



Covariant hamiltonian spin dynamics in curved space–time



G. d'Ambrosi^{a,*}, S. Satish Kumar^b, J.W. van Holten^{a,b}

^a Nikhef, Science Park 105, Amsterdam, Netherlands

^b Lorentz Institute, Leiden University, Niels Bohrweg 2, Leiden, Netherlands

ARTICLE INFO

Article history:

Received 23 January 2015

Received in revised form 4 March 2015

Accepted 5 March 2015

Available online 10 March 2015

Editor: A. Ringwald

ABSTRACT

The dynamics of spinning particles in curved space–time is discussed, emphasizing the hamiltonian formulation. Different choices of hamiltonians allow for the description of different gravitating systems. We give full results for the simplest case with minimal hamiltonian, constructing constants of motion including spin. The analysis is illustrated by the example of motion in Schwarzschild space–time. We also discuss a non-minimal extension of the hamiltonian giving rise to a gravitational equivalent of the Stern–Gerlach force. We show that this extension respects a large class of known constants of motion for the minimal case.

© 2015 The Authors. Published by Elsevier B.V. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>). Funded by SCOAP³.

1. Spinning-particle dynamics

The dynamics of angular momentum and spin of gravitating compact bodies has been a subject of great interest and intense investigation since the early days of relativity theory [1–12]; for recent overviews see [13–15]. As argued in [16] there are two complementary approaches to the subject. One approach starts from the covariant divergence-free energy–momentum tensor of matter, which makes it possible to keep track of aspects of the structure of the body. The energy–momentum vector and the angular-momentum tensor can be constructed by computing integrals of components of the energy–momentum tensor and their first moments over the volume of the body, using suitable boundary conditions. Equations of motion for these quantities are then derived by applying the conservation law for the energy–momentum tensor of matter [4,5,7].

The other approach is to construct effective equations of motion for point-like objects, which is an idealization of a compact body, at the price of neglecting details of the internal structure by assigning the point-like object an overall position, momentum and spin. This is also known as the spinning-particle approximation, and is used for the semi-classical description of elementary particles as well. A large variety of models for spinning particles is found in the literature [3,17–28].

In this letter we take the second point of view for the description of spinning test masses in curved space–time, using an

effective hamiltonian formalism similar to the one introduced in Ref. [29]. One of the advantages of this description is that it can be applied to compact bodies with different types of spin dynamics, such as different gravimagnetic ratios. In this way specific aspects of the structure can still be accounted for.

2. Covariant phase-space structure

Hamiltonian dynamical systems are specified by three sets of ingredients: the phase space, identifying the dynamical degrees of freedom, the Poisson–Dirac brackets defining a symplectic structure, and the hamiltonian generating the evolution of the system with given initial conditions by specifying a curve in the phase space passing through the initial point. The parametrization of phase-space is not unique, as is familiar from the Hamilton–Jacobi theory of dynamical systems. Changes in the parametrization can be compensated by redefining the brackets and the hamiltonian. A convenient starting point for models with gauge–field interactions is the use of covariant, i.e. kinetic, momenta rather than canonical momenta; see [30] and references cited there for a general discussion, and [29] for the application to spinning particles.

The spin degrees of freedom are described by an antisymmetric tensor $\Sigma^{\mu\nu}$, which can be decomposed into two space-like four-vectors by introducing a time-like unit vector u : $u_\mu u^\mu = -1$, and defining

$$S^\mu = \frac{1}{2\sqrt{-g}} \varepsilon^{\mu\nu\kappa\lambda} u_\nu \Sigma_{\kappa\lambda}, \quad Z^\mu = \Sigma^{\mu\nu} u_\nu. \quad (1)$$

* Corresponding author.

E-mail addresses: gdambros@nikhef.nl (G. d'Ambrosi), satish@lorentz.leidenuniv.nl (S. Satish Kumar), t32@nikhef.nl (J.W. van Holten).

<http://dx.doi.org/10.1016/j.physletb.2015.03.007>

0370-2693/© 2015 The Authors. Published by Elsevier B.V. This is an open access article under the CC BY license (<http://creativecommons.org/licenses/by/4.0/>). Funded by SCOAP³.

By construction both four-vectors S and Z are space-like:

$$S^\mu u_\mu = 0, \quad Z^\mu u_\mu = 0. \quad (2)$$

In the following we take u to be the proper four-velocity of the particle. Then S is the Pauli–Lubanski pseudo-vector, from which a magnetic dipole moment can be constructed, whilst the components of Z , which will be referred to as the Pirani vector, can be used to define an electric [31] or mass dipole moment [32,33]. Observe that we can invert the relations (1) to write

$$\Sigma^{\mu\nu} = -\frac{1}{\sqrt{-g}} \varepsilon^{\mu\nu\kappa\lambda} u_\kappa S_\lambda + u^\mu Z^\nu - u^\nu Z^\mu. \quad (3)$$

Therefore, if the Pirani vector vanishes: $Z = 0$ [34], the full spin tensor can be reconstructed from S . However, in non-flat space-time this is generally not the case.

The full set of phase-space co-ordinates of a spinning particle thus consists of the position co-ordinate x^μ , the covariant momentum π_μ and the spin tensor $\Sigma^{\mu\nu}$, with anti-symmetric Dirac-Poisson brackets

$$\begin{aligned} \{x^\mu, \pi_\nu\} &= \delta_\nu^\mu, & \{\pi_\mu, \pi_\nu\} &= \frac{1}{2} \Sigma^{\kappa\lambda} R_{\kappa\lambda\mu\nu}, \\ \{\Sigma^{\mu\nu}, \pi_\lambda\} &= \Gamma_{\lambda\kappa}^\mu \Sigma^{\nu\kappa} - \Gamma_{\lambda\kappa}^\nu \Sigma^{\mu\kappa}, \\ \{\Sigma^{\mu\nu}, \Sigma^{\kappa\lambda}\} &= g^{\mu\kappa} \Sigma^{\nu\lambda} - g^{\mu\lambda} \Sigma^{\nu\kappa} - g^{\nu\kappa} \Sigma^{\mu\lambda} + g^{\nu\lambda} \Sigma^{\mu\kappa}. \end{aligned} \quad (4)$$

The brackets imply that π represents the generator of covariant translations, whilst the spin degrees of freedom Σ generate internal rotations and Lorentz transformations. It is straightforward to check that these brackets are closed in the sense that they satisfy the Jacobi identities for triple bracket expressions. Thus they define a consistent symplectic structure on the phase space.¹

To get a well-defined dynamical system we need to complete the phase-space structure with a hamiltonian generating the proper-time evolution of the system. In principle a large variety of covariant expressions can be constructed; however if we impose the additional condition that the particle interacts only gravitationally and that in the limit of vanishing spin the motion reduces to geodesic motion, the variety is reduced to hamiltonians

$$H = H_0 + H_\Sigma, \quad H_0 = \frac{1}{2m} g^{\mu\nu} \pi_\mu \pi_\nu, \quad (5)$$

where $H_\Sigma = 0$ whenever $\Sigma^{\mu\nu} = 0$. In this letter we focus first on the dynamics generated by the minimal hamiltonian H_0 . However, we also consider an extension with [21]

$$H_\Sigma = \frac{\kappa}{4} R_{\mu\nu\kappa\lambda} \Sigma^{\mu\nu} \Sigma^{\kappa\lambda}, \quad (6)$$

The choice of hamiltonians can be enlarged further by including charges coupling the particle to vector fields like the electromagnetic field [29,31].

3. Equations of motion

Eqs. (4) and (5) specify a complete and consistent dynamical scheme for spinning particles. Note that the choice of hamiltonian is fixed by further physical requirements, and can differ for different compact objects. In that sense the hamiltonian is an *effective* hamiltonian, suitable to describe the motion of various types of objects in so far as the role of other internal degrees of freedom can

be restricted to their effects on overall position, linear momentum and spin.

The simplest model is obtained by restricting the hamiltonian to the minimal geodesic term H_0 . By itself this hamiltonian generates the following set of proper-time evolution equations:

$$\dot{x}^\mu = \{x^\mu, H_0\} \Rightarrow \pi_\mu = m g_{\mu\nu} \dot{x}^\nu, \quad (7)$$

stating that the covariant momentum π is a tangent vector to the world line, proportional to the proper four-velocity $u = \dot{x}$. Next

$$\begin{aligned} \dot{\pi}_\mu &= \{\pi_\mu, H_0\} \\ \Rightarrow D_\tau \pi_\mu &\equiv \dot{\pi}_\mu - \dot{x}^\lambda \Gamma_{\lambda\mu}^\nu \pi_\nu = \frac{1}{2m} \Sigma^{\kappa\lambda} R_{\kappa\lambda\mu}{}^\nu \pi_\nu, \end{aligned} \quad (8)$$

which specifies how the world line curves in terms of the evolution of its tangent vector. Finally the rate of change of the spin tensor is

$$\begin{aligned} \dot{\Sigma}^{\mu\nu} &= \{\Sigma^{\mu\nu}, H_0\} \\ \Rightarrow D_\tau \Sigma^{\mu\nu} &\equiv \dot{\Sigma}^{\mu\nu} + \dot{x}^\lambda \Gamma_{\lambda\kappa}^\mu \Sigma^{\kappa\nu} + \dot{x}^\lambda \Gamma_{\lambda\kappa}^\nu \Sigma^{\mu\kappa} = 0. \end{aligned} \quad (9)$$

In these equations the overdot denotes an ordinary derivative w.r.t. proper time τ , whereas D_τ denotes the pull-back of the covariant derivative along the world line $x^\mu(\tau)$. By substitution of Eq. (7) into Eq. (8) one finds that

$$D_\tau^2 x^\mu = \ddot{x}^\mu + \Gamma_{\lambda\nu}^\mu \dot{x}^\lambda \dot{x}^\nu = \frac{1}{2m} \Sigma^{\kappa\lambda} R_{\kappa\lambda}{}^\mu{}_\nu \dot{x}^\nu, \quad (10)$$

which reduces to the geodesic equation in the limit $\Sigma = 0$. The world line is the solution of the combined Eqs. (10) and (9) satisfying some initial conditions. This world line is a curve in space-time along which the spin tensor is covariantly constant. It has been remarked by many authors [15,29,35,36], that the spin-dependent force (8) exerted by the space-time curvature on the particle is similar to the Lorentz force with spin replacing the electric charge and curvature replacing the electromagnetic field strength. In this analogy the covariant conservation of spin along the world line is the natural equivalent of the conservation of charge.

Even though the spin tensor is covariantly constant, this does not hold for the Pauli–Lubanski and Pirani vectors S and Z individually. Indeed, due to the gravitational Lorentz force

$$\begin{aligned} D_\tau S^\mu &= \frac{1}{4m\sqrt{-g}} \varepsilon^{\mu\nu\kappa\lambda} \Sigma_{\kappa\lambda} \Sigma^{\alpha\beta} R_{\alpha\beta\nu\rho} u^\rho, \\ D_\tau Z^\mu &= \frac{1}{2m} \Sigma^{\mu\nu} \Sigma^{\alpha\beta} R_{\alpha\beta\nu\rho} u^\rho, \end{aligned} \quad (11)$$

where $\Sigma^{\mu\nu}$ is the linear expression in terms of S^μ and Z^μ given in Eq. (3). We observe that the rate of change of both spin vectors is of order $\mathcal{O}[\Sigma^2]$. In particular, as Z is not conserved in non-flat space-times the condition $Z = 0$ cannot be imposed during the complete motion in general. Indeed, the evolution of the system is completely determined by Eqs. (7), (8), (9), and leaves no room for additional constraints.

We close this section by remarking that the gravitational Lorentz force for unit mass $1/2 \Sigma^{\kappa\lambda} R_{\kappa\lambda}{}^\mu{}_\nu u^\nu$ can be interpreted geometrically as the change in the unit vector u^μ generated by transporting it around a closed loop with area projection in the x^κ – x^λ -plane equal to $\Sigma^{\kappa\lambda}$.

4. Conservation laws

By construction the time-independent hamiltonian represented by (5), (6) is a constant of motion for the spinning body, irrespective of the specific geometry of the space-time manifold. In particular for the minimal geodesic hamiltonian H_0 we have

$$H_0 = -\frac{m}{2}. \quad (12)$$

¹ We have not found this complete set of brackets in curved space-time in the literature. However, other sets of brackets have been proposed [9] based on a larger set of degrees of freedom, some of which are subsequently removed by supplementary constraints.

Another obvious constant of motion is the total spin:

$$I = \frac{1}{2} g_{\kappa\mu} g_{\lambda\nu} \Sigma^{\kappa\lambda} \Sigma^{\mu\nu} = S_\mu S^\mu + Z_\mu Z^\mu. \quad (13)$$

In addition, there may exist conserved quantities $J(x, \pi, \Sigma)$ resulting from symmetries of the background geometry, as implied by Noether's theorem [7,10,37]. They are solutions of the generic equation

$$\{J, H_0\} = \frac{1}{m} g^{\mu\nu} \pi_\nu \left[\frac{\partial J}{\partial x^\mu} + \Gamma_{\mu\lambda}^\kappa \pi_\kappa \frac{\partial J}{\partial \pi_\lambda} + \frac{1}{2} \Sigma^{\alpha\beta} R_{\alpha\beta\lambda\mu} \frac{\partial J}{\partial \pi_\lambda} + \Gamma_{\mu\alpha}^\kappa \Sigma^{\lambda\alpha} \frac{\partial J}{\partial \Sigma^{\kappa\lambda}} \right] = 0. \quad (14)$$

It follows that any constants of motion linear in momentum [37] are of the form

$$J = \alpha^\mu \pi_\mu + \frac{1}{2} \beta_{\mu\nu} \Sigma^{\mu\nu}, \quad (15)$$

with

$$\nabla_\mu \alpha_\nu + \nabla_\nu \alpha_\mu = 0, \quad \nabla_\lambda \beta_{\mu\nu} = R_{\mu\nu\lambda}^\kappa \alpha_\kappa. \quad (16)$$

These equations imply that α is a Killing vector on the space-time, and β is its anti-symmetrized gradient:

$$\beta_{\mu\nu} = \frac{1}{2} (\nabla_\mu \alpha_\nu - \nabla_\nu \alpha_\mu). \quad (17)$$

Similarly constants of motion quadratic in momentum [38] are of the form:

$$J = \frac{1}{2} \alpha^{\mu\nu} \pi_\mu \pi_\nu + \frac{1}{2} \beta_{\mu\nu}^\lambda \Sigma^{\mu\nu} \pi_\lambda + \frac{1}{8} \gamma_{\mu\nu\kappa\lambda} \Sigma^{\mu\nu} \Sigma^{\kappa\lambda}, \quad (18)$$

where the coefficients have to satisfy the ordinary partial differential equations

$$\begin{aligned} \nabla_\lambda \alpha_{\mu\nu} + \nabla_\mu \alpha_{\nu\lambda} + \nabla_\nu \alpha_{\lambda\mu} &= 0, \\ \nabla_\mu \beta_{\kappa\lambda\nu} + \nabla_\nu \beta_{\kappa\lambda\mu} &= R_{\kappa\lambda\mu}^\rho \alpha_{\nu\rho} + R_{\kappa\lambda\nu}^\rho \alpha_{\mu\rho}, \\ \nabla_\rho \gamma_{\mu\nu\kappa\lambda} &= R_{\mu\nu\rho}^\sigma \beta_{\kappa\lambda\sigma} + R_{\kappa\lambda\rho}^\sigma \beta_{\mu\nu\sigma}. \end{aligned} \quad (19)$$

Thus α is a symmetric rank-two Killing tensor, and the coefficients (β, γ) satisfy a hierarchy of inhomogeneous Killing-like equations determined by the $\alpha_{\mu\nu}$. In the case of Grassmann-valued spin tensors $\Sigma^{\mu\nu} = i\psi^\mu \psi^\nu$ the coefficient γ is completely anti-symmetric and the equations are known to have a solution in terms of Killing-Yano tensors [39].

The constants of motion (15) linear in momentum are special in that they define a Lie algebra: if J and J' are two such constants of motion, then their bracket is a constant of motion of the same type. This follows from the Jacobi identity

$$\{\{J, J'\}, H_0\} = \{\{J, H_0\}, J'\} - \{\{J', H_0\}, J\} = 0. \quad (20)$$

Thus, if $\{e_i\}_{i=1}^r$ is a complete basis for Killing vectors:

$$\alpha^\mu = \alpha^i e_i^\mu, \quad e_j^\nu \nabla_\nu e_i^\mu - e_i^\nu \nabla_\nu e_j^\mu = f_{ij}^k e_k^\mu,$$

the constants of motion define a representation of the same algebra:

$$J_i = e_i^\mu \pi_\mu + \frac{1}{2} \nabla_\mu e_{i\nu} \Sigma^{\mu\nu} \Rightarrow \{J_i, J_j\} = f_{ij}^k J_k. \quad (21)$$

Evidently such constants of motion are helpful in the analysis of spinning particle dynamics [10,12,40].

5. Schwarzschild space-time

The dynamics of spinning bodies can be illustrated by the motion in a static and spherically symmetric Schwarzschild space-time, for which the hamiltonian H_0 in Droste co-ordinates is given by

$$2mH_0 = -\frac{1}{1 - \frac{2M}{r}} \pi_t^2 + \left(1 - \frac{2M}{r}\right) \pi_r^2 + r^2 \pi_\theta^2 + r^2 \sin^2 \theta \pi_\varphi^2. \quad (22)$$

The space-time manifold admits four Killing vectors, for time-translations and rotations. They give rise to the conservation of kinetic energy:

$$-E = \pi_t + \frac{M}{r^2} \Sigma^{tr}, \quad (23)$$

and angular momentum:

$$\begin{aligned} J_1 &= -\sin \varphi \pi_\theta - \cotan \theta \cos \varphi \pi_\varphi \\ &\quad - r \sin \varphi \Sigma^{r\theta} - r \sin \theta \cos \theta \cos \varphi \Sigma^{r\varphi} + r^2 \sin^2 \theta \cos \varphi \Sigma^{\theta\varphi}, \\ J_2 &= \cos \varphi \pi_\theta - \cotan \theta \sin \varphi \pi_\varphi \\ &\quad + r \cos \varphi \Sigma^{r\theta} - r \sin \theta \cos \theta \sin \varphi \Sigma^{r\varphi} + r^2 \sin^2 \theta \sin \varphi \Sigma^{\theta\varphi}, \\ J_3 &= \pi_\varphi + r \sin^2 \theta \Sigma^{r\varphi} + r^2 \sin \theta \cos \theta \Sigma^{\theta\varphi}. \end{aligned} \quad (24)$$

It is straightforward to check that these satisfy the usual algebra of time-translations and spatial rotations:

$$\{E, J_i\} = 0, \quad \{J_i, J_j\} = \varepsilon_{ijk} J_k. \quad (25)$$

As usual, the conservation of total angular momentum and the spherical symmetry of the space-time geometry allow one to take the angular momentum \mathbf{J} as the direction of the z -axis, such that

$$\mathbf{J} = (0, 0, J). \quad (26)$$

For spinless particles, for which the angular momentum is strictly orbital, this implies that the orbital motion is in a plane perpendicular to the angular momentum 3-vector; with our choice of the z -axis this is the equatorial plane $\theta = \pi/2$.

In the presence of spin the result no longer holds in general, as the precession of spin can be compensated by precession of the orbital angular momentum, resulting in a non-planar orbit [41]. However, one can ask under which conditions planar motion is still possible. As in that case the directions of orbital and spin angular momentum are separately preserved, it means that necessary conditions for motion in the equatorial plane are

$$J_1 = J_2 = 0, \quad \pi_\theta = 0, \quad (27)$$

and therefore also

$$\Sigma^{r\theta} = \Sigma^{\theta\varphi} = 0. \quad (28)$$

Furthermore the absence of acceleration perpendicular to the equatorial plane expressed by $D_\tau \pi_\theta = 0$ implies that

$$\Sigma^{t\theta} = 0. \quad (29)$$

Thus planar motion requires alignment of the spin with the orbital angular momentum; it is straightforward to show that the reverse statement also holds [42,43].

In terms of the four-velocity components we are now left with relevant constants of motion

$$E = m \left(1 - \frac{2M}{r}\right) u^t - \frac{M}{r^2} \Sigma^{tr}, \quad (30)$$

and

$$J = mr^2 u^\varphi + r \Sigma^{r\varphi}, \quad (31)$$

in addition to the hamiltonian constraint

$$\left(1 - \frac{2M}{r}\right) u^{t2} = 1 + \frac{u^{r2}}{1 - \frac{2M}{r}} + r^2 u^{\varphi2}, \quad (32)$$

and the conservation of total spin I , or equivalently:

$$\Sigma^{t\varphi2} = -\frac{1}{r^2} \frac{I + \Sigma^{tr2}}{1 - \frac{2M}{r}} + \frac{\Sigma^{r\varphi2}}{\left(1 - \frac{2M}{r}\right)^2}. \quad (33)$$

These equations show, that once the orbital velocities are known, all the non-vanishing spin components can be calculated from Eqs. (30), (31) and (33).

The simplest type of planar orbit is the circular orbit $r = R = \text{constant}$, $u^r = 0$. In this case the symmetry of the orbit implies that (u^t, u^φ) are constant in time, and that $\Sigma^{t\varphi} = 0$. This can be shown as follows. First, absence of radial acceleration $D_\tau u^r = 0$ gives, upon using the conservation laws for E and J :

$$\begin{aligned} &\left(1 - \frac{2M}{R}\right) \left(2 - \frac{3M}{R}\right) m u^{t2} - \left(1 - \frac{3M}{R}\right) m R^2 u^{\varphi2} \\ &= 2E \left(1 - \frac{2M}{R}\right) u^t + \frac{JM}{R} u^\varphi, \end{aligned} \quad (34)$$

whilst the hamiltonian constraint (32) simplifies to

$$\left(1 - \frac{2M}{R}\right) u^{t2} = 1 + R^2 u^{\varphi2}. \quad (35)$$

These two equations can be solved for u^t and u^φ in terms of (R, E, J) , implying that they are constant. An immediate consequence is, that Σ^{tr} , $\Sigma^{r\varphi}$ and $\Sigma^{t\varphi}$ are constant as well, and actually $\Sigma^{t\varphi}$ vanishes. This follows directly from the absence of four-acceleration:

$$\frac{du^t}{d\tau} = \frac{M}{mR} u^\varphi \Sigma^{t\varphi} = 0, \quad \frac{du^\varphi}{d\tau} = \frac{M}{mR^3} \left(1 - \frac{2M}{R}\right) u^t \Sigma^{t\varphi} = 0. \quad (36)$$

Then also the rate of change of $\Sigma^{t\varphi}$ must vanish:

$$\begin{aligned} &-\frac{M}{R} \left(1 - \frac{2M}{R}\right) \frac{d\Sigma^{t\varphi}}{d\tau} \\ &= \left(1 - \frac{M}{R}\right) \left(1 - \frac{3M}{R}\right) m u^t u^\varphi + \frac{JM^2}{R^4} u^t - E \left(1 - \frac{2M}{R}\right) u^\varphi \\ &= 0. \end{aligned} \quad (37)$$

Now from Eqs. (34) and (35) it follows that

$$\frac{2E}{m} \left(1 - \frac{2M}{R}\right) u^t = 2 - \frac{3M}{R} - \frac{JM}{mR} u^\varphi + R^2 u^{\varphi2}. \quad (38)$$

These equations then allow the elimination of E and u^t , with the result that

$$\begin{aligned} &\frac{JM}{mR^2} \left(\frac{2M}{R} + R^2 u^{\varphi2}\right) \\ &= R u^\varphi \left[\frac{M}{R} - \left(1 - \frac{6M}{R} + \frac{6M^2}{R^2}\right) R^2 u^{\varphi2}\right]. \end{aligned} \quad (39)$$

As for the total spin, for circular orbits the expression (33) can be written as

$$\begin{aligned} I &= -\Sigma^{tr2} + \frac{R^2 \Sigma^{r\varphi2}}{1 - \frac{2M}{R}} \\ &= -\frac{R^4}{M^2} \left[\left(1 - \frac{2M}{R}\right) m u^t - E\right]^2 + \frac{1}{\left(1 - \frac{2M}{R}\right)^2} \left[J - m R^2 u^\varphi\right]^2. \end{aligned} \quad (40)$$

Thus for circular orbits u^φ and u^t are constants which can be expressed in terms of R and J , in turn fixing E and I as well.

6. Non-minimal hamiltonians

So far we have studied the dynamics of compact spinning objects generated by the minimal geodesic hamiltonian H_0 . In this section we consider the non-minimal extension (6)

$$H_\Sigma = \frac{\kappa}{4} R_{\mu\nu\kappa\lambda} \Sigma^{\mu\nu} \Sigma^{\kappa\lambda},$$

including the spin-spin interaction via space-time curvature. It is straightforward to derive the equations of motion:

$$\begin{aligned} \dot{x}^\mu &= \{x^\mu, H\} \Rightarrow \pi_\mu = m g_{\mu\nu} \dot{x}^\nu, \\ \dot{\pi}_\mu &= \{\pi_\mu, H\} \\ &\Rightarrow D_\tau \pi_\mu = \frac{1}{2m} \Sigma^{\kappa\lambda} R_{\kappa\lambda\mu}{}^\nu \pi_\nu - \frac{\kappa}{4} \Sigma^{\kappa\lambda} \Sigma^{\rho\sigma} \nabla_\mu R_{\kappa\lambda\rho\sigma}, \\ \dot{\Sigma}^{\mu\nu} &= \{\Sigma^{\mu\nu}, H\} \\ &\Rightarrow D_\tau \Sigma^{\mu\nu} = \kappa \Sigma^{\kappa\lambda} (R_{\kappa\lambda}{}^\mu{}_\sigma \Sigma^{\nu\sigma} - R_{\kappa\lambda}{}^\nu{}_\sigma \Sigma^{\mu\sigma}). \end{aligned} \quad (41)$$

Comparing again with the electro-magnetic force, the middle equation implies that in addition to the gravitational Lorentz force there is a gravitational Stern-Gerlach force, coupling spin to the gradient of the curvature. Therefore the coupling parameter κ has been termed the gravimagnetic ratio [22,44]. Like in the electro-magnetic case [45] the Pauli-Lubanski and Pirani-vectors are affected by this Stern-Gerlach force:

$$\begin{aligned} D_\tau S^\mu &= \frac{1}{4m\sqrt{-g}} \varepsilon^{\mu\nu\kappa\lambda} \Sigma_{\kappa\lambda} \Sigma^{\alpha\beta} (R_{\alpha\beta\nu\sigma} u^\sigma \\ &\quad - \frac{\kappa}{2} \Sigma^{\rho\sigma} \nabla_\nu R_{\rho\sigma\alpha\beta}), \\ D_\tau Z^\mu &= -\kappa \Sigma^{\kappa\lambda} R_{\kappa\lambda}{}^\mu{}_\nu Z^\nu + \left(\kappa + \frac{1}{2m}\right) \Sigma^{\mu\nu} \Sigma^{\kappa\lambda} R_{\kappa\lambda\nu\sigma} u^\sigma \\ &\quad - \frac{\kappa}{4m} \Sigma^{\mu\nu} \Sigma^{\kappa\lambda} \Sigma^{\rho\sigma} \nabla_\nu R_{\kappa\lambda\rho\sigma}. \end{aligned} \quad (42)$$

The second equation simplifies strongly for the special value

$$\kappa = -\frac{1}{2m}. \quad (43)$$

In that case an initial condition $Z^\mu = 0$ is conserved up to terms of cubic order in spin.

For the extended hamiltonian the conditions for the existence of constants of motion are modified. The total spin I defined in (13) is still conserved, but the conserved hamiltonian now is of course $H = H_0 + H_\Sigma$. Finally we prove that the constants of motion J of the form (15) are preserved under this modification of the hamiltonian. To see this, observe that

$$\{J, H_\Sigma\} = -\kappa \Sigma^{\mu\nu} \Sigma^{\rho\sigma} \left(\frac{1}{4} \alpha^\lambda \nabla_\lambda R_{\mu\nu\rho\sigma} + \beta_{\mu\lambda} R^\lambda{}_{\nu\rho\sigma}\right). \quad (44)$$

For the Killing-vector solutions (16) the right-hand side takes the form

$$\begin{aligned}
& \Sigma^{\mu\nu} \Sigma^{\rho\sigma} \left(\frac{1}{4} \alpha^\lambda \nabla_\lambda R_{\mu\nu\rho\sigma} + \beta_{\mu\lambda} R^\lambda_{\nu\rho\sigma} \right) \\
&= \frac{1}{2} \Sigma^{\mu\nu} \Sigma^{\rho\sigma} (\nabla_\mu \nabla_\rho \nabla_\sigma + \nabla_\rho \nabla_\mu \nabla_\sigma) \alpha_\nu \\
&= \frac{1}{2} \Sigma^{\mu\nu} \Sigma^{\rho\sigma} (\nabla_\mu \nabla_\rho + \nabla_\rho \nabla_\mu) \beta_{\sigma\nu} = 0, \tag{45}
\end{aligned}$$

due to the anti-symmetry of the tensor $\beta_{\sigma\nu}$. Therefore in particular the expressions (23) and (24) also define constants of motion in Schwarzschild space-time in the presence of Stern–Gerlach forces, as described by the non-minimal hamiltonian (6).

7. Conclusions

In the context of general relativity the notion of point-masses is troublesome; any non-zero mass has a characteristic scale, typified by its Schwarzschild radius, describing its minimal size as defined by the corresponding horizon [46]. Therefore the approximation of a gravitating compact body as a point-like massive object in curved space-time requires the body to be small compared to the radius of curvature of the background space-time [12,47]. In addition, the mass must be small enough to ignore its effect on the space-time geometry at large. In the existing literature much effort has been put into obtaining effective equations of motion for compact objects by defining a position variable which can be interpreted effectively as the material center [10,33]. One then computes the momentum and angular momentum in terms of a mass and momentum distribution in a finite neighborhood of this point. However, such a position variable is not unique, and moreover it often traces out a complicated world line, as shown for example by the well-known helical motion that is a solution of the Mathisson–Papapetrou–Dixon equations [4,32,48].

In situations where a point-particle approximation of a spinning and gravitating body is appropriate, a complementary approach suggests itself by constructing a lagrangian or hamiltonian mechanics for a mass-point carrying spin in a curved space-time. In this letter we have chosen the hamiltonian point of view, as in our opinion this is most transparent in its results and application. In particular, the closed set of Dirac–Poisson brackets (4) provides a unique and unambiguous starting point for the derivation of equations of motion for any representation of the spin degrees of freedom, allowing for a large class of physical implementations as fixed by the choice of hamiltonian. Two such choices, a minimal and a non-minimal one, have been presented and analyzed in this letter.

The minimal choice of hamiltonian is the one which also describes the geodesic motion of spinless particles. With this choice of hamiltonian the spin is covariantly constant along the world line, which is no longer geodesic due to spin–orbit coupling. It naturally provides a different implementation of the notion of position of the body, one for which now the Pirani vector Z is no longer taken to vanish. The advantage is that in terms of this choice of position variable the motion becomes tractable in non-trivial situations of practical interest; the motion in Schwarzschild space-time, as analyzed in Section 5, provides a case in point. In addition, non-minimal hamiltonians can provide more complicated dynamics, as required for example for objects with non-vanishing gravimagnetic ratios [22,35]. In this case the spin is subject to a kind of gravitational Larmor force, making it precess around field-lines of constant curvature.

The question which effective hamiltonian to use for which physical system now becomes a matter of phenomenology. One should either derive the correct effective hamiltonian from first principles, connecting the formalism to the specific energy–momentum tensor, or determine it from experiments or observations.

For the particular case of rotating black holes it could presumably be measured by observing gravitational waves from Extreme Mass Ratio binary systems involving a stellar-mass black hole; for a review see [49].

Acknowledgements

For G.d'A. and J.W.v.H. this work is part of the research programme *Gravitational Physics* of the Netherlands Foundation for Fundamental Research on Matter (FOM).

References

- [1] W. de Sitter, Einstein's theory of gravitation and its astronomical consequences, first paper, *Mon. Not. R. Astron. Soc.* 76 (1916) 699–728.
- [2] L. Thomas, The motion of a spinning electron, *Nature* 117 (1926) 514.
- [3] J. Frenkel, Die Elektrodynamik des rotierenden Elektrons, *Z. Phys.* 37 (1926) 243–262.
- [4] M. Mathisson, Neue mechanik materieller systeme, *Acta Phys. Pol.* 6 (1937) 163–200.
- [5] A. Papapetrou, Spinning test particles in general relativity, 1, *Proc. R. Soc. Lond. Ser. A, Math. Phys. Sci.* 209 (1951) 248–258.
- [6] V. Fock, D. Ivanenko, Über eine mögliche geometrische Deutung der relativistischen Quantentheorie, *Z. Phys.* 54 (1929) 798–802.
- [7] W. Dixon, Dynamics of extended bodies in general relativity. I. Momentum and angular momentum, *Proc. R. Soc. Lond. Ser. A, Math. Phys. Sci.* 314 (1970) 499–527.
- [8] R.M. Wald, Gravitational spin interaction, *Phys. Rev. D* 6 (1972) 406–413.
- [9] A.J. Hanson, T. Regge, The relativistic spherical top, *Ann. Phys.* 87 (1974) 498.
- [10] O. Semerak, Spinning test particles in a Kerr field, I, *Mon. Not. R. Astron. Soc.* 308 (1999) 863–875.
- [11] K. Kyrián, O. Semerak, Spinning test particles in a Kerr field, II, *Mon. Not. R. Astron. Soc.* 382 (2007) 1922–1932.
- [12] R. Plyatsko, O. Stefanyshyn, M. Fenyk, Mathisson–Papapetrou–Dixon equations in the Schwarzschild and Kerr backgrounds, *Class. Quantum Gravity* 28 (2011) 195025, arXiv:1110.1967 [gr-qc].
- [13] G. Schafer, Gravitomagnetic effects, *Gen. Relativ. Gravit.* 36 (2004) 2223, arXiv:gr-qc/0407116.
- [14] J. Steinhoff, Canonical formulation of spin in general relativity, *Ann. Phys.* 523 (2011) 296–353, arXiv:1106.4203 [gr-qc].
- [15] L.F.O. Costa, J. Natário, M. Zilhao, Spacetime dynamics of spinning particles – exact gravito-electromagnetic analogies, arXiv:1207.0470 [gr-qc].
- [16] I. Bailey, W. Israel, Lagrangian dynamics of spinning particles and polarized media in general relativity, *Commun. Math. Phys.* 42 (1975) 65–82.
- [17] A.O. Barut, *Electrodynamics and Classical Theory of Fields and Particles*, MacMillan, New York, 1964.
- [18] C. Duval, H.H. Fliche, J.M. Souriau, *C. R. Acad. Sci. Paris* 274 (1972) 1082.
- [19] L. Brink, P. Di Vecchia, P.S. Howe, A lagrangian formulation of the classical and quantum dynamics of spinning particles, *Nucl. Phys. B* 118 (1977) 76.
- [20] A.O. Barut, N. Zanghi, *Phys. Rev. Lett.* 52 (1984).
- [21] I. Khriplovich, A. Pomeransky, Gravitational interaction of spinning bodies, center-of-mass coordinate and radiation of compact binary systems, *Phys. Lett. A* 216 (1996) 7, arXiv:gr-qc/9602004.
- [22] I. Khriplovich, A. Pomeransky, Equations of motion of spinning relativistic particle in external fields, *Surv. High Energy Phys.* 14 (1999) 145–173, arXiv:gr-qc/9809069.
- [23] A. Barducci, R. Casalbuoni, L. Lusanna, Supersymmetries and the pseudoclassical relativistic electron, *Nuovo Cimento A* 35 (1976) 377.
- [24] F. Berezin, M. Marinov, Particle spin dynamics as the Grassmann variant of classical mechanics, *Ann. Phys.* 104 (1977) 336.
- [25] P. Salomonson, Supersymmetric actions for spinning particles, *Phys. Rev. D* 18 (1978) 1868–1880.
- [26] R. Rietdijk, J. van Holten, Spinning particles in Schwarzschild space-time, *Class. Quantum Gravity* 10 (1993) 575–594.
- [27] E. Barausse, E. Racine, A. Buonanno, Hamiltonian of a spinning test-particle in curved spacetime, *Phys. Rev. D* 80 (2009) 104025, arXiv:0907.4745 [gr-qc].
- [28] W. Guzmán Ramírez, A.A. Deriglazov, A.M. Pupasov-Maksimov, Frenkel electron and a spinning body in a curved background, *JHEP* 1403 (2014) 109, arXiv:1311.5743 [hep-th].
- [29] I. Khriplovich, Particle with internal angular momentum in a gravitational field, *Sov. Phys. JETP* 69 (1989) 217–219.
- [30] J.W. van Holten, Covariant hamiltonian dynamics, *Phys. Rev. D* 75 (2007) 025027, arXiv:hep-th/0612216.
- [31] J.W. van Holten, On the electrodynamics of spinning particles, *Nucl. Phys. B* 356 (1991) 3–26.

- [32] L.F.O. Costa, C.A. Herdeiro, J. Natário, M. Zilhao, Mathisson's helical motions for a spinning particle: are they unphysical? *Phys. Rev. D* 85 (2012) 024001, arXiv:1109.1019 [gr-qc].
- [33] L.F. Costa, J. Natário, Center of mass, spin supplementary conditions, and the momentum of spinning particles, arXiv:1410.6443 [gr-qc].
- [34] F. Pirani, On the physical significance of the Riemann tensor, *Acta Phys. Pol.* 15 (1956) 389–405.
- [35] I. Khriplovich, Spinning relativistic particles in external fields, arXiv:0801.1881 [gr-qc].
- [36] J.W. van Holten, Relativistic dynamics of spin in strong external fields, arXiv: hep-th/9303124.
- [37] R. Rüdiger, Conserved quantities of spinning test particles in general relativity I, *Proc. R. Soc. Ser. A* 375 (1761) (1981) 185–193.
- [38] R. Rüdiger, Conserved quantities of spinning test particles in general relativity II, *Proc. R. Soc. Ser. A* 385 (1788) (1983) 229–239.
- [39] G. Gibbons, R. Rietdijk, J. van Holten, SUSY in the sky, *Nucl. Phys. B* 404 (1993) 42–64, arXiv:hep-th/9303112.
- [40] J.W. Van Holten, Gravitational waves and black holes: an introduction to general relativity, *Fortschr. Phys.* 45 (1997) 439–516, arXiv:gr-qc/9704043.
- [41] D. Bini, F. de Felice, A. Geralico, R.T. Jantzen, Spin precession in the Schwarzschild spacetime: circular orbits, *Class. Quantum Gravity* 22 (2005) 2947–2970, arXiv:gr-qc/0506017.
- [42] D. Bini, A. Geralico, R.T. Jantzen, Spin-geodesic deviations in the Schwarzschild spacetime, *Gen. Relativ. Gravit.* 43 (2011) 959, arXiv:1408.4946 [gr-qc].
- [43] G. d'Ambrosi, S. Satish Kumar, J.W. van Holten, in preparation, 2015.
- [44] I. Khriplovich, A. Pomeransky, Equations of motion of spinning relativistic particle in external fields, *J. Exp. Theor. Phys.* 86 (1998) 839–849, arXiv:gr-qc/9710098.
- [45] K. Heinemann, On Stern–Gerlach forces allowed by special relativity and the special case of the classical spinning particle of Derbenev–Kondratenko, arXiv:physics/9611001.
- [46] C.W. Misner, K.S. Thorne, J.A. Wheeler, *Gravitation*, 1973.
- [47] E. Poisson, The motion of point particles in curved space–time, *Living Rev. Relativ.* 7 (2004) 6, arXiv:gr-qc/0306052.
- [48] L.F.O. Costa, J. Natário, M. Zilhao, Mathisson's helical motions demystified, *AIP Conf. Proc.* 1458 (2011) 367–370, arXiv:1206.7093 [gr-qc].
- [49] S. Babak, J.R. Gair, R.H. Cole, Extreme mass ratio inspirals: perspectives for their detection, arXiv:1411.5253 [gr-qc].