

UNIRATIONALITY OF DEL PEZZO SURFACES OF DEGREE TWO OVER FINITE FIELDS

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ABSTRACT. We prove that every del Pezzo surface of degree two over a finite field is unirational, building on the work of Manin and an extension by Salgado, Testa, and Várilly-Alvarado, who had proved this for all but three surfaces. Over general fields of characteristic not equal to two, we state sufficient conditions for a del Pezzo surface of degree two to be unirational.

1. INTRODUCTION

A *del Pezzo surface* is a smooth, projective, geometrically integral surface X of which the anticanonical divisor $-K_X$ is ample. We define the *degree* of a del Pezzo surface X as the self intersection number of K_X , that is, $\deg X = K_X^2$. If k is an algebraically closed field, then every del Pezzo surface of degree d over k is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ (with $d = 8$), or to \mathbb{P}^2 blown up in $9 - d$ points in general position.

Over arbitrary fields, the situation is more complicated and del Pezzo surfaces need not be birationally equivalent with \mathbb{P}^2 . We therefore look at the weaker notion of unirationality. We say that a variety X of dimension n over a field k is *unirational* if there exists a dominant rational map $\mathbb{P}^n \dashrightarrow X$, defined over k . We prove the following theorem.

Theorem 1.1. *Every del Pezzo surface of degree 2 over a finite field is unirational.*

The analog for higher degree holds over any field. Works of B. Segre, Yu. Manin, J. Kollár, M. Pieropan, and A. Knecht prove that every del Pezzo surface of degree $d \geq 3$, defined over any field k , is unirational, provided that the set $X(k)$ of rational points is non-empty. For references, see [Seg43, Seg51] for $k = \mathbb{Q}$ and $d = 3$, see [Man86, Theorem 29.4 and 30.1] for $d \geq 3$ with the extra assumption for $d \in \{3, 4\}$ that k has enough elements. See [Kol02, Theorem 1.1] for $d = 3$ and a general ground field. The earliest reference we could find for $d = 4$ and a general ground field is [Pie12, Proposition 5.19]. Independently, for $d = 4$, [Kne15, Theorem 2.1] covers all finite fields. Since all del Pezzo surfaces over finite fields have a rational point (see [Man86, Corollary 27.1.1]), this implies that every del Pezzo surface of degree at least 3 over a finite field is unirational.

Most of the work to prove Theorem 1.1 was already done. Building on work by Manin (see [Man86, Theorem 29.4]), C. Salgado, D. Testa, and A. Várilly-Alvarado prove that all del Pezzo surfaces of degree 2 over a finite field are unirational, except possibly for three isomorphism classes of surfaces (see [STVA14, Theorem 1]). In Section 3, we will present the three difficult surfaces and show that these are also unirational, thus proving Theorem 1.1.

Before that, in Section 2, we will recall the basics about del Pezzo surfaces of degree 2, including the fact that the linear system associated to the anti-canonical divisor induces a finite morphism to \mathbb{P}^2 of degree 2. We call this morphism the *anti-canonical morphism* associated to X . This allows us to state the second main theorem.

Theorem 1.2. *Suppose k is a field of characteristic not equal to 2. Let X be a del Pezzo surface of degree 2 over k , and let $\pi: X \rightarrow \mathbb{P}^2$ be its anti-canonical morphism. Assume that X has a k -rational point, say P . Let $C \subset \mathbb{P}^2$ be a geometrically integral curve over k of degree $d \geq 2$ and suppose that $\pi(P)$ is a point of multiplicity $d - 1$ on C . Suppose, moreover, that C intersects the branch locus B of the morphism π with even multiplicity everywhere. Then the following statements hold.*

- (1) If $\pi(P)$ is not contained in B , then X is unirational.
(2) If $\pi(P)$ is contained in B , and it is an ordinary singular point on C and we have $d \in \{3, 4\}$, then there exists a field extension ℓ of k of degree at most 2 for which the preimage $\pi^{-1}(C_\ell)$ is birationally equivalent with \mathbb{P}^1_ℓ ; for each such field ℓ , the surface X_ℓ is unirational.

The main tool for the proof of both theorems is Lemma 3.2 (that is, [STVA14, Theorem 17]), which states that, outside characteristic 2, a del Pezzo surface of degree 2 is unirational if it contains a rational curve. We prove Theorem 1.2 in Section 4 by showing that, under the hypotheses of Theorem 1.2, the pull-back of the curve C to X contains a rational component. Manin's original construction, and the generalisation by Salgado, Testa, and Várilly-Alvarado, produces a rational curve that corresponds to case (1) of Theorem 1.2, with $4 - d$ equal to the number of exceptional curves that P lies on. In particular, their construction requires a point on the del Pezzo surface that does not map to the branch locus of the anti-canonical morphism. The three remaining surfaces do not have such a point: for each of them, all the rational points lie on the ramification locus. For these surfaces, we use case (2) of Theorem 1.2 (see Remark 4.1) to prove unirationality. Here we benefit from the fact that if k is a finite field, then any curve that becomes birationally equivalent with \mathbb{P}^1 over an extension of k , already is birationally equivalent with \mathbb{P}^1 over k itself.

For interesting examples and more details about the proof of Theorem 1.2, Manin's construction, as well as a generalisation of Theorem 1.2, we refer the reader to an extended version of this paper [FvL14].

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2. DEL PEZZO SURFACES OF DEGREE TWO

The statements in this section are well known and we will use them freely. Let X be a del Pezzo surface of degree 2 over a field k with canonical divisor K_X . The Riemann-Roch spaces $\mathcal{L}(-K_X)$ and $\mathcal{L}(-2K_X)$ have dimension 3 and 7, respectively. Let x, y, z be generators of $\mathcal{L}(-K_X)$ and choose an element $w \in \mathcal{L}(-2K_X)$ that is not contained in the image of the natural map $\text{Sym}^2 \mathcal{L}(-K_X) \rightarrow \mathcal{L}(-2K_X)$. Then X embeds into the weighted projective space $\mathbb{P} = \mathbb{P}(1, 1, 1, 2)$ with coordinates x, y, z , and w . We will identify X with its image in \mathbb{P} , which is a smooth surface of degree 4. Conversely, every smooth surface of degree 4 in \mathbb{P} is a del Pezzo surface of degree 2. There are homogeneous polynomials $f, g \in k[x, y, z]$ of degrees 2 and 4, respectively, such that $X \subset \mathbb{P}$ is given by

$$(1) \quad w^2 + fw = g.$$

If the characteristic of k is not 2, then after completing the square on the left-hand side, we may assume $f = 0$. For more details and proofs of these facts, see [Kol96, Section III.3, Theorem III.3.5] and [Man86, Section IV.24].

The restriction to X of the 2-uple embedding $\mathbb{P} \rightarrow \mathbb{P}^6$ corresponds to the complete linear system $|-2K_X|$. Every hyperplane section of $X \subset \mathbb{P}$ is linearly equivalent with $-K_X$. The projection $\mathbb{P} \dashrightarrow \mathbb{P}^2$ onto the first three coordinates restricts to a finite, separable morphism $\pi_X: X \rightarrow \mathbb{P}^2$ of degree 2, which corresponds to the complete linear system $|-K_X|$. This is the anti-canonical morphism mentioned in the introduction.

The morphism π_X is ramified above the branch locus $B_X \subset \mathbb{P}^2$ given by $f^2 + 4g = 0$. If the characteristic of k is not 2, then B_X is a smooth curve. We denote the ramification locus $\pi_X^{-1}(B_X)$ of π_X by R_X . As for every double cover, the morphism π_X induces an involution $\iota_X: X \rightarrow X$ that sends a point $P \in X$ to the unique second point in the fiber $\pi_X^{-1}(\pi_X(P))$, or to P itself if π_X is ramified at P . If X is clear from the context, then we sometimes leave out the subscripts and write π, ι, B , and R for π_X, ι_X, B_X , and R_X , respectively.

3. PROOF OF THE FIRST MAIN THEOREM

Set $k_1 = k_2 = \mathbb{F}_3$ and $k_3 = \mathbb{F}_9$. Let $\gamma \in k_3$ denote an element satisfying $\gamma^2 = \gamma + 1$. Note that γ is not a square in k_3 . For $i \in \{1, 2, 3\}$, we define the surface X_i in $\mathbb{P} = \mathbb{P}(1, 1, 1, 2)$ with

coordinates x, y, z, w over k_i by

$$\begin{aligned} X_1 : -w^2 &= (x^2 + y^2)^2 + y^3z - yz^3, \\ X_2 : -w^2 &= x^4 + y^3z - yz^3, \\ X_3 : \gamma w^2 &= x^4 + y^4 + z^4. \end{aligned}$$

These surfaces are smooth, so they are del Pezzo surfaces of degree 2. C. Salgado, D. Testa, and A. Várilly-Alvarado proved the following result.

Theorem 3.1. *Let X be a del Pezzo surface of degree 2 over a finite field. If X is not isomorphic to X_1, X_2 , and X_3 , then X is unirational.*

Proof. See [STVA14, Theorem 1]. □

We will use the following lemma to prove the complementary statement, namely that X_1, X_2 , and X_3 are unirational as well.

Lemma 3.2. *Let X be a del Pezzo surface of degree 2 over a field k . Suppose that $\rho: \mathbb{P}^1 \rightarrow X$ is a nonconstant morphism; if the characteristic of k is 2 and the image of ρ is contained in the ramification divisor R_X , then assume also that the field k is perfect. Then X is unirational.*

Proof. See [STVA14, Theorem 17]. □

For $i \in \{1, 2, 3\}$, we define a morphism $\rho_i: \mathbb{P}^1 \rightarrow X_i$ by extending the map $\mathbb{A}^1(t) \rightarrow X_i$ given by

$$t \mapsto (x_i(t) : y_i(t) : z_i(t) : w_i(t)),$$

where

$$\begin{aligned} x_1(t) &= t^2(t^2 - 1), & x_2(t) &= t(t^2 + 1)(t^4 - 1), & x_3(t) &= (t^4 + 1)(t^2 - \gamma^3), \\ y_1(t) &= t^2(t^2 - 1)^2, & y_2(t) &= -t^4, & y_3(t) &= (t^4 - 1)(t^2 + \gamma^3), \\ z_1(t) &= t^8 - t^2 + 1, & z_2(t) &= t^8 + 1, & z_3(t) &= (t^4 + \gamma^2)(t^2 - \gamma), \\ w_1(t) &= t(t^2 - 1)(t^4 + 1)(t^8 + 1), & w_2(t) &= t^2(t^2 + 1)(t^{10} - 1), & w_3(t) &= \gamma^2 t(t^8 - 1)(t^2 + \gamma). \end{aligned}$$

It is easy to check for each i that the morphism ρ_i is well defined, that is, the polynomials x_i, y_i, z_i , and w_i satisfy the equation of X_i , and that ρ_i is non-constant.

Theorem 3.3. *The del Pezzo surfaces X_1, X_2 , and X_3 are unirational.*

Proof. By Lemma 3.2, the existence of ρ_1, ρ_2 , and ρ_3 implies that X_1, X_2 , and X_3 are unirational. □

Proof of Theorem 1.1. This follows from Theorems 3.1 and 3.3. □

4. PROOF OF THE SECOND MAIN THEOREM

If C is a plane curve with an ordinary singularity Q and \tilde{C} is the normalisation of C , then we can think of the points of \tilde{C} above Q as corresponding with the branches of C through Q . The intersection multiplicity of C with another plane curve B at Q is then the sum of the intersection multiplicities of B with all the branches of C through Q . This point of view is used in the following proof. For more technical details about this approach, see the extended version of this paper [FvL14].

Proof of Theorem 1.2. Let $\iota: X \rightarrow X$ denote the involution associated to the double cover π . Set $Q = \pi(P)$. Projection away from the point $Q \in C \subset \mathbb{P}^2$ yields a birational map $C \dashrightarrow \mathbb{P}^1$ whose inverse $\vartheta: \mathbb{P}^1 \rightarrow C$ can be identified with the normalisation map of C . The map ϑ restricts to an isomorphism $\mathbb{P}^1 \setminus \vartheta^{-1}(Q) \rightarrow C \setminus \{Q\}$, and C is smooth away from Q . Let $D = \pi^{-1}(C)$ be the inverse image of C under π , and let \tilde{D} be its normalisation. Then π induces a double cover $\tilde{\pi}: \tilde{D} \rightarrow \mathbb{P}^1$.

Let $S \in \mathbb{P}^1$ be a point and set $T = \vartheta(S) \in C$. The curve B is given locally around T by the vanishing of a rational function on \mathbb{P}^2 that is regular at T . We let h denote the image of such a function in the local ring $\mathcal{O}_{C,T}$ of T in C .

If $T \neq Q$, then T is a smooth point of C , so the ring $\mathcal{O}_{C,T} = \mathcal{O}_{\mathbb{P}^1,S}$ is a discrete valuation ring. In this case, the valuation of h equals the intersection multiplicity of B and C at T , which is even. Since the characteristic of k is not 2, this implies that adjoining a square root of h to $\mathcal{O}_{C,T}$ yields an unramified extension, so the morphism $\tilde{\pi}: \tilde{D} \rightarrow \mathbb{P}^1$ is not ramified above S when $T \neq Q$.

Suppose that Q is not contained in B . Then for $T = Q$, the element h is a unit in the local ring $\mathcal{O}_{C,T}$, and therefore also in the ring extension $\mathcal{O}_{\mathbb{P}^1,S}$. Hence, as before, since the characteristic of k is not 2, this implies that the morphism $\tilde{\pi}$ is not ramified above S . This means that $\tilde{\pi}$ is unramified. Since $\mathbb{P}_{\bar{k}}^1$ has no nontrivial unramified covers, this means that the curve \tilde{D} , and hence the curve $D \subset X$, splits into two components over some quadratic extension ℓ of k . Exactly one of the components of D contains the rational point P and the other component contains $\iota(P)$. This implies that the Galois group $\text{Gal}(\ell/k)$ sends each component to itself, so these components are defined over k . Each maps isomorphically to C , so X contains a curve that is birationally equivalent to \mathbb{P}^1 and therefore X is unirational by Lemma 3.2. This proves (1).

Suppose that Q is contained in B and that it is an ordinary singular point on C . Then $\vartheta^{-1}(Q)$ consists of exactly $d - 1$ points over \bar{k} , each corresponding to the tangent direction of one of the $d - 1$ branches of C at Q . At most one of the tangent directions is tangent to B , so at least $d - 2$ of the branches intersect B with multiplicity 1. The total intersection multiplicity of B and C at Q is even. If d is odd, then the contribution $(d - 2) \cdot 1$ of the $d - 2$ branches with intersection multiplicity 1 is odd, so the last branch intersects B with odd multiplicity as well; hence all $d - 1$ branches intersect B with odd multiplicity, which implies that $\tilde{\pi}: \tilde{D} \rightarrow \mathbb{P}^1$ is ramified above all $d - 1$ points above Q . If d is even, then the contribution of the $d - 2$ branches of C that intersect B with multiplicity 1 is even as well, so the last branch intersects B with even multiplicity; as before, this means that $\tilde{\pi}$ is not ramified above the point in $\vartheta^{-1}(Q) \subset \mathbb{P}^1$ that corresponds to this last branch, so $\tilde{\pi}$ is ramified above exactly $d - 2$ of the $d - 1$ points above Q . For $d \in \{3, 4\}$, these two cases (d odd or even) imply that the map $\tilde{\pi}: \tilde{D} \rightarrow \mathbb{P}^1$ is ramified at exactly two points, so \tilde{D} is a geometrically integral curve of genus 0 by the theorem of Riemann-Hurwitz. Indeed, this implies that there is a field extension ℓ of k of degree at most 2 for which \tilde{D}_ℓ , and thus $D_\ell = \pi^{-1}(C_\ell)$, is birationally equivalent with \mathbb{P}_ℓ^1 . For each such field, the surface X_ℓ is unirational by Lemma 3.2. This proves (2). \square

Remark 4.1. Let the surfaces X_1, X_2, X_3 and the morphisms ρ_1, ρ_2, ρ_3 be as in the previous section. Take any $i \in \{1, 2, 3\}$. Set $A_i = \rho_i(\mathbb{P}^1)$ and $C_i = \pi_i(A_i)$, where $\pi_i = \pi_{X_i}: X_i \rightarrow \mathbb{P}^2$ is as described in the previous section. By Remark 2 of [STVA14], the surface X_i is minimal, and the Picard group $\text{Pic } X_i$ is generated by the class of the anticanonical divisor $-K_{X_i}$. The same remark states that the linear system $|-nK_{X_i}|$ does not contain a geometrically integral curve of geometric genus zero for $n \leq 3$ if $i \in \{1, 2\}$, nor for $n \leq 2$ if $i = 3$. For $i \in \{1, 2\}$, the curve A_i has degree 8, so it is contained in the linear system $|-4K_{X_i}|$. The curve A_3 has degree 6, so it is contained in the linear system $|-3K_{X_3}|$. This means that for $i \in \{1, 2, 3\}$, the curve A_i has minimal degree among all rational curves on X_i . The restriction of π_i to A_i is a double cover $A_i \rightarrow C_i$. The curve $C_i \subset \mathbb{P}^2$ has degree 4 for $i \in \{1, 2\}$ and degree 3 for $i = 3$, and C_i is given by the vanishing of h_i , with

$$\begin{aligned} h_1 &= x^4 + xy^3 + y^4 - x^2yz - xy^2z, \\ h_2 &= x^4 - x^2y^2 - y^4 + x^2yz + yz^3, \\ h_3 &= x^2y + xy^2 + x^2z - xyz + y^2z - xz^2 - yz^2 - z^3. \end{aligned}$$

For $i \in \{1, 2\}$, the curve C_i has an ordinary triple point Q_i , with $Q_1 = (0 : 0 : 1)$, $Q_2 = (0 : 1 : 1)$. The curve C_3 has an ordinary double point at $Q_3 = (1 : 1 : 1)$. For all i , the point Q_i lies on the branch locus $B_i = B_{X_i}$.

Using the polynomial h_i , one can check that the curve C_i intersects the branch locus B_i with even multiplicity everywhere. In fact, had we defined C_i by the vanishing of h_i , then one would

easily check that C_i satisfies the conditions of part (2) of Theorem 1.2. This gives an alternative proof of unirationality of X_i without the need of the explicit morphism ρ_i ; here we may use the fact that if k is a finite field, then any curve that becomes birationally equivalent to \mathbb{P}^1 over an extension of k , already is birationally equivalent with \mathbb{P}^1 over k . Indeed, in practice we first found the curves C_1 , C_2 , and C_3 , and then constructed the parametrisations ρ_1, ρ_2, ρ_3 , which allow for the more direct proof that we gave of Theorem 3.3 in the previous section.

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