with the limsup replaced by the liminf. It follows that $\lambda_0^{1\!\!1}(\kappa)$ exists and equals $\lambda_0^{\delta_0}(\kappa)$. A further monotonicity argument shows that the same is true for $\lambda_0^{u_0}(\kappa)$ for any initial condition u_0 subject to (2.1.4). ∎

In view of Lemma 2.4.1, our target is to prove (2.4.3). We fix M subject to (2.4.8), N of the form $N = CA^{R_0 d}$, $\varepsilon > 0$ small, and write t as $t = rA^R$, $r, R \in \mathbb{N}$, $A > 2$. Note that the choice of M implies that it is enough to concentrate on path with at most Mt jumps.

We proceed by introducing space-time blocks and dividing them into good blocks and bad blocks, respectively, into N-sufficient blocks and N-insufficient blocks (compare with (2.1.18) and Fig. 2.2).

Definition 2.4.2. *For $x \in \mathbb{Z}^d$, $k \in \mathbb{N}$ and $b, c \in \mathbb{N}_0$, define (see Fig. 2.4)*

$$\hat{B}_R^A(x,k;b,c) = \left(\prod_{j=1}^{d} [(x(j)-1-b)4MA^R, (x(j)+1+b)4MA^R) \cap \mathbb{Z}^d \right) \times [(k-c)A^R, (k+1)A^R) \tag{2.4.13}$$

and abbreviate $\hat{B}_R^A(x,k) = \hat{B}_R^A(x,k;0,0)$.

For $S \subset \mathbb{Z}^d$, let ∂S denote the inner boundary of S. For $S \times S' \subset \mathbb{Z}^d \times \mathbb{R}$, let $\Pi_1(S \times S')$ denote the projection of $S \times S'$ onto the first d coordinates (the spatial coordinates).

Definition 2.4.3. *The subpedestal of $\hat{B}_R^A(x,k)$ is defined as*

$$\hat{B}_R^{A,\mathrm{sub}}(x,k) = \Big\{ y \in \Pi_1(\hat{B}_R^A(x,k))\colon\ |y(j)-z(j)| \geq 2MA^R,\ j \in \{1,2,\dots,d\}\, \forall\, z \in \partial\Pi_1(\hat{B}_R^A(x,k)) \Big\} \times \{kA^R\}. \tag{2.4.14}$$

Definition 2.4.4. *A block $\hat{B}_R^A(x,k)$ is called N-sufficient when, for every $y \in \Pi_1(\hat{B}_R^{A,\mathrm{sub}}(x,k))$ (see Fig. 2.4),*

$$E_y\left(e^{\overline{\mathcal{I}}_N^{\kappa}(0,A^R,kA^R)} \mathbb{1}\big\{N(X^\kappa, A^R) \leq MA^R\big\} \right) \leq e^{(\overline{\lambda}_0^{\mathbb{1}}(\kappa)+\varepsilon)A^R}. \tag{2.4.15}$$

Otherwise $\hat{B}_R^A(x,k)$ is called N-insufficient. A subpedestal is called N-sufficient/N-insufficient when its corresponding block is N-sufficient/N-insufficient.

The notion of good/bad is similar as in Definition 2.1.5 with the only difference that $B_R^A(x,k)$ is replaced by $\hat{B}_R^A(x,k)$ and $B_R^A(x,k;b,c)$ by $\hat{B}_R^A(x,k;b,c)$. Similarly as in (2.3.5), define $\hat{\Xi}_R^{A,j}$ to be the maximal number of bad R-blocks a path with j jumps can cross.

2.4.2 Three lemmas

For the proof of Theorem 2.1.15 we need Lemmas 2.4.5–2.4.7 below. The first says that each block is N-sufficient with a large probability (and is comparable with [CMS02], Lemma 4.3), the second controls the number of bad blocks (and is comparable with Lemma 2.3.5), the third estimates the number of N-insufficient blocks that are good and are visited by a typical random walk path (see [CMS02], Lemma 4.4 and [KS03], Lemma 11).

Lemma 2.4.5. *Fix $A \in \mathbb{N}$. For every $\widetilde{\delta} > 0$ there is an $R_0 = R_0(A, \widetilde{\delta}) \in \mathbb{N}$ such that*

$$\mathbb{P}(\hat{B}_R^A(x,k) \text{ is } N\text{-sufficient}\,) \geq 1 - \widetilde{\delta} \qquad \forall\, R \geq R_0,\ x \in \mathbb{Z}^d,\ k \in \mathbb{N}. \tag{2.4.16}$$

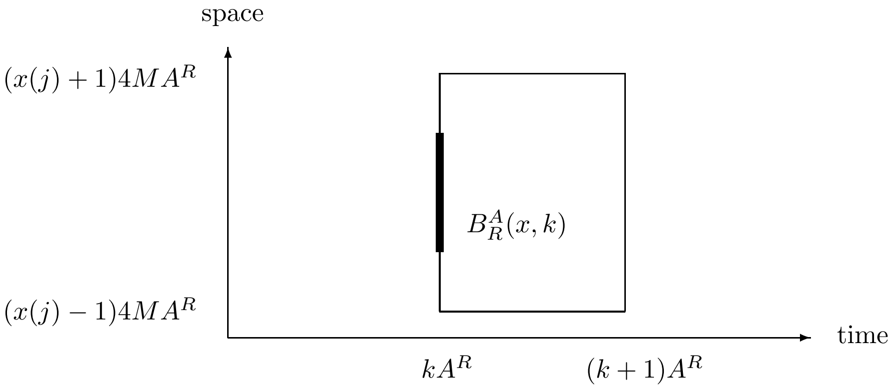

Figure 2.4: The thick line is the subpedestal.

Lemma 2.4.6. *For every $A_0 > 2$ there is an $A \in \mathbb{N}$ with $A \geq A_0$ such that, for some $C > 0$ independent of A,*

$$\mathbb{P}\Big(\hat{\Xi}_R^{A,j} \geq C(t+j)(A^{(1+2d)})^{-R} \text{ for some } j \in \mathbb{N}_0 \text{ and } R \in \mathbb{N}\Big) \tag{2.4.17}$$

is summable on $t \in \mathbb{N}$.

Lemma 2.4.7. *Let $C(Mt, \eta t/A^R)$ be the event that there is a path Φ with $\Phi(0) = 0$ and $N(\Phi,t) \leq Mt$ that up to time t crosses more than $\eta t/A^R$ N-insufficient subpedestals of a good R-block. Then, under the Gärtner-mixing type-II condition, for every $\eta > 0$ there is an A (which can be chosen as in Lemma 2.4.6) and $R_0 \in \mathbb{N}$ such that, for some $c_1 > 0$,*

$$\mathbb{P}\left(C(Mt, \eta t/A^R)\right) \leq e^{-c_1 \eta t/A^R}, \qquad \forall R \geq R_0. \tag{2.4.18}$$

2.4.3 Proof of the independence from the initial condition

Proof. The proof comes in two steps. Fix $0 < \eta < \varepsilon$, and choose $A, R \geq R_0$ according to Lemmas 2.4.5–2.4.7.

1. Fix a $\mathbb{Z}^d$-valued sequence of vertices $x_0, x_1, \dots, x_{t/A^R-1}$ such that $x_0 = 0$ and such that there is a path starting in 0, makes $0 \leq j \leq Mt$ jumps, and is in the subpedestals $\hat{B}_R^{A,\mathrm{sub}}(x_k,k)$ at times kA^R for all $k \in \{0,1,\dots,t/A^R-1\}$. By the Markov property of X^κ applied at times kA^R, $k \in \{1,2,\dots,t/A^R\}$,

$$\begin{aligned} &E_0\left(e^{\overline{\mathcal{I}}_N^\kappa(0,t,0)} \prod_{k=1}^{t/A^R-1} 1\!\!1\Big\{X^\kappa(kA^R) \in \hat{B}_R^{A,\mathrm{sub}}(x_k,k)\Big\}\right) \\ &\leq \prod_{k=0}^{t/A^R-1} \sup_{y \in \Pi_1(\hat{B}_R^{A,\mathrm{sub}}(x_k,k))} E_y\left(e^{\overline{\mathcal{I}}_N^\kappa(0,A^R,kA^R)}\right). \end{aligned} \tag{2.4.19}$$

Let I and S be the sets of indices k such that (x_k, kA^R) is the index for an N-insufficient, respectively, N-sufficient subpedestal. Then the right-hand side of (2.4.19) can be rewritten as

$$\prod_{k\in I} \sup_{y\in\pi_1(\hat{B}_R^{A,\mathrm{sub}}(x_k,k))} E_y\left(e^{\overline{\mathcal{I}}_N^\kappa(0,A^R,kA^R)}\right) \prod_{k\in S} \sup_{y\in\pi_1(\hat{B}_R^{A,\mathrm{sub}}(x_k,k))} E_y\left(e^{\overline{\mathcal{I}}_N^\kappa(0,A^R,kA^R)}\right). \tag{2.4.20}$$

Because of Lemmas 2.4.6 and 2.4.7, there is a measurable set, independent of j and of ξ-probability at least $1-e^{-c_1\eta t/A^R}$, such that

$$|I| \le \eta t/A^R + C(t+j)(A^{(1+2d)})^{-R}. \tag{2.4.21}$$

Since $\xi_N \le N$, on a set of that probability the first term in (2.4.20) can be estimated from above by $e^{N\eta t}\exp\{NC(t+j)/A^{2Rd}\}$.

2. Pick a realization of ξ that satisfies (2.4.21). To bound the second term in (2.4.20), we split this term up as

$$\begin{aligned}\prod_{k\in S}\Bigg[&\sup_{y\in\Pi_1(\hat{B}_R^{A,\mathrm{sub}}(x_k,k))} E_y\left(e^{\overline{\mathcal{I}}_N^\kappa(0,A^R,kA^R)}\mathbb{1}\{N(X^\kappa,A^R)\le MA^R\}\right)\\ &+\sup_{y\in\Pi_1(\hat{B}_R^{A,\mathrm{sub}}(x_k,k))} E_y\left(e^{\overline{\mathcal{I}}_N^\kappa(0,A^R,kA^R)}\mathbb{1}\{N(X^\kappa,A^R)> MA^R\}\right)\Bigg],\end{aligned} \tag{2.4.22}$$

which can be written as

$$\begin{aligned}\sum_{J\subset S}\Bigg[&\prod_{k\in J}\sup_{y\in\Pi_1(\hat{B}_R^{A,\mathrm{sub}}(x_k,k))} E_y\left(e^{\overline{\mathcal{I}}_N^\kappa(0,A^R,kA^R)}\mathbb{1}\{N(X^\kappa,A^R)\le MA^R\}\right)\\ &\times\prod_{k\notin J}\sup_{y\in\Pi_1(\hat{B}_R^{A,\mathrm{sub}}(x_k,k))} E_y\left(e^{\overline{\mathcal{I}}_N^\kappa(0,A^R,kA^R)}\mathbb{1}\{N(X^\kappa,A^R)> MA^R\}\right)\Bigg].\end{aligned} \tag{2.4.23}$$

Take $c \gg 1$. Then, for M large enough, $P_0\left(N(X^\kappa,A^R) > MA^R\right) \le e^{-cA^R}$. Hence the second term in (2.4.23) can be bounded from above by $e^{A^R(-c+N)(t/A^R-|J|)}$. Recall the definition of an N-sufficient block, to bound the sum in (2.4.23) by

$$e^{t(-c+N)}\left(1+e^{A^R(\overline{\lambda}_0^{\mathbf{1}}(\kappa)+\varepsilon+c-N)}\right)^{t/A^R}. \tag{2.4.24}$$

Summing over all possible values $(x_0, x_1, \dots, x_{t/A^R-1})$ compatible with a path Φ such that $\Phi(0)=0$ and $N(\Phi,t)\le Mt$, fixing $\eta\le\varepsilon$, and using the estimation of the last inequality of [CMS02], Lemma 4.2, to bound the number of such sequences in the last inequality, we obtain

$$\begin{aligned}
&E_0\left(e^{\overline{\mathcal{I}}^\kappa_N(0,t,0)}1\!\!1\{N(X^\kappa,t)\le Mt\}\right)\\
&\le \sum_{x_1,x_2,\dots,x_{t/A^R-1}} E_0\left(e^{\overline{\mathcal{I}}^\kappa_N(0,t,0)}\prod_{k=1}^{t/A^R-1}1\!\!1\Big\{X^\kappa(kA^R)\in \hat{B}_R^{A,\mathrm{sub}}(x_k,k)\Big\}\right)\\
&\le \exp\left\{C'\frac{t}{A^R}\right\}\exp\left\{t\left(N\eta+\frac{NC(M+1)}{A^{2Rd}}-c+N\right)\right\}\left(1+e^{A^R(\overline{\lambda}_0^{1\!\!1}(\kappa)+\varepsilon+c-N)}\right)^{t/A^R}\\
&\stackrel{\mathrm{def}}{=} C_1^N(t,A^R,\varepsilon),
\end{aligned} \tag{2.4.25}$$

for some constant $C'>0$. Thus, we have shown that there is an $A>2$ such that for each $R\ge R_0$

$$\mathbb{P}\left[E_0\left(e^{\overline{\mathcal{I}}^\kappa_N(0,rA^R,0)}1\!\!1\big\{N(X^\kappa,rA^R)\le MrA^R\big\}\right)>C_1^N(rA^R,A^R,\varepsilon)\right]\le e^{-r\eta c_1}, \tag{2.4.26}$$

which is summable on $r\in\mathbb{N}$. By the boundedness of ξ_N, the same is true without the indicator in the expectation (after a possible enlargement of C_1^N by ε). Further, note that

$$\begin{aligned}
&\frac{1}{rA^R}\log C_1^N(rA^R,A^R,\varepsilon)\\
&=\frac{C'}{A^R}+N\eta+\frac{NC(M+1)}{A^{2Rd}}+\overline{\lambda}_0^{1\!\!1}(\kappa)+\varepsilon+\frac{1}{A^R}\log\left(e^{-A^R(\overline{\lambda}_0^{1\!\!1}(\kappa)+\varepsilon+c-N)}+1\right),
\end{aligned} \tag{2.4.27}$$

so that $C_1^N(rA^R,A^R,\varepsilon)$ is indeed of the form $\overline{\lambda}_0^{1\!\!1}(\kappa)+\varepsilon$. Thus, we have proved Lemma 2.4.1 and hence Theorem 2.1.15. ∎

2.4.4 Proof of the three lemmas

2.4.4.1 Proof of Lemma 2.4.5

Proof. The proof comes in three steps and uses ideas from Cranston, Mountford and Shiga [CMS02], Lemma 4.3, and Drewitz, Gärtner, Ramirez and Sun [DGRS12], Lemma 4.3.

1. Suppose that we already showed that, ξ-a.s. and independently of the realization of ξ, for all $\varepsilon>0$ there is an $\eta'>0$ such that, for all $0<\eta<\eta'$,

$$\begin{aligned}
\limsup_{R\to\infty}\sup_{\substack{x,y\in[-2MA^R,2MA^R]^d\cap\mathbb{Z}^d\\ \|x-y\|\le\eta A^R}}\frac{1}{A^R}\Bigg|&\log E_x\left(e^{\overline{\mathcal{I}}^\kappa_N(0,A^R,0)}1\!\!1\Big\{N(X^\kappa,A^R)\le MA^R\Big\}\right)\\
&-\log E_y\left(e^{\overline{\mathcal{I}}^\kappa_N(0,A^R,0)}1\!\!1\Big\{N(X^\kappa,A^R)\le MA^R\Big\}\right)\Bigg|\le\varepsilon.
\end{aligned} \tag{2.4.28}$$

We show how (2.4.28) can be used to obtain the claim.

2. By Proposition 2.1.3 for fixed $\widetilde{\delta} > 0$ there is an $R_0 = R_0(A, \widetilde{\delta}) \in \mathbb{N}$ such that, for all $R \geq R_0$,

$$\mathbb{P}\left(E_0 \left(e^{\overline{\mathcal{I}}^\kappa(0,A^R,0)} 1\!\!1\left\{ N(X^\kappa, A^R) \leq MA^R \right\} \right) \leq e^{(\overline{\lambda}_0^{\mathbf{1}}(\kappa)+\varepsilon)A^R} \right) \geq 1 - \widetilde{\delta}. \tag{2.4.29}$$

To extend this to $\overline{\mathcal{I}}_N$, note that for each $t \in \mathbb{N}$

$$\int_0^t \xi_N(X^\kappa(s), s)\, ds \leq \int_0^t \xi(X^\kappa(s), s)\, ds + \int_0^t -\xi(X^\kappa(s), s) 1\!\!1\{\xi(X^\kappa(s), s) < -N\}\, ds. \tag{2.4.30}$$

Thus, given a realization of X^κ with no more than Mt jumps, by the fact that ξ is Gärtner-negative-hyper-mixing, (2.3.53) and the arguments given in the proof of Lemma 2.3.4, the second term in the right-hand side is at most

$$(M+1)tA^d C' \sum_{R=R_0}^{\infty} A^{-Rd}. \tag{2.4.31}$$

Hence (2.4.29) remains true when we replace $\overline{\mathcal{I}}$ by $\overline{\mathcal{I}}_N$. According to (2.4.28), this estimate also holds when we replace 0 by any x with $\|x\| \leq \eta A^R$ for η small enough, independently of the realization of ξ. Consequently, for any $\widetilde{\delta} > 0$ there is an $R_0 \in \mathbb{N}$ such that for all $R \geq R_0$,

$$\mathbb{P}\left(\sup_{x\colon\ \|x\| \leq \eta A^R} E_x \left(e^{\overline{\mathcal{I}}_N^\kappa(0,A^R,0)} 1\!\!1\left\{ N(X^\kappa, A^R) \leq MA^R \right\} \right) \leq e^{(\overline{\lambda}_0^{\mathbf{1}}(\kappa)+\varepsilon)A^R} \right) \geq 1 - \widetilde{\delta}. \tag{2.4.32}$$

Next, note that $[-2MA^R, 2MA^R]^d$ can be divided into K boxes, with $K \sim 4^d M^d/\eta^d$, of the form $x_i(A^R) + B_{\eta A^R}$, where the $x_i(A^R)$'s are separated by ηA^R. By the stationarity of ξ in space, we have

$$\begin{aligned} &\mathbb{P}\left(E_0 \left(e^{\overline{\mathcal{I}}_N^\kappa(0,A^R,0)} 1\!\!1\left\{ N(X^\kappa, A^R) \leq MA^R \right\} \right) \leq e^{(\overline{\lambda}_0^{\mathbf{1}}(\kappa)+\varepsilon)A^R} \right) \\ &= \mathbb{P}\left(E_{x_i(A^R)} \left(e^{\overline{\mathcal{I}}_N^\kappa(0,A^R,0)} 1\!\!1\left\{ N(X^\kappa, A^R) \leq MA^R \right\} \right) \leq e^{(\overline{\lambda}_0^{\mathbf{1}}(\kappa)+\varepsilon)A^R} \right). \end{aligned} \tag{2.4.33}$$

Thus, for the same choice of R_0 as in (2.4.32),

$$\begin{aligned} &\mathbb{P}\left(\sup_{y\colon\ y \in x_i(A^R)+B_{\eta A^R}} E_y \left(e^{\overline{\mathcal{I}}_N^\kappa(0,A^R,0)} 1\!\!1\left\{ N(X^\kappa, A^R) \leq MA^R \right\} \right) \leq e^{(\overline{\lambda}_0^{\mathbf{1}}(\kappa)+\varepsilon)A^R} \right) \\ &\quad \geq 1 - \widetilde{\delta} \qquad R \geq R_0. \end{aligned} \tag{2.4.34}$$

Since K is independent of R_0, we may conclude that

$$\mathbb{P}\left(B_R^A(0,0) \text{ is } N\text{-sufficient}\right) \geq 1 - \widetilde{\delta}. \tag{2.4.35}$$

By the stationarity of ξ in space and time, the same statement holds for any block $B_R^A(x,k)$, which proves the claim. It therefore remains to prove (2.4.28).

3. Since for M large the event $\Big\{N(X^\kappa, A^R) > MA^R\Big\}$ does not contribute on an exponential scale, in order to prove (2.4.28) it suffices to show that

$$\limsup_{R\to\infty} \sup_{\substack{x,y\in[-2MA^R,2MA^R]^d\cap\mathbb{Z}^d\\ \|x-y\|\leq\eta A^R}} \frac{1}{A^R}\left|\log\frac{E_x\left(e^{\overline{\mathcal{I}}_N^\kappa(0,A^R,0)}\right)}{E_y\left(e^{\overline{\mathcal{I}}_N^\kappa(0,A^R,0)}\right)}\right| \leq \varepsilon. \tag{2.4.36}$$

To that end, we show that we can restrict ourself to contributions coming from random walk paths that stay within a certain distance of $[-2M,2M]^d\cap\mathbb{Z}^d$. More precisely, ξ-a.s. there is a box $B_L = [-L,L]^d\cap\mathbb{Z}^d$, independent of $0<\eta<1$ and containing $[-2M,2M]^d\cap\mathbb{Z}^d$, such that

$$\begin{aligned}
&\sup_{x\in[-2MA^R,2MA^R]^d\cap\mathbb{Z}^d} \sum_{\substack{w\in\mathbb{Z}^d\\ w\notin A^RB_L}} E_x\left(e^{\overline{\mathcal{I}}_N^\kappa(0,\eta A^R,0)}\delta_w(X^\kappa(\eta A^R))\right) E_w\left(e^{\overline{\mathcal{I}}_N^\kappa(0,(1-\eta)A^R,\eta A^R)}\right)\\
&\qquad\leq e^{A^R} \sup_{x\in[-2MA^R,2MA^R]^d\cap\mathbb{Z}^d} P_x\left(X^\kappa(\eta A^R)\notin A^RB_L\right)\\
&\qquad= e^{A^R} P_0\Big(X^\kappa(\eta A^R)\notin A^RB_{L-2M}\Big)\\
&\qquad\leq e^{A^R}\exp\left\{-A^R(L-2M)\left(\log\left(\frac{L-2M}{\kappa d}\right)-1\right)\right\},
\end{aligned} \tag{2.4.37}$$

where the last inequality follows from Gärtner and Molchanov [GM90], Lemma 4.3. Consequently, we may concentrate in (2.4.36) on the contribution coming from paths that stay inside $A^RB_L\cap\mathbb{Z}^d$. Next, note that, ξ-a.s. and uniformly in $x,y\in[-2MA^R,2MA^R]^d\cap\mathbb{Z}^d$,

$$\begin{aligned}
&\frac{\sum_{w\in A^RB_L} E_x\left(e^{\overline{\mathcal{I}}_N^\kappa(0,A^R,0)}\delta_w(X^\kappa(\eta A^R))\right)}{\sum_{w\in A^RB_L} E_y\left(e^{\overline{\mathcal{I}}_N^\kappa(0,A^R,0)}\delta_w(X^\kappa(\eta A^R))\right)}\\
&\leq \sup_{w\in A^RB_L}\frac{E_x\left(e^{\overline{\mathcal{I}}_N^\kappa(0,\eta A^R,0)}\delta_w(X^\kappa(\eta A^R))\right)}{E_y\left(e^{\overline{\mathcal{I}}_N^\kappa(0,\eta A^R,0)}\delta_w(X^\kappa(\eta A^R))\right)}\\
&\leq e^{2N\eta A^R}\sup_{w\in A^RB_L}\frac{P_0\left(X^\kappa(\eta A^R)=w-x\right)}{P_0\left(X^\kappa(\eta A^R)=w-y\right)}, \qquad 0<\eta<1.
\end{aligned} \tag{2.4.38}$$

To obtain (2.4.28), it remains to estimate the probabilities in the last line. This can be done by applying bounds on probabilities for simple random walks (see [DGRS12], Lemma 4.3 for details). ∎

2.4.4.2 Proof of Lemma 2.4.6

The only difference with the situation in the proof of Lemma 2.3.5 is that we replaced the R-blocks $B_R^A(x,k)$ by the R-blocks $\hat{B}_R^A(x,k)$. However, this does not affect the proof. Thus, the proof of Lemma 2.3.5 yields the claim.

2.4.4.3 Proof of Lemma 2.4.7

Proof. The proof comes in two steps and is essentially a copy of the proof of Lemma 2.3.6. Throughout the proof, $x, x' \in \mathbb{Z}^d$, $k, k' \in \mathbb{N}$ and $\delta > 0$ is fixed.

1. Pick $a_1, a_2 \in \mathbb{N}$ according to our main assumption. We say that (x,k) and (x',k') are equivalent if and only if

$$x \equiv x' \mod a_1 \quad \text{and} \quad k \equiv k' \mod a_2. \tag{2.4.39}$$

This equivalence relation divides $\mathbb{Z}^d \times \mathbb{N}$ into $a_1^d a_2$ equivalence classes. We write $\sum_{(x^*,k^*)}$ to denote the sum over all equivalence classes. Furthermore, we define

$$\hat{\chi}^A(x,k) = \mathbb{1}\Big\{\hat{B}_R^A(x,k) \text{ is good, but has an } N\text{-insufficient subpedestal}\Big\}. \tag{2.4.40}$$

Henceforth we assume that all blocks under consideration intersect $[-Mt, Mt] \times [0,t]$. Then we have

$$\begin{aligned} &\mathbb{P}(C(Mt, \eta t/A^R)) \\ &\leq \sum_{(x^*,k^*)} \mathbb{P}\left(\begin{array}{c} \exists \text{a path with no more than } Mt \text{ jumps that intersects at least} \\ \eta t/A^R a_1^d a_2 \text{ blocks } \hat{B}_R^A(x,k) \text{ with } \hat{\chi}^A(x,k) = 1,\ (x,k) \equiv (x^*,k^*) \end{array} \right). \end{aligned} \tag{2.4.41}$$

2. To proceed we fix $j \leq Mt$ and an equivalence class. Let $\nu = \lceil \delta^{-1/(1+d)} \rceil$ and define the space block

$$\overline{B}_R^A(x,k) = \left(\prod_{j=1}^d [\nu(x(j)-1)4MA^R, \nu(x(j)+1)4MA^R) \cap \mathbb{Z}^d \right) \times [\nu k A^R, \nu(k+1)A^R). \tag{2.4.42}$$

As in the proof of Lemma 3.6 we see that a path Φ with j jumps crosses at most

$$\mu(j) = 3^d \left(\frac{t+j}{\nu A^R} + 2 \right) \tag{2.4.43}$$

blocks $\overline{B}_R^A(x,k)$. We write

$$\bigcup_{(x_i,k_i)} \overline{B}_R^A(x_i,k_i) \quad \text{and} \quad \sum_{\overline{B}_R^A(x_i,k_i)} \tag{2.4.44}$$

to denote the union over at most $\mu(j)$ blocks $\overline{B}_R^A(x_i,k_i)$, $0 \le i \le \mu(j)-1$, and to denote the sum over all possible sequences of blocks $\overline{B}_R^A(x_i,k_i)$, that may be crossed by a path Φ with j jumps, respectively. To further estimate, define for any sequence $\overline{B}_R^A(x_0,k_0),\dots \overline{B}_R^A(x_{\mu(j)-1},k_{\mu(j)-1})$ of blocks (2.4.44) and any $n\in\mathbb{N}$ the event

$$\begin{aligned}
&\hat{\mathcal{A}}^n((x_0,k_0),\dots,(x_{\mu(j)-1},k_{\mu(j)-1}))\\
&\qquad=\left\{\begin{array}{l}\text{the union in (2.4.44) contains } n \text{ blocks } \hat{B}_R^A(x,k)\\ \text{with } \hat{\chi}^A(x,k)=1,\ (x,k)\equiv(x^*,k^*)\end{array}\right\}.
\end{aligned}\tag{2.4.45}$$

By our assumption, there is $R_0\in\mathbb{N}$ such that the probability of the event in (2.4.45) may be bounded from above by $K\delta^n$. We conclude in the same way as in Lemma 3.6, see also Claim 2.3.8, that for some $C>0$

$$\begin{aligned}
&\sum_{\overline{B}_R^A(x_i,k_i)}\sum_{n=\frac{\eta t}{A^R a_1^d a_2}}^{L}\mathbb{P}\Big(\hat{\mathcal{A}}^n((x_0,k_0),\dots,(x_{\mu(j)-1},k_{\mu(j)-1}))\Big)\\
&\qquad\qquad\le e^{C\mu(j)}(1-\delta)^{-L}K\mathbb{P}\Big(T_L\ge\frac{\eta t}{A^R a_1^d a_2}\Big),
\end{aligned}\tag{2.4.46}$$

where $T_L=\mathrm{BIN}(L,\delta)$, $L=\nu^{(1+d)}\mu(j)$. The same arguments as in Lemma 2.3.6 yield that the binomially distributed random variable may be bounded from above by

$$\exp\left\{-C'\eta t/A^R a_1^d a_2\right\}\tag{2.4.47}$$

for some $C'>0$. Similarly as in Lemma 2.3.6, if $\delta^{-1/1+d}\in\mathbb{N}$, the second term on the right hand side of (2.4.46) may be bounded from above by

$$\frac{3^d(M+1)t\delta^{1/1+d}}{(1-\delta)A^R}+\frac{3^d 2}{1-\delta}.\tag{2.4.48}$$

Since δ tends to zero, if R tends to infinity, for t large enough the second term on the right hand side of (2.4.46) does not contribute. The same can be seen to be true for the first term on the right hand side of (2.4.46). Finally, to estimate (2.4.41), insert (2.4.47) into (2.4.41), to obtain

$$\mathbb{P}\left(C(Mt,\eta t/A^R)\right)\le Ka_1^d a_2 Mt\exp\left\{-C'\eta t/A^R\right\},\tag{2.4.49}$$

which yields the claim. ∎

2.5 Continuity at $\kappa = 0$

The proof of Theorem 2.1.16 is given in Section 2.5.3. It is based on Lemmas 2.5.1–2.5.3 below, which are stated in Section 2.5.1 and are proved in Section 2.5.2.

2.5.1 Three lemmas

Fix $b \in (0,1)$, and define the set of paths

$$A^{\kappa}_{nt} = \Big\{ \Phi\colon [0,nt] \to \mathbb{Z}^d\colon N(\Phi,nt) \le \frac{1}{\log(1/\kappa)^b}\, nt, \forall\, 1 \le j \le n$$
$$\exists\, x_j \in \mathbb{Z}^d\colon \|x_j\| \le \frac{1}{\log(1/\kappa)^b}\, nt,\ \Phi(s) = x_j\ \forall\, s \in [(j-1)t+1, jt) \Big\}, \tag{2.5.1}$$

i.e., paths of length nt that do not jump in time intervals of length $t-1$ and whose number of jumps is bounded by $\frac{1}{\log(1/\kappa)^b}\, nt$. Note that $\kappa \mapsto A^{\kappa}_{nt}$ is non-decreasing in the sense that if $\kappa_1 \le \kappa_2$ then $A^{\kappa_1}_{nt} \subseteq A^{\kappa_2}_{nt}$.

Lemma 2.5.1. *Suppose that ξ satisfies condition (b) in Definition 2.1.9. Then, ξ-a.s., for any sequence of positive numbers $(a_m)_{m\in\mathbb{N}}$ tending to zero there exists a strictly positive and non-increasing sequence $(\kappa_m)_{m\in\mathbb{N}}$ such that, for all $m \in \mathbb{N}$ and $0 < \kappa \le \kappa_m$, there exists a $t_m = t_m(\kappa_m)$ such that, for all $t \in \mathbb{Q} \cap [t_m,\infty)$, there exists an $n_m = n_m(\kappa_m, t)$ such that*

$$\sup_{\Phi\in A^{\kappa}_{nt}} \sum_{j=1}^{n} \int_{(j-1)t+1}^{jt} \xi(\Phi((j-1)t+1), s)\, ds \le a_m nt \qquad \forall\, n \ge n_m. \tag{2.5.2}$$

We say that two paths Φ_1 and Φ_2 on $[0,nt]$ are equivalent, written $\Phi_1 \sim \Phi_2$, if and only if

$$\Phi_1|_{[(j-1)t+1,jt)} = \Phi_2|_{[(j-1)t+1,jt)} \qquad \forall\, 1 \le j \le n. \tag{2.5.3}$$

This defines an equivalence relation $\sim$, and we denote by $A^{\kappa,\sim}_{nt}$ the set of corresponding equivalent classes. The following lemma provides an estimation of the cardinality of $A^{\kappa,\sim}_{nt}$.

Lemma 2.5.2. $|A^{\kappa,\sim}_{nt}| \le (nt/\log(1/\kappa)^b) 2^n (2d)^{nt/\log(1/\kappa)^b} + 1$.

Lemma 2.5.3. *Suppose that ξ is Gärtner-positive-hyper-mixing. Then there are $A, C > 0$ such that ξ-a.s. for $t \in \mathbb{Q}$ large enough and any choice of disjoint subintervals $\mathcal{I}_1, \mathcal{I}_2, \cdots, \mathcal{I}_k$, $k \in \mathbb{N}$, of $[0,t]$ such that $|\mathcal{I}_i| = |\mathcal{I}_l|$, $i,l \in \{1,2,\cdots,k\}$, and each $R_0 \in \mathbb{N}$ and each path $\Phi \in B_t$*

$$\sum_{i=1}^{k} \int_{\mathcal{I}_i} \xi(\Phi(s),s)\, ds \le k|\mathcal{I}_1| C A^{R_0 d} + (t + N(\Phi,t)) C' A^d \sum_{R=R_0}^{\infty} A^{-Rd}, \tag{2.5.4}$$

for some constant $C' > 0$.

2.5.2 Proof of the three lemmas

2.5.2.1 Proof of Lemma 2.5.2

Proof. Fix an integer $k \leq nt$. We start by estimating the number of possible arrangements of the jumps in paths with k jumps. Since we do not distinguish between two paths that coincide on the intervals $[(j-1)t+1, jt)$, $1 \leq j \leq n$, only the last jumps before the times $(j-1)t+1$, $1 \leq j \leq n$, need to be considered.

First, the number of arrangements with jumps in $0 \leq l \leq k$ different intervals is $\binom{n}{l}$. Since the number of different intervals cannot exceed n, the number of different arrangements is bounded from above by $\sum_{l=1}^{n} \binom{n}{l} \leq 2^n$. Next, there are $(2d)^k$ different points in $\mathbb{Z}^d$ that can be visited by a path with k jumps. Therefore

$$|A_{nt}^{\kappa,\sim}| \leq \sum_{k=1}^{nt/\log(1/\kappa)^b} 2^n (2d)^k + 1 \leq (nt/\log(1/\kappa)^b) 2^n (2d)^{nt/\log(1/\kappa)^b} + 1, \tag{2.5.5}$$

which proves the claim. ∎

2.5.2.2 Proof of Lemma 2.5.1

Proof. Choose κ_1 such that

$$\frac{\log(2d)}{\log(1/\kappa_1)^b} < \delta_2(a_1), \tag{2.5.6}$$

and $t_1 = t_1(\kappa_1)$ such that

$$t_1 \left(-\delta_2 + \frac{\log(2d)}{\log(1/\kappa_1)^b} \right) < -\log 2. \tag{2.5.7}$$

Then, by condition (b) in Definition 2.1.9 and Lemma 2.5.2, for all $t \geq t_0 \vee t_1$ we have

$$\begin{aligned} &\mathbb{P}\left(\sup_{\Phi \in A_{nt}^{\kappa_1}} \sum_{j=1}^{n} \int_{(j-1)t+1}^{jt} \xi(\Phi((j-1)t+1), s)\, ds \geq a_1 nt \right) \\ &\quad \leq \sum_{\Phi \in A_{nt}^{\kappa_1,\sim}} \mathbb{P}\left(\sum_{j=1}^{n} \int_{(j-1)t+1}^{jt} \xi(\Phi((j-1)t+1), s)\, ds \geq a_1 nt \right) \\ &\quad \leq \left[(nt/\log(1/\kappa)^b) 2^n (2d)^{nt/\log(1/\kappa)^b} + 1 \right] e^{-\delta_2 nt}, \end{aligned} \tag{2.5.8}$$

which is summable on n. Hence, by the Borel-Cantelli lemma, there exists a set $B_{\kappa_1,t}$ with $\mathbb{P}(B_{\kappa_1,t}) = 1$ for which there exists an $n_0 = n_0(\xi, \kappa_1, t)$ such that

$$\sup_{\Phi \in A_{nt}^{\kappa_1}} \sum_{j=1}^{n} \int_{(j-1)t+1}^{jt} \xi(\Phi((j-1)t+1), s)\, ds \leq a_1 nt \qquad \forall\, n \geq n_0. \tag{2.5.9}$$

Since $\kappa \mapsto A^{\kappa}_{nt}$ is non-decreasing, (2.5.9) is true for all $0 < \kappa \leq \kappa_1$. Define

$$B_1 = \bigcap_{t \geq t_1, t \in \mathbb{Q}} B_{\kappa_1, t}, \tag{2.5.10}$$

for which still $\mathbb{P}(B_1) = 1$. Similarly, we can construct sets B_m, $m \in \mathbb{N} \setminus \{1\}$, with $\mathbb{P}(B_m) = 1$ such that on B_m there exist κ_m and $t_m = t_m(\kappa_m)$ such that for all $0 < \kappa \leq \kappa_m$ and $t \geq t_m$ with $t \in \mathbb{Q}$ there exists an $n_0 = n_0(\xi, \kappa_m, t)$ such that

$$\sup_{\Phi \in A^{\kappa}_{nt}} \sum_{j=1}^{n} \int_{(j-1)t+1}^{jt} \xi(\Phi((j-1)t+1), s)\, ds \leq a_m nt \qquad \forall\, n \geq n_0. \tag{2.5.11}$$

Hence $B = \cap_{m \in \mathbb{N}} B_m$ is the desired set. Note that we can control the value of t_m by choosing κ_m small enough. Indeed, with the right choice of κ_m, it follows that $t_{m-1}(\kappa_{m-1}) = t_m(\kappa_m)$ for all $m \in \mathbb{N}$. ∎

2.5.2.3 Proof of Lemma 2.5.3

Proof. Fix A, C as in Section 2.3, $R_0 \in \mathbb{N}$, a path $\Phi \in B_t$ and disjoint subintervals $\mathcal{I}_1, \mathcal{I}_2, \cdots, \mathcal{I}_k$, $k \in \mathbb{N}$, of $[0,t]$ with equal length. Note that

$$\begin{aligned}\sum_{j=1}^{k} \int_{\mathcal{I}_j} \xi(\Phi(s), s)\, ds \leq \sum_{j=1}^{k} \int_{\mathcal{I}_j} &\xi(\Phi(s), s) \mathbb{1}\{\xi(\Phi(s), s) \leq C A^{R_0 d}\}\, ds \\ + \int_0^t &\xi(\Phi(s), s) \mathbb{1}\{\xi(\Phi(s), s) > C A^{R_0 d}\}\, ds.\end{aligned} \tag{2.5.12}$$

By (2.3.53) and Lemma 2.3.4, for $t \in \mathbb{Q}$ sufficiently large, the second term on the right hand side in (2.5.12) may be bounded from above by

$$(t + N(\Phi, t)) C' A^d \sum_{R=R_0}^{\infty} A^{-Rd}. \tag{2.5.13}$$

Inserting (2.5.13) into (2.5.12) yields the claim. ∎

2.5.3 Proof of the continuity

In this section we prove Theorem 2.1.16 with the help of Lemmas 2.5.1–2.5.3. The proof comes in three steps, organized as Sections 2.5.3.1–2.5.3.3.

2.5.3.1 Estimation of the Feynman-Kac representation on A^{κ}_{nT}

Consider the case $u_0(x) = \delta_0(x)$, $x \in \mathbb{Z}^d$. Recall (2.1.13) and (2.1.30), and estimate

$$\lambda_0^{\delta_0}(\kappa) \leq \lim_{n \to \infty} \frac{1}{nT} \log E_0 \left(e^{\overline{\mathcal{I}}\kappa(0, nT, 0)} \right) < \infty, \qquad T > 0, \tag{2.5.14}$$

where we reverse time, use that X^κ is a reversible dynamics, and remove the constraint $X^\kappa(nT) = 0$. Recalling (2.5.1) and (2.5.3), we have

$$\begin{aligned} E_0\left(e^{\overline{\mathcal{I}}_\kappa(0,nT,0)}1\!\!1_{A^\kappa_{nT}}\right) &= \sum_{\Phi\in A^{\kappa,\sim}_{nT}} E_0\left(e^{\overline{\mathcal{I}}_\kappa(0,nT,0)}1\!\!1_{A^\kappa_{nT}}1\!\!1_{\left\{X|_{[0,nT]}\sim\Phi\right\}}\right)\\ &= \sum_{\Phi\in A^{\kappa,\sim}_{nT}} E_0\left(\exp\left\{\sum_{j=1}^{n}\overline{\mathcal{I}}^\kappa((j-1)T+1,jT,0)\right\}\right.\\ &\qquad\left.\times\exp\left\{\sum_{j=0}^{n-1}\overline{\mathcal{I}}^\kappa(jT,jT+1,0)\right\}1\!\!1_{A^\kappa_{nT}}1\!\!1_{\left\{X|_{[0,nT]}\sim\Phi\right\}}\right). \end{aligned} \tag{2.5.15}$$

By the Hölder inequality with $p,q>1$ and $1/p+1/q=1$, we have

$$\begin{aligned} &E_0\left(e^{\overline{\mathcal{I}}_\kappa(0,nT,0)}1\!\!1_{\{A^\kappa_{nT}\}}\right)\\ &\quad\le\sum_{\Phi\in A^{\kappa,\sim}_{nT}} E_0\left(\exp\left\{p\sum_{j=1}^{n}\overline{\mathcal{I}}^\kappa((j-1)T+1,jT,0)\,ds\right\}1\!\!1_{A^\kappa_{nT}}1\!\!1_{\left\{X|_{[0,nT]}\sim\Phi\right\}}\right)^{1/p}\\ &\qquad\times E_0\left(\exp\left\{q\sum_{j=0}^{n-1}\overline{\mathcal{I}}^\kappa(jT,jT+1,0)\right\}1\!\!1_{A^\kappa_{nT}}\right)^{1/q}. \end{aligned} \tag{2.5.16}$$

Next, fix $(a_m)_{m\in\mathbb{N}}$, $(\kappa_m)_{m\in\mathbb{N}}$, t_m as in Lemma 2.5.1, choose $T>0$ such that

$$t_m \le T = T(\kappa_m) = K\lfloor\log(1/\kappa_m)\rfloor, \qquad m\gg 1, \tag{2.5.17}$$

where K is a constant to be chosen later. For all $0<\kappa\le\kappa_m$ and $n\ge n_m(\kappa_m,T(\kappa))$, by Lemma 2.5.1 we have

$$\begin{aligned} &\sum_{\Phi\in A^{\kappa,\sim}_{nT}} E_0\left(\exp\left\{p\sum_{j=1}^{n}\overline{\mathcal{I}}^\kappa((j-1)T+1,jT,0)\right\}1\!\!1_{A^\kappa_{nT}}1\!\!1_{\left\{X|_{[0,nT]}\sim\Phi\right\}}\right)^{1/p}\\ &\quad\le\sum_{\Phi\in A^{\kappa,\sim}_{nT}} e^{a_m nT}P_0\left(A^\kappa_{nT},X|_{[0,nT]}\sim\Phi\right)^{1/p}\le e^{a_m nT}|A^{\kappa,\sim}_{nT}|, \end{aligned} \tag{2.5.18}$$

while by Lemma 2.5.3 we have

$$\begin{aligned} &E_0\left(\exp\left\{q\sum_{j=0}^{n-1}\overline{\mathcal{I}}^\kappa(jT,jT+1,0)\right\}1\!\!1_{A^\kappa_{nT}}\right)^{1/q}\\ &\quad\le\exp\left\{nCA^{R_0 d}+nT\left(1+\frac{1}{\log(1/\kappa)^b}\right)C'A^d\sum_{R=R_0}^{\infty}A^{-Rd}\right\}. \end{aligned} \tag{2.5.19}$$

From Lemma 2.5.2 we know that $\frac{1}{nT}\log|A^{\kappa,\sim}_{nT}|$ tends to zero if we let first $n\to\infty$ and then $\kappa\downarrow 0$. Therefore, combining (2.5.16–2.5.19) and using that $\lim_{\kappa\downarrow 0} T = \lim_{\kappa\downarrow 0} T(\kappa) = 0$, we get

$$\limsup_{\kappa\downarrow 0}\limsup_{n\to\infty}\frac{1}{nT}\log E_0\left(e^{\overline{\mathcal{I}}\kappa(0,nT,0)}1\!\!1_{A^\kappa_{nT}}\right) \le \max\left\{a_m, C'A^d\sum_{R=R_0}^{\infty} A^{-Rd}\right\}. \quad (2.5.20)$$

2.5.3.2 Estimation of the Feynman-Kac representation on $[A^\kappa_{nT}]^{\mathrm{c}}$

The proof comes in three steps.

1. We start by estimating the corresponding Feynman-Kac term on $[A^\kappa_{nT}]^{\mathrm{c}}$. Split

$$[A^\kappa_{nT}]^{\mathrm{c}} = B^\kappa_{nT}\cup C^\kappa_{nT}, \quad (2.5.21)$$

with

$$B^\kappa_{nT} = \left\{N(X^\kappa, nT) > \frac{1}{\log(1/\kappa)^b}\,nT\right\} \quad (2.5.22)$$

and

$$C^\kappa_{nT} = \Big\{\exists\, 1\le j\le n\colon \text{ for all } x\in\mathbb{Z}^d \text{ with } \|x\|\le \frac{1}{\log(1/\kappa)^b}\,nT \\ \text{there exists an } s_j\in[(j-1)T+1, jT) \text{ such that } X^\kappa(s_j)\ne x\Big\}, \quad (2.5.23)$$

i.e., C^κ_{nT} is the set of paths such that there is a time-interval $[(j-1)T+1, jT)$, $1\le j\le n$, that are not constant equal to any $x\in\mathbb{Z}^d$ with $\|x\|\le\frac{1}{\log(1/\kappa)^b}\,nT$ during this time interval. Then

$$P_0\big(B^\kappa_{nT}\big) \le \exp\big\{\big[-J_\kappa(1/\log(1/\kappa)^b) + o_n(1)\big]\,nT\big\}, \quad (2.5.24)$$

where

$$J_\kappa(x) = x\log(x/2d\kappa) - x + 2d\kappa \quad (2.5.25)$$

is the large deviation rate function of the rate-$2d\kappa$ Poisson process. Thus, by the Hölder inequality with $p,q>1$ and $1/p+1/q=1$, we have

$$E_0\left(e^{\overline{\mathcal{I}}\kappa(0,nT,0)}1\!\!1_{B^\kappa_{nT}}\right) \\ \le E_0\left(\exp\left\{p\overline{\mathcal{I}}\kappa(0,nT,0)\right\}\right)^{1/p}\exp\big\{1/q\big[-J_\kappa(1/\log(1/\kappa)^b)+o_n(1)\big]\,nT\big\}. \quad (2.5.26)$$

Recalling Theorem 2.1.14 and using that $\lim_{\kappa\downarrow 0} J_\kappa(1/\log(1/\kappa)^b) = \infty$, we get

$$\lim_{\kappa\downarrow 0}\lim_{n\to\infty}\frac{1}{nT}\log E_0\left(e^{\overline{\mathcal{I}}\kappa(0,nT,0)}1\!\!1_{B^\kappa_{nT}}\right) = -\infty. \quad (2.5.27)$$

2. Note that

$$C^\kappa_{nT}\subseteq B^\kappa_{nT}\cup D^\kappa_{nT} \quad\text{with}\quad D^\kappa_{nT} = \left(C^\kappa_{nT}\cap[B^\kappa_{nT}]^{\mathrm{c}}\right). \quad (2.5.28)$$

Since we have just proved that the Feynman-Kac representation on B^κ_{nT} is not contributing, we only have to look at the contribution coming from D^κ_{nT}, namely,

$$E_0\left(e^{\overline{\mathcal{I}}\kappa(0,nT,0)}1\!\!1_{D^\kappa_{nT}}\right). \tag{2.5.29}$$

On the event D^κ_{nT}, the random walk X^κ stays inside the box of radius $nT/\log(1/\kappa)^b$, and jumps during the time intervals $[(j-1)T+1, jT)$, $1 \le j \le n$, defined in (2.5.23). By the Hölder inequality with $p, q > 1$ and $1/p + 1/q = 1$, we have

$$E_0\left(e^{\overline{\mathcal{I}}\kappa(0,nT,0)}1\!\!1_{D^\kappa_{nT}}\right) \le I \times II, \tag{2.5.30}$$

where

$$\begin{aligned} I &= \left[E_0\left(\exp\left\{p\sum_{j=1}^{n}\overline{\mathcal{I}}\kappa((j-1)T+1, jT, 0)\right\}1\!\!1_{D^\kappa_{nT}}\right)\right]^{1/p}, \\ II &= \left[E_0\left(\exp\left\{q\sum_{j=0}^{n-1}\overline{\mathcal{I}}^\kappa(jT, jT+1, 0)\right\}1\!\!1_{D^\kappa_{nT}}\right)\right]^{1/q}. \end{aligned} \tag{2.5.31}$$

Define

$$J = \Big\{1 \le j \le n\colon\ N(X^\kappa, [(j-1)T+1, jT)) \ge 1\Big\}, \tag{2.5.32}$$

where $N(X^\kappa, \mathcal{I})$ is the number of jumps of the random walk X^κ during the time interval $\mathcal{I}$. Using that X^κ is not jumping in the time intervals $[(j-1)T+1, jT))$, $j \in J^c$, we may write

$$\begin{aligned} &\sum_{j=1}^{n}\overline{\mathcal{I}}^\kappa((j-1)T+1, jT, 0) \\ &\quad = \sum_{j\in J}\overline{\mathcal{I}}^\kappa((j-1)T+1, jT, 0) + \sum_{j\in J^c}\int_{(j-1)T+1}^{jT}\xi(X^\kappa((j-1)T+1), s)\,ds. \end{aligned} \tag{2.5.33}$$

3. To estimate the second term in the right-hand side of (2.5.33), pick any $\Phi^{X^\kappa} \in A^\kappa_{nT}$ such that $\Phi^{X^\kappa} = X^\kappa$ on $\cup_{j\in J^c}[(j-1)T+1, jT)$ and apply Lemma 2.5.1, to get

$$\begin{aligned} &\sum_{j\in J^c}\int_{(j-1)T+1}^{jT}\xi(X^\kappa((j-1)T+1), s)\,ds \\ &\qquad \le a_m nT - \sum_{j\in J}\int_{(j-1)T+1}^{jT}\xi(\Phi^{X^\kappa}(s), s)\,ds \quad \xi\text{-a.s.} \end{aligned} \tag{2.5.34}$$

Note that ξ is Gärtner-negative-hyper-mixing, so that by Lemma 2.5.3 we may estimate the second term on the right hand side of (2.5.34) by

$$|J|(T-1)CA^{R_0 d} + nT\left(1 + \frac{1}{\log(1/\kappa)^b}\right)C'A^d\sum_{R=R_0}^{\infty}A^{-Rd}. \tag{2.5.35}$$

To estimate the first term in the right-hand side of (2.5.33), apply Lemma 2.5.3, to get

$$\sum_{k=1}^{n}\left[E_0\left(\exp\left\{p\sum_{j\in J}\overline{\mathcal{I}}^{\kappa}((j-1)T+1,jT,0)\right\}1\!\!1_{D^{\kappa}_{nT}}1\!\!1_{\{|J|=k\}}\right)e^{pk(T-1)CA^{R_0d}}\right]$$
$$\leq \exp\left\{pnT\left(1+\frac{1}{\log(1/\kappa)^b}\right)C'A^d\sum_{R=R_0}^{\infty}A^{-Rd}\right\}$$
$$\times\sum_{k=1}^{n}\exp\left\{2pk(T-1)CA^{R_0d}\right\}P_0\left(D^{\kappa}_{nT},|J|=k\right). \tag{2.5.36}$$

The distribution of $|J|$ is BIN$(n,1-e^{-2d\kappa(T-1)})$. Hence the sum on the right hand side of (2.5.36) is bounded from above by

$$\begin{aligned}&\sum_{k=1}^{n}\binom{n}{k}\left(1-e^{-2d\kappa(T-1)}\right)^k e^{-2d\kappa(T-1)(n-k)}e^{2pk(T-1)CA^{R_0d}}\\&\leq\left((1-e^{-2d\kappa(T-1)})e^{2p(T-1)CA^{R_0d}}+e^{-2d\kappa(T-1)}\right)^n.\end{aligned} \tag{2.5.37}$$

Combining (2.5.34–(2.5.37)), we arrive at

$$\begin{aligned}I \leq & e^{a_m nT}\exp\left\{2nT\left(1+\frac{1}{\log(1/\kappa)^b}\right)C'A^d\sum_{R=R_0}A^{-Rd}\right\}\\&\times\left((1-e^{-2d\kappa(T-1)})e^{2p(T-1)CA^{R_0d}}+e^{-2d\kappa(T-1)}\right)^{n/p}.\end{aligned} \tag{2.5.38}$$

On the other hand, by Lemma 2.5.3, we have

$$II \leq e^{nCA^{R_0d}}\exp\left\{nT\left(1+\frac{1}{\log(1/\kappa)^b}\right)C'A^d\sum_{R=R_0}A^{-Rd}\right\}. \tag{2.5.39}$$

Therefore, combining (2.5.38–2.5.39), we finally obtain

$$\begin{aligned}&E_0\left(e^{\overline{\mathcal{I}}^{\kappa}(0,nT,0)}1\!\!1_{D^{\kappa}_{nT}}\right)\\&\quad\leq e^{a_m nT}e^{nCA^{R_0d}}\exp\left\{3nT\left(1+\frac{1}{\log(1/\kappa)^b}\right)C'A^d\sum_{R=R_0}A^{-Rd}\right\}\\&\quad\times\left((1-e^{-2d\kappa(T-1)})e^{2p(T-1)CA^{R_0d}}+e^{-2d\kappa(T-1)}\right)^{n/p}.\end{aligned} \tag{2.5.40}$$

2.5.3.3 Final estimation

By (2.5.40), we have

$$\frac{1}{nT}\log E_0\left(e^{\overline{\mathcal{I}}^\kappa(0,nT,0)}\mathbb{1}_{D^\kappa_{nT}}\right) \\ \leq a_m + \frac{2d\kappa(T-1)}{pT}\,e^{2p(T-1)CA^{R_0d}} + \frac{CA^{R_0d}}{T} + 3\left(1+\frac{1}{\log(1/\kappa)^b}\right)C'A^d\sum_{R=R_0}A^{-Rd}. \tag{2.5.41}$$

Here, the term $\frac{2d\kappa(T-1)}{pT}\,e^{2p(T-1)CA^{R_0d}}$ results from the estimates

$$1-e^{-2d\kappa(T-1)}\leq 2d\kappa(T-1),\quad e^{-2d\kappa(T-1)}\leq 1\quad\text{and}\quad\log(1+x)\leq x\quad\text{for all } x. \tag{2.5.42}$$

Abbreviate $M_2 = 2pCA^{R_0d}$ and recall (2.5.17). Then the right-hand side of (2.5.41) is asymptotically equivalent to

$$a_m + \frac{2d\kappa}{p}(1/\kappa)^{M_2K} + 3\left(1+\frac{1}{\log(1/\kappa)^b}\right)C'A^d\sum_{R=R_0}A^{-Rd},\qquad \kappa\downarrow 0. \tag{2.5.43}$$

Choosing $K\leq 1/(2M_2)$, $K\in\mathbb{Q}$, and recalling (2.5.20) we finally arrive at

$$\lambda_0^{\delta_0}(\kappa)\leq a_m + \frac{2d\sqrt{\kappa}}{p} + 3\left(1+\frac{1}{\log(1/\kappa)^b}\right)C'A^d\sum_{R=R_0}A^{-Rd}, \tag{2.5.44}$$

which tends to zero as $\kappa\downarrow 0$, $R_0\to\infty$ and $m\to\infty$.

2.6 No Lipschitz continuity at $\kappa = 0$

In this section we prove Theorem 2.1.17. The proof is very close to that of Gärtner, den Hollander and Maillard [GdHM12], Theorem 1.2(iii), where it is assumed that ξ is bounded from below. For completeness we will repeat the main steps in that proof.

Proof. Fix $C_1 > 0$, let e be a nearest neighbors of 0, which we fix from now on, write (see (2.3.1)),

$$\lambda_0^{\delta_0}(\kappa) \\ = \lim_{n\to\infty}\frac{1}{nT+1}\log E_0\left(e^{\overline{\mathcal{I}}^\kappa(0,nT+1,0)}\delta_0(X^\kappa(nT+1))\mathbb{1}\left\{X^\kappa([0,nT+1])\subseteq[C_1]_{nT+1}\right\}\right) \tag{2.6.1}$$

and abbreviate

$$I_j^\xi(x) = \int_{(j-1)T+1}^{jT}\xi(x,s)\,ds,\qquad Z_j^\xi = \operatorname*{argmax}_{x\in\{0,e\}} I_j^\xi(x),\qquad 1\leq j\leq n. \tag{2.6.2}$$

We intend to find a lower bound of $\lambda_0^{\delta_0}(\kappa)$ in terms of $I_j^{\xi}(x)$, $\|x\| \le 1$. Consider the event

$$A^{\xi} = \left[\bigcap_{j=1}^{n}\{X^{\kappa}(t) = Z_j^{\xi}\ \forall\, t \in [(j-1)T+1, jT)\}\right] \cap \{X^{\kappa}(nT+1) = 0\}. \qquad (2.6.3)$$

We have

$$\begin{aligned} &E_0\left(\exp\left\{\overline{\mathcal{I}}^{\kappa}(0, nT+1, 0)\right\}\delta_0(X^{\kappa}(nT+1))\right) \\ &\ge E_0\left(\exp\left\{\sum_{j=1}^{n+1}\overline{\mathcal{I}}^{\kappa}((j-1)T, (j-1)T+1, 0) + \sum_{j=1}^{n}\overline{\mathcal{I}}^{\kappa}((j-1)T+1, jT, 0)\right\}1\!\!1_{A^{\xi}}\right). \end{aligned} \qquad (2.6.4)$$

Using the reverse Hölder inequality with $q < 0 < p < 1$ and $1/q + 1/p = 1$, we have

$$\begin{aligned} &E_0\left(\exp\left\{\sum_{j=1}^{n+1}\overline{\mathcal{I}}^{\kappa}((j-1)T, (j-1)T+1, 0) + \sum_{j=1}^{n}\overline{\mathcal{I}}^{\kappa}((j-1)T+1, jT, 0)\right\}1\!\!1_{A^{\xi}}\right) \\ &\ge \left[E_0\left(\exp\left\{q\sum_{j=1}^{n+1}\overline{\mathcal{I}}^{\kappa}((j-1)T, (j-1)T+1, 0)\right\}1\!\!1\left\{X^{\kappa}([0, nT+1]) \subseteq [C_1]_{nT+1}\right\}\right)\right]^{1/q} \\ &\times \left[E_0\left(\exp\left\{p\sum_{j=1}^{n}\overline{\mathcal{I}}^{\kappa}((j-1)T+1, jT, 0)\right\}1\!\!1_{A^{\xi}}1\!\!1\left\{X^{\kappa}([0, nT+1]) \subseteq [C_1]_{nT+1}\right\}\right)\right]^{1/p}. \end{aligned} \qquad (2.6.5)$$

To estimate the first term on the right hand side of (2.6.5), fix $R_0 \in \mathbb{N}$ and choose $A, C > 0$ such that all results of Section 2.3 are satisfied for $q\xi$. Moreover, note that by a refinement of the arguments given in the proof of Lemma 2.3.4 and (2.3.53), one has ξ-a.s. for $nT+1 \in \mathbb{N}$ sufficiently large

$$\begin{aligned} &\sum_{j=1}^{n+1}\int_{(j-1)T}^{(j-1)T+1} q\xi(X^{\kappa}(s), s)1\!\!1\left\{q\xi(X^{\kappa}(s), s) > -qCA^{R_0 d}\right\}ds \\ &\le -\frac{nT+1+N(X^{\kappa}, nT+1)}{T}qC'A^d\sum_{R=R_0}^{\infty}A^{-Rd}. \end{aligned} \qquad (2.6.6)$$

Consequently,

$$\begin{aligned} &E_0\left(\exp\left\{q\sum_{j=1}^{n+1}\overline{\mathcal{I}}^{\kappa}((j-1)T, (j-1)T+1, 0)\right\}1\!\!1\left\{X^{\kappa}([0, nT+1]) \subseteq [C_1]_{nT+1}\right\}\right) \\ &\le e^{-q(n+1)CA^{R_0 d}}E_0\left(\exp\left\{-\frac{nT+1+N(X^{\kappa}, nT+1)}{T}qC'A^d\sum_{R=R_0}^{\infty}A^{-Rd}\right\}\right), \end{aligned} \qquad (2.6.7)$$

which equals

$$e^{-q(n+1)CA^{R_0 d}} \exp\left\{-\frac{nT+1}{T} qC'A^d \sum_{R=R_0}^{\infty} A^{-Rd}\right\} \times \exp\left\{2d\kappa(nT+1)\Big(e^{-\frac{q}{T}C'A^d \sum_{R=R_0}^{\infty} A^{-Rd}} - 1\Big)\right\}. \tag{2.6.8}$$

As in the proof of [GdHM12], Theorem 1.2(iii), we have

$$\left[E_0\left(\exp\left\{p\sum_{j=1}^{n} \overline{\mathcal{I}}^{\kappa}((j-1)T+1, jT, 0)\right\} 1\!\!1_{A^{\xi}}\right)\right]^{1/p} \geq \left[\exp\left\{[1+o_n(1)]\, np\, \mathbb{E}\left(\max\{I_1^{\xi}(0), I_1^{\xi}(e)\}\right)\right\} [p_1^{\kappa}(e)]^{n+1}\, e^{-2d\kappa n(T-1)}\right]^{1/p}. \tag{2.6.9}$$

Combining (2.6.4–2.6.9), we arrive at

$$\begin{aligned}
&\frac{1}{nT+1}\log E_0\left(e^{\overline{\mathcal{I}}^{\kappa}(0,nT+1,0)}\, \delta_0(X^{\kappa}(nT+1))\right)\\
&\geq -\frac{1}{(nT+1)} CA^{R_0 d}(n+1) - \frac{1}{T}C'A^d \sum_{R=R_0}^{\infty} A^{-Rd} + \frac{2d\kappa}{q}\Big(e^{-\frac{q}{T}C'A^d \sum_{R=R_0}^{\infty} A^{-Rd}} - 1\Big)\\
&\qquad + \frac{1}{p(nT+1)}\left[[1+o_n(1)]\, pn\, \mathbb{E}\left(\max\left\{I_1^{\xi}(0), I_1^{\xi}(e)\right\}\right)\right] - \frac{2d\kappa n(T-1)}{p(nT+1)}\\
&\qquad + \frac{n+1}{p(nT+1)} \log p_1^{\kappa}(e).
\end{aligned} \tag{2.6.10}$$

Using that $p_1^{\kappa}(e) = \kappa[1+o_{\kappa}(1)]$ as $\kappa \downarrow 0$, the fact that $I_1^{\xi}(0)$ and $I_1^{\xi}(e)$ have zero expectation and letting $n \to \infty$, we get that

$$\begin{aligned}
&\lambda_0^{\delta_0}(\kappa)\\
&\geq -\frac{1}{(nT+1)} CA^{R_0 d}(n+1) - \frac{1}{T}C'A^d \sum_{R=R_0}^{\infty} A^{-Rd} + \frac{2d\kappa}{q}\Big(e^{-\frac{q}{T}C'A^d \sum_{R=R_0}^{\infty} A^{-Rd}} - 1\Big)\\
&\qquad - \frac{2d\kappa(T-1)}{pT} + \frac{1}{T}\,[1+o_{\kappa}(1)]\, \left(\tfrac{1}{2}\, \mathbb{E}(|I_1^{\xi}(0) - I_1^{\xi}(e)|) - \frac{1}{p}\, \log(1/\kappa)\right).
\end{aligned} \tag{2.6.11}$$

At this point we can copy the rest of the proof of [GdHM12], Theorem 1.2(iii), with a few minor adaptations of constants. ∎

2.7 Examples

In Section 2.7.1 we prove Corollary 2.1.19, in Section 2.7.2 we prove Corollary 2.1.20.

2.7.1 Examples of potentials for the finiteness and initial condition result

In Section 2.7.1.1 we settle Part (1), in Section 2.7.1.2 we settle Part (2).

2.7.1.1 Proof of Corollary 2.1.19 (1)

1.1 The first condition in Definition 2.1.7 is satisfied by our assumption on ξ.

1.2 We show that ξ is type-I Gärtner-mixing. Fix $A > 1$, pick $b = c = 0$ and $a_1 = a_2 = 2$ (see (2.3.26)), and define

$$B_R^{A,\mathrm{sub}}(x,k) = \left(\prod_{j=1}^{d} [(x(j)-1)A^R, (x(j)+1)A^R) \cap \mathbb{Z}^d\right) \times \{kA^R\}. \tag{2.7.1}$$

We start by estimating the probability of the event

$$\mathrm{B}(x,k) \stackrel{\mathrm{def}}{=} \big\{B_{R+1}^A(x,k) \text{ is good, but contains a bad } R\text{-block}\big\}. \tag{2.7.2}$$

Note that each $R+1$-block contains at most $2^d A^{(1+d)}$ R-blocks. For each such R-block $B_R^A(y,l)$ there are no more than A^{Rd} blocks

$$Q_R^A(z_1) = \prod_{j=1}^{d} [(z_1(j), z_1(j) + A^R) \tag{2.7.3}$$

contained in it. For any such block we may estimate, for $C_1 > 0$,

$$\begin{aligned}
&\mathbb{P}\left(\exists s \in [lA^R, (l+1)A^R)\colon \sum_{z_2 \in Q_R^A(z_1)} \xi(z_2, s) > C_1 A^{Rd}\right) \\
&\le \sum_{k=0}^{\lfloor A^R \rfloor} \mathbb{P}\left(\sum_{z_2 \in Q_R^A(z_1)} \sup_{s\in[k,k+1)} X_s(z_2) > C_1 A^{Rd}\right) \\
&\le (A^R+1)\exp\big\{-C_1 A^{Rd}\big\}\, \mathbb{E}\left(\exp\left\{\sup_{s\in[0,1)} X_s(0)\right\}\right)^{A^{Rd}},
\end{aligned} \tag{2.7.4}$$

where we use the time stationarity in the first inequality, and the time stationarity and the space independence in the second inequality. Thus, for C_1 sufficiently large, there is a $C_1' > 0$ such that

$$\mathbb{P}\big(\mathrm{B}(x,k)\big) \le e^{-C_1' A^{Rd}}. \tag{2.7.5}$$

Moreover, for space-time blocks that are disjoint in space, the corresponding events in (2.7.2) are independent. Hence we may assume that $B_{R+1}^A(x_1,k_1), \ldots, B_{R+1}^A(x_n,k_n)$ are equal in space but disjoint in time. Since

$$\mathbb{P}\left(\bigcap_{i=1}^{n} \mathrm{B}(x_i,k_i)\right) = \mathbb{P}\left(\mathrm{B}(x_n,k_n) \,\Big|\, \bigcap_{i=1}^{n-1} \mathrm{B}(x_i,k_i)\right)\mathbb{P}\left(\bigcap_{i=1}^{n-1} \mathrm{B}(x_i,k_i)\right), \tag{2.7.6}$$

it is enough to show that there is a constant $K<\infty$, independent of R, such that the conditional probability in (2.7.6) may be estimated from above by $K\mathbb{P}(\mathrm{B}(x_n,k_n))$. To do this, we apply the Markov property to obtain

$$\begin{aligned}\mathbb{P}\left(\mathrm{B}(x_n,k_n) \,\Big|\, \bigcap_{i=1}^{n-1} \mathrm{B}(x_i,k_i)\right) &\le \mathbb{P}\left(\mathrm{B}(x_n,k_n) \,\Big|\, \bigcap_{i=1}^{n-1} \mathrm{B}(x_i,k_i), B_{R+1}^{A,\mathrm{sub}}(x_n,k_n) \text{ is good}\right)\\ &= \mathbb{P}\left(\mathrm{B}(x_n,k_n) \,\Big|\, B_{R+1}^{A,\mathrm{sub}}(x_n,k_n) \text{ is good}\right).\end{aligned} \tag{2.7.7}$$

Thus, the left-hand side of (2.7.7) is at most

$$\frac{\mathbb{P}(\mathrm{B}(x_n,k_n))}{\mathbb{P}(B_{R+1}^{A,\mathrm{sub}}(x_n,k_n) \text{ is good})}. \tag{2.7.8}$$

Since $\lim_{R\to\infty}\mathbb{P}(B_{R+1}^{A,\mathrm{sub}}(x_n,k_n) \text{ is good}) = 1$, we obtain that ξ is type-I Gärtner-mixing.

1.3 Condition (a3) in Definition 2.1.7 follows from the calculations in (2.7.4).

2. The same strategy as above works to show that ξ is type-II Gärtner-mixing. If X has exponential moments of all negative orders, then the same calculations as in the first part show that ξ is Gärtner-negative-hyper-mixing. All requirements of Theorems 2.1.14–2.1.15 are thus met.

2.7.1.2 Proof of Corollary 2.1.19(2)

Let ξ be the zero-range process as described in Corollary 2.1.19(2). We will use that each particle, independently of all the other particles, carries an exponential clock of parameter one. If there are k particles at a site x and one of these clocks rings, then the corresponding particle jumps to y with probability $\frac{g(k)}{2dk}$ and it stays at y with probability $1-\frac{g(k)}{k}$.

1.1 By Andjel [A82], Theorem 1.9, the product measures in (2.1.32) are extremal for ξ. Thus, $\mathbb{E}\left[e^{q\xi(0,0)}\right]<\infty$ for all $q\ge 0$. Consequently, to show that $\mathbb{E}\left[e^{q\sup_{s\in[0,1]}\xi(0,s)}\right]<\infty$, it suffices to prove that there is a constant $K>0$ such that, for all $k\in\mathbb{N}$ sufficiently large,

$$\mathbb{P}\left(\sup_{s\in[0,1]}\xi(0,s)\ge k\right) \le K\mathbb{P}\left(\xi(0,1)\ge \frac{e^{-1}}{2}k\right). \tag{2.7.9}$$

Write $\mathrm{NR}(\frac{e^{-1}}{2}k,\tau)$ for the event that there are at least $\frac{e^{-1}}{2}k$ exponential clocks of particles located at zero that do not ring in the time interval $[\tau,\tau+1)$. Then we may estimate

$$\begin{aligned}\mathbb{P}\bigg(\xi(0,1)\geq\frac{e^{-1}}{2}k\;\bigg|\;\exists\,\tau\in[0,1]\colon\;\xi(0,\tau)\geq k\bigg)\\ \geq\mathbb{P}\bigg(\mathrm{NR}\Big(\frac{e^{-1}}{2}k,\tau\Big)\;\bigg|\;\exists\,\tau\in[0,1]\colon\;\xi(0,\tau)\geq k\bigg).\end{aligned}\tag{2.7.10}$$

Since the probability that a clock does not ring within a time interval of length one is equal to e^{-1}, and all clocks are independent, we may estimate the right-hand side of (2.7.10) from below by

$$\mathbb{P}\Big(T\geq\frac{e^{-1}}{2}k\Big),\qquad T=\mathrm{BIN}(k,e^{-1}).\tag{2.7.11}$$

Finally, note that the probability in (2.7.11) is bounded away from zero. Thus, inserting (2.7.10) and (2.7.11) into (2.7.9), we get the claim.

1.2. We show that ξ is type-I Gärtner-mixing. Fix $A>3$, choose $b=3$, $c=1$, $a_1=13$, $a_2=2$ (see (2.3.26), and introduce additional space-time blocks

$$\begin{aligned}\widetilde{B}_R^{A,+}(x,k)&=\left(\prod_{j=1}^d[(x(j)-2)A^R,(x(j)+2)A^R)\cap\mathbb{Z}^d\right)\times[kA^R-A^{R-1},(k+1)A^R)\\ \widetilde{B}_R^{A,\mathrm{sub}}(x,k)&=\left(\prod_{j=1}^d[(x(j)-4)A^R,(x(j)+4)A^R)\cap\mathbb{Z}^d\right)\times\{(k-1)A^R\}.\end{aligned}\tag{2.7.12}$$

Given $S\subseteq\mathbb{Z}^d$ and $S'\subseteq\mathbb{N}_0$, write ∂S to denote the inner boundary of S and $\Pi_1(S\times S')$ to denote the projection onto the spatial coordinates. Furthermore, e_j, $j\in\{1,2,\cdots,d\}$ denotes the j-th unit vector (and we agree that $\frac{0}{0}=0$).

We call a space-time block $B_R^A(x,k)$ *contaminated* if there is a particle at some space-time point $(y,s)\in\widetilde{B}_R^{A,+}(x,k)$ that has been outside $\Pi_1(\prod_{j=1}^d[(x(j)-4)A^R,(x(j)+4)A^R))$ at a time s' such that $(k-1)A^R\leq s'<s$. The reason for introducing this notion is that events depending on non-contaminated blocks that are equal in time but disjoint in space are all independent.

Contaminated blocks. For $L>0$, define

$$\chi(x,k)=1\!\!1\Big\{B_{R+1}^A(x,k)\text{ is good, but contaminated and intersects }[-L,L]^{d+1}\Big\}\tag{2.7.13}$$

and fix $(x^*,k^*)\in\mathbb{Z}^d\times\mathbb{N}$.

Claim 2.7.1. *There is a $C'>0$ independent of L such that $(\chi(x,k))_{(x,k)\equiv(x^*,k^*)}$ is stochastically dominated by independent Bernoulli random variables $(Z(x,k))_{(x,k)=(x^*,k^*)}$ with success probability $e^{-C'A^{R+1}}$.*

Proof. We use a discretization scheme. More precisely, we construct a discrete-time version of the zero-range process where particles are allowed to jump at times k/n, $k \in \mathbb{N}_0$ only. Here, n is an integer that will later tend to infinity, and we will denote by $\xi^n(x,s)$ the number of particles at site x at time s. To construct this process, we take a family $X^n(x,s,q_1,q_2)$ of independent random variables with index set $\mathbb{Z}^d \times \frac{1}{n}\mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{N}_0$ whose distribution is defined via

$$\begin{aligned} &\mathbb{P}\Big(X^n(\cdot,\cdot,\cdot,q_2) = 0\Big) = 1 - \frac{g(q_2)}{nq_2}, \\ &\mathbb{P}\Big(X^n(\cdot,\cdot,\cdot,q_2) = \pm e_j\Big) = \frac{g(q_2)}{2dnq_2}, \qquad j \in \{1,2,\dots,d\}. \end{aligned} \tag{2.7.14}$$

With this family in hand, we proceed as follows. At time zero start with an initial configuration that comes from the invariant measure π_ρ. Attach to each particle σ a uniform-$[0,1]$ random variable $\mathcal{U}(\sigma)$. Take all these random variables independent of each other and of $X^n(x,s,q_1,q_2)$ for all choices of $(x,s,q_1,q_2) \in \mathbb{Z}^d \times \frac{1}{n}\mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{N}_0$. For each site x, order all particles present at x at time zero so that their uniform random variables are increasing. To the q_1-th variable attach $X^n(x,0,q_1,\xi^n(x,0))$, i.e., the position of the q_1-th particle in this ordering at time $\frac{1}{n}$ is $x + X^n(x,0,q_1,\xi^n(x,0))$. In this way we obtain the configuration of the system at time $\frac{1}{n}$. To construct the process $\frac{1}{n}$ time units further, repeat the first step, but let the particles jump according to $X^n(\cdot,\frac{1}{n},\cdot,\xi^n(\cdot,\frac{1}{n}))$. Thus, our construction is such that each particle chooses at each step uniformly at random, but dependent on the number of particles at the same location, a new jump distribution. In what follows we will use the phrase "at level n" to emphasize that we refer to the discrete-time version of the process. For instance, we say that $B_R(x,k)$ is good at level n if

$$\sum_{z \in Q_R^A(y)} \xi^n(z,s) \le CA^{Rd} \qquad \forall\, y \in \mathbb{Z}^d, s \ge 0 \text{ s.t. } Q_R(y) \times \{s\} \subseteq \widetilde{B}_R^A(x,k). \tag{2.7.15}$$

Next, we introduce

$$\begin{aligned} &\chi^n(x,k) \\ &= \mathbb{1}\Big\{B_{R+1}^A(x,k) \text{ is good, but is contaminated at level } n \text{ and intersects } [-L,L]^{d+1}\Big\}. \end{aligned} \tag{2.7.16}$$

It is not hard to show that the joint distribution of χ^n converges weakly to the joint distribution of χ (use that only finitely many particles can enter a fixed region in space-time, so that the above family of random variables may be approximated by a function depending on finitely many particles only). Thus, to estimate the joint distribution of χ, it is enough to analyze the joint distribution of χ^n, as long as the estimates are uniform in n. In what follows, $s, s', s'' \in \frac{1}{n}\mathbb{N}_0$.

Let $B_{R+1}^A(x,k)$ be a good block that is contaminated at level n. Then there is a particle at a site $y \in \partial\Pi_1(\widetilde{B}_{R+1}^A(x,k))$ at a time $s \in [(k-1)A^{R+1},(k+1)A^{R+1})$ (see below 2.4.13)) that is at a site $y' \in \partial\Pi_1(\widetilde{B}_{R+1}^{A,+}(x,k))$ at a time $s' \in [(k-1)A^{R+1},(k+1)A^{R+1})$, $s < s'$.

Furthermore, for all s'' such that $s < s'' < s'$, the particle is inside $\Pi_1(\widetilde{B}^A_{R+1}(x,k))$. This implies, when $X^n(y,s,q_1,\xi^n(y,s))$ denotes the random variable attached to σ at time s, that $X^n(y,s,q_1,\xi^n(y,s)) \neq 0$. Pick any such particle. Since this particle travels over a distance larger than $2A^{R+1}$, there is at least one coordinate direction along which it makes at least $2A^{R+1}$ steps. We call this direction $e_j(\sigma)$, and say that each step in this direction is a success. Note the uniform estimate

$$\mathbb{P}\big(\sigma \text{ has a success at time } s''\big) \leq \frac{1}{2dn} \qquad s'' \in [(k-1)A^{R+1},(k+1)A^{R+1}). \tag{2.7.17}$$

Thus, if σ contaminates $B^A_{R+1}(x,k)$ at level n, then from time s' up to time $(k+1)A^{R+1}$ it has at least $2A^{R+1}$ successes in $\Pi_1(\widetilde{B}^A_{R+1}(x,k))$. We write $\mathrm{S}(y,s,q_1,q_2)$ for the event just described, provided the particle was attached to $X^n(y,s,q_1,q_2)$ when it entered $\widetilde{B}^A_{R+1}(x,k)$. Since $B^A_{R+1}(x,k)$ is good, at each space-time point $(y'',s'') \in \widetilde{B}^A_{R+1}(x,k)$ there are at most $CA^{(R+1)d}$ particles that can contaminate $B^A_{R+1}(x,k)$. We therefore obtain

$$\begin{aligned}
&\Big\{B^A_{R+1}(x,k) \text{ is good, but contaminated at level } n\Big\}\\
&\subseteq \bigcup_{\substack{y\in\partial\Pi_1(\widetilde{B}^A_{R+1}(x,k))\\ s\in[(k-1)A^{R+1},(k+1)A^{R+1}),\, s\in\frac{1}{n}\mathbb{N}_0\\ q_1\leq CA^{(R+1)d}\\ q_2\leq CA^{(R+1)d}}} \Big\{X^n(y,s,q_1,q_2)\neq 0, \mathrm{S}(y,s,q_1,q_2)\Big\}\\
&\stackrel{\text{def}}{=} \mathrm{C}^n(x,k).
\end{aligned} \tag{2.7.18}$$

Next, note that the event $\mathrm{C}^n(x,k)$ depends on the $X^n(y,s,q_1,q_2)$ with $(y,s) \in \widetilde{B}^A_{R+1}(x,k)$ only. Hence, for $x \in \mathbb{Z}^d$, $k \in \mathbb{N}_0$, the family $(\mathrm{C}^n(x,k))_{(x,k)\equiv(x^*,k^*)}$ consists of independent events. We estimate $\mathbb{P}(\mathrm{C}^n(x,k))$. By (2.7.17), the probability inside the union may be bounded from above by

$$\frac{1}{n}\mathbb{P}\big(T \geq 2A^{R+1}\big), \tag{2.7.19}$$

where $T = \mathrm{BIN}(2A^{R+1}n, \frac{1}{2dn})$. Note that the event in (2.7.19) is a large deviation event, so that Bernstein's inequality guarantees the existence of a constant $C' > 0$ such that (2.7.19) is at most

$$\frac{1}{n}\exp\left\{-C'A^{R+1}\right\}. \tag{2.7.20}$$

Recall the definition of $\mathrm{C}^n(x,k)$ to see that, for a possibly different constant $C'' > 0$,

$$\mathbb{P}(\mathrm{C}^n(x,k)) \leq \exp\left\{-C''A^{R+1}\right\}. \tag{2.7.21}$$

Hence there is a family of independent Bernoulli random variables $(Z^n(x,k))_{(x,k)\equiv(x^*,k^*)}$ that stochastically dominates $(\chi^n(x,k))_{(x,k)\equiv(x^*,k*)}$ and has success probability $e^{-C''A^{R+1}}$.

Thus, if f is a positive and bounded function that is increasing in all of its arguments, then

$$\mathbb{E}\Big(f(\chi^n(x_1,k_1),\dots,\chi^n(x_n,k_n))\Big) \le \mathbb{E}\Big(f(Z^n(x_1,k_1),\dots,Z^n(x_n,k_n))\Big). \tag{2.7.22}$$

As the right-hand side does not depend on n, we obtain, by letting $n \to \infty$,

$$\mathbb{E}\Big(f(\chi(x_1,k_1),\dots,\chi(x_n,k_n))\Big) \le \mathbb{E}\Big(f(Z(x_1,k_1),\dots,Z(x_n,k_n))\Big), \tag{2.7.23}$$

which proves Claim 2.7.1. In particular, since all estimates are independent of L, we may even set $L=\infty$, to get that the whole field of good but contaminated $(R+1)$-blocks may be dominated by an independent family of Bernoulli random variables with the same success probability as above. This field will be denoted by $Z(x,k)$ as well. ∎

Non-contaminated blocks. We begin by estimating the probability of the event

$$\Big\{B^A_{R+1}(x,k) \text{ is good, but contains a bad } R\text{-block}\Big\}. \tag{2.7.24}$$

Let $B^A_R(y,l)$ be an R-block that is contained in $B^A_{R+1}(x,k)$. We bound the probability that this block is bad. For that we take $z_1 \in \mathbb{Z}^d$ such that

$$Q^A_R(z_1) = \prod_{j=1}^d [z_1(j), z_1(j)+A^R) \subseteq \widetilde{B}^A_R(y,l). \tag{2.7.25}$$

Use the time stationarity of ξ and the fact that $[(l-1)A^R,(l+1)A^R)$ may be divided into at most $2A^R+1$ time intervals of length one, to obtain

$$\begin{aligned}&\mathbb{P}\Bigg(\exists s \in [(l-1)A^R,(l+1)A^R)\colon \sum_{z_2\in Q^A_R(z_1)} \xi(z_2,s) > CA^{Rd}\Bigg)\\ &\le (2A^R+1)\mathbb{P}\Bigg(\sup_{s\in[0,1]} \sum_{z_2\in Q^A_R(z_1)} \xi(z_2,s) > CA^{Rd}\Bigg).\end{aligned} \tag{2.7.26}$$

In the same way as in Step 1.1 of this proof, we may now show that for some constant $K>0$,

$$\mathbb{P}\Bigg(\sup_{s\in[0,1]} \sum_{z_2\in Q^A_R(z_1)} \xi(z_2,s) > CA^{Rd}\Bigg) \le K\mathbb{P}\Bigg(\sum_{z_2\in Q^A_R(z_2)} \xi(z_2,1) \ge \frac{e^{-1}}{2}CA^{Rd}\Bigg). \tag{2.7.27}$$

Next, note that under the invariant measure π_ρ the sum in the right-hand side of (2.7.27) is a sum of i.i.d. random variables with finite exponential moments. Hence (2.7.27) is bounded from above by

$$K\exp\Big\{-\frac{e^{-1}}{2}CA^{Rd}\Big\}\mathbb{E}\left[e^{\xi(0,0)}\right]^{A^{Rd}}. \tag{2.7.28}$$

Now choose C large enough so that (2.7.26) decays superexponentially fast in R.

In order to estimate the joint distribution of non-contaminated blocks, we let $B^{A_1}_{R+1}(x_1,k_1),\ldots,B^{A_1}_{R+1}(x_n,k_n)$ be space-time blocks whose indices increase in the lexicographic order of $\mathbb{Z}^d\times\mathbb{N}$ and belong to the same equivalence class. We abbreviate

$$N^{\mathrm{sub}}(x_i,k_i)=\Big\{\widetilde{B}^{A,\mathrm{sub}}_{R+1}(x_i,k_i)\text{ is good, }B^A_{R+1}(x_i,k_i)\text{ contains a bad }R\text{-block, but is not contaminated}\Big\}. \tag{2.7.29}$$

Note that $N^{\mathrm{sub}}(x_i,k_i)$ and $N^{\mathrm{sub}}(x_j,k_j), i\neq j$, are independent when they depend on blocks that coincide in time but are disjoint in space. This observation, together with the Markov property, leads to

$$\begin{aligned}&\mathbb{P}\Bigg(N^{\mathrm{sub}}(x_n,k_n)\ \Bigg|\ \bigcap_{i=1}^{n-1}N^{\mathrm{sub}}(x_i,k_i)\Bigg)\\&\leq\mathbb{P}\Bigg(N^{\mathrm{sub}}(x_n,k_n)\ \Bigg|\ \bigcap_{i=1}^{n-1}N^{\mathrm{sub}}(x_i,k_i),\widetilde{B}^{A,\mathrm{sub}}_{R+1}(x_n,k_n)\text{ is good}\Bigg)\\&=\mathbb{P}\Big(N^{\mathrm{sub}}(x_n,k_n)\ \Big|\ \widetilde{B}^{A,\mathrm{sub}}_{R+1}(x_n,k_n)\text{ is good}\Big).\end{aligned} \tag{2.7.30}$$

Thus, the left-hand side of (2.7.30) is at most

$$\frac{\mathbb{P}(B^A_{R+1}(x,k)\text{ is good, but contains a bad }R\text{-block})}{\mathbb{P}(\widetilde{B}^{A_1,\mathrm{sub}}_{R+1}(x_n,k_n)\text{ is good})}. \tag{2.7.31}$$

Note that the denominator tends to one as $R\to\infty$. This comes from the fact that, for all $t\geq 0$, $(\xi(x,t))_{x\in\mathbb{Z}^d}$ is an i.i.d. field of random variables distributed according to π_ρ. Thus, from (2.7.26) and the lines below, we infer that (2.7.31) decays superexponentially fast in R.

Finally, write

$$\Big\{B^A_{R+1}(x_i,k_i)\text{ is good, but contains a bad }R\text{-block }\Big\}\subseteq C(x_i,k_i)\cup N^{\mathrm{sub}}(x_i,k_i), \tag{2.7.32}$$

where we denote by $C(x_i,k_i)$ the event that $B^A_{R+1}(x_i,k_i)$ is good, contains a bad R-block, and is contaminated. Then

$$\begin{aligned}&\mathbb{P}\Bigg(\bigcap_{i=1}^{n}\Big\{B^{A_1}_{R+1}(x_i,k_i)\text{ is good, but contains a bad }R\text{-block}\Big\}\Bigg)\\&\qquad\leq\mathbb{P}\Bigg(\bigcap_{i=1}^{n}\big(C(x_i,k_i)\cup N^{\mathrm{sub}}(x_i,k_i)\big)\Bigg).\end{aligned} \tag{2.7.33}$$

If we denote by C the subset of all $i\in\{1,2,\ldots,n\}$ for which $C(x_i,k_i)$ occurs, then

(2.7.33) may be rewritten as

$$\sum_{C\subseteq\{1,2,\dots,n\}} \mathbb{P}\Bigg(\bigcap_{i\in C} C(x_i,k_i)\cap\bigcap_{i\notin C} N^{\mathrm{sub}}(x_i,k_i)\Bigg). \tag{2.7.34}$$

Note that either $|C|\geq n/2$ or $|\{1,2,\dots,n\}\backslash C|\geq n/2$, so that by Claim 2.7.1 and (2.7.28) there is a $C''>0$ such that the expression in (2.7.34) is at most $2^n\exp\{-C''A^{R+1}n/2\}$. A comparison with the right-hand side of (2.1.21) shows that ξ is type-I Gärtner-mixing.

1.3 From the previous calculations we infer that ξ satisfies condition (a3) of Definition 2.1.7.

2. Since ξ is bounded from below it is Gärtner-negative-hyper-mixing. Hence it remains to show that ξ is type-II Gärtner-mixing. To that end, fix the same constants as in the proof of the first part of the corollary. Furthermore, fix $\delta>0$ and let $B^A_{R+1}(x_1,k_1),\dots,B^A_{R+1}(x_n,k_n)$ be space-time blocks whose indices increase in the lexicographic order of $\mathbb{Z}^d\times\mathbb{N}$, and belong to the same equivalence class. Take events $\mathcal{A}_i^R\in\sigma(B^A_{R+1}(x_n,k_n))$, $i\in\{1,2,\dots,n\}$, that are invariant under shifts in space and time, and satisfy

$$\lim_{R\to\infty}\mathbb{P}(\mathcal{A}_i^R)=0. \tag{2.7.35}$$

As in part 1, we divide space-time blocks into contaminated and non-contaminated blocks. With the help of Claim 2.7.1, we may control contaminated blocks. To treat non-contaminated blocks, we introduce

$$N^{\mathrm{sub}}(x_i,k_i,\mathcal{A}_i^R)=\Big\{\widetilde{B}^{A,\mathrm{sub}}_{R+1}(x_i,k_i)\text{ is good, but contaminated, }\mathcal{A}_i^R\text{ occurs}\Big\} \tag{2.7.36}$$

and proceed as in the lines following (2.7.29), to finish the proof.

2.7.2 Examples of potentials for the continuity result

In this section we prove Corollary 2.1.20. Suppose that

- ▶ ξ is Markov with initial distribution ν and generator L defined on a domain $D(L)\subset L^2(d\nu)$. (2.7.37)

Denote by

$$\bar{\varepsilon}(f,g)=\tfrac12\big[\langle -Lf,g\rangle+\langle -Lg,f\rangle\big],\qquad f,g\in D(L), \tag{2.7.38}$$

its symmetrized Dirichlet form, assume that $(\bar{\varepsilon},D(L))$ is closable, and denote its closure by $(\bar{\varepsilon},D(\bar{\varepsilon}))$. Furthermore, for $V\colon\,\Omega\to\mathbb{R}$, define

$$J_V(r)=\inf\left\{\bar{\varepsilon}(f,f)\colon\ f\in D(\bar{\varepsilon})\cap L^2(|V|d\nu),\ \int f^2\,d\nu=1,\ \int Vf^2\,d\nu=r\right\},\qquad r\in\mathbb{R}, \tag{2.7.39}$$

note that $r \mapsto J_V(r)$ is convex, and let I_V be its lower semi-continuous regularization. For $W\colon \Omega \times [0,\infty) \to \mathbb{R}$, define

$$\Gamma_t^L = \sup_{\substack{f\in D(L)\cap L^2(|W(\cdot,t)|d\nu)\\ \|f\|_2=1}} \left\{\langle Lf,f\rangle + \langle W(\cdot,t), f^2\rangle\right\}, \qquad t \geq 0. \tag{2.7.40}$$

In the particular case of a static $W_0\colon \Omega \to \mathbb{R}$, we denote the corresponding variational expression by Γ_0^L.

Lemma 2.7.2. *Suppose that W is bounded from below, piecewise continuous in the time-coordinate, and ν-integrable in the space-coordinate. Suppose further that:*
(i) $\Gamma_t^L < \infty$ for all $t > 0$.
(ii) ξ is reversible in time and càdlàg.
Then

$$E_\nu\left(\exp\left\{\int_0^t W(\xi(s),s)\,ds\right\}\right) \leq \exp\left\{\int_0^t \Gamma_s^L\,ds\right\} \qquad \forall\, t \geq 0. \tag{2.7.41}$$

Proof. The proof is based on ideas in Kipnis and Landim [KL99], Appendix 1.7, and comes in three steps.

1. Suppose that W is piecewise continuous, not necessarily bounded from above, and define $W^n = W \wedge n$. Then, by an argument similar to that in [KL99], Appendix 1, Lemma 7.2, we have

$$\mathbb{E}_\nu\left(\exp\left\{\int_0^t W^n(\xi(s),s)\,ds\right\}\right) \leq \exp\left\{\int_0^t \Gamma_s^{L,n}\,ds\right\}, \tag{2.7.42}$$

where $\Gamma^{L,n}$ is defined as in (2.7.40) but with W replaced by W^n. It is here that we use that ξ is reversible and càdlàg, since under this condition we have a Feynman-Kac representation for the parabolic Anderson equation with potential W^n and with Δ replaced by L. Because W^n is bounded, we have that $L^2(|W^n(\cdot,t)|d\nu) = L^2(d\nu)$. Hence

$$\Gamma_s^{L,n} = \sup_{\substack{f\in D(L)\\ \|f\|_2=1}} \left\{\langle Lf,f\rangle + \langle W^n(\cdot,s), f^2\rangle\right\}. \tag{2.7.43}$$

Since, by monotone convergence,

$$\lim_{n\to\infty} \mathbb{E}_\nu\left(\exp\left\{\int_0^t W^n(\xi(s),s)\,ds\right\}\right) = \mathbb{E}_\nu\left(\exp\left\{\int_0^t W(\xi(s),s)\,ds\right\}\right), \tag{2.7.44}$$

it suffices to show that

$$\lim_{n\to\infty} \Gamma_s^{L,n} = \Gamma_s^L, \qquad s \geq 0. \tag{2.7.45}$$

2. To show that the left-hand side in (2.7.45) is an upper bound, fix $\varepsilon > 0$ and pick $f \in D(L) \cap L^2(|W(\cdot,t)|d\nu)$ such that

$$\langle Lf,f\rangle + \langle W(\cdot,t), f^2\rangle + \varepsilon \geq \Gamma_s^L. \tag{2.7.46}$$

Then, by monotone convergence,

$$\lim_{n\to\infty} \big\{\langle Lf,f\rangle + \langle W^n(\cdot,t),f^2\rangle\big\} = \langle Lf,f\rangle + \langle W(\cdot,t),f^2\rangle. \tag{2.7.47}$$

3. To prove that the left-hand side in (2.7.45) is also a lower bound, we need to assume that L is self-adjoint. Then, by Wu [Wu94], Remark on p. 209, we have

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_\nu \left(\exp\left\{ \int_0^t W^n(\xi(s),0)\, ds \right\}\right) = \Gamma_0^{L,n}, \tag{2.7.48}$$

and the same is true when W^n is replaced by W. But, obviously,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_\nu \left(\exp\left\{ \int_0^t W^n(\xi(s),0)\, ds \right\}\right) \le \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_\nu \left(\exp\left\{ \int_0^t W(\xi(s),0)\, ds \right\}\right), \tag{2.7.49}$$

which shows that $\Gamma_0^{L,n} \le \Gamma_0^L$. Note that the time point 0 does not play any special role. Hence we obtain (2.7.45). ∎

Proposition 2.7.3. *Suppose that* ξ *satisfies* (2.7.37) *and condition (ii) in Lemma 2.7.2. Then, for all paths* Φ *(recall* (2.1.26)*),*

$$\mathbb{P}_\nu \left(\frac{1}{n(T-1)} \sum_{j=1}^n \int_{(j-1)T+1}^{jT} \big[\xi(\Phi(s),s)\, ds - \rho\big] > \delta \right) \le \exp\big\{ -n(T-1) I_{V_0}(\rho+\delta)\big\}, \tag{2.7.50}$$

where $V_0(\eta) = \eta(0)$ *and* $\rho = \mathbb{E}(\xi(0,0))$.

Proof. First note that by Lemma 2.7.2,

$$\mathbb{E}_\nu \left(\exp\left\{ \sum_{j=1}^n \int_{(j-1)T+1}^{jT} \xi(\Phi(s),s)\, ds \right\}\right) \le \exp\big\{n(T-1)\Gamma_0^L\big\}, \tag{2.7.51}$$

where Γ_0^L is defined with $W(\cdot,0) = V_0$. To see that, take a path Φ which starts in zero and define $W\colon \mathbb{R}^{\mathbb{Z}^d} \times [0,\infty) \to \mathbb{R}$ by

$$W(\eta,t) = \begin{cases} \eta(\Phi(t)), & \text{if } t \in [(j-1)T+1, jT) \text{ for some } 1 \le j \le n, \\ 0, & \text{otherwise.} \end{cases} \tag{2.7.52}$$

Let $x \in \mathbb{Z}^d$ be such that $\Phi(t) = x$, and denote by τ_x the space-shift over x. Then, using the fact that ν is shift-ergodic, we get

$$\begin{aligned} \langle W(\cdot,t), f^2\rangle &= \int_{\mathbb{R}^{\mathbb{Z}^d}} \eta(x) f^2(\eta)\, d\nu(\eta) \\ &= \int_{\mathbb{R}^{\mathbb{Z}^d}} \tau_x\eta(0) f^2(\eta)\, d\nu(\eta) \\ &= \int_{\mathbb{R}^{\mathbb{Z}^d}} \eta(0) (\tau_{-x}f)^2(\eta)\, d\nu(\eta) = \langle W(\cdot,0), (\tau_{-x}f)^2\rangle, \end{aligned} \tag{2.7.53}$$

which yields $\Gamma_t^L = \Gamma_0^L$. Since the space point 0 does not play any special role, Lemma 2.7.2 leads to (2.7.51) for any path Φ. Next apply the Chebyshev inequality to the left-hand side of (2.7.50). After that it remains to solve an optimization problem. See Wu [Wu00] for details. ∎

Note that, for a reversible dynamics, I_{V_0} is the large deviation rate function for the occupation time

$$T_t = \int_0^t \xi(0,s)\,ds, \qquad t \geq 0. \tag{2.7.54}$$

We are now ready to give the proof of Corollary 2.1.20.

Proof. All three dynamics in (1)–(3) satisfy condition (a) in Definition 2.1.9. The proof of (b) below consists of an application of Proposition 2.7.3, combined with a suitable analysis of I_{V_0}.

(1) Redig and Völlering [RV11], Theorem 4.1, shows that for all $\delta_1 > 0$ there is a $\delta_2 = \delta_2(\delta_1)$ such that

$$\mathbb{P}_\nu\left(\int_0^{nt} \xi(0,s)\,ds \geq \delta_1 nt\right) \leq e^{-\delta_2 nt}. \tag{2.7.55}$$

A straightforward extension of this result implies that condition (b) in Definition 2.1.9 is satisfied. All requirements in Theorem 2.1.16 are thus met.

(2) By Landim [L92], Theorem 4.2, the rate function of the simple exclusion process is non-degenerate (i.e., it has a unique zero at ρ). Hence condition (b) in Definition 2.1.9 is satisfied. Thus, all requirements of Theorem 2.1.16 are met.

(3) By Cox and Griffeath [CG84], Theorem 1, the rate function for independent simple random walks is non-degenerate. Hence condition (b) in Definition 2.1.9 is satisfied. All requirements in Theorem 2.1.16 are thus met. ∎

3 Space-time ergodicity for the quenched Lyapunov exponent

This chapter is based on:
D. Erhard, F. den Hollander, G. Maillard. *The parabolic Anderson model in a dynamic random environment: space-time ergodicity for the quenched Lyapunov exponent.* Posted on *arXiv:1304.2274v2*, to appear in Probability Theory and Related Fields.

Abstract

We continue our study of the parabolic Anderson equation $\partial u(x,t)/\partial t = \kappa\Delta u(x,t) + \xi(x,t)u(x,t)$, $x \in \mathbb{Z}^d$, $t \geq 0$, where $\kappa \in [0,\infty)$ is the diffusion constant, Δ is the discrete Laplacian, and ξ plays the role of a *dynamic random environment* that drives the equation. The initial condition $u(x,0) = u_0(x)$, $x \in \mathbb{Z}^d$, is taken to be non-negative and bounded. The solution of the parabolic Anderson equation describes the evolution of a field of particles performing independent simple random walks with binary branching: particles jump at rate $2d\kappa$, split into two at rate $\xi \vee 0$, and die at rate $(-\xi) \vee 0$.

We assume that ξ is stationary and ergodic under translations in space and time, is not constant and satisfies $\mathbb{E}(|\xi(0,0)|) < \infty$, where $\mathbb{E}$ denotes expectation w.r.t. ξ. Our main object of interest is the *quenched Lyapunov exponent* $\lambda_0(\kappa) = \lim_{t\to\infty} \frac{1}{t} \log u(0,t)$. In earlier work [GdHM12], [EdHM14a] we established a number of basic properties of $\kappa \mapsto \lambda_0(\kappa)$ under certain mild space-time mixing and noisiness assumptions on ξ. In particular, we showed that the limit exists ξ-a.s., is finite and continuous on $[0,\infty)$, is globally Lipschitz on $(0,\infty)$, is not Lipschitz at 0, and satisfies $\lambda_0(0) = \mathbb{E}(\xi(0,0))$ and $\lambda_0(\kappa) > \mathbb{E}(\xi(0,0))$ for $\kappa \in (0,\infty)$.

In the present chapter we show that $\lim_{\kappa\to\infty} \lambda_0(\kappa) = \mathbb{E}(\xi(0,0))$ under an additional space-time mixing condition on ξ we call Gärtner-hyper-mixing. This result, which completes our study of the quenched Lyapunov exponent for general ξ, shows that the parabolic Anderson model exhibits space-time ergodicity in the limit of large diffusivity. This fact is interesting because there are choices of ξ that are Gärtner-hyper-mixing for which the *annealed Lyapunov exponent* $\lambda_1(\kappa) = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}(u(0,t))$ is infinite on $[0,\infty)$, a situation that is referred to as *strongly catalytic behavior.* Our proof is based on a *multiscale analysis* of ξ, in combination with discrete rearrangement inequalities for local times of simple random walk and spectral bounds for discrete Schrödinger operators.

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Key words and phrases. Parabolic Anderson equation, quenched Lyapunov exponent, large deviations, Gärtner-hyper-mixing, multiscale analysis, rearrangement inequalities,

spectral bounds.

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3.1 Introduction and main theorem

A fair amount is known about the behavior as a function of underlying parameters of the *annealed Lyapunov exponents* for the parabolic Anderson model in a dynamic random environment. For an overview we refer the reader to [GdHM08]. The main motivation behind the present paper is to understand the behavior of the *quenched Lyapunov exponent*, which is much harder to deal with. Our ultimate goal is to arrive at a full qualitative picture of the quenched Lyapunov exponent for *general* dynamic random environments subject to certain mild space-time mixing and noisiness assumptions.

Section 3.1.1 defines the parabolic Anderson model and recalls the main results from [GdHM12], [EdHM14a]. Section 3.1.2 contains our main theorem, which states that the quenched Lyapunov exponent converges to the average value of the environment in the limit of large diffusivity. Section 3.1.3 contains definitions, while Section 3.1.4 discusses the main theorem, provides the necessary background, and gives a brief outline of the rest of the paper.

3.1.1 Parabolic Anderson model

The parabolic Anderson model is the partial differential equation

$$\frac{\partial}{\partial t}u(x,t) = \kappa\Delta u(x,t) + \xi(x,t)u(x,t), \qquad x \in \mathbb{Z}^d,\ t \geq 0. \tag{3.1.1}$$

Here, the u-field is $\mathbb{R}$-valued, $\kappa \in [0,\infty)$ is the diffusion constant, Δ is the discrete Laplacian acting on u as

$$\Delta u(x,t) = \sum_{\substack{y\in\mathbb{Z}^d \\ \|y-x\|=1}} [u(y,t) - u(x,t)] \tag{3.1.2}$$

($\|\cdot\|$ is the l_1-norm), while

$$\xi = (\xi_t)_{t\geq 0} \text{ with } \xi_t = \{\xi(x,t)\colon\ x \in \mathbb{Z}^d\} \tag{3.1.3}$$

is an $\mathbb{R}$-valued random field playing the role of a *dynamic random environment* that drives the equation. As initial condition for (3.1.1) we take

$$u(x,0) = u_0(x),\ x \in \mathbb{Z}^d, \text{ with } u_0 \text{ non-negative, not identically zero, and bounded.} \tag{3.1.4}$$

One interpretation of (3.1.1) and (3.1.4) comes from *population dynamics*. Consider the special case where $\xi(x,t) = \gamma\xi_*(x,t) - \delta$ with $\delta, \gamma \in (0,\infty)$ and ξ_* an $\mathbb{N}_0$-valued random field. Consider a system of two types of particles, A (catalyst) and B (reactant), subject to:

– A-particles evolve autonomously according to a prescribed dynamics with $\xi_*(x,t)$ denoting the number of A-particles at site x at time t;

- B-particles perform independent simple random walks at rate $2d\kappa$ and split into two at a rate that is equal to γ times the number of A-particles present at the same location at the same time;
- B-particles die at rate δ;
- the average number of B-particles at site x at time 0 is $u_0(x)$.

Then

$$u(x,t) \;=\; \begin{array}{l}\text{the average number of } B\text{-particles at site } x \text{ at time } t\\ \text{conditioned on the evolution of the } A\text{-particles.}\end{array} \tag{3.1.5}$$

The ξ-field is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Throughout the paper we assume that

$$\begin{array}{ll}\blacktriangleright & \xi \text{ is } \textit{stationary} \text{ and } \textit{ergodic} \text{ under translations in space and time.}\\ \blacktriangleright & \xi \text{ is } \textit{not constant} \text{ and } \mathbb{E}(|\xi(0,0)|) < \infty.\end{array} \tag{3.1.6}$$

The formal solution of (3.1.1) is given by the *Feynman-Kac formula*

$$u(x,t) = E_x\left(\exp\left\{\int_0^t \xi(X^\kappa(s), t-s)\,ds\right\} u_0(X^\kappa(t))\right), \tag{3.1.7}$$

where $X^\kappa = (X^\kappa(t))_{t\geq 0}$ is the continuous-time simple random walk jumping at rate $2d\kappa$ (i.e., the Markov process with generator $\kappa\Delta$), and P_x is the law of X^κ when $X^\kappa(0) = x$. In [EdHM14a] we proved the following:

(0) Subject to the assumption that ξ-a.s. $s \mapsto \xi(x,s)$ is locally integrable for every x and that $\mathbb{E}(e^{q\xi(0,0)}) < \infty$ for all $q \geq 0$, (3.1.7) is finite for all x, t and is the solution of (3.1.1).

The *quenched Lyapunov exponent* associated with (3.1.1) is defined as

$$\lambda_0(\kappa) = \lim_{t\to\infty} \frac{1}{t} \log u(0,t). \tag{3.1.8}$$

In [GdHM12] we showed that $\lambda_0(0) = \mathbb{E}(\xi(0,0))$ and $\lambda_0(\kappa) > \mathbb{E}(\xi(0,0))$ for $\kappa \in (0,\infty)$ as soon as the limit in (3.1.8) exists. In [EdHM14a] we proved the following:

(1) Subject to certain *space-time mixing assumptions* on ξ, the limit in (3.1.8) exists ξ-a.s. and in $L^1(\mathbb{P})$, is ξ-a.s. constant, is finite, and does not depend on u_0 satisfying (3.1.4).

(2) Subject to certain additional *noisiness assumptions* on ξ, $\kappa \mapsto \lambda_0(\kappa)$ is continuous on $[0,\infty)$, is globally Lipschitz on $(0,\infty)$, and is not Lipschitz at 0.

3.1.2 Main theorem and examples

Our main result is the following.

Theorem 3.1.1. *If $u_0 = \delta_0$ and ξ is Gärtner-hyper-mixing, then*

$$\lim_{\kappa\to\infty} \lambda_0(\kappa) = \mathbb{E}(\xi(0,0)). \tag{3.1.9}$$

The definition of Gärtner-hyper-mixing is given in Definitions 3.1.3–3.1.5 below. A weaker form of these definitions was introduced and exploited in [EdHM14a]. Here are two examples of ξ-fields that are Gärtner-hyper-mixing.

Example 3.1.2. (See [EdHM14a])
(e1) *Let $Y = (Y_t)_{t\geq 0}$ be a stationary and ergodic $\mathbb{R}$-valued Markov process satisfying*

$$\mathbb{E}\left[e^{q \sup_{t\in[0,1]} |Y_t|}\right] < \infty \qquad \forall\, q \geq 0. \tag{3.1.10}$$

Let $(Y(x))_{x\in\mathbb{Z}^d}$ be a field of independent copies of Y. Then ξ given by $\xi(x,t) = Y_t(x)$ is Gärtner-hyper-mixing.
(e2) *Let ξ be the zero-range process with rate function $g\colon \mathbb{N}_0 \to (0,\infty)$ given by $g(k) = k^\beta$, $\beta \in (0,1]$, and transition probabilities given by simple random walk on $\mathbb{Z}^d$. If ξ starts from the product measure π_ρ, $\rho \in (0,\infty)$, with marginals*

$$\forall\, x \in \mathbb{Z}^d\colon \qquad \pi_\rho\big\{\eta \in \mathbb{N}_0^{\mathbb{Z}^d}\colon\, \eta(x) = k\big\} = \begin{cases} \gamma\, \frac{\rho^k}{\prod_{l=1}^k g(l)}, & \text{if } k > 0,\\ \gamma, & \text{if } k = 0, \end{cases} \tag{3.1.11}$$

where $\gamma \in (0,\infty)$ is a normalization constant, then ξ is Gärtner-hyper-mixing.

Example (e1) includes independent spin-flips, example (e2) includes independent random walks.

3.1.3 Definitions

Throughout the rest of this paper we assume without loss of generality that $\mathbb{E}(\xi(0,0)) = 0$.

For $a_1, a_2, N \in \mathbb{N}$, denote by $\Delta_N(a_1,a_2)$ the set of $(\mathbb{Z}^d\times\mathbb{N})$-valued sequences $\{(x_i,k_i)\}_{i=1}^N$ that are increasing with respect to the lexicographic ordering of $\mathbb{Z}^d \times \mathbb{N}$ and are such that, for all $1 \leq i < j \leq N$,

$$x_j \equiv x_i \,(\operatorname{mod} a_1), \qquad k_j \equiv k_i \,(\operatorname{mod} a_2). \tag{3.1.12}$$

For $A \geq 1$, $\alpha > 0$, $R \in \mathbb{N}$, $x \in \mathbb{Z}^d$ and $k,b,c \in \mathbb{N}_0$, define the *space-time blocks* (see Fig. 3.1)

$$\begin{aligned} &\widetilde{B}_R^{A,\alpha}(x,k;b,c)\\ &= \left(\prod_{j=1}^d \big[(x(j)-1-b)\alpha A^R, (x(j)+1+b)\alpha A^R\big) \cap \mathbb{Z}^d\right) \times [(k-c)A^R, (k+1)A^R). \end{aligned} \tag{3.1.13}$$

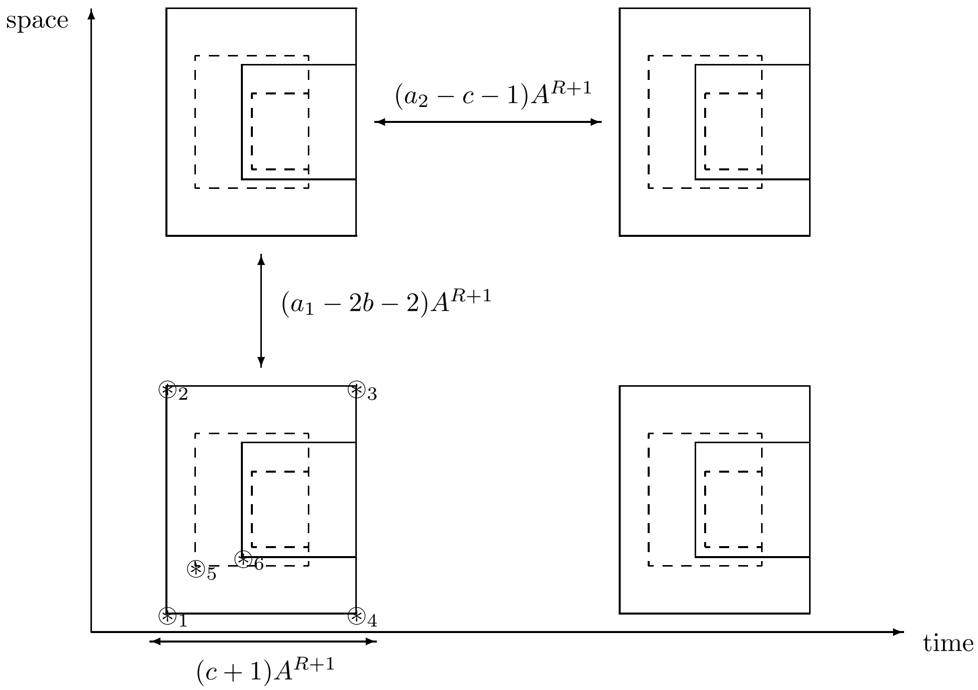

Figure 3.1: The dashed blocks are R-blocks, i.e., $B_R^A(x,k)$ (inner) and $\widetilde{B}_R^A(x,k;b,c)$ (outer) for some choice of A,x,k,b,c. The solid blocks are $(R+1)$-blocks, i.e., $B_{R+1}^A(y,l)$ (inner) and $\widetilde{B}_{R+1}^A(y,l;b,c)$ (outer) for some choice of A,y,l,b,c such that these $(R+1)$-blocks contain the corresponding R-blocks. All these blocks belong to the same equivalence class. The symbols $\{\circledast_i\}_{i=1,2,3,4,5,6}$ represents the space-time coordinates $\circledast_1 = ((y-1-b)A^{R+1}, (l-c)A^{R+1})$, $\circledast_2 = ((y+1+b)A^{R+1}, (l-c)A^{R+1})$, $\circledast_3 = ((y+1+b)A^{R+1}, (l+1)A^{R+1})$, $\circledast_4 = ((y-1-b)A^{R+1}, (l+1)A^{R+1})$, $\circledast_5 = ((x-1-b)A^R, (k-c)A^R)$, $\circledast_6 = ((y-1)A^{R+1}, lA^{R+1})$.

Abbreviate $B_R^{A,\alpha}(x,k) = \widetilde{B}_R^{A,\alpha}(x,k;0,0)$ and $B_R^A(x,k) = B_R^{A,1}(x,k)$, and define the *space-blocks*

$$Q_R^{A,\alpha}(x) = x + [0, \alpha A^R)^d \cap \mathbb{Z}^d. \tag{3.1.14}$$

Definition 3.1.3. [Good and bad blocks]
For $A \geq 1$, $\alpha > 0$, $R \in \mathbb{N}$, $x \in \mathbb{Z}^d$, $m > 0$, $k \in \mathbb{N}_0$, $\delta \in [0, \operatorname{ess\,sup}[\xi(0,0)]]$ and $b, c \in \mathbb{N}_0$, the R-block $B_R^{A,\alpha}(x,k)$ is called (δ, b, c, m)-good for the potential ξ when, for all $s \in [(k-c)A^R, (k+1)A^R - 1/m)$,

$$\frac{1}{|Q_R^{A,\alpha}(y)|} \sum_{z \in Q_R^{A,\alpha}(y)} \sup_{r \in [s, s+1/m)} \xi(z,r) \leq \delta \qquad \forall\, y \in \mathbb{Z}^d \colon\ Q_R^{A,\alpha}(y) \times \{s\} \subseteq \widetilde{B}_R^{A,\alpha}(x,k;b,c), \tag{3.1.15}$$

and is called (δ, b, c)-bad otherwise.

For $A \geq 1$, $\alpha > 0$, $R \in \mathbb{N}$, $x \in \mathbb{Z}^d$, $m > 0$, $k \in \mathbb{N}_0$, $\delta \in [0, \operatorname{ess\,sup}[\xi(0,0)]]$ and $b, c \in \mathbb{N}_0$, let

$$\begin{aligned} &\mathcal{A}_R^{A,\alpha,\delta,m}(x,k;b,c) \\ &\quad = \big\{ B_{R+1}^{A,\alpha}(x,k) \text{ is } (\delta,b,c)\text{-good, but contains an } R\text{-block that is } (\delta,b,c,m)\text{-bad} \big\}. \end{aligned} \tag{3.1.16}$$

Definition 3.1.4. [Gärtner-mixing]
The ξ-field is called $(A, \alpha, \delta, m, b, c)$-Gärtner-mixing when there are $a_1, a_2 \in \mathbb{N}$ such that for all $R \in \mathbb{N}$, $N \in \mathbb{N}$,

$$\sup_{(x_i,k_i)_{i=1}^N \in \Delta_N(a_1,a_2)} \mathbb{P}\left(\bigcap_{i=1}^N \mathcal{A}_R^{A,\alpha,\delta,m}(x_i,k_i;b,c) \right) \leq \left(A^{-4d(2d+1)(d+1)R} \right)^N. \tag{3.1.17}$$

In the rest of this document we use the abbreviation ξ_K for $\xi \mathbb{1}\{\xi \geq K\}$.

Definition 3.1.5. [Gärtner-hyper-mixing]
The ξ-field is called Gärtner-hyper-mixing when the following conditions are satisfied:
(a1) *There are $b, c \in \mathbb{N}_0$ and $K \geq 0$ such that for every $\delta > 0$ there are $A_0 > 1$ and $m_0 > 0$ such that ξ_K and ξ are $(A, \alpha, \delta, m, b, c)$-Gärtner-mixing for all $A \geq A_0$, $m \geq m_0$ and all $\alpha \geq 1$, with a_1, a_2 in Definition* 3.1.4 *not depending on A, m and α.*
(a2) $\mathbb{E}[e^{q \sup_{s \in [0,1]} \xi(0,s)}] < \infty$ *for all $q \geq 0$.*
(a3) *There are $R_0 \in \mathbb{N}$ and $C_1 \in [0, \operatorname{ess\,sup}[\xi(0,0)]] \cap \mathbb{R}$ such that*

$$\mathbb{P}\left(\sup_{s \in [0,1]} \frac{1}{|B_R|} \sum_{y \in B_R} \xi(y,s) \geq C \right) \leq |B_R|^{-\alpha} \quad \forall\, R \geq R_0, C \geq C_1, \tag{3.1.18}$$

for some $\alpha > [2d(2d+1)+1](d+2)/d$, where $B_R = [-R,R]^d \cap \mathbb{Z}^d$.

Remark 3.1.6. The proof in [EdHM14a] that Examples 3.1.2(e1–e2) are Gärtner-hyper-mixing uses (3.1.15) without the supremum, but easily carries over by inspection.

Remark 3.1.7. (1) Gärtner-hyper-mixing requires that space averages of ξ taken in space-time blocks of a suitable size are close to their mean with a large probability. It also requires that the configurations of ξ restricted to the space-time blocks for which this closeness is realized are almost independent.
(2) For those examples where $\xi(x,t)$ represents "the number of particles at site x at time t", we may view Gärtner-hyper-mixing as a consequence of the fact that there are not enough particles in the blocks $\widetilde{B}_R^{A,\alpha}(x_i,k_i;b,c)$ that manage to travel to the blocks $\widetilde{B}_R^{A,\alpha}(x_j,k_j;b,c)$ with $j \neq i$. Indeed, if there is a bad block on scale R that is contained in a good block on scale $R+1$, then in some neighborhood of this bad block the particle density cannot be too large. This also explains why we must work with the extended blocks $\widetilde{B}_R^{A,\alpha}(x,k;b,c)$ instead of with the original blocks $B_R^{A,\alpha}(x,k;0,0)$. Indeed, the surroundings of a bad block on scale R can be bad when it is located near the boundary of a good block on scale $R+1$.
(3) We expect that most interacting particle systems are Gärtner-hyper-mixing, including such classical systems as the stochastic Ising model, the contact process, the voter model and the exclusion process. Since these are bounded random fields, conditions (a2) and (a3) in Definition 3.1.5 below are redundant and only condition (a1) needs to be verified. We will not tackle this problem in the present paper.

3.1.4 Discussion

1. Theorem 3.1.1 yields a partial answer to the question: Which random walk paths give the main contribution to the Feynman-Kac formula in (3.1.7)? Indeed, Theorem 3.1.1 shows that, for large κ and any *dynamic* ξ that is Gärtner-hyper-mixing, the main contribution comes from those paths that spend most of their time in regions where ξ looks typical. This is in sharp contrast with what is known for the parabolic Anderson model with a *static* i.i.d. random environment $\xi = \{\xi(x)\colon\ x \in \mathbb{Z}^d\}$. In this case the main contribution to the Feynman-Kac formula in (3.1.7) comes from those paths that are localized, in the sense that they spend almost all of their time in regions where ξ is large. The latter implies that for bounded ξ the quenched Lyapunov exponent equals $\operatorname{ess\,sup} \xi(0)$ instead of $\mathbb{E}(\xi(0))$.

2. What is interesting about Theorem 3.1.1 is that it reveals a sharp contrast with what is known for the *annealed Lyapunov exponent*

$$\lambda_1(\kappa) = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}(u(0,t)). \tag{3.1.19}$$

Indeed, there are choices of ξ for which $\kappa \mapsto \lambda_1(\kappa)$ is everywhere infinite on $[0,\infty)$, a property referred to as *strongly catalytic* behavior. For instance, as shown in [GdH06], if ξ is γ times a field of independent simple random walks starting in a Poisson equilibrium with arbitrary density, then this uniform divergence occurs in $d=1,2$ for $\gamma \in (0,\infty)$ and in $d \geq 3$ for $\gamma \in [1/G_d,\infty)$, with G_d the Green function of simple random walk at the origin. By Example 3.1.2(e2) (with $\beta = 1$), this choice of ξ is Gärtner-hyper-mixing.

3. The annealed Lyapunov exponents

$$\lambda_p(\kappa) = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}([u(0,t)]^p), \qquad p \in \mathbb{N}, \tag{3.1.20}$$

were studied in detail in a series of papers where ξ was chosen to evolve according to four specific interacting particle systems in equilibrium: independent Brownian motions, independent simple random walks, the simple symmetric exclusion process, and the voter model (for an overview, see [GdHM08]). Their behavior turns out to be very different from that of $\lambda_0(\kappa)$. In [EdHM14a] it was conjectured that

$$\lim_{\kappa\to\infty} [\lambda_p(\kappa) - \lambda_0(\kappa)] = 0 \qquad \forall\, p \in \mathbb{N} \tag{3.1.21}$$

because ξ is ergodic in space and time. For the case where $\lambda_p(\kappa) \equiv \infty$ this statement is to be read as saying that $\lim_{\kappa\to\infty} \lambda_0(\kappa) = \infty$. The heuristic behind this conjecture is that in the annealed setting the regions where ξ is large are close to the origin, so that the random walk can easily find them. In the quenched setting, however, these regions are far away from the origin, but since κ is large the random walk is still able to easily find them. This conjecture was furthermore supported by the fact that there are examples of ξ for which

$$\lim_{\kappa\to\infty} \lambda_p(\kappa) = \mathbb{E}(\xi(0,0)) \tag{3.1.22}$$

(see [GdH06], [GdHM07] and [GdHM10]). Nonetheless, Theorem 3.1.1 shows that this conjecture is false. The reason is that, because of the large diffusivity, the random walk is unlikely to spend a large time in the regions where ξ is large. Thus, for a ξ that is Gärtner-hyper-mixing and satisfies conditions (0) and (2) in Section 3.1.1, the qualitative behavior of $\kappa \mapsto \lambda_0(\kappa)$ is as in Fig. 3.2.

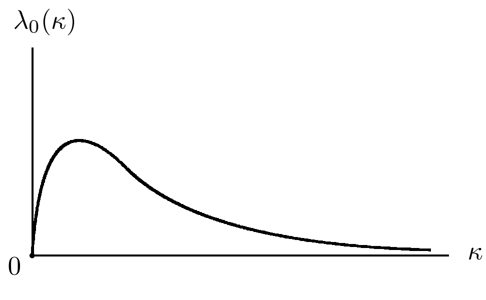

Figure 3.2: Qualitative behavior of $\kappa \mapsto \lambda_0(\kappa)$.

4. Our proof of Theorem 3.1.1 is based on a *multiscale analysis* of ξ, in the spirit of [KS03] and consists of two major steps:

(I) We look at the bad R-blocks for all $R \in \mathbb{N}$ and show that bad R-blocks are rare for large R. Since these blocks are located randomly in space-time, it is a non-trivial task to control the time the random walk spends inside them. Therefore we search for the optimal set in space-time, with the same space-time volume as the union of the bad R-blocks, that maximizes the expected time the random walk spends inside. For that we make use of *discrete rearrangement inequalities* for local times of simple random walk. These show that the contribution to the expectation in (3.1.7) coming from bad R-blocks increases when we move these blocks towards the origin. Therefore this contribution can be bounded from above by an expectation that pretends the bad R-blocks to be rearranged in a *deterministic* space-time cylinder around the origin. Since bad R-blocks are rare, this cylinder is narrow. Afterwards, because simple random walk is unlikely to spend a lot of time in a narrow space-time cylinder, we are able to control the contribution coming from bad R-blocks to the expectation (3.1.7) uniformly in t and κ.

(II) We look at the good R-blocks for all $R \in \mathbb{N}$. We control their contribution by using an *eigenvalue expansion* of (3.1.7). An analysis of the largest eigenvalue in this expansion concludes the argument.

5. A related model is that of directed polymers in random environment. Here, time is discrete and the random environment $\xi = \{\xi(x,n)\colon\ x \in \mathbb{Z}^d, n \in \mathbb{N}_0\}$ is i.i.d. in space and time. Thus, at every unit of time ξ is updated in space in an i.i.d. manner, so that this choice of ξ satisfies a discrete-time version of the Gärtner-hyper-mixing assumption. A choice of ξ in continuous time that is similar in spirit is $\xi = \dot{W}$, where $\dot{W}$ is space-time white noise. This model was studied in [CM94], [GdH07] and it was conjectured that all Lyapunov exponents merge as $\kappa \to \infty$. Note that $\dot{W}$ is not a function, so that this model does not fall into the class of models considered in the present paper, and Theorem 3.1.1 does not apply directly. However, due to the space-time independence of $\dot{W}$, we may expect that a suitable notion of Gärtner-hyper-mixing exists for $\dot{W}$ and that similar techniques work.

The remainder of this paper is organized as follows. In Section 3.2 we formulate three key propositions and use these to prove Theorem 3.1.1. The three propositions are proved in Sections 3.3–3.7, respectively. In Appendix 3.8 we prove two technical lemmas that are needed in Section 3.4, while in Appendix 3.9 we prove a spectral bound that is needed in Section 3.7.

3.2 Three key propositions and proof of Theorem 3.1.1

To state our three key propositions we need some definitions. Fix $k_* \in \mathbb{N}$, and $t > 0$. We say that $\Phi\colon\ [0,t] \to \mathbb{Z}^d$ is a path when it is right-continuous and

$$\|\Phi(s) - \Phi(s-)\| \leq 1 \qquad \forall\, s \in [0,t]. \tag{3.2.1}$$

Define the set of paths

$$\Pi(k_*, t, A) = \big\{\Phi\colon\ [0,t] \to \mathbb{Z}^d\colon\ \Phi \text{ crosses } k_* \text{ 1-blocks}\big\}. \tag{3.2.2}$$

For $\Lambda \subseteq \mathbb{Z}^d \times [0,\infty)$, let $\Pi_1(\Lambda)$ denote the projection of Λ onto its first spatial coordinate, i.e.,

$$\Pi_1(\Lambda) = \{(x_1, s) \in \mathbb{Z} \times [0,\infty)\colon \text{ there is } y \in \mathbb{Z}^d \text{ such that } y_1 = x_1 \text{ and } (y,s) \in \Lambda\}. \tag{3.2.3}$$

When Λ can be written as $\Lambda = \widetilde{\Lambda} \times \mathcal{I}$, where $\widetilde{\Lambda} \subseteq \mathbb{Z}^d$ and $\mathcal{I} \subseteq [0,\infty)$, then for $y \in \mathbb{Z}$ we sometimes write $y \in \Pi_1(\Lambda)$, when there is a $s \in \mathcal{I}$ such that $(y,s) \in \Pi_1(\Lambda)$. Moreover, we denote by $\Pi_1(X^\kappa)(s)$ the first coordinate of the random walk X^κ at time s. For $\Lambda \subseteq \mathbb{Z}^d \times [0,t]$ we denote $\Lambda(s) = \Lambda \cap (\mathbb{Z}^d \times \{s\})$ and let $l_t(\Lambda)$ be the local time of X^κ in Λ up to time t, i.e.,

$$l_t(\Lambda) = \int_0^t 1\!\!1\{X^\kappa(s) \in \Lambda(s)\}\, ds. \tag{3.2.4}$$

In a similar fashion we let $l_t(\Pi_1(\Lambda))$ be the local time of $\Pi_1(X^\kappa)$ in $\Pi_1(\Lambda)$ up to time t. Furthermore, we let $l_t(\mathrm{BAD}^\delta_R(\xi_K))$ and $l_t(\mathrm{BAD}^\delta(\xi))$ denote the local time of X^κ in (δ,b,c)-bad R-blocks up to time t for the potential ξ_K and in (δ,b,c)-bad 1-blocks up to time t for the potential ξ, respectively. Here and in the rest of the paper a bad R-block is (δ,b,c)-bad for a choice of K,δ,b,c and some $A \geq A_0$, $m \geq m_0$, according to Definition 3.1.5.

In what follows, when we write sums like $\sum_{0\leq k<t/A^R}$ or $\sum_{R=1}^{\varepsilon \log t}$ we will pretend that t/A^R and $\varepsilon \log t$ are integer in order not to burden the notation with round off brackets. From the context it will always be clear where to place the brackets.

Proposition 3.2.1. *There is a $C_2 > 0$ such that for every $\varepsilon > 0$ and $\delta > 0$ there is an $A = A(\varepsilon,\delta) > 3$, satisfying $\lim_{\varepsilon\downarrow 0} A(\varepsilon,\delta) = \infty$, such that ξ-a.s. for all $\kappa > 0$ and all $t > 0$ large enough,*

$$\begin{aligned} E_0\bigg(\exp\bigg\{\int_0^t \xi(X^\kappa(s), t-s)\, ds\bigg\}\bigg) &\leq e^{-t} + E_0\bigg(\exp\bigg\{\int_0^t \overline{\xi}(X^\kappa(s), t-s)\, ds\bigg\}\bigg)^{1/2} \\ &\quad \times E_0\Bigg(\exp\Bigg\{2\delta A^d l_t(\mathrm{BAD}^\delta(\xi)) + 2\sum_{R=1}^{\varepsilon \log t} \delta A^{(R+1)d} l_t\big(\mathrm{BAD}^\delta_R(\xi_K)\big)\Bigg\} \\ &\quad \times 1\!\!1\big\{\exists\, k_* \leq C_2\kappa t\colon\ X^\kappa \in \Pi(k_*, t, A)\big\}\Bigg)^{1/2}, \end{aligned} \tag{3.2.5}$$

where

$$\overline{\xi}(x,s) = 2\xi(x,s) 1\!\!1\big\{\xi(x,s) < \delta A^d, (x,s) \textit{ is in a good } 1\textit{-block of } \xi\big\}. \tag{3.2.6}$$

Proposition 3.2.2. *There is a $C_2 > 0$ such that for every $\varepsilon, \widetilde{\varepsilon} > 0$ and $\delta > 0$ there is an $A = A(\varepsilon, \widetilde{\varepsilon}, \delta) > 3$, satisfying $\lim_{\widetilde{\varepsilon}\downarrow 0} A(\varepsilon, \widetilde{\varepsilon}, \delta) = \infty$, such that ξ-a.s. for all $\kappa > 0$ and* *all $t > 0$ large enough,*

$$E_0\Bigg(\exp\Bigg\{ 2\delta A^d l_t(\mathrm{BAD}^\delta(\xi)) + 2\sum_{R=1}^{\varepsilon \log t} \delta A^{(R+1)d} l_t\big(\mathrm{BAD}_R^\delta(\xi_K)\big) \Bigg\} \\ \times 1\!\!1\big\{\exists\, k_* \leq C_2\kappa t\colon\ X^\kappa \in \Pi(k_*, t, A)\big\} \Bigg) \leq e^{\widetilde{\varepsilon} t}. \tag{3.2.7}$$

Proposition 3.2.3. *There is a constant $C_3 > 0$ such that, for every $A > 1$ and $\delta > 0$,*

$$\limsup_{\kappa\to\infty} \limsup_{n\to\infty} \frac{1}{An} \log E_0\Bigg(\exp\Bigg\{ \int_0^{An} \overline{\xi}(X^\kappa(s), An - s)\, ds \Bigg\} \Bigg) \leq \frac{C_3}{A} + 5\delta \qquad \xi - a.s. \tag{3.2.8}$$

Proposition 3.2.1 estimates the Feynman-Kac formula in (3.1.7) in terms of bad blocks and good blocks, Proposition 3.2.2 controls the contribution of bad block, while Proposition 3.2.3 controls the contribution of good blocks.

We are now ready to prove Theorem 3.1.1.

Proof. Note that by Theorem 1.2(i) in [GdHM12], for all $\kappa \geq 0$ we have the lower bound $\lambda_0(\kappa) \geq 0$. Thus, it suffices to show the inequality in the reverse direction. To that end, note that for any $A > 0$

$$\lambda_0(\kappa) \leq \limsup_{n\to\infty} \frac{1}{An} \log u(0, An). \tag{3.2.9}$$

Indeed, when $t \in [An, A(n+1))$, $n \in \mathbb{N}$, then inserting the indicator on the event $\{X^\kappa(s) = 0 \text{ for all } s \in [0, A(n+1) - t]\}$ in (3.1.7) and an application of the Markov property at time $A(n+1) - t$ yield that

$$\begin{aligned} &u(0,t) \\ &\leq P_0(X^\kappa(s) = 0 \,\text{for all}\ s \in [0, A(n+1) - t])^{-1} \times e^{-\int_t^{A(n+1)} \xi(0,s)\, ds} \times u(0, A(n+1)) \\ &\leq e^{2d\kappa(A(n+1)-t)} \times e^{-\int_t^{A(n+1)} \xi(0,s)\, ds} \times u(0, A(n+1)). \end{aligned} \tag{3.2.10}$$

Moreover, by the time ergodicity of ξ in time it can be shown that

$$\lim_{t\to\infty} \frac{1}{t} \int_t^{A(n+1)} \xi(0,s)\, ds = 0, \tag{3.2.11}$$

so that (3.2.9) can be deduced from (3.2.10) (using also that $t \in [An, A(n+1))$), (3.2.11) and the fact that $(A(n+1) - t)/t \to 0$ as $t \to \infty$. Hence, it is sufficient to estimate $u(0, An)$, $n \in \mathbb{N}$. To continue, fix $C_2, C_3 > 0$ according to Propositions 3.2.1–3.2.3, and fix $\varepsilon, \widetilde{\varepsilon}, \delta > 0$. According to Proposition 3.2.2, there is an $A = A(\varepsilon, \widetilde{\varepsilon}, \delta)$ such that, ξ-a.s. for all $\kappa > 0$ and all t of the form $t = An$ with $n \in \mathbb{N}$ large enough, the term in the left-hand side of (3.2.7) is bounded from above by $e^{\widetilde{\varepsilon}An}$. According to Proposition 3.2.3, we have

$$E_0\left(\exp\left\{\int_0^{An}\overline{\xi}(X^\kappa(s),An-s)\,ds\right\}\right)\le e^{5\delta An+C_3n+\chi(\kappa,n)} \tag{3.2.12}$$

with $\limsup_{\kappa\to\infty}\limsup_{n\to\infty}\chi(\kappa,n)/n=0$. Proposition 3.2.1 therefore yields that, for all $\varepsilon,\widetilde{\varepsilon},\delta>0$,

$$\limsup_{\kappa\to\infty}\lambda_0(\kappa)\le\frac{C_3}{2A}+\frac{5}{2}\delta+\frac{\widetilde{\varepsilon}}{2}. \tag{3.2.13}$$

Since $\lim_{\widetilde{\varepsilon}\downarrow 0}A(\varepsilon,\widetilde{\varepsilon},\delta)=\infty$ by Proposition 3.2.2, we get that for all $\delta>0$,

$$\limsup_{\kappa\to\infty}\lambda_0(\kappa)\le\frac{5}{2}\delta. \tag{3.2.14}$$

Let $\delta\downarrow 0$ to get the claim. ∎

3.3 Proof of Proposition 3.2.1

The proof is given in Section 3.3.1 subject to Lemmas 3.3.1–3.3.2 below. The proof of these lemmas is given in Section 3.3.2.

3.3.1 Proof of Proposition 3.2.1 subject to two lemmas

Recall (3.2.2). For $A\ge 1$, $R\in\mathbb{N}$ and $\Phi\in\Pi(k_*,t,A)$, define

$$\begin{aligned}\Xi_R^A(\Phi)&=\text{number of bad }R\text{-blocks crossed by }\Phi,\\ \Xi_R^{A,k_*}&=\sup_{\Phi\in\Pi(k_*,t,A)}\Xi_R^A(\Phi).\end{aligned} \tag{3.3.1}$$

Lemma 3.3.1. *For every $\varepsilon>0$ there is an $A=A(\varepsilon)>3$ satisfying $\lim_{\varepsilon\downarrow 0}A(\varepsilon)=\infty$ such that*

$$\mathbb{P}\Big(\Xi_R^{A,k_*}>0 \text{ for some } R\ge\varepsilon\log t \text{ and some } k_*\in\mathbb{N}\Big) \tag{3.3.2}$$

is summable over $t\in\mathbb{N}$. A possible choice is $A=e^{1/a\varepsilon[2d(2d+1)+1]}$ for some $a>1$.

Lemma 3.3.2. *There is a $C_2>0$ such that ξ-a.s. for all $A>1$, all $t>0$ and all $\kappa>0$ large enough,*

$$E_0\left(\exp\left\{\int_0^t\xi(X^\kappa(s),t-s)\,ds\right\}\mathbb{1}\big\{\exists\,k_*>C_2\kappa t\colon\ X^\kappa\in\Pi(k_*,t,A)\big\}\right)\le e^{-t}. \tag{3.3.3}$$

We are now ready to prove Proposition 3.2.1.

Proof. Fix C_2 in accordance with Lemma 3.3.2 and $\varepsilon > 0$. Let $\delta > 0$ and fix $A > 1$ according to Lemma 3.3.1 such that $\delta A^d \geq K$, see Definition 3.1.5. Note that

$$\begin{aligned}
&E_0\left(\exp\left\{\int_0^t \xi(X^\kappa(s), t-s)\, ds\right\} 1\!\!1\big\{\exists\, k_* \leq C_2\kappa t\colon\ X^\kappa \in \Pi(k_*, t, A)\big\}\right)\\
&= E_0\Bigg(\exp\left\{\sum_{i=1}^{N(X^\kappa,t)} \int_{s_{i-1}}^{s_i} \xi(x_{i-1}, t-u)\, du + \int_{s_{N(X^\kappa,t)}}^{t} \xi(x_{N(X^\kappa,t)}, t-u)\, du\right\}\\
&\qquad \times 1\!\!1\big\{\exists\, k_* \leq C_2\kappa t\colon\ X^\kappa \in \Pi(k_*, t, A)\big\}\Bigg),
\end{aligned} \tag{3.3.4}$$

where $N(X^\kappa, t)$ is the number of jumps by X^κ up to time t, $0 = x_0, x_1, \ldots, x_{N(X^\kappa,t)}$ are the nearest-neighbor sites visited, and $0 = s_0 < s_1 < \cdots < s_{N(X^\kappa,t)} \leq t$ are the jump times. To analyze (3.3.4), define

$$\begin{aligned}
\Lambda_t(\mathrm{BAD}_R^\delta) = &\bigcup_{i=1}^{N(X^\kappa,t)} \Big\{u \in [s_{i-1}, s_i)\colon\ \delta A^{Rd} < \xi(x_{i-1}, t-u) \leq \delta A^{(R+1)d}\Big\}\\
&\bigcup \Big\{u \in [s_{N(X^\kappa,t)}, t)\colon\ \delta A^{Rd} < \xi(x_{N(X^\kappa,t)}, t-u) \leq \delta A^{(R+1)d}\Big\}.
\end{aligned} \tag{3.3.5}$$

Then the contribution to the exponential in (3.3.4) may be bounded from above by

$$\int_0^t \xi(X^\kappa(s), t-s) 1\!\!1\{\xi(X^\kappa(s), t-s) < \delta A^d\}\, ds + \sum_{R\in\mathbb{N}} \delta A^{(R+1)d} \big|\Lambda_t\big(\mathrm{BAD}_R^\delta\big)\big|. \tag{3.3.6}$$

By Definition 3.1.3 and the fact that $\delta A^d \geq K$ (see the line preceding (3.3.4)), if $\delta A^{Rd} < \xi(x_{i-1}, t-u) \leq \delta A^{(R+1)d}$, then $(x_{i-1}, t-u)$ belongs to a bad R-block for the potential ξ_K. Hence

$$\big|\Lambda_t\big(\mathrm{BAD}_R^\delta\big)\big| \leq l_t\big(\mathrm{BAD}_R^\delta(\xi_K)\big). \tag{3.3.7}$$

To continue we write the indicator in (3.3.5)

$$\begin{aligned}
&1\!\!1\{\xi(X^\kappa(s), t-s) < \delta A^d, (X^\kappa(s), t-s) \text{ is in a good 1-block of } \xi\}\\
&+ 1\!\!1\{\xi(X^\kappa(s), t-s) < \delta A^d, (X^\kappa(s), t-s) \text{ is in a bad 1-block of } \xi\}.
\end{aligned} \tag{3.3.8}$$

By Lemma 3.3.1 and our choice of A at the beginning of the proof, ξ-a.s. for t large enough there are no bad R-blocks with $R > \varepsilon \log t$. Thus, the expectation in the right-hand side of (3.3.4) may be estimated from above by

$$
\begin{aligned}
&E_0\Bigg(\exp\bigg\{ \int_0^t \xi(X^\kappa(s), t-s) \\
&\quad \times \mathbb{1}\big\{\xi(X^\kappa(s), t-s) < \delta A^d, (X^\kappa(s), t-s) \text{ is in a good 1-block of } \xi\big\}\, ds\bigg\} \\
&\quad \times \exp\bigg\{\delta A^d l_t(\mathrm{BAD}^\delta(\xi)) + \sum_{R=1}^{\varepsilon \log t} \delta A^{(R+1)d} l_t(\mathrm{BAD}_R^\delta(\xi_K))\bigg\} \\
&\quad \times \mathbb{1}\big\{\exists\, k_* \leq C_2 \kappa t\colon\; X^\kappa \in \Pi(k_*, t, A)\big\}\Bigg).
\end{aligned}
\tag{3.3.9}
$$

Recall (3.2.6). An application of the Cauchy-Schwarz inequality yields the following upper bound for (3.3.9):

$$
\begin{aligned}
&E_0\bigg(\exp\bigg\{ \int_0^t \overline{\xi}(X^\kappa(s), t-s)\, ds\bigg\}\bigg)^{1/2} \\
&\times E_0\Bigg(\exp\bigg\{2\delta A^d l_t(\mathrm{BAD}^\delta(\xi)) + 2\sum_{R=1}^{\varepsilon \log t} \delta A^{(R+1)d} l_t(\mathrm{BAD}_R^\delta(\xi_K))\bigg\} \\
&\qquad\qquad \times \mathbb{1}\big\{\exists\, k_* \leq C_2 \kappa t\colon\; X^\kappa \in \Pi(k_*, t, A)\big\}\Bigg)^{1/2}.
\end{aligned}
\tag{3.3.10}
$$

The claim in (3.2.5) therefore follows by combining (3.3.4), (3.3.6–3.3.7) and (3.3.9)–(3.3.10) with Lemma 3.3.2. ∎

3.3.2 Proof of Lemmas 3.3.1–3.3.2

Proof. For the proof of Lemma 3.3.1, see Lemma 2.3.3 or [EdHM14a, Lemma 3.3]. To prove Lemma 3.3.2, use Cauchy-Schwarz to estimate the expectation in (3.3.3) from above by

$$
\bigg[E_0\bigg(\exp\bigg\{ 2\int_0^t \xi(X^\kappa(s), t-s)\, ds\bigg\}\bigg)\bigg]^{1/2} \bigg[P_0\Big(\exists\, k_* > C_2 \kappa t\colon\; X^\kappa \in \Pi(k_*, t, A)\Big)\bigg]^{1/2}.
\tag{3.3.11}
$$

To bound the first term in (3.3.11), note that by [EdHM14a, Eq.(3.54)] there is a $C > 0$ such that ξ-a.s. for all $t, \kappa > 0$,

$$
E_0\Big(e^{2\int_0^t \xi(X^\kappa(s), t-s)\, ds}\Big) \leq e^{tC(\kappa+1)}.
\tag{3.3.12}
$$

To bound the second term in (3.3.11) we use a similar strategy as for the proof of Lemma 3.4.4. Given $l_1, \ldots, l_{t/A} \in \mathbb{N}$, we say that X^κ has label $(l_1, \ldots, l_{t/A})$ when X^κ

crosses l_i 1-blocks in the time interval $[(i-1)A, iA)$, $i \in \{1, \ldots, t/A\}$. Fix $C_2 > 0$ and write

$$\begin{aligned} &P_0\Big(X^\kappa \in \Pi(k_*, t, A) \text{ for some } k_* > C_2\kappa t\Big) \\ &= \sum_{j=1}^{\infty} P_0\Big(X^\kappa \in \Pi(k_*, t, A) \text{ for some } k_* \in (jC_2\kappa t, (j+1)C_2\kappa t]\Big). \end{aligned} \tag{3.3.13}$$

For $j \in \mathbb{N}$, write $\sum_{(l_1^j, \ldots, l_{t/A}^j)}$ to denote the sum over all sequences $(l_1^j, \ldots, l_{t/A}^j) \in \mathbb{N}^{t/A}$ with $jC_2\kappa t < \sum_{i=1}^{t/A} l_i^j \le (j+1)C_2\kappa t$. Then each summand in (3.3.13) may, by an application of the Markov property, be rewritten as

$$\sum_{(l_1^j, \ldots, l_{t/A}^j)} E_0\Big(1\!\!1\{X^\kappa \text{ has label } (l_1^j, \ldots, l_{t/A-1}^j)\} P_{X^\kappa(t-A)}\big(X^\kappa \text{ has label } l_{t/A}^j\big)\Big). \tag{3.3.14}$$

Note that the number of jumps of a path Φ that visits l_i^j 1-blocks is at least $(l_i^j/2^d - 1)A$. This is because for each 1-block there are $(2^d - 1)$ 1-blocks with the same time coordinate at l^∞-distance one. Hence, we may estimate (3.3.14) from above by

$$\sum_{(l_1^j, \ldots, l_{t/A}^j)} E_0\Big(1\!\!1\{X^\kappa \text{ has label } (l_1^j, \ldots, l_{t/A-1}^j)\}\Big) P_0\Big(N(X^\kappa, A) \ge (l_i^j/2^d - 1)A\Big), \tag{3.3.15}$$

where $N(X^\kappa, A)$ denotes the number of jumps of X^κ in the time interval $[0, A)$. An iteration of the arguments in (3.3.14–3.3.15), together with the tail estimate $P(\text{POISSON}(\lambda) \ge k) \le e^{-\lambda}(\lambda e)^k/k^k$, $k > 2\lambda + 1$, for Poisson random variables with mean λ, yields that for $C' > 0$ large enough each summand in (3.3.13) is bounded from above by

$$\begin{aligned} &\sum_{(l_1^j, \ldots, l_{t/A}^j)} \prod_{i : l_i^j \ge \kappa C'} P_0\Big(N(X^\kappa, A) \ge (l_i^j/2^d - 1)A\Big) \\ &\le \sum_{(l_1^j, \ldots, l_{t/A}^j)} \prod_{i : l_i^j \ge \kappa C'} e^{-A2d\kappa} \exp\Big\{-(l_i^j/2^d - 2)A \log([\kappa C'/2^d - 2]/2d\kappa e)\Big\}. \end{aligned} \tag{3.3.16}$$

(It suffices to pick C' such that $(C'/2^d - 2)A \ge 4eAd\kappa + 1$, which for $A > 1$ and $\kappa > 1$ is fulfilled when $C' \ge 2^d(4de + 3)$.) Now note that if $C_2 > 2C'$, then for all $j \in \mathbb{N}$,

$$\sum_{i : l_i^j \ge \kappa C'} l_i^j \ge \frac{jtC_2\kappa}{2}. \tag{3.3.17}$$

Hence, inserting (3.3.17) into (3.3.16), choosing C_2 large enough, and using the fact that for some $a, b \in (0, \infty)$ there are no more than $ae^{b\sqrt{C_2\kappa t}}$ such sequences $(l_1^j, \ldots, l_{t/A}^j)$ (see [HR18] or [E42]), we get that for some $C'' > 0$ the left-hand side of (3.3.13) is bounded from above by $e^{-C''\kappa t}$. Inserting this bound into (3.3.11), using that $C'' \to \infty$ as $C_2 \to \infty$, and using (3.3.12), we get the claim. ∎

3.4 Proof of Proposition 3.2.2

Proposition 3.2.2 is proved in Section 3.4.2 subject to Propositions 3.4.1–3.4.2 below, which are stated in Section 3.4.1 and proved in Sections 3.5–3.6.

3.4.1 Two more propositions

Endow $\mathbb{Z}$ with the ordering $0 \prec 1 \prec -1 \prec 2 \prec -2 \prec 3 \prec \cdots$. We say that two functions $f, g\colon \mathbb{Z} \to \mathbb{R}$ are equimeasurable when

$$|\{x \in \mathbb{Z}\colon\ f(x) > \lambda\}| = |\{x \in \mathbb{Z}\colon\ g(x) > \lambda\}| \qquad \forall\, \lambda \geq 0. \tag{3.4.1}$$

The *symmetric decreasing rearrangement* of a function $f\colon \mathbb{Z} \to \mathbb{R}$ is defined to be the unique non-increasing function $f^\sharp\colon \mathbb{Z} \to \mathbb{R}$ that is equimeasurable with f. Given $A \subseteq \mathbb{Z}$, $A^\sharp \subseteq \mathbb{Z}$ is defined to be the unique set such that $(\mathbb{1}_A)^\sharp = \mathbb{1}_{A^\sharp}$.

Recall the definition of Π_1 in (3.2.3). The one-dimensional symmetric decreasing rearrangement of $B \subseteq \mathbb{Z}^d \times [0,\infty)$ is the set

$$\Pi_1(B)^\sharp = \bigcup_{s \in [0,\infty)} \Big(\big\{x \in \mathbb{Z}\colon\ (x,s) \in \Pi_1(B)\big\}^\sharp \times \{s\} \Big). \tag{3.4.2}$$

For $A \geq 1$ and $R \in \mathbb{N}$, an *R-interval* is a time-interval of the form $[kA^R, (k+1)A^R)$, $0 \leq k < t/A^R$. To make the proof more accessible, we no longer distinguish between badness referring to ξ_K (where ξ_K was defined below (3.2.4)) and badness referring to ξ. Since both potentials satisfy the same mixing assumption (a1) it will be clear from the proof that this does not affect the result.

Proposition 3.4.1. *Let $\Phi \in \Pi(k_*, t, A)$. Then for all A large enough there is a sequence $(\delta_R)_{R\in\mathbb{N}}$ in $(0,\infty)$ satisfying*

$$\sum_{R \in \mathbb{N}} A^{Rd} \sqrt{\delta_R} < \infty \tag{3.4.3}$$

such that ξ-a.s. the number of R-intervals in which Φ crosses more than $\delta_R k_*/(t/A)$ bad R-blocks is bounded from above by $\sqrt{\delta_R}\, t/A^R$. A possible choice is $\delta_R = K_1 A^{-8d^2/3} A^{-4d(2d+1)R/3}$ for some $K_1 > 0$ not depending on A and R.

Proposition 3.4.2. *For every $\varepsilon, t > 0$, every sequence $(B_R)_{R\in\mathbb{N}}$ in $\mathbb{Z}^d \times [0,t]$ and every sequence $(C_R)_{R\in\mathbb{N}}$ in $[0,\infty)$ (see Fig. 3.3),*

$$E_0\left(\exp\left\{ \sum_{R=1}^{\varepsilon \log t} C_R l_t(B_R) \right\} \right) \leq E_0\left(\exp\left\{ \sum_{R=1}^{\varepsilon \log t} C_R l_t(\Pi_1(B_R)^\sharp) \right\} \right). \tag{3.4.4}$$

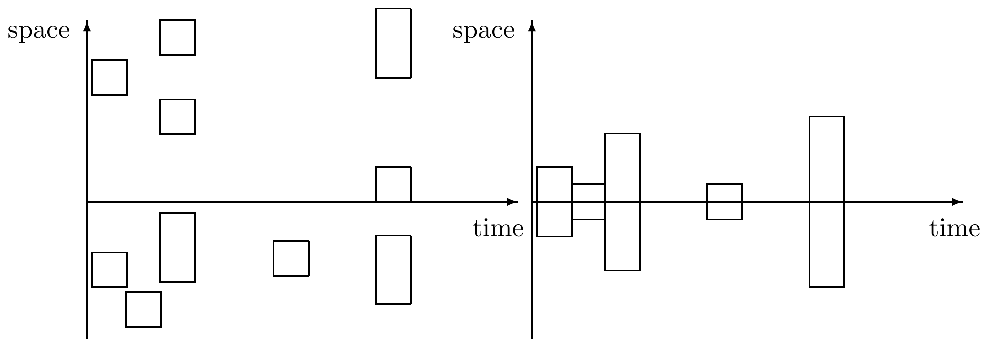

Figure 3.3: The picture on the left shows a configuration of space-time blocks before its rearrangement, the picture on the right after its rearrangement. Note that in each time-interval the total space volume of the blocks is the same in both configurations.

3.4.2 Proof of Proposition 3.2.2 subject to two propositions

Proof. Fix $\varepsilon > 0$ and $A \geq 1$ according to Propositions 3.2.1 and 3.4.1, and fix $\widetilde{\varepsilon} > 0$. The proof comes in 6 steps.

1. We begin by introducing some more notation. Define the space-time blocks

$$B_R^A(x,k;\kappa) = \left(\prod_{j=1}^{d} [\sqrt{\kappa}(x(j)-1)A^R, \sqrt{\kappa}(x(j)+1)A^R) \cap \mathbb{Z}^d \right) \times [kA^R, (k+1)A^R), \quad (3.4.5)$$

which we call (κ, R)-blocks. These blocks are the same as $\widetilde{B}_R^{A,\alpha}(x,k;0,0) = B_R^{A,\alpha}(x,k)$ in (3.1.13) with $\alpha = \sqrt{\kappa}$. For $k_* \in \mathbb{N}$ and $(x_i,k_i)_{0 \leq i < k_*} \in (\mathbb{Z}^d \times \mathbb{N})^{k_*}$ define

$$\begin{aligned} &\mathrm{BAD}_R^\delta((x_i,k_i)_{0\leq i<k_*}) \\ &\quad = \Big\{ B_R^A(x,k)\colon\ B_R^A(x,k) \text{ is bad and there is a } 0 \leq i < k_* \text{ such that} \\ &\qquad\qquad B_R^A(x,k) \text{ intersects } B_1^A(x_i,k_i;\kappa) \Big\}. \end{aligned} \quad (3.4.6)$$

2. We write $l_t(\mathrm{BAD}_R^\delta)$ for the local time of X^κ in (δ,b,c)-bad R-blocks up to time t, where badness refers either to ξ_K or ξ. By (3.2.7), it is enough to show that for all κ and t large enough,

$$E_0\left(\exp\left\{ 4 \sum_{R=1}^{\varepsilon \log t} \delta A^{(R+1)d} l_t\big(\mathrm{BAD}_R^\delta\big) \right\} 1\!\!1\big\{\exists\, k_* \leq C_2\kappa t\colon\ X^\kappa \in \Pi(k_*,t,A)\big\} \right) \leq e^{\widetilde{\varepsilon} t}. \quad (3.4.7)$$

Recall (3.2.2) to note that the left-hand side of (3.4.7) equals

$$\sum_{k_*=t/A}^{C_2\kappa t} E_0\left(\exp\left\{4\sum_{R=1}^{\varepsilon\log t}\delta A^{(R+1)d} l_t\big(\mathrm{BAD}_R^\delta\big)\right\}1\!\!1\big\{X^\kappa \text{ crosses } k_* \text{ 1-blocks}\big\}\right). \tag{3.4.8}$$

To prove (3.4.7), we attempt to apply Proposition 3.4.2. To that end, for each k_* we must sum over all configurations of k_* 1-blocks that may be crossed by X^κ. However, this sum is difficult to control, and therefore we do an additional *coarse-graining of space-time* by considering (κ, R)-blocks instead of R-blocks. To that end we first note that there is a $C_4 > 0$ such that if X^κ crosses k_* 1-blocks then it crosses at most $C_4 k_*/\sqrt{\kappa} + 2t/A$ $(\kappa, 1)$-blocks (see Lemma 3.5.6 in Section 3.5.4 for a similar statement). To see why, note that if X^κ crosses $l_i \le \sqrt{\kappa}$ 1-blocks in the time-interval $[(i-1)A, iA)$, $1 \le i \le t/A$, then it crosses $l_i^\kappa \le 2$ $(\kappa, 1)$-blocks in the same time-interval. Moreover, if $j\sqrt{\kappa}+1 \le l_i \le (j+1)\sqrt{\kappa}$ for some $j \in \mathbb{N}$, then $l_i^\kappa \le j+2 \le (j+2)l_i/j\sqrt{\kappa}$. Hence, the total number of $(\kappa, 1)$-blocks that may be crossed by X^κ is bounded from above by

$$\sum_{i=1}^{t/A} l_i^\kappa \le \sum_{\substack{1\le i\le t/A\\ l_i\le\sqrt{\kappa}}} 2 + \sum_{\substack{1\le i\le t/A\\ l_i\ge\sqrt{\kappa}+1}} \frac{3l_i}{\sqrt{\kappa}} \le 2t/A + \frac{3k_*}{\sqrt{\kappa}}. \tag{3.4.9}$$

Thus, (3.4.8) is bounded from above by

$$\sum_{k_*=t/A}^{C_2C_4\sqrt{\kappa}t+2t/A} E_0\left(\exp\left\{4\sum_{R=1}^{\varepsilon\log t}\delta A^{(R+1)d} l_t\big(\mathrm{BAD}_R^\delta\big)\right\}1\!\!1\big\{X^\kappa \text{ crosses } k_*\ (\kappa,1)\text{-blocks}\big\}\right). \tag{3.4.10}$$

To analyze (3.4.10), we fix $k_* \in [t/A, C_2C_4\sqrt{\kappa}t + 2t/A]$ and we write

$$\begin{aligned} &1\!\!1\big\{X^\kappa \text{ crosses } k_*\ (\kappa,1)\text{-blocks}\big\}\\ &\quad= \sum_{(B_1^A(x_i,k_i;\kappa))_{0\le i<k_*}} 1\!\!1\big\{X^\kappa \text{ crosses } B_1^A(x_i,k_i;\kappa), 0\le i<k_*\big\}. \end{aligned} \tag{3.4.11}$$

Here, the symbol $\sum_{(B_1^A(x_i,k_i;\kappa))_{0\le i<k_*}}$ denotes the sum over all sequences of blocks $(B_1^A(x_i,k_i;\kappa))_{0\le i<k_*}$, which may be crossed by a realization of X^κ. In particular, in (3.4.11), the sum is taken over a subset of sequences of blocks, which have spatial distance one (since X^κ is a nearest-neighbor random walk it follows that two consecutive blocks, which are visited by X^κ, necessarily have spatial distance one) and are such that the events in the indicators on the right hand side of (3.4.11) are disjoint. Recalling (3.4.6), we may estimate each summand in (3.4.10) by

$$\begin{aligned} \sum_{(B_1^A(x_i,k_i;\kappa))_{0\le i<k_*}} E_0\Bigg(&\exp\left\{4\sum_{R=1}^{\varepsilon\log t}\delta A^{(R+1)d} l_t\Big(\mathrm{BAD}_R^\delta\big((x_i,k_i)_{0\le i<k_*}\big)\Big)\right\}\\ &\times 1\!\!1\big\{X^\kappa \text{ crosses } B_1^A(x_i,k_i;\kappa), 0\le i<k_*\big\}\Bigg). \end{aligned} \tag{3.4.12}$$

By Cauchy-Schwarz, (3.4.12) is at most

$$\sum_{(B_1^A(x_i,k_i;\kappa))_{0\le i<k_*}} \left[E_0\left(\exp\left\{8\sum_{R=1}^{\varepsilon\log t}\delta A^{(R+1)d} l_t\Big(\mathrm{BAD}_R^\delta\big((x_i,k_i)_{0\le i<k_*}\big)\Big)\right\}\right)\right]^{1/2} \times\left[P_0\Big(X^\kappa \text{ crosses } B_1^A(x_i,k_i,\kappa), 0\le i<k_*\Big)\right]^{1/2}. \tag{3.4.13}$$

3. By Proposition 3.4.2, the first factor in the summand of (3.4.13) is not more than

$$\left[E_0\left(\exp\left\{8\sum_{R=1}^{\varepsilon\log t}\delta A^{(R+1)d} l_t\Big(\Pi_1\big(\mathrm{BAD}_R^\delta((x_i,k_i)_{0\le i<k_*})\big)^\sharp\Big)\right\}\right)\right]^{1/2}. \tag{3.4.14}$$

Next, if X^κ crosses k_* $(\kappa,1)$-blocks $B_1^A(x,k;\kappa)$, then a trivial counting estimate yields that X^κ crosses at most $k_*\sqrt{\kappa}$ 1-blocks. Therefore, by Proposition 3.4.1, the number of R-intervals in which X^κ crosses more than $\delta_R k_*\sqrt{\kappa}A/t$ bad R-blocks is bounded from above by $\sqrt{\delta_R}t/A^R$. We call these R-intervals R-atypical. Similarly, an R-interval is called R-typical, if the number of bad R-blocks crossed by X^κ is bounded by $\delta_R k_*\sqrt{\kappa}A/t$. Define

$$R^*(k_*) = \max\big\{R\in\mathbb{N}\colon\ \delta_R k_*\sqrt{\kappa}A/t\ge 1\big\}. \tag{3.4.15}$$

If $R>R^*(k_*)$, then there are no bad R-blocks in R-typical intervals. (By the choice of R, their number is strictly less than one and therefore is zero.) Hence the local time in bad R-blocks is determined by the local time in bad R-blocks, which lie in R-atypical intervals. Consequently,

$$l_t\Big(\Pi_1\big(\mathrm{BAD}_R^\delta((x_i,k_i)_{0\le i<k_*})\big)^\sharp\Big)\le(\sqrt{\delta_R}t/A^R)A^R=\sqrt{\delta_R}t. \tag{3.4.16}$$

On the other hand, if $1\le R\le R^*(k_*)$ (see Fig. 3.4), then there is a contribution coming from R-typical intervals as well, and so

$$l_t\Big(\Pi_1\big(\mathrm{BAD}_R^\delta((x_i,k_i)_{0\le i<k_*})\big)^\sharp\Big)\le\sqrt{\delta_R}t+l_t(\widetilde{B}_R(k_*)), \tag{3.4.17}$$

where

$$\widetilde{B}_R(k_*)=\Big(\Big[-\tfrac12 A^R\delta_R k_*\sqrt{\kappa}A/t,\tfrac12 A^R\delta_R k_*\sqrt{\kappa}A/t\Big)\cap\mathbb{Z}\Big)\times[0,t]. \tag{3.4.18}$$

Hence, (3.4.14) is bounded from above by

$$\left[E_0\left(\exp\left\{8\sum_{R=1}^{R^*(k_*)}\delta A^{(R+1)d} l_t\big(\widetilde{B}_R(k_*)\big)\right\}\right)\right]^{1/2}\exp\left\{4\sum_{R\in\mathbb{N}}\delta A^{(R+1)d}\sqrt{\delta_R}t\right\}. \tag{3.4.19}$$

For A large enough, by Proposition 3.4.1 and the specific choice of $(\delta_R)_{R\in\mathbb{N}}$ in Propostion 3.4.1, the sum in the second term is $\le\widetilde{\varepsilon}t/2$.

4. To estimate the first factor in (3.4.19) and control the second factor in the summand of (3.4.13), we need the following two lemmas whose proof is deferred to Appendix 3.8.

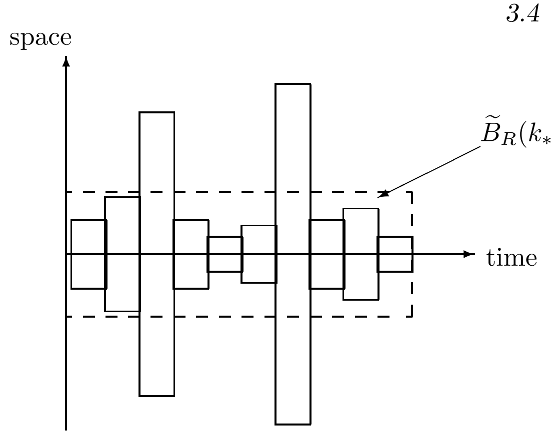

Figure 3.4: The picture shows a possible configuration of bad R-blocks after its rearrangement. There are two time-intervals in which the number of bad R-blocks is atypically large, i.e., larger than $\delta_R k_* \sqrt{\kappa} A/t$. The local time in these bad R-blocks can be bounded from above by the total length of these time-intervals, which is at most $\sqrt{\delta_R} t$. The local time of the bad R-blocks in the other time-intervals can be bounded from above by the local time of the enveloping dashed block, i.e., $\widetilde{B}_R(k_*)$.

Lemma 3.4.3. *Let X^κ be simple random walk on $\mathbb{Z}$ with step rate 2κ. There is a $K_2 > 0$ such that for all $\kappa > 0$, all $n \in \mathbb{N}$, all $\beta_1, \beta_2, \dots, \beta_n \geq 0$ and all nested finite intervals $\emptyset = I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \mathbb{Z}$,*

$$\log E_0 \left(\exp\left\{ \sum_{i=1}^n \beta_i \sum_{x \in I_i} l_t(X^\kappa, x) \right\} \right) \leq \frac{K_2 t}{\sqrt{\kappa}} \sum_{i=1}^n \left[|I_i \setminus I_{i-1}| \left(\sum_{j=i}^n \beta_j \right)^{3/2} \right] + o(t), \tag{3.4.20}$$

where $l_t(X^\kappa, x)$ is the local time of X^κ at site x up to time t.

Lemma 3.4.4. *There are $C_5, C_6 > 0$ such that for all $\kappa, t > 0$ large enough, all $A > 0$ and all $k_* \geq C_5 t/A$,*

$$P_0 \Big(X^\kappa \text{ crosses } k_* \ (\kappa, 1)\text{-blocks} \Big) \leq e^{-C_6 A k_*}. \tag{3.4.21}$$

Note that $A^{R+1}\delta_{R+1} < A^R \delta_R$, and so $\widetilde{B}_{R+1}(k_*) \subseteq \widetilde{B}_R(k_*)$ for all $R \in \mathbb{N}$. Moreover, for $k_* \leq C_2 C_4 \sqrt{\kappa} t + 2t/A$ and $1 \leq R \leq R^*(k_*)$ we have that the cardinality of the spatial part of the blocks defined in (3.4.18) satisfies $|\widetilde{B}_R(k_*)| \leq |\widetilde{B}_1(k_*)| \leq A^2 \delta_1 C_2 C_4 \kappa + 2A\delta_1 \sqrt{\kappa}$, which is *bounded uniformly in* t. To apply Lemma 3.4.3, we choose t_0 (which may depend on κ) such that for each family of intervals $I_1, \dots, I_{R^*(k_*)}$, $k_* \in [t/A, C_2 C_4 \sqrt{\kappa} t + 2t/A]$, with the property that $|I_i| \in [A, A^2 \delta_1 C_2 C_4 \kappa + 2A\delta_1 \sqrt{\kappa}]$ for all $i \in \{1, \dots, R^*(k_*)\}$ the assertion of Lemma 3.4.3 holds uniformly in $t \geq t_0$. Then, for all $t \geq t_0$, the expectation in the left-hand side of (3.4.19) is at most

$$\exp\left\{\frac{K_2 t}{\sqrt{\kappa}}\sum_{R=1}^{R^*(k_*)}\left[\left|\widetilde{B}_R(k_*)\setminus\widetilde{B}_{R+1}(k_*)\right|\left(\sum_{j=1}^{R}8\delta A^{(j+1)d}\right)^{3/2}\right]+o(t)\right\}, \tag{3.4.22}$$

where $\widetilde{B}_{R^*(k_*)+1}(k_*)=\emptyset$. Next, note that

$$|\widetilde{B}_R(k_*)\setminus\widetilde{B}_{R+1}(k_*)|\leq\frac{A^R\delta_R k_*\sqrt{\kappa}A}{t}. \tag{3.4.23}$$

Therefore the first term in the exponent of (3.4.22), may be estimated from above by

$$\begin{aligned}&\frac{K_2 t}{\sqrt{\kappa}}\sum_{R=1}^{R^*(k_*)}\left[\frac{A^R\delta_R k_*\sqrt{\kappa}A}{t}\left(\sum_{j=1}^{R}8\delta A^{(j+1)d}\right)^{3/2}\right]\\&\qquad\leq(8\delta)^{3/2}K_2k_*A^{3d/2+1}\sum_{R=1}^{R^*(k_*)}\left[A^R\delta_R\left(\sum_{j=1}^{R}A^{jd}\right)^{3/2}\right].\end{aligned} \tag{3.4.24}$$

Furthermore,

$$\sum_{j=1}^{R}A^{jd}=\frac{A^d}{A^d-1}(A^{Rd}-1)\leq CA^{Rd}, \tag{3.4.25}$$

where $C>0$ does not depend on A. Hence, the right-hand side of (3.4.24) is at most

$$(8\delta)^{3/2}CK_2k_*A^{3d/2+1}\sum_{R=1}^{R^*(k_*)}A^R\delta_R A^{3Rd/2}. \tag{3.4.26}$$

Recalling our choice of δ_R in Proposition 3.4.1, we can estimate the sum in (3.4.26) from above by

$$K_1A^{-8d^2/2}A^{-D(d)}\left[\frac{1-A^{-R^*(k_*)D(d)}}{1-A^{-D(d)}}\right]. \tag{3.4.27}$$

with $D(d)=(16d^2-d-6)/6>0$. Since $A>3$ by Proposition 3.2.1, the last term in (3.4.27) is bounded uniformly in A and $R^*(k_*)$. Inserting (3.4.27) into (3.4.26), we see that there is a $C_7>0$, not depending on A, such that the exponent in (3.4.22) is bounded from above by $(8\delta)^{3/2}C_7A^{-D'(d)}k_*+o(t)$ with $D'(d)=(16d^2-5d-6)/3>0$, where $o(t)$ is uniform in $t\geq t_0$ for all $k_*\in[t/A, C_2C_4\sqrt{\kappa}t+2t/A]$.

5. It remains to estimate (recall (3.4.13))

$$\sum_{(B_1^A(x_i,k_i;\kappa))_{0\leq i<k_*}}\left[P_0\Big(X^\kappa\text{ crosses }B_1^A(x_i,k_i,\kappa),0\leq i<k_*\Big)\right]^{1/2}. \tag{3.4.28}$$

Let $|\sum_{(B_1^A(x_i,k_i;\kappa))_{0\le i<k_*}}|$ denote the cardinality of the sum in (3.4.28), i.e., the number of sequences of $(\kappa,1)$-blocks of length k_*, which may be crossed by a realization of X^κ. By the Cauchy Schwarz inequality, (3.4.28) is not more than

$$\begin{aligned}
&\Bigg|\sum_{(B_1^A(x_i,k_i;\kappa))_{0\le i<k_*}}\Bigg|^{1/2}\Bigg[\sum_{(B_1^A(x_i,k_i;\kappa))_{0\le i<k_*}} P_0\Big(X^\kappa \text{ crosses } B_1^A(x_i,k_i;\kappa), 0\le i<k_*\Big)\Bigg]^{1/2}\\
&=\Bigg|\sum_{(B_1^A(x_i,k_i;\kappa))_{0\le i<k_*}}\Bigg|^{1/2}\Bigg[P_0\Big(X^\kappa \text{ crosses } k_*\ (\kappa,1)\text{-blocks}\Big)\Bigg]^{1/2}.
\end{aligned}\tag{3.4.29}$$

To estimate the first term in the right-hand side of (3.4.29), note that $|\sum_{(B_1^A(x_i,k_i;\kappa))_{0\le i<k_*}}|$ equals the number of different ways to visit k_* $(\kappa,1)$-blocks. Hence, there is a $C_8>0$ such that $|\sum_{(B_1^A(x_i,k_i;\kappa))_{0\le i<k_*}}|$ is bounded from above by $e^{C_8k_*}$ (see also Lemma 3.5.5 in Section 3.5.4). Therefore, by Lemma 3.4.4, for $k_*\ge C_5t/A$ and κ large enough, the right-hand side of (3.4.29) may be estimated from above by

$$e^{C_8k_*}\,e^{-C_6Ak_*}. \tag{3.4.30}$$

If $k_*\le C_5t/A$, then, bounding each term in the sum in (3.4.28) by one and using the same arguments as above, we may conclude that in this case (3.4.28) is bounded by

$$e^{C_8C_5t/A}. \tag{3.4.31}$$

6. We are now in a position to complete the proof of (3.4.7). Combining the estimates in (3.4.14), (3.4.19) and (3.4.24–3.4.30), we get for $t\ge t_0$ (see the lines following (3.4.27)),

$$\begin{aligned}
&E_0\Bigg(\exp\Bigg\{4\sum_{R=1}^{\varepsilon\log t}\delta A^{(R+1)d}l_t\big(\mathrm{BAD}_R^\delta\big)\Bigg\}\mathbb{1}\{\exists\, k_*\le C_2\kappa t\colon\ X^\kappa\in\Pi(k_*,t,A)\}\Bigg)\\
&\quad\le e^{\widetilde{\varepsilon}t/2}\sum_{k_*=t/A}^{C_5t/A-1}e^{(8\delta)^{3/2}C_7A^{-D'(d)}k_*+o(t)}e^{C_8C_5t/A}\\
&\qquad+e^{\widetilde{\varepsilon}t/2}\sum_{k_*=C_5t/A}^{C_2C_4\sqrt{\kappa}t+2t/A}e^{(8\delta)^{3/2}C_7A^{-D'(d)}k_*+o(t)}\,e^{C_8k_*}e^{-C_6Ak_*}\\
&\quad\le e^{\widetilde{\varepsilon}t/2}\frac{C_5t}{A}\,e^{(8\delta)^{3/2}C_7C_5A^{-D'(d)}t+o(t)}e^{C_8C_5t/A}+e^{\widetilde{\varepsilon}t/2}C_9\\
&\quad\le e^{2\widetilde{\varepsilon}t},
\end{aligned}\tag{3.4.32}$$

where we use that the sum in the third line of (3.4.32) is finite for A large enough (which requires that ε is small enough; recall Proposition 3.2.1). This settles (3.4.7) and completes the proof of Proposition 3.2.2. ∎

3.5 Proof of Proposition 3.4.1

The proof is given in Section 3.5.1 subject to Lemma 3.5.1 below. This lemma is stated in Section 3.5.1 and proved in Sections 3.5.2–3.5.5. Recall the definition of Ξ_R^{A,k_*} in (3.3.1). Throughout this section we abbreviate

$$\widetilde{\delta}_R = A^{-2d(2d+1)R}. \tag{3.5.1}$$

3.5.1 Proof of Proposition 3.4.1 subject to a further lemma

Lemma 3.5.1. *There is a $C > 0$ such that ξ-a.s. for all A and m large enough, all $R \in \mathbb{N}$ and all $k_* \in \mathbb{N}$,*

$$\Xi_R^{A,k_*} \leq C A^{-(4d^2-1)} A^{-R} \widetilde{\delta}_R k_*. \tag{3.5.2}$$

We are now ready to prove Proposition 3.4.1.

Proof. Let $\Phi \in \Pi(k_*, t, A)$ and $R \in \mathbb{N}$. Suppose that there is a $\delta_R > 0$ such that there are at least $\sqrt{\delta_R} t / A^R$ R-intervals in which Φ crosses more than $\delta_R k_* / (t/A)$ bad R-blocks. In all of these R-intervals Φ crosses at least

$$\frac{\sqrt{\delta_R} t}{A^R} \frac{\delta_R k_*}{(t/A)} = \delta_R^{3/2} A^{-(R-1)} k_* \tag{3.5.3}$$

bad R-blocks. Lemma 3.5.1 implies that ξ-a.s. $\delta_R^{3/2} A^{-(R-1)} \leq C A^{-(4d^2-1)} A^{-R} \widetilde{\delta}_R$, which is the same as $\delta_R^{3/2} \leq C A^{-4d^2} A^{-2d(2d+1)R}$. This yields the claim below (3.4.3) with $K_1 = C^{2/3}$. ∎

3.5.2 Proof of Lemma 3.5.1 subject to two further lemmas

The proof of Lemma 3.5.1 is a modification of the proof of Lemma 2.3.5 (and also of [EdHM14a, Lemma 3.5]) and is based on Lemmas 3.5.2–3.5.3 below, which count bad R-blocks. The proof of the second lemma is deferred to Section 3.5.3.

For $A \geq 1$, $R \in \mathbb{N}$ and $\Phi \in \Pi(k_*, t, A)$, define

$$\begin{aligned} \Psi_R^A(\Phi) &= \text{number of good } (R+1)\text{-blocks crossed by } \Phi \text{ containing a bad } R\text{-block}, \\ \Psi_R^{A,k_*} &= \sup_{\Phi \in \Pi(k_*, t, A)} \Psi_R^A(\Phi). \end{aligned} \tag{3.5.4}$$

Lemma 3.5.2. *There is a $C' > 0$ such that for all A and m large enough*

$$\mathbb{P}\Big(\Psi_R^{A,k_*} \geq C' A^{-R} \widetilde{\delta}_R k_* \text{ for some } R \in \mathbb{N} \text{ and some } k_* \in \mathbb{N}_0\Big) \tag{3.5.5}$$

is summable over $t \in \mathbb{N}$. A possible choice is $C' = 3$.

Lemma 3.5.3. *For all $\varepsilon > 0$ there is an $A = A(\varepsilon) > 3$ such that ξ-a.s. there is a $t_0 > 0$ such that for all $R \in \mathbb{N}$, all $k_* \in \mathbb{N}$ and all $t \geq t_0$,*

$$\Xi_R^{A,k_*} \leq A^{d+1} \sum_{i=1}^{\varepsilon \log t - R - 1} 2^{di} A^{(d+1)i} \Psi_{R+i}^{A,k_*}. \tag{3.5.6}$$

Proof. Lemma 3.5.3 is the same as Lemma 2.3.7. The idea is to look at a bad R-block and check whether it is contained in a good $(R+1)$-block or in a bad $(R+1)$-block. An iteration over R, combined with a simple counting argument and Lemma 3.3.1, yields the claim. ∎

We are now ready to prove Lemma 3.5.1.

Proof. By Lemma 3.5.2, ξ-a.s. for t large enough $\Psi_R^{A,k_*} \leq C' A^{-R} \widetilde{\delta}_R k_*$ for all $R \in \mathbb{N}$ and all $k_* \in \mathbb{N}$. By Lemma 3.5.3, recalling that $\widetilde{\delta}_R = A^{-2d(2d+1)R}$, we may estimate

$$\begin{aligned} \Xi_R^{A,k_*} &\leq A^{d+1} \sum_{i \geq 1} 2^{di} A^{(d+1)i} C' A^{-(R+i)} \widetilde{\delta}_{R+i} k_* \\ &= C' A^{d+1} A^{-R} \widetilde{\delta}_R k_* \sum_{i \geq 1} 2^{di} A^{(d+1)i} A^{-i} A^{-2d(2d+1)i} \\ &= C' A^{d+1} A^{-R} \widetilde{\delta}_R k_* \frac{2^d A^d A^{-2d(2d+1)}}{1 - 2^d A^d A^{-2d(2d+1)}}. \end{aligned} \tag{3.5.7}$$

Note that for $A \geq A_0 > 1$ there is a $C > 0$, depending on A_0 but not on A, such that the term in the right-hand side of (3.5.7) is bounded from above by

$$C A^{-(4d^2-1)} A^{-R} \widetilde{\delta}_R k_*, \tag{3.5.8}$$

which yields the claim. ∎

3.5.3 Proof of Lemma 3.5.2 subject to a further lemma

The proof of Lemma 3.5.2 is based on Lemma 3.5.4 below. Let $x \in \mathbb{Z}^d$ and $k, R \in \mathbb{N}$. Abbreviate

$$\chi^A(x,k) = 1\!\!1\big\{B_{R+1}^A(x,k) \text{ is good but contains a bad } R\text{-block}\big\}. \tag{3.5.9}$$

Lemma 3.5.4. *There is a $C > 0$ such that for all A and m large enough, all $R \in \mathbb{N}$ and all $k_* \in \mathbb{N}$,*

$$\mathbb{P}\left(\begin{array}{l} \text{there is a path that crosses } k_* \text{ 1-blocks and intersects} \\ \text{at least } 3A^{-R}\widetilde{\delta}_R k_* \text{ blocks } B_{R+1}^A(x,k) \text{ with } \chi^A(x,k) = 1 \end{array} \right) \leq \exp\big\{ -CA^{-R}\widetilde{\delta}_R k_* \big\}. \tag{3.5.10}$$

We are now ready to prove Lemma 3.5.2.

Proof. First note that $k_* \geq t/A$ and that, ξ-a.s. for t large enough, $1 \leq R \leq \varepsilon \log t$, by Lemma 3.3.1. For each such R, we have by Lemma 3.5.4,

$$\begin{aligned} &\mathbb{P}\Big(\begin{array}{l} \text{there is a path that crosses } k_* \text{ 1-blocks and intersects at least} \\ 3A^{-R}\widetilde{\delta}_R k_* \text{ blocks } B^A_{R+1}(x,k) \text{ with } \chi^A(x,k)=1 \text{ for some } k_* \geq t/A \end{array} \Big) \\ &\leq \sum_{k_* \geq t/A} \exp\{-CA^{-R}\widetilde{\delta}_R k_*\} \leq \frac{\exp\{-CA^{-R}\widetilde{\delta}_R t/A\}}{1-\exp\{-CA^{-R}\widetilde{\delta}_R\}}. \end{aligned} \tag{3.5.11}$$

Because $1 \leq R \leq \varepsilon \log t$ and $R \mapsto A^{-R}\widetilde{\delta}_R$ is non-increasing, the numerator in the right-hand side of (3.5.11) is bounded from above by $\exp\{-CA^{-\varepsilon \log t}\widetilde{\delta}_{\varepsilon \log t} t/A\}$ while the denominator is bounded from below by $1-\exp\{-CA^{-\varepsilon \log t}\widetilde{\delta}_{\varepsilon \log t}\}$. Using the choice of A in Lemma 3.3.1, we see that (3.5.11) is bounded from above by

$$\frac{\exp\{-Ct^{1-a^{-1}}/A\}}{1-\exp\{-Ct^{-a^{-1}}\}}, \qquad a>1. \tag{3.5.12}$$

Note that this is of order $\exp\{-C' t^{\widetilde{\varepsilon}}\}$ for some $C', \widetilde{\varepsilon} > 0$, and so the probability in (3.5.5) is bounded from above by $(\varepsilon \log t) \exp\{-C' t^{\widetilde{\varepsilon}}\}$, which is summable in $t \in \mathbb{N}$. ∎

3.5.4 Proof of Lemma 3.5.4 subject to two further lemmas

The proof of Lemma 3.5.4 is based on Lemmas 3.5.5–3.5.6 below, which are proved in Section 3.5.5.

Proof. Our first further lemma reads:

Lemma 3.5.5. *There is a $C > 0$ such that for all $l \in \mathbb{N}$ and $R \in \mathbb{N}$ there are no more than e^{Cl} possible ways for Φ to visit at most l R-blocks.*

Fix $R \in \mathbb{N}$. We divide blocks into equivalence classes such that blocks belonging to the same equivalence class can essentially be treated as independent. To that end, we take $a_1, a_2 \in \mathbb{N}$ according to condition (a1) in Definition 3.1.5 and say that (x,k) and (x',k') are equivalent when

$$x = x' \,(\mathrm{mod}\, a_1), \qquad k = k' \,(\mathrm{mod}\, a_2). \tag{3.5.13}$$

We denote the set of corresponding representants by $([x],[k])$, and write $\sum_{([x],[k])}$ to denote the sum over all equivalence classes. Note that the left-hand side of (3.5.10) is bounded from above by

$$\sum_{([x],[k])} \mathbb{P}\left(\begin{array}{l} \text{there is a path that crosses } k_* \text{ 1-blocks and intersects} \\ \text{at least } 3A^{-R}\widetilde{\delta}_R k_*/a_1^d a_2 \text{ blocks } B^A_{R+1}(x,k) \\ \text{with } \chi^A(x,k)=1 \text{ and } (x,k) \equiv ([x],[k]) \end{array} \right). \tag{3.5.14}$$

Fix an equivalence class. Put $\rho_R = A^{-4d(2d+1)(d+1)R}$ (recall (3.1.17)). To control the cardinality of the number of different ways to visit a given number of $(R+1)$-blocks, we consider enlarged blocks, namely, we let

$$L = L(R) = (1/\rho_R)^{1/(d+1)} \tag{3.5.15}$$

and define

$$\widetilde{B}_R^A(x,k) = \left(\prod_{j=1}^{d} \left[Lx(j)A^R, L(x(j)+1)A^R \right) \cap \mathbb{Z}^d \right) \times [LkA^R, L(k+1)A^R). \tag{3.5.16}$$

Our second further lemma reads:

Lemma 3.5.6. *If* Φ *crosses* k_* 1*-blocks, then for all* $R \in \mathbb{N}$ *it crosses no more than* $l_R = 3k_*/A^{R-1}L$ *blocks* $\widetilde{B}_R^A(x,k)$.

We write

$$\bigcup_{(x_i,k_i)_{0\le i<l_{R+1}}} \widetilde{B}_{R+1}^A(x_i,k_i) \tag{3.5.17}$$

to denote the union over at most l_{R+1} blocks $\widetilde{B}_R^A(x,k)$ and

$$\sum_{(\widetilde{B}_{R+1}^A(x_i,k_i))_{0\le i<l_{R+1}}} \tag{3.5.18}$$

to denote the sum over all possible sequences of at most l_{R+1} blocks $\widetilde{B}_{R+1}^A(x_i,k_i)$ that can be crossed by a path Φ. Since each block $B_{R+1}^A(x,k)$ that may be crossed by Φ lies in the union of (3.5.17), we may estimate the probability under the sum in (3.5.14) from above by

$$\sum_{(\widetilde{B}_{R+1}^A(x_i,k_i))_{0\le i<l_{R+1}}} \mathbb{P}\left(\begin{array}{l} \text{the union in (3.5.17) contains at least } 3A^{-R}\widetilde{\delta}_R k_*/a_1^d a_2 \\ \text{blocks } B_{R+1}^A(x,k) \text{ with } \chi^A(x,k)=1 \text{ and } (x,k)\equiv([x],[k]) \end{array} \right). \tag{3.5.19}$$

Next, note that the union in (3.5.17) contains at most $l_{R+1}L^{d+1}$ $(R+1)$-blocks and that there are $\binom{l_{R+1}L^{d+1}}{n}$ ways of choosing n blocks $B_{R+1}^A(x,k)$ with $\chi^A(x,k)=1$ from $l_{R+1}L^{d+1}$ $(R+1)$-blocks. Hence, by the mixing condition in (3.1.17) for A and m large enough, each summand in (3.5.19) is bounded from above by

$$\sum_{n=\delta_R k_{R+1}/a_1^d a_2}^{l_{R+1}L^{d+1}} \binom{l_{R+1}L^{d+1}}{n} (\rho_R)^n \le (1-\rho_R)^{-l_{R+1}L^{d+1}} \mathbb{P}\big(T \ge 3A^{-R}\widetilde{\delta}_R k_*/a_1^d a_2\big), \tag{3.5.20}$$

where $T = \mathrm{BINOMIAL}(l_{R+1}L^{d+1}, \rho_R)$. Since

$$\begin{aligned} &\mathbb{E}(T) \\ &= \rho_R l_{R+1} L^{d+1} = l_{R+1} = 3k_*/A^R L = 3A^{-R}A^{-4d(2d+1)R}k_* = 3A^{-R}\widetilde{\delta}_R^2 k_* \ll 3A^{-R}\widetilde{\delta}_R k_*, \end{aligned} \tag{3.5.21}$$

we can apply standard large deviation estimates to bound the right-hand side of (3.5.20). Indeed, by Bernstein's inequality, there is a $C'>0$ (depending on a_1 and a_2 only) such that for all A and m large enough,

$$\mathbb{P}\big(T \geq 3A^{-R}\widetilde{\delta}_R k_*/a_1^d a_2\big) \leq e^{-C' 3A^{-R}\widetilde{\delta}_R k_*/a_1^d a_2}. \tag{3.5.22}$$

Moreover, there is a $C''>0$ (not depending on A, provided A is large enough) such that

$$(1-\rho_R)^{-l_{R+1}L^{d+1}} \leq e^{\rho_R l_{R+1}L^{d+1}/(1-\rho_R)} \leq e^{C'' 3A^{-R}\widetilde{\delta}_R^2 k_*}. \tag{3.5.23}$$

Furthermore, by Lemma 3.5.5, and after a possible increase of C'', the sum in (3.5.18) contains at most $e^{C'' l_{R+1}} = e^{C'' 3A^{-R}\widetilde{\delta}_R^2 k_*}$ elements. Hence, combining (3.5.14), (3.5.19–3.5.20) and (3.5.22–3.5.23), we see that the left-hand side of (3.5.10) is bounded from above by $e^{-CA^{-R}\widetilde{\delta}_R k_*}$, with C such that $CA^{-R}\widetilde{\delta}_R k_* \geq (C'/a_1^d a_2 - \widetilde{\delta}_R 2C'')3A^{-R}\widetilde{\delta}_R k_*$, which yields the claim in (3.5.10). ∎

3.5.5 Proof of Lemmas 3.5.5–3.5.6

Proof. For the proof of Lemma 3.5.5, see the proof of Claim 2.3.8. The proof of Lemma 3.5.6 goes as follows. Let $R \in \mathbb{N}$. Divide time into intervals of length LA^R. Let l_i^L and l_i be the number of blocks $\widetilde{B}_R^A(x,k)$, respectively, 1-blocks, crossed by X^κ in the i-th time interval $[(i-1)LA^R, iLA^R)$, $1 \leq i \leq t/LA^R$. Note that $l_i \geq LA^{R-1}$ because the length of the time-interval of each block $\widetilde{B}_R^A(x,k)$ is LA^R, which may be divided into LA^{R-1} time-intervals of length A. Moreover X^κ has to cross at least one 1-block in each such interval of length A. Also note that if $l_i = LA^{R-1}$, then $l_i^L \leq 2 = 2l_i/l_i \leq 2l_i^1/LA^{R-1}$. If $LA^{R-1}+1 \leq l_i \leq 2LA^{R-1}$, then $l_i^L \leq 3$, because X^κ may start at an interface between two blocks $\widetilde{B}_R^A(x,k)$ and immediately jump from one such block to another. However, to afterwards reach the next block $\widetilde{B}_R^A(x,k)$ it has to cross at least LA^{R-1} 1-blocks, and so $l_i^L \leq 3l_i/l_i \leq 3l_i^1/LA^{R-1}$. Furthermore, for $j\in\mathbb{N}$, if $jLA^{R-1}+1 \leq l_i \leq (j+1)LA^{R-1}$, then

$$l_i^L \leq (j+2)l_i/l_i \leq (j+2)l_i/jLA^{R-1}. \tag{3.5.24}$$

Therefore we have

$$k_* = \sum_{i=1}^{t/LA^{R-1}} l_i \geq \frac{LA^{R-1}}{3} \sum_{i=1}^{t/LA^{R-1}} l_i^L, \tag{3.5.25}$$

or $\sum_{i=1}^{t/LA^{R-1}} l_i^L \leq (3/LA^{R-1})k_* = l_R$, which completes the proof. ∎

3.6 Proof of Proposition 3.4.2

In Section 3.6.1 we reduce the problem to one dimension and recall two discrete rearrangement inequalities from the literature (Propositions 3.6.3–3.6.4 below). In Section 3.6.2 we use the latter to give the proof of Proposition 3.4.2.

3.6.1 Reduction to one dimension and discrete rearrangement inequalities

Recall the definition of Π_1 in (3.2.3) and the lines following (3.2.3).

Lemma 3.6.1. *Let $B \subseteq \mathbb{Z}^d \times [0,t]$. Then, for all $C \geq 0$,*

$$E_0\big(e^{Cl_t(B)}\big) \leq E_0\big(e^{Cl_t(\Pi_1(B))}\big). \tag{3.6.1}$$

Proof. A d-dimensional simple random walk with jump rate $2d\kappa$ is a vector of d independent one-dimensional simple random walks, each having jump rate 2κ. Hence, given any set $B \subseteq \mathbb{Z}^d \times [0,t]$,

$$\forall\, s \geq 0\colon \qquad X^\kappa(s) \in B \quad \Longrightarrow \quad \Pi_1(X^\kappa)(s) \in \Pi_1(B)(s). \tag{3.6.2}$$

This in turn implies that $l_t(B) \leq l_t(\Pi_1(B))$, which proves the claim. ∎

To prove Proposition 3.4.2 we need two discrete rearrangement inequalities [P96], [P98]. For an overview on continuous rearrangement inequalities we refer the reader to [LL01, Chapter 3].

Definition 3.6.2. *A function $L\colon \mathbb{Z} \times \mathbb{Z} \to [0,\infty)$ is called of Riesz-type when, for all pairs of functions $f, g\colon \mathbb{Z} \to [0,\infty)$,*

$$\sum_{x,y\in\mathbb{Z}} f(x)L(x,y)g(y) \leq \sum_{x,y\in\mathbb{Z}} f^\sharp(x)L(x,y)g^\sharp(y). \tag{3.6.3}$$

Proposition 3.6.3. [P96, Theorem 2.2], [P98]) *Let $K\colon\ [0,\infty) \to [0,\infty)$ be non-increasing. Then $L\colon \mathbb{Z} \times \mathbb{Z} \to [0,\infty)$ given by $L(x,y) = K(|x-y|)$ is of Riesz-type.*

Note that $(x,y) \mapsto p_s^\kappa(x,y)$ with $p_s^\kappa(x,y)$ the transition kernel of one-dimensional simple random walk with jump rate 2κ is of Riesz-type. Indeed, $p_s^\kappa(x,y) = p_s^\kappa(x-y,0) = p_s^\kappa(|x-y|,0)$ is a non-increasing function of $|x-y|$.

The following multiple-sum version of Proposition 3.6.3 will be needed also.

Proposition 3.6.4. ([P96, Lemma 9.1 in Chapter 2], [P98]) *Fix $n \in \mathbb{N}$. Let $L_0, L_1, \dots, L_{n-1}$ be a collection of Riesz-type functions on $\mathbb{Z} \times \mathbb{Z}$, and let $S_0, S_1, \dots, S_n$ be a collection of non-negative functions on $\mathbb{Z}$. Then*

$$\begin{aligned}&\sum_{x_0,x_1,\dots,x_n\in\mathbb{Z}} \left(\prod_{i=0}^{n-1} S_i(x_i)L_i(x_i,x_{i+1})\right) S_n(x_n)\\ &\qquad \leq \sum_{x_0,x_1,\dots,x_n\in\mathbb{Z}} \left(\prod_{i=0}^{n-1} S_i^\sharp(x_i)L_i(x_i,x_{i+1})\right) S_n^\sharp(x_n).\end{aligned} \tag{3.6.4}$$

3.6.2 Proof of Proposition 3.4.2

Proof. Let $(B_R)_{R\in\mathbb{N}}$ be a sequence in $\mathbb{Z}\times[0,t]$ (recall Lemma 3.6.1) and $(C_R)_{R\in\mathbb{N}}$ a sequence of nonnegative numbers. Write

$$E_0\left(\exp\left\{\sum_{R\in\mathbb{N}} C_R l_t(B_R)\right\}\right) = \sum_{n\in\mathbb{N}_0}\frac{1}{n!}E_0\left(\left\{\sum_{R\in\mathbb{N}} C_R l_t(B_R)\right\}^n\right). \tag{3.6.5}$$

The n-th moments in (3.6.5) may be rewritten as

$$\sum_{R_1,\dots,R_n\in\mathbb{N}}\left(\prod_{i=1}^n C_{R_i}\right)E_0\left(\prod_{i=1}^n l_t(B_{R_i})\right). \tag{3.6.6}$$

Write out

$$\prod_{i=1}^n l_t(B_{R_i}) = \int_0^t ds_1\dots\int_0^t ds_n\ 1\!\!1\big\{(X^\kappa(s_1),s_1)\in B_{R_1},\dots(X^\kappa(s_n),s_n)\in B_{R_n}\big\}, \tag{3.6.7}$$

so that the second factor under the sum in (3.6.6) equals

$$\int_0^t ds_1\dots\int_0^t ds_n\ P_0\Big((X^\kappa(s_1),s_1)\in B_{R_1},\dots(X^\kappa(s_n),s_n)\in B_{R_n}\Big). \tag{3.6.8}$$

Fix a choice of $(s_1,\dots,s_n)\in[0,t]^n$. Without loss of generality we may assume that $s_1<s_2<\dots<s_n$, so that the probability in (3.6.8) becomes ($x_0=0$, $s_0=0$)

$$\sum_{x_1,\dots,x_n\in\mathbb{Z}}\left(\prod_{i=0}^n 1\!\!1\Big\{(x_i,s_i)\in B_{R_i}\Big\}\right)\left(\prod_{i=0}^{n-1} p^\kappa_{s_{i+1}-s_i}(x_i,x_{i+1})\right). \tag{3.6.9}$$

An application of Proposition 3.6.4 gives that (3.6.9) is bounded from above by

$$\sum_{x_1,\dots x_n\in\mathbb{Z}}\left(\prod_{i=0}^n 1\!\!1\Big\{(x_i,s_i)\in B^\sharp_{R_i}\Big\}\right)\left(\prod_{i=0}^{n-1} p^\kappa_{s_{i+1}-s_i}(x_i,x_{i+1})\right), \tag{3.6.10}$$

recall also the first lines of Section 3.4.1. Then, by (3.6.8),

$$E_0\left(\prod_{i=1}^n l_t(B_{R_i})\right)\le E_0\left(\prod_{i=1}^n l_t(B^\sharp_{R_i})\right). \tag{3.6.11}$$

Inserting this back into (3.6.5) and (3.6.6), we get the claim. ∎

3.7 Proof of Proposition 3.2.3

In Section 3.7.1 we introduce some notation and state two more propositions, Propositions 3.7.3–3.7.4 below, whose proof is given in Sections 3.7.3–3.7.4. In Section 3.7.2 we give the proof of Proposition 3.2.3 subject to these propositions.

3.7.1 Two more propositions

Henceforth we assume that α in (3.1.13) takes the form $\alpha = 4M\kappa$ with M a constant that will be determined later on. Recall the definition of Π_1 below (3.2.2) and of $\bar{\xi}$ in (3.2.6).

Definition 3.7.1. *The subpedestal of $B_1^{A,4M\kappa}(x,k)$ is (see Fig. 3.5)*

$$B_{1,\mathrm{sub}}^{A,4M\kappa}(x,k) = \Big\{ y \in \Pi_1\big(B_1^{A,4M\kappa}(x,k)\big)\colon\ |y(j) - z(j)| \geq 2M\kappa A, \\ j \in \{1,2,\dots,d\}\, \forall\, z \in \partial\Pi_1\big(B_1^{A,4M\kappa}(x,k)\big)\Big\} \times \{kA\}. \tag{3.7.1}$$

Definition 3.7.2. *Let $\varepsilon > 0$, and $k,n \in \mathbb{N}_0$ such that $n \geq k$. A block $B_1^{A,4M\kappa}(x,k)$ is called ε-sufficient at level n when, for every $y \in \Pi_1(B_{1,\mathrm{sub}}^{A,4M\kappa}(x,k))$,*

$$E_y\left(\exp\left\{\int_0^A \bar{\xi}(X^\kappa(s), A(n-k)-s)\,ds\right\} 1\!\!1\big\{N(X^\kappa, A) \leq M\kappa A\big\}\right) \leq e^{\varepsilon A}. \tag{3.7.2}$$

Otherwise it is called ε-insufficient at level n. A subpedestal is called ε-(in)sufficient at level n when its corresponding block is ε-(in)sufficient at level n.

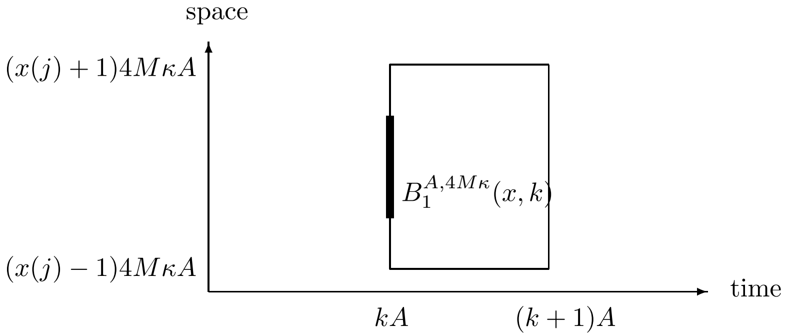

Figure 3.5: The thick line is the subpedestal.

Proposition 3.7.3. *Let $A > 1$. There is a constant $C_3 > 0$ such that for all $n \in \mathbb{N}$ the number of different sequences of subpedestals $B_{1,\mathrm{sub}}^{A,4M\kappa}(0,0), B_{1,\mathrm{sub}}^{A,4M\kappa}(x_1,1),\dots,$ $B_{1,\mathrm{sub}}^{A,4M\kappa}(x_{n-1},n-1)$ with the property that there is a path $\Phi\colon [0,An] \to \mathbb{Z}^d$ with at most $M\kappa An$ jumps satisfying $\Phi(kA) \in B_{1,\mathrm{sub}}^{A,4M\kappa}(x_k,k)$, $k \in \{0,1,\dots,n-1\}$, is bounded from above by $e^{C_3 n}$.*

Proposition 3.7.4. *Fix $\varepsilon > 0$. Let $\delta = \frac{1}{5}\varepsilon$ in the definition of $\bar{\xi}$ and $A > 1$. Then there is a $\kappa_0 > 0$ such that, for all $\kappa \geq \kappa_0$ and ξ-a.s. for all $n \in \mathbb{N}$, all blocks $B_1^{A,4M\kappa}(x,k)$, $x \in \mathbb{Z}^d$, $k \in \mathbb{N}$, $k \leq n$, are ε-sufficient at level n.*

3.7.2 Proof of Proposition 3.2.3 subject to two propositions

Proof. The proof comes in 2 Steps.

1. Fix $\varepsilon > 0$ and put $\delta = \frac{1}{5}\varepsilon$. Choose $\kappa \geq \kappa_0$ according to Proposition 3.7.4. Then the tail estimate $P\big(\text{POISSON}(\lambda) \geq k\big) \leq e^{-\lambda}(\lambda e)^k/k^k$, $k \geq 2\lambda + 1$, for Poisson-distributed random variables with mean λ shows that, for $M > 0$ large enough,

$$\begin{aligned} &E_0\Bigg(\exp\bigg\{\int_0^{An} \overline{\xi}(X^\kappa(s), An-s)\,ds\bigg\}\mathbb{1}\big\{N(X^\kappa, An) > M\kappa An\big\}\Bigg) \\ &\qquad \leq e^{2\delta A^{d+1}n} e^{-2d\kappa An} \exp\big\{-M\kappa An[\log(M/2d) - 1]\big\}, \end{aligned} \tag{3.7.3}$$

where we use (3.2.6). Since we later let $\kappa \to \infty$, (3.7.3) shows that it is enough to concentrate on contributions coming from paths with at most $M\kappa An$ jumps. To that end, fix a $\mathbb{Z}^d$-valued sequence of vertices $x_0, x_1, \ldots, x_{n-1}$ such that $x_0 = 0$ and such that there is a path that starts in 0, makes $0 \leq j \leq M\kappa An$ jumps, and is in the subpedestal $B^{A,4M\kappa}_{1,\text{sub}}(x_k, k)$ at time kA for $k \in \{0, 1, \ldots, n-1\}$. By the Markov property of X^κ applied at times kA, $k \in \{1, 2, \ldots, n-1\}$,

$$\begin{aligned} &E_0\Bigg(\exp\bigg\{\int_0^{An} \overline{\xi}(X^\kappa(s), An-s)\,ds\bigg\}\mathbb{1}\big\{N(X^\kappa, An) \leq M\kappa An\big\} \\ &\qquad\qquad \times \prod_{k=1}^{n-1} \mathbb{1}\Big\{X^\kappa(kA) \in B^{A,4M\kappa}_{1,\text{sub}}(x_k, k)\Big\}\Bigg) \\ &\quad \leq \prod_{k=0}^{n-1} \sup_{y \in \Pi_1\left(B^{A,4M\kappa}_{1,\text{sub}}(x_k,k)\right)} E_y\Bigg(\exp\bigg\{\int_0^A \overline{\xi}(X^\kappa(s), A(n-k)-s)\,ds\bigg\}\Bigg). \end{aligned} \tag{3.7.4}$$

This is at most

$$\begin{aligned} &\prod_{k=0}^{n-1}\Bigg[\sup_{y \in \Pi_1\left(B^{A,4M\kappa}_{1,\text{sub}}(x_k,k)\right)} E_y\Bigg(\exp\bigg\{\int_0^A \overline{\xi}(X^\kappa(s), A(n-k)-s)\,ds\bigg\}\mathbb{1}\big\{N(X^\kappa, A) \leq M\kappa A\big\}\Bigg) \\ &+ \sup_{y \in \Pi_1\left(B^{A,4M\kappa}_{1,\text{sub}}(x_k,k)\right)} E_y\Bigg(\exp\bigg\{\int_0^A \overline{\xi}(X^\kappa(s), A(n-k)-s)\,ds\bigg\}\mathbb{1}\big\{N(X^\kappa, A) > M\kappa A\big\}\Bigg)\Bigg] \\ &= \sum_{J \subset \{0,1,\ldots,n-1\}}\Bigg[\\ &\prod_{k \in J} \sup_{y \in \Pi_1(B^{A,4M\kappa}_{1,\text{sub}}(x_k,k))} E_y\Bigg(\exp\bigg\{\int_0^A \overline{\xi}(X^\kappa(s), A(n-k)-s)\,ds\bigg\}\mathbb{1}\big\{N(X^\kappa, A) \leq M\kappa A\big\}\Bigg) \\ &\times \prod_{k \notin J} \sup_{y \in \Pi_1(B^{A,4M\kappa}_{1,\text{sub}}(x_k,k))} E_y\Bigg(\exp\bigg\{\int_0^A \overline{\xi}(X^\kappa(s), A(n-k)-s)\,ds\bigg\}\mathbb{1}\big\{N(X^\kappa, A) > M\kappa A\big\}\Bigg)\Bigg]. \end{aligned} \tag{3.7.5}$$

Now, by the Poisson tail estimate mentioned above and the fact that $\overline{\xi} < 2\delta A^d$, the second factor under the sum in (3.7.5) may be bounded from above by

$$\Big(e^{2\delta A^{d+1}} e^{-2d\kappa A} \exp\big\{-M\kappa A[\log(M/2d)-1]\big\}\Big)^{n-|J|}. \tag{3.7.6}$$

Since, by Proposition 3.7.4 and our choice of κ (see the observation made prior to (3.7.3)), all blocks $B_1^{A,4M\kappa}(x,k)$, $x \in \mathbb{Z}^d$, $k \in \mathbb{N}_0$, $k \le n$, are ε-sufficient at level n, we may conclude that all $y \in \Pi_1(B_{1,\mathrm{sub}}^{A,4M\kappa}(x_k,k))$ with $k \in J$ are in an ε-sufficient subpedestal at level n. Hence, using the binomial formula, we may estimate (3.7.5) from above by

$$\begin{aligned}\sum_{J\subset\{0,1,\dots,n-1\}} & e^{A\varepsilon|J|}\Big(e^{2\delta A^{d+1}} e^{-2d\kappa A} \exp\big\{-M\kappa A[\log(M/2d)-1]\big\}\Big)^{n-|J|} \\ &= \Big(e^{A\varepsilon} + e^{2\delta A^{d+1}} e^{-2d\kappa A} \exp\big\{-M\kappa A[\log(M/2d)-1]\big\}\Big)^{n}.\end{aligned} \tag{3.7.7}$$

2. Summing over all possible sequences $(x_i)_{i\in\{1,2,\dots,n-1\}}$ compatible with a path Φ such that $\Phi(0) = 0$ and $N(\Phi, An) \le M\kappa An$, and using (3.7.4–3.7.7) and Proposition 3.7.3, we obtain

$$\begin{aligned}&E_0\bigg(\exp\Big\{\int_0^{An}\overline{\xi}(X^\kappa(s), An-s)\,ds\Big\}1\!\!1\big\{N(X^\kappa, An) \le M\kappa An\big\}\bigg)\\ &\le \sum_{x_1,x_2,\dots,x_{n-1}} E_0\bigg(\exp\Big\{\int_0^{An}\overline{\xi}(X^\kappa(s), An-s)\,ds\Big\}1\!\!1\big\{N(X^\kappa, An) \le M\kappa An\big\}\\ &\qquad\qquad \times \prod_{k=1}^{n-1} 1\!\!1\Big\{X^\kappa(kA) \in B_{1,\mathrm{sub}}^{A,4M\kappa}(x_k,k)\Big\}\bigg)\\ &\le \sum_{x_1,x_2,\dots,x_{n-1}} \prod_{k=0}^{n-1} \sup_{y\in B_{1,\mathrm{sub}}^{A,4M\kappa}(x_k,k)} E_y\bigg(\exp\Big\{\int_0^{An}\overline{\xi}(X^\kappa(s), An-s)\,ds\Big\}\bigg)\\ &\le e^{C_3 n}\Big(e^{A\varepsilon} + e^{2\delta A^{d+1}} e^{-2d\kappa A} \exp\big\{-M\kappa A[\log(M/2d)-1]\big\}\Big)^{n}.\end{aligned} \tag{3.7.8}$$

Combining (3.7.3–3.7.8), we get

$$\begin{aligned}&\limsup_{\kappa\to\infty}\limsup_{n\to\infty}\frac{1}{An}\log E_0\bigg(\exp\Big\{\int_0^t \overline{\xi}(X^\kappa(s), t-s)\,ds\Big\}\bigg)\\ &\le \frac{C_3}{A} + \frac{1}{A}\limsup_{\kappa\to\infty}\log\Big(e^{A\varepsilon} + e^{2\delta A^{d+1}} e^{-2d\kappa A} \exp\big\{-M\kappa A[\log(M/2d)-1]\big\}\Big) = \frac{C_3}{A} + \varepsilon.\end{aligned} \tag{3.7.9}$$

Since $\varepsilon = 5\delta$, this yields the claim. ∎

3.7.3 Proof of Proposition 3.7.3

Proof. Write $\|\cdot\|$ for the ℓ^1-norm on $\mathbb{Z}^d$. Let $B_{1,\mathrm{sub}}^{A,4M\kappa}(0,0), B_{1,\mathrm{sub}}^{A,4M\kappa}(x_1,1),\ldots,$ $B_{1,\mathrm{sub}}^{A,4M\kappa}(x_{n-1},\, n-1)$ be a sequence of subpedestals that may be crossed by a path Φ with at most $M\kappa An$ jumps. Since Φ needs at least $(\|x_k-x_{k-1}\|-d)_+4M\kappa A$ jumps to go from $B_{1,\mathrm{sub}}^{A,4M\kappa}(x_{k-1},k-1)$ to $B_{1,\mathrm{sub}}^{A,4M\kappa}(x_k,k)$, $k\in\{1,2,\ldots,n-1\}$, we obtain the bound

$$\sum_{k=1}^{n-1}(\|x_k-x_{k-1}\|-d)_+ \le \frac{M\kappa An}{4M\kappa A}=\frac{n}{4}, \tag{3.7.10}$$

which implies that

$$\sum_{k=1}^{n-1}\|x_k-x_{k-1}\| \le \frac{(1+4d)n}{4}. \tag{3.7.11}$$

As shown in Hardy and Ramanujan [HR18] and Erdös [E42], there are $a,b>0$ such that the number of unordered integer-valued sequences $(a_k)_{k\in\mathbb{N}}$ such that $\sum_{k\in\mathbb{N}}a_k\le(1+4d)n/4$ is bounded from above by $ane^{b\sqrt{n}}$. From this it follows that there is a $c>0$ such that there are at most $n^{c\sqrt{n}}ane^{b\sqrt{n}}$ ordered integer-valued sequences $(a_k)_{1\le k\le n-1}$ satisfying (3.7.11). To conclude, define $a_k=\|x_k-x_{k-1}\|$, $k\in\{1,2,\ldots,n-1\}$, and note that the sequence $(a_k)_{k\in\{1,2,\ldots,n-1\}}$ determines the sequence $(x_k)_{k\in\{0,1,\ldots,n-1\}}$ uniquely when it is known for all $k\in\{1,2,\ldots,n-1\}$ and all $j\in\{1,2,\ldots,d\}$ whether $x_k(j)-x_{k-1}(j)$ is positive, zero or negative. Consequently, the number of different subpedestals $B_{1,\mathrm{sub}}^{A,4M\kappa}(0,0), B_{1,\mathrm{sub}}^{A,4M\kappa}(x_1,1),\ldots,B_{1,\mathrm{sub}}^{A,4M\kappa}(x_{n-1},n-1)$ that may be crossed by a path Φ with at most $M\kappa An$ jumps is bounded from above by $3^{dn}n^{c\sqrt{n}}ane^{b\sqrt{n}}\le e^{C_3n}$ for some $C_3>0$. ∎

3.7.4 Proof of Proposition 3.7.4

The proof of Proposition 3.7.4 is given in Section 3.7.5 subject to Lemmas 3.7.5–3.7.6 below, which are stated in Sections 3.7.4.1–3.7.4.2. The proof of the first lemma is given in Section 3.7.4.1, the proof of the second lemma is deferred to Appendix 3.9.

3.7.4.1 A time-dependent Feynman-Kac estimate

Recall (3.2.6). Abbreviate

$$Q^{\kappa\log\kappa}=(-\kappa\log\kappa,\kappa\log\kappa)^d\cap\mathbb{Z}^d. \tag{3.7.12}$$

Lemma 3.7.5. *Fix $A>1$ and $m>0$ such that $Am\in\mathbb{N}$ and let $\kappa>0$ be written in the* *form* $\kappa = \kappa_1\kappa_2$ *with* $\kappa_1 > 1$. *There is a* $\kappa_0 = \kappa_0(M, A)$ *such that,* ξ*-a.s. for all* $x \in \mathbb{Z}^d$,

$$\begin{aligned}\log E_x\Bigg(\exp\bigg\{\int_0^A \overline{\xi}(X^\kappa(s), A-s)\,ds\bigg\}1\!\!1\big\{N(X^\kappa; A) \le M\kappa A\big\}\Bigg)& \\ \le \frac{Am}{\kappa_1}\log\big((2\kappa\log\kappa - 1)^{d/2}\big) + \sum_{k=1}^{Am}\frac{\kappa_2}{m}\lambda_1(\overline{\xi}_k/\kappa_2), \qquad \kappa \ge \kappa_0,&\end{aligned} \tag{3.7.13}$$

where $\lambda_1(\overline{\xi}_k/\kappa_2)$ *is the top of the spectrum of the operator* $\Delta + \frac{1}{\kappa_2}\sup_{r\in[(k-1)/m,k/m)}\overline{\xi}(\cdot, r)$, $k \in \{1, 2, \ldots, Am\}$.

Proof. We give the proof for $x = 0$. The proof for $x \in \mathbb{Z}^d\backslash\{0\}$ goes along the same lines. First note that we may rewrite the expectation in the left-hand side of (3.7.13) as

$$E_0\Bigg(\exp\bigg\{\frac{1}{\kappa}\int_0^{\kappa A}\overline{\xi}(X(s), A - s/\kappa)\,ds\bigg\}1\!\!1\big\{N(X, \kappa A) \le M\kappa A\big\}\Bigg), \tag{3.7.14}$$

where X is simple random walk with step rate $2d$. Furthermore, there is a $\kappa_0 = \kappa_0(M, A)$ such that $M\kappa A \le \kappa\log\kappa$ for all $\kappa \ge \kappa_0$. Hence, by the Markov property of X applied at times $k\kappa/m$, $k \in \{1, 2, \ldots, Am\}$, we may estimate (3.7.14) from above by

$$\begin{aligned}&E_0\Bigg(\exp\bigg\{\frac{1}{\kappa}\int_0^{\kappa A}\overline{\xi}(X(s), A - s/\kappa)\,ds\bigg\}1\!\!1\big\{X([0,\kappa A]) \subseteq Q^{\kappa\log\kappa}\big\}\Bigg) \\ &\le \prod_{k=1}^{Am}\sup_{\substack{x\in\mathbb{Z}^d \\ \|x\|_\infty<\kappa\log\kappa}} E_x\Bigg(\exp\bigg\{\frac{1}{\kappa}\int_0^{\kappa/m}\overline{\xi}(X(s), k/m - s/\kappa)\,ds\bigg\}1\!\!1\big\{X([0,\kappa/m]) \subseteq Q^{\kappa\log\kappa}\big\}\Bigg).\end{aligned} \tag{3.7.15}$$

Next, for $k \in \{1, 2, \ldots, Am\}$ define

$$\overline{\xi}_k(x) = \sup_{r\in[(k-1)/m,k/m)}\overline{\xi}(x, r), \qquad x \in \mathbb{Z}^d. \tag{3.7.16}$$

Then (3.7.15) is at most

$$\prod_{k=1}^{Am}\sup_{\substack{x\in\mathbb{Z}^d \\ \|x\|_\infty<\kappa\log\kappa}} E_x\Bigg(\exp\bigg\{\frac{1}{\kappa}\int_0^{\kappa/m}\overline{\xi}_k(X(s))\,ds\bigg\}1\!\!1\{X([0,\kappa/m]) \subseteq Q^{\kappa\log\kappa}\}\Bigg). \tag{3.7.17}$$

From now on we write κ as $\kappa = \kappa_1\kappa_2$, $\kappa_1 > 1$. Then, by Jensen's inequality, (3.7.17) may be estimated from above by

$$\prod_{k=1}^{Am}\sup_{\substack{x\in\mathbb{Z}^d \\ \|x\|_\infty<\kappa\log\kappa}} E_x\Bigg(\exp\bigg\{\frac{1}{\kappa_2}\int_0^{\kappa/m}\overline{\xi}_k(X(s))\,ds\bigg\}1\!\!1\{X([0,\kappa/m]) \subseteq Q^{\kappa\log\kappa}\}\Bigg)^{1/\kappa_1}. \tag{3.7.18}$$

For each $k \in \{1, 2, \dots, Am\}$, each expectation under the product in (3.7.18) is a solution of the equation

$$\begin{cases} \frac{\partial u_k}{\partial t}(x,t) = \Big[(\Delta + \frac{1}{\kappa_2}\overline{\xi}_k(x))u_k\Big](x,t), \\ u_k(x,0) = 1, \end{cases} \qquad \|x\|_\infty < \kappa \log \kappa, t \geq 0, \tag{3.7.19}$$

with Dirichlet boundary conditions evaluated at time κ/m. However, on any finite subset of $\mathbb{Z}^d$ the operator $\Delta + \frac{1}{\kappa_2}\overline{\xi}_k$ is a self-adjoint matrix. Therefore, by the spectral representation theorem, we may rewrite each expectation under the product in (3.7.17) as

$$\sum_{j=1}^{|Q^{\kappa \log \kappa}|} e^{(\kappa/m)\lambda_j^D(\overline{\xi}_k/\kappa_2)} \langle v_j^k, 1\!\!1_{Q^{\kappa \log \kappa}} \rangle \, v_j^k(x), \tag{3.7.20}$$

where $\lambda_j^D(\overline{\xi}_k/\kappa_2)$ is the j-th largest eigenvalue of $\Delta + \overline{\xi}_k/\kappa_2$ with Dirichlet boundary conditions on $Q^{\kappa \log \kappa}$, $j \in \{1, 2, \dots, |Q^{\kappa \log \kappa}|\}$, and the v_j^k, $j \in \{1, 2, \dots, |Q^{\kappa \log \kappa}|\}$, form an orthonormal system of eigenvectors such that, for all $k \in \{1, 2, \dots, Am\}$,

$$\mathbb{R}^{|Q^{\kappa \log \kappa}|} = \ker\big(e^{\Delta + \overline{\xi}_k/\kappa_2}\big) \oplus \operatorname{span}\big\{v_j^k, j \in \{1, 2, \dots, |Q^{\kappa \log \kappa}|\}\big\}. \tag{3.7.21}$$

(Since $e^{\Delta + \overline{\xi}_k/\kappa_2}$ is a strictly positive operator, $\ker(e^{\Delta + \overline{\xi}_k/\kappa_2}) = \{0\}$.) In particular, for each $k \in \{1, 2, \dots, Am\}$, we may estimate using Parseval's identity in the penultimate inequality

$$\begin{aligned} &\sum_{j=1}^{|Q^{\kappa \log \kappa}|} e^{(\kappa/m)\lambda_j^D(\overline{\xi}_k/\kappa_2)} \langle v_j^k, 1\!\!1_{Q^{\kappa \log \kappa}} \rangle \, v_j^k(x) \\ &\leq \Bigg(\sum_{j=1}^{|Q^{\kappa \log \kappa}|} e^{(\kappa/m)\lambda_j^D(\overline{\xi}_k/\kappa_2)} \langle v_j^k, 1\!\!1_{Q^{\kappa \log \kappa}} \rangle^2 \Bigg)^{1/2} \Bigg(\sum_{j=1}^{|Q^{\kappa \log \kappa}|} e^{(\kappa/m)\lambda_j^D(\overline{\xi}_k/\kappa_2)} \langle v_j^k, \delta_x \rangle^2 \Bigg)^{1/2} \\ &\leq e^{(\kappa/m)\lambda_1^D(\overline{\xi}_k/\kappa_2)} \| 1\!\!1_{Q^{\kappa \log \kappa}} \|_2 \|\delta_x\|_2 \leq e^{(\kappa/m)\lambda_1^D(\overline{\xi}_k/\kappa_2)} \sqrt{|Q^{\kappa \log \kappa}|}. \end{aligned} \tag{3.7.22}$$

Combining (3.7.14)–(3.7.22), we get

$$\begin{aligned} \log E_x \Bigg(\exp\bigg\{ \int_0^A \overline{\xi}(X^\kappa(s), A-s)\, ds \bigg\} 1\!\!1\big\{N(X^\kappa; A) \leq M\kappa A\big\} \Bigg) \\ \leq \frac{Am}{\kappa_1} \log((2\kappa \log \kappa - 1)^{d/2}) + \sum_{k=1}^{Am} \frac{\kappa_2}{m} \lambda_1^D(\overline{\xi}_k/\kappa_2). \end{aligned} \tag{3.7.23}$$

Finally, by the Rayleigh-Ritz principle we have that $\lambda_1^D(\overline{\xi}_k/\kappa_2) \leq \lambda_1(\overline{\xi}_k/\kappa_2)$, where $\lambda_1(\overline{\xi}_k/\kappa_2)$ is the top of the spectrum of $\Delta + \overline{\xi}_k/\kappa_2$. ∎

3.7.4.2 A spectral estimate

Let $(B(x))_{x\in\mathbb{Z}^d}$ be an arbitrary partition of $\mathbb{Z}^d$ into finite boxes. Let $\langle\cdot,\cdot\rangle$ be the scalar product on $\mathbb{R}^B$ and on $\ell^2(\mathbb{Z}^d)$. Let $V\colon\ \mathbb{Z}^d\to\mathbb{R}$ be bounded such that there is a $\delta>0$ for which

$$\frac{1}{|B(x)|}\sum_{y\in B(x)} V(y)\le 2\delta, \qquad x\in\mathbb{Z}^d. \tag{3.7.24}$$

The proof of the following lemma is deferred to Appendix 3.9.

Lemma 3.7.6. *Subject to* (3.7.24)*, there is a* $\kappa_0>0$ *such that, for all* $\kappa\ge\kappa_0$*,*

$$\sup_{\substack{f\in\ell^2(\mathbb{Z}^d)\\ \|f\|_2=1}} \Big\langle\Big(\Delta+\frac{1}{\kappa}V\Big)f,f\Big\rangle\le 4\frac{1}{\kappa}\delta. \tag{3.7.25}$$

Lemma 3.7.6 and the Rayleigh-Ritz principle yield that the top of the spectrum of $\Delta+\frac{1}{\kappa}V$ is bounded from above by $4\frac{1}{\kappa}\delta$ for $\kappa\ge\kappa_0$.

3.7.5 Completion of the proof of Proposition 3.7.4

Proof. Fix $\delta>0$, $A>1$ and $m>1$. By Lemma 3.7.5, for $\kappa=\kappa_1\kappa_2$, $\kappa_1>1$, there is a $\kappa_0>0$ such that, for all $\kappa\ge\kappa_0$ and ξ-a.s. for all $x\in\mathbb{Z}^d$,

$$\begin{aligned}&\log E_x\left(\exp\left\{\int_0^A\overline{\xi}(X^\kappa(s),A-s)\,ds\right\}\mathbb{1}\big\{N(X^\kappa;A)\le M\kappa A\big\}\right)\\ &\quad\le\frac{Am}{\kappa_1}\log\big((2\kappa\log\kappa-1)^{d/2}\big)+\sum_{k=1}^{Am}\frac{\kappa_2}{m}\lambda_1(\overline{\xi}_k/\kappa_2),\qquad \kappa\ge\kappa_0=\kappa_0(M,A).\end{aligned} \tag{3.7.26}$$

Next, by Lemma 3.7.6 with $V=\overline{\xi}_k$, $k\in\{1,2,\dots,Am\}$ (recall (3.7.16)) and $B(x)=\Pi_1(B_1^A(x,0))$ (recall (3.1.13); Π_1 denotes the projection onto the spatial coordinates), there is a $\widetilde{\kappa}_2>0$ such that, for all $\kappa_2\ge\widetilde{\kappa}_2$ and all $k\in\{1,2,\dots,Am\}$,

$$\lambda_1(\overline{\xi}_k/\kappa_2)\le 4\frac{1}{\kappa_2}\delta. \tag{3.7.27}$$

We fix $\widetilde{\kappa}_2$. Then, there is a $\widetilde{\kappa}_1=\widetilde{\kappa}_1(m,\widetilde{\kappa}_2)$ such that

$$\frac{m}{\kappa_1}\log\big((2\kappa\log\kappa-1)^{d/2}\big)\le\delta,\qquad \forall\,\kappa_1\ge\widetilde{\kappa}_1. \tag{3.7.28}$$

This shows that, ξ-a.s. for $\kappa\ge\widetilde{\kappa}_1\widetilde{\kappa}_2$, any block $B_1^{A,4M\kappa}(x,0)$, $x\in\mathbb{Z}^d$, is ε-sufficient at level 1. The stationarity of ξ in time completes the proof. ∎

Appendix

3.8 Proof of Lemmas 3.4.3–3.4.4

In this section we prove two lemmas that were used in Section 3.4.

3.8.1 Proof of Lemma 3.4.3

Proof. Our first observation is that

$$\limsup_{t\to\infty}\frac{1}{t}\log E_0\left(\exp\left\{\sum_{i=1}^{n}\beta_i\sum_{x\in I_i}l_t(X^\kappa,x)\right\}\right)\leq\mu, \tag{3.8.1}$$

where

$$\mu=\sup_{\substack{f\in\ell^2(\mathbb{Z})\colon\\ \|f\|_2=1,\,f\geq 0}}\mu(f) \tag{3.8.2}$$

with

$$\mu(f)=\left\langle f,\left[\kappa\Delta+\sum_{i=1}^{n}\beta_i\mathbb{1}_{I_i}\right]f\right\rangle=\sum_{i=1}^{n}\beta_i\sum_{x\in I_i}f^2(x)-\kappa\sum_{x\in\mathbb{Z}}[f(x+1)-f(x)]^2. \tag{3.8.3}$$

Indeed, this follows from the large deviation principle for the occupation time measure of one-dimensional simple random walk on $\mathbb{Z}$ with jump rate 2κ (which is the continuous-time Markov process with generator $\kappa\Delta$) in combination with Varadhan's lemma (see [dH00, Chapters 3–4]). A formal proof proceeds by truncating $\mathbb{Z}$ to a large finite torus, wrapping the random walk around the torus, deriving the claim for a fixed torus size, letting the torus size tend to infinity, and showing that the variational formula on the finite torus converges to the variational formula on $\mathbb{Z}$. The details are standard and are left to the reader (see [dH00, Chapter 8]).

We claim that μ is the largest eigenvalue of $\kappa\Delta+\sum_{i=1}^{n}\beta_i\mathbb{1}_{I_i}$. Indeed, by [HMO11, Theorem 2.2] the operator $\kappa\Delta+\sum_{i=1}^{n}\beta_i\mathbb{1}_{I_i}$ has at least one eigenvalue. Consequently, the min-max principle [RS4, Theorem XIII.1] yields the claim. The inequality in (3.4.20) now follows from [S10, Corollary 1.4]. ∎

3.8.2 Proof of Lemma 3.4.4

Proof. Let $t,\kappa>0$ and let X^κ be one-dimensional random walk with step rate $2\kappa>0$. We first show that for all $C>0$,

$$P_0\left(\sup_{0\leq s\leq t}|X^\kappa(s)|\geq C\sqrt{\kappa t}\right)\leq 2e^{-\frac{C^2\sqrt{\kappa t}}{2(C+3\sqrt{\kappa t})}}. \tag{3.8.4}$$

The proof is based on a discretization argument in combination with Bernstein's inequality. Fix $n \in \mathbb{N}$ with $n \gg \kappa$, and define

$$q^n(y) = \begin{cases} 1 - \frac{2\kappa}{n}, & y = 0 \\ \frac{\kappa}{n}, & y = \pm 1 \\ 0, & \text{elsewhere.} \end{cases} \tag{3.8.5}$$

Let $X^{(n)} = (X^{(n)}(t))_{t\geq 0}$ be the discrete-time random walk with jump distribution q^n and jump times k/n, $k \in \mathbb{N}$. Then, for each $t > 0$, $(X^{(n)}(s))_{0\leq s\leq t}$ converges weakly as $n \to \infty$ to $(X^\kappa(s))_{0\leq s\leq t}$ in the Skorokhod space $D([0,t],\mathbb{Z})$. Since $X^{(n)}$ is unlikely to move in a short time interval, uniformly in n, it is enough to prove (3.8.4) for $X^{(n)}$ with n fixed. To that end, let $k \in \mathbb{N}$ be such that $k/n \leq t < (k+1)/n$. Note that, because $X^{(n)}$ is a martingale, Doob's maximal inequality and Bernstein's inequality yield

$$P_0\Big(\sup_{0\leq s\leq t} X^{(n)}(s) \geq C\sqrt{\kappa t}\Big) \leq \exp\Big\{ - \frac{C^2 \kappa t}{2(C\sqrt{\kappa t} + 3\kappa t)} \Big\} \qquad \forall\, C > 0. \tag{3.8.6}$$

The same inequality is valid for the probability of $\sup_{0\leq s\leq t}[-X^{(n)}(s)] \geq C\sqrt{\kappa t}$, which yields the claim in (3.8.4).

Next, note that if X^κ leaves the spatial part of a space-time block $B_1(x,k;\kappa)$, then there is at least one coordinate $j \in \{1,\dots,d\}$ such that $\pi_j(X^\kappa(s)) \notin [\sqrt{\kappa}x(j)A, \sqrt{\kappa}(x(j)+1)A)$ for some $s \in [kA,(k+1)A)$, where $\pi_j(X^\kappa)$ denotes the projection of X^κ onto the j-th coordinate. In particular, if X^κ visits l_i^κ $(\kappa,1)$-blocks with $l_i^\kappa > d$ in the time interval $[(i-1)A, iA)$, then there is at least one coordinate that visits at least l_i^κ/d one-dimensional $(\kappa,1)$-blocks, i.e., blocks of the form $[\sqrt{\kappa}xA, \sqrt{\kappa}(x+1)A) \times [kA,(k+1)A)$, $x \in \mathbb{Z}$. Consequently, without loss of generality we may assume that X^κ is one-dimensional simple random walk with step rate 2κ. Given $l_1^\kappa,\dots,l_{t/A}^\kappa \in \mathbb{N}$, we say that X^κ has label $(l_1^\kappa,\dots,l_{t/A}^\kappa)$ when X^κ crosses l_i^κ $(\kappa,1)$-blocks in the time interval $[(i-1)A, iA)$, $1 \leq i \leq t/A$.

Next, fix $C_7 > 0$ and let $k_* \geq C_7 t$, and note that

$$P_0\Big(X^\kappa \text{ crosses } k_* \ (\kappa,1)\text{-blocks}\Big) = \sum_{\substack{(l_i^\kappa)_{i=1}^{t/A}:\\ \sum l_i^\kappa = k_*}} P_0\Big(X^\kappa \text{ has label } (l_1^\kappa,\dots,l_{t/A}^\kappa)\Big). \tag{3.8.7}$$

Using the Markov property and the fact that a path crossing l_i^κ $(\kappa,1)$-blocks has to travel a distance at least $(l_i^\kappa - 2)\sqrt{\kappa}A/2$, we may further estimate each summand in the right-hand side of (3.8.7) by

$$\begin{aligned} &E_0\Big(1\!\!1\{X^\kappa \text{ has label } (l_1^\kappa,\dots,l_{t/A-1}^\kappa)\} P_{X^\kappa(t-A)}\big(X^\kappa \text{ has label } l_{t/A}^\kappa\big)\Big) \\ &\leq E_0\Big(1\!\!1\{X^\kappa \text{ has label } (l_1^\kappa,\dots,l_{t/A-1}^\kappa)\} P_{X^\kappa(t-A)}\Big(\sup_{0\leq s\leq A} |X^\kappa(s)| \geq (l_i^\kappa - 2)\sqrt{\kappa}A/2\Big)\Big). \end{aligned} \tag{3.8.8}$$

To proceed, note that, by (3.8.4),

$$P_{X^\kappa(t-A)}\Big(\sup_{0\le s\le A}|X^\kappa(s)|\ge (l_i^\kappa-2)\sqrt{\kappa}A/2\Big)\le\begin{cases}2\exp\Big\{-\frac14\frac{(l_i^\kappa-2)^2A\sqrt{\kappa A}}{(l_i^\kappa-2)\sqrt{A}+6\sqrt{\kappa A}}\Big\}, & l_i^\kappa\ge 3\\ 1, & l_i^\kappa\le 2.\end{cases} \tag{3.8.9}$$

An iteration of the estimates in (3.8.8) yields that the left-hand side of (3.8.7) is bounded from above by

$$\sum_{\substack{(l_i^\kappa)_{i=1}^{t/A}:\\ \sum l_i^\kappa=k_*}}\prod_{\substack{i=1\\ l_i^\kappa\ge 3}}^{t/A}2\exp\Big\{-\frac14\frac{(l_i^\kappa-2)^2A\sqrt{\kappa A}}{(l_i^\kappa-2)\sqrt{A}+6\sqrt{\kappa A}}\Big\}. \tag{3.8.10}$$

We have

$$\sum_{\substack{i=1:\\ l_i^\kappa\ge 3}}^{t/A}\frac{(l_i^\kappa-2)^2}{(l_i^\kappa-2)\sqrt{A}+6\sqrt{\kappa A}}\ge\sum_{\substack{i=1:\\ l_i^\kappa\ge 3}}^{t/A}\frac{(l_i^\kappa-2)}{\sqrt{A}+6\sqrt{\kappa A}}. \tag{3.8.11}$$

Moreover,

$$\sum_{\substack{i=1:\\ l_i^\kappa\ge 3}}^{t/A}(l_i^\kappa-2)\ge k_*-\frac{4t}{A}. \tag{3.8.12}$$

Inserting (3.8.11–3.8.12) into (3.8.10), noting that [HR18, E42]

$$\exists\, a,b>0:\ \ \text{number of summands in right-hand side of (3.8.7)}\ \le ae^{b\sqrt{k_*}}/k_*, \tag{3.8.13}$$

and choosing C_5 and κ large enough ($C_5>4$ is sufficient), we get that the right-hand side of (3.8.7) is at most $e^{-C_6Ak_*}$. Here, C_6 is such that for all $k_*\ge C_5t/A$ the inequality $(k_*-4t/A)/(1/\sqrt{\kappa}+6)\ge C_6k_*$ holds, which for $\kappa\ge 1$ is fulfilled when $C_5(1-7C_6)\ge 4$. This yields the claim in (3.4.21). ∎

3.9 Proof of Lemma 3.7.6

In Sections 3.9.1–3.9.3 we prove a lemma that was used in Section 3.7. The proof is inspired by [A10, Theorem 12].

3.9.1 Neumann boundary conditions

In this section we recall the definition and some properties of the discrete Laplacian with Neumann boundary conditions. For further details we refer the reader to [K07].

Fix $x\in\mathbb{Z}^d$, $A>1$ and define the matrix $M_{B(x)}$ as

$$M_{B(x)}(y,z)=\begin{cases}1, & \text{if } y,z\in B(x), \|y-z\|=1,\\ 0, & \text{otherwise,}\end{cases} \tag{3.9.1}$$

and the number of neighbors of y in $B(x)$ as

$$n_{B(x)}(y) = |\{z \in B(x)\colon\ \|y-z\| = 1\}|. \tag{3.9.2}$$

Definition 3.9.1. *The Neumann Laplacian $\Delta_{B(x)}$ on $B(x)$ is defined via the formula*

$$\Delta_{B(x)} = M_{B(x)} - n_{B(x)}, \tag{3.9.3}$$

where $n_{B(x)}$ is the multiplication operator with the function $n_{B(x)}$.

Remark 3.9.2. *The quadratic form associated with $\Delta_{B(x)}$ is given by*

$$\left\langle \Delta_{B(x)} f, g \right\rangle = -\frac{1}{2} \sum_{\substack{y,z \in B(x) \\ \|y-z\|=1}} [f(y)-f(z)]\,[g(y)-g(z)], \qquad f, g \in \mathbb{R}^{B(x)}. \tag{3.9.4}$$

($\Delta_{B(x)}$ does not see that $B(x)$ is imbedded in $\mathbb{Z}^d$, which is why it is sometimes referred to as the graph Laplacian on $B(x)$.)

Lemma 3.9.3. *The following properties hold for all $x \in \mathbb{Z}^d$ and $A > 1$.*
(a) $\langle \Delta_{B(x)} f, f\rangle \leq 0$ for all $f \in \mathbb{R}^{B(x)}$.
(b) $\Delta_{B(x)}$ is self-adjoint.
(c) $\ker(\Delta_{B(x)}) = \mathbb{R}\mathbb{1}$, where $\mathbb{1}$ is the vector in $\mathbb{R}^{B(x)}$ with all entries equal to one.
(d) For all $f \in \ell^2(\mathbb{Z}^d)$,

$$\langle \Delta f, f\rangle \leq \sum_{x \in \mathbb{Z}^d} \left\langle \Delta_{B(x)} f^{B(x)}, f^{B(x)} \right\rangle, \tag{3.9.5}$$

where $f^{B(x)}$ is the restriction of f to $B(x)$.

Proof. Fix $x \in \mathbb{Z}^d$ and $A > 1$.
(a) and (b) are consequences of Remark 3.9.2.
(c) From Remark 3.9.2 it is clear that constant functions are in the kernel of $\Delta_{B(x)}$. For the reverse direction, let $f \in \ker(\Delta_{B(x)})$. Again by Remark 3.9.2,

$$0 = \left\langle \Delta_{B(x)} f, f \right\rangle = -\frac{1}{2} \sum_{\substack{y,z \in B(x) \\ \|y-z\|=1}} [f(y)-f(z)]^2. \tag{3.9.6}$$

Hence, for all $y \in B(x)$ we have that $f(y) = f(z)$ for all z such that $\|y-z\| = 1$, $z \in B(x)$.
(d) Let $f \in \ell^2(\mathbb{Z}^d)$. Then

$$\begin{aligned} -2\langle \Delta f, f\rangle &= \sum_{\substack{y,z \in \mathbb{Z}^d \\ \|y-z\|=1}} [f(y)-f(z)]^2 = \sum_{x \in \mathbb{Z}^d} \sum_{y \in B(x)} \sum_{\substack{z \in \mathbb{Z}^d \\ \|y-z\|=1}} [f(y)-f(z)]^2 \\ &\geq \sum_{x \in \mathbb{Z}^d} \sum_{y \in B(x)} \sum_{\substack{z \in B(x) \\ ||y-z||=1}} [f(y)-f(z)]^2 = -2 \sum_{x \in \mathbb{Z}^d} \left\langle \Delta_{B(x)} f^{B(x)}, f^{B(x)} \right\rangle, \end{aligned} \tag{3.9.7}$$

where the second equality uses that $(B(x))_{x\in\mathbb{Z}^d}$ is a partition of $\mathbb{Z}^d$, while the third equality follows from Remark 3.9.2. ∎

3.9.2 Proof of Lemma 3.7.6 subject to a further lemma

Let $\|\cdot\|_2$ stand for both the Euclidean norm on $\mathbb{R}^B$ and the ℓ^2-norm on $\ell^2(\mathbb{Z}^d)$.

Lemma 3.9.4. *Subject to* (3.7.24)*, there is a* $\kappa_0 > 0$ *such that, for all* $\kappa \geq \kappa_0$*, all* $f \in \mathbb{R}^{B(x)}$ *and all* $x \in \mathbb{Z}^d$*,*

$$\Big\langle (\Delta_{B(x)} + \frac{1}{\kappa} V) f, f \Big\rangle \leq 4 \frac{1}{\kappa} \delta \|f\|_2^2. \tag{3.9.8}$$

The proof of Lemma 3.9.4 is deferred to Section 3.9.3. First we complete the proof of Lemma 3.7.6 subject to Lemma 3.9.4.

Proof. Let $f \in \ell^2(\mathbb{Z}^d)$ and $\kappa \geq \kappa_0$, where κ_0 is chosen according to Lemma 3.9.4. Then, by Lemma 3.9.3(d) and the fact that $(B(x))_{x\in\mathbb{Z}^d}$ is a partition of $\mathbb{Z}^d$, we may estimate

$$\Big\langle (\Delta + \frac{1}{\kappa} V) f, f \Big\rangle \leq \sum_{x\in\mathbb{Z}^d} \Big\langle \Big(\Delta_{B(x)} + \frac{1}{\kappa} V\Big) f^{B(x)}, f^{B(x)} \Big\rangle. \tag{3.9.9}$$

Since $f = \sum_{x\in\mathbb{Z}^d} f^{B(x)}$, we have $\|f\|^2 = \sum_{x\in\mathbb{Z}^d} \|f^{B(x)}\|_2^2$. Combining (3.9.8–3.9.9), we get the claim. ∎

3.9.3 Proof of Lemma 3.9.4

Proof. Fix $x \in \mathbb{Z}^d$ and $A > 1$. First recall that, by Lemma 3.9.3(c), $\ker(\Delta_{B(x)}) = \mathbb{R}1\!\!1$, so that we can write each $f \in \mathbb{R}^{B(x)}$ as $f = \alpha 1\!\!1 + g$, $\alpha \in \mathbb{R}$, $g \in \big(\mathbb{R}1\!\!1\big)^\perp$. Therefore, using that $\Delta_{B(x)}$ is self-adjoint, see Lemma 3.9.3(b) and hence symmetric, we obtain that

$$\Big\langle \Big(\Delta_{B(x)} + \frac{1}{\kappa} V\Big) f, f \Big\rangle = \Big\langle \Delta_{B(x)} g, g \Big\rangle + \frac{1}{\kappa} \big[\alpha^2 \langle V1\!\!1, 1\!\!1\rangle + 2\alpha \langle V1\!\!1, g\rangle + \langle Vg, g\rangle\big]. \tag{3.9.10}$$

Using that the unit sphere intersected with $\big(\mathbb{R}1\!\!1\big)^\perp$ is compact, that $\Delta_{B(x)}$ is negative on $\big(\mathbb{R}1\!\!1\big)^\perp$, see Lemma 3.9.3(a), and that the scalar product is continuous, we deduce that there is an $\eta > 0$ such that $\langle \Delta_{B(x)} h, h\rangle \leq -\eta$ for all $h \in \big(\mathbb{R}1\!\!1\big)^\perp$ with $\|h\|_2 = 1$. Hence we may estimate the right-hand side of (3.9.10) from above by

$$-\eta \|g\|_2^2 + \frac{1}{\kappa} \big[\alpha^2 \langle V1\!\!1, 1\!\!1\rangle + 2\alpha \langle V1\!\!1, g\rangle + \langle Vg, g\rangle\big]. \tag{3.9.11}$$

Next, by (3.7.24), we have $\langle V1\!\!1, 1\!\!1\rangle = \sum_{y\in B(x)} V(y) \leq 2\delta |B(x)|$. An additional application of the Cauchy-Schwarz inequality shows that (3.9.11) is at most

$$-\eta \|g\|_2^2 + \frac{1}{\kappa} \big[\alpha^2 2\delta |B(x)| + 2\alpha \|V1\!\!1\|_2 \|g\|_2 + \|Vg\|_2 \|g\|_2\big]. \tag{3.9.12}$$

Using the bound $\|Vg\|_2 \le \|V\|_\infty \|g\|_2$ (which also holds for g replaced by $\mathbb{1}$) and the assumption that V is bounded, we may further estimate (3.9.12) from above by

$$-\eta\|g\|_2^2 + \frac{1}{\kappa}\big[\alpha^2 2\delta|B(x)| + 2\alpha\|V\|_\infty\|\mathbb{1}\|_2\|g\|_2 + \|V\|_\infty\|g\|_2^2\big]. \tag{3.9.13}$$

For any $a, b \in \mathbb{R}$ and $\gamma > 0$ we have the inequality $2ab \le \gamma a^2 + b^2/\gamma$. Pick $a = \alpha\|V\|_\infty\|\mathbb{1}\|_2$ and $b = \|g\|_2$. Then we may further estimate (3.9.13) from above by

$$\begin{aligned} &-\eta\|g\|_2^2 + \frac{1}{\kappa}\big[\alpha^2 2\delta|B(x)| + \gamma\alpha^2\|V\|_\infty^2\|\mathbb{1}\|_2^2 + \|g\|_2^2/\gamma + \|g\|_2^2\big] \\ &= \|g\|_2^2\Big[-\eta + \frac{1}{\kappa}\Big(1 + \frac{1}{\gamma}\Big)\Big] + \frac{1}{\kappa}\big[2\delta + \gamma\|V\|_\infty^2\big]\alpha^2\|\mathbb{1}\|_2^2, \end{aligned} \tag{3.9.14}$$

where we use that $\|\mathbb{1}\|_2^2 = |B(x)|$. Now pick $\gamma = 2\delta/\|V\|_\infty^2$ and note that, for κ large enough so that $\frac{1}{\kappa}(1 + 1/\gamma - 4\delta) \le \eta$, we have $-\eta + \frac{1}{\kappa}(1 + 1/\gamma) \le 4\frac{1}{\kappa}\delta$. Therefore, for κ large enough we may estimate the right-most term in (3.9.14) from above by

$$4\frac{1}{\kappa}\delta\Big[\|g\|_2^2 + \alpha^2\|\mathbb{1}\|_2^2\Big] = 4\frac{1}{\kappa}\delta\|f\|_2^2. \tag{3.9.15}$$

∎

Part II Long–Range Percolation

4 Introduction to Part II

This chapter serves as an introduction to Chapters 5 and 6, where two long–range percolation models are studied. In Section 4.1 we give a short introduction to Bernoulli bond percolation, which is the most standard of all percolation models. In Section 4.2 the first model is presented and results of Chapter 5 are listed. Section 4.3 contains a short introduction to the second model, random interlacements, and moreover the result of Chapter 6 is listed.

4.1 Bernoulli bond percolation

This section is based on Chapter 1 of Grimmett [G00].

The standard introductory example of percolation theory is the following: Suppose that a large porous stone is immersed into a bucket of water. What is the probability that the center of the stone gets wet? This can be modelled in terms of a simple stochastic model, nowadays known as percolation model, due to Broadbent and Hammersley [BH57]. Let $p \in [0, 1]$ and declare each edge of $\mathbb{Z}^d$ to be open, independently of all other edges, with probability p and closed otherwise. The edges of $\mathbb{Z}^d$ represent the passageways of the stone, and the parameter p is the proportion of passages that are broad enough to allow water to pass along them. The stone is modelled as a large finite subset of $\mathbb{Z}^d$ and a vertex x inside the stone is wet when there is a path of open edges from x to the boundary of the stone. Percolation theory is mainly concerned with the study of such open paths. Questions of interest are for instance: (1) *Is there an infinite cluster of open paths?* (2) *How many of such infinite clusters are there?* (3) *Consider the graph whose edges are precisely the open edges and whose vertices are the endpoints of those edges. What geometric properties does this graph have?*

The above described model is referred to as Bernoulli bond percolation and it is the most studied of all percolation models. The motivations are manyfold. (1) The model is easy to formulate but not unrealistic in qualitative predictions for random media. (2) It is a simple model in which the phenomenon of a phase transition may be observed, i.e., when $d \geq 2$ it can be shown that there is a $p_c \in (0, 1)$ such that for all $p < p_c$ all clusters of open edges are finite, whereas for all $p > p_c$ there is with probability one a (unique) infinite cluster of open edges. (3) It serves as a jumpboard to develop techniques for more complicated models and even leads to new branches of mathematics. A beautiful example is the theory of the Schramm-Loewner evolution, which arises as the scaling limit of general two-dimensional percolation models in statistical physics at criticality including Bernoulli percolation itself. (4) It leads to many beautiful conjectures that are easy to state but hard to prove. (5) Partial results for models with a complex dependency structure can be obtained by comparing them with Bernoulli bond percolation. Grimmett [G00] states the following example. Consider a physical model having a parameter T, that is interpreted as the temperature of the system. It may be expected that there exists a critical value T_c marking a phase transition. While this fact may itself be unproven, it may be possible to prove by comparison with Bernoulli bond percolation that the behaviour of the process for small T is qualitatively different from that for large T.

4.2 Brownian paths homogeneously distributed in space

The model

For $\lambda > 0$, let $(\Omega, \mathcal{A}, \mathbb{P}_\lambda)$ be a probability space on which a Poisson point process $\mathcal{E}$ with intensity $\lambda \times \mathrm{Leb}_d$ is defined, where Leb_d is the d-dimensional Lebesgue measure. Conditionally on $\mathcal{E}$, we fix a collection of independent Brownian motions $\{(B_t^x)_{t\geq 0},\ x \in \mathcal{E}\}$ such that $B_0^x = x$ for each $x \in \mathcal{E}$ and such that $(B_t^x - x)_{t\geq 0}$ is independent of $\mathcal{E}$. For $t, r \geq 0$ we study the *occupied* set

$$\mathcal{O}_{t,r} := \bigcup_{x\in\mathcal{E}} \bigcup_{0\leq s\leq t} \mathcal{B}(B_s^x, r), \tag{4.2.1}$$

where $\mathcal{B}(y, r)$ denotes the ball with respect to the euclidean norm around $y \in \mathbb{R}^d$ with radius r. If $d \leq 3$, then we put $r = 0$. The reason for that will become clearer when discussing the results. In the remainder of this section we write $\mathcal{O}_t$ instead of $\mathcal{O}_{t,0}$.
We are interested in the percolative properties of $\mathcal{O}_{t,r}$: Is there an unbounded cluster for large t? Is it unique? What happens for small t? Since an elementary monotonicity argument shows that $t \mapsto \mathcal{O}_{t,r}$ is non-decreasing, the first and the third question may be rephrased as follows: Is there a percolation transition in t?

Motivation and related models

The model described above fits into the class of continuum percolation models, which have been studied intensively by both mathematicians and physicists. Their first appearance can be traced back (at least) to Gilbert [G61] under the name of random plane networks. Gilbert was interested in modeling infinite communication networks of stations with range $R > 0$. He did this by connecting each two points of a Poisson point process on $\mathbb{R}^2$ whenever their distance is less than R. Another application, mentioned in his work is the modeling of a contagious infection. Here, each individual gets infected when it has distance less than R to an infected individual.

A subclass of continuum percolation models follows the following recipe: Consider a point process (e.g. a Poisson point process) and attach to each of its points a geometric object, like a disk of random radius (Boolean model) or a segment of random length and random orientation (Poisson sticks model or needle percolation). Our model also falls into this class: to each point of a Poisson point process we attach a Brownian path (a path of a Wiener sausage when $d \geq 4$). This can be seen as a model of defects that are randomly distributed in a material and are propagating at random. We can think for example of an (infinite) piece of wood containing (homogeneously distributed) worms, where each worm eats its way through the piece of wood at random (see Menshikov, Molchanov and Sidorenko [MMS88] for other physical motivations of continuum percolation). The informal description above is reminiscent of (and actually borrowed from) the problem of the disconnection of a cylinder by a random walk, which itself is linked to interlacement percolation [S10]. The latter is defined as the random subset obtained when looking at the trace of a simple random walk on the torus $(\mathbb{Z}/N\mathbb{Z})^d$, starting from the uniform distribution and running up to time uN^d in the limit as $N \uparrow \infty$. Here, u plays the role of an intensity parameter for the interlacement set. However, even though the model of random interlacements and our model seem to share some similarities, there is an important difference: in the interlacement model the number of trajectories that enter a ball of radius R scales like cR^{d-2} for some $c > 0$, whereas in our model it is at least of order R^d.

Another motivation for studying our model is that it should arise as the scaling limit of a class of discrete dependent percolation models, namely a system of independent finite-time random walks homogeneously distributed on $\mathbb{Z}^d$. The latter can also be seen as a system of non-interacting ideal polymer chains.

Results

Fix $\lambda > 0$.

Theorem 4.2.1. *[**No percolation for** $d = 1$] Let $d = 1$. Then, for all $t \geq 0$, the set $\mathcal{O}_t$ has almost surely no unbounded cluster.*

Theorem 4.2.2. *[**Percolation phase transition and uniqueness for** $d = 2, 3$] Let $d = 2, 3$. Then there exists a $t_c = t_c(\lambda, d) > 0$ such that, for $t < t_c$, $\mathcal{O}_t$ has almost surely no unbounded cluster whereas, for $t > t_c$, $\mathcal{O}_t$ has almost surely a unique unbounded cluster.*

Let $d \geq 4$ and $r > 0$. We denote by $\lambda_c(r)$ the critical value such that for all $\lambda < \lambda_c(r)$ the set $\mathcal{O}_{0,r}$ almost surely does not contain an unbounded cluster, whereas for $\lambda > \lambda_c(r)$ it does. Gouéré [G08] showed that $\lambda_c(r) > 0$ for $r > 0$ and $\lim_{r \to 0} \lambda_c(r) = \infty$.

Theorem 4.2.3. *[**Percolation phase transition and uniqueness for** $d \geq 4$] Let $d \geq 4$, and let $r > 0$ be such that $\lambda < \lambda_c(\delta_r)$. Then there exists a $t_c = t_c(\lambda, d, r) > 0$ such that, for $t < t_c$, $\mathcal{O}_{t,r}$ has almost surely no unbounded cluster whereas, for $t > t_c$, $\mathcal{O}_{t,r}$ has almost surely a unique unbounded cluster.*

Comments on the results

Theorems 4.2.1–4.2.3 describe a phase transition in t. It would be possible to play with the intensity λ instead. Indeed, when we multiply the intensity λ by a factor η

we change the typical distance between two Poisson points by a factor $\eta^{-1/d}$. By scale invariance of Brownian motion, the percolative behaviour of the model is the same when we consider the Brownian paths up to time $\eta^{-2/d}t$ instead. Hence, tuning λ boils down to tuning t.

It is worthwhile to mention that Theorem 4.2.2 is stated only in the case $r = 0$, which is the case of interest to us. The result is the same when $r > 0$, up to minor modifications. However, if $d \geq 4$, then the paths of two independent d-dimensional Brownian motions starting at different points do not intersect and r has to be chosen positive, otherwise no percolation phase transition occurs.

To sum up, the above settle the first questions typically asked when studying a new percolation model. Many challenges are open. One may wonder, for instance, how fast is the decay of the probability (in the supercritical regime) that a ball of a certain size, centered at the origin, is contained in the vacant set. Moreover, it would be interesting to investigate the scaling behaviour of t_c in dimension $d \geq 4$ as r tends to zero. One could ask for sharp upper and lower bounds on t_c. Finally, it is not clear whether percolation occurs at t_c or not.

4.3 Random interlacements

Introduction to the model

The model of random interlacements has been introduced by Sznitman [S10] as a family of random subsets of $\mathbb{Z}^d$ denoted by $\mathcal{I}^u$, $u \geq 0$, where u plays the role of an intensity parameter. $\mathcal{I}^u$ locally "looks like" the trace of a simple random walk on the discrete torus $(\mathbb{Z}/N\mathbb{Z})^d$ run up to time uN^d (see Windisch [W08], Teixeira and Windisch [TW11]). With the help of the inclusion-exclusion formula the distribution of the set $\mathcal{I}^u$ can also be characterized as

$$\mathbb{P}[K \cap \mathcal{I}^u = \emptyset] = e^{-u\,\mathrm{cap}(K)}, \qquad K \subset\subset \mathbb{Z}^d.$$

Here, $\mathrm{cap}(K)$ denotes the capacity of the compact set K, defined as

$$\mathrm{cap}(K) := \sum_{x \in K} e_K(x), \quad \text{with} \quad \mathrm{e}_K(x) := P_x\big[\widetilde{H}_K = \infty\big]\,\mathbb{1}_{\{x \in K\}}, \tag{4.3.1}$$

where P_x denotes the law of simple random walk $w = (w(n))_{n \in \mathbb{N}_0}$ started at $x \in \mathbb{Z}^d$ and $\widetilde{H}_K$ denotes the first hitting time of the set K by the random walk:

$$\widetilde{H}_K(w) = \inf\{n \in \mathbb{N} \colon\ w(n) \in K\}. \tag{4.3.2}$$

In a more constructive fashion, random interlacements at level u can also be obtained by considering the trace of the elements in the support of a Poisson point process with intensity parameter u, taking values in the space of locally finite measures on doubly infinite simple random walk trajectories modulo time shifts. This constructive definition suggests that the model exhibits long–range dependence. Indeed the asymptotics

$$\mathrm{Cov}(\mathbb{1}_{x\in\mathcal{I}^u}, \mathbb{1}_{y\in\mathcal{I}^u}) \sim c(u)|x-y|_2^{-(d-2)}, \tag{4.3.3}$$

where $|\cdot|_2$ denotes the euclidean norm on $\mathbb{Z}^d$ (and similarly for $\mathcal{I}^u$ replaced by $\mathcal{V}^u = \mathbb{Z}^d \setminus \mathcal{I}^u$) holds for $|x-y|_2 \to \infty$, as can be deduced from (0.11) in Sznitman [S10]. As a consequence, standard techniques from Bernoulli percolation do not apply. For example, due to (4.3.3) the Peierls argument and the van den Berg-Kesten inequality break down. The long–range dependence also entails that random interlacements neither stochastically dominate nor can be dominated by Bernoulli percolation (see Remark 1.6 (1) of [S10]). Moreover random interlacements do not fulfill the finite energy property (see Remark 2.2 (3) of [S10]). These features make the model more appealing, and at the same time more complicated to investigate.

State of the art and motivation

During the past couple of years there has been intensive research on random interlacements. Basic properties such as the shift-invariance, ergodicity and connectedness of $\mathcal{I}^u$ have been established in Sznitman [S10]. Since then, a deeper understanding has been obtained of the geometry of random interlacements. Ráth and Sapozhnikov [RS11] have shown transience for random interlacements $\mathcal{I}^u$ throughout the whole range of parameters $u \in (0,\infty)$. Ráth and Sapozhnikov [RS10] and Procaccia and Tykesson [PT11] have shown that any two points of the set $\mathcal{I}^u$ can be connected by using at most $\lceil d/2 \rceil$ trajectories from the constructive definition described above. Recently, with the help of extensions of the techniques in [RS10], this result has been generalized to an arbitrary number of points by Lacoin and Tykesson [LT12]. Another step in showing that the geometry of random interlacements resembles that of $\mathbb{Z}^d$ has been undertaken by Černý and Popov [CP12], who prove that the chemical distance (also called graph distance or internal distance) in the set $\mathcal{I}^u$ is comparable to that of $\mathbb{Z}^d$. Using this result, they prove a shape theorem for balls in $\mathcal{I}^u$ with respect to the metric induced by the chemical distance.

It is particularly interesting to obtain a deeper understanding of the vacant set $\mathcal{V}^u$ and its geometry. On the one hand, this is more challenging than the investigation of $\mathcal{I}^u$, in the sense that one cannot directly take advantage of the many tools available for simple random walk, that have proven to be very helpful in understanding the set $\mathcal{I}^u$. On the other hand, it has been shown by Sznitman [S10] and Sidoravicius and Sznitman [SS09] that there exists a non-trivial percolation phase transition for $\mathcal{V}^u$ at some $u_*(d) \in (0,\infty)$ in the following sense: For $u > u_*(d)$ the vacant set $\mathcal{V}^u$ as a subgraph of $\mathbb{Z}^d$ contains only finite connected components (subcritical phase), whereas for $u \in [0, u_*(d))$ it contains an infinite connected component almost surely (supercritical phase). Using the techniques of Burton and Keane [BK89], and taking care of the difficulties that arise from the lack of the finite energy property for random interlacements, Teixeira [T09] has shown uniqueness of the infinite connected component of $\mathcal{V}^u$ (denoted by $\mathcal{V}^u_\infty$) in the supercritical

phase.
To the best of our knowledge, our result is the first that is valid throughout most of the supercritical phase.

The result

Recall that a connected graph $G = (V, E)$ with finite degree, with vertex set V and edge set E, is called transient if simple random walk on G is transient.

Theorem 4.3.1. *Let $\varepsilon \in (0, 1)$. There is a $d_0 = d_0(\varepsilon) \in \mathbb{N}$ such that, for all $d \geq d_0$ and all $u \leq (1 - \varepsilon)u_*(d)$, the unique infinite connected component $\mathcal{V}^u_\infty$ of the vacant set $\mathcal{V}^u$ of random interlacements in $\mathbb{Z}^d$ is transient $\mathbb{P}$-a.s.*

Comments on the result

Theorem 4.3.1 provides a rough geometric description of the infinite connected component of the vacant set that is valid throughout most of the supercritical phase when d is large enough.

Besides the challenge to extend our result to the entire supercritical phase, it would be interesting to obtain a more precise understanding of $\mathcal{V}^u_\infty$. Results in this direction have been obtained in Drewitz, Ráth, Sapozhnikov and Procaccia, Rosenthal, Sapozhnikov [DRS12b, PRS13]. A key assumption in these papers is a local uniqueness property (in our context of $\mathcal{V}^u_\infty$), which roughly states that with high probability the second largest component in a macroscopic box is small compared to the largest connected component in the same box. This local uniqueness property has so far only been established for a non-degenerate part of the supercricital phase, and obtaining its validity throughout the whole supercritical phase would be an interesting topic for further investigation.

5 Brownian percolation

This chapter is based on:
D. Erhard, J. Martínez, J. Poisat. *Brownian paths homogeneously distributed in space: percolation phase transition and uniqueness of the unbounded cluster.*
Posted on *arXiv:1311.2907v1*, submitted to Electronic Journal of Probability.

Abstract

We consider a continuum percolation model on $\mathbb{R}^d$, $d \geq 1$. For $t, \lambda \in (0, \infty)$ and $d \in \{1, 2, 3\}$, the occupied set is given by the union of independent Brownian paths running up to time t whose initial points form a Poisson point process with intensity $\lambda > 0$. When $d \geq 4$, the Brownian paths are replaced by Wiener sausages with radius $r > 0$.
We establish that, for $d = 1$ and all choices of t, no percolation occurs, whereas for $d \geq 2$, there is a non-trivial percolation transition in t, provided λ and r are chosen properly. The last statement means that λ has to be chosen to be strictly smaller than the critical percolation parameter for the occupied set at time zero (which is infinite when $d \in \{2, 3\}$, but finite and dependent on r when $d \geq 4$). We further show that for all $d \geq 2$, the unbounded cluster in the supercritical phase is unique.
Along the line a finite box criterion for non-percolation in the Boolean model is extended to radius distributions with an exponential tail. This may be of independent interest.

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5.1 Introduction

Notation. For every $d \geq 1$, we denote by Leb_d the Lebesgue measure on $\mathbb{R}^d$. $||\cdot||$ and $||\cdot||_\infty$ stand for the Euclidean norm and supremum norm on $\mathbb{R}^d$, respectively. For any set A, the symbol $A^{\complement}$ refers to the complement set of A. The open ball with center z and radius r with respect to the Euclidean norm is denoted by $\mathcal{B}(z,r)$, whereas $\mathcal{B}_\infty(z,r)$ stands for the same ball with respect to the supremum norm. Furthermore, for every $0 < r < r'$, we denote by $\mathcal{A}(r,r') = \mathcal{B}(0,r') \setminus \overline{\mathcal{B}}(0,r)$ and $\mathcal{A}_\infty(r,r') = \mathcal{B}_\infty(0,r') \setminus \overline{\mathcal{B}}_\infty(0,r)$ the annulus delimited by the balls of radii r and r' with respect to the Euclidean norm and supremum norm, respectively. Given a d-dimensional Brownian motion $(B_t)_{t\geq 0}$, we denote its i-th component by $(B_{t,i})_{t\geq 0}$, for $i \in \{1, 2, \dots, d\}$. For all $I \subseteq \mathbb{R}^+$, we denote by B_I the set $\{B_t,\, t \in I\}$. The symbol $\mathbb{P}^a$ denotes the law of a Brownian motion starting in a. Finally, $\mathbb{P}^{a_1,a_2}$ denotes the law of two independent Brownian motions starting in a_1 and a_2, respectively.

5.1.1 Overview and motivation

For $\lambda > 0$, let $(\Omega_p, \mathcal{A}_p, \mathbb{P}_\lambda)$ be a probability space on which a Poisson point process $\mathcal{E}$ with intensity $\lambda \times \text{Leb}_d$ is defined. Conditionally on $\mathcal{E}$, we fix a collection of independent Brownian motions $\{(B^x_t)_{t\geq 0},\, x \in \mathcal{E}\}$ such that for each $x \in \mathcal{E}$, $B^x_0 = x$ and such that $(B^x_t - x)_{t\geq 0}$ is independent of $\mathcal{E}$. A more rigorous definition is provided in Section 5.1.3 below, where ergodic properties are obtained along. We study for $t, r \geq 0$ the *occupied* set (see Figure 5.1 below):

$$\mathcal{O}_{t,r} := \bigcup_{x\in\mathcal{E}} \bigcup_{0\leq s\leq t} \mathcal{B}(B^x_s, r). \tag{5.1.1}$$

In the rest of the chapter, we write $\mathcal{O}_t$ instead of $\mathcal{O}_{t,0}$.

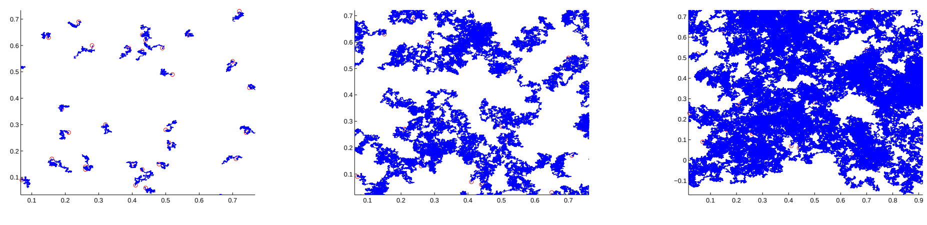

Figure 5.1: Simulations of $\mathcal{O}_t$ in the case $d = 2$, at a small time, intermediate and large time.

Two points x and y of $\mathbb{R}^d$ are said to be *connected* in $\mathcal{O}_{t,r}$ if and only if there exists a continuous function $\gamma : [0,1] \mapsto \mathcal{O}_{t,r}$ such that $\gamma(0) = x$ and $\gamma(1) = y$. A subset of $\mathcal{O}_{t,r}$ is connected if and only if all of its points are pairwise connected. In the following a connected subset of $\mathcal{O}_{t,r}$ is called a component. A component $\mathcal{C}$ is bounded if there exists $R > 0$ such that $\mathcal{C} \subseteq \mathcal{B}(0, R)$. Otherwise, the component is said to be unbounded. A *cluster* is a connected component which is maximal in the sense that it is not strictly contained in another connected component.

We are interested in the percolative properties of the occupied set: is there an unbounded cluster for large t? Is it unique? What happens for small t? Since an elementary monotonicity argument shows that $t \mapsto \mathcal{O}_{t,r}$ is non-decreasing, the first and the third question may be rephrased as follows: is there a percolation transition in t?

5.1.2 Results

We fix $\lambda > 0$.

Theorem 5.1.1. *[**No percolation for** $d = 1$] Let $d = 1$. Then, for all $t \geq 0$, the set $\mathcal{O}_t$ has almost surely no unbounded cluster.*

Theorem 5.1.2. *[**Percolation phase transition and uniqueness for** $d = 2, 3$] Let $d = 2, 3$. Then, there exists $t_c = t_c(\lambda, d) > 0$ such that for $t < t_c$, $\mathcal{O}_t$ has almost surely no unbounded cluster, whereas for $t > t_c$, $\mathcal{O}_t$ has almost surely a unique unbounded cluster.*

Let $d \geq 4$, $r > 0$ and let δ_r be the Dirac measure concentrated on r. We denote by $\lambda_c(\delta_r)$ the critical value for $\mathcal{O}_{0,r}$ such that for all $\lambda < \lambda_c(\delta_r)$ the set $\mathcal{O}_{0,r}$ almost surely does not contain an unbounded cluster, and such that for $\lambda > \lambda_c(\delta_r)$ it does, see also (5.2.5). It follows from Theorem 5.2.1, that $\lambda_c(\delta_r) > 0$ and $\lim_{r\to 0} \lambda_c(\delta_r) = \infty$.

Theorem 5.1.3. *[**Percolation phase transition and uniqueness for** $d \geq 4$] Let $d \geq 4$ and let $r > 0$ be such that $\lambda < \lambda_c(\delta_r)$. Then, there exists $t_c = t_c(\lambda, d, r) > 0$ such that for $t < t_c$, $\mathcal{O}_{t,r}$ has almost surely no unbounded cluster, whereas for $t > t_c$, it has almost surely a unique unbounded cluster.*

5.1.3 Construction and an ergodic property

In this section we briefly outline how to construct the model described in Section 5.1.1 and we state an ergodic theorem. The construction is very close to the construction of the Boolean percolation model, in which balls of random radii are placed around each point of a Poisson point process. We refer the reader to Section 1.4 of [MR96], where a more detailed description of the Boolean percolation model is given (see also Section 5.2 in the present work).

Construction. Let $\mathcal{E}$ be a Poisson point process with intensity $\lambda \times \mathrm{Leb}_d$ defined on $(\Omega_p, \mathcal{A}_p, \mathbb{P}_\lambda)$. Consider the family of binary cubes

$$K(n, z) = \prod_{i=1}^{d} (z_i 2^{-n}, (z_i + 1)2^{-n}], \qquad \forall n \in \mathbb{N},\ z = (z_i)_{1\leq i\leq d} \in \mathbb{Z}^d, \tag{5.1.2}$$

so that for each $n \in \mathbb{N}$, $\{K(n,z), z \in \mathbb{Z}^d\}$ is a partition of $\mathbb{R}^d$. In particular, for each $x \in \mathcal{E}$ and $n \in \mathbb{N}$, there exists a unique $z(n,x)$ such that $x \in K(n, z(n,x))$. Consequently, $\mathbb{P}_\lambda$-a.s., for each $x \in \mathcal{E}$,

$$n_0(x) := \inf\{n \geq 1 \ : \ K(n, z(n,x)) \cap \mathcal{E} = \{x\}\} \tag{5.1.3}$$

is well defined. Let $\mathcal{B}(C([0,\infty),\mathbb{R}^d))$ be the Borel σ-algebra on $C([0,\infty),\mathbb{R}^d)$ with respect to the supremum norm. To continue define $\Omega_B = C([0,\infty),\mathbb{R}^d)^{\mathbb{N}\times\mathbb{Z}^d}$, equip Ω_B with the product σ-algebra $\mathcal{A}_B = \mathcal{B}(C([0,\infty),\mathbb{R}^d))^{\mathbb{N}\times\mathbb{Z}^d}$ and let $\mathbb{P}_B = W_B^{\otimes \mathbb{N}\times\mathbb{Z}^d}$, where W_B is the Wiener measure on $C([0,\infty),\mathbb{R}^d)$. The Brownian path associated to $x \in \mathcal{E}$ is defined to be

$$w_B(n_0(x), z(n_0(x), x)), \quad w_B \in \Omega_B. \tag{5.1.4}$$

Finally, we set $\Omega = \Omega_p \times \Omega_B$, $\mathcal{A} = \mathcal{A}_p \times \mathcal{A}_B$ and $\mathbb{P} = \mathbb{P}_\lambda \times \mathbb{P}_B$, so that the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ corresponds to the model described in Section 5.1.1.

Ergodicity. For $x \in \mathbb{Z}^d$ let $T_x : \mathbb{R}^d \to \mathbb{R}^d$ be the translation defined by $T_x(y) = y + x$, $y \in \mathbb{R}^d$. This induces a translation S_x on Ω_p via the equation $(S_x\omega_p)(A) = \omega_p(T_x^{-1}A)$, $A \in \mathcal{A}_p$. A translation on Ω_B is given by the formula $(U_x\omega_B)(n,z) = \omega_B(n, z-x)$, so that we finally can define the translation $\widetilde{T}_x$ on the product space Ω as $\widetilde{T}_x\omega = (S_x\omega_p, U_x\omega_B)$. A simple adaption of the proof of Proposition 2.8 in [MR96] yields the following result.

Proposition 5.1.4. *For all $t, r \geq 0$ the set $\mathcal{O}_{t,r}$ defined in (5.1.1) is ergodic with respect to the family of translations $\{\widetilde{T}_x,\ x \in \mathbb{Z}^d\}$.*

5.1.4 Discussion

Motivation and related models. Our model fits into the class of continuum percolation models, which have been studied by both mathematicians and physicists. Their first appearance can be traced back (at least) to Gilbert [G61] under the name of random plane networks. Gilbert was interested in modeling infinite communication networks of stations with range $R > 0$. He did this by connecting each two points of a Poisson point process on $\mathbb{R}^2$, whenever their distance is less than R. Another application, which is mentioned in his work is the modeling of a contagious infection. Here, each individual gets infected when it has distance less than R to an infected individual.

A subclass of continuum percolation models follows the following recipe: first throw a point process (e.g. Poisson point process) and attach to each of its points a geometric object, like a disk of random radius (Boolean model) or a segment of random length and random orientation (Poisson sticks model or needle percolation). Our model also falls into this class: we attach to each point of a Poisson point process a Brownian path (a path of a Wiener sausage when $d \geq 4$). This can be seen as a model of defects that are randomly distributed in a material and are propagating at random. We can think for example of an (infinite) piece of wood containing (homogeneously distributed) worms, where each worm eats its way through the piece of wood at random (see Menshikov, Molchanov and Sidorenko [MMS88] for other physical motivations of continuum percolation). The informal description above is reminiscent of (and actually, borrowed from) the problem of the disconnection of a cylinder by a random walk, which itself is linked to interlacement percolation [S10]. The latter is given by the random subset obtained when looking at the trace of a simple random walk on the torus $(\mathbb{Z}/N\mathbb{Z})^d$, when started from the uniform distribution and running up to time uN^d, as $N \uparrow \infty$. Here u plays the role of an intensity parameter for the interlacements set. However, even though the model of random interlacements and our model seem to share some similarities, there is an important difference: in the interlacement model, the number of trajectories which enter a ball of radius R scales like cR^{d-2} for some $c > 0$, whereas in our case it is at least of order R^d.

Another motivation for studying such a model is that it should arise as the scaling limit of a certain class of discrete dependent percolation models. More precisely, percolation models for a system of independent finite-time random walks homogeneously distributed on $\mathbb{Z}^d$. This could also be seen as a system of non-interacting ideal polymer chains.

Comments on the results. First of all notice that we investigated a phase transition in t. It would also be possible to play with the intensity λ instead. Indeed, multiplying the intensity λ by a factor η changes the typical distance between two Poisson points by a factor $\eta^{-1/d}$. Thus, by scale invariance of Brownian motion, the percolative behaviour of the model is the same when we consider the Brownian paths up to time $\eta^{-2/d}t$ instead. Hence, tuning λ boils down to tuning t.

Moreover, it is worthwhile mentioning that Theorem 5.1.2 is stated only in the case $r = 0$, which is the case of interest to us. The result is the same when $r > 0$, up to minor modifications. However, if $d \geq 4$ the paths of two independent d-dimensional Brownian motions starting at different points do not intersect. Hence, in this case r has to be chosen positive, otherwise no percolation phase transition occurs.

Besides, we draw the reader's attention to Lemma 5.2.3, which is useful in proving the continuity result in Proposition 5.2.2. This lemma provides a finite-box criterion for non-percolation for the Boolean model. It is stated in the case of radius distributions with exponential tail. To our knowledge such a criterion was only proved for bounded radii.

To sum up, the results proven in this article answer the first questions typically asked when studying a new percolation model. However, there are still many challenges left open. One may wonder for instance how fast the decay of the probability is (in the supercritical regime), that there is a ball of a certain size centered in the origin, which is contained in the vacant set. Moreover, it would be interesting to investigate the scaling behaviour of t_c in dimension $d \geq 4$ as r tends to zero. In the same line one could ask for sharp upper and lower bounds for t_c. Finally, it is not clear whether percolation occurs at t_c.

5.1.5 Outline of the chapter

We shortly describe the organization of the article. In Section 5.2 we introduce the Boolean percolation model and list and prove some of its properties. In Section 5.3 we prove Theorem 5.1.1. The proofs of Theorems 5.1.2 and 5.1.3 are given in Sections 5.4–5.6. Section 5.4 (resp. 5.5) deals with the existence of a non-percolation (resp. percolation) phase. In Section 5.6 the uniqueness of the unbounded cluster is established. The appendix provides proofs of technical lemmas, which are needed in Sections 5.2 and 5.6.

5.2 Preliminaries on Boolean percolation

The model of Boolean percolation has been discussed in great detail in Meester and Roy [MR96] and we refer to this source for a discussion which goes beyond the description we are giving here.

5.2.1 Introduction of the model

Let ρ be a probability measure on $[0,\infty)$ and let χ be the Poisson point process on $\mathbb{R}^d \times [0,\infty)$ with intensity $(\lambda \times \mathrm{Leb}_d) \otimes \rho$. We denote the corresponding probability measure by $\mathbb{P}_{\lambda,\rho}$. A point $(x, r(x)) \in \chi$ is interpreted to be the open ball in $\mathbb{R}^d$ with center x and radius $r(x)$. Furthermore, we let $\mathcal{E}$ be the projection of χ onto $\mathbb{R}^d$. Boolean percolation deals with properties of the random set

$$\Sigma = \bigcup_{x \in \mathcal{E}} \mathcal{B}(x, r(x)). \tag{5.2.1}$$

Moreover, $C(y)$, $y \in \mathbb{R}^d$, denotes the cluster of Σ which contains y. If $y \notin \Sigma$, then $C(y) = \emptyset$.

Theorem 5.2.1 (Gouéré, [G08], Theorem 2.1)**.** *For all probability measures ρ on $(0,\infty)$ the following assertions are equivalent:*
(a)

$$\int_0^\infty x^d \, \rho(dx) < \infty. \tag{5.2.2}$$

(b) There exists $\lambda_0 \in (0,\infty)$ such that for all $\lambda < \lambda_0$,

$$\mathbb{P}_{\lambda,\rho}\big(C(0) \text{ is unbounded}\big) = 0. \tag{5.2.3}$$

Moreover, if (a) holds, then, for some $C = C(d) > 0$, (5.2.3) is satisfied for all

$$\lambda < C\Big(\int_0^\infty x^d \rho(dx)\Big)^{-1}. \tag{5.2.4}$$

It is immediate from Theorem 5.2.1, that

$$\lambda_c(\rho) := \inf\big\{\lambda > 0 : \mathbb{P}_{\lambda,\rho}\big(C(0) \text{ is unbounded}\big) > 0\big\} > 0. \tag{5.2.5}$$

Moreover, from the remark on page 52 of [MR96] it also follows that $\lambda_c(\rho) < \infty$ if $\rho((0,\infty)) > 0$. A more geometric fashion to characterize (5.2.5) is via crossing probabilities. For that fix $N_1, N_2, \dots, N_d > 0$ and let $\mathrm{CROSS}(N_1, N_2, \dots, N_d)$ be the event that the set $[0, N_1] \times [0, N_2] \times \cdots \times [0, N_d]$ contains a component $\mathcal{C}$ such that $\mathcal{C} \cap \{0\} \times [0, N_2] \times \cdots \times [0, N_d] \neq \emptyset$ and $\mathcal{C} \cap \{N_1\} \times [0, N_2] \times \cdots \times [0, N_d] \neq \emptyset$. The critical value λ_{CROSS} with respect to this event is defined by

$$\lambda_{\mathrm{CROSS}}(\rho) = \inf\left\{\lambda > 0 : \limsup_{N\to\infty} \mathbb{P}_{\lambda,\rho}\left(\mathrm{CROSS}(N, 3N, \dots, 3N)\right) > 0\right\}. \tag{5.2.6}$$

Assuming that ρ has compact support, Menshikov, Molchanov and Sidorenko [MMS88] proved that

$$\lambda_c(\rho) = \lambda_{\mathrm{CROSS}}(\rho). \tag{5.2.7}$$

5.2.2 Continuity of $\lambda_c(\rho)$

Given two probability measures ν and μ on a predefined probability space we write $\nu \preceq \mu$, if μ stochastically dominates ν.

Proposition 5.2.2. *Let ρ be a probability measure on $[0,\infty)$ with bounded support and let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of probability measures on $[0,\infty)$ such that $\rho_n \to \rho$ weakly as $n \to \infty$ and such that $\rho \preceq \rho_n$ for each $n \in \mathbb{N}$. Moreover, assume that*

- *there are $C > 0$ and $R_0 > 0$ such that for all $n \in \mathbb{N}$, $\rho_n([R,\infty)) \le e^{-CR}$ for all $R \ge R_0$;*
- *there is a probability measure ρ' on $[0,\infty)$ with finite moments of order d such that $\rho_n \preceq \rho'$ for all $n \in \mathbb{N}$.*

Then,

$$\lim_{n\to\infty} \lambda_c(\rho_n) = \lambda_c(\rho). \tag{5.2.8}$$

The proof of Proposition 5.2.2 relies on the following two lemmas whose proofs are given in the appendix and at the end of this section, respectively.

Lemma 5.2.3. *Let $N \in \mathbb{N}$, $\lambda > 0$ and let ρ be a probability measure on $[0,\infty)$ such that there are constants $C = C(\rho) > 0$ and $R_0 > 0$ such that $\rho([R,\infty)) \le e^{-CR}$ for all $R \ge R_0$. There is an $\varepsilon = \varepsilon(C, d) > 0$ such that if*

$$\mathbb{P}_{\lambda,\rho}(\mathrm{CROSS}(N, 3N, \dots, 3N)) \le \varepsilon, \tag{5.2.9}$$

then $\mathbb{P}_{\lambda,\rho}(\exists\, y \in \mathbb{R}^d : \mathrm{Leb}_d(C(y)) = \infty) = 0$.

Lemma 5.2.4. *Choose $\eta > 0$ and ρ' according to Proposition 5.2.2, then for all $N \in \mathbb{N}$*

$$\lim_{M\to\infty} \mathbb{P}_{\lambda,\rho'}\Big(\exists\, y \in \mathcal{B}_\infty(0,M)^{\complement} \cap \mathcal{E} \text{ s.t. } \mathcal{B}(y,r(y)) \cap [0,N]\times[0,3N]^{d-1} \neq \emptyset\Big) = 0. \tag{5.2.10}$$

Remark 5.2.5. *We expect that our proof of Lemma 5.2.3 still works when ρ has a polynomial tail (of sufficiently large order) instead of an exponential tail. However, since we do not need Lemma 5.2.3 in this stronger version, we did not verify all the details needed for that.*

We start with the proof of Proposition 5.2.2 subject to Lemmas 5.2.3–5.2.4.

Proof of Proposition 5.2.2. The idea of the proof is due to Penrose [Pen95]. First, note that

$$\limsup_{n\to\infty} \lambda_c(\rho_n) \leq \lambda_c(\rho), \tag{5.2.11}$$

since $\rho \preceq \rho_n$ for all $n \in \mathbb{N}$. Thus, we may focus on the reversed direction in (5.2.11). Second, fix $\lambda < \lambda_c(\rho)$ and let $\varepsilon > 0$ be chosen according to Lemma 5.2.3. By (5.2.7) there is an $N \in \mathbb{N}$ such that

$$\mathbb{P}_{\lambda,\rho}\left(\mathrm{CROSS}(N,3N,\ldots,3N)\right) \leq \varepsilon/3. \tag{5.2.12}$$

We consider $(\hat{\Omega},\hat{\mathbb{P}})$ the following coupling of $\{\mathbb{P}_{\lambda,\rho_n}\}_{n\in\mathbb{N}}$ and $\mathbb{P}_{\lambda,\rho}$:

- the points of $\mathcal{E}$ are sampled according to $\mathbb{P}_\lambda$;
- for each point $x \in \mathcal{E}$, by Skorokhod's embedding theorem, the radii $\{r_n(x)\}_{n\in\mathbb{N}}$ and $r(x)$ can be chosen such that they have respective distributions $\{\rho_n\}_{n\in\mathbb{N}}$ and ρ and are coupled such that $r_n(x) \xrightarrow[n\to\infty]{} r(x)$ a.s.

The configurations obtained via this coupling are denoted by

$$\Sigma_n := \bigcup_{x\in\mathcal{E}} \mathcal{B}(x,r_n(x)),\ n\in\mathbb{N}, \quad \text{and} \quad \Sigma_\infty := \bigcup_{x\in\mathcal{E}} \mathcal{B}(x,r(x)). \tag{5.2.13}$$

Let $M > 0$ and consider the events

$$E_n = \{\hat{\Sigma} := (\Sigma_k)_{k\in\mathbb{N}\cup\{\infty\}} \;:\; \Sigma_n \in \mathrm{CROSS}^M\}, \quad n \in \mathbb{N}\cup\{\infty\},$$

where

$$\mathrm{CROSS}^M = \left\{ \begin{array}{c} \mathrm{CROSS}(N,3N,\ldots,3N) \text{ happens by open balls} \\ \text{whose centers are in } \mathcal{B}_\infty(0,M) \end{array} \right\}.$$

Since the number of points in $\mathcal{B}_\infty(0,M)\cap\mathcal{E}$ is finite a.s., we may conclude that

$$\lim_{n\to\infty} \mathbb{1}_{E_n} = \mathbb{1}_{E_\infty} \qquad a.s. \tag{5.2.14}$$

(Note that the convergence in (5.2.14) is not true for every possible realization, but indeed on a set of probability one.) Hence, by the dominated convergence theorem,

$$\lim_{n\to\infty} \hat{\mathbb{P}}(E_n) = \hat{\mathbb{P}}(E_\infty).$$

Therefore,

$$\lim_{n\to\infty} \mathbb{P}_{\lambda,\rho_n}(\mathrm{CROSS}^M) = \mathbb{P}_{\lambda,\rho}(\mathrm{CROSS}^M),$$

so that for all $n \in \mathbb{N}$ large enough,

$$\mathbb{P}_{\lambda,\rho_n}(\mathrm{CROSS}^M) \le 2\varepsilon/3. \tag{5.2.15}$$

Whence, Lemma 5.2.4 and the fact that $\rho_n \preceq \rho'$ for all $n \in \mathbb{N}$, yields that there is $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$,

$$\mathbb{P}_{\lambda,\rho_n}\big(\mathrm{CROSS}(N, 3N, \dots, 3N)\big) \le \varepsilon. \tag{5.2.16}$$

Thus, an application of Lemma 5.2.3 yields that there is no unbounded component under $\mathbb{P}_{\lambda,\rho_n}$ for all $n \ge n_0$. Consequently, $\lambda < \lambda_c(\rho_n)$ for all $n \ge n_0$, from which Proposition 5.2.2 follows. ∎

The proof of Lemma 5.2.3 is given in Appendix 5.7.

Proof of Lemma 5.2.4. Recall that $\mathcal{A}_\infty(K, K+1)$ denotes the annulus $\mathcal{B}_\infty(0, K+1) \setminus \overline{\mathcal{B}}_\infty(0, K)$. Then, by summing over the positions of all Poisson points,

$$\begin{aligned}
&\mathbb{P}_{\lambda,\rho'}\Big(\exists\, y \in \mathcal{B}_\infty(0, M)^{\complement} \cap \mathcal{E}\colon\ \mathcal{B}(y, r(y)) \cap [0, N] \times [0, 3N]^{d-1} \ne \emptyset\Big)\\
&= \sum_{K=M}^{\infty} \mathbb{P}_{\lambda,\rho'}\Big(\exists\, y \in \mathcal{A}_\infty(K, K+1) \cap \mathcal{E}\colon\ \mathcal{B}(y, r(y)) \cap [0, N] \times [0, 3N]^{d-1} \ne \emptyset\Big)\\
&\le \sum_{K=M}^{\infty} \mathbb{P}_{\lambda,\rho'}\Big(\exists\, y \in \mathcal{A}_\infty(K, K+1) \cap \mathcal{E}\colon\ r(y) \ge K - 3N\Big)\\
&= \sum_{K=M}^{\infty} \sum_{\ell=1}^{\infty} \mathbb{P}_{\lambda,\rho'}\Big(|\mathcal{A}_\infty(K, K+1) \cap \mathcal{E}| = \ell,\ \exists\, y \in \mathcal{A}_\infty(K, K+1) \cap \mathcal{E} : \ r(y) \ge K - 3N\Big).
\end{aligned} \tag{5.2.17}$$

Using that for some constant $c = c(d) > 0$ and all $K \in \mathbb{N}$, $\mathrm{Leb}_d(\mathcal{A}_\infty(K+1, K)) = cK^{d-1}$, the last term in (5.2.17) may be estimated from above by

$$\sum_{K=M}^{\infty} \sum_{\ell=1}^{\infty} e^{-\lambda c K^{d-1}} \frac{(\lambda c K^{d-1})^\ell}{\ell!} \ell \rho'([K - 3N, \infty)) \le \mathrm{Cst} \sum_{K=M-3N}^{\infty} K^{d-1} \rho'([K, \infty)), \tag{5.2.18}$$

which goes to 0 as M goes to infinity since ρ' has moments of order d. ∎

5.3 Proof of Theorem 5.1.1: no percolation in $d = 1$

Let $t > 0$. Note that

$$\Sigma_t := \bigcup_{x \in \mathcal{E}} \mathcal{B}\Big(x, \sup_{0 \le s \le t} \|B^x_s - x\|\Big) \tag{5.3.1}$$

has the same law as the occupied set in the Boolean percolation model with radius distribution

$$\rho_t([L,\infty)) = \mathbb{P}^0\bigg(\sup_{0\le s\le t} ||B_s|| \ge L\bigg). \tag{5.3.2}$$

Note that ρ_t has finite moments of order d. Indeed, for all $L>0$,

$$\rho_t([L,\infty)) \le 2\mathbb{P}^0\bigg(\sup_{0\le s\le t} B_s \ge L\bigg) \le 4\mathbb{P}^0\bigg(B_t \ge L\bigg) \le \frac{4}{L}\sqrt{\frac{t}{2\pi}}e^{-L^2/2t}, \tag{5.3.3}$$

where we used the reflexion principle in the second inequality. Thus, by Theorem 3.1 in [MR96], almost-surely, the set Σ_t does not contain an unbounded cluster. Finally, the relation $\mathcal{O}_t \subseteq \Sigma_t$ yields the result.

5.4 Theorems 5.1.2-5.1.3: no percolation for small times

In this section we show that there is a $t_c = t_c(\lambda, d) > 0$ ($t_c = t_c(\lambda, d, r) > 0$ when $d \ge 4$) such that $\mathcal{O}_t$ ($\mathcal{O}_{t,r}$ when $d \ge 4$) does not percolate when $t < t_c$. The proof for $d \in \{2,3\}$ comes in Section 5.4.1, whereas the proof for $d \ge 4$ comes in Section 5.4.2. Both proofs heavily rely on the results of Section 5.2.

5.4.1 No percolation for $d = 2, 3$

Let $t > 0$ and define Σ_t and ρ_t as in Section 5.3, but with the one-dimensional Brownian motions of Section 5.3 replaced by its d-dimensional counterparts. As in Section 5.3 it is sufficient to show the existence of a $t_c > 0$ such that for all $t < t_c$ the set Σ_t almost surely does not have an unbounded component. For that we intend to apply Theorem 5.2.1. For all $\varepsilon > 0$,

$$\int_0^\infty x^d \rho_t(dx) \le \varepsilon^d \int_0^\varepsilon \rho_t(dx) + \int_\varepsilon^\infty x^d \rho_t(dx) \le \varepsilon^d + \int_{\varepsilon^d}^\infty \rho_t([y^{1/d},\infty))dy. \tag{5.4.1}$$

A calculation similar to the one in (5.3.3) shows that the second term on the right-hand side of (5.4.1) is bounded by

$$4d\sqrt{\frac{td}{2\pi}}\int_{\varepsilon^d}^\infty \frac{1}{y^{1/d}}e^{-y^{2/d}/2td}\,dy, \tag{5.4.2}$$

which tends towards zero, as $t \to 0$. Thus, by (5.4.1)–(5.4.2) we see that

$$\lim_{t\to 0}\int_0^\infty x^d\,\rho_t(dx) = 0. \tag{5.4.3}$$

An application of equation (5.2.4) in Theorem 5.2.1 yields the claim.

5.4.2 No percolation for $d \geq 4$

Let $t > 0$ and let $\rho_{r,t}$ be the probability measure on $[r, \infty)$ defined via

$$\rho_{t,r}([a,b]) = \mathbb{P}^0\left(\sup_{0 \leq s \leq t} \|B_s\| \in [a-r, b-r] \right), \qquad r \leq a \leq b. \tag{5.4.4}$$

Note that $\rho_{t,r} \to \delta_r$ weakly as $t \to 0$. Thus, by similar calculations as in (5.3.3) and Proposition 5.2.2 (with $\rho' = \rho_{1,r}$), $\lambda_c(\rho_{t,r}) \to \lambda_c(\delta_r)$ as $t \to 0$. Hence, there is a $t_0 > 0$ such that $\lambda < \lambda_c(\rho_{t,r})$ holds for all $t < t_0$. Finally, observe that the set

$$\Sigma_{t,r} = \bigcup_{x \in \mathcal{E}} \mathcal{B}\left(x, \sup_{0 \leq s \leq t} \|B_s^x - x\| + r \right), \qquad \forall t \geq 0, \tag{5.4.5}$$

is generated by the Poisson point process with intensity measure $(\lambda \times \mathrm{Leb}_d) \otimes \rho_{t,r}$ and contains $\mathcal{O}_{t,r}$, see (5.1.1). This is enough to conclude the claim.

5.5 Theorems 5.1.2–5.1.3: percolation for large times

In this section we establish that $\mathcal{O}_t$ ($\mathcal{O}_{t,r}$ when $d \geq 4$) percolates, when t is sufficiently large. The proof for $d \in \{2, 3\}$ comes in Section 5.5.1, whereas the proof for $d \geq 4$ comes in Section 5.5.2.

5.5.1 Proof of the percolation phase in $d = 2, 3$

We use a coarse-graining argument to prove existence of a percolation phase. More precisely, we divide $\mathbb{R}^d$ into boxes which are indexed by $\mathbb{Z}^d$ and we consider an edge percolation model on the coarse-grained graph whose vertices are identified with the centers of the boxes and the edges connect nearest-neighbours. An edge connecting nearest-neighbours, say x and x', in $\mathbb{Z}^d$, is said to be open if (i) both boxes associated to x and x' contain at least one point of the Poisson point process, and (ii) the Brownian motions which correspond to the point of the Poisson point process which are the closest to the centers of their respective boxes, intersect each other. Some technical computations and a domination result by Liggett, Schonmann and Stacey [LSS97] finally show that percolation in that coarse-grained model occurs if one suitably chooses the size of the boxes and let time run long enough. This implies percolation of our original model.

We now define this coarse-grained model more rigorously. Let $R > 0$ and $t > 0$ to be chosen later. For $x \in \mathbb{Z}^d$, we define

$$\mathcal{B}_x^{(R)} := \mathcal{B}_\infty(2Rx, R) \tag{5.5.1}$$

and the random variable

$$N^{(R)}(x) := |\ \mathcal{E} \cap \mathcal{B}_x^{(R)}\ | \,. \tag{5.5.2}$$

When $N^{(R)}(x) \geq 1$, we define the point $z^{(R,x)}$, which is almost surely uniquely determined, via

$$\|z^{(R,x)} - 2Rx\| = \inf_{z \in \mathcal{E} \cap \mathcal{B}_x^{(R)}} \|z - 2Rx\|. \tag{5.5.3}$$

Note that $z^{(R,x)}$ is the point which is the closest to the center of the box $\mathcal{B}_x^{(R)}$ among all Poisson points of $\mathcal{B}_x^{(R)}$. We denote by $B^{(R,x)}$ the Brownian motion starting from $z^{(R,x)}$. For all couples of nearest-neighbours $(x,y) \in \mathbb{Z}^d \times \mathbb{Z}^d$, we say that the edge (x,y), which connects x and y, is open if

$$\text{(i)} \quad N^{(R)}(x) \geq 1, \tag{5.5.4}$$

$$\text{(ii)} \quad N^{(R)}(y) \geq 1, \tag{5.5.5}$$

$$\text{(iii)} \quad B^{(R,x)}_{[0,t]} \cap B^{(R,y)}_{[0,t]} \neq \emptyset. \tag{5.5.6}$$

We let $X^{R,t}_{(x,y)}$ be the random variable which takes value 1 if the edge (x,y) is open, and 0 otherwise. In what follows, to not burden the notation, we write $X_{(x,y)}$ instead of $X^{R,t}_{(x,y)}$.

Lemma 5.5.1. *Let $\epsilon > 0$. There exists $R > 0$ and $t > 0$ such that for any couple of nearest-neigbours $(x,y) \in \mathbb{Z}^d \times \mathbb{Z}^d$, $\mathbb{P}(X_{(x,y)} = 1) \geq 1 - \epsilon$.*

The proof of Lemma 5.5.1 is deferred to the end of this section. We first show how one may deduce the existence of a percolation phase from it.

Proof of the existence of a percolation phase. Note that if (x,x') and (y,y') is a couple of nearest-neighbour points in $\mathbb{Z}^d$ such that $\{x,x'\} \cap \{y,y'\} = \emptyset$, then $X_{(x,x')}$ and $X_{(y,y')}$ are independent. Therefore, the coarse-grained percolation model is a 2-dependent percolation model. Thus, Theorem 0.0 of Liggett, Schonmann and Stacey [LSS97] yields that we may bound the coarse-grained percolation model from below by Bernoulli bond percolation, whose parameter, say p^*, can be chosen to be arbitrarily close to 1, when $\mathbb{P}(X_{(x,y)} = 1)$ is sufficiently close to 1. Let $p_c(\mathbb{Z}^d)$ be the critical percolation parameter for Bernoulli bond percolation. Then, by Lemma 5.5.1, there are $R_0 > 0$ and $t_0 > 0$ such that $p^* > p_c(\mathbb{Z}^d)$ for all $R \geq R_0$ and $t \geq t_0$. In that case, the coarse-grained model percolates, and so does $\mathcal{O}_t$. ∎

Consequently, it remains to prove Lemma 5.5.1. For that we need an additional lemma. It states that the probability that two independent Brownian motions, starting at points $x, y \in \mathbb{R}^d$ have a non-empty intersection up to time t increases, when we move the starting points towards each other.

Lemma 5.5.2. *Let $t > 0$. Then,*

$$(x,y) \mapsto \mathbb{P}^{x,y}\Big(B^{(1)}_{[0,t]} \cap B^{(2)}_{[0,t]} \neq \emptyset\Big), \qquad (x,y) \in \mathbb{R}^d \times \mathbb{R}^d, \tag{5.5.7}$$

is a non-increasing function of $\|x-y\|$.

We first prove Lemma 5.5.1 subject to Lemma 5.5.2. The proof of Lemma 5.5.2 comes afterwards.

Proof of Lemma 5.5.1. By independence of the events in (i)–(iii), we have

$$\mathbb{P}(X_{(x,y)} = 1) = \mathbb{P}(N^{(R)}(x) \geq 1)^2 \times \mathbb{P}\Big(B^{(R,x)}_{[0,t]} \cap B^{(R,y)}_{[0,t]} \neq \emptyset\Big). \tag{5.5.8}$$

To proceed, we fix $R > 0$ large enough such that

$$\mathbb{P}(N^{(R)}(x) \geq 1) = 1 - e^{-\lambda(2R)^d} \geq 1 - \epsilon. \tag{5.5.9}$$

Furthermore, by Lemma 5.5.2, $\mathbb{P}(B^{(R,x)}_{[0,t]} \cap B^{(R,y)}_{[0,t]} \neq \emptyset)$ decreases, when $\|z^{(R,x)} - z^{(R,y)}\|$ increases. However, note that $\|z^{(R,x)} - z^{(R,y)}\| \leq R\sqrt{4(d-1)+16}$, when $\|x - y\| = 1$. Thus,

$$\begin{aligned}\mathbb{P}\Big(B^{(R,x)}_{[0,t]} \cap B^{(R,y)}_{[0,t]} \neq \emptyset\Big) &\geq \mathbb{P}\Big(B^{(R,x)}_{[0,t]} \cap B^{(R,y)}_{[0,t]} \neq \emptyset \Big| \|z^{(R,x)} - z^{(R,y)}\| = R\sqrt{4(d-1)+16}\Big) \\ &= \mathbb{P}^{z_1,z_2}\Big(B^{(1)}_{[0,t]} \cap B^{(2)}_{[0,t]} \neq \emptyset\Big),\end{aligned} \tag{5.5.10}$$

for any choice of z_1 and z_2 such that $\|z_1 - z_2\| = R\sqrt{4(d-1)+16}$. Using Theorem 9.1 (b) in Mörters and Peres [MP10], there exists t large enough such that for all such choices of z_1 and z_2,

$$\mathbb{P}^{z_1,z_2}\Big(B^{(1)}_{[0,t]} \cap B^{(2)}_{[0,t]} \neq \emptyset\Big) \geq 1 - \epsilon, \tag{5.5.11}$$

which is enough to deduce the claim. ∎

We now prove Lemma 5.5.2.

Proof of Lemma 5.5.2. Note that it is enough to prove the claim for the function

$$y \mapsto \mathbb{P}^{0,y}\Big(B^{(1)}_{[0,t]} \cap B^{(2)}_{[0,t]} \neq \emptyset\Big). \tag{5.5.12}$$

We fix $R' > R > 0$ and $y, y' \in \mathbb{R}^d$ such that $||y|| = R$ and $||y'|| = R'$, respectively. Using rotational invariance of Brownian motion in the first equality and scale invariance of Brownian motion in the last equality, we may write

$$\begin{aligned}\mathbb{P}^{0,y'}\Big(B^{(1)}_{[0,t]} \cap B^{(2)}_{[0,t]} \neq \emptyset\Big) &= \mathbb{P}^{0,(R'/R)y}\Big(B^{(1)}_{[0,t]} \cap B^{(2)}_{[0,t]} \neq \emptyset\Big) \\ &\leq \mathbb{P}^{0,(R'/R)y}\Big(B^{(1)}_{[0,(R'/R)^2 t]} \cap B^{(2)}_{[0,(R'/R)^2 t]} \neq \emptyset\Big) \\ &= \mathbb{P}^{0,y}\Big(B^{(1)}_{[0,t]} \cap B^{(2)}_{[0,t]} \neq \emptyset\Big).\end{aligned} \tag{5.5.13}$$

This yields the claim. ∎

5.5.2 Proof of the percolation phase for $d \geq 4$

Throughout the proof, z always denotes the d-th coordinate of $x = (\xi, z) \in \mathbb{R}^d$. We further define

$$\mathcal{H}_0 = \{(\xi, z) \in \mathbb{R}^d : z = 0\}. \tag{5.5.14}$$

The main idea is to reduce the problem to a Boolean percolation problem on $\mathcal{H}_0$. More precisely, we use that for each $x \in \mathcal{E}$, B^x will eventually hit $\mathcal{H}_0$. From this we deduce that for t large enough, the traces of the Wiener sausages which hit $\mathcal{H}_0$ dominate a supercritical $(d-1)$-dimensional Boolean percolation model, and therefore percolate.

We now formalize this strategy. For each $k \in \mathbb{N}$, let

$$\mathcal{S}_k := \{(\xi, z) \in \mathbb{R}^d \ : \ k-1 < z \leq k\}, \tag{5.5.15}$$

so that $(\mathcal{S}_k)_{k \in \mathbb{Z}}$ is a partition of $\mathbb{R}^{d-1} \times (0, \infty)$. We fix $k \in \mathbb{N}$ and consider

$$\mathcal{E}_k = \{\xi : \exists\, z \in \mathbb{R} \text{ s.t. } (\xi, z) \in \mathcal{S}_k \cap \mathcal{E}\}. \tag{5.5.16}$$

Note that $(\mathcal{E}_k)_{k \geq 0}$ are i.i.d. Poisson point processes with parameter $\lambda \times \mathrm{Leb}_{d-1}$. Given $\mathcal{E}_k$, we construct a random set C_t^k in the following way:

- Thinning: each $\xi \in \mathcal{E}_k$ is kept if $\tau_0(z^\xi) \leq t$, where z^ξ is such that $(\xi, z^\xi) \in \mathcal{S}_k \cap \mathcal{E}$ (there is almost-surely only one choice), and $\tau_0(z)$ is the first hitting time of the origin of an one-dimensional Brownian motion starting at z. We choose all Brownian motions, which are associated to some $\xi \in \mathcal{E}_k$, to be independent. Otherwise ξ is discarded.
- Translation: each $\xi \in \mathcal{E}_k$ that was not removed after the previous step is translated by $\bar{B}(\tau_0(z^\xi))$, where $\bar{B}$ is $(d-1)$-dimensional Brownian motion starting at the origin, which is independent of all the previous variables.

Note that z^ξ is uniformly distributed in $(k-1, k)$. Moreover, z^ξ, $\tau_0(z^\xi)$ and $\bar{B}$ are independent of ξ. Thus, C_t^k is the result of a thinning and a translation of $\mathcal{E}_k$, and both operations depend on random variables, which are independent of $\mathcal{E}_k$. Therefore, $(C_t^k)_{k \geq 0}$ is a collection of i.i.d. Poisson point processes with parameter $\lambda p_t^k \times \mathrm{Leb}_{d-1}$, where

$$p_t^k = \int_{k-1}^{k} \mathbb{P}^z\Big(\inf_{0 \leq s \leq t} B_s \leq 0\Big)\, dz \geq \mathbb{P}^0\Big(\sup_{0 \leq s \leq t} B_s \geq k\Big). \tag{5.5.17}$$

By independence of the C_t^k's, the set $\mathcal{C}_t := \bigcup_{k=1}^{\infty} C_t^k$ is thus a Poisson point process with parameter $\lambda \sum_{k \geq 1} p_t^k \times \mathrm{Leb}_{d-1}$.

Let us now consider the Boolean model generated by $\mathcal{C}_t$ with deterministic radius r. Observe that,

$$\sum_{k=1}^{\infty} p_t^k \geq \sum_{k=0}^{\infty} \mathbb{P}^0\Big(\sup_{0 \leq s \leq t} B_s \geq k\Big) - \mathbb{P}^0\Big(\sup_{0 \leq s \leq t} B_s \geq 0\Big) \geq \mathbb{E}^0\Big[\sup_{0 \leq s \leq t} B_s\Big] - 1. \tag{5.5.18}$$

Note that the right-hand side of (5.5.18) tends to infinity as $t \to \infty$. Thus, by the remark on page 52 in [MR96], there exists $t_0 > 0$ large enough such that the Boolean model generated by $\mathcal{C}_t$ percolates for all $t \geq t_0$. Finally, note that $\mathcal{C}_t$ is stochastically dominated by $\mathcal{O}_t \cap \mathcal{H}_0$, in the sense that $\mathcal{C}_t$ has the same distribution as a subset of $\mathcal{O}_t \cap \mathcal{H}_0$. This completes the proof.

5.6 Theorems 5.1.2–5.1.3: uniqueness of the unbounded cluster

We fix $t, r, \lambda \geq 0$ such that $t > t_c(\lambda, d, r)$. In the following we denote by N_∞ the number of unbounded clusters in $\mathcal{O}_{t,r}$, which is almost-surely a constant as a consequence of Proposition 5.1.4. For all $d \geq 2$, the proof of uniqueness consists of (i) excluding the case $N_\infty = k$ with $k \in \mathbb{N} \setminus \{1\}$ and of (ii) excluding the case $N_\infty = \infty$. This section is organized as follows. In Section 5.6.1, we give a short heuristic of (i) in the case $d = 2$, which we use as a guideline for the proofs in all other cases. Section 5.6.2 contains the proof of uniqueness for Wiener sausages ($r > 0$) in $d \geq 4$, which is also on a technical level close to the heuristics in Section 5.6.1. This is not true anymore in dimension $d = 3$, which is due to the fact that there is no simple way under which the paths of two independent three-dimensional Brownian motions intersect each other. Therefore, when $d = 3$, the strategy described in Section 5.6.1 needs to be adapted, which requires a certain number of technical steps. Since the proof for $d = 3$ works for $d = 2$ as well, we decided to give a unified proof for both cases in Section 5.6.3.

5.6.1 Heuristics

Let $d = 2$ and $r = 0$. We proceed by contradiction and assume that almost-surely, $N_\infty = k$ with $k \in \mathbb{N} \setminus \{1\}$. For $R_2 > R_1 > 0$, we introduce the event (see Fig. 5.2 below):

$$E_{R_1,R_2} = \left\{ \begin{array}{c} \mathcal{B}(0, R_2) \text{ intersects all } k \text{ unbounded clusters} \\ \text{without using paths starting in } \mathcal{B}(0, R_1) \end{array} \right\}. \tag{5.6.1}$$

We fix $R_1 > 0$. First, note that by monotonicity in R_2,

$$\mathbb{P}(E_{R_1,R_2}) \geq \mathbb{P}(E_{R_1,R_2} \cap \{\mathcal{E} \cap \mathcal{B}(0, R_1) = \emptyset\}) \overset{R_2 \to \infty}{\longrightarrow} \mathbb{P}(\mathcal{E} \cap \mathcal{B}(0, R_1) = \emptyset) > 0. \tag{5.6.2}$$

Therefore, we can find an $R_2 > 0$ such that $\mathbb{P}(E_{R_1,R_2}) > 0$. Let us fix such an R_2 and observe that E_{R_1,R_2} is independent from the points in $\mathcal{E} \cap \mathcal{B}(0, R_1)$ and the Brownian motions starting from them. Next, one can show that the event

$$L_{R_1,R_2} = \left\{ \begin{array}{c} |\mathcal{B}(0, R_1) \cap \mathcal{E}| = 1 \text{ and for } x \in \mathcal{E} \cap \mathcal{B}(0, R_1), \\ B^x_{[0,t]} \text{ contains a “loop” in } \mathcal{A}(R_2, R_2 + 1) \end{array} \right\} \tag{5.6.3}$$

has positive probability. Finally, the contradiction is a consequence of

$$\mathbb{P}(N_\infty = 1) \geq \mathbb{P}(E_{R_1,R_2} \cap L_{R_1,R_2}) = \mathbb{P}(E_{R_1,R_2})\mathbb{P}(L_{R_1,R_2}) > 0, \tag{5.6.4}$$

since we assumed that $\mathbb{P}(N_\infty = k) = 1$, $k \in \mathbb{N} \setminus \{1\}$.

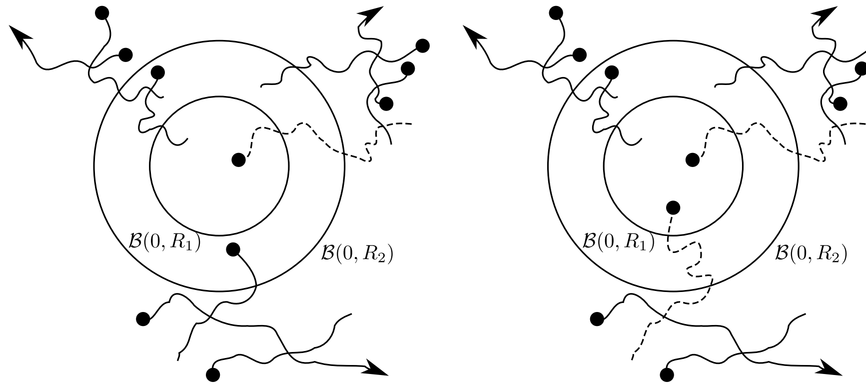

Figure 5.2: The plot on the left hand side represents a configuration of the event E_{R_1,R_2} with $k = 3$. The symbol • represents the points of $\mathcal{E}$, whereas ▶ represents connectivity with infinity. Finally, the dashed line emphasizes the fact that points starting inside $\mathcal{B}(0, R_1)$ are not considered for the intersection condition in (5.6.1). Because of that, the configuration represented on the right hand side does not belong to E_{R_1,R_2}.

Remark 5.6.1. *The above heuristics also shows how to create trifurcation points. In combination with Lemma 5.6.3, the strategy alluded to above will be used to exclude the possibility of having infinitely many unbounded clusters.*

5.6.2 Uniqueness in $d \geq 4$

5.6.2.1 Excluding $2 \leq N_\infty < \infty$

Again we proceed by contradiction. Let us assume that N_∞ is almost-surely equal to a constant $k \in \mathbb{N} \setminus \{1\}$. For simplicity, we further assume that $k = 2$, the extension of the argument to other values of k being straightforward.

For $R_2 > R_1 > 0$, let us define E_{R_1,R_2} as follows

$$E_{R_1,R_2} = \left\{ \begin{array}{c} \mathcal{B}(0, R_2) \text{ intersects at least one path of each of the two} \\ \text{unbounded clusters, without using paths starting in } \mathcal{B}(0, R_1) \end{array} \right\}. \quad (5.6.5)$$

First, we note that there exist R_1 and R_2 such that

$$\mathbb{P}(E_{R_1,R_2}) > 0, \quad (5.6.6)$$

which can be seen as in the lines following (5.6.2). Next, we consider the event analogous to (5.6.3),

$$L_{R_1,R_2} = \left\{ \begin{array}{c} |\mathcal{B}(0, R_1) \cap \mathcal{E}| = 1 \text{ and for } x \in \mathcal{B}(0, R_1) \cap \mathcal{E}, \\ \overline{\mathcal{A}}(R_2 - 3r/2, R_2 - r/2) \subset \cup_{0 \leq s \leq t} \mathcal{B}(B_s^x, r) \subset \mathcal{B}(0, R_2) \end{array} \right\}, \quad (5.6.7)$$

which is independent of E_{R_1,R_2} and has positive probability, see Remark 5.6.2 below. The independence is due to the fact that E_{R_1,R_2} and L_{R_1,R_2} depend on different points of $\mathcal{E}$ and on different Brownian paths. Note that on $E_{R_1,R_2} \cap L_{R_1,R_2}$ the two unbounded clusters, are only connected inside $\mathcal{B}(0, R_2)$.

The contradiction now follows as in (5.6.4).

Remark 5.6.2. *A sketch of the proof that L_{R_1,R_2} has positive probability goes as follows. Let $\epsilon \in (0, r/8)$. By compactness, $\overline{\mathcal{A}}(R - 3r/2, R_2 - 3r/2 + \epsilon)$ can be covered by a finite number of balls of radius ϵ. Moreover, a Brownian motion starting in $\mathcal{B}(0, R_1)$ has a positive probability of visiting all these balls before time t and before leaving $\mathcal{B}(0, R_2 - r)$. Consequently, on the aforementioned event, L_{R_1,R_2} is satisfied.*

5.6.2.2 Excluding $N_\infty = \infty$

We assume that $N_\infty = \infty$. We show that this assumption leads to a contradiction. The proof is based on ideas of Meester and Roy [MR94, Theorem 2.1], who extended a technique developed by Burton and Keane [BK89] to a continuous percolation model. In the proof we use the following counting lemma, which is due to Gandolfi, Keane and Newman [GKN92]. It will yield a contradiction to the existence of trifurcation points, which will be constructed in the first step of the proof.

Lemma 5.6.3 (Lemma 4.2 in [GKN92])**.** *Let S be a set, R be a non-empty finite subset of S and $K > 0$. Suppose that*
(a) for all $z \in R$, there is a family $(C_z^1, C_z^2, \dots, C_z^{n_z})$, $n_z \geq 3$, of disjoint non-empty subsets of S, which do not contain z and are such that $|C_z^i| \geq K$, for all z and for all $i \in \{1, 2, \dots, n_z\}$,
(b) for all $z, z' \in R$ one of the following cases occurs (where we abbreviate $C_z = \cup_{i=1}^{n_z} C_z^i$ for all $z \in R$):
(i) $(\{z\} \cup C_z) \cap (\{z'\} \cup C_{z'}) = \emptyset$;
(ii) there are $i, j \in \{1, 2, \dots, n_z\}$ such that $\{z'\} \cup C_{z'} \setminus C_{z'}^j \subseteq C_z^i$ and $\{z\} \cup C_z \setminus C_z^i \subseteq C_{z'}^j$;
(iii) there is $i \in \{1, 2, \dots, n_z\}$ such that $\{z'\} \cup C_{z'} \subseteq C_z^i$;
(iv) there is $j \in \{1, 2, \dots, n_{z'}\}$ such that $\{z\} \cup C_z \subseteq C_{z'}^j$.
Then $|S| \geq K(|R| + 2)$.

STEP 1. Balls containing a trifurcation point. Again, we define E_{R_1,R_2} and L_{R_1,R_2} as in (5.6.5) ,(5.6.7), respectively. By means of these events, in the same manner as in Subsection 5.6.2.1, one can show that there are $\delta > 0$ and $R \in \mathbb{N}$ such that the event

$$E_R(0) := \left\{ \begin{array}{l} \exists \text{ an unbounded cluster } C \text{ such that } C \cap \mathcal{B}_\infty(0,R)^{\complement} \text{ contains at least} \\ \text{three unbounded clusters, } |C \cap \mathcal{B}_\infty(0,R) \cap \mathcal{E}| \geq 1 \text{ and each cluster which} \\ \text{intersects } \mathcal{B}_\infty(0,R) \text{ belongs to } C. \end{array} \right\}, \tag{5.6.8}$$

has probability at least δ. Note that $E_R(0)$ implies that each $x \in \mathcal{B}_\infty(0, R)$ which belongs to an infinite cluster also belongs to C. We call each unbounded cluster in $C \cap \mathcal{B}_\infty(0, R)^{\complement}$

a branch. To proceed, we fix $K > 0$ and choose $M > 0$ such that the event

$$E_{R,M}(0) = E_R(0) \cap \left\{ \begin{array}{l} \text{there are at least three different branches of } \mathcal{B}_\infty(0,R) \text{ which con-} \\ \text{tain at least } K \text{ points in } \mathcal{E} \cap (\mathcal{B}_\infty(0,RM) \setminus \mathcal{B}_\infty(0,R)) \end{array} \right\}, \tag{5.6.9}$$

has probability at least $\delta/2$ (see Fig. 5.3 below). For $z \in \mathbb{Z}^d$, the events $E_{R,M}(2Rz)$ and $E_R(2Rz)$ are defined in a similar manner as $E_{R,M}(0)$ and $E_R(0)$, except that the balls in the definitions are centered around $2Rz$.

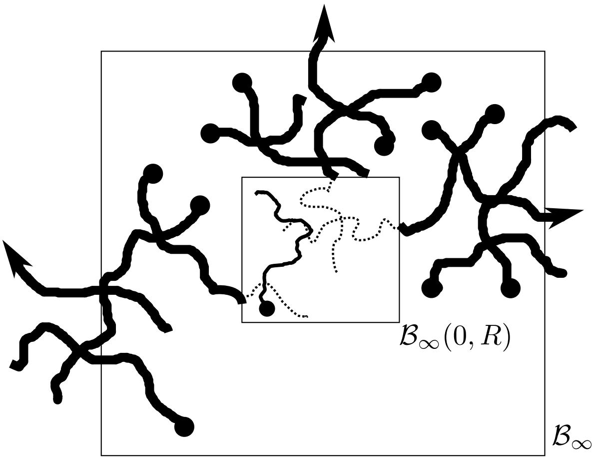

Figure 5.3: The plot represents a configuration in $E_{R,M}(0)$ with $K = 3$ (see (5.6.8)-(5.6.9)). The thick lines belong to the branches. As in the previous figure, ► represents connection to infinity.

Let $L > M + 2$ and define the set

$$\mathcal{R} = \{z \in \mathbb{Z}^d : \mathcal{B}_\infty(2Rz, RM) \subseteq \mathcal{B}_\infty(0, LR),\ E_{R,M}(2Rz) \text{ occurs}\}. \tag{5.6.10}$$

Note that

$$|\{z \in \mathbb{Z}^d : \mathcal{B}_\infty(2Rz, RM) \subseteq \mathcal{B}_\infty(0, LR)\}| \geq (L - M - 2)^d, \tag{5.6.11}$$

so that we obtain by stationarity

$$\mathbb{E}(|\mathcal{R}|) \geq \frac{(L - M - 2)^d \delta}{2}. \tag{5.6.12}$$

STEP 2. Application of Lemma 5.6.3 and contradiction. We identify each $z \in \mathcal{R}$ with a Poisson point in $\mathcal{B}_\infty(2Rz, R) \cap C$, which is contained in the corresponding infinite cluster. In what follows we write Λ_z instead of $\mathcal{B}_\infty(2Rz, R)$, for simplicity of notation. Let n_z be the total number of branches of Λ_z, which contain at least K Poisson points in $\mathcal{B}_\infty(2Rz, R)$. For $i \in \{1, \dots, n_z\}$, let $\mathbf{B}_z^i$ be the branch which is the ith-closest to $2Rz$ among all branches of $\mathcal{B}_\infty(2Rz, R)$, see Equation (5.6.9).

A point x is said to be connected to a set A *through* the set Λ if there exists a continuous function $\gamma : [0,1] \mapsto \Lambda \cap \mathcal{O}_{t,r}$ such that $\gamma(0) = x$ and $\gamma(1) \in A$. We denote it briefly by $x \xleftrightarrow{\Lambda} A$. Finally, we define

$$C_z^i = \mathcal{E} \cap \mathcal{B}(0, LR) \cap \mathbf{B}_z^i = \left\{ x \in \mathcal{E} \cap \mathcal{B}_\infty(0, LR) \colon\ x \xleftrightarrow{\Lambda_z^c} \mathbf{B}_z^i \right\} \qquad \forall i \in \{1, \dots, n_z\}. \tag{5.6.13}$$

Now we proceed to check that the conditions of Lemma 5.6.3 are fulfilled. Here $S = \mathcal{B}_\infty(0, LR) \cap \mathcal{E}$. First note that, by definition of a branch, we have that for all $z \in \mathcal{R}$:

- $|C_z^i| \geq K$,
- $C_z^i \cap C_z^j = \emptyset$ for all $i, j \in \{1, \dots, n_z\}$ with $i \neq j$,
- $z \notin C_z$.

Hence, assumption (a) of Lemma 5.6.3 is met.

We now claim that the collection $\{C_z^i\}_{z \in R, i \in \{1,\dots,n_z\}}$ satisfies also assumption (b) of Lemma 5.6.3. At this point we would like to stress some facts to be used later:

Fa. Due to (5.6.8), $z \xleftrightarrow{\Lambda_z} C_z^i$ for all $i \in \{1, \dots, n_z\}$.

Fb. If $\widetilde{C}$ is an unbounded cluster such that $\widetilde{C} \cap \Lambda_z \neq \emptyset$, then $z \xleftrightarrow{\Lambda_z} \widetilde{C}$.

Suppose that $(\{z\} \cup C_z) \cap (\{z'\} \cup C_{z'}) \neq \emptyset$. We consider three different cases:

1. If $z' \in C_z$ then there exists a unique $i \in \{1, \dots, n_z\}$ such that $z' \in C_z^i$. We consider two sub-cases:
    - If $z \in C_{z'}$, then there exists a unique $i' \in \{1, \dots, n_{z'}\}$ such that $z \in C_{z'}^{i'}$, and we claim that $\{z'\} \cup C_{z'} \setminus C_{z'}^{i'} \subseteq C_z^i$ and $\{z\} \cup C_z \setminus C_z^i \subseteq C_{z'}^{i'}$. Indeed, pick $x' \in C_{z'} \setminus C_{z'}^{i'}$. Then there exists a unique $j' \neq i'$ such that $x' \xleftrightarrow{\Lambda_{z'}^c} C_{z'}^{j'}$. It is crucial to note that $x' \overset{\Lambda_{z'}^c \cap \Lambda_z^c}{\longleftrightarrow} C_{z'}^{j'}$ since otherwise, due to **Fb.**, $z \xleftrightarrow{\Lambda_{z'}^c} C_{z'}^{j'}$ (by first connecting z to x' in $\Lambda_{z'}^c$ and then x' to $C_{z'}^{j'}$ in $\Lambda_{z'}^c$), which contradicts the uniqueness of i'.

      Finally, we have that $x' \xleftrightarrow{\Lambda_z^c} C_{z'}^{j'}$, $z' \overset{\Lambda_{z'} \subset \Lambda_z^c}{\longleftrightarrow} C_{z'}^{j'}$, $z' \xleftrightarrow{\Lambda_z^c} C_z^i$. A concatenation of all these paths gives $x' \xleftrightarrow{\Lambda_z^c} C_z^i$, that is $x' \in C_z^i$. This proves the first inclusion that we claimed. The second inclusion follows by symmetry.
    - If $z \notin C_{z'}$, then we claim: $\{z'\} \cup C_{z'} \subseteq C_z^i$.

      Indeed, take $x' \in C_{z'}$, then there exists a unique j' such that $x' \xleftrightarrow{\Lambda_{z'}^c} C_{z'}^{j'}$. As before we have that $x' \overset{\Lambda_{z'}^c \cap \Lambda_z^c}{\longleftrightarrow} C_{z'}^{j'}$ (this time the contradiction follows from $z \notin C_{z'}$). The conclusion follows in the same way as in the previous case.
2. If $z \in C_{z'}$, then one may conclude as in 1.

3. Suppose that there exist i, i' such that $C_z^i \cap C_{z'}^{i'} \neq \emptyset$. Take $x' \in C_z^i \cap C_{z'}^{i'}$, then we have that $x' \overset{\Lambda_z^c}{\longleftrightarrow} C_z^i$ and $x' \overset{\Lambda_{z'}^c}{\longleftrightarrow} C_{z'}^{i'}$. There are two cases:
   - The path $x' \overset{\Lambda_z^c}{\longleftrightarrow} C_z^i$ intersects $\Lambda_{z'}$: Due to **Fb.** we have that $z' \overset{\Lambda_z^c}{\longleftrightarrow} C_z^i$. Hence $z' \in C_z$, which reduces this case to a previous one.
   - In the second case, $x' \overset{\Lambda_z^c \cap \Lambda_{z'}^c}{\longleftrightarrow} C_z^i$: Due to **Fa.**, we have $z \overset{\Lambda_z \subset \Lambda_{z'}^c}{\longleftrightarrow} C_z^i$. Finally, a concatenation of the previous two paths with $x' \overset{\Lambda_{z'}^c}{\longleftrightarrow} C_{z'}^{i'}$ yields that $z \in C_{z'}$, which reduces this case again to a previous one.

Hence, by Lemma 5.6.3

$$\mathbb{E}\big(|\mathcal{B}_\infty(0, LR) \cap \mathcal{E}|\big) \geq K(\mathbb{E}(|\mathcal{R}|) + 2), \tag{5.6.14}$$

so that, by (5.6.12),

$$\mathbb{E}\big(|\mathcal{B}_\infty(0, LR) \cap \mathcal{E}|\big) \geq K((L - M - 2)^d \delta/2 + 2). \tag{5.6.15}$$

On the other hand, since $\mathcal{E}$ is a Poisson point process with intensity measure $\lambda \times \mathrm{Leb}_d$,

$$\mathbb{E}\big(|\mathcal{B}_\infty(0, LR) \cap \mathcal{E}|\big) = \lambda(2LR)^d. \tag{5.6.16}$$

Thus, combining (5.6.15) and (5.6.16), yields

$$\forall L > M + 2, \quad K((L - M - 2)^d \delta/2 + 2) \leq \lambda(2LR)^d. \tag{5.6.17}$$

Note that M depends on K, so in order to get a contradiction one can choose $L = 2M$ and let K go to ∞ in the inequality above.

5.6.3 Uniqueness in $d = 2, 3$

5.6.3.1 Excluding $\{2 \leq N_\infty < \infty\}$

As in the heuristic of Section 5.6.1, we proceed by contradiction: we assume that $\mathbb{P}(N_\infty = k) = 1$ for some $k \in \mathbb{N} \setminus \{1\}$ and prove that $\mathbb{P}(N_\infty = 1) > 0$, which is absurd. To make the proof more accessible, we assume that $k = 2$ (see Remark 5.6.7 below).

Remark: The previous heuristic does not work verbatim for $d = 3$ because of clear geometrical reasons: a three-dimensional Brownian motion travelling around an annulus, which is crossed by the two unbounded clusters, does not necessarily connect them. Let us first briefly describe how we adapt this strategy. For R large enough and ϵ small enough, we show that with positive probability, both unbounded clusters intersect $\mathcal{B}(0, R)$ in such a way, such that each of them contains a Brownian path crossing $\mathcal{A}(R - \epsilon, R + \epsilon)$. Afterwards, we show that, still with positive probability, we can reroute the (let us say first) excursions inside $\mathcal{A}(R - \epsilon, R + \epsilon)$ of each of these two Brownian paths such that they intersect each other and, as a consequence, merge the two unbounded clusters into a single one. This leads to the desired contradiction, since our construction provides a set of configurations of positive probability on which $N_\infty = 1$.

We now give the proof in full detail. Let $R > 0$ and denote by N_∞^R the number of unbounded clusters in $\mathcal{O}_t \setminus \mathcal{B}(0,R)$. In the case that N_∞^R is not zero, we denote those by $\{\mathcal{C}_i(R), 1 \le i \le N_\infty^R\}$ (though it has little relevance, let us agree that clusters are indexed according to the order in which one finds them by radially exploring the occupied set from 0). We also consider the 'extended' clusters, defined by

$$\mathcal{C}_i^{\text{ext}}(R) = \bigcup_{x \in \mathcal{E} \,:\, B^x_{[0,t]} \cap \mathcal{C}_i(R) \neq \emptyset} B^x_{[0,t]}, \tag{5.6.18}$$

i.e. $\mathcal{C}_i^{\text{ext}}(R)$ is the union of all Brownian paths up to time t, which have a non-empty intersection with $\mathcal{C}_i(R)$ (see Fig. 5.4 below).

We further define in five steps a notion of good extended clusters and prove that those occur with positive probability.

Good extended clusters in five steps.

STEP 1. Intersection with a large ball. We use the abbreviations $\mathcal{C}_1^{\text{ext}} := \mathcal{C}_1^{\text{ext}}(R)$ and $\mathcal{C}_2^{\text{ext}} := \mathcal{C}_2^{\text{ext}}(R)$ for the two extended unbounded clusters and define

$$E_R := \{N_\infty^R = 2\} \cap \{\mathcal{C}_1^{\text{ext}} \cap \mathcal{B}(0,R) \neq \emptyset\} \cap \{\mathcal{C}_2^{\text{ext}} \cap \mathcal{B}(0,R) \neq \emptyset\}. \tag{5.6.19}$$

One way of having exactly two unbounded clusters in $\mathcal{O}_t \setminus \mathcal{B}(0,R)$ is to have exactly two unbounded clusters in total (i.e. on the whole configuration), hence

$$\mathbb{P}(E_R) \ge \mathbb{P}(N_\infty = 2,\, \mathcal{C}_1^{\text{ext}} \cap \mathcal{B}(0,R) \neq \emptyset,\, \mathcal{C}_2^{\text{ext}} \cap \mathcal{B}(0,R) \neq \emptyset). \tag{5.6.20}$$

Since the event on the right-hand side of (5.6.20) is increasing in R, its probability converges, as R tends to ∞, to $\mathbb{P}(N_\infty = 2)$, which equals 1 by our initial assumption. Therefore, we may choose R large enough such that $\mathbb{P}(E_R) \ge 1/2$.

STEP 2. Choice of a path in each cluster. For $i \in \{1,2\}$, define

$$\text{Cross}(i) = \{x \in \mathcal{E} \cap \mathcal{C}_i^{\text{ext}} \;:\; \exists s \in [0,t],\, (\|x\| - R)(\|B_s^x\| - R) < 0\}, \tag{5.6.21}$$

that is the set of points in $\mathcal{E} \cap \mathcal{C}_i^{\text{ext}}$, whose associated Brownian motion crosses $\partial\mathcal{B}(0,R)$. Note that $\text{Cross}(i) \neq \emptyset$ on E_R. For $i \in \{1,2\}$ we denote by x_i the almost-surely uniquely defined $x_i \in \text{Cross}(i)$, such that

$$\|x_i\| = \inf_{y \in \text{Cross}(i)} \|y\|. \tag{5.6.22}$$

Note that this way of picking x_i is arbitrary. Any other way would serve our purpose as well.

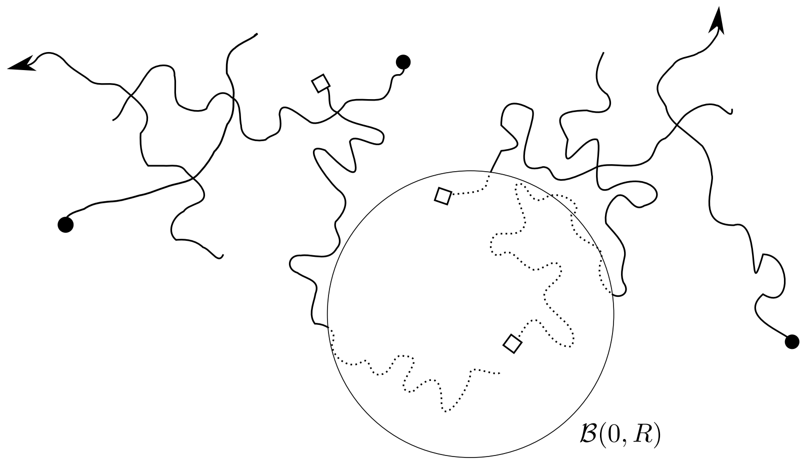

Figure 5.4: The regular lines are a realization of $\mathcal{C}_i,\ i=1,2$. In addition with the dotted lines they form the extended clusters $\mathcal{C}_i^{ext},\ i=1,2$. The points marked with $\square$ are the ones in Cross$(i),\ i=1,2$.

STEP 3. First excursion through an annulus centered around $\mathcal{B}(0,R)$. For some $\epsilon>0$ to be determined, let us consider the annulus $\mathcal{A}_{R,\epsilon} := \mathcal{A}(R-\epsilon, R+\epsilon)$. Further, define for each $x\in\mathcal{E}$,

$$I(x) := \mathbb{1}\{\inf\{s\geq 0 : \|B_s^x\| = R+\epsilon\} < \inf\{s\geq 0 : \|B_s^x\| = R-\epsilon\}\}, \tag{5.6.23}$$

in the case when at least one of the infima is finite. Otherwise, we set $I(x)=0$. We will see later that the latter case is of no importance. For $i\in\{1,2\}$, we introduce the following entrance and exit times:

$$\begin{aligned}\sigma_i^{\text{out}} &= \inf\{s\geq 0 : \|B_s^{x_i}\| = R+(-1)^{I(x_i)}\epsilon\},\\ \sigma_i^{\text{in}} &= \sup\{s\leq \sigma_i^{\text{out}} : \|B_s^{x_i}\| = R-(-1)^{I(x_i)}\epsilon\},\end{aligned} \tag{5.6.24}$$

i.e. $B^{x_i}_{[\sigma_i^{\text{in}},\sigma_i^{\text{out}}]}$ is the first excursion through $\mathcal{A}_{R,\epsilon}$ of B^{x_i} (see Fig. 5.5 below). The reason for this at a first glance strange definition is, that we do not want to exclude the possibility that x_1 or x_2 is located inside $\mathcal{B}(0,R)$. By choosing ϵ small enough we guarantee that the Brownian motions started at x_1 and x_2 cross $\mathcal{A}_{R,\epsilon}$, that is, $\sigma_i^{\text{in}}\leq\sigma_i^{\text{out}}\leq t$ for $i\in\{1,2\}$. Later in the proof we will merge $\mathcal{C}_1^{\text{ext}}$ and $\mathcal{C}_2^{\text{ext}}$ into a single unbounded cluster by "replacing" $B^{x_1}_{[\sigma_1^{\text{in}},\sigma_1^{\text{out}}]}$ and $B^{x_2}_{[\sigma_2^{\text{in}},\sigma_2^{\text{out}}]}$ with suitable excursions. However, this operation should not disconnect $B^{x_i}_{[0,t]}$ from $\mathcal{C}_i^{\text{ext}}$. For that reason, we consider the event on which $B^{x_i}_{[0,\sigma_i^{\text{in}})}$ or $B^{x_i}_{(\sigma_i^{\text{out}},t]}$ is already connected to $\mathcal{C}_i^{\text{ext}}$, i.e. we introduce for $i\in\{1,2\}$

$$E^{\text{conn}}_{\epsilon,i} := \left\{\left(B^{x_i}_{[0,\sigma_i^{\text{in}})}\cup B^{x_i}_{(\sigma_i^{\text{out}},t]}\right)\cap\mathcal{C}_i^{\text{ext}}\neq\emptyset\right\}. \tag{5.6.25}$$

Summing everything up, we restrict ourselves to configurations in the set

$$E_{R,\epsilon} = E_R \bigcap_{i=1,2} \{\sigma_i^{\text{in}} \le \sigma_i^{\text{out}} \le t\} \cap E_{\epsilon,i}^{\text{conn}}. \tag{5.6.26}$$

By monotonicity in ϵ, $\mathbb{P}(E_{R,\epsilon})$ converges to $\mathbb{P}(E_R) > 1/2$ as ϵ tends to 0. Therefore, we may fix for the rest of the proof $\epsilon > 0$ such that $\mathbb{P}(E_{R,\epsilon}) \ge 1/4$.

STEP 4. Restriction on the time spent to cross the annulus. As has been explained above, our goal is to restrict ourselves to some specific excursions of $B^{x_1}_{[\sigma_1^{\text{in}},\sigma_1^{\text{out}}]}$ and $B^{x_2}_{[\sigma_2^{\text{in}},\sigma_2^{\text{out}}]}$. The probability of those turn out to be easier to control when we have a deterministic lower bound on the random time lengths $\sigma_i^{\text{out}}-\sigma_i^{\text{in}}$. Therefore, we introduce for $T \in (0,t)$ the following event:

$$E_{R,\varepsilon,T} = E_{R,\epsilon} \bigcap_{i=1,2} \{\sigma_i^{\text{out}} - \sigma_i^{\text{in}} \ge T\}. \tag{5.6.27}$$

Again, by monotonicity in T, we can choose the latter small enough such that $\mathbb{P}(E_{R,\epsilon,T}) \ge \mathbb{P}(E_{R,\epsilon})/2 \ge 1/8$.

STEP 5. Staying away from the boundary of the annulus during the excursion. To obtain a configuration with a unique unbounded cluster, we restrict ourselves to configurations in the set $E_{R,\epsilon,T}$ and we reroute $B^{x_1}_{[\sigma_1^{\text{in}},\sigma_1^{\text{out}}]}$ and $B^{x_2}_{[\sigma_2^{\text{in}},\sigma_2^{\text{out}}]}$ such that they intersect each other. Since σ_i^{in} is not a stopping time, the law of $B^{x_i}_{[\sigma_i^{\text{in}},\sigma_i^{\text{out}}]}$ is not the one of a Brownian motion. Conditioned on both endpoints, $\big(B^{x_i}_{[\sigma_i^{\text{in}},\sigma_i^{\text{out}}]}\big)$, $i \in \{1,2\}$, are instead Brownian excursions, the law of which is not absolutely continuous with respect to the one of a Brownian motion. As a consequence, we cannot directly use our knowledge on the intersection probabilities of two Brownian motions. This is why we will work with $(B^{x_i}_{[\sigma_i^{\text{in}}+\delta,\sigma_i^{\text{out}}-\delta]})$, $i \in \{1,2\}$, for some $\delta \in (0,T/8)$ instead (the restriction to consider the Brownian motions only up to time $\sigma_i^{\text{out}} - \delta$ is just for esthetic reasons) . These subpaths, when conditioned on both endpoints, are Brownian bridges conditioned to stay in $\mathcal{A}_{R,\epsilon}$, and indeed the density of a Brownian bridge with respect to a Brownian motion is explicit and tractable. To be more precise, the latter property holds only on time intervals excluding neighbourhoods of the endpoints, so we need to work with $B^{x_i}_{[\sigma_i^{\text{in}}+2\delta,\sigma_i^{\text{out}}-2\delta]}$ instead. To get a uniform lower bound on the intersection probability (see (5.6.37)), we consider for some $\overline{\epsilon} \in (0,\epsilon)$ in addition the events

$$\widetilde{E}_{R,\epsilon,T,\overline{\epsilon}} := E_{R,\varepsilon,T} \bigcap_{i=1,2} \left\{B^{x_i}_{\sigma_i^{\text{in}}+\delta}, B^{x_i}_{\sigma_i^{\text{in}}+2\delta}, B^{x_i}_{\sigma_i^{\text{out}}-2\delta}, B^{x_i}_{\sigma_i^{\text{out}}-\delta} \in \mathcal{A}_{R,\overline{\epsilon}}\right\}, \quad \text{and} \tag{5.6.28}$$

$$E_{R,\epsilon,T,\overline{\epsilon}} := E_{R,\varepsilon,T} \bigcap_{i=1,2} \left\{B^{x_i}_{\sigma_i^{\text{in}}+\delta}, B^{x_i}_{\sigma_i^{\text{out}}-\delta} \in \mathcal{A}_{R,\overline{\epsilon}}\right\}. \tag{5.6.29}$$

Again, by monotonicity of $E_{R,\epsilon,T,\overline{\epsilon}}$ w.r.t. $\overline{\epsilon}$, as $\overline{\epsilon}$ converges to ε, $\mathbb{P}(E_{R,\epsilon,T,\overline{\epsilon}})$ converges to $\mathbb{P}(E_{R,\epsilon,T}) \ge 1/8$. Hence, we may choose $\overline{\epsilon}$ such that $\mathbb{P}(E_{R,\epsilon,T,\overline{\epsilon}}) \ge 1/16 > 0$. Finally, we call a configuration which lies in $E_{R,\epsilon,T,\bar{\epsilon}}$ a configuration of good extended clusters.

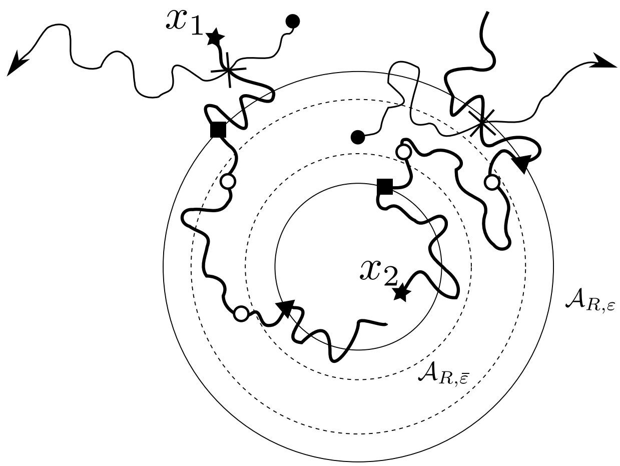

Figure 5.5: In this picture the points marked with ★ are $x_i,\ i = 1, 2$. The symbols ■, ▲ refer to the times σ^{in} and σ^{out}, respectively. The symbol ∘ represents the times $\sigma^{\text{in}} + \delta$ and $\sigma^{\text{out}} - \delta$, respectively. Finally, the symbol × stresses the fact that condition (5.6.25) is fulfilled.

Additional notation. At this point we would like to introduce some notation in order to avoid repetitions of complicated expressions.
First, let us introduce the events of interest. Let $s > r \geq 0$. For a set $D \subset \mathbb{R}^d$, we denote by

$$\mathcal{S}_{[r,s]}(D) := \{\Pi \in C([0,\infty), \mathbb{R}^d) : \Pi_{[r,s]} \subseteq D\}, \tag{5.6.30}$$

the set of all continuous paths, which *stay in the set* D during the whole time interval $[r, s]$, and by

$$\mathcal{L}_{r,s}(D) := \{\Pi \in C([0,\infty), \mathbb{R}^d) : \Pi_r, \Pi_s \in D\}, \tag{5.6.31}$$

the set of all continuous paths , which *belong to the set* D at times r, s.
In the same fashion we also define for $s_1 > r_1 \geq 0$ and $s_2 > r_2 \geq 0$

$$\mathcal{I}_{[s_1,r_1],[s_2,r_2]} := \{\Pi^{(1)}, \Pi^{(2)} \in C([0,\infty), \mathbb{R}^d) : \Pi^{(1)}_{[s_1,r_1]} \bigcap \Pi^{(2)}_{[s_2,r_2]} \neq \emptyset\}, \tag{5.6.32}$$

the set of all pairs of continuous paths $\Pi^{(1)}$ and $\Pi^{(2)}$ whose traces, when restricted to the time intervals $[r_1, s_1]$ and $[r_2, s_2]$, respectively, have a non-empty intersection.
Secondly, we modify our previous notation a bit: $\mathbb{P}^a_t$ now denotes the law of Brownian motion starting at a and running from time 0 up to time t. If we consider Brownian bridges instead of Brownian motions we substitute the letter a by $\mathbf{a} = (\underline{a}; \overline{a})$ containing the starting and ending position of the Brownian bridge. In case of considering two independent copies of a Brownian motion (Brownian bridge) we will add a superscript/subscript, ie. $\mathbb{P}^{a_1,a_2}_{t_1,t_2}$ ($\mathbb{P}^{\mathbf{a_1},\mathbf{a_2}}_{t_1,t_2}$). Finally, we will refer to a Brownian bridge as W.

Connecting $\mathcal{C}_1^{\rm ext}$ and $\mathcal{C}_2^{\rm ext}$ inside the annulus. Step 1–Step 5 translates into the following lower bound:

$$\mathbb{P}(N_\infty = 1) \geq \mathbb{P}(\widetilde{E}_{R,\epsilon,T,\overline{\epsilon}} \bigcap \{B^{x_1}_{[\sigma_1^{\rm in}+\delta,\sigma_1^{\rm out}-\delta]} \cap B^{x_2}_{[\sigma_2^{\rm in}+\delta,\sigma_2^{\rm out}-\delta]} \neq \emptyset\}), \tag{5.6.33}$$

which equals

$$\begin{aligned}&\mathbb{P}(E_{R,\epsilon,T,\overline{\epsilon}})\\ &\times \mathbb{P}\Big(\Big\{B^{x_1}_{[\sigma_1^{\rm in}+\delta,\sigma_1^{\rm out}-\delta]} \cap B^{x_2}_{[\sigma_2^{\rm in}+\delta,\sigma_2^{\rm out}-\delta]} \neq \emptyset\Big\} \bigcap_{i=1,2} \Big\{B^{x_i}_{\sigma_i^{\rm in}+2\delta}, B^{x_i}_{\sigma_i^{\rm out}-2\delta} \in \mathcal{A}_{R,\overline{\epsilon}}\Big\} \mid E_{R,\epsilon,T,\overline{\epsilon}}\Big).\end{aligned} \tag{5.6.34}$$

Observation: For $i \in \{1,2\}$, conditionally on $T_i := \sigma_i^{\rm out} - \sigma_i^{\rm in}$ and the endpoints $(B^{x_i}_{\sigma_i^{\rm in}+\delta}, B^{x_i}_{\sigma_i^{\rm out}-\delta}) = (a_i, b_i)$, $B^{x_i}_{[\sigma_i^{\rm in}+\delta,\sigma_i^{\rm out}-\delta]}$ is a Brownian bridge running from a_i to b_i in a time interval of length $\tau_i := T_i - 2\delta \geq \frac{3T}{4}$, conditioned to stay in $\mathcal{A}_{R,\epsilon}$ (recall the definitions of $\sigma_i^{\rm in}$ and $\sigma_i^{\rm out}$, $i \in \{1,2\}$).

The observation above yields,

$$\mathbb{P}(N_\infty = 1) \geq \mathbb{P}(E_{R,\epsilon,T,\overline{\epsilon}}) \inf_{\substack{\tau_1,\tau_2 \geq 3T/4\\ \mathbf{a_1},\mathbf{a_2} \in \mathcal{A}^2_{R,\overline{\epsilon}}}} \mathcal{P}_\cap(\mathbf{a_1},\mathbf{a_2},\tau_1,\tau_2), \tag{5.6.35}$$

where

$$\mathcal{P}_\cap(\mathbf{a_1},\mathbf{a_2},\tau_1,\tau_2) := \mathbb{P}^{\mathbf{a_1},\mathbf{a_2}}_{\tau_1,\tau_2}\Big(\bigcap_{i=1,2} \{\mathcal{L}^i_{\delta,\tau_i-\delta}(\mathcal{A}_{R,\bar{\varepsilon}})\ ,\ \mathcal{S}^i_{[0,\tau_i]}(\mathcal{A}_{R,\varepsilon})\},\quad \mathcal{I}_{[0,\tau_1],[0,\tau_2]}\Big) \tag{5.6.36}$$

and the superscript i, $i \in \{1,2\}$, on the events in (5.6.36) refers to the i-th copy of the corresponding processes. Since $\mathbb{P}(E_{R,\epsilon,T,\overline{\epsilon}}) > 0$, by Step 1–Step 5, it is enough to prove that

$$\inf_{\substack{\tau_1,\tau_2 \geq 3T/4\\ \mathbf{a_1},\mathbf{a_2} \in \mathcal{A}^2_{R,\overline{\epsilon}}}} \mathcal{P}_\cap(\mathbf{a_1},\mathbf{a_2},\tau_1,\tau_2) > 0. \tag{5.6.37}$$

Proof of Inequality (5.6.37). We fix $\mathbf{a_1},\mathbf{a_2} \in \mathcal{A}_{R,\overline{\epsilon}}$ and $\tau_1,\tau_2 \geq 3T/4$. The right-hand side of (5.6.36) may be bounded from below by

$$\mathbb{P}^{\mathbf{a_1},\mathbf{a_2}}_{\tau_1,\tau_2}\Big(\bigcap_{i=1,2} \{\mathcal{L}^i_{\delta,\tau_i-\delta}(\mathcal{A}_{R,\bar{\varepsilon}})\ ,\ \mathcal{S}^i_{[0,\tau_i]}(\mathcal{A}_{R,\varepsilon})\},\quad \mathcal{I}_{[0,\tau_1-\delta],[0,\tau_2-\delta]}\Big), \tag{5.6.38}$$

which equals, by the Markov property applied at time $\tau_i - \delta$, $i \in \{1,2\}$,

$$\mathbb{E}^{\mathbf{a_1},\mathbf{a_2}}_{\tau_1,\tau_2}\Big(\prod_{i=1,2} \mathbb{1}\big\{\mathcal{L}^i_{\delta,\tau_i-\delta}(\mathcal{A}_{R,\bar{\varepsilon}})\ ,\ \mathcal{S}^i_{[0,\tau_i-\delta]}(\mathcal{A}_{R,\varepsilon})\big\}\ .\ \mathbb{1}\big\{\mathcal{I}_{[0,\tau_1-\delta],[0,\tau_2-\delta]}\big\}\Phi_\delta((W^{(i)}_{\tau_i-\delta};\overline{a}_i))\Big) \tag{5.6.39}$$

where

$$\Phi_\delta(\mathbf{a}) := \mathbb{P}_\delta^{\mathbf{a}}(\mathcal{S}_{[0,\delta]}(\mathcal{A}_{R,\epsilon})), \qquad \mathbf{a} = (\underline{a}, \overline{a}) \in (\mathbb{R}^d)^2. \tag{5.6.40}$$

is the probability that a Brownian bridge, going from $\underline{a}$ to $\overline{a}$ within the time interval $[0, \delta]$, stays in $\mathcal{A}_{R,\epsilon}$. To bound (5.6.39) from below we use the following three lemmas, whose proofs are postponed to the appendix.

Lemma 5.6.4. ***[Positive probability for a Brownian bridge to stay inside the annulus]*** *There exists $c > 0$ such that for all $\mathbf{a} \in \mathcal{A}_{R,\overline{\epsilon}}^2$, $\Phi_\delta(\mathbf{a}) \geq c$.*

Lemma 5.6.5. ***[Substitution of the Brownian bridge by a Brownian motion]*** *Let $\tau > 0$ and $\delta \in (0, \tau)$. There exists $c > 0$ such that for all $\mathbf{a} \in \mathcal{A}_{R,\overline{\epsilon}}^2$, $\mathbf{a} = (\underline{a}, \overline{a})$,*

$$\frac{\mathrm{d}\mathbb{P}_\tau^{\mathbf{a}}(W_{[0,\tau-\delta]} \in \cdot\ ,\ \mathcal{L}_{\delta,\tau-\delta}(\mathcal{A}_{R,\overline{\epsilon}}))}{\mathrm{d}\mathbb{P}_\tau^{\underline{a}}(B_{[0,\tau-\delta]} \in \cdot\ ,\ \mathcal{L}_{\delta,\tau-\delta}(\mathcal{A}_{R,\overline{\epsilon}}))} \geq c. \tag{5.6.41}$$

Lemma 5.6.6. ***[Two Brownian motions restricted to be inside the annulus do intersect]*** *Let $\tau_1, \tau_2 > 0$ and $0 < \delta < \frac{\tau_1 \wedge \tau_2}{2}$. There exists $c > 0$ such that for all $a_1, a_2 \in \mathcal{A}_{R,\overline{\epsilon}}$*

$$\mathbb{P}_{\tau_1,\tau_2}^{a_1,a_2}\Big(\bigcap_{i=1,2} \{\mathcal{L}_{\delta,\tau_i-\delta}^i(\mathcal{A}_{R,\bar{\varepsilon}})\ ,\ \mathcal{S}_{[0,\tau_i-\delta]}^i(\mathcal{A}_{R,\varepsilon})\}, \quad \mathcal{I}_{[0,\tau_1-\delta],[0,\tau_2-\delta]}\Big) \geq c. \tag{5.6.42}$$

We now explain how to get (5.6.37) by applying Lemmas 5.6.4–5.6.6 to (5.6.39). Since the $W_{\tau_i-\delta}$, $i \in \{1, 2\}$, appearing in (5.6.39) are in $\mathcal{A}_{R,\overline{\epsilon}}$, Lemma 5.6.4 yields that, for some $c > 0$, (5.6.39) is not smaller than

$$c^2 \cdot\ \mathbb{P}_{\tau_1,\tau_2}^{\mathbf{a_1},\mathbf{a_2}}\Big(\bigcap_{i=1,2} \{\mathcal{L}_{\delta,\tau_i-\delta}^i(\mathcal{A}_{R,\bar{\varepsilon}})\ ,\ \mathcal{S}_{[0,\tau_i-\delta]}^i(\mathcal{A}_{R,\varepsilon})\}, \quad \mathcal{I}_{[0,\tau_1-\delta],[0,\tau_2-\delta]}\Big). \tag{5.6.43}$$

Next, a change of measure argument together with the bound on the Radon-Nikodym derivative as provided in Lemma 5.6.5 yields, for a possibly different constant $c > 0$, that (5.6.43) is at least

$$c \cdot\ \mathbb{P}_{\tau_1,\tau_2}^{\underline{a_1},\underline{a_2}}\Big(\bigcap_{i=1,2} \{\mathcal{L}_{\delta,\tau_i-\delta}^i(\mathcal{A}_{R,\bar{\varepsilon}})\ ,\ \mathcal{S}_{[0,\tau_i-\delta]}^i(\mathcal{A}_{R,\varepsilon})\}, \quad \mathcal{I}_{[0,\tau_1-\delta],[0,\tau_2-\delta]}\Big), \tag{5.6.44}$$

which is positive by Lemma 5.6.6. To deduce Equation (5.6.37) from it, it is enough to note that all the previous estimates were uniform in $\mathbf{a_1}, \mathbf{a_2} \in \mathcal{A}_{R,\overline{\epsilon}}$. This finally yields the claim.

Remark 5.6.7. *If $k > 2$, then one follows the same scheme. The notion of good extended clusters is easily generalized and one ends up connecting k excursions in an annulus. Using the same proof as for two excursions, one can connect $B^{x_1}_{[\sigma_1^{\mathrm{in}},\sigma_1^{\mathrm{out}}]}$ to $B^{x_i}_{[\sigma_i^{\mathrm{in}},\sigma_i^{\mathrm{out}}]}$ during the time interval $[\sigma_1^{\mathrm{in}} + (i-1)\delta/k, \sigma_1^{\mathrm{in}} + i\delta/k]$, where $\delta \in (0, T)$, for all $1 < i \leq k$.*

5.6.3.2 Excluding $N_\infty = \infty$

Let us assume that the number N_∞ of unbounded clusters in $\mathcal{O}_t$ is almost-surely equal to infinity. In the same fashion as in Subsection 5.6.2.2 we show that this leads to a contradiction. We define the event

$$E_R(0) := \left\{ \begin{array}{l} \exists \text{ an unbounded cluster } C \text{ such that } C \cap \mathcal{B}_\infty(0,R)^{\complement} \text{ contains at least three} \\ \text{unbounded clusters and each unbounded cluster which has a non-empty} \\ \text{intersection with } \mathcal{B}_\infty(0,R) \text{ equals } C. \end{array} \right\}, \tag{5.6.45}$$

The fact that there is R large enough such that $E_R(0)$ has positive probability can be seen as follows. First, note that for R large enough, with positive probability the event

$$E_R^1(0) = \bigcup_{k \geq 3} \left\{ \exists\, k \text{ unbounded clusters in } \mathcal{B}_\infty(0,R)^{\complement} \text{ which intersect } \mathcal{B}(0,R) \right\} \tag{5.6.46}$$

happens. As a consequence, there is $k^* \geq 3$ such that the event inside the union in (5.6.46) occurs for $k = k^*$ with positive probability. Moreover, we may write

$$\begin{aligned} E_R(0) &= \bigcup_{k \geq 3} \left\{ \begin{array}{c} \exists\, k \text{ unbounded clusters in } \mathcal{B}_\infty(0,R)^{\complement}, \text{ which intersect} \\ \mathcal{B}_\infty(0,R) \text{ and all of them are connected inside } \mathcal{B}_\infty(0,R) \end{array} \right\} \\ &\supseteq \left\{ \begin{array}{c} \exists\, k^* \text{ unbounded clusters in } \mathcal{B}_\infty(0,R)^{\complement}, \text{ which intersect} \\ \mathcal{B}_\infty(0,R) \text{ and all of them are connected inside } \mathcal{B}_\infty(0,R) \end{array} \right\}. \end{aligned} \tag{5.6.47}$$

Remark 5.6.7 and the lines preceding (5.6.47) yield that the last event in (5.6.47) has positive probability and consequently, so does $E_R(0)$. From now on, the proof works similarly as the proof in Section 5.6.2.2. Thus, to avoid repetitions we just point out the differences with the proof in Section 5.6.2.2.
The identification done in **STEP 2.** of Section 5.6.2.2 has to be changed. For each $z \in \mathbb{Z}^d$, we replace the Poisson point inside $\mathcal{B}_\infty(2Rz,R)$ that was used to connect the "external" clusters by what we call an *intersection point*, which is just an arbitrarily chosen point $\widetilde{z} \in \mathcal{B}_\infty(2Rz,R)$ contained in all the clusters. Finally, at the moment of applying Lemma 5.6.3, we consider

$$C_z^i = \left\{ x \in \{\mathcal{E} \cap \mathcal{B}_\infty(0,LR)\} \cup \{\text{intersection points}\} \colon\ x \overset{\Lambda_z^c}{\longleftrightarrow} \mathbf{B}_z^i \right\} \qquad i = 1, \ldots, n_z,$$

and

$$S = \mathcal{B}_\infty(0,LR) \cap (\mathcal{E} \cup \{\text{intersection points}\}).$$

This choice generates a minor difference at the moment of getting the contradiction in (5.6.17). Indeed, we have that

$$\mathbb{E}\big(|S|\big) \geq K((L-M-2)^d \delta/2 + 2) \tag{5.6.48}$$

but, taking into account the intersection points we have that,

$$\mathbb{E}\big(|S|\big) \leq \mathbb{E}(|\mathcal{B}_\infty(0,LR) \cap \mathcal{E}|) + \mathbb{E}(|\mathcal{R}|) \leq \lambda(2LR)^d + (L-M+2)^d. \tag{5.6.49}$$

In the last inequality we used that

$$|\mathcal{R}| \leq |\{z \in \mathbb{Z}^d : \mathcal{B}_\infty(2Rz, R) \subseteq \mathcal{B}_\infty(0, LR)\}| \leq (L - M + 2)^d. \tag{5.6.50}$$

Thus, combining (5.6.48) and (5.6.49) yields

$$\forall L > M + 2, \quad K((L - M - 2)^d \delta/2 + 2) \leq \lambda(2LR)^d + (L - M + 2)^d, \tag{5.6.51}$$

from which we obtain the desired contradiction in the same way as in the case $d \geq 4$.

Appendix

5.7 Proof of Lemma 5.2.3: a finite box criterion for subcriticality

The proof consists of two steps. In the first step a coarse-graining procedure is introduced, which reduces the problem of showing subcriticality of a continuous percolation model to showing subcriticality of an infinite range site percolation model on $\mathbb{Z}^d$. This coarse-graining was essentially already introduced in [MR96, Lemma 3.3], where ρ was supposed to have a compact support. To overcome the additional difficulties arising from the long range dependencies in the coarse-grained model we use a renormalization scheme, which is very similar to the one in Sznitman [S10, Theorem 3.5].

STEP 1. Coarse-graining.
We fix $N \in \mathbb{N}$. For $n \in \mathbb{N}$, a sequence of vertices $z_0, z_1, \ldots, z_{n-1}$ in $\mathbb{Z}^d$ is called a $*$-path, when $\|z_i - z_{i-1}\|_\infty = 1$ for all $i \in \{1, 2, \ldots, n-1\}$. Furthermore, a site $z = (z(j), 1 \leq j \leq d) \in \mathbb{Z}^d$ is called open when there is an occupied cluster Λ of Σ such that

$$(i)\ \Lambda \cap \prod_{j=1}^{d} [z(j)N, (z(j)+1)N) \neq \emptyset \text{ and } (ii)\ \Lambda \cap \left(\prod_{j=1}^{d} [(z(j)-1)N, (z(j)+2)N) \right)^{\mathsf{c}} \neq \emptyset. \tag{5.7.1}$$

Otherwise z is called closed. It was shown in [MR96, Lemma 3.3] that to obtain Lemma 5.2.3 it suffices to show that

$$\mathbb{P}_{\lambda,\rho}\Big(0 \text{ is contained in an infinite } *\text{-path of open sites}\Big) = 0. \tag{5.7.2}$$

To prove (5.7.2) we introduce a renormalization scheme.

STEP 2. Renormalization.
• **New notation and a first bound.** We start by introducing a fair amount of new notation. We fix integers $R > 1$ and $L_0 > 1$, both to be determined and we introduce an increasing sequence of scales via

$$\forall n \in \mathbb{N}_0, \quad L_{n+1} = R^{n+1} L_n. \tag{5.7.3}$$

Moreover, for $i \in \mathbb{Z}^d$, we introduce a sequence of increasing boxes via

$$\begin{aligned} C_n(i) &= \prod_{j=1}^{d} [i(j)L_n, (i(j)+1)L_n) \cap \mathbb{Z}^d \quad \text{and} \\ \widetilde{C}_n(i) &= \prod_{j=1}^{d} [(i(j)-1)L_n, (i(j)+2)L_n) \cap \mathbb{Z}^d. \end{aligned} \tag{5.7.4}$$

We further abbreviate $C_n = C_n(0)$ and $\widetilde{C}_n = \widetilde{C}_n(0)$. Thus, $\widetilde{C}_n(i)$ is the union of boxes $C_n(j)$ such that $\|j - i\|_\infty \le 1$. Moreover, for $n \in \mathbb{N}$, we introduce the events

$$A_n(i) = \Big\{ \text{there is a } *\text{-path of open sites from } C_n(i) \text{ to } \partial_{\mathrm{int}} \widetilde{C}_n(i) \Big\}, \tag{5.7.5}$$

and we write A_n instead of $A_n(0)$. Here, $\partial_{\mathrm{int}} B$ refers to the inner boundary of a set $B \subseteq \mathbb{Z}^d$ with respect to the $\|\cdot\|_\infty$-norm. The idea of the renormalization scheme is to bound the probability of A_{n+1} in terms of the probability of the intersection of events $A_n(i)$ and $A_n(k)$, where $i \in \mathbb{Z}^d$ and $k \in \mathbb{Z}^d$ are far apart. By our assumption on the radius distribution ρ, the events $A_n(i)$ and $A_n(k)$ can then be treated as being basically independent. This will result in a recursion inequality, which relates the events A_n, $n \in \mathbb{N}$, at different scales to each other. For that, we fix $n \in \mathbb{N}$ and let

$$\begin{aligned} \mathcal{H}_1 &= \Big\{ i \in \mathbb{Z}^d \,:\, C_n(i) \subseteq C_{n+1}, C_n(i) \cap \partial_{\mathrm{int}} C_{n+1} \neq \emptyset \Big\} \quad \text{and} \\ \mathcal{H}_2 &= \Big\{ k \in \mathbb{Z}^d \,:\, C_n(k) \cap \Big\{ z \in \mathbb{Z}^d \,:\, \mathrm{dist}(z, C_{n+1}) = \frac{L_{n+1}}{2} \Big\} \neq \emptyset \Big\}. \end{aligned} \tag{5.7.6}$$

Here, $\mathrm{dist}(z, C_{n+1})$ denotes the distance of z from the set C_{n+1} with respect to the supremum norm. Note that here and in the rest of the proof, for notational convenience, we pretend that expressions like $L_{n+1}/2$ are integers. Observe that if A_{n+1} occurs, then there are $i \in \mathcal{H}_1$ and $k \in \mathcal{H}_2$ such that both $A_n(i)$ and $A_n(k)$ occur. Hence,

$$\begin{aligned} \mathbb{P}_{\lambda,\rho}(A_{n+1}) &\le \sum_{i \in \mathcal{H}_1, k \in \mathcal{H}_2} \mathbb{P}_{\lambda,\rho}(A_n(i) \cap A_n(k)) \\ &\le c_1 R^{2(d-1)(n+1)} \sup_{i \in \mathcal{H}_1, k \in \mathcal{H}_2} \mathbb{P}_{\lambda,\rho}(A_n(i) \cap A_n(k)), \end{aligned} \tag{5.7.7}$$

where $c_1 = c_1(d) > 0$ is a constant which only depends on the dimension.

•Partition of $A_n(i) \cap A_n(k)$. We fix $i \in \mathcal{H}_1$ and $k \in \mathcal{H}_2$. Let $z \in \widetilde{C}_n(i)$ and note that to decide if z is open, it suffices to look at the trace of the Boolean percolation model on

$$\prod_{j=1}^{d} [(z(j)-1)N, (z(j)+2)N). \tag{5.7.8}$$

In a similar fashion one sees that the area which determines if $A_n(i)$ occurs is given by

$$\begin{aligned}&\prod_{j=1}^{d}[((i(j)-1)L_n-1)N,((i(j)+2)L_n+2)N]\\ &\subseteq\prod_{j=1}^{d}[(i(j)-2)L_nN,(i(j)+3)L_nN)\stackrel{\text{def}}{=}\mathrm{DET}(\widetilde{C}_n(i))\end{aligned}\tag{5.7.9}$$

and likewise for $A_n(k)$ with i replaced by k. Here, we used that by our choice of R and L_0 the relation $L_n \geq 2$ holds for all $n \in \mathbb{N}$. We introduce

$$\mathcal{D}(x,r(x)) := \{\mathcal{B}(x,r(x))\cap \mathrm{DET}(\widetilde{C}_n(i)) \neq \emptyset,\, \mathcal{B}(x,r(x))\cap \mathrm{DET}(\widetilde{C}_n(k)) \neq \emptyset\}\tag{5.7.10}$$

and

$$B_n(i,k) := \bigcup_{x\in\mathcal{E}} \mathcal{D}(x,r(x))\tag{5.7.11}$$

so that,

$$\begin{aligned}\mathbb{P}_{\lambda,\rho}(A_n(i)\cap A_n(k)) &= \mathbb{P}_{\lambda,\rho}(A_n(i)\cap A_n(k)\big|B_n(i,k)^{\complement})\times\mathbb{P}_{\lambda,\rho}(B_n(i,k)^{\complement})\\ &\quad+\mathbb{P}_{\lambda,\rho}(A_n(i)\cap A_n(k)\big|B_n(i,k))\times\mathbb{P}_{\lambda,\rho}(B_n(i,k)).\end{aligned}\tag{5.7.12}$$

•Analysis of the first term on the right hand side of (5.7.12). We claim that under $\mathbb{P}_{\lambda,\rho}(\cdot\big|B_n(i,k)^{\complement})$ the events $A_n(i)$ and $A_n(k)$ are independent. To see that, note that the Poisson point process χ on $\mathbb{R}^d\times[0,\infty)$ with intensity measure $\nu=(\lambda\times\mathrm{Leb}_d)\otimes\rho$ (see Section 5.2.1) is a Poisson point process under $\mathbb{P}_{\lambda,\rho}(\cdot|B_n(i,k)^{\complement})$, with intensity measure

$$1\!\!1\{\text{there is no } (x,r(x))\in\chi \text{ such that } \mathcal{D}(x,r(x)) \text{ occurs}\}\times\nu.\tag{5.7.13}$$

However, on $B_n(i,k)^{\complement}$, the events $A_n(i)$ and $A_n(k)$ depend on disjoint subsets of $\mathbb{R}^d\times[0,\infty)$. Consequently, they are independent under $\mathbb{P}_{\lambda,\rho}(\cdot\big|B_n(i,k)^{\complement})$. Hence,

$$\begin{aligned}&\mathbb{P}_{\lambda,\rho}(A_n(i)\cap A_n(k)\big|B_n(i,k)^{\complement})\times\mathbb{P}_{\lambda,\rho}(B_n(i,k)^{\complement})\\ &\quad=\mathbb{P}_{\lambda,\rho}(A_n(i)\big|B_n(i,k)^{\complement})\mathbb{P}_{\lambda,\rho}(A_n(k)\big|B_n(i,k)^{\complement})\times\mathbb{P}_{\lambda,\rho}(B_n(i,k)^{\complement})\\ &\quad\leq\mathbb{P}_{\lambda,\rho}(A_n)^2\times\mathbb{P}_{\lambda,\rho}(B_n(i,k)^{\complement})^{-1}.\end{aligned}\tag{5.7.14}$$

For the last inequality in (5.7.14) we also used the fact that $\mathbb{P}_{\lambda,\rho}(A_n(i))$ does not depend on $i\in\mathbb{Z}^d$.

•Analysis of the second term on the right hand side of (5.7.12). To bound the second term on the right hand side of (5.7.12) it will be enough to bound $\mathbb{P}_{\lambda,\rho}(B_n(i,k))$, since the other term may be bounded by one. Note that

$$\mathbb{P}_{\lambda,\rho}(B_n(i,k))\leq\sum_{\ell\in3\mathbb{Z}^d}\mathbb{P}_{\lambda,\rho}\begin{pmatrix}\exists x\in\mathcal{E}\cap\widetilde{C}_{n+1}(\ell)N: & \mathcal{B}(x,r(x))\cap\mathrm{DET}(\widetilde{C}_n(i))\neq\emptyset\\ \text{and} & \mathcal{B}(x,r(x))\cap\mathrm{DET}(\widetilde{C}_n(k))\neq\emptyset\end{pmatrix}.\tag{5.7.15}$$

Here, the set $\widetilde{C}_{n+1}(\ell)N$ is the set $\{x \in \mathbb{R}^d : x = zN, z \in \widetilde{C}_{n+1}(\ell)\}$. To warm up, we first treat the term $\ell = 0$ in the sum (5.7.15). Note that,

$$\mathrm{dist}(\mathrm{DET}(\widetilde{C}_n(i)), \mathrm{DET}(\widetilde{C}_n(k)) \geq \Big(\frac{L_{n+1}}{2} - 8L_n\Big)N \geq \frac{L_{n+1}}{3}N, \tag{5.7.16}$$

where the last inequality holds for all $n \in \mathbb{N}$, provided R and L_0 are chosen accordingly. Thus, if there is a Poisson point whose corresponding ball intersects $\mathrm{DET}(\widetilde{C}_n(i))$ and $\mathrm{DET}(\widetilde{C}_n(k))$, then its radius is at least $L_{n+1}N/6$. This yields

$$\begin{aligned} &\mathbb{P}_{\lambda,\rho}\left(\begin{array}{cc} \exists x \in \mathcal{E} \cap \widetilde{C}_{n+1}N : & \mathcal{B}(x, r(x)) \cap \mathrm{DET}(\widetilde{C}_n(i)) \neq \emptyset \\ \text{and} & \mathcal{B}(x, r(x)) \cap \mathrm{DET}(\widetilde{C}_n(k)) \neq \emptyset \end{array}\right) \\ &\qquad \leq \mathbb{P}_{\lambda,\rho}\Big(\exists x \in \mathcal{E} \cap \widetilde{C}_{n+1}N : r(x) \geq L_{n+1}N/6\Big). \end{aligned} \tag{5.7.17}$$

We may rewrite (5.7.17) as

$$\begin{aligned} 1 - &\sum_{m=0}^{\infty} \mathbb{P}_{\lambda,\rho}\Big(\forall x \in \mathcal{E} \cap \widetilde{C}_{n+1}, r(x) < L_{n+1}N/6 \Big| |\mathcal{E} \cap \widetilde{C}_{n+1}N| = m\Big) \\ &\qquad\qquad\qquad\qquad \times \mathbb{P}_{\lambda,\rho}(|\mathcal{E} \cap \widetilde{C}_{n+1}N| = m) \\ = 1 - &\sum_{m=0}^{\infty} [1 - \rho([L_{n+1}N/6, \infty))]^m \times \frac{(\lambda \mathrm{Leb}_d(\widetilde{C}_{n+1}N))^m}{m!} \times e^{-\lambda \mathrm{Leb}_d(\widetilde{C}_{n+1}N)} \\ = 1 - &\exp\big\{-\lambda \mathrm{Leb}_d(\widetilde{C}_{n+1}N)\rho([L_{n+1}N/6, \infty))\big\}, \end{aligned} \tag{5.7.18}$$

which is at most $\lambda \mathrm{Leb}_d(\widetilde{C}_{n+1}N)\rho([L_{n+1}N/6, \infty))$. By our assumption on the radius distribution, for R and L_0 large enough, there is a constant $c_2 = c_2(\rho) > 0$ such that the last term may be bounded by $\lambda(3L_{n+1}N)^d e^{-c_2 L_{n+1}N/6}$. Similar arguments show that the left hand side of (5.7.15) is at most

$$\lambda(3L_{n+1}N)^d e^{-c_2 L_{n+1}N/6} + \sum_{m=1}^{\infty} \sum_{\substack{\ell \in 3\mathbb{Z}^d \\ \|\ell\|_\infty = m}} \lambda(3L_{n+1}N)^d \times e^{-c_2(3(m-1)+1/2)L_{n+1}N}. \tag{5.7.19}$$

This may be bounded by

$$c_3\lambda(3L_{n+1}N)^d e^{-c_2 L_{n+1}N/6}, \tag{5.7.20}$$

for some constant $c_3 > 0$ which is independent of R, L_0 and N. Hence, we have bounded the second term on the right hand side of (5.7.12). In particular, we may deduce that for all $n \in \mathbb{N}$, again for a suitable choice of R and L_0, $\mathbb{P}_{\lambda,\rho}(B_n(i,k)^{\complement}) \geq 1/2$.

•**Analysis of the recursion scheme.** Equation (5.7.7) in combination with (5.7.12) and the arguments following it show that

$$\mathbb{P}_{\lambda,\rho}(A_{n+1}) \leq 2c_1 R^{2(d-1)(n+1)}\Big(\mathbb{P}_{\lambda,\rho}(A_n)^2 + c_3\lambda(3L_{n+1}N)^d e^{-c_2 L_{n+1}N/6}\Big). \tag{5.7.21}$$

To deduce the desired result, we first show with the help of (5.7.21) that $\mathbb{P}_{\lambda,\rho}(A_n)$ being small implies that $\mathbb{P}_{\lambda,\rho}(A_{n+1})$ is small as well. As a final step it then remains to show that $\mathbb{P}_{\lambda,\rho}(A_0)$ is already small. We now make this idea more precise. We put

$$\forall n \in \mathbb{N}, \quad a_n = 2c_1 R^{2(d-1)n}\mathbb{P}_{\lambda,\rho}(A_n). \tag{5.7.22}$$

Claim 5.7.1. *For R large enough, for all $n \in \mathbb{N}$ and for all $L_0 \geq 2R^{4(d-1)+1}$, the inequality $a_n \leq L_n^{-1}$ implies that $a_{n+1} \leq L_{n+1}^{-1}$.*

Proof. To prove the claim, let $n \in \mathbb{N}$ and assume that $a_n \leq L_n^{-1}$. Then,

$$\begin{aligned} a_{n+1} &= 2c_1 R^{2(d-1)(n+1)}\mathbb{P}_{\lambda,\rho}(A_{n+1}) \\ &\leq 4c_1^2 R^{4(d-1)(n+1)}\Big[\mathbb{P}_{\lambda,\rho}(A_n)^2 + c_3\lambda(3L_{n+1}N)^d e^{-c_2 L_{n+1}N/6}\Big] \\ &= a_n^2 R^{4(d-1)} + 4c_1^2 c_3 R^{4(d-1)(n+1)}\lambda(3L_{n+1}N)^d e^{-c_2 L_{n+1}N/6}. \end{aligned} \tag{5.7.23}$$

Thus, it is enough to show that

$$a_n^2 R^{4(d-1)} \leq (2L_{n+1})^{-1} \quad \text{and} \quad 4c_1^2 c_3 R^{4(d-1)(n+1)}(3L_{n+1}N)^d e^{-c_2 L_{n+1}N/6} \leq (2L_{n+1})^{-1}. \tag{5.7.24}$$

For that, note that by our assumption on a_n

$$a_n^2 R^{4(d-1)} 2L_{n+1} \leq 2L_n^{-2} R^{4(d-1)} L_{n+1} = 2R^{4(d-1)}\frac{R^{n+1}}{R^n L_{n-1}} \leq 2R^{4(d-1)+1}L_0^{-1}. \tag{5.7.25}$$

Thus, choosing $L_0 \geq 2R^{4(d-1)+1}$ yields the first desired inequality. The second term on the right hand side of (5.7.23) may be bounded using similar considerations. This yields Claim 5.7.1. ∎

Hence, to use the claim, we need that $\mathbb{P}_{\lambda,\rho}(A_0) \leq L_0^{-1}$. For that observe that

$$\begin{aligned} \mathbb{P}_{\lambda,\rho}(A_0) &= \mathbb{P}_{\lambda,\rho}\Big(\text{There is a } *\text{-path of open sites from } [0,L_0)^d \text{ to } \partial_{\text{int}}[-L_0,2L_0)^d.\Big) \\ &\leq \mathbb{P}_{\lambda,\rho}\Big(\text{There is } z \in \partial_{\text{int}}[-L_0,2L_0)^d, \text{ which is open.}\Big) \\ &\leq c_4 L_0^{d-1}\mathbb{P}_{\lambda,\rho}(0 \text{ is open}), \end{aligned} \tag{5.7.26}$$

where $c_4 = c_4(d) > 0$ does only depend on the dimension. Equation (3.64) of [MR96] shows that

$$\mathbb{P}_{\lambda,\rho}(0 \text{ is open}) \leq 2d\mathbb{P}_{\lambda,\rho}(\text{CROSS}(N, 3N, \ldots, 3N)). \tag{5.7.27}$$

Therefore, if the right hand side of (5.7.27) is smaller than $(4dc_1c_4L_0^d)^{-1}$, we get from (5.7.26) that $\mathbb{P}_{\lambda,\rho}(A_0) \leq (2c_1L_0)^{-1}$, which is the same as saying that $a_0 \leq L_0^{-1}$. This, in combination with Claim 5.7.1 and the observation that an infinite $*$-path of open sites containing zero implies the events A_n for all $n \in \mathbb{N}$, finally yields

$$\mathbb{P}_{\lambda,\rho}\Big(0 \text{ is contained in an infinite } *\text{-path of open sites}\Big) \leq \lim_{n\to\infty}\mathbb{P}_{\lambda,\rho}(A_n) = 0. \tag{5.7.28}$$

Consequently, we have shown that Lemma 5.2.3 is satisfied for $\varepsilon \leq (4dc_1c_4L_0^{d+1})^{-1}$.

5.8 Proofs of Lemmas 5.6.4–5.6.6: properties of Brownian motions

5.8.1 Proof of Lemma 5.6.4: positive probability for a Brownian bridge to stay inside an annulus

Proof. Let $\mathbf{a} \in \overline{\mathcal{A}}^2_{R,\overline{\epsilon}}$. First, note that

$$\Phi_\delta(\mathbf{a}) > 0. \tag{5.8.1}$$

Indeed, since for all $\bar{\delta} < \delta$ the path of a Brownian motion $B_{[0,\bar{\delta}]}$ starting in $\underline{a}$ is absolutely continuous with respect to that of the Brownian bridge $W_{[0,\bar{\delta}]}$,

$$\mathbb{P}^{\mathbf{a}}_\delta(W_{[0,\delta/2]} \subseteq \mathcal{A}_{R,\epsilon},\, W_{\delta/2} \in \mathcal{B}(\overline{a}, \epsilon')) > 0, \tag{5.8.2}$$

where $\epsilon' > 0$ is chosen so small that $\mathcal{B}(\overline{a}, \epsilon') \subseteq \mathcal{A}_{R,\epsilon}$. From the representation

$$\forall s \in [0, \delta], \quad W_s = B_s - \frac{s}{\delta}(B_\delta - \overline{a}) \tag{5.8.3}$$

and the fact that a Brownian motion stays with a positive probability in an arbitrary small ball around its starting point within finite time intervals, we have the following:

$$\forall a' \in \mathcal{B}(\overline{a}, \epsilon'), \quad \mathbb{P}^{\mathbf{a}}_\delta(W_{[\delta/2,\delta]} \subseteq \mathcal{A}_{R,\epsilon} \mid W_{\delta/2} = a') > 0. \tag{5.8.4}$$

Equation (5.8.1) then follows from (5.8.2), the Markov property applied at time $\delta/2$ and (5.8.4). Second, the representation in (5.8.3) shows that the map

$$\mathbf{a} \mapsto \mathbb{P}^{\mathbf{a}}_\delta\Big(W_{[0,\delta]} \in \cdot\Big), \qquad \mathbf{a} \in \overline{\mathcal{A}}^2_{R,\overline{\epsilon}}, \tag{5.8.5}$$

is weakly continuous. Moreover, the probability for a Brownian bridge to hit the boundary of $\mathcal{A}_{R,\varepsilon}$ but to stay inside $\mathcal{A}_{R,\varepsilon}$ is zero. Thus, an application of the Portemanteau Theorem yields that the function

$$\mathbf{a} \mapsto \mathbb{P}^{\mathbf{a}}_\delta\Big(W_{[0,\delta]} \in \mathcal{A}_{R,\varepsilon}\Big), \qquad \mathbf{a} \in \overline{\mathcal{A}}^2_{R,\overline{\epsilon}}, \tag{5.8.6}$$

is continuous. This fact together with (5.8.1) is enough to conclude the claim. ∎

5.8.2 Proof of Lemma 5.6.5: substitution of the Brownian bridge by a Brownian motion

Proof. First, for $\mathbf{a} = (\underline{a}, \overline{a}) \in \mathcal{A}_{R,\overline{\epsilon}}$ we have that (see Exercise 1.5 in [MP10])

$$\frac{\mathrm{d}\mathbb{P}^{\mathbf{a}}_\tau(W_{[0,\tau-\delta]} \in \cdot\,)}{\mathrm{d}\mathbb{P}^{\underline{a}}_\tau(B_{[0,\tau-\delta]} \in \cdot\,)} = \frac{p(\delta, W_{\tau-\delta}, \overline{a})}{p(\tau, \underline{a}, \overline{a})}, \tag{5.8.7}$$

where

$$p(s,x,y) := \frac{1}{(2\pi s)^{d/2}} \exp\left(-\frac{\|x-y\|^2}{2s}\right). \tag{5.8.8}$$

Moreover, there exist constants c_1 and c_2 such that

$$0 < c_1 \leq \inf_{\substack{\delta \leq s \leq \tau \\ x,y \in \overline{\mathcal{A}}_{R,\overline{\epsilon}}}} p(s,x,y) \leq \sup_{\substack{\delta \leq s \leq \tau \\ x,y \in \overline{\mathcal{A}}_{R,\overline{\epsilon}}}} p(s,x,y) \leq c_2 < \infty. \tag{5.8.9}$$

Therefore,

$$\frac{\mathrm{d}\mathbb{P}^{\mathbf{a}}_{\tau}(W_{[0,\tau-\delta]} \in \cdot, \mathcal{L}_{\delta,\tau-\delta}(\mathcal{A}_{R,\overline{\epsilon}}))}{\mathrm{d}\mathbb{P}^{\underline{a}}_{\overline{\tau}}(B_{[0,\tau-\delta]} \in \cdot, \mathcal{L}_{\delta,\tau-\delta}(\mathcal{A}_{R,\overline{\epsilon}}))} \geq \left(\frac{c_1}{c_2}\right) > 0. \tag{5.8.10}$$

■

5.8.3 Proof of Lemma 5.6.6: two Brownian motions restricted to be inside an annulus do intersect

Proof. To achieve the intersection event, we use the following strategy:

- before time δ, both paths enter a ball inside $\mathcal{A}_{R,\overline{\epsilon}}$, and from this moment, stay in a slightly bigger ball;
- the two paths intersect each other between time δ and $\tau_1 \wedge \tau_2 - \delta$, while staying in a larger ball contained in $\mathcal{A}_{R,\overline{\epsilon}}$.

More precisely, let us choose arbitrarily $z \in \mathcal{A}_{R,\overline{\epsilon}}$. Let $\epsilon_4 > \epsilon_3 > \epsilon_2 > \epsilon_1 > 0$ to be determined later. For the moment we only assume that $\mathcal{B}(z,\epsilon_4) \subset \mathcal{A}_{R,\overline{\epsilon}}$. For $i \in \{1,2\}$, let us define

$$\sigma_1^{(i)} = \inf\{s \geq 0 : B_s^{a_i} \in \overline{\mathcal{B}}(z,\epsilon_1)\} \tag{5.8.11}$$

$$\sigma_2^{(i)} = \inf\{s \geq \sigma_1^{(i)} : B_s^{a_i} \notin \mathcal{B}(z,\epsilon_2)\}. \tag{5.8.12}$$

First note that with $\hat{\tau} := \tau_1 \wedge \tau_2$ and $\check{\tau} := \tau_1 \vee \tau_2$

$$\left\{ \bigcap_{i=1,2} \{\mathcal{L}^i_{\delta,\tau_i-\delta}(\mathcal{A}_{R,\bar{\varepsilon}})\ ,\ \mathcal{S}^i_{[0,\tau_i-\delta]}(\mathcal{A}_{R,\varepsilon})\},\quad \mathcal{I}_{[0,\tau_1-\delta],[0,\tau_2-\delta]} \right\} \supseteq \tag{5.8.13}$$

$$\left\{ \begin{array}{c} \sigma_1^{(1)} \vee \sigma_1^{(2)} < \delta,\ \sigma_2^{(1)} \wedge \sigma_2^{(2)} > \delta,\ \bigcap_{i=1,2} \mathcal{S}^i_{[0,\delta]}(\mathcal{A}_{R,\epsilon}), \\ \bigcap_{i=1,2} \left\{ \mathcal{S}^i_{[\delta,\hat{\tau}-\delta]}(\mathcal{B}(z,\epsilon_3)), \mathcal{S}^i_{[\hat{\tau}-\delta,\check{\tau}-\delta]}(\mathcal{B}(z,\epsilon_4)) \right\},\ \mathcal{I}_{[\delta,\hat{\tau}-\delta],[\delta,\hat{\tau}-\delta]} \end{array} \right\}.$$

An application of the Markov property at time δ shows that it is enough to establish that

$$\inf_{a_1,a_2 \in \mathcal{A}_{R,\overline{\epsilon}}} \mathbb{P}^{a_1,a_2}_{\delta,\delta}\left(\sigma_1^{(1)} \vee \sigma_1^{(2)} < \delta,\ \sigma_2^{(1)} \wedge \sigma_2^{(2)} > \delta,\ \bigcap_{i=1,2} \mathcal{S}^i_{[0,\delta]}(\mathcal{A}_{R,\epsilon})\right) > 0, \tag{5.8.14}$$

and

$$\inf_{x,y\in\mathcal{B}(z,\epsilon_2)} \mathbb{P}^{x,y}_{\check\tau-2\delta,\check\tau-2\delta}\Big(\mathcal{I}_{[0,\hat\tau-2\delta],[0,\hat\tau-2\delta]} \bigcap_{i=1,2} \mathcal{S}^i_{[0,\hat\tau-2\delta]}(\mathcal{B}(z,\epsilon_3)), \mathcal{S}^i_{[\hat\tau-2\delta,\check\tau-2\delta]}(\mathcal{B}(z,\epsilon_4))\Big) > 0. \tag{5.8.15}$$

Let us first prove (5.8.14). The probability in the infimum is clearly positive for all a_1, a_2 in the compact set $\overline{\mathcal{A}}_{R,\overline{\epsilon}}$. Furthermore, one can use the same arguments as in the proof of Lemma 5.6.4 to show that it is continuous in (a_1, a_2) on $\overline{\mathcal{A}}_{R,\overline{\epsilon}} \times \overline{\mathcal{A}}_{R,\overline{\epsilon}}$, hence the infimum is also positive.

Now we proceed to prove (5.8.15). Again, an application of the Markov property at time $\hat\tau - 2\delta$ shows that it is enough to prove that

$$\inf_{x,y\in\mathcal{B}(z,\epsilon_2)} \mathbb{P}^{x,y}_{\hat\tau-2\delta,\hat\tau-2\delta}\Big(\mathcal{I}_{[0,\hat\tau-2\delta],[0,\hat\tau-2\delta]} \bigcap_{i=1,2} \mathcal{S}^i_{[0,\hat\tau-2\delta]}(\mathcal{B}(z,\epsilon_3))\Big) > 0, \tag{5.8.16}$$

and

$$\inf_{x,y\in\mathcal{B}(z,\epsilon_3)} \mathbb{P}^{x,y}_{\check\tau-\hat\tau,\check\tau-\hat\tau}\Big(\bigcap_{i=1,2} \mathcal{S}^i_{[0,\check\tau-\hat\tau]}(\mathcal{B}(z,\epsilon_4))\Big) > 0. \tag{5.8.17}$$

Now we focus on (5.8.16). For all $\tau_0 > 0$ and $R_0 > 1$, let us consider

$$\begin{aligned}\rho(\tau_0, R_0) &:= \inf_{x,y\in\mathcal{B}(0,1)} \mathbb{P}^{x,y}_{\tau_0,\tau_0}\Big(\mathcal{I}_{[0,\tau_0],[0,\tau_0]} \bigcap_{i=1,2} \mathcal{S}^i_{[0,\tau_0]}(\mathcal{B}(0,R_0))\Big) \\ &\geq \inf_{x,y\in\mathcal{B}(0,1)} \mathbb{P}^{x,y}_{\tau_0,\tau_0}\Big(\mathcal{I}_{[0,\tau_0],[0,\tau_0]}\Big) - 2\sup_{x\in\mathcal{B}(0,1)} \mathbb{P}^{x}_{\tau_0}\Big(\sup_{s\in[0,\tau_0]} \|B_s\| > R_0\Big).\end{aligned} \tag{5.8.18}$$

By using the monotonicity argument in Lemma 5.5.2 and Theorem 9.1 in [MP10], the last infimum can be made arbitrarily close to 1 by choosing τ_0 large enough, whereas standard estimates yield that the supremum goes to 0 as R_0 goes to infinity. Therefore, there is a choice of τ_0 and R_0 leading to $\rho(\tau_0, R_0) > 0$. By the scale invariance of Brownian motion,

$$\forall u > 0, \quad \inf_{x,y\in\mathcal{B}(0,u)} \mathbb{P}^{x,y}_{u^2\tau_0,u^2\tau_0}\Big(\mathcal{I}_{[0,u^2\tau_0],[0,u^2\tau_0]} \bigcap_{i=1,2} \mathcal{S}^i_{[0,u^2\tau_0]}(\mathcal{B}(0,uR_0))\Big) = \rho(\tau_0, R_0) > 0. \tag{5.8.19}$$

We may now choose $u_0 > 0$ such that

$$\tau' := u_0^2\tau_0 < \hat\tau - 2\delta, \quad 2u_0R_0 < \mathrm{dist}(z, \mathcal{A}_{R,\epsilon}), \tag{5.8.20}$$

and we set

$$\epsilon_2 := u_0, \quad \epsilon_3 := 2u_0R_0. \tag{5.8.21}$$

Note that we may choose R_0 such that $\epsilon_3/2 > \epsilon_2$. Hence, an application of the Markov property at time τ' to the left hand side of (5.8.16) yields,

$$\begin{aligned}&\text{l.h.s. of (5.8.16)} \\ &\geq \rho(\tau_0, R_0) \inf_{x,y\in\mathcal{B}(0,\epsilon_3/2)} \mathbb{P}^{x,y}_{\hat\tau-2\delta-\tau',\hat\tau-2\delta-\tau'}\Big(\bigcap_{i=1,2} \mathcal{S}^i_{[0,\hat\tau-2\delta-\tau']}(\mathcal{B}(0,\epsilon_3))\Big) > 0.\end{aligned} \tag{5.8.22}$$

The positivity of the second factor of (5.8.22) and of (5.8.17) may be shown by using similar arguments as in the proof of Lemma 5.6.4. This finally yields the claim. ∎

6 Random interlacements: transience of the vacant set

This chapter is based on:
A. Drewitz, D. Erhard. *Transience of the vacant set for near-critical random interlacements in high dimensions.* Posted on *arXiv:1312.2980v1*, to appear in Annales de l'Institut Henri Poincaré Probabilités et Statistiques.

Abstract

The model of random interlacements is a one-parameter family $\mathcal{I}^u$, $u \geq 0$, of random subsets of $\mathbb{Z}^d$, which locally describes the trace of simple random walk on a d-dimensional torus run up to time u times its volume. Its complement, the so-called vacant set $\mathcal{V}^u$, has been shown to undergo a non-trivial percolation phase-transition in u; i.e., there exists $u_*(d) \in (0, \infty)$ such that for $u \in [0, u_*(d))$ the vacant set $\mathcal{V}^u$ contains a unique infinite connected component $\mathcal{V}^u_\infty$, while for $u > u_*(d)$ it consists of finite connected components. It is known [S11a, SS11b] that $u_*(d) \sim \log d$, and in this article we show the existence of $u(d) > 0$ with $\frac{u(d)}{u_*(d)} \to 1$ as $d \to \infty$ such that $\mathcal{V}^u_\infty$ is transient for all $u \in [0, u(d))$.

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6.1 Introduction and the main result

6.1.1 Introduction

The model of random interlacements has been introduced by Sznitman [S10] as a family of random subsets of $\mathbb{Z}^d$ denoted by $\mathcal{I}^u$, $u \geq 0$, where u plays the role of an intensity parameter. It locally describes the trace of simple random walk on the discrete torus $(\mathbb{Z}/N\mathbb{Z})^d$ run up to time uN^d (see Windisch [W08] as well as Teixeira and Windisch [TW11]). Using the inclusion-exclusion formula the distribution of the set $\mathcal{I}^u$ can be neatly characterized via the equalities

$$\mathbb{P}[K \cap \mathcal{I}^u = \emptyset] = e^{-u \operatorname{cap}(K)}, \quad \forall K \subset\subset \mathbb{Z}^d.$$

Here, $\mathrm{cap}(K)$ is used to denote the capacity of the set K (see (6.2.3) for the definition of capacity). In a more constructive fashion, random interlacements at level u can also be obtained by considering the trace of the elements in the support of a Poisson point process with intensity parameter $u \geq 0$, which itself takes values in the space of locally finite measures on doubly infinite simple random walk trajectories modulo time shift (see Section 6.2.2 for further details).

This constructive definition already suggests that the model exhibits long range dependence, and indeed the asymptotics

$$\mathrm{Cov}(\mathbb{1}_{x \in \mathcal{I}^u}, \mathbb{1}_{y \in \mathcal{I}^u}) \sim c(u)|x-y|_2^{-(d-2)}, \tag{6.1.1}$$

(and similarly for $\mathcal{I}^u$ replaced by $\mathcal{V}^u$) holds for $|x-y|_2 \to \infty$, as can be deduced from (0.11) in [S10]. As a consequence, standard techniques from Bernoulli percolation do not apply anymore. For example, due to (6.1.1) Peierl's argument and the van den Berg-Kesten inequality break down. The long range dependence also entails that random interlacements neither stochastically dominates nor can be dominated by Bernoulli percolation (cf. Remark 1.6 1) of [S10]). Moreover from the constructive definition of random interlacements alluded to above, one can infer that the model does not fulfill the finite energy property (see Remark 2.2 3) of [S10]). These features make the model both, more appealing and more complicated to investigate.

During the past couple of years there has been intensive research on random interlacements. Basic properties such as e.g. the shift-invariance, ergodicity and connectedness of $\mathcal{I}^u$ have been established in the seminal paper [S10]. Since then, one has obtained a deeper understanding of the geometry of random interlacements. In fact, Ráth and Sapozhnikov [RS11] have shown the transience for random interlacements $\mathcal{I}^u$ itself throughout the whole range of parameters $u \in (0, \infty)$. The same authors in [RS10], as well as Procaccia and Tykesson [PT11] have shown by essentially different methods (using ideas from the field of potential theory on the one hand, and stochastic dimension on the other hand) that any two points of the set $\mathcal{I}^u$ can be connected by using at most $\lceil d/2 \rceil$ trajectories from the constructive definition described above. Recently, using in parts extensions of the techniques in [RS10], this result has been generalized to an arbitrary number of points by Lacoin and Tykesson [LT12]. Another step in showing that the geometry of random interlacements resembles that of $\mathbb{Z}^d$ has been undertaken by Černý and Popov [CP12], where the authors prove that the chemical distance (also called graph distance or internal distance) in the set $\mathcal{I}^u$ is comparable to that of $\mathbb{Z}^d$. Using this result they proceed to prove a shape theorem for balls in $\mathcal{I}^u$ with respect to the metric induced by the chemical distance.

It is particularly interesting to obtain a deeper understanding of the vacant set $\mathcal{V}^u$ and its geometry also. Indeed, on the one hand, this is more challenging than the investigation of $\mathcal{I}^u$ in the sense that one cannot directly take advantage of the many tools available for simple random walk, which have proven to be very helpful in understanding the set $\mathcal{I}^u$. On the other hand, it has been shown by Sznitman [S10] as well as Sidoravicius and Sznitman [SS09] that there exists a non-trivial percolation phase-transition for $\mathcal{V}^u$ at some $u_*(d) \in (0, \infty)$ in the following sense: For $u > u_*(d)$ the vacant set $\mathcal{V}^u$ as a subgraph of $\mathbb{Z}^d$ contains only finite connected components (subcritical phase), whereas for $u \in [0, u_*(d))$ it has an infinite connected component almost surely (supercritical phase). Using a strategy inspired by that of the seminal paper of Burton and Keane [BK89], and taking care of the difficulties arising from the lack of the finite energy property for random interlacements, Teixeira [T09] has shown the uniqueness of the infinite connected component of $\mathcal{V}^u$ (denoted by $\mathcal{V}^u_\infty$) in the supercritical phase.

While for random interlacements itself many results have been shown to be valid for any $u > 0$, the situation is more complicated when investigating the vacant set $\mathcal{V}^u$. In fact, while there are few results concerning the vacant set in the first place so far, the ones which describe geometric properties such as Teixeira [T11], Drewitz, Ráth and Sapozhnikov [DRS12a], Popov and Teixeira [PT12] (dealing with the size distribution of finite clusters of the vacant set and local uniqueness properties of $\mathcal{V}^u_\infty$) and Drewitz, Ráth and Sapozhnikov [DRS12b] as well as Procaccia, Rosenthal and Sapozhnikov [PRS13] (providing chemical distance results as well as heat kernel estimates in a more general context) are valid for some non-degenerate fraction of the supercritical phase only. To the best of our knowledge, our main result Theorem 6.1.1 is the first one concerning geometric properties of the vacant set which is valid throughout most, and asymptotically all, of the supercritical phase for $\mathcal{V}^u$.

6.1.2 Main result

Here we formulate our main result. For this purpose recall that a connected graph with finite degree $G = (V, E)$ with vertex set V and edge set E is called transient if simple random walk on G is transient. For the rest of this article V will usually denote a subset of $\mathbb{Z}^d$ and E will be the set of nearest neighbor edges in $\mathbb{Z}^d$ which have both ends contained in V.

Theorem 6.1.1. *Let $\varepsilon \in (0,1)$. There is $d_0 = d_0(\varepsilon) \in \mathbb{N}$, such that for all $d \geq d_0$ and all $u \leq (1-\varepsilon)u_*(d)$, the unique infinite connected component $\mathcal{V}^u_\infty$ of the vacant set $\mathcal{V}^u$ of random interlacements in $\mathbb{Z}^d$ is transient $\mathbb{P}$-a.s.*

Recall here that $u_*(d) \sim \log d$, see [S11a, SS11b], where log denotes the natural logarithm. We refer to Section 6.2 for a rigorous definition of the terms appearing in Theorem 6.1.1.

6.1.3 Discussion

Theorem 6.1.1 provides a rough geometrical description of the infinite connected component of the vacant set, which is valid throughout most of the supercritical phase when d is large enough. To establish this result we introduce a classification of vertices in $\mathbb{Z}^3 \times \{0\}^{d-3}$ into "good" ones and "bad" ones, where "good" refers to having good local connectivity properties. This way the problem will be reduced to showing the transience of an infinite connected component of good vertices in $\mathbb{Z}^3$. Our construction of this infinite cluster will employ results of Sznitman [S11a, S12], whereas the proof of the actual transience of this component uses ideas of Angel, Benjamini, Berger and Peres [ABP06]. Besides making the attempt to extend our result to the entire supercritical phase it would be interesting to obtain a more precise understanding of $\mathcal{V}_\infty^u$. Results in this direction have been obtained in [DRS12b, PRS13]. A key assumption in these papers was a local uniqueness property (in our context of $\mathcal{V}_\infty^u$), which roughly states that with high probability the second largest component in a predetermined macroscopic box is small compared to the largest connected component in the same box. However, this local uniqueness property has so far only been established for a non-degenerate part of the supercricital phase, and obtaining its validity throughout the whole supercritical phase would be an interesting topic for further investigations.

The rest of this chapter is organized as follows. In Section 6.2 we introduce further notation, give a more detailed description of the model and provide a decoupling inequality tailored to our needs (Proposition 6.2.3). The proof of Theorem 6.1.1 is carried out in Section 6.3. Sections 6.4 and 6.5 contain the proofs of auxiliary results employed when proving Theorem 6.1.1.

6.2 Notation and introduction to the model

Section 6.2.1 introduces notation used in this article, Section 6.2.2 defines random interlacements, while Section 6.2.3 states a decoupling inequality. Throughout the article we assume that $d \geq 3$.

6.2.1 Basic notation

In the rest of this article we will tacitly identify $\mathbb{Z}^3$ with $\mathbb{Z}^3 \times \{0\}^{d-3}$ via the bijection $(x_1, x_2, x_3) \mapsto (x_1, x_2, x_3, 0, \dots, 0)$, if no confusion arises.

For a subset $K \subseteq \mathbb{Z}^d$ we write $K \subset\subset \mathbb{Z}^d$, if its cardinality $|K|$ is finite or equivalently if K is compact. We denote by $|\cdot|_1$ the ℓ^1-norm, by $|\cdot|_2$ the Euclidean norm, whereas $|\cdot|_\infty$ stands for the ℓ^∞-norm on $\mathbb{Z}^d$. Sites x, x' in $\mathbb{Z}^d$ are said to be nearest neighbors ($*$-neighbors), if $|x - x'|_1 = 1$ ($|x - x'|_\infty = 1$). A sequence $x_0, x_1, \dots, x_n$ in $\mathbb{Z}^d$ is called a nearest neighbor path ($*$-path), if x_i and x_{i+1} are nearest neighbors ($*$-neighbors), for all $0 \leq i \leq n-1$; in this case we say that the path has length $n+1$. A set $K \subseteq \mathbb{Z}^d$ is said to be connected ($*$-connected), if for any pair $x_1, x_2 \in K$ there exists a nearest neighbor ($*$-neighbor) path $x_1, y_1, y_2, \dots, y_n, x_2$ such that these vertices are contained in K. For $K \subseteq \mathbb{Z}^d$ we introduce the following notions of boundaries

$$\begin{aligned} \partial_{\text{int}} K &= \{x \in K : x \text{ has a nearest neighbor in } K^c\}, \\ \partial^*_{\text{int}} K &= \{x \in K : x \text{ has a } *\text{-neighbor in } K^c\}, \\ \partial K &= \{x \in K^c : x \text{ has a nearest neighbor in } K\}, \\ \partial^* K &= \{x \in K^c : x \text{ has a } *\text{-neighbor in } K\}, \end{aligned} \tag{6.2.1}$$

to which we refer as interior boundary (interior $*$-boundary) and boundary ($*$-boundary), respectively. Moreover, the exterior boundary (exterior $*$-boundary), denoted by $\partial_{\text{ext}} K$ ($\partial^*_{\text{ext}} K$), is the set of vertices in the boundary ($*$-boundary), which are the starting point of an infinite non-intersecting nearest neighbor path with no vertex inside K.
The closure of a set $K \subseteq \mathbb{Z}^d$ is defined by $\overline{K} = K \cup \partial K$. If $x \in \mathbb{Z}^d$ or $x \in \mathbb{Z}^3$ and $L \geq 0$, we write

$$B_i(x, L) = \{y \in \mathbb{Z}^d : \; |x-y|_i \leq L\} \quad \text{and} \quad B_i^3(x, L) = \{y \in \mathbb{Z}^3 : \; |x-y|_i \leq L\},$$

respectively, for $i \in \{1, 2, \infty\}$. Given a set $K \subseteq \mathbb{Z}^d$ and $w : \mathbb{N}_0 \to \mathbb{Z}^d$, we denote by

$$H_K(w) = \inf\{n \geq 0 : \; w(n) \in K\} \quad \text{and} \quad \widetilde{H}_K(w) = \inf\{n \geq 1 : \; w(n) \in K\} \tag{6.2.2}$$

the entrance time in and the hitting time of K, respectively. For $x \in \mathbb{Z}^d$, let P_x denote the law of simple random walk on $\mathbb{Z}^d$ with starting point x. If $K \subset\subset \mathbb{Z}^d$, we write e_K for the equilibrium measure of K, i.e.,

$$\mathrm{e}_K(x) = P_x\big[\widetilde{H}_K = \infty\big] \mathbb{1}_{\{x \in K\}} \quad \text{and} \quad \mathrm{cap}(K) = \sum_{x \in K} e_K(x) \tag{6.2.3}$$

for the total mass of e_K, which is usually referred to as the capacity of K. From this one immediately obtains the subadditivity of the capacity; i.e., for all $K, K' \subset\subset \mathbb{Z}^d$ one has

$$\mathrm{cap}(K \cup K') \leq \mathrm{cap}(K) + \mathrm{cap}(K'). \tag{6.2.4}$$

We denote by $g : \mathbb{Z}^d \times \mathbb{Z}^d \to [0, \infty)$ the Green function of simple random walk on $\mathbb{Z}^d$, which is defined via

$$g(x, x') = \sum_{n \in \mathbb{N}_0} P_x\big[X_n = x'\big], \quad \text{for } x, x' \in \mathbb{Z}^d,$$

and we write $g(0) = g(0, 0)$. Finally, let us explain the convention we use concerning constants. Throughout the article, small letters such as c, c', $c_1, c_2, \cdots$, denote constants which are independent of d. Capital letters, such as C and C_1 might depend on the dimension. Constants that come with an index are fixed from their first appearance on (modulo changes of the dimension if they are capital letter constants), whereas constants without index may change from place to place.

6.2.2 Definition of random interlacements

The model of random interlacements has been introduced in [S10], and we refer to this source for a discussion that goes beyond the description we are giving here. We write

$$W_+ = \Big\{w : \mathbb{N}_0 \to \mathbb{Z}^d : \; |w(n) - w(n+1)|_1 = 1 \; \forall \, n \in \mathbb{N}_0, \text{ and } \lim_{n \to \infty} |w(n)|_1 = \infty\Big\}$$

for the set of infinite nearest neighbor paths tending to infinity and

$$W = \Big\{w : \mathbb{Z} \to \mathbb{Z}^d : \; |w(n) - w(n+1)|_1 = 1 \; \forall \, n \in \mathbb{Z}, \text{ and } \lim_{n \to \pm\infty} |w(n)|_1 = \infty\Big\}$$

for the set of doubly infinite nearest neighbor paths tending to infinity at positive and negative infinite times. W_+ is endowed with the σ-algebra $\mathcal{W}_+$ generated by the canonical coordinate maps X_n, $n \in \mathbb{N}_0$. Similarly, we will write $\mathcal{W}$ and X_n, $n \in \mathbb{Z}$, for the canonical σ-algebra and the canonical coordinate process on W. We denote by W^* the space of equivalence classes of trajectories in W modulo time-shifts, i.e.,

$$W^* = W/\sim, \text{ where } w \sim w' \text{ iff } w(\cdot) = w'(\cdot + k) \text{ for some } k \in \mathbb{Z}.$$

We let $\pi^* : W \to W^*$ be the canonical projection and endow W^* with the σ-algebra induced by π^* via

$$\mathcal{W}^* = \big\{A \subset W^* \colon (\pi^*)^{-1}(A) \in \mathcal{W}\big\}.$$

We furthermore introduce for $K \subset\subset \mathbb{Z}^d$ the subsets

$$\begin{aligned} W_K &= \big\{w \in W \colon \text{there is } k \in \mathbb{Z} \text{ such that } w(k) \in K\big\}, \\ W_K^* &= \pi^*(W_K) \end{aligned}$$

of W and W^*, respectively. Note that $W_K \in \mathcal{W}$ and $W_K^* \in \mathcal{W}^*$. For $A, B \in \mathcal{W}_+$, $K \subset\subset \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$ we define a finite measure Q_K on W via

$$Q_K\big[(X_{-n})_{n\geq 0} \in A, X_0 = x, (X_n)_{n\geq 0} \in B\big] = P_x[A \mid \widetilde{H}_K = \infty]\, \mathrm{e}_K(x) P_x[B].$$

According to Theorem 1.1 in [S10] there exists a unique σ-finite measure ν on $(W^*, \mathcal{W}^*)$ such that for all $K \subset\subset \mathbb{Z}^d$ and $E \in \mathcal{W}^*$ with $E \subseteq W_K^*$, the equation

$$\nu[E] = Q_K\big[(\pi^*)^{-1}(E)\big]$$

is fulfilled. We will also need the space

$$\Omega = \Big\{\omega = \sum_{i\geq 0} \delta_{(w_i^*, u_i)} \text{ with } (w_i^*, u_i) \in W^* \times [0, \infty), \text{ for } i \geq 0, \\ \text{and } \omega[W_K^* \times [0, u]] < \infty \text{ for any } K \subset\subset \mathbb{Z}^d \text{ and } u \geq 0\Big\}$$

of locally finite point measures on $W^* \times [0, \infty)$. Let $\mathcal{B}([0, \infty))$ be the Borel σ-algebra on $[0, \infty)$ and let $\mathcal{A}$ be the σ-algebra on Ω which is generated by the family of evaluation maps $\omega \mapsto \omega[D]$, $D \in \mathcal{W}^* \otimes \mathcal{B}([0, \infty))$. We denote by $\mathbb{P}$ the law of the Poisson point process on $(\Omega, \mathcal{A})$ with intensity measure $\nu \otimes \mathrm{d}u$. This process is usually referred to as the interlacement Poisson point process. Random interlacements at level u is then defined as the subset of $\mathbb{Z}^d$ given by

$$\mathcal{I}^u(\omega) = \bigcup_{u_i \leq u} \text{range}(w_i^*), \qquad \text{where } \omega = \sum_{i\geq 0} \delta_{(w_i^*, u_i)} \in \Omega,$$

and $\text{range}(w^*) = \big\{w(n) : n \in \mathbb{Z}\big\}$ for arbitrary $w \in \pi^{-1}(\{w^*\})$. The vacant set at level $u \geq 0$ is defined by

$$\mathcal{V}^u(\omega) = \mathbb{Z}^d \setminus \mathcal{I}^u(\omega), \quad \omega \in \Omega.$$

As has been shown in [S10] and [SS09], in any dimension $d \geq 3$ there exists a $u_*(d) \in (0, \infty)$ such that for $u \in [0, u_*(d))$ the vacant set $\mathcal{V}^u$ contains an infinite connected component, whereas for $u \in (u_*(d), \infty)$ it consists of finite connected components.

6.2.3 Cascading events and a decoupling inequality

In this section we give a slightly refined version of a decoupling inequality of [S12, Theorem 3.4]. This is a fundamental tool to deal with the dependence structure inherent to the model. Since the constants appearing in the decoupling inequality depend implicitly on the dimension d, we have to pay special attention to their behavior for large d. Proposition 6.2.3 below states all these dependencies explicitly. We write Ψ_x, $x \in \mathbb{Z}^d$, for the canonical coordinates on $\{0,1\}^{\mathbb{Z}^d}$. Let us recall Definition 3.1 of [S12] of so-called cascading events.

Definition 6.2.1 (Cascading events). *Let $\lambda > 0$. A family $\mathcal{G} = (G_{x,L})_{x \in \mathbb{Z}^d, L \geq 1 \text{ integer}}$ of events in $\{0,1\}^{\mathbb{Z}^d}$ cascades with complexity at most λ, if*

$$G_{x,L} \text{ is } \sigma\left(\Psi_{x'}, x' \in B_2(x, 10\sqrt{d}L)\right) - \text{measurable for each } x \in \mathbb{Z}^d,\ L \geq 1,$$

and for each multiple l of 100, $x \in \mathbb{Z}^d$, $L \geq 1$, there exists $\Lambda \subseteq \mathbb{Z}^d$ and a constant $C_1 = C_1(\mathcal{G}, \lambda)$ such that

$$\begin{aligned}
&\Lambda \subseteq B_2(x, 9\sqrt{d}lL),\\
&|\Lambda| \leq C_1 l^{\lambda},\\
&G_{x,lL} \subseteq \bigcup_{x',x'' \in \Lambda : |x'-x''|_2 \geq \frac{l}{100}\sqrt{d}L} G_{x',L} \cap G_{x'',L}.
\end{aligned}$$

Remark 6.2.2. *Note that the cascading events are defined with respect to the ℓ^2-norm instead of the more common ℓ^∞-norm. Since we are working in a high dimensional setting, this makes the constants appearing in Proposition 6.2.3 easier to control. This again is due to the fact that, see (1.22) and (1.23) of [SS11b], there are constants $c_2, c_3 > 0$, which do not depend on d, such that for all $L \geq d$,*

$$\left(\frac{c_2 L}{\sqrt{d}}\right)^{d-2} \leq \mathrm{cap}(B_2(0,L)) \leq \left(\frac{c_3 L}{\sqrt{d}}\right)^{d-2}. \tag{6.2.5}$$

The notions introduced below pertain to the so-called sprinkling technique. The idea is that with high probability the mutual dependencies of the events under consideration can be dominated by considering random interlacements at two different levels $u_\infty^- < u$. For this purpose we introduce for l_0 positive the quantity

$$f(l_0) = \prod_{k \geq 0}\left(1 + 32e^2 c_1^d \frac{1}{(k+1)^{\frac{3}{2}}} l_0^{-(d-3)/2}\right). \tag{6.2.6}$$

The constant c_1 will be chosen according to the formulation of Proposition 6.2.3 below. Furthermore, we define for $u > 0$, $u_\infty^- = u_\infty^-(u) = \frac{u}{f(l_0)}$, as well as for $L_0 \geq 1$,

$$\varepsilon(u) = \frac{2e^{-uL_0^{d-2}l_0}}{1 - e^{-uL_0^{d-2}l_0}}. \tag{6.2.7}$$

We let $L_0 \geq 1$ and define the scales $L_n = l_0^n L_0$, $n \in \mathbb{N}_0$. L_n and l_0 will play from now on the role of L and l in Definition 6.2.1. Finally, for a subset $A \subset \{0,1\}^{\mathbb{Z}^d}$ and $u \geq 0$ we write

$$A^u := \Big\{\omega \in \Omega : \ \mathbb{1}_{\mathcal{I}^u(\omega)} \in A\Big\}.$$

Also, $A \subset \{0,1\}^{\mathbb{Z}^d}$ is called increasing if the following holds: For all $\xi \in A$ and $\xi' \in \{0,1\}^{\mathbb{Z}^d}$ such that $\xi_x \leq \xi'_x$ holds for all $x \in \mathbb{Z}^d$, one has that $\xi' \in A$ also.

A refinement of the arguments in [S12], proof of Theorem 2.6 (with a special emphasis on the dependence of the constants on the dimension), leads to the following result.

Proposition 6.2.3 (Decoupling inequality)**.** *Let $\lambda > 0$. Consider $\mathcal{G} = (G_{x,L})_{x\in\mathbb{Z}^d, L\geq 1 \text{ integer}}$ a collection of increasing events on $\{0,1\}^{\mathbb{Z}^d}$ that cascades with complexity at most λ. Then there are $c_0, c_1 > 1$ (the latter one comes into play in (6.2.6)) such that for all $l_0 \geq 10^6\sqrt{d}c_0$, all $L_0 \geq \sqrt{d}$ and all $n \in \mathbb{N}_0$, one has*

$$\sup_{x\in\mathbb{Z}^d} \mathbb{P}\Big[G^{u_\infty^-}_{x,L_n}\Big] \leq \Big(C_1 l_0^{2\lambda}\Big)^{2^n} \left(\sup_{x\in\mathbb{Z}^d} \mathbb{P}\Big[G^{u_0}_{x,L_0}\Big] + \varepsilon(u_\infty^-)\right)^{2^n}. \tag{6.2.8}$$

See Appendix 6.6 for the proof.

6.3 Proof of the main result: transience of the vacant set

In this section we introduce a classification of vertices in $\mathbb{Z}^3$ into "good" (exhibiting good connectivity properties, see Definition 6.3.1 below) and "bad" vertices. Subsequently, we give two auxiliary results on the existence of an infinite connected component of good vertices (Proposition 6.3.3) which is transient as a subset of $\mathbb{Z}^3$ (Proposition 6.3.5). From the latter result we deduce Theorem 6.1.1.

6.3.1 Auxiliary results: Classification into good and bad vertices

Let $\mathcal{C}_y = 2y + \{0,1\}^d$, $y \in \mathbb{Z}^d$, and $\mathcal{C} = \mathcal{C}_0$.

Definition 6.3.1. *Let $u \geq 0$. A vertex $y \in \mathbb{Z}^3$ is defined to be u-good (with respect to $\omega \in \Omega$) if*

$$\omega \in \mathcal{G}_{y,u} := \Bigg\{\omega \in \Omega : \ \forall z \in \mathbb{Z}^3 \text{ with } |y-z|_1 \leq 1, \ \text{ the set } \mathcal{V}^u(\omega) \cap \mathcal{C}_z \text{ contains a connected component } \mathfrak{C}_{y,z} \text{ with } |\overline{\mathfrak{C}_{y,z}} \cap \mathcal{C}_z| \geq (1-d^{-2})|\mathcal{C}_z|, \text{ and these components are connected in } \mathcal{V}^u(\omega) \cap \Bigg(\bigcup_{z\in\mathbb{Z}^3 : |y-z|_1\leq 1} \mathcal{C}_z\Bigg)\Bigg\}. \tag{6.3.1}$$

Otherwise, y is called u-bad (with respect to $\omega \in \Omega$).

Remark 6.3.2. *Lemma 2.1 in [S11a] states that there is a $d_0 \in \mathbb{N}$ such that if $d \geq d_0$, then any subset $V \subset \mathcal{C}$ contains at most one connected component $\mathfrak{C}$ of V such that $|\overline{\mathfrak{C}} \cap \mathcal{C}| \geq (1 - d^{-2})|\mathcal{C}|$. Thus, for $d \geq d_0$, if a connected component $\mathfrak{C}_{y,z}$ as in* (6.3.1) *exists, then it is necessarily unique.*

Denote by

$$\mathcal{G}^u(\omega) := \{y \in \mathbb{Z}^3 : \omega \in \mathcal{G}_{y,u}\} \quad \text{and} \quad \mathcal{B}^u(\omega) := \mathbb{Z}^3 \backslash \mathcal{G}^u(\omega) \tag{6.3.2}$$

the set of u-good and u-bad vertices given ω. We can now state the auxiliary results alluded to above.

Proposition 6.3.3 (Existence of an infinite connected component of good vertices)**.** *Fix $\varepsilon \in (0, 1)$. There is $d_0 = d_0(\varepsilon) \in \mathbb{N}$ such that for all $d \geq d_0$ and $u \leq (1 - \varepsilon)u_*(d)$, $\mathbb{P}$-a.s. there exists an infinite connected component in $\mathcal{G}^u$.*

Remark 6.3.4. *Using Proposition 6.4.1 below, it is not hard to establish the uniqueness of this infinite connected component. However, since we do not need this uniqueness, we will not give a proof of this fact.*

In the forthcoming proposition all parameters are chosen according to Proposition 6.3.3 above. From now on, $\mathcal{G}^u_\infty$ will denote an arbitrary infinite connected component of $\mathcal{G}^u$.

Proposition 6.3.5 (Transience of $\mathcal{G}^u_\infty$)**.** *Fix $\varepsilon \in (0, 1)$. There is $d_0 = d_0(\varepsilon) \in \mathbb{N}$ such that for all $d \geq d_0$ and $u \leq (1 - \varepsilon)u_*(d)$, one has that $\mathcal{G}^u_\infty$ is transient $\mathbb{P}$-a.s.*

The above two results will be proven in Sections 6.4 and 6.5.

6.3.2 Proof of the main result given the two auxiliary propositions

In this section we show how Theorem 6.1.1 can be deduced from Propositions 6.3.3 and 6.3.5. For a connected subset G of $\mathbb{Z}^d$, let $\Pi_y(G)$, $y \in G$, be the set of infinite non-intersection (which we will also call simple for the sake of brevity) nearest neighbor paths on G starting in y. We recall the following characterization of the transience of G.

Lemma 6.3.6. *The following are equivalent:*

(a) The graph G (with connectivity structure induced by $\mathbb{Z}^d$) is transient.

(b) There is $y \in G$ such that there is a probability measure μ on $\Pi_y(G)$ fulfilling

$$\sum_{x \in V} \mu^2\big[\pi \in \Pi_y(G) : x \in \pi\big] < \infty. \tag{6.3.3}$$

Remark 6.3.7. *A version similar to Lemma 6.3.6 may be found in [P97, Theorem 10.1]. Note that (b) is equivalent to the fact that there is $y \in G$ and a probability measure μ on $\Pi_y(G)$ such that if two path are chosen independently according to μ from $\Pi_y(G)$, then the expected number of intersections of these two paths is finite. Thus, Lemma 6.3.6* *states that transient graphs must be large enough to find such two paths $\Pi_y(G)$. We refer the reader also to [LP01, Chapter 2] where further transience and recurrence criteria may be found. Note that in [P97, Theorem 10.1] the sum in (6.3.3) is taken over nearest neighbor edges whose ends both lie in G, rather than over the vertices. However, using the fact that $\mathbb{Z}^d$ has uniformly bounded degree, one can deduce Lemma 6.3.6 from the corresponding edge-based version without problems. We will omit the proof of this fact.*

The strategy to prove Theorem 6.1.1 is as follows: Since by Propositions 6.3.3 and 6.3.5 the subset $\mathcal{G}_\infty^u$ of $\mathbb{Z}^3$ is transient for d and u as in the assumptions, Lemma 6.3.6 provides us with a measure μ on the simple nearest neighbor paths in $\mathcal{G}_\infty^u$ fulfilling (6.3.3) for $G = \mathcal{G}_\infty^u$. We then map simple nearest neighbor paths in $\mathcal{G}_\infty^u$ to simple nearest neighbor paths in $\mathcal{V}_\infty^u$, in a way that does not blow up the lengths of the paths too much, cf. (6.3.5) below. The pushforward of μ under this mapping then supplies us with a probability measure supported on infinite simple nearest neighbor paths in $\mathcal{V}_\infty^u$ that still satisfies condition (6.3.3) (cf. Claim 6.3.8). We now make this strategy precise.

Proof of Theorem 6.1.1. We fix $\varepsilon \in (0,1)$ and choose $d_0 = d_0(\varepsilon)$ such that the implications of both Propositions 6.3.3 and 6.3.5, hold true. Write

$$\Gamma\colon\ \mathcal{G}_\infty^u \to \bigcup_{y \in \mathcal{G}_\infty^u} \mathfrak{C}_{y,y}$$

for the mapping that sends $y \in \mathcal{G}_\infty^u$ to the element $z \in \mathfrak{C}_{y,y}$ that has minimal lexicographical order among all elements of $\mathfrak{C}_{y,y}$. Moreover, by the definition of $\mathcal{G}_\infty^u$, for each

$$x, y \in \mathcal{G}_\infty^u \text{ with } |x-y|_1 = 1 \tag{6.3.4}$$

there is a simple nearest neighbor path $(\widetilde{\pi}_y^x(k))_{k=0}^{n-1}$ on

$$\mathcal{V}_\infty^u \cap \{\mathcal{C}_z :\ z \in \mathbb{Z}^3 \text{ and } |y-z|_1 \le 1\}$$

such that

$$\begin{aligned} &\bullet\ \widetilde{\pi}_y^x(0) = \Gamma(y) \text{ and } \widetilde{\pi}_y^x(n-1) = \Gamma(x); \\ &\bullet\ n \le \Big| \bigcup_{z \in \mathbb{Z}^3\,:\,|y-z|_1 \le 1} \mathfrak{C}_{y,z} \Big| \le 7 \times 2^d. \end{aligned} \tag{6.3.5}$$

For any pair of points x and y as in (6.3.4) we choose and fix a path $\widetilde{\pi}_y^x$ with the above properties. Given an infinite simple nearest neighbor path π on $\mathcal{G}_\infty^u$, we obtain an infinite nearest neighbor path $\widetilde{\pi}$ on $\mathcal{V}_\infty^u$ starting in $\Gamma(\pi(0))$ by concatenating the paths $\widetilde{\pi}_{\pi(k)}^{\pi(k+1)}$, $k = 0, 1, 2, \ldots$. Finally, we denote by φ the map that sends π to the loop-erasure of $\widetilde{\pi}$ (note that the latter is an infinite simple nearest neighbor path in $\mathcal{V}_\infty^u$).

Now due to Proposition 6.3.5 and Lemma 6.3.6 there exists a probability measure μ on $\Pi_y(\mathcal{G}_\infty^u)$, for some $y \in \mathcal{G}_\infty^u$, fulfilling (6.3.3). Hence, Theorem 6.1.1 is a consequence of the claim below and Lemma 6.3.6. ∎

Claim 6.3.8. *If a measure μ on $\Pi_y(\mathcal{G}^u_\infty)$ fulfills (6.3.3), then so does the measure $\widetilde{\mu} := \mu \circ \varphi^{-1}$ on $\Pi_{\Gamma(y)}(\mathcal{V}^u_\infty)$.*

Proof. In a slight abuse of notation, for $x \in \mathcal{V}^u_\infty \subset \mathbb{Z}^d$ define $\Gamma^{-1}(x)$ to be the $z \in \mathbb{Z}^3$ (unique, if it exists) such that $x \in \mathcal{C}_z$. If no such z exists, then let $\Gamma^{-1}(x) = \infty$ and define $|\Gamma^{-1}(x) - w|_1 = \infty$ for all $w \in \mathbb{Z}^3$ in this case. We will see that the latter case is of no importance, since the construction of φ is such that it restricts all paths on $\mathcal{V}^u_\infty$ to such x for which $\Gamma^{-1}(x) \in \mathbb{Z}^3$. Then for $x \in \mathcal{V}^u_\infty$, one has

$$\mu \circ \varphi^{-1}[\pi \in \Pi_{\Gamma(y)}(\mathcal{V}^u_\infty) \,:\, x \in \pi] \leq \sum_{z \in \mathbb{Z}^3 \,:\, |\Gamma^{-1}(x) - z|_1 \leq 1} \mu[\pi \in \Pi_y(\mathcal{G}^u_\infty) \,:\, z \in \pi].$$

Thus, an application of the Cauchy-Schwary inequality yields that

$$\begin{aligned} &\sum_{x \in \mathbb{Z}^d} \mu \circ \varphi^{-1}[\pi \in \Pi_{\Gamma(y)}(\mathcal{V}^u_\infty) \,:\, x \in \pi]^2 \\ &\leq \sum_{x \in \mathbb{Z}^d} \Bigg[\Bigg(\sum_{z \in \mathbb{Z}^3 \,:\, |\Gamma^{-1}(x) - z|_1 \leq 1} \mu[\pi \in \Pi_y(\mathcal{G}^u_\infty) \,:\, z \in \pi]^2 \Bigg)^{1/2} \\ &\qquad\qquad \times |z \in \mathbb{Z}^3 \,:\, |\Gamma^{-1}(x) - z|_1 \leq 1|^{1/2} \Bigg]^2 . \end{aligned} \tag{6.3.6}$$

Note that for any $x \in \mathbb{Z}^d$, by the definition of Γ^{-1}, one has $|\{z \in \mathbb{Z}^3 \,:\, |\Gamma^{-1}(x) - z|_1 \leq 1\}| = 7$. Hence, the right hand-side of (6.3.6) is bounded by

$$7 \sum_{x \in \mathbb{Z}^d} \sum_{z \in \mathbb{Z}^3 \,:\, |\Gamma^{-1}(x) - z|_1 \leq 1} \mu[\pi \in \Pi_y(\mathcal{G}^u_\infty) \,:\, z \in \pi]^2. \tag{6.3.7}$$

By (6.3.5) (or the facts that $|\mathcal{C}_y| = 2^d$ and again using that for $x \in \mathbb{Z}^d$ one has $|\{z \in \mathbb{Z}^3 \,:\, |\Gamma^{-1}(x) - z|_1 \leq 1\}| = 7$), we finally may bound (6.3.7) from above by

$$2^d \cdot 7^2 \sum_{z \in \mathbb{Z}^3} \mu^2[\pi \in \Pi_y(\mathcal{G}^u_\infty) \,:\, z \in \pi] < \infty, \tag{6.3.8}$$

where the finiteness follows from the assumptions. This concludes the proof. ∎

6.4 Proof of Proposition 6.3.3: existence of an infinite connected component of good vertices

In the proof of this proposition we exploit the fact that as $d \to \infty$, certain averaging effects occur which (in combination with so-called "sprinkling") imply that with high probability and for slightly supercritical intensities u, such hypercubes are u-good in the sense of Definition 6.3.1 (a big chunk of this work is done by Theorem 4.2 in [S11a] and in Lemma 6.4.2 we neatly adapt this result to our purposes). By identifying hypercubes with vertices this will lead to a dependent percolation problem on $\mathbb{Z}^3$. This is where we will take advantage of the decoupling inequality (6.2.8) in order to deduce that $*$-connected components of u-bad vertices are sufficiently small, and hence an infinite connected component of u-good vertices exists.

6.4.1 Proof of Proposition 6.3.3 given an auxiliary result

The result below provides an estimate on the size of $*$-connected components of u-bad vertices. Its proof is postponed to Section 6.4.2.

Proposition 6.4.1 ($*$-connected components of u-bad vertices are small)**.** *Fix $\varepsilon \in (0,1)$. There is $d_0 = d_0(\varepsilon) \in \mathbb{N}$ such that for all $d \geq d_0$, there are $C_2, C_3 > 0$ such that for all $u \leq (1-\varepsilon)u_*(d)$ and $N \in \mathbb{N}$*

$$\sup_{x\in\mathbb{Z}^3} \mathbb{P}\big[x \textit{ is contained in a simple } *\textit{-path of } u\textit{-bad vertices of length at least } N\big] \\ \leq C_2 e^{-N^{C_3}}.$$

Before we proceed, recall the notion of exterior boundary below (6.2.1). We now prove Proposition 6.3.3.

Proof of Proposition 6.3.3. For $x \in \mathbb{Z}^3$ define

$$\mathcal{G}_x = \begin{cases} \text{the connected component of } u\text{-good vertices containing } x, & \text{if } x \text{ is } u\text{-good,} \\ \emptyset, & \text{otherwise.} \end{cases}$$

Now assume that there is $N \in \mathbb{N}$ such that $\mathcal{G}_x$ has finite cardinality for all $x \in \mathbb{Z}^3$ with $|x|_\infty \leq N$. We claim that

$$\begin{gathered}\text{in this case there is a } y \in \mathbb{Z}^3 \text{ with } |y|_\infty \geq N, \text{ such that } y \text{ is} \\ \text{connected to } B^3_\infty(y, |y|_\infty)^c \text{ by a } *\text{-path of } u\text{-bad vertices.}\end{gathered} \tag{6.4.1}$$

Let us for a moment assume that the claim is correct. Then by Proposition 6.4.1 and using a union bound in combination with the fact that $\partial |B^3_\infty(y,k)| \leq 6(2k+1)^2$, one has

$$\mathbb{P}\Big[\mathcal{G}_x \text{ is finite for all } |x|_\infty \leq N\Big] \leq C_2' \sum_{k=N}^{\infty} k^2 e^{-k^{C_3}}, \tag{6.4.2}$$

which is smaller than one if N is large enough. Consequently there is, with positive $\mathbb{P}$-probability, an infinite connected component in $\mathcal{G}^u$. Since the existence of an infinite connected component in $\mathcal{G}^u$ is an event that is invariant under shifts in $\mathbb{Z}^3$, and since $\mathbb{P}$ is ergodic with respect to these shifts (see [S10, Theorem 2.1]), we obtain that, $\mathbb{P}$-a.s. there is an infinite connected component in $\mathcal{G}^u$.

We now prove (6.4.1). If $\partial_{\text{ext}}\mathcal{G}_x = \emptyset$ for all $x \in B_\infty^3(0,N)$, then due to the finiteness assumption on the $\mathcal{G}_x$ we get $\mathcal{G}_x = \emptyset$ for all such x, and hence all such x are u-bad, which would yield the claim. Therefore, assume otherwise, and let $y' \in \partial_{\text{ext}}\mathcal{G}_x$, with $|x|_\infty \le N$, be such that it has maximal first coordinate among all such vertices y' fulfilling $y'_2, y'_3 \in [-N,N]$ (where y'_i, $i \in \{1,2,3\}$, denotes the i-th coordinate of y'). Choose one (of the possibly several) x such that $y' \in \partial_{\text{ext}}\mathcal{G}_x$ and denote it by $x^{(0)}$. We aim to find a u-bad $z \in \mathbb{Z}^3$, such that $|z-y'|_\infty > |y'|_\infty$ and such that there is a $*$-path of u-bad vertices which connects y' to z. For this purpose, we distinguish two cases:

(i) If $|y'|_\infty \le N$, then observe that as a consequence of the definition of y', all vertices in $\partial_{\text{int}} B_\infty^3(0,N) \cap (\{N\} \times \mathbb{Z}^2)$ are u-bad. Hence, one can immediately choose $y, z \in \partial_{\text{int}} B_\infty^3(0,N) \cap (\{N\} \times \mathbb{Z}^2)$ fulfilling the required properties.

(ii) Assume now that $|y'|_\infty > N$. Then we have $|y'|_\infty = y'_1 > N$, and we set $y^{(0)} := y := y'$ and $x^{(0)} := x$. Define a nearest neighbor path via $\Phi(n) = (y_1^{(0)} - n, y_2^{(0)}, y_3^{(0)})$ for $n \ge 0$. In addition let $z^{(0)} \in \partial\mathcal{G}_{x^{(0)}}$ be such that there is no vertex in range$(\Phi) \cap \partial\mathcal{G}_{x^{(0)}}$ which has a smaller first coordinate than $z^{(0)}$, and define m_0 via $\Phi(m_0) := z^{(0)}$. In particular, by definition we have $z^{(0)} \in \partial_{\text{ext}}\mathcal{G}_{x^{(0)}}$. In addition, let

$$n_0 = \max\{n \ge m_0 \,:\, \Phi(m) \text{ is } u\text{-bad for all } m_0 \le m \le n\} \wedge 2y_1^{(0)},$$

and set $y^{(1)} = \Phi(n_0)$. By Timár [T13, Lemma 2], the set $\partial_{\text{ext}}\mathcal{G}_{x^{(0)}}$ is $*$-connected. Now if $y_1^{(1)} < 0$, then this $*$-connectivity of $\partial_{\text{ext}}\mathcal{G}_{x^{(0)}}$ is enough to deduce the claim. In fact, in this case we may connect $y^{(0)}$ to $y^{(1)}$ via a $*$-path of u-bad vertices of length more than $|y^{(0)}|$, by first connecting $y^{(0)}$ to $z^{(0)}$ via a $*$-path contained in $\partial_{\text{ext}}\mathcal{G}_{x^{(0)}}$ and by then connecting $z^{(0)}$ to $y^{(1)}$ along Φ; this would finish the proof. If, on the other hand, $y_1^{(1)} \ge 0$, then observe that $y^{(1)} \in \partial_{\text{ext}}\mathcal{G}_{\Phi(n_0+1)}$ (to see this, use that $y^{(1)} \in \partial\mathcal{G}_{\Phi(n_0+1)}$ and that it is connected to $y^{(0)}$ along a $*$-path of u-bad vertices, and that $y^{(0)}$ has maximal first coordinate among all elements $z \in \partial_{\text{ext}}\mathcal{G}_x$, for some $|x|_\infty \le N$, and such that $z_2, z_3 \in [-N,N]$). We can now repeat the procedure started in *(ii)* with $y^{(1)}$ taking the role of $y^{(0)}$ in order to obtain a $y^{(2)}$ taking the role of the previous $y^{(1)}$, and so on. I.e., we construct a sequence $y^{(0)}, y^{(1)}, \ldots, y^{(n)}$ (up to the smallest $n \in \mathbb{N}$ such that $y_1^{(n)} < 0$) such that $y^{(k)}$ may be connected to $y^{(k+1)}$ by a $*$-path of u-bad vertices for all $k \in \{0,1,\ldots,n-1\}$. In particular, since $y_1^{(k)} \le y_1^{(k-1)} - 2$ for all $k \le n$, after at most $|y'|_\infty/2 + 1$ iterations (and taking the loop-erasure of the path connecting $y^{(0)}$ to $z := y^{(n)}$) we will have found the desired $z \in \mathbb{Z}^3$, which finally yields the claim.

∎

6.4.2 Proof of the auxiliary result: bad components are small

The proof will be divided into several lemmas. For this purpose fix $\varepsilon \in (0,1)$ and define

$$\widetilde{u}_0 = (1-\varepsilon)u_*(d). \tag{6.4.3}$$

The following estimate will serve as a seed estimate for the decoupling inquality of Proposition 6.2.3 and as such be employed in Lemma 6.4.4.

Lemma 6.4.2. *There is $d_0 \in \mathbb{N}$ such that for all $y \in \mathbb{Z}^3$,*

$$\mathbb{P}\Big[\mathcal{G}^c_{y,\widetilde{u}_0}\Big] \le d^{-7}/5 \qquad \text{for all } d \ge d_0,\ d \in \mathbb{N},$$

where $\mathcal{G}_{y,\widetilde{u}_0}$ was defined in (6.3.1).

Proof. We will derive the result using Theorem 4.2 of [S11a]. For this purpose identify $\mathbb{Z}^2$ with $\mathbb{Z}^2 \times \{0\}^{d-2}$ and set

$$\widetilde{\mathbb{Z}}^2 = \big\{x \in \mathbb{Z}^d :\ x_i = 0 \text{ for all } i \notin \{2,3\}\big\}.$$

Furthermore, define $\mathcal{G}^2_{y,u}$ and $\widetilde{\mathcal{G}}^2_{y,u}$, respectively, by $\mathcal{G}_{y,u}$ as in (6.3.1), but with $\mathbb{Z}^3$ replaced by $\mathbb{Z}^2$ and $\widetilde{\mathbb{Z}}^2$, respectively. By Remark 6.3.2, there is $d_0 \in \mathbb{N}$ such that for $d \ge d_0$, the hypercube $\mathcal{C}$ contains at most one connected component $\mathfrak{C}$ with $|\overline{\mathfrak{C}} \cap \mathcal{C}| \ge (1 - d^{-2})|\mathcal{C}|$. As a consequence we deduce

$$\mathcal{G}^2_{0,\widetilde{u}_0} \cap \widetilde{\mathcal{G}}^2_{0,\widetilde{u}_0} \subseteq \mathcal{G}_{0,\widetilde{u}_0}. \tag{6.4.4}$$

Finally, it remains to apply Theorem 4.2 in [S11a]. Note that the intensity parameter in that result equals $(1-\varepsilon)g(0)\log d$, where $g(0)$ was the Green function at the origin; however since the main result of [SS11b] supplies us with $u_*(d) \le (1+\varepsilon)\log d$ for $\varepsilon > 0$ arbitrary d large enough, and since $g(0) \to 1$ as $d \to \infty$ (see e.g. Lemma 1.2 in [S11a]), we can apply it with intensity $\widetilde{u}_0$ also, if d large enough. Hence, we infer that for $d \ge d_0$

$$\mathbb{P}\Big[\mathcal{G}^2_{0,\widetilde{u}_0}\Big] \ge 1 - d^{-7}/10.$$

As the same is true for $\widetilde{\mathcal{G}}^2_{0,\widetilde{u}_0}$, in combination with (6.4.4) we obtain the claim for $y = 0$. Since $\mathbb{P}$ is invariant under shifts in space, we obtain the result for every $y \in \mathbb{Z}^3$. ∎

We define for $x \in \mathbb{Z}^3$ and $L \ge 1$, L integer,

$$A_{x,L} = \Big\{\Psi \in \{0,1\}^{\mathbb{Z}^d} :\ B^3_\infty(x,L) \text{ is connected to } \partial_{\text{int}} B^3_\infty(x,2L) \text{ by a } *\text{-path on } \mathbb{Z}^3 \text{ along which } \Psi \text{ equals one}\Big\}.$$

If $x \notin \mathbb{Z}^3$, then $A_{x,L} = \emptyset$. We will denote "bad" crossing events by

$$\begin{aligned} &B^u_{x,L} \\ &= \Big\{\omega \in \Omega :\ \mathbb{1}_{\mathcal{B}^u(\omega)} \in A_{x,L}\Big\} \\ &= \Big\{\omega \in \Omega :\ B^3_\infty(x,L) \text{ is connected to } B^3_\infty(x,2L) \text{ by a } *\text{-path on } \mathbb{Z}^3 \text{ of } u\text{-bad vertices}\Big\}, \end{aligned}$$

where we recall that $\mathcal{B}^u$ had been defined in (6.3.2). Also, recall Definition 6.2.1 of cascading events.

Lemma 6.4.3. *$\mathcal{A} = (A_{x,L})_{x\in\mathbb{Z}^d, L\geq 1 \text{ integer}}$ is a family of increasing events which cascades with complexity at most* 3*. Moreover $C_1 = C_1(\mathcal{A}, 3)$ as introduced in Definition 6.2.1 does not depend on d.*

Proof. The proof is similar to the proof of (3.10) in [S12], except that one additionally has to make use of the fact that $|\cdot|_2 \leq \sqrt{d}|\cdot|_\infty$. We omit the details.

∎

The family of events $(B^u_{x,L})_{x\in\mathbb{Z}^d, L\geq 1 \text{ integer}}$ is shift invariant in the following sense: Let

$$\omega = \sum_{i\geq 0} \delta_{(w_i^*, u_i)} \in \Omega,$$

and define

$$\tau_x : \Omega \mapsto \Omega, \quad \omega \mapsto \sum_{i\geq 0} \delta_{(w_i^* + x, u_i)},$$

where $w^* + x = \pi(w(\cdot) + x)$, any $w \in \pi^{-1}(w^*)$. Then for all $x, y \in \mathbb{Z}^3$ one has

$$\omega \in B_{x,L} \quad \text{if and only if} \quad \tau_y(\omega) \in B_{x+y,L}. \tag{6.4.5}$$

We are now in the position to apply the decoupling inequality (6.2.8). For this purpose l_0 and L_0 are such that they satisfy the relations

$$l_0 \geq 10^6\sqrt{d}c_0 \qquad \text{and} \qquad L_0 = \lceil\sqrt{d}\rceil. \tag{6.4.6}$$

We further recall the definition of u_∞^-, see the lines following (6.2.6), as well as the definition of $\widetilde{u}_0$, in (6.4.3).

Lemma 6.4.4. *There is $d_0 \in \mathbb{N}$ such that for all $d \geq d_0$, $d \in \mathbb{N}$, there is l_0 satisfying* (6.4.6) *such that for all $u \leq u_\infty^- = u_\infty^-(\widetilde{u}_0)$, one has*

$$\mathbb{P}\big[B^u_{0,L_n}\big] \leq e^{-2^n}. \tag{6.4.7}$$

Proof. By Proposition 6.2.3, Lemmas 6.4.2–6.4.3, (6.4.5) and the fact that $\mathbb{P}$ is invariant under shifts in $\mathbb{Z}^3$, we get

$$\mathbb{P}\Big[B^{u_\infty^-}_{0,L_n}\Big] \leq \Big(C_1 l_0^6\Big)^{2^n}\Big(\mathbb{P}\big[B^{\widetilde{u}_0}_{0,L_0}\big] + \varepsilon(u_\infty^-)\Big)^{2^n}. \tag{6.4.8}$$

To estimate the probability on the right-hand side of (6.4.8), note that

$$\mathbb{P}\big[B^{\widetilde{u}_0}_{0,L_0}\big] \leq \mathbb{P}\big[\text{there is } x \in \partial^*_{\text{int}} B^3_\infty(0, L_0) \text{ which is } \widetilde{u}_0\text{-bad}\big] \leq cd\mathbb{P}\Big[\mathcal{G}^c_{y,\widetilde{u}_0}\Big] \leq cd^{-6},$$

where we used a union bound in combination with the fact that there is a constant $c > 0$ such that the cardinality of $\partial^*_{\text{int}} B^3_\infty(0, L_0)$ is bounded by cd to get the second inequality. The last inequality is a consequence of Lemma 6.4.2.

Hence, in order to prove the desired decay of the right-hand side, it is enough to determine l_0 such that

$$cC_1 l_0^6 d^{-6} \leq \frac{1}{2e} \qquad \text{and} \qquad C_1 l_0^6 \varepsilon(u_\infty^-) \leq \frac{1}{2e}. \tag{6.4.9}$$

The first inequality in (6.4.9) is indeed satisfied for all d large enough, subject to the choice of l_0 in (6.4.6). To show the second inequality in (6.4.9), observe that

$$\lim_{d\to\infty} \frac{c_1^d}{l_0^{(d-3)/2}} = 0, \qquad \text{for } l_0 \geq 10^6\sqrt{d}c_0.$$

Employing this equality in the definition of u_∞^- in (6.2.6), we obtain that $u_\infty^- \geq (1-2\varepsilon)u_*(d)$, if d large enough. Using this inequality and the fact that by the main result of [S11a] one has that for d large enough $u_*(d) \geq (1-\varepsilon)\log d$, the definition of $\varepsilon(u_\infty^-)$ leads to the desired estimate. This shows that (6.4.7) is true for $u = u_\infty^-$. The claim for every other $u \leq u_\infty^-$ follows by the fact that $B_{x,L}^u$ is increasing in u. ∎

As a direct consequence of this result we obtain the following corollary.

Corollary 6.4.5. *If* (6.4.7) *holds true, then for some $C = C(d) < \infty$, all $u \leq u_\infty^-$ and $N \geq 1$,*

$$\mathbb{P}\Big[\textit{There is a } *\textit{-path of } u\textit{-bad vertices from the origin to } \partial_{\text{int}} B_\infty^3(0,N)\Big] \leq Ce^{-N^{\frac{1}{C}}}.$$

Using this corollary, we can now prove Proposition 6.4.1.

Proof of Proposition 6.4.1. For all l_0 subject to (6.4.6) using similar arguments as in the proof of Lemma 6.4.3 we see that there is d_0 such that for all $d \geq d_0$ one has $u_\infty^- \geq (1-2\varepsilon)u_*(d)$. Fix $u \leq u_\infty^-$. Due to the shift-invariance of $\mathbb{P}$ it is enough to prove the result for $x = 0$. Assume that 0 is in a $*$-connected component of u-bad vertices of length at least N. Consequently, there is a $*$-path of u-bad vertices from 0 to $\partial_{\text{int}} B_\infty^3(0,(N/3)^{1/d})$ in $\mathbb{Z}^3$. Thus, by Corollary 6.4.5,

$$\begin{aligned}
&\mathbb{P}\big[0 \text{ is contained in a } *\text{-path of } u\text{-bad vertices of length at least } N\big] \\
&\leq \mathbb{P}\Big[\text{There is a } *\text{-path of } u\text{-bad vertices from the origin to } \partial_{\text{int}} B_\infty^3(0, cN^{\frac{1}{d}})\Big] \\
&\leq C \sum_{k=N/3}^{\infty} e^{-k^{1/(Cd)}} \leq C_2 e^{-N^{C_3}},\ C_2, C_3 > 0,
\end{aligned}$$

which proves the claim. ∎

6.5 Proof of Proposition 6.3.5: transience of $\mathcal{G}_\infty^u$

In this section we take advantage of the relations between simple random walk and electrical network theory in order to deduce that $\mathcal{G}_\infty^u$ is transient for u as in (6.4.1) and d large enough (see Proposition 6.3.5).

6.5.1 Rerouting paths around bad vertices

The following is inspired by methods of [ABP06]. Assume that the almost sure event of Proposition 6.3.3 occurs. Since $\mathbb{Z}^3$ is transient, Lemma 6.3.6 supplies us with the existence of a probability measure μ on infinite simple nearest neighbor paths in $\mathbb{Z}^3$ starting in some $y \in \mathbb{Z}^3$, and fulfilling (6.3.3). The idea now is to map infinite simple nearest neighbor paths π on $\mathbb{Z}^3$ via a function $\widehat{\varphi}$ to infinite simple nearest neighbor paths $\widehat{\varphi}(\pi)$ on $\mathcal{G}_\infty^u \subset \mathbb{Z}^3$ in such a way that $\mu \circ \widehat{\varphi}^{-1}$ still satisfies condition (6.3.3) and hence, again by Lemma 6.3.6, this supplies us with the transience of $\mathcal{G}_\infty^u$. This mapping will be constructed by cutting out pieces of a path π on $\mathbb{Z}^3$ which are not in $\mathcal{G}_\infty^u$ and afterwards replacing them by finite simple nearest neighbor paths of vertices on $\partial_{\text{int}}\mathcal{G}_\infty^u$. These sequences are chosen in such a way that they connect all parts of the path which are inside $\mathcal{G}_\infty^u$. In order to ensure that $\mathbb{P}$-a.s., the measure $\mu \circ \widehat{\varphi}^{-1}$ still satisfies condition (6.3.3), we will have to ensure that $*$-connected components of u-bad vertices are not too large. This is the content of the following lemma.

Lemma 6.5.1. *Let u and d be as in Proposition* (6.4.1)*. Then there is $C_4 > 0$ such that $\mathbb{P}$-a.s. one finds $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ the event*

$$\Big\{ \textit{there is } x \in B_\infty^3(0,N) \textit{ such that } x \textit{ is contained in a simple } *\textit{-path} \\ \textit{of } u\textit{-bad vertices of length at least } (\log N)^{C_4} \Big\}$$

does not occur.

Proof. This follows from Proposition 6.4.1 and an application of the Borel-Cantelli Lemma. ∎

In the rest of this section we describe the mapping $\widehat{\varphi}$ that will send infinite simple nearest neighbor paths on $\mathbb{Z}^3$ to infinite simple nearest neighbor paths $\widehat{\varphi}(\pi)$ on $\mathcal{G}_\infty^u$ as alluded to above. Let π be an infinite simple nearest neighbor path on $\mathbb{Z}^3$. We use the following notation for the sequence of successive returns to and departures from $\mathcal{G}_\infty^u$:

$$D_0 = \min\big\{k \geq 0\colon\ \pi(k) \in \mathbb{Z}^3 \setminus \mathcal{G}_\infty^u\big\},\ R_0 = \min\big\{k > D_0\colon\ \pi(k) \in \mathcal{G}_\infty^u\big\},$$
$$D_n = \min\big\{k > R_{n-1}\colon\ \pi(k) \in \mathbb{Z}^3 \setminus \mathcal{G}_\infty^u\big\},\ R_n = \min\big\{k > D_n\colon\ \pi(k) \in \mathcal{G}_\infty^u\big\},\ \text{for } n \in \mathbb{N}.$$

We modify the path π on $\mathbb{Z}^3$ in the following way:

1. if $D_0 = 0$, we erase the segment $(\pi(0), \dots, \pi(R_0 - 1))$;
2. for each n with $0 < D_n < \infty$ we replace the segment $(\pi(D_n), \dots, \pi(R_n - 1))$ by a finite shortest simple nearest neighbor path on $\mathcal{G}_\infty^u$ which connects $\pi(D_n - 1)$ to $\pi(R_n)$.

Finally, let $\widehat{\varphi}(\pi)$ be the loop-erasure of the path obtained this way, which is an infinite simple nearest neighbor path on $\mathcal{G}_\infty^u$. Below we will use the notation

$$\mathcal{B}_{x,u} = \begin{cases} \text{the } *\text{-connected component of } x \in \mathbb{Z}^3 \setminus \mathcal{G}_\infty^u \text{ of } u\text{-bad vertices}, & \text{if } x \text{ is } u\text{-bad}, \\ \emptyset, & \text{if } x \text{ is } u\text{-good}. \end{cases}$$

Remark 6.5.2. *Step* (b) *in the above construction is* $\mathbb{P}$*-a.s. well-defined. In fact, if* $D_n < \infty$*, then by Lemma 6.5.1,* $\mathcal{B}_{\pi(D_n),u}$ *is of finite cardinality, and* π *has to hit* $\partial^*_{\mathrm{ext}}\mathcal{B}_{\pi(D_n),u}$ *in finite time. By definition,* $\partial^*_{\mathrm{ext}}\mathcal{B}_{\pi(D_n),u}$ *consists of u-good vertices only; in addition, due to [T13, Theorem 4], it is connected, and since it contains* $\pi(D_n - 1) \in \mathcal{G}^u_\infty$*, we get* $\partial^*_{\mathrm{ext}}\mathcal{B}_{\pi(D_n),u} \subset \mathcal{G}^u_\infty$*. As a consequence,* R_n*,* $n \geq 1$*, coincides with the first hitting time of* $\partial^*_{\mathrm{ext}}\mathcal{B}_{\pi(D_n),u}$ *after time* D_n *and is finite. If* $D_0 > 0$*, then the same arguments show that* R_0 *is finite. To see that this is also true in the case that* $D_0 = 0$ *note that one may connect* $\pi(D_0)$ *by a finite nearest neighbor path to* $\mathcal{G}^u_\infty$*. This allows to apply the previous arguments to deduce the finiteness of* R_0 *also in this case. In particular, a finite shortest simple nearest neighbor path as postulated in* (b) *exists.*

6.5.2 Rerouting paths preserves finite energy

In this section we show that $\widehat{\varphi}(\pi)$ induces a probability measure as in condition (b) of Lemma 6.3.6. In fact, since $\mathbb{Z}^3$ is transient, Lemma 6.3.6 implies that there is $z \in \mathbb{Z}^3$ and a probability measure μ on $\Pi_z(\mathbb{Z}^3)$ which satisfies the finite energy condition (6.3.3), i.e., we have

$$\sum_{x \in \mathbb{Z}^3} \mu^2[\pi \in \Pi_z(\mathbb{Z}^3) \, : \, x \in \pi] < \infty. \tag{6.5.1}$$

By Lemma 6.3.6, in order to prove that $\mathcal{G}^u_\infty$ is transient a.s., we only need to show that $\mu \circ \widehat{\varphi}^{-1}$ satisfies (6.3.3), i.e., we have

$$\sum_{x \in \mathbb{Z}^3} \mu^2[x \in \widehat{\varphi}(\pi)] < \infty, \quad \mathbb{P} - \text{a.s.}^1 \tag{6.5.2}$$

We set for $x, y \in \mathbb{Z}^3$

$$S(x) = \begin{cases} \partial^*_{\mathrm{int}}\mathbb{Z}^3 \backslash \mathcal{B}_{x,u}, & \text{if } x \in \mathbb{Z}^3 \backslash \mathcal{G}^u_\infty, \\ \{x\}, & \text{if } x \in \mathcal{G}^u_\infty, \end{cases}$$

and

$$T(y) = \{x : \, y \in S(x)\}.$$

Using the definition of $\widehat{\varphi}$, we obtain the first inequality in

$$\mu^2[x \in \widehat{\varphi}(\pi)] \leq \left(\sum_{y \in T(x)} \mu[y \in \pi] \right)^2 \leq |T(x)| \sum_{y \in T(x)} \mu^2[y \in \pi],$$

and the second inequality in this chain is due to the Cauchy-Schwarz inequality. Hence,

$$\sum_{x \in \mathbb{Z}^3} \mu^2[x \in \widehat{\varphi}(\pi)] \leq \sum_{x \in \mathbb{Z}^3} |T(x)| \sum_{y \in T(x)} \mu^2[y \in \pi] = \sum_{y \in \mathbb{Z}^3} \mu^2[y \in \pi] \sum_{x \in S(y)} |T(x)|.$$

[1] In fact, note that since by Lemma 6.5.1 we have $|\mathcal{B}_{z,u}| < \infty$ a.s., there exists $z' \in \mathcal{G}^u_\infty$ such that $\mu \circ \widehat{\varphi}^{-1}$ puts positive mass on $\Pi_{z'}(\mathcal{G}^u_\infty)$. Restricting μ to this latter set and normalizing it puts us into the exact context of Lemma 6.3.6.

Therefore, in order to establish (6.5.2), by (6.5.1) it suffices to show that

$$\sup_{x\in\mathbb{Z}^3} \mathbb{E}\left[\sum_{y\in S(x)} |T(y)|\right] < \infty. \tag{6.5.3}$$

Lemma 6.5.3. *The term in (6.5.3) is finite.*

Proof. By shift invariance of $\mathbb{P}$ it suffices to prove the claim for $x = 0$. Note that

$$z \in \bigcup_{y\in S(0)} T(y) \Longleftrightarrow S(0) \cap S(z) \neq \emptyset,$$

which yields

$$\mathbb{E}\left[\sum_{y\in S(0)} |T(y)|\right] = \mathbb{P}[S(0) \neq \emptyset] + \sum_{z\neq 0} \mathbb{P}[S(0)\cap S(z) \neq \emptyset]. \tag{6.5.4}$$

To estimate the second term on the right-hand side of (6.5.4) note that if $S(0)\cap S(z) \neq \emptyset$, then for $y \in S(0) \cap S(z)$,

$$\begin{aligned} &(1)\ \text{there is } x_0 \in \mathcal{B}_{0,u} \text{ such that } |y - x_0|_\infty = 1;\\ &(2)\ \text{and there is } x_1 \in \mathcal{B}_{z,u} \text{ such that } |y - x_1|_\infty = 1. \end{aligned} \tag{6.5.5}$$

Since $\mathcal{B}_{0,u}$ and $\mathcal{B}_{z,u}$ are $*$-connected, there is a $*$-path of u-bad vertices starting in 0 and ending in x_0, and a $*$-path of u-bad vertices starting in x_1 and ending in z. Since $|x_0 - x_1|_\infty \leq 2$, we infer that at least one of these two paths must have length at least $\lfloor |z|_\infty - 1\rfloor/2$, and hence either 0 is contained in a $*$-path of u-bad vertices of length at least $\lfloor |z|_\infty - 1\rfloor/2$, or this property holds for z. Proposition 6.4.1 and the shift invariance of $\mathbb{P}$ yield

$$\begin{aligned} \mathbb{P}[S(0)\cap S(z)\neq\emptyset] &\leq 2\mathbb{P}\left[\begin{array}{l} 0 \text{ is contained in a simple } *\text{-path of } u\text{-bad vertices} \\ \text{of length at least } \lfloor |z|_\infty - 1\rfloor/2 \end{array}\right] \\ &\leq C_5 e^{-|z|_\infty^{C_6}}, \qquad C_5, C_6 > 0. \end{aligned} \tag{6.5.6}$$

∎

Appendix

6.6 Proof of Proposition 6.2.3: a decoupling inequality

In this appendix we prove Proposition 6.2.3. The proof is essentially the same as the proof of Theorems 2.1 and 3.4 in [S12]. While the proof of the latter one goes through in exactly the same way, we restrict ourselves to giving the main modifications of the proof of Theorem 2.1 in [S12]. Note that the setting in [S12] differs sligthly from the setting of the current work. Indeed, in [S12] more general graphs are considered and the norm in [S12] is different from the Euclidean norm we are considering here. Nevertheless, as stated in the first paragraph in [S12] up to a change of constants the results of [S12] stay true when working in the setting of this article.

• **Notation in [S12].** Let $l_0 > 1$ be a constant to be chosen later on, $L_0 \geq 1$, and define the geometric scales $L_n = l_0^n L_0$, $n \in \mathbb{N}_0$. For $n \in \mathbb{N}_0$, we denote the dyadic tree of depth n by $T_n = \bigcup_{0 \leq k \leq n} \{1,2\}^k$ and the set of vertices of the tree at depth k by $T_{(k)} = \{1,2\}^k$. Given a mapping $\mathcal{T} \colon T_n \to \mathbb{Z}^d$, we define

$$x_{m,\mathcal{T}} = \mathcal{T}(m),\ \widetilde{C}_{m,\mathcal{T}} = B_2(x_{m,\mathcal{T}}, 10\sqrt{d} L_{n-k}),\ \text{for } m \in T_{(k)},\ 0 \leq k \leq n. \tag{6.6.1}$$

For any $0 \leq k < n$, $m \in T_{(k)}$, we say that m_1, m_2 are the two descendants of m in $T_{(k+1)}$, if they are obtained by concatenating 1 and 2 to m, respectively. We say that $\mathcal{T}$ is a permitted embedding if for any $0 \leq k < n$ and $m \in T_{(k)}$,

$$\widetilde{C}_{m_1,\mathcal{T}} \cup \widetilde{C}_{m_2,\mathcal{T}} \subseteq \widetilde{C}_{m,\mathcal{T}}, \qquad |x_{m_1,\mathcal{T}} - x_{m_2,\mathcal{T}}|_2 \geq \frac{\sqrt{d}}{100} L_{n-k}. \tag{6.6.2}$$

The set of all permitted embeddings is denoted by Λ_n. Given $n \in \mathbb{N}_0$ and $\mathcal{T} \in \Lambda_n$, we say that a family A_m, $m \in T_{(n)}$, of events of measurable subsets of $\{0,1\}^{\mathbb{Z}^d}$ is $\mathcal{T}$-adapted if

$$A_m \text{ is } \sigma\big(\Psi_x, x \in \widetilde{C}_{m,\mathcal{T}}\big) - \text{measurable for each } m \in T_{(n)}.$$

For $n \in \mathbb{N}_0$ and $\mathcal{T} \in \Lambda_{n+1}$, we denote by $\mathcal{T}_i$, $i \in \{1,2\}$, the embeddings of T_n such that $\mathcal{T}_i(m) = \mathcal{T}((i, i_1, \ldots, i_k))$, for $m = (i_1, i_2, \ldots, i_k)$ in $T_{(k)}$. Given a $\mathcal{T}$-adapted collection A_m, $m \in T_{(n+1)}$, we define $\mathcal{T}_i$-adapted collections, $A_{m,i}$, $i \in \{1,2\}$, via

$$A_{m,i} = A_{(i,i_1,\ldots,i_n)},\ \text{for } m = (i_1, i_2, \ldots, i_n) \in T_{(n)}.$$

• **The Proof.** Recall (6.2.6–6.2.7) and the convention we made about constants in the introduction. We now adapt Theorem 2.1 in [S12] to our setting.

Theorem 6.6.1. *There are $c_0, c_1 > 1$, such that for $l_0 \geq 10^6 \sqrt{d} c_0$ and $L_0 \geq \sqrt{d}$, for all $n \in \mathbb{N}_0$, $\mathcal{T} \in \Lambda_{n+1}$, for all $\mathcal{T}$-adapted collections A_m, $m \in T_{(n+1)}$, of increasing events on $\{0,1\}^{\mathbb{Z}^d}$, and for all $u > u' > 0$ such that*

$$u \geq \left(1 + 32 e^2 c_1^d \frac{1}{(n+1)^{3/2}} l_0^{-(d-3)/2}\right) u',$$

one has

$$\mathbb{P}\left[\bigcap_{m \in T_{(n+1)}} A_m^{u'}\right] \leq \mathbb{P}\left[\bigcap_{\overline{m}_1 \in T_{(n)}} A_{\overline{m}_1,1}^{u}\right] \mathbb{P}\left[\bigcap_{\overline{m}_2 \in T_{(n)}} A_{\overline{m}_2,2}^{u}\right] + 2e^{-2u' \frac{1}{(n+1)^3} L_n^{d-2} l_0}.$$

Proof. The proof is analogous to that of Theorem 2.1 in [S12]. Thus, we only point out the modifications which are necessary to adapt the proof of [S12] to our setting.
First replace Lemma 1.2 in [S12], which is used in equation (2.31) in [S12], by Proposition 1.3 in [SS11b], which reads as follows.

Proposition 6.6.2. *There exist $c_0, c_4 > 1$, such that if $L \geq d$ and if h is a non-negative function defined on $\overline{B_2(0, c_0 L)}$ and harmonic in $B_2(0, c_0 L)$, one has*

$$\max_{x \in B_2(0,L)} h(x) \leq c_4^d \min_{x \in B_2(0,L)} h(x).$$

Second, define similarly as in [S12], (2.13)–(2.14), for $i \in \{1, 2\}$ and $\mathcal{T} \in \Lambda_{n+1}$

$$U = U_1 \cup U_2 \qquad \text{with} \qquad U_i = B_2\Big(\mathcal{T}(i), \frac{\sqrt{d} L_{n+1}}{1000}\Big).$$

as well as

$$\widetilde{B}_i = B_2\Big(\mathcal{T}(i), \frac{\sqrt{d} L_{n+1}}{2000M}\Big) \tag{6.6.3}$$

for a constant $1 \leq M \leq l_0/(2 \cdot 10^4)$ to be determined. Note in particular that, by (6.6.2), one has $U_1 \cap U_2 = \emptyset$. Moreover, from the definition of the scales L_n we infer $\widetilde{C}_{i,\mathcal{T}} \in U_i$, $i \in \{1, 2\}$.

The forthcoming lemma replaces Lemma 2.3 in [S12] and provides bounds on the probability that a random walk starting in $\partial U \cup \partial_{\text{int}} U$ enters a strict subset $\widetilde{W}$ of U in finite time. It is applied in equations (2.33) and (2.36) in [S12]. Before stating the lemma we recall the definition of the entrance time H_K in (6.2.2) and we moreover define

$$P_{e_U} = \sum_{x \in U} e_U(x) P_x.$$

Lemma 6.6.3. *Let $l_0 \geq 10^6 \sqrt{d} c_0$ and $L_0 \geq \sqrt{d}$. For any $\widetilde{W} \subseteq B_2\big(\mathcal{T}(1), \sqrt{d} L_{n+1}/2000\big) \cup B_2\big(\mathcal{T}(2), \sqrt{d} L_{n+1}/2000\big)$, $x \in \partial U \cup \partial_{\text{int}} U$, $x' \in \widetilde{W}$, one has for some constants $c_5, c_6 > 0$,*

$$c_5^d L_{n+1}^{-(d-2)} e_{\widetilde{W}}(x') \leq P_x\big[H_{\widetilde{W}} < \infty, X_{H_{\widetilde{W}}} = x'\big] \leq c_6^d L_{n+1}^{-(d-2)} e_{\widetilde{W}}(x').$$

Proof. The proof follows the lines of the proof of Lemma 2.3 in [S12] with a special attention to the dependence of constants on the dimension. First, since $\widetilde{W} \subseteq U$, one has the sweeping identity

$$e_{\widetilde{W}}(x') = P_{e_U}\big[H_{\widetilde{W}} < \infty, X_{H_{\widetilde{W}}} = x'\big],$$

from which one infers that

$$\begin{aligned} \operatorname{cap}(U) \inf_{x \in \partial_{\text{int}} U} P_x\big[H_{\widetilde{W}} < \infty, X_{H_{\widetilde{W}}} = x'\big] &\leq e_{\widetilde{W}}(x') \\ \leq \operatorname{cap}(U) \sup_{x \in \partial_{\text{int}} U} P_x\big[H_{\widetilde{W}} < \infty, X_{H_{\widetilde{W}}} = x'\big]&. \end{aligned} \tag{6.6.4}$$

Next, we claim that using (6.6.1–6.6.2) one can find $c_7 > 0$ such that any two points $x_1, x_2 \in \partial_{\text{int}} U$ may be connected by not more than c_7 overlapping balls $B_2\big(x', \sqrt{d}L_{n+1}/4000c_0\big)$, $x' \in \partial_{\text{int}} U \cup U^c$. In fact, along the lines of Lemma 2.2 of [SS11b], any two points on $\partial_{\text{int}} U_i$ can be connected "along" the great circle centered in $\mathcal{T}(i)$ with radius $\frac{\sqrt{d}L_{n+1}}{1000}$, by $c_7/3$ such overlapping balls; on the other hand, from (6.6.1) one can deduce that the same is true for two points y_1, y_2 such that $y_i \in \partial_{\text{int}} U_i$, and such that they have minimal distance among any such pair of points, whence the claim follows. Since the function $h(x) = P_x\big[H_{\widetilde{W}} < \infty, X_{H_{\widetilde{W}}} = x'\big]$ is non-negative and harmonic on $B_2\big(x', \sqrt{d}L_{n+1}/4000\big) \subseteq \widetilde{W}^c$, for all $x' \in \partial_{\text{int}} U \cup U^c$, and since $\sqrt{d}L_{n+1}/4000c_0 \geq d$, we obtain by Proposition 6.6.2 that

$$\begin{aligned}\sup_{x \in \partial_{\text{int}} U} P_x\big[H_{\widetilde{W}} < \infty, X_{H_{\widetilde{W}}} = x'\big] &\leq c_4^{dc_7} \inf_{x \partial_{\text{int}} U} P_x\big[H_{\widetilde{W}} < \infty, X_{H_{\widetilde{W}}} = x'\big] \\ &= c^d \inf_{x \in \partial_{\text{int}} U} P_x\big[H_{\widetilde{W}} < \infty, X_{H_{\widetilde{W}}} = x'\big].\end{aligned} \tag{6.6.5}$$

Finally, note that by (6.2.5) and the subadditivity of capacity (see (6.2.4)) we have

$$\left(\frac{c_2 L_{n+1}}{1000}\right)^{d-2} \leq \operatorname{cap}(U) \leq 2(c_3 L_{n+1})^{d-2}. \tag{6.6.6}$$

Inserting (6.6.5) and (6.6.6) into (6.6.4), yields the claim for $x \in \partial_{\text{int}} U$. The extension to $x \in \partial U$ follows from the fact that $P_x[X_1 = y] = 1/(2d)$ for all $x, y \in \mathbb{Z}^d$ with $|x - y|_1 = 1$. ∎

Since for all $n \in \mathbb{N}_0$ the inequality $\sqrt{d}L_n \geq d$ holds one can apply (6.2.5) to all balls in the Euclidean norm whose radius is larger than $\sqrt{d}L_n$. Using this fact repeatedly, from that moment on, the proof works similarly as the proof of [S12, Theorem 2.1]. In particular M as introduced in (6.6.3), which is determined in equation (2.36) in [S12], satisfies

$$c_6^d L_{n+1}^{-(d-2)} \operatorname{cap}(\widetilde{B}_1 \cup \widetilde{B}_2) \leq 2c_6^d L_{n+1}^{-(d-2)} \Big(\frac{c_3 L_{n+1}}{2000M}\Big)^{d-2} \leq (2e)^{-1}.$$

Thus, M does not depend on d. To conclude Proposition 6.2.3 from Theorem 6.6.1 one proceeds as in the proof of [S12, Theorem 3.4]. ∎

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Samenvatting

Dit proefschrift bestaat uit twee onderdelen.

Deel I behandelt het parabolische Anderson model (PAM). Dit is de partiële differentiaalvergelijking $\partial u(x,t)/\partial t = \kappa\Delta u(x,t) + \xi(x,t)u(x,t)$, $x \in \mathbb{Z}^d$, $t \geq 0$, waar u en ξ waarden in $\mathbb{R}$ aannemen, $\kappa \in [0,\infty)$ is de diffusieconstante, Δ is de discrete Laplace-operator, en ξ speelt de rol van een dynamische toevallige omgeving. De beginvoorwaarde $u(x,0) = u_0(x)$, $x \in \mathbb{Z}^d$, is niet-negatief en begrensd. De oplossing van de parabolische Anderson vergelijking beschrijft de evolutie van deeltjes op $\mathbb{Z}^d$ die onafhankelijke random wandelingen met binaire vertakking uitvoeren: deeltjes springen met snelheid $2d\kappa$, splitsen in twee met snelheid $\xi \vee 0$ en sterven met snelheid $(-\xi) \vee 0$. Dit proefschrift behandelt de vraag hoe de exponentiële groei van de deeltjespopulatie afhangt van de diffusieconstante κ. Om deze vraag te beantwoorden bestuderen we de "quenched Lyapunov exponent" λ_0. In hoofdstuk 1 geven we een samenvatting van de bestaande literatuur voor zowel het PAM waarbij ξ niet afhangt van de tijd als het PAM waarbij ξ wel afhangt van de tijd. In hoofdstuk 2 bewijzen we enkele fundamentele eigenschappen van de parabolische Anderson vergelijking, zoals existentie en uniciteit van de oplossing. Bovendien bewijzen we, onder bepaalde ruimte-tijd-mengingsvoorwaarden, dat $\lambda_0 < \infty$ en niet afhangt van de beginvoorwaarde u_0. Onder een extra ruimte-tijd-mengingsvoorwaarde en een "noisiness" voorwaarde tonen we aan dat λ_0 een continue maar niet Lipschitz-continue functie van κ is. Aan het einde van hoofdstuk 2 geven we enkele voorbeelden van dynamische toevallige omgevingen ξ die voldoen aan onze voorwaarden. Hoofdstuk 3 behandelt de vraag hoe λ_0 zich gedraagt voor grote κ. We bewijzen, onder vergelijkbare voorwaarden als in hoofdstuk 2, dat λ_0 convergeert naar $\mathbb{E}(\xi(0,0))$ als $\kappa \to \infty$.

Deel II behandelt twee verschillende percolatie-modellen. In het eerste model worden deeltjes in $\mathbb{R}^d$ geplaatst volgens een Poisson-punt-proces met intensiteit $\lambda > 0$. Vervolgens voert ieder deeltje (onafhankelijk van de andere deeltjes) een d-dimensionale Brownse beweging uit. Een van de belangrijke vragen is of dit netwerk van paden een onbegrensde cluster heeft of niet, en zo ja, of deze cluster uniek is. Het tweede model is random interlacement. Random interlacement met intensiteit $u > 0$ is een random deelverzameling $\mathcal{I}^u$ van $\mathbb{Z}^d$, $d \geq 3$, die er uit ziet als een pad van een symmetrische random wandeling op de torus $(\mathbb{Z}/N\mathbb{Z})^d$ gedurende tijd uN^d in de limiet $N \to \infty$. Het is bekend dat er een kritieke waarde $u_*(d) \in (0,\infty)$ is zodanig dat de vacante verzameling $\mathcal{V}^u = \mathbb{Z}^d \setminus \mathcal{I}^u$ voor alle $0 < u < u_*(d)$ een unieke onbegrensde samenhangende cluster $\mathcal{V}^u_\infty$ heeft, maar dat er geen onbegrensde cluster is voor $u > u_*(d)$. In hoofdstuk 4 geven we een korte samenvatting van de bestaande literatuur over percolatie en definiëren we de percolatie-modellen van hoofdstuk 5 en hoofdstuk 6. In hoofdstuk 5 geven we voor het model van Brownse percolatie een compleet antwoord op de vragen van existentie en uniciteit van de onbegrensde cluster voor alle dimensies. In hoofdstuk 6 bewijzen we voor het model van random interlacement dat er een kritieke waarde $u(d)$ is zodanig dat $\mathcal{V}_\infty^u$ transient is voor alle $0 < u < u(d)$, en dat $u(d)/u_*(d) \to 1$ als $d \to \infty$. Dit is het eerste resultaat over de meetkunde van $\mathcal{V}_\infty^u$ voor u dichtbij $u_*(d)$.

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Curriculum Vitae

Dirk Erhard was born in Berlin on 20 January 1985. He studied at the Technische Universität Berlin, completing his diploma in mathematics in 2010 under the supervision of Prof. Dr. Wolfgang König. From September 2008 until July 2009 he was an Erasmus student at the Université Paris-VII in Paris. In March 2011 he moved to Leiden to start a PhD under the supervision of Prof. Dr. Frank den Hollander (Universiteit Leiden) and Dr. Grégory Maillard (Université d'Aix Marseille). In September 2014 he will start post-doctoral research at the Mathematics Institute of the University of Warwick under the supervison of Prof. Dr. Martin Hairer.