Conformal field theories and deep inelastic scattering

Zohar Komargodski,[1] Manuela Kulaxizi,[2] Andrei Parnachev,[2,3] and Alexander Zhiboedov[4]

[1]*Weizmann Institute of Science, Rehovot 76100, Israel*
[2]*School of Mathematics, Trinity College Dublin, Dublin D2, Ireland*
[3]*Institute Lorentz, Leiden University, Leiden 2300RA, The Netherlands*
[4]*Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA*

We consider deep inelastic scattering thought experiments in unitary conformal field theories. We explore the implications of the standard dispersion relations for the operator product expansion data. We derive positivity constraints on the operator product expansion coefficients of minimal-twist operators of even spin $s \geq 2$. In the case of $s = 2$, when the leading-twist operator is the stress tensor, we reproduce the Hofman-Maldacena bounds. For $s > 2$, the bounds are new.

I. INTRODUCTION AND SUMMARY

Conformal field theories (CFTs) in d spacetime dimensions are described first and foremost by correlation functions of local operators. The operator product expansion (OPE) fixes these in terms of the spectrum of local operators and their three-point functions. Conformal symmetry determines the three-point functions up to a set of numbers. The spectrum of unitary CFTs is constrained by unitarity bounds, which follow from the operator-state correspondence and the requirement that states have positive norm [1–3]. There are, however, less obvious bounds coming from, for example, positivity of energy correlators [4], deep inelastic scattering sum rules [5,6], and causality [7].

In the case of energy correlators [4], one demands positivity of the energy flux at infinity integrated over all times. In the simplest case of a state created by a local operator with a given momentum, this leads to new constraints on the three-point functions of operators with spin and the stress-energy tensor of the type $\langle \mathcal{O}^{\dagger}_{\mu_1...\mu_s} T_{\mu\nu} \mathcal{O}_{\nu_1...\nu_s} \rangle$. The positivity of the integrated energy flux is a plausible assumption, but one may wonder whether there is an independent argument for it. There have been a couple of proposals in the literature. In Ref. [8], the energy flux positivity has been derived from nontrivial assumptions about the OPE and the spectrum of nonlocal operators. Understanding the properties of these nonlocal operators and their OPE in unitary CFTs is an open problem. Another proposal has been put forward in Ref. [9], where the OPE of two stress-energy tensors has been extrapolated beyond the region of its validity to argue the energy flux positivity.

In Refs. [5,6], it was shown that by considering a setup where a particle with spin is scattered off a massive state, one can relate (using the optical theorem) the positivity of the inclusive cross section (unitarity requires the cross section to be positive) with the OPE data, thereby placing constraints on the latter.[1] This leads to the convexity property of the minimal-twist operators which appear in the OPE of two Hermitian-conjugate operators. In this paper, we use a similar deep inelastic scattering (DIS) setup to derive the positivity of the energy flux and related constraints on the OPE data for operators with spin. The idea of using DIS together with scale invariance is not new—for example, see Ref. [10]. We also discuss how to formulate the DIS experiment purely in a CFT without considering a flow to a gapped phase.

The results of our paper can be summarized as positivity constraints on the coefficients of the operator product expansion[2]

$$\mathcal{O}^{\dagger}_j \mathcal{O}_j \sim \sum_m a^*_{s,m} \mathcal{O}_{\tau^*,s} + \cdots, \qquad (1.1)$$

where $\mathcal{O}_{\tau^*,s}$ is the minimal-twist operator of even spin s (the twist is defined as $\tau = \Delta - s$ where Δ is the conformal dimension and s is the spin), the index m refers to the different tensor structures which appear in the DIS sum rules, and the dots stand for the contribution of higher-twist operators. Then, the coefficients $a^*_{s,m}$ satisfy the following conditions:

$$a^*_{2,m} \geq 0, \qquad m = 0, ..., j,$$
$$a^*_{s,m_1} a^*_{s,m_2} \geq 0, \qquad m_1, m_2 = 0, ..., j. \qquad (1.2)$$

---

[1]The assumption in Ref. [6] involves the existence of a relevant operator which induces a renormalization group flow terminating in a gapped phase; the scattering experiment involves the lightest particle in that gapped theory. Here, we will argue that this additional structure is not necessary.

[2]Three-point functions of operators with spin were analyzed in Ref. [11]. $a^*_{s,m}$ are certain linear combinations of the structures from that paper as explained in detail in the main body of the present paper.

In the case of $s = 2$, the bounds above (1.2) are the familiar Hofman-Maldacena bounds [4,12] because the minimal-twist spin-2 operator is always a stress tensor.

In fact, we can obtain more general bounds by considering a four-point function of the type

$$\langle \mathcal{O}_j^\dagger \mathcal{O}_j \tilde{\mathcal{O}}_{\tilde{j}}^\dagger \tilde{\mathcal{O}}_{\tilde{j}} \rangle. \qquad (1.3)$$

As before, we denote the minimal-twist operators which appear in the expansion of the correlator (1.3) in the s-channel by $\mathcal{O}_{\tau^*,s}$ and the corresponding OPE coefficients by $a_{s,m}^*$ and $\tilde{a}_{s,m}^*$. Then, a more general set of constraints derived from DIS can be formulated as

$$a_{s,m_1}^* \tilde{a}_{s,m_2}^* \geq 0, \qquad m_1 = 0, ..., j, \qquad m_2 = 0, ..., \tilde{j}. \qquad (1.4)$$

In the course of deriving the DIS sum rules, we *assumed* a certain behavior for the scattering amplitudes in the Regge limit. This translates into the lowest-spin s_c for which we can trust the sum rules. Thus, strictly speaking, our argument implies (1.4) only for $s \geq s_c$, where s_c is some unknown number which depends both on the theory and the external operators. However, when the external operators are energy-momentum tensors, then there is evidence that $s_c = 2$. We will discuss this point in detail below.

The ideas explained here have various applications for holographic theories, but we will pursue them elsewhere [13]. One simple example that makes contact with Refs. [7,14] is to consider a quartic scalar coupling in the bulk $\sim \lambda (\partial \varphi)^4$. Denote the operator dual to φ by $\mathcal{O}$. The interaction $\sim \lambda (\partial \varphi)^4$ shifts the dimension of the spin-2 operator $\mathcal{O} \partial_\mu \partial_\nu \mathcal{O}$ to $2 + 2\Delta_\mathcal{O} - \lambda$. Therefore, convexity is obeyed only if $\lambda > 0$, as demanded by causality in the bulk [15].

The rest of the paper is organized as follows. In Sec. II, we consider the DIS experiment with gravitons and derive constraints on the OPE coefficients of two stress tensors in a unitary CFT. In Sec. III, we generalize these considerations to the case of generic operators. In Sec. IV, we derive a relation between the bounds obtained from the positivity of the energy flux and the DIS experiment. In Sec. V, we comment on how one can set up the DIS experiment without flowing to the gapped phase. Many technical details are collected in the Appendixes.

II. DEEP INELASTIC SCATERING

Deep inelastic scattering probes the internal structure of matter. The scattering process consists of bombarding a target with a highly energetic quantum and examining the final state. DIS was first used to probe the structure of hadronic particles. The setup is depicted in Fig. 1. A lepton emits a virtual photon which strikes a hadron. In principle, to investigate the structure of the target $|P\rangle$, one may shoot
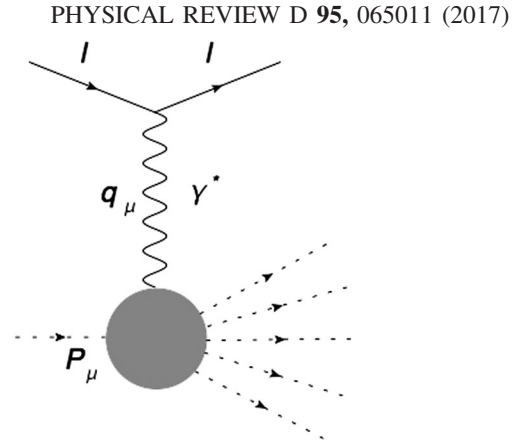

FIG. 1. A lepton emits a virtual photon which strikes a hadron. The hadron breaks up into a complicated final state.

different particles at it. A natural choice is particles which couple to conserved currents. The options depend on the theory, and the symmetries it preserves. A universal choice to consider is the graviton. We can couple the stress-energy tensor of the theory to the background graviton and perform the DIS experiment. More generally, we can couple a source to any operator of the theory.

We also have to specify the state $|P\rangle$. For that, we imagine that our theory is gapped, and we denote with $|P\rangle$ the lightest, massive, one-particle state in the system which we assume to be a scalar.

In the standard treatment of DIS, one can relate the deep Euclidean (i.e., ultraviolet) data to the positive-definite total cross section using dispersion relations. While our presentation is aimed at being self-contained, one can consult, for instance, the reviews [16,17].

It was already demonstrated in Refs. [5,6] that the ideas of DIS can lead to nontrivial consequences for unitary CFTs. There, it was argued that the minimal twist of operators which appear in the OPE of Hermitian conjugate operators is a monotonic, nonconcave function of spin starting from some $s \geq s_c$.

In what follows, we will discuss the DIS experiment with gravitons and restrict to the case of a scalar target $|P\rangle$. Later, we will argue that it is not necessary to make this series of assumptions. Meanwhile, we make these assumptions in order to simplify the presentation.

A. DIS experiment with gravitons

Let us consider the DIS experiment for the case of the stress-energy tensor operator $T_{\mu\nu}(x)$. A background graviton $\delta g_{\mu\nu}(x)$ couples to the theory via $\sim \int d^d x T^{\mu\nu}(x) \delta g_{\mu\nu}(x)$. We imagine that some physical particle emits an off-shell graviton which strikes a state of the theory. So we have in mind the setup of Fig. 1, only with the photon replaced by the graviton.

A useful intermediate object to consider is the "DIS amplitude." For that, we imagine an exclusive process,
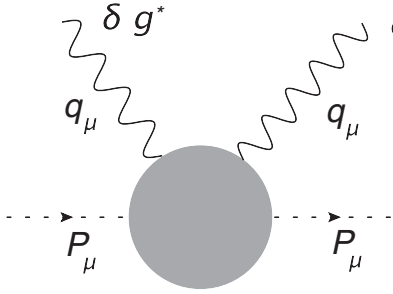
FIG. 2.   The deep inelastic scattering amplitude. δg^* stands for a virtual graviton with momentum q_μ.

FIG. 3.   The analytic structure in the x plane.

where the graviton strikes the state $|P\rangle$ and the out states are again a graviton (with the same polarization and momentum) and the same initial state, $\langle P|$. This is depicted in Fig. 2.

The amplitude for the "graviton-DIS" process depicted in Fig. 2 is given by

$$\mathcal{A}(q_\mu, P_\nu) = \int d^4 y e^{-iqy} \langle P|\mathcal{T}(T(\epsilon^\star, y)T(\epsilon, 0))|P\rangle, \quad (2.1)$$

where the momentum of the target $|P\rangle$ is denoted by P_μ; $T(\epsilon, y) \equiv T_{\mu\nu}(y)\epsilon^{\mu\nu}$, and $\epsilon^{\mu\nu}$ is a polarization tensor ($\epsilon^\star$ is the conjugate polarization tensor). We can shift $\epsilon^{\mu\nu} \to \epsilon^{\mu\nu} + q^\nu l^\mu + q^\mu l^\nu$ with arbitrary l^ν. This would not affect the two-point function in the vacuum because of energy-momentum conservation. But here we are dealing with a two-point function in a nontrivial state so contact terms may contribute. We therefore do not impose $\epsilon.q = 0$. However, note that if we were to take $\epsilon^{\mu\nu} \sim \eta^{\mu\nu}$, then we would be studying the scattering of the conformal mode of the metric, i.e., the dilaton. These scattering amplitudes are suppressed at large q because the trace of the energy-momentum tensor vanishes in a conformal field theory. We therefore take the tensor $\epsilon^{\mu\nu}$ to be traceless.

We imagine a general massive (nonconformal, gapped) theory, in which the lightest state is $|P\rangle$. The above amplitude depends on the mass scales of the theory, on the polarization $\epsilon^{\mu\nu}$, and on two kinematical invariants, i.e., q^2 and $x = \frac{q^2}{2q.P}$. We promote x to a complex variable and study the amplitude for fixed *spacelike* momentum $q^2 > 0$. Since $|P\rangle$ is the lightest particle of the theory (which we assume to be a scalar for simplicity), the above amplitude will have a branch cut discontinuity for $-1 \leq x \leq 1$, as depicted in Fig. 3. The optical theorem relates the discontinuity across the cut in the x plane to the square of the forward amplitude, which is positive definite.

For large (compared to the mass scale) and spacelike $q^2 > 0$, we can compute the DIS amplitude (2.1) with the help of the OPE, which is determined in the ultraviolet conformal field theory. The resulting expression is a series expansion around $x \to \infty$, valid for fixed and large $q^2 > 0$.
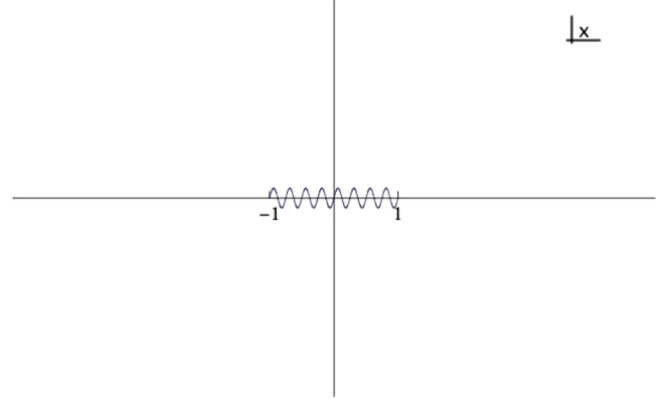
To isolate the coefficient of the sth power of x in the expansion, one computes the "s-moment" defined as $\mu_s(q^2) = \oint dx x^{s-1} \mathcal{A}(x, q^2)$. As long as the amplitude vanishes sufficiently quickly for small x, we can pull the contour from infinity to the branch cut and write

$$\oint dx x^{s-1} \mathcal{A}(x, q^2) = 2\int_0^1 dx x^{s-1} \text{Im}[\mathcal{A}(x, q^2)], \quad (2.2)$$

where we assumed that s is even. For odd s, the contribution from the left and the right cuts cancel each other.

For (2.2) to be valid for all $s \geq 2$, we need to assume that

$$\lim_{x\to 0} \mathcal{A}(q^2, x) < x^{-2}. \quad (2.3)$$

In general, we only know that $\mathcal{A}$ is bounded by some x^{-N} in this limit. (This is discussed in Ref. [18]; for a recent discussion and references, see also Ref. [19].) However, there are some pieces of evidence that (2.3) indeed holds for graviton deep inelastic scattering. One is that the convexity theorems derived from it in Ref. [6] hold in all known examples. The other piece of evidence is that, as we will show below, by assuming (2.3), we get precisely the bounds of Ref. [4] if we focus on $s = 2$. We will therefore take (2.3) as an assumption in this section and revisit it in the next section when we discuss more general DIS gedanken experiments.[3]

Unitarity implies

$$\text{Im}[\mathcal{A}(x, q^2)] \geq 0, \quad (2.4)$$

which leads, via (2.2), to

---

[3]As mentioned in the Introduction, the bounds following from $s = 2$ constrain the allowed effective theories in anti–de Sitter. For example, the bound on the sign of λ in the $\lambda(\partial\phi)^4$ theory in anti-de Sitter recently discussed in Refs. [7,14] immediately follows from the convexity of anomalous dimensions, assuming the $s = 2$ sum rule converges.

$$\oint dx x^{s-1} \mathcal{A}(x, q^2) \geq 0, \qquad (2.5)$$

imposing positivity relations on the coefficients of the OPE. These constraints are in addition to the nonconcavity of the minimal-twist function.

B. OPE in DIS kinematics

Our objective is to evaluate (2.1) with the help of the OPE and investigate the positivity constraints one obtains from (2.5). We start with the operator product expansion for two energy-momentum tensors,

$$T(\epsilon^\star, y) T(\epsilon, 0) = \sum_{s=0,2,4\ldots} \sum_\alpha \hat{f}_s^{(\alpha),\mu_1\ldots\mu_s}(y, \epsilon, \epsilon^\star) \mathcal{O}_{\mu_1\ldots\mu_s}^{(\alpha)}(0),$$

$$(2.6)$$

where s denotes the spin of the operator and α labels operators of the same spin. Actually, there could be operators in other representations in (2.6), for example, operators in mixed symmetric-antisymmetric representations (see, e.g., Ref. [20]). Since we are ultimately interested in using the OPE to evaluate (2.1), we can

ignore the representations which have some of their indices antisymmetrized because the corresponding expectation values in the (scalar) state $|P\rangle$ vanish. For a similar reason, we do not include descendants in (2.6); they give a vanishing contribution because $\partial_\mu \langle P | \mathcal{O}(x) | P \rangle = 0$.

For the operators $\mathcal{O}_{\mu_1\mu_2\ldots\mu_s}^{(\alpha)}$ in the even s symmetric traceless representation which appear in the OPE (2.6), the expectation values of $\mathcal{O}_{\mu_1\mu_2\ldots\mu_s}^{(\alpha)}$ in the state $|P\rangle$ are parametrized as

$$\langle P | \mathcal{O}_{\mu_1\mu_2\ldots\mu_s}^{(\alpha)}(0) | P \rangle = B_s^{(\alpha)} P_{\mu_1} P_{\mu_2} \ldots P_{\mu_s} - \cdots, \qquad (2.7)$$

where the $B_s^{(\alpha)}$ are some dimensionful coefficients and the dots stand for trace terms (terms involving the metric tensor), which we will not need to specify. For example, in the case of the stress tensor expectation value in a one-particle state, we famously have [21][4]

$$\langle P | T_{\mu_1\mu_2}(0) | P \rangle = P_{\mu_1} P_{\mu_2}. \qquad (2.8)$$

Therefore, the corresponding coefficient B_T is determined to be 1.

Conformal symmetry fixes the form of the leading OPE coefficients for small enough y to be[5]

$$\hat{f}_{\mu_1\ldots\mu_s}^{(\alpha)}(\epsilon^*, \epsilon, y) = y^{(\tau_s^{(\alpha)} - 2d)} [\hat{a}_{s,0}^{(\alpha)}(\epsilon^\star.\epsilon) y_{\mu_1} \ldots y_{\mu_s} + \hat{a}_{s,1}^{(\alpha)}(\epsilon^\star.\epsilon)^{\lambda\kappa} y_\lambda y_\kappa y_{\mu_1} \ldots y_{\mu_s}(y^2)^{-1}$$

$$+ \hat{a}_{s,2}^{(\alpha)}((\epsilon^\star)^{\kappa_1\kappa_2} y_{\kappa_1} y_{\kappa_2})(\epsilon^{\lambda_1\lambda_2} y_{\lambda_1} y_{\lambda_2}) y_{\mu_1} \ldots y_{\mu_s}(y^2)^{-2} + \cdots], \qquad (2.9)$$

where the dots denote terms which contain polarization tensors with noncontracted indices as well as terms subleading in powers of y. Both of these will turn out to be subleading in the kinematics we are considering.

We now substitute (2.9) in (2.1) and take the Fourier transform, leading to

$$\mathcal{A}(q_\mu, P_\mu) = \sum_{s=0,2,4,\ldots} \sum_\alpha (\epsilon^\star.\epsilon) \left(\hat{a}_{s,0}^{(\alpha)} B_s^{(\alpha)} \left(i \frac{\partial}{\partial q}.P \right)^s - \text{traces} \right) f_{s,0}^{(\alpha)}(q)$$

$$+ (\epsilon^\star.\epsilon)^{\lambda\kappa} \left(i \frac{\partial}{\partial q^\lambda} \right) \left(i \frac{\partial}{\partial q^\kappa} \right) \left(\hat{a}_{s,1}^{(\alpha)} B_s^{(\alpha)} \left(i \frac{\partial}{\partial q}.P \right)^s - \text{traces} \right) f_{s,1}^{(\alpha)}(q)$$

$$+ \left(\frac{\partial}{\partial q^{\lambda_1}} \frac{\partial}{\partial q^{\lambda_2}} (\epsilon^\star)^{\lambda_1\lambda_2} \right) \left(\frac{\partial}{\partial q^{\kappa_1}} \frac{\partial}{\partial q^{\kappa_2}} \epsilon^{\kappa_1\kappa_2} \right) \left(\hat{a}_{s,2}^{(\alpha)} B_s^{(\alpha)} \left(i \frac{\partial}{\partial q}.P \right)^s - \text{traces} \right) f_{s,2}^{(\alpha)}(q) + \cdots \qquad (2.10)$$

Here, the functions $f_{s,m}^{(\alpha)}(q)$ are Fourier transformations of the "Feynman" propagators, defined as follows,

$$f_{s,m}^{(\alpha)}(q) = \int d^d y e^{-iqy} (y^2 + i\varepsilon)^{\frac{1}{2}\tau_s^{(\alpha)} - d - m}, \qquad (2.11)$$

and "traces" stands for terms of the form $P^{2n} (P. \frac{\partial}{\partial q})^{s-2n} (\frac{\partial}{\partial q}. \frac{\partial}{\partial q})^n$ and $2 \leq 2n \leq s$. We will soon see that these terms are negligible in the limit we consider.

---

[4]We normalize the one-particle states as follows: $\langle P' | P \rangle = (2\pi)^{d-1} E_{\vec{P}} \delta^{(d-1)}(\vec{P}' - \vec{P})$.
[5]Here, $\epsilon^\star.\epsilon = \epsilon_{\alpha\beta}^\star \epsilon^{\alpha\beta}$ and $(\epsilon^\star.\epsilon)^{\lambda\kappa} = \epsilon_\alpha^{\star\lambda} \epsilon^{\kappa\alpha}$.

At this point, it is convenient to express the amplitude in terms of the kinematical invariants, $q^2, x \equiv \frac{q^2}{2P \cdot q}$. We are interested in the regime of large spacelike $q^2 > 0$, but we work to all orders in x. Therefore, for a given power of x, we keep only the leading terms in the limit $q^2 \to \infty$. We obtain

$$
\begin{aligned}
\mathcal{A}(q^2, x) &= \sum_s (q^2)^{-\tau_{s,0}^*/2+d/2} C_{s,0}^* x^{-s} (\epsilon^\star . \epsilon)^2 \\
&+ \sum_s (q^2)^{-\tau_{s,1}^*/2+d/2} C_{s,1}^* x^{-s} \frac{(\epsilon^\star . \epsilon)^{\lambda\kappa} q_\lambda q_\kappa}{q^2} \\
&+ \sum_s (q^2)^{-\tau_{s,2}^*/2+d/2} C_{s,2}^* x^{-s} \\
&\times \frac{(\epsilon^\star)^{\lambda_1\lambda_2} q_{\lambda_1} q_{\lambda_2} \epsilon^{\kappa_1\kappa_2} q_{\kappa_1} q_{\kappa_2}}{(q^2)^2},
\end{aligned} \tag{2.12}
$$

where $\tau_{s,i}^*$ denotes the twist of the minimal-twist operator which contributes to the corresponding polarization tensor structure. *A priori* we do not have to impose $\tau_{s,i}^* = \tau_s^*$, but generically we do expect this to be the case since there is no symmetry principle that sets some of the tensor structures to zero. Below, we assume that

$$
\tau_{s,m}^* = \tau_s^*, \tag{2.13}
$$

unless stated otherwise.[6]

The trace terms have been consistently neglected by invoking the monotonicity of the twists [6]. Similarly, one can verify that terms containing $\epsilon . P$ are irrelevant for our consideration[7] Among the set of operators of a given spin s, only the one with the smallest twist, τ_s^*, has been retained in (2.12). The corresponding coefficients, $C_{s,m}^*$, are given by

$$
C_{s,m}^* = 2^{\tau_s^*-d-2m} \pi^{\frac{d}{2}} \frac{\Gamma(\frac{\tau_s^*-d}{2}+m+s)}{\Gamma(m+\frac{2d-\tau_s^*}{2})} B_s^* a_{s,m}^*, \tag{2.14}
$$

which can be derived using the Fourier transform (2.11). Explicit expressions for the $a_{s,m}^*$ in terms of the $\hat{a}_{s,m}^*$ which appear in (2.10) are given in Appendix A.

---

[6]One may worry that the large momentum limit of the DIS amplitude is not correctly captured by the Fourier transform of the OPE [22,23]. We expect this issue not to be relevant here, because the terms which dominate over the Fourier transform of the OPE in the large momentum limit come from "semilocal terms" [23] in position space. It would be interesting to show that this is indeed the case.

[7]Both $\epsilon . P$ and trace terms behave like x^{m-s} with $m > 0$ for small x. As a result, they only contribute to (2.5) for spins $s' < s$. Their contribution behaves like $q^{-2\tau_s^*}$ (or higher power) in the large q^2 limit. However, the fact that the twist is a monotonically increasing function of the spin [6], namely that $\tau_s > \tau_{s'}$, implies that it is subleading compared to the contribution $q^{-2\tau_{s'}^*}$ coming from the leading-twist operator in the s' sector.

As long as we can deform the contour in the complex plane as explained above (2.2), we can substitute (2.12) into (2.5) to obtain various positivity relations as required by unitarity, i.e.,

$$
C_{s,m}^* \geq 0, \qquad m = 0, 1, 2. \tag{2.15}
$$

These three inequalities for each spin are achieved by judicious choices of the polarization tensor. First, we choose a convenient reference frame for the spacelike momentum $q^\mu = (0, 0, ..., 0, k)$. We then organize the polarization tensor $\epsilon_{\mu\nu}$ according to its properties under the subgroup of rotations which leave q^μ invariant. There are three possibilities, and each produces a single constraint:

(i) We can take $\epsilon_{01} = \epsilon_{10} = 1$ and let all the other components vanish. Then, only the first line in (2.12) remains.

(ii) We take $\epsilon_{01} = \epsilon_{10} = \epsilon_{1(d-1)} = \epsilon_{(d-1)1} = 1$, and all the other components vanish. Only the second line in (2.12) remains nonzero.

(iii) We take $\epsilon_{00} = \epsilon_{(d-1)(d-1)} = \epsilon_{0(d-1)} = \epsilon_{(d-1)0} = 1$ with the rest of the components set to zero. In this case, only the last line in (2.12) is nonvanishing.

It is instructive to consider in detail the case $s = 2$. In this case, the operator of the smallest twist is none other but the stress-energy tensor. Unitarity sets a lower bound on the twist of all spin-s operators $\tau_s \geq d - 2$ (and when the inequality is saturated, we get a conserved current) [1,3]. Hence, the energy-momentum tensor is the minimal-twist operator with $s = 2$ unless the theory has more than one conserved spin-2 current. For the energy-momentum tensor, we know from (2.8) that $B_T = 1$. It follows that (2.15) directly imposes bounds on the OPE coefficients of the CFT.

Remarkably, these bounds coincide with the energy flux constraints obtained in Ref. [4]. To make this explicit, we should relate the $a_{T,m}$ to the independent OPE coefficients of $TT \sim T$ using the formalism of Refs. [11,24]. A similar computation in $d = 4$ was done in Ref. [9]. For generic d, we get[8]

$$
a_{T,0} = -\frac{d(2b-c) + a(d^2 + 4d - 4)}{4(-2b - c(1+d) + a(-6 + d + d^2))} \sim n_v \geq 0,
$$

$$
a_{T,1} = \frac{1}{8}\frac{a(d^2 + 6d - 8) - b(2 - 3d) - 2dc}{(-2b - c(1+d) + a(d^2 + d - 6))} \sim n_f \geq 0,
$$

$$
a_{T,2} = -\frac{1}{32}\frac{d(d-2)(4a + 2b - c)}{(a(d^2 + d - 6) - 2b - c(1+d))} \sim n_s \geq 0,
$$

$$\tag{2.16}$$

---

[8]See Appendix B for details on the derivation of these constraints.

where (a, b, c) denote the parameters which determine the three-point function of the stress-energy tensor in the notations of Ref. [24] and (n_s, n_f, n_v) in the basis of structures generated by free field theories [12]. Equation (2.16) holds in any $d \geq 4$. In $d = 4$, it yields the familiar expressions in four dimensions [4].

In $d = 3$ dimensions, there are only two independent conformal free theories (those of free scalars and of free fermions), and the number of independent parameters in the three-point function of the stress-energy tensor is accordingly reduced to two. In this case, explicit computation leads to two constraints $n_s \geq 0$, $n_f \geq 0$.[9]

Deep inelastic scattering allows for a clean separation between infrared physics and ultraviolet physics. This is a key ingredient in our arguments. In Ref. [9], an attempt to use the OPE beyond its regime of validity has been discussed. We are circumventing this conceptual difficulty by the DIS analysis, which relates ultraviolet and infrared data by a contour argument.

Let us now discuss the case of higher spins, $s > 2$. As explained in Refs. [6,27,28], in this case, $d - 2 \leq \tau_s^* < 2(d - 2)$, and because of this, the ratio of gamma functions that appears in $C_{s,m}^*$ is positive definite. For spins $s > 2$, we do not know the sign of B_s^*, but we can still get some mileage out the constraints above since B_s^* does not depend on m. Assuming that for the minimal-twist operator $B_s^* \neq 0$, we get an infinite set of new bounds for unitary CFTs,

$$a_{s,m_1}^* a_{s,m_2}^* \geq 0. \tag{2.17}$$

This product appears naturally in the OPE of the four-point function of stress-energy tensors. For this reason, it seems reasonable to hope that the prediction (2.17) can be tested in future studies of the conformal bootstrap for operators with spin.

III. DEEP INELASTIC SCATTERING FOR GENERIC OPERATORS

In the previous section, we considered the DIS of gravitons which couple to the stress tensor $T_{\mu\nu}$. One can naturally generalize this to any source that couples to some operator $\mathcal{O}_j(\epsilon, x)$,[10] which is a symmetric, traceless CFT operator of spin j and conformal dimension $\Delta_{\mathcal{O}}$ that satisfies the unitarity bound $\Delta_{\mathcal{O}} - j \geq d - 2$.

For a generic external operator $\mathcal{O}_j(\epsilon, x)$, we do not commit on the rate of the decay of the amplitude for small x (except that it is bounded by some power). We will be more precise about this issue below.

The DIS amplitude of interest is

$$\mathcal{A}(q, P) = \int d^d y \, e^{-iqy} \langle P | \mathcal{T}(\mathcal{O}_j^\dagger(\epsilon^\star, y) \mathcal{O}_j(\epsilon, 0)) | P \rangle. \tag{3.1}$$

It is convenient to choose the polarization tensor as follows [11],

$$\epsilon^{a_1 \dots a_j} = \epsilon^{a_1} \dots \epsilon^{a_j}, \tag{3.2}$$

where $\epsilon^2 = 0$. So, we consider

$$\mathcal{O}_j(\epsilon, y) \equiv \mathcal{O}_{a_1 \dots a_j}(y) \epsilon^{a_1} \dots \epsilon^{a_j}. \tag{3.3}$$

By a straightforward generalization of the previous analysis, we obtain an expression for the DIS amplitude (3.1) in the limit of large q^2,

$$\mathcal{A}(q^2, x) = \sum_s (q^2)^{-\tau^*/2 + \Delta_{\mathcal{O}} - d/2} x^{-s}$$

$$\times \sum_{m=0}^{j} C_{s,m}^* (\epsilon^* . \epsilon)^{j-m} \frac{(\epsilon^* . q)^m (\epsilon . q)^m}{(q^2)^m}, \tag{3.4}$$

where the constants are defined as $C_{s,m}^* \propto a_{s,m}^* B_s^*$ with a proportionality coefficient derived from the Fourier transform (we will give explicit expressions soon) and the asterisk represents the lowest twist for each spin-s operator in the OPE. Substituting (3.4) into (2.5) leads (after appropriate choices of the polarization tensor) to positivity constraints on the coefficients of the expansion,

$$C_{s,m}^* \geq 0, \qquad m = 0, 1, \dots, j. \tag{3.5}$$

Focusing upon the stress-energy operator ($s = 2$) on the right-hand side of (3.4), we find positivity requirements for the OPE coefficients in unitary CFTs. There are in total $(j + 1)$ positivity conditions.

It is time to discuss to what extent we can trust (3.5) for all $s \geq 2$. The validity of (2.5) is dependent upon the behavior of the DIS amplitude for fixed q^2 and small x, or equivalently, large $\nu = 2P.q$. If we assume

$$\lim_{x \to 0} \mathcal{A}(q^2, x) \leq x^{-N} \tag{3.6}$$

for some integer N, the DIS sum rules and the bounds (3.5) would be justified for $s \geq N$.

We can try to obtain some information about N indirectly as follows. As previously mentioned, $C_{s,m}^*$ is proportional to the OPE coefficient times the expectation value B_s^* up to an overall number derived from a Fourier transform, as in (2.11). For the case at hand, of generic external operators of spin-j and conformal dimension $\Delta_{\mathcal{O}}$, the relevant Fourier transform is

---

[9]Two out of the three structures in (2.12) yield $n_s \geq 0$, and the other one yields $n_f \geq 0$. In three dimensions, an additional parity odd structure in the three-point function of the stress-energy tensor is allowed [25,26]. Here and in the rest of this paper, we restrict the discussion to parity even structures.

[10]Here, $\mathcal{O}_j(\epsilon, x) = \mathcal{O}_{\mu_1 \dots \mu_j}(x) \epsilon^{\mu_1 \dots \mu_j}$.

$$\int d^d y e^{iqy} (y^2 + i\epsilon)^{\frac{1}{2}\tau_s^* - \Delta_{\mathcal{O}} - m} = \frac{\pi^{\frac{d}{2}} \Gamma[d/2 + \tau^*/2 - \Delta_{\mathcal{O}} - m]}{\Gamma[-\tau^*/2 + \Delta_{\mathcal{O}} + m]}$$
$$\times (q^2/4 - i\epsilon)^{-\tau_*/2 + \Delta_{\mathcal{O}} + m - d/2}. \tag{3.7}$$

The precise expression for $C_{s,m}$ in terms of B_s^* and the OPE coefficients $a_{s,m}^*$ is

$$C_{s,m} = 4^{-\beta} \frac{\pi^{\frac{d}{2}} \Gamma[s + m + \frac{\tau^*}{2} + \frac{d}{2} - \Delta_{\mathcal{O}}]}{\Gamma[\Delta_{\mathcal{O}} + m - \frac{\tau^*}{2}]} a_{s,m}^* B_s^*,$$
$$\beta = -\frac{\tau^*}{2} + \Delta_{\mathcal{O}} + m - d/2 \tag{3.8}$$

and is obtained by differentiating (3.7) $(s + 2m)$ times with respect to q_μ. The $a_{s,m}^*$ are specific linear combinations of the position space $\hat{a}_{s,m}^*$, similarly to what happens in the graviton DIS.

Let us consider now the case of the stress tensor exchange, i.e., $\tau_T^* = d - 2$, $s = 2$. In this case, $B_T = 1$, which leads to $a_{T,m} \geq 0$ as long as the numerical factor in (3.8) is positive definite. For $\tau_T^* = d - 2$ and $s = 2$, the arguments of the gamma functions in the numerator/denominator of (3.8) are equal to $d + m + 1 - \Delta_{\mathcal{O}}$ and $\Delta_{\mathcal{O}} + m + 1 - d/2$ respectively. The latter is positive definite by unitarity, but the former is not necessarily positive. We get that it is positive definite only for[11]

$$\Delta_{\mathcal{O}} \leq d + 1. \tag{3.9}$$

Equivalently, for $\Delta_{\mathcal{O}} > d + 1$, the Fourier transform above is divergent, and we define it by an analytic continuation. Assuming that the energy flux bounds still hold, we would get an apparent contradiction for the $\Delta_{\mathcal{O}}$ for which the ratio of the Γ-functions changes sign (see Appendix C for an explicit computation in the case of scalar, external operators). We think that this signals the need for subtraction in the sum rule.[12] We consider the example of a free scalar field in Appendix D, where the scenario just described is explicitly realized. Summarizing, for generic operators of conformal dimension $\Delta_{\mathcal{O}}$ and spin j, we expect to trust the x sum rule and the derived constraints (3.5), in the spin-s sector with $s \geq s_c \geq \Delta_{\mathcal{O}} - \frac{\tau_{s_c}^*}{2} - \frac{d}{2}$, where s_c is the first spin for which it holds that $s \geq \Delta_{\mathcal{O}} - \frac{\tau_s^*}{2} - \frac{d}{2}$.

---

[11]When $(d - \Delta_{\mathcal{O}} + 1)$ is a negative integer, the Fourier transform of the integral should be regulated by adding a local term to cancel the Γ-function pole. The result for the overall coefficient is still a number of alternating sign.

[12]It can be easily seen that doing subtractions in the x sum rule used in the previous section automatically projects out all low-spin operators from the OPE.

IV. DIS VS ENERGY CORRELATOR: ARE THE CONSTRAINTS ALWAYS EQUIVALENT?

In this section, we show that the constraints one gets from the positivity of the energy flux in a state produced by a given local operator $\mathcal{O}_j$ of spin j imply the constraints obtained from the DIS $s = 2$ sum rule. More precisely, for the case of external operators which are conserved currents, we show that the constraints derived from the DIS sum rule and those obtained from the positivity of the energy correlators are equivalent. On the other hand, for generic operators, the energy correlators constraints are stronger than the ones which follow from the DIS $s = 2$ sum rule. The bounds that are associated to $s > 2$ in DIS do not follow in any simple way from the positivity of the energy flux.

Consider now the energy flux operator, defined as in Ref. [4],

$$\mathcal{E}(n) = \lim_{r \to \infty} r^{d-2} \int_{-\infty}^{\infty} dt T_{0i} n^i(t, rn^i), \tag{4.1}$$

and $n = (1, \vec{n})$, or equivalently one can define the calorimeter operator in a manifestly covariant way [29]. The expectation value of the energy flux on the state

$$|\mathcal{O}_j(\epsilon, k)\rangle = \int d^d y e^{ikx} \mathcal{O}_j(\epsilon, x)|0\rangle, \tag{4.2}$$

obtained by acting with the operator $\mathcal{O}_j$ carrying momentum k on the vacuum, is fixed by rotational invariance up to a few parameters;

$$\langle \mathcal{E}(n) \rangle_{\mathcal{O}.\epsilon(k)} \sim \langle \mathcal{O}_j(\epsilon, k) | \mathcal{E}(n) | \mathcal{O}_j(\epsilon, k) \rangle$$
$$= \frac{(k^2)^\Delta}{(k.n)^{d-1}} \sum_{\ell=0}^{j} D_\ell (\epsilon^*.\epsilon)^{j-\ell} \frac{(\epsilon^*.k)^\ell (\epsilon.k)^\ell}{(k^2)^\ell}. \tag{4.3}$$

Here, we imposed the transversality condition $\epsilon.n = 0$. Notice that usually the polarization tensor is chosen such that $\epsilon.k = 0$ (see, for example, Ref. [4]); however, for the purpose of comparison with DIS, the choice above is more convenient. Conformal invariance determines the three-point correlation functions up to a few numbers, and thus the D_ℓ can be expressed as linear combinations of those numbers. Requiring positivity of the energy, $\langle \mathcal{E}(n) \rangle \geq 0$, leads to $(j + 1)$ linear constraints on the parameters $D_\ell \geq 0$ or, equivalently, on the constants which determine the three-point functions $\langle \mathcal{O}T\mathcal{O} \rangle$.

Below, we show that the constraints obtained from the computation above in the $\epsilon.n = 0$ "gauge" are identical to the ones derived from DIS, assuming that we can trust the $s = 2$ dispersion relation integral.

A. Computing the energy correlator

Consider the three-point function $\langle \mathcal{O}^\dagger(x_1, \epsilon^*) T_{\mu\nu}(x_2) \times \mathcal{O}(x_3, \epsilon) \rangle$. Together with the two-point function

$\langle \mathcal{O}^\dagger(x, \epsilon^*) \mathcal{O}(x, \epsilon) \rangle$, it can be used to compute two objects: the one-point energy correlator and the OPE coefficient in $\mathcal{O}^\dagger(x, \epsilon^*) \mathcal{O}(0, \epsilon) \sim C^{\mu\nu}(x, \epsilon) T_{\mu\nu}(0)$. The latter is useful to obtain the DIS constraints as discussed in the previous sections. It was observed in Sec. II that in some cases, the constraints obtained via the two methods coincide.

In this section, we show that the two always produce the same constraints provided that $\epsilon.n = 0$. First, we consider the energy correlator as defined in Ref. [29]. We will use the formalism of Ref. [11] and restrict our discussion to operators which are symmetric and traceless tensors. The three-point function we are interested in is

$$
\langle \mathcal{O}_j^\dagger(x_1, \epsilon^*) T(x_2, \bar{n}) \mathcal{O}_j(x_3, \epsilon) \rangle
$$
$$
= \frac{\sum \alpha_i V_1^{v_1} V_2^{v_2} V_3^{v_3} H_{12}^{h_{12}} H_{13}^{h_{13}} H_{23}^{h_{23}}}{x_{12}^{d+2} x_{13}^{2\bar{\tau}-(d+2)} x_{23}^{d+2}}, \qquad (4.4)
$$

where $\bar{\tau} = \Delta + j$ and $\bar{n} = (1, -\vec{n})$ and the exponents v_i and h_{ij} obey the following constraints:

$$
v_1 + h_{12} + h_{13} = j,
$$
$$
v_2 + h_{12} + h_{23} = 2,
$$
$$
v_3 + h_{13} + h_{23} = j. \qquad (4.5)
$$

So, different structures are labelled by the $\{h_{12}, h_{23}, h_{13}\}$. Of course, on top of these constraints, one should impose the conservation condition or—possibly—permutation symmetry. For our argument, imposing those is not necessary.

The method we are using is the one of Ref. [30]. The relevant notation is introduced in Appendix E. We start by expressing the energy correlator one-point function using (4.1) and (4.4). We then take the limit for the stress tensor approaching null infinity with the help of Appendix F. The result can be expressed as follows,

$$
\frac{\sum \alpha_{\{h_{12}, h_{13}, h_{23}\}} \hat{V}_1^{v_1} \hat{V}_2^{v_2} \hat{V}_3^{v_3} \hat{H}_{12}^{h_{12}} \hat{H}_{13}^{h_{13}} \hat{H}_{23}^{h_{23}}}{(x_{21}.n)^{\frac{d+2}{2}} (x_{13}^2)^{\bar{\tau} - \frac{(d+2)}{2}} (x_{23}.n)^{\frac{d+2}{2}}}, \qquad (4.6)
$$

where we introduced

$$
\hat{V}_1 = -\frac{x_{13}.\epsilon^* x_{12}.n - \epsilon^*.n \frac{x_{13}^2}{2}}{x_{23}.n}, \qquad \hat{V}_2 = \frac{x_{13}.n}{x_{13}^2},
$$

$$
\hat{V}_3 = -\frac{x_{13}.\epsilon x_{23}.n - \epsilon.n \frac{x_{13}^2}{2}}{x_{12}.n},
$$

$$
\hat{H}_{12} = -\epsilon^*.n, \qquad \hat{H}_{13} = \epsilon^*.\epsilon x_{13}^2 - 2 x_{13}.\epsilon^* x_{13}.\epsilon,
$$

$$
\hat{H}_{23} = -\epsilon.n. \qquad (4.7)
$$

Setting $\epsilon.n = 0$ leads to further simplifications. The three-point function then reduces to

$$
\sum_{h_{13}=0}^{j} \alpha_{\{0, h_{13}, 0\}} \frac{(\epsilon^*.x_{13} \epsilon.x_{13})^{j-h_{13}} \hat{H}_{13}^{h_{13}} (x_{13}.n)^2}{x_{12}.n^{\frac{d+2}{2}} x_{23}.n^{\frac{d+2}{2}} x_{13}^{2\bar{\tau}-(d-2)}}. \qquad (4.8)
$$

Next, we integrate over the position of the detector $\int_{-\infty}^{\infty} d(x_2.n)$. This boils down to the replacement $(x_{12}.n)^{-\frac{d+2}{2}} \times (x_{23}.n)^{-\frac{d+2}{2}} \to (x_{13}.n)^{-(d+1)}$ in the formula above (see Appendix F for the precise formula). Notice that after this replacement, the dimensionality of the object (4.8) is $(1 + 2\Delta)$, as it should be for a correlator which measures energy.[13]

The final step in the computation of the energy correlator is the Fourier transform, which implements the insertion of an operator with a given momentum. This leads to the following expression for the energy flux one-point function,

$$
\langle \mathcal{E}(n) \rangle_{\mathcal{O}.\epsilon(k)} \sim \int_0^\infty ds \, s^d \int d^d x_{13} e^{-i(k-sn).x_{13}}
$$
$$
\times \frac{\sum_{h_{13}=0}^{j} \alpha_{\{0, h_{13}, 0\}} (\epsilon^*.x_{13} \epsilon.x_{13})^{j-h_{13}} (x_{13}.n)^2 \hat{H}_{13}^{h_{13}}}{(x_{13}^2)^{\bar{\tau} - \frac{d-2}{2}}} \geq 0, \qquad (4.9)
$$

where we ignored an overall positive constant. Recall that in the formula above, the propagator is the Wightman one and the integral has nonzero support only for $(k - sn)$ timelike and having positive energy. We will not need to compute this integral explicitly.

B. Computation on the DIS side

Let us repeat the computation on the DIS side. We start with the analysis of the OPE. The relevant formula is the following [11]:

$$
\mathcal{O}(\epsilon^*, x_{13}) \mathcal{O}(\epsilon, 0)
$$
$$
\sim \mathcal{O}(0, \partial_z) t(x_{13}, \epsilon^*, z, \epsilon) x_{13}^{-(\Delta_1 + \Delta_3 - \Delta_2 + s_1 + s_2 + s_3)}. \qquad (4.10)
$$

The polynomial $t(x_{13}, \epsilon^*, z, \epsilon)$ is fixed by the three-point function to be

$$
t(x_{13}, \epsilon^*, z, \epsilon) = \sum \alpha_{\{h_{12}, h_{13}, h_{23}\}} (x_{13}^2)^{v_2 + h_{12} + h_{23}} (-1)^{v_1 + v_3}
$$
$$
\times (\epsilon^*.x_{13})^{v_1} \left(\frac{x_{13}.z}{x_{13}^2} \right)^{v_2}
$$
$$
\times (\epsilon.x_{13})^{v_3} (\epsilon^*.z)^{h_{12}} \hat{H}_{13}^{h_{13}} (z.\epsilon)^{h_{23}}. \qquad (4.11)
$$

This leading contribution to the sum rules comes from the term $h_{12} = h_{23} = 0$ as explained before.[14] Moreover, we are interested in the case $\mathcal{O}(0, \partial_z) \to T_{\mu\nu}$. For this case, we get

$$
\langle P | \mathcal{O}(\epsilon^*, x_{13}) \mathcal{O}(\epsilon, 0) | P \rangle
$$
$$
\sim \frac{\langle P | T(0, \partial_z) | P \rangle}{(x_{13}^2)^{\bar{\tau} - \frac{d-2}{2}}} \sum_{h_{13}=0}^{j} \alpha_{\{0, h_{13}, 0\}} (\epsilon^*.x_{13} \epsilon.x_{13})^{j-h_{13}}
$$
$$
\times (x_{13}.z)^2 \hat{H}_{13}^{h_{13}}, \qquad (4.12)
$$

---

[13]The 2Δ piece cancels when we divide by the two-point function, which is given by $\frac{H_{13}^j}{(x_{13}^2)^{\bar{\tau}}}$.

[14]Effectively, this is equivalent to setting $\epsilon^*.P = \epsilon.P = 0$.

where $\sim$ denotes that we neglected the contribution of all the other operators present in the OPE. Now, it is trivial to act with ∂_z, which boils down to $z \to p$,

$$\langle P|\mathcal{O}(\epsilon^*, x_{13})\mathcal{O}(\epsilon, 0)|P\rangle$$
$$\sim 2\frac{\sum_{h_{13}=0}^{j}\alpha_{\{0,h_{13},0\}}(\epsilon^*.x_{13}\epsilon.x_{13})^{j-h_{13}}(x_{13}.P)^2\hat{H}_{13}^{h_{13}}}{(x_{13}^2)^{\bar{\tau}-\frac{d-2}{2}}}.$$

(4.13)

Finally, we must take the Fourier transform with respect to x_{13}, which, assuming we can trust the dispersion integral, leads to the following constraint,

$$\int d^4x e^{-iqx}$$
$$\times \frac{\sum_{h_{13}=0}^{j}\alpha_{\{0,h_{13},0\}}(\epsilon^*.x_{13}\epsilon.x_{13})^{s_3-h_{13}}(x_{13}.P)^2\hat{H}_{13}^{h_{13}}}{(x_{13}^2)^{\bar{\tau}-\frac{d-2}{2}}} \geq 0,$$

(4.14)

where q is spacelike and x_{13}^2 is the usual time-ordered propagator.

C. Relation between energy correlators and DIS

We will now use the results of Secs. IV A and IV B to find a precise relation between the energy correlator and the DIS amplitude. Combining Eqs. (4.9) and (4.14), we can express the energy correlator one-point function as follows,

$$\langle \mathcal{E}(n)\rangle_{\mathcal{O}.\epsilon(q)} \sim \int_0^\infty ds s^d \text{Im}_{q^2}[\mathcal{A}_2(q^2, \epsilon.q, p.q)]_{q\to q-P;P\to sn},$$

(4.15)

where $\mathcal{A}_2(q^2, \epsilon.q, P.q)$ is defined as the term in the full DIS amplitude $\mathcal{A}(q^2, \epsilon.q, p.q)$,

$$\mathcal{A}(q^2, \epsilon.q, P.q) = \int d^d y e^{-iqy}\langle P|\mathcal{T}(\mathcal{O}_j(\epsilon^*, y)\mathcal{O}_j(\epsilon, 0))|P\rangle,$$

(4.16)

derived from the OPE coefficient of the stress-energy tensor operator. Recall that we consider polarization tensors satisfying $\epsilon.n = 0$.

Equation (3.4) allows us to write $\mathcal{A}_2(q^2, \epsilon.q, P.q)$ in the following form,

$$\mathcal{A}_2(q^2, \epsilon.q, p.q) = (2P.q)^2 \sum_{m=0}^{j}(q^2 - i\epsilon)^{-d-1+\Delta_j-m}$$
$$\times C_{T,m}(2\epsilon.q)^m(2\epsilon^*.q)^m(\epsilon^*.\epsilon)^{j-m},$$

(4.17)

where the Fourier transform has been obtained following the Feynmann -$i\epsilon$ prescription. The coefficients $C_{T,m}$ are defined as

$$C_{T,m} = 4^{-\beta}\frac{\pi^{\frac{d}{2}}\Gamma[-\Delta_\mathcal{O} + m + d + 1]}{\Gamma[\Delta_\mathcal{O} + m + 1 - d/2]}a_{T,m}$$
$$\beta = \Delta_\mathcal{O} + m + 1 - d,$$

(4.18)

where $a_{T,m}$ denote the constant OPE coefficients of the energy-momentum tensor. Observe that the Γ-function in the denominator of (4.18) is positive definite by unitarity but the one in the numerator is not necessarily positive definite, as discussed in detail in Sec. III.

The energy flux expectation value in (4.15) depends on $\text{Im}_{q^2}\mathcal{A}_2(q^2, \epsilon.q, P.q)$, which is equal to

$$\text{Im}_{q^2}\mathcal{A}_2(q^2, \epsilon.q, p.q) = (2P.q)^2\theta(q^0)\theta(-q^2)$$
$$\times \sum_{m=0}^{j}\tilde{C}_{T,m}(2\epsilon.q)^m(2\epsilon^*.q)^m$$
$$\times (\epsilon^*.\epsilon)^{j-m}(-q^2)^{-d-1+\Delta_j-m},$$

(4.19)

where $\tilde{C}_m$ are equal to

$$\tilde{C}_{T,m} = \frac{4^{-\beta}\pi^{\frac{d}{2}+1}}{\Gamma[\Delta_\mathcal{O} + m + 1 - d/2]\Gamma[\Delta_\mathcal{O} - m - d]}a_{T,m}.$$

(4.20)

Note that the product of Γ-functions appearing in $\tilde{C}_{T,m}$ is not positive definite, either. For operators of spin j, unitarity implies that $\Delta_\mathcal{O} - m - d \geq (j - m) - 2$, whereas for scalars, i.e., $j = m = 0$, unitarity leads to $\Delta_\mathcal{O} - d \geq -\frac{d}{2} - 1$. Substituting (4.19) into (4.15) yields

$$\langle\mathcal{E}(n)\rangle \sim \sum_{m=0}^{j}\tilde{C}_m(2\epsilon.q)^m(2\epsilon^*.q)^m(\epsilon^*.\epsilon)^{j-m}(2n.q)^2$$
$$\times \int_0^\infty ds s^{d+2}\theta(q^0 - s)\theta(-(q^2 - 2sq.n))(-(q^2 - 2sq.n))^{\Delta_\mathcal{O}-d-m-1}$$
$$= \sum_{m=0}^{j}a_m(2\epsilon.q)^m(2\epsilon^*.q)^m(\epsilon^*.\epsilon)^{j-m}(2n.q)^2\theta(q^0)\theta(-q^2)(-q^2)^{\Delta_\mathcal{O}-d-m-1}\left(\frac{q^2}{2q.n}\right)^{d+3}$$
$$\times \frac{\Gamma(d+3)}{\Gamma(\Delta_\mathcal{O} - m + 3)\Gamma[\Delta_\mathcal{O} + m + 1 - d/2]}.$$

(4.21)

Positivity of (4.21) is equivalent to $(j + 1)$ positivity relations, one for each value of m, obtained by appropriately choosing the polarization tensors (ϵ^*, ϵ). In other words,

$$\mathcal{E}(n) \geq 0 \Leftrightarrow a_m \frac{\Gamma(d + 3)}{\Gamma(\Delta_{\mathcal{O}} - m + 3)\Gamma[\Delta_{\mathcal{O}} + m + 1 - d/2]} \geq 0.$$
(4.22)

Notice, that the Γ-functions in (4.22) are now positive definite by unitarity, i.e.,

$$\Delta_{\mathcal{O}} \geq d - 2 + j \geq d - 2 + m \geq m - 3,$$

$$\Delta_{\mathcal{O}} \geq d - 2 + j \geq d/2 - m - 1,$$
(4.23)

allowing us to write

$$\mathcal{E}(n) \geq 0 \Leftrightarrow a_m \geq 0, \qquad m = 0, 1, \ldots, j.$$
(4.24)

Equation (4.24) establishes the equivalence between the constraints obtained from DIS and those derived from the positivity of the energy correlators for a certain class of transverse polarizations. This class of polarizations exhausts all possible choices for conserved currents. A generic operator, however, may also have longitudinal polarizations. To examine in detail what happens for generic operators, we consider below the case of a nonconserved spin-1 current.

D. Nonconserved spin-1 current

Let us consider the three-point function which involves two operators of spin 1, with a generic twist. Its general form is [31]

$$\langle \mathcal{O}(x_1, z_1) T(x_2, z_2) \mathcal{O}(x_3, z_3) \rangle = \frac{1}{x_{12}^{d+2} x_{13}^{2\bar{\tau}-(d+2)} x_{23}^{d+2}}$$
$$\times \left\{ a_1 V_2^2 H_{13} + \frac{(d-2)^2 a_2}{2} \left(V_2^2 V_1 V_3 - V_2 \frac{2V_1 H_{23} + 2V_3 H_{12} + V_2 H_{13}}{d-2} + \frac{2H_{12} H_{23}}{(d-2)^2} \right) \right.$$
$$\left. - 2(d-1)a_3 \left(V_2^2 V_1 V_3 + \frac{V_2}{2} [V_2 H_{13} + V_1 H_{23} + V_3 H_{12}] \right) \right\}.$$
(4.25)

Here, (z_1, z_2, z_3) denote the correponding polarization tensors. The coefficients a_2 and a_3 are proportional to the structures $\langle JTJ \rangle$ generated in the theory of a free boson and free fermion respectively for conserved spin-1 current J. We can compute the energy correlator as explained above. Stress tensor Ward identities relate this three-point function to the two-point function $\langle \mathcal{O}(z_1, x_1) \mathcal{O}(z_3, x_3) \rangle$ [24]. This can be translated into a relation between the parameters (a_1, a_2, a_3) appearing in (4.25). To find this relation, we require instead that the two-point function is correctly reproduced after integrating the energy flux correlator over the position of the detector. This leads to the following constraint:

$$a_1 = -(\Delta - d + 1)(a_2 + a_3).$$
(4.26)

Notice that for $\Delta = d - 1$, which corresponds to the case of a conserved current, $a_1 = 0$, as it should (in this case, only two independent structures are expected to appear). Reflection positivity of the two-point function then yields

$$\langle \mathcal{O} \mathcal{O} \rangle \sim a_2 + a_3 > 0.$$
(4.27)

Computing the energy flux correlator after imposing (4.26), and requiring it to be positive definite, results in the following two conditions:

$$a_2 < 0,$$

$$a_3 \geq -\frac{a_2}{(\Delta - d + 1)} \frac{(\Delta + 1)(2\Delta + (d - 2)(d - 1))}{2\Delta + d - 2},$$

$$a_2 \geq 0,$$

$$a_3 > -\frac{2(\Delta - d + 1)(\Delta - 1)}{2(\Delta - d + 1)(\Delta - 1) + d\Delta} a_2.$$
(4.28)

We should stress here that in deriving (4.28), we did not require $\epsilon.n = 0$. Notice that when $\Delta \to d - 1$, the solution which corresponds to the first line of (4.28) disappears, whereas the second line approaches the bounds of conserved currents [4]. Indeed, for $\Delta = d - 1$, we recover the usual $a_2 \geq 0$ and $a_3 \geq 0$ conditions.

On the other hand, we can compute the DIS bound or, equivalently, restrict our consideration to $\epsilon.n = 0$ in the energy correlator computation. The result is the same for $a_2 \geq 0$, but for the other case, we get

$$a_2 < 0,$$

$$a_3 > -\frac{a_2}{(\Delta - d + 1)} \frac{4(\Delta - 1) + (d - 4)(d - 2)}{4}.$$
(4.29)

It is easy to see that the bounds derived from the positivity of the energy correlator are stronger than those obtained from DIS, for any $\Delta > d - 1$.

E. Nonconserved spin-2 current

Similarly, we computed the energy correlator for a generic nonconserved spin-2 current. There are six different structures that appear in the three-point function. Matching to the two-point function after integrating over the position of the detector fixes one of the constants in terms of the others. We again find that the constraints from the energy correlator are stronger than the ones from DIS for non-conserved spin-2 operator. In the limit $\Delta \to d$, the constraints derived from DIS become equivalent to those required by the positivity of the energy correlator, as predicted by the general argument above.

V. DIS IN A CFT

A consistent unitary CFT should produce correlation functions that are reflection positive. As is often the case, it is easier to analyze the constraints following from reflection positivity in the Lorentzian signature. Obviously, these constraints should hold independently of whether or not the CFT admits an renormalization group flow to a gapped phase. In this section, we reformulate the sum rules studied in the previous sections purely in the CFT language. Instead of a proton, we consider the state $|P\rangle$ defined as follows,

$$|P\rangle \equiv \int d^d y\, e^{iPy} \mathcal{O}(y)|0\rangle, \qquad (5.1)$$

where $\mathcal{O}(x)$ is an arbitrary, scalar operator.

The expectation value of the stress-energy tensor on the state $|P\rangle$ is determined by Lorentz invariance up to two numbers,

$$\langle P|T_{\mu\nu}|P\rangle = c_1 P_\mu P_\nu + c_2 \eta_{\mu\nu} P^2, \qquad (5.2)$$

where c_1, c_2 are some dimensionful coefficients. Conformal invariance allows us to further express c_2 in terms of c_1. We will not consider the second term on the right-hand side of (G2) since it belongs to the so-called trace terms, the contribution to the OPE of which is negligible for large, spacelike momentum. Instead, we show in Appendix G that c_1, up to an overall divergent term, is positive definite. The divergence can be easily regularized; for example, we can imagine making the norm finite by considering $e^{\frac{y_0^2 + \vec{y}^2}{\sigma^2}}$ for the wave function.

We next consider the DIS amplitude defined as follows,

$$\mathcal{A}(q^2, \epsilon.q, P.q)$$

$$= \int d^d y\, e^{-iqy} \langle P|\mathcal{T}(\mathcal{O}_j(\epsilon^*, y)\mathcal{O}_j(\epsilon, 0))|P\rangle_{\text{conn}}$$

$$= \int d^d y\, e^{-iqy} (\langle P|\mathcal{T}(\mathcal{O}_j(\epsilon^*, y)\mathcal{O}_j(\epsilon, 0))|P\rangle$$

$$\quad - \langle P|P\rangle\langle 0|\mathcal{T}(\mathcal{O}_j(\epsilon^*, y)\mathcal{O}_j(\epsilon, 0))|0\rangle), \qquad (5.3)$$

where operators are ordered as written and $\mathcal{T}(\ldots)$ stands for time ordering.

The imaginary part of (5.3) is positive definite. To see this, recall that the imaginary part of the full correlator is given by the positive-definite Wightman function and that the imaginary part of the disconnected piece is independent of $x = \frac{q^2}{2P.q}$ and vanishes for spacelike q^2.

Let us recall the analytic structure of $\mathcal{A}(q^2, \epsilon.q, P.q)$. It has discontinuities for $(P+q)^2 < 0$ and $(P-q)^2 < 0$. These can be rewritten as

$$-\frac{1}{1 + \frac{P^2}{q^2}} \le x \le \frac{1}{1 + \frac{P^2}{q^2}}, \qquad (5.4)$$

where $x \equiv \frac{q^2}{2P.q}$. We can then proceed as before. We have to assume a certain behavior at infinity to use the dispersion relations, but otherwise all the formulas are identical to the ones in the previous sections. Formulated in this language, B_s^* from Secs. II and III are simply proportional to the corresponding three-point couplings. This is why (1.4) follows.

VI. CONCLUSIONS

In this paper, we considered the DIS experiment in a unitary CFT. The basic object under consideration is the scattering amplitude (3.1). Using it, one can write the standard sum rules (2.2) which relate the OPE data to the integrated positive-definite cross section.

An interesting case to consider is the graviton DIS in a CFT which flows to a gapped phase. In this case, the structure of the amplitude is given by (2.12), and the positivity of the cross section leads to the constraints (2.16) which are the well-known Hofman-Maldacena constraints. More general constraints exist in each even spin sector (2.17). These can be therefore viewed as generalized Hofman-Maldacena constraints.

We studied the general DIS experiment with some spinning external operators and elucidated the relation between the bounds produced by DIS and energy correlator considerations. Our first conclusion is that the $s = 2$ DIS bounds are equivalent to the energy flux constraints computed for a subclass of polarization tensors. This follows from the relation (4.15) which is the result of an explicit computation.

Considering the DIS experiment which involves non-conserved spin-1 and spin-2 currents, we found that, generically, the constraints obtained from the energy flux positivity are stronger than those coming from the DIS $s = 2$ sum rule, as explained in Sec. 3. 4. The difference between the two methods disappears in the limit when the operators become conserved. Understanding better the origin of this difference is an important open problem.

Finally, we reformulated the DIS experiment purely in CFT terms. The role of the DIS amplitude is played by the four-point correlation function (5.3), with the particular

ordering of operators and a special choice of external wave functions. Positivity of the cross section translates to positivity of the norm, and the usual problem with the dominance of the unit operator does not appear because of the choice of ordering and kinematics.

In writing the sum rules, we assumed a certain behavior of the amplitude in the Regge limit. We expect that this behavior depends both on the details of the theory and on the properties of the operators involved. This can be easily seen to be the case for the free scalar theory (see Appendix D). In the bulk of the paper, we simply assumed that we can write the sum rules and derived the consequences. However, for the graviton DIS, it is legitimate to assume that all the sum rules for $s \geq 2$ converge. This produces various results that are satisfied in all the examples known to us.

Our analysis indicates that there are infinitely more constraints on the three-point function of spinning operators than have been known before. It would be very interesting to see what can be learned about them using other methods. These include integrability [32], numerical [33] and analytic bootstrap [28], and casuality [7].

There are several directions in which our analysis can be generalized. One direction involves considering operators in generic representations of the Lorentz group both for the probe operator and for the target, including the parity odd structures in $d = 3$. One may also consider the odd spin sum rules. In this case, the OPE data for the minimal-twist odd spin operators are related to the difference of cross sections of the type $\sigma_{pn} - \sigma_{p\bar{n}}$ where $\bar{n}$ refers to an antiparticle. This difference is not known to be positive definite. The recent conjecture for bounds on the $\langle JJJ \rangle$ three-point coupling put forward in Ref. [28], together with the convergence of the $s = 1$ sum rule, would imply the sign definiteness of (roughly) $\int_0^\infty dx (\sigma_{pn} - \sigma_{p\bar{n}}) \geq 0$. It would be interesting to investigate this further.

Another interesting open avenue is bounding the non-integrated expectation value of the stress tensor in a given state and deriving the consequences for the OPE. In a classical theory, the expectation value of the stress-energy tensor is non-negative. In a quantum theory, however, the expectation value can be locally negative, but as reviewed in this paper, the integrated over time expectation value is expected to be non-negative. A more refined version of this statement is that there are bounds on how negative the local expectation value of the stress tensor in a given state could be [34,35]. This was recently discussed in Refs. [36,37] and in Ref. [38] where bounds on the three-point function of the stress tensor in a unitary four-dimensional CFT were obtained. It would be interesting to understand if these bounds could be strengthened and what constraints on the spectrum and the three-point functions of the CFT they imply.

One of the puzzling features of AdS/CFT is the emergence of locality on the sub–anti-de Sitter scale. It is believed [39] that CFTs with large central charge N and a large gap in the spectrum of higher-spin currents $\Delta_{\text{gap}} \gg 1$ are described at low energies by Einstein's theory in AdS_{d+1} with all higher-derivative corrections suppressed by the gap Δ_{gap}^{-1}. Proving this using purely CFT methods seems to be a necessary and important step in our understanding of the AdS/CFT correspondence and more generally quantum gravity. An even more ambitious goal is to show that every theory with such properties is a string theory. We hope that methods developed in this paper could be useful to make progress in this direction.

In Ref. [40], it was shown, using bulk arguments, that the picture described above follows from causality in the case of the simplest possible observable, namely, the graviton self-coupling. Showing this for all correlation functions and using purely CFT methods is still an open problem.

APPENDIX A: RELATION BETWEEN $a_{s,m}^*$ AND $\hat{a}_{s,m}^*$ IN THE $TT \sim T$ OPE

In this Appendix, we present the relation between the Fourier transformed OPE coefficients $\hat{a}_{s,i}^*$ and the coefficients $a_{s,i}^*$ which characterize the OPE in position space, as defined in (2.9),

$$a_{s,0}^* = i^s \left(\hat{a}_{s,0}^* + \frac{\hat{a}_{s,1}^*}{2d - \tau_s^*} + \frac{2\hat{a}_{s,2}^*}{(2d - \tau_s^*)(2d - \tau_s^* + 2)} \right),$$

$$a_{s,1}^* = -i^s 4 \left(\hat{a}_{s,1}^* + \frac{4\hat{a}_{s,2}^*}{2d - \tau_s^* + 2} \right),$$

$$a_{s,2}^* = i^s \hat{a}_{s,2}^*. \tag{A1}$$

APPENDIX B: DIS FOR THE STRESS-ENERGY TENSOR OPERATOR

Here, we explicitly evaluate the $a_{T,m}$ and derive (2.16). Our starting point is Eq. (6.38) of Ref. [24]. Requiring that $\epsilon^2 = (\epsilon^*)^2 = \epsilon.P = \epsilon^*.P = 0$ and neglecting trace terms leads us to consider only the following terms from Eq. (6.38) of Ref. [24]:

$$\hat{A}^1_{\mu\nu\sigma\rho\alpha\beta}(y)C_T \to \frac{d-2}{d+2}(4a+2b-c)H^1_{\alpha\beta\mu\nu\sigma\rho}(y)$$
$$+\frac{da+b-c}{d}H^2_{\alpha\beta\mu\nu\rho\sigma}(y)$$
$$+\frac{2da+2b-c}{d(d-2)}H^3_{\alpha\beta\mu\nu\sigma\rho}(y). \tag{B1}$$

Evaluating the Fourier transform together with the appropriate contractions yields

$$\int d^dy e^{-iqy}\epsilon^{*,\mu}\epsilon^{*,\nu}H^1_{\alpha\beta\mu\nu\sigma\rho}\epsilon^\sigma\epsilon^\rho P^\alpha P^\beta$$
$$= -\frac{1}{8}\frac{\pi^{d/2}}{\Gamma(d/2+1)}\frac{(2\epsilon^*.q)^2(2\epsilon.q)^2}{(q^2)^2}(2P.q)^2q^{-2}$$
$$+\frac{1}{2}\frac{\pi^{d/2}}{\Gamma(d/2+1)}\frac{(\epsilon^*.\epsilon)(2\epsilon^*.q)(2\epsilon.q)}{q^2}(2P.q)^2q^{-2}$$
$$-\frac{1}{2}\frac{\pi^{d/2}}{\Gamma(d/2+1)}(\epsilon^*.\epsilon)^2(2P.q)^2q^{-2} \tag{B2}$$

$$\int d^dy e^{-iqy}\epsilon^{*,\mu}\epsilon^{*,\nu}H^2_{\alpha\beta\mu\nu\sigma\rho}\epsilon^\sigma\epsilon^\rho P^\alpha P^\beta$$
$$= \frac{\pi^{d/2}}{\Gamma(d/2)}\frac{(\epsilon^*.\epsilon)(2\epsilon^*.q)(2\epsilon.q)}{q^2}(2P.q)^2q^{-2}$$

$$\int d^dy e^{-iqy}\epsilon^{*,\mu}\epsilon^{*,\nu}H^3_{\alpha\beta\mu\nu\sigma\rho}\epsilon^\sigma\epsilon^\rho P^\alpha P^\beta$$
$$= -2\frac{\pi^{d/2}}{\Gamma(d/2-1)}(\epsilon^*.\epsilon)^2(2P.q)^2q^{-2}. \tag{B3}$$

Collecting the appropriate factors from (B2)–(B3) leads to (2.16).

APPENDIX C: SCALAR DIS

Here, we consider the simplest example of a DIS experiment, where the external operator is a scalar. Following Ref. [24], we consider

$$\langle T_{\mu\nu}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\rangle = \frac{1}{x_{12}^d x_{23}^{2\Delta-d} x_{31}^d}\mathcal{I}_{\mu\nu,\sigma\rho(x_{13})}t_{\sigma\rho}(X_{12}),$$
$$\tag{C1}$$

where

$$\langle\mathcal{O}(x)\mathcal{O}(0)\rangle = \frac{N}{x^{2\Delta}}$$

$$\langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle = \mathcal{C}_T\frac{\mathcal{I}_{\mu\nu,\rho\sigma}(x)}{x^{2d}}$$

$$\mathcal{I}_{\mu\nu,\rho\sigma} = \frac{1}{2}(I_{\mu\sigma}(x)I_{\nu\rho}(x)+I_{\mu\rho}(x)I_{\nu\sigma}(x))-\frac{\eta_{\mu\nu}\eta_{\rho\sigma}}{d}$$

$$I_{\mu\nu} = \eta_{\mu\nu}-2\frac{x_\mu x_\nu}{x^2} \tag{C2}$$

and

$$t_{\mu\nu} = a\left(\hat{X}_\mu\hat{X}_\nu-\frac{1}{d}\eta_{\mu\nu}\right) \qquad \hat{X}_\mu = \frac{X_\mu}{\sqrt{X^2}} \tag{C3}$$

and X_{ij} is defined as

$$X_{ij} = -X_{ji} = \frac{x_{ik}}{x_{ik}^2}-\frac{x_{jk}}{x_{jk}^2},$$

$$X_{ij}^2 = \frac{x_{ij}^2}{x_{ik}^2 x_{jk}^2}$$

$$i = 1,2,3, i\neq j, j\neq k, i\neq k. \tag{C4}$$

The three-point function (C1) in the limit $y^\mu \equiv (x_2 - x_3)^\mu \to 0$ yields

$$\langle T_{\mu\nu}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\rangle \simeq_{y\sim 0} \frac{\mathcal{I}_{\mu\nu,\rho\sigma}(x_{13})}{x_{13}^{2d}}(-)^d\frac{t_{\rho\sigma}(y)}{y^{2\Delta-d}}$$
$$\simeq C_{\rho\sigma}(y)C_T\frac{\mathcal{I}_{\mu\nu,\rho\sigma}(x_{13})}{x_{13}^{2d}} \Rightarrow C_{\rho\sigma}(y)C_T$$
$$= (-)^d\frac{t_{\rho\sigma}(y)}{s^{2\Delta-d}}, \tag{C5}$$

leading to

$$\mathcal{O}(y)\mathcal{O}(0) \sim \cdots + C_{\mu\nu}(y)T^{\mu\nu}(0) + \cdots, \tag{C6}$$

where the dots represent the contributions of other operators in the OPE. We therefore express $C_{\mu\nu}(y)$ as follows:

$$C_{\mu\nu}(y) = \frac{a}{C_T}\left[\frac{1}{4(\Delta-\frac{d}{2}-1)(\Delta-\frac{d}{2})}\partial_\mu\partial_\nu\frac{1}{y^{\Delta-\frac{d}{2}-1}}\right.$$
$$\left.-\eta_{\mu\nu}\frac{\Delta-d}{d(\Delta-\frac{d}{2})}\frac{1}{y^{\Delta-\frac{d}{2}}}\right]. \tag{C7}$$

$C_{\mu\nu}$ does not contribute to conformal Ward identities as can be immediately seen from Eqs. (13.a) and (13.b) of Ref. [24]. This is to be contrasted with the OPE coefficient $\hat{C}_{\mu\nu}$ in the OPE $\mathcal{O}(s)T_{\mu\nu}(0) \sim \hat{C}_{\mu\nu}(s)\mathcal{O}(0)$. In the latter case, the conformal Ward identities relate the three-point function coefficient a with the coefficient N of the two-point function $\langle\mathcal{O}(x)\mathcal{O}(0)\rangle$ in (C2) as follows (see Eq. (6.20) of Ref. [24]):

$$a = -\frac{dN\Delta}{(d-1)S_d}. \qquad (C8)$$

For a unitary CFT, Eq. (C8) implies that $a \le 0$.

To obtain the contribution of the stress-energy tensor in the scalar DIS amplitude $\mathcal{A}(x, q^2) = \int d^d y e^{-iqy} \langle P | \mathcal{O}(y) \mathcal{O}(0) | P \rangle$, we simply need to take the Fourier transform of the OPE coefficient (C7), taking into acount the expectation value of the stress-energy tensor and disregarding the trace terms. Explicitly, we have that

$$\mathcal{A}(x, q^2) = \cdots - B_T \frac{a}{C_T} \frac{1}{4(\Delta - \frac{d}{2} - 1)(\Delta - \frac{d}{2})} (P.q)^2 \int d^d y e^{-iqy} y^{-\Delta + d/2 + 1} + \cdots$$

$$= \cdots - B_T \frac{a}{C_T} (2q.P)^2 \frac{1}{16} \frac{\pi^{d/2+1} \Gamma(d - \Delta + 1)}{\pi \Gamma(\Delta - d/2 + 1)} (q^2/4 - i\varepsilon)^{\Delta - d - 1} + \cdots$$

$$= \cdots - B_T \frac{a}{C_T} \frac{\pi^{d/2+1} \Gamma(d - \Delta + 1)}{\pi \Gamma(\Delta - d/2 + 1)} (q^2/4 - i\varepsilon)^{\Delta - d + 1} x^{-2} + \cdots \qquad (C9)$$

Notice that the sign of the term in (C9) is equal to the sign of the Γ-function in the numerator. This is because $a < 0$ from the Ward identities and Eq. (C8), $B_T = 1$, and unitarity requires that $C_T > 0$ and $\Delta - d/2 + 1 > 0$. As long as $\Delta < d + 1$, the Γ-function is positive definite and the positivity constraints from DIS are trivially satisfied. On the other hand, when $\Delta > d + 1$, the Fourier integral is divergent, and the moments are not well defined. Naively applying the DIS positivity relations (2.5) for this case would lead to inconsistencies due to the periodically alternating sign of the Γ-function.

APPENDIX D: DIS IN THE FREE FIELD THEORY

Let us consider a free massless scalar $\phi(x)$ and the DIS amplitude for $\phi^{n+1}(x)$,

$$\mathcal{A}(P, q) = \int d^d x e^{-iqx} \langle P | \phi^{n+1}(x) \phi^{n+1}(0) | P \rangle = \int d^d x \frac{e^{-i(q-P)x} + e^{-i(q+P)x}}{(x^2 + i\varepsilon)^{n\frac{d-2}{2}}}$$

$$\sim [((q - P)^2 - i\varepsilon)^{n\frac{d-2}{2} - \frac{d}{2}} + ((q + P)^2 - i\varepsilon)^{n\frac{d-2}{2} - \frac{d}{2}}] \frac{\Gamma(\frac{d}{2} - n\frac{d-2}{2})}{\Gamma(n\frac{d-2}{2})}$$

$$= (q^2)^{n\frac{d-2}{2} - \frac{d}{2}} \left[\left(1 - \frac{1}{x} - \frac{M^2}{q^2} - i\varepsilon \right)^{n\frac{d-2}{2} - \frac{d}{2}} + \left(1 + \frac{1}{x} - \frac{M^2}{q^2} - i\varepsilon \right)^{n\frac{d-2}{2} - \frac{d}{2}} \right] \times \frac{\Gamma(\frac{d}{2} - n\frac{d-2}{2})}{\Gamma(n\frac{d-2}{2})}. \qquad (D1)$$

We see explicitly what is happening in this example. The amplitude has an imaginary part exactly where expected, for $-(1 - \frac{M^2}{q^2})^{-1} \le x \le (1 - \frac{M^2}{q^2})^{-1}$. Moreover, the imaginary part is positive definite.

To write the dispersion relations, we have to explore the behavior at $x \to 0$. The amplitude behaves as

$$\mathcal{A}(P, q) \sim x^{-(n\frac{d-2}{2} - \frac{d}{2})}, \qquad (D2)$$

and for the sum rule to converge, we thus get

$$s_0 > n \frac{d-2}{2} - \frac{d}{2} = \Delta - d + 1. \qquad (D3)$$

The Γ-function which appears in the Fourier transform of the OPE that becomes negative has the argument

$$\Gamma\left(s_0 - \left(\Delta - d + 1 - \frac{\tau^* - (d-2)}{2}\right)\right). \qquad (D4)$$

For the case at hand, $\tau^* = d - 2$, and we see that the convergence of the sum rule goes in parallel with the positivity of the Γ-function.

APPENDIX E: KINEMATICS OF THE THREE-POINT FUNCTIONS

Here, we collect some of the notation that we used in the bulk of the paper:

$$n = (1, \vec{n}), \qquad \bar{n} = (-1, \vec{n}), \qquad x.n = -t + \vec{x}.\vec{n},$$
$$x_{ij}^2 = -(t_i - t_j)^2 + (\vec{x}_i - \vec{x}_j)^2,$$
$$x_{\pm;ij}^2 = -(t_i - t_j \pm i\varepsilon)^2 + (\vec{x}_i - \vec{x}_j)^2,$$
$$-2P_i.P_j = x_{ij}^2, \qquad Z_i.Z_j = z_i.z_j, \qquad P_i.Z_j = x_{ij}.z_j. \qquad (E1)$$

The conformal covariants, as defined in Ref. [11], are given by

$$V_1 = V_{1,23} = \frac{Z_1.P_2 P_1.P_3 - Z_1.P_3 P_1.P_2}{P_2.P_3} = -\frac{x_{12}.z_1 x_{13}^2 - x_{13}.z_1 x_{12}^2}{x_{23}^2},$$

$$V_2 = V_{2,31} = \frac{Z_2.P_3 P_2.P_1 - Z_2.P_1 P_2.P_3}{P_1.P_3} = -\frac{x_{23}.z_2 x_{12}^2 + x_{12}.z_2 x_{23}^2}{x_{13}^2},$$

$$V_3 = V_{3,12} = \frac{Z_3.P_1 P_3.P_2 - Z_3.P_2 P_3.P_1}{P_1.P_2} = -\frac{x_{23}.z_3 x_{13}^2 - x_{13}.z_3 x_{23}^2}{x_{12}^2},$$

$$H_{12} = -2(Z_1.Z_2 P_1.P_2 - Z_1.P_2 Z_2.P_1) = z_1.z_2 x_{12}^2 - 2x_{12}.z_1 x_{12}.z_2,$$

$$H_{13} = -2(Z_1.Z_3 P_1.P_3 - Z_1.P_3 Z_3.P_1) = z_1.z_3 x_{13}^2 - 2x_{13}.z_1 x_{13}.z_3,$$

$$H_{23} = -2(Z_2.Z_3 P_2.P_3 - Z_2.P_3 Z_3.P_2) = z_2.z_3 x_{23}^2 - 2x_{23}.z_2 x_{23}.z_3. \tag{E2}$$

APPENDIX F: COMPUTING THE ENERGY CORRELATOR

To compute the energy correlator, we have to first take the limit $\lim_{x_2.\bar{n}\to\infty}(x_2.\bar{n})^{d-2}$ of (4.4). Here, we write down some useful formulas which allow us to analyze the limit easily[15]:

$$\lim_{x_2.\bar{n}\to\infty} x_{12}^2 = -x_2.\bar{n} x_{12}.n, \qquad \lim_{x_2.\bar{n}\to\infty} x_{23}^2 = x_2.\bar{n} x_{23}.n,$$

$$\lim_{x_2.\bar{n}\to\infty} x_{12}.z_1 = -\frac{1}{2} x_2.\bar{n} z_1.n, \qquad \lim_{x_2.\bar{n}\to\infty} x_{23}.z_3 = \frac{1}{2} x_2.\bar{n} z_3.n,$$

$$\lim_{x_2.\bar{n}\to\infty} x_{13}.z_1 = x_{13}.z_1, \qquad \lim_{x_2.\bar{n}\to\infty} x_{13}.z_3 = x_{13}.z_3,$$

$$\lim_{x_2.\bar{n}\to\infty} x_{23}.z_2 = x_2.\bar{n}, \qquad \lim_{x_2.\bar{n}\to\infty} x_{12}.z_2 = -x_2.\bar{n}. \tag{F1}$$

Using (F1), we find that the covariant structures of (E2) can be expressed in this limit as follows:

$$V_1 \to -\frac{x_{13}.z_1 x_{12}.n - z_1.n \frac{x_{13}^2}{2}}{x_{23}.n}, \qquad V_2 \to (x_2.\bar{n})^2 \frac{x_{13}.n}{x_{13}^2}, \qquad V_3 \to -\frac{x_{13}.z_3 x_{23}.n - z_3.n \frac{x_{13}^2}{2}}{x_{12}.n},$$

$$H_{12} \to -(x_2.\bar{n})^2 z_1.n, \qquad H_{13} \to z_1.z_3 x_{13}^2 - 2x_{13}.z_1 x_{13}.z_3, \qquad H_{23} \to -(x_2.\bar{n})^2 z_3.n. \tag{F2}$$

Thus, in the limit, we get

$$\frac{1}{(x_{12}^2)^{\frac{d+2}{2}}(x_{13}^2)^{\bar{\tau}-\frac{(d+2)}{2}}(x_{23}^2)^{\frac{d+2}{2}}} \to \frac{1}{(x_2.\bar{n})^{d+2}} \frac{1}{(x_{21}.n)^{\frac{d+2}{2}}(x_{13}^2)^{\bar{\tau}-\frac{(d+2)}{2}}(x_{23}.n)^{\frac{d+2}{2}}}$$

$$\sum \alpha_i V_1^{v_1} V_2^{v_2} V_3^{v_3} H_{12}^{h_{12}} H_{13}^{h_{13}} H_{23}^{h_{23}} \to (-(x_2.\bar{n})^2)^{v_2+h_{12}+h_{23}} \sum \alpha_i \hat{V}_1^{v_1} \hat{V}_2^{v_2} \hat{V}_3^{v_3} \hat{H}_{12}^{h_{12}} \hat{H}_{13}^{h_{13}} \hat{H}_{23}^{h_{23}}. \tag{F3}$$

Notice that due to (E2), $v_2 + h_{12} + h_{23} = s_2 = 2$, and the correlator has the expected asymptotic behavior $\frac{1}{(x_2.\bar{n})^{d-2}}$.

Gathering the results above, we conclude that after taking the limit, the correlator reduces to

$$\frac{\sum \alpha_{\{h_{12},h_{13},h_{23}\}} \hat{V}_1^{v_1} \hat{V}_2^{v_2} \hat{V}_3^{v_3} \hat{H}_{12}^{h_{12}} \hat{H}_{13}^{h_{13}} \hat{H}_{23}^{h_{23}}}{(x_{21}.n)^{\frac{d+2}{2}}(x_{13}^2)^{\bar{\tau}-\frac{(d+2)}{2}}(x_{23}.n)^{\frac{d+2}{2}}}, \tag{F4}$$

where we introduced

---

[15]$x_2 = \frac{x_2.n}{2}\bar{n} + \frac{x_2.\bar{n}}{2}n.$

$$\hat{V}_1 = -\frac{x_{13}.z_1 x_{12}.n - z_1.n\frac{x_{13}^2}{2}}{x_{23}.n}, \qquad \hat{V}_2 = \frac{x_{13}.n}{x_{13}^2}, \qquad \hat{V}_3 = -\frac{x_{13}.z_3 x_{23}.n - z_3.n\frac{x_{13}^2}{2}}{x_{12}.n},$$

$$\hat{H}_{12} = -z_1.n, \qquad \hat{H}_{13} = z_1.z_3 x_{13}^2 - 2x_{13}.z_1 x_{13}.z_3, \qquad \hat{H}_{23} = -z_3.n. \tag{F5}$$

For the integration over the position of the detector, recall the following formula:

$$\int_{-\infty}^{\infty} d(x_2.n)\frac{1}{(x_{12}.n)^a (x_{23}.n)^b} = \frac{2\pi i}{(x_{13}.n)^{a+b-1}}\frac{\Gamma(a+b-1)}{\Gamma(a)\Gamma(b)}. \tag{F6}$$

APPENDIX G: POSITIVITY OF THE EXPECTATION VALUE OF THE STRESS-ENERGY TENSOR IN A CFT

Let us define a state $|P\rangle$ as follows,

$$|P\rangle \equiv \int d^d x e^{iPx}\mathcal{O}(x)|0\rangle, \tag{G1}$$

where $\mathcal{O}(x)$ is an arbitrary, scalar operator. The expectation value of the stress-energy tensor on the state $|P\rangle$ is determined by Lorentz invariance up to two numbers,

$$\langle P|T_{\mu\nu}|P\rangle = c_1 P_\mu P_\nu + c_2 \eta_{\mu\nu}P^2, \tag{G2}$$

where c_1, and c_2 are some dimensionful coefficients. Conformal invariance allows us to further express c_2 in terms of c_1. Here, we will not consider the second term on the right-hand side of (G2) since it belongs to the so-called trace terms, the contribution in the OPE of which is negligible for large, spacelike momentum. We would like instead to determine explicitly c_1 and show that, up to an overall divergent term which can be easily regularized, it is positive definite.

To this end, we consider the following three-point function in the CFT,

$$\int d^d x d^d y e^{iPx}e^{-iPy}\langle \mathcal{O}(y)T_{\mu\nu}n^\mu n^\nu \mathcal{O}(x)\rangle, \tag{G3}$$

with n^μ some null vector and $\mathcal{O}(x)$ the scalar operator of conformal dimension Δ associated to the state $|P\rangle$.

For convenience, will work in light-cone coordinates with

$$ds^2 = -dx^+ dx^- + \delta_{ij}dx^i dx^j \tag{G4}$$

and choose $n^\mu = (0, 1, 0, ..., 0)$ so that we only need to compute $\langle P|T_{--}|P\rangle$. With this choice, we expect that $\langle P|T_{\mu\nu}|P\rangle = c_1(P.n)^2 = \frac{c_1}{4}(P^+)^2$, disregarding for the moment possible divergences.

We start from the general form of the three-point function of two scalar operators and the stress-energy tensor, as given in (III.1) of Ref. [24],

$$\langle \mathcal{O}(y)T_{\mu\nu}(0)\mathcal{O}(x)\rangle = \frac{1}{y^d x^d (y-x)^{2\Delta-d}}t_{\mu\nu}(X_{23}), \tag{G5}$$

where

$$t_{\mu\nu}(X) = a\left(\hat{X}_\mu \hat{X}_\nu - \frac{1}{d}\eta_{\mu\nu}\right), \qquad \hat{X}_\mu = \frac{X_\mu}{\sqrt{X^2}}. \tag{G6}$$

X_{ij} is defined as follows:

$$X_{ij} = -X_{ji} = \frac{x_{ik}}{x_{ik}^2} - \frac{x_{jk}}{x_{jk}^2},$$

$$X_{ij}^2 = \frac{x_{ij}^2}{x_{ik}^2 x_{jk}^2} \qquad i \neq j, j \neq k, k \neq i. \tag{G7}$$

Note that the overall coefficient a is completely determined by Ward identities (see Eq. (6.20) of Ref. [24]),

$$a = -\frac{dN\Delta}{(d-1)S_d} < 0, \tag{G8}$$

with N the normalization constant of the two-point function of $\mathcal{O}(x)$ and S_d the volume of the d-dimensional sphere. Using (G5)–(G7) leads to

$$\langle P|T_{--}|P\rangle = \frac{a}{4}\int d^d x d^d y \frac{(y^+ x^2 - x^+ y^2)^2 e^{iP(x-y)}}{(y^2 + i\epsilon y^0)^{\frac{d}{2}+1}(x^2 - i\epsilon x^0)^{d/2+1}[(y-x)^2 + i\epsilon(y^0 - x^0)]^{\Delta-d/2+1}}, \tag{G9}$$

where the $-i\epsilon$ presiption is the appropriate one for the Wightman correlator. We split the integral in (G9) into three separate integrals by expanding the square in the numerator. Each integral is of the form

$$I_{m,\ell}(P) = \int d^d x d^d y \frac{(y^+ x^2)^m (x^+ y^2)^\ell e^{iP(x-y)}}{(y^2 + i\epsilon y^0)^{\frac{d}{2}+1}(x^2 - i\epsilon x^0)^{d/2+1}[(y-x)^2 + i\epsilon(y^0 - x^0)]^{\Delta-d/2+1}}, \tag{G10}$$

where $m, \ell = 0, 1, 2$ and $m + \ell = 2$.

First, we express each factor in the denominator as follows:

$$x^2 - i\epsilon = (x^+ + i\epsilon)\left(-x^- + \sum_i \frac{(x^i)^2}{x^+} - i\epsilon\right)$$

$$y^2 + i\epsilon = (y^+ - i\epsilon)\left(-y^- + \sum_i \frac{(y^i)^2}{y^+} + i\epsilon\right)$$

$$(y-x)^2 + i\epsilon = (y^+ - x^+ - i\epsilon)\left(-y^- + x^- + \sum_i \frac{(y^i - x^i)^2}{y^+ - x^+} + i\epsilon\right). \tag{G11}$$

Then, we introduce Schwinger parameters

$$\left(-x^- + \sum_i \frac{(x^i)^2}{x^+} - i\epsilon\right)^{d/2+1-m} = \frac{i^{d/2+1-m}}{\Gamma(d/2+1-m)}\int_0^\infty ds_1 s_1^{d/2-m} e^{-is_1(-x^- + \sum_i \frac{(x^i)^2}{x^+} - i\epsilon)}$$

$$\left(-y^- + \sum_i \frac{(y^i)^2}{y^+} + i\epsilon\right)^{d/2+1-\ell} = \frac{(-i)^{d/2+1-\ell}}{\Gamma(d/2+1-\ell)}\int_0^\infty ds_2 s_2^{d/2-\ell} e^{is_2(-y^- + \sum_i \frac{(y^i)^2}{y^+} + i\epsilon)}$$

$$\left(-y^- + x^- + \sum_i \frac{(y^i - x^i)^2}{y^+ - x^+} + i\epsilon\right)^{\Delta - d/2+1} = \frac{(-i)^{\Delta - d/2+1}}{\Gamma(\Delta - d/2 + 1)}\int_0^\infty ds_3 s_3^{\Delta - d/2} e^{is_3(-y^- + x^- + \sum_i \frac{(y^i - x^i)^2}{y^+ - x^+})} \tag{G12}$$

and over (x^-, y^-) to obtain $(2\pi)^2 \delta(s_1 + s_3 - p^+/2)\delta(-s_2 - s_3 + p^+/2)\theta(\frac{P^+}{2} - s_3)$. The integration over (s_1, s_2) then becomes trivial; this amounts to setting $s_1 = s_2 = \frac{P^+}{2} - s_3$ and multiplying with an overall factor of $(1/2)^2$—note that the integrations of the δ-fucntions are from zero to infinity.

In what follows, we find it convenient to keep the variables s_1 and s_2 and perform the relevant substitution later. The next step is to perform the integration over the (x^i) and then over the (y^i). The former reads

$$\int\left(\prod_{i=1}^{d-2} dx^i\right)\exp\left(iP^i x^i - is_1 \frac{(x^i)^2}{x^+} + is_3 \frac{(x^i)^2}{y^+ - x^+} - 2is_3 \frac{y^i x^i}{y^+ - x^+}\right) = \frac{\pi^{\frac{d-2}{2}}}{(-iA)^{\frac{d-2}{2}}}\exp\left\{-i\frac{\sum_i (P^i - 2s_3 \frac{y^i}{y^+ - x^+})^2}{4A}\right\} \tag{G13}$$

and the latter reads

$$\int\left(\prod_i^{d-2} dy^i\right)\exp\left(-iP^i y^i + is_2 \frac{(y^i)^2}{y^+} + is_3 \frac{(y^i)^2}{y^+ - x^+} - i\frac{s_3^2}{A}\frac{(y^i)^2}{(y^+ - x^+)^2} + i\frac{s_3}{A}P^i \frac{y^i}{y^+ - x^+}\right)$$

$$= \frac{\pi^{\frac{d-2}{2}}}{(-iB)^{\frac{d-2}{2}}}\exp\left\{-i\frac{\sum_i (P^i)^2(-1 + \frac{s_3}{A(y^+ - x^+)})^2}{4B}\right\}, \tag{G14}$$

where

$$A \equiv -\frac{s_1}{x^+} + \frac{s_3}{y^+ - x^+}, \qquad B \equiv \frac{s_2}{y^+} + \frac{s_3}{y^+ - x^+} - \frac{s_3^2}{A(y^+ - x^+)^2}. \tag{G15}$$

Next follows the integration over the variables x^+, y^+, i.e.,

$$\int dx^+ dy^+ \frac{(-AB)^{\frac{2-d}{2}}\exp\left\{-i\frac{\vec{P}^2}{4}\left(\frac{(-1 + \frac{s_3}{A(y^+ - x^+)})^2}{B} + \frac{1}{A}\right) - i\frac{P^-}{2}x^+ + i\frac{P^-}{2}y^+\right\}}{(x^+ + i\epsilon)^{d/2+1-\ell-m}(y^+ - i\epsilon)^{d/2+1-m-\ell}(y^+ - x^+ - i\epsilon)^{\Delta - d/2+1}}. \tag{G16}$$

A little bit of algebra combined with the substitution $s_1 = s_2 = \frac{P^+}{2} - s_3$ yields

$$-AB = \frac{P^+}{2}\frac{(\frac{P^+}{2} - s_3)}{x^+ y^+}\frac{(-1 + \frac{s_3}{A(y^+ - x^+)})^2}{B} + \frac{1}{A} = \frac{2(y^+ - x^+)}{P^+}, \tag{G17}$$

which allows us to express (G10) as follows,

$$I_{m,\ell}(P) = C\theta\left(\frac{P^+}{2}\right) \int_0^{P^+/2} ds\, s^{\Delta-d/2}\left(\frac{P^+}{2} - s\right)^{d/2-m-\ell+1}$$

$$\times \int dx^+ dy^+ \frac{e^{-i\frac{P^2}{2P^+}(y^+-x^+)}}{(x^+ + i\epsilon)^{2-\ell-m}(y^+ - i\epsilon)^{2-\ell-m}(y^+ - x^+ - i\epsilon)^{\Delta-d/2+1}}, \tag{G18}$$

where

$$C \equiv \left(\frac{P^+}{2}\right)^{1-d/2} \frac{\pi^2 i^{-m}(-i)^{-\ell+\Delta-d/2+1}\pi^{d-2}}{\Gamma(d/2+1-m)\Gamma(d/2+1-\ell)\Gamma(\Delta-d/2+1)}. \tag{G19}$$

The integral over s is equal to

$$\int_0^{P^+/2} ds\, s^{\Delta-d/2}\left(\frac{P^+}{2} - s\right)^{d/2-m-\ell+1} = \left(\frac{P^+}{2}\right)^{\Delta+2-m-\ell} \int_0^1 du\, u^{\Delta-d/2}(1-u)^{d/2-m-\ell+1}$$

$$= \left(\frac{P^+}{2}\right)^{\Delta+2-m-\ell} \frac{\Gamma(\Delta-d/2+1)\Gamma(d/2-m-\ell+2)}{\Gamma(\Delta-m-\ell+3)}. \tag{G20}$$

The integral over x^+, y^+ yields

$$\int dx^+ dy^+ \frac{e^{-i\frac{P^2}{2P^+}(y^+-x^+)}}{(x^+ + i\epsilon)^{2-\ell-m}(y^+ - i\epsilon)^{2-\ell-m}(y^+ - x^+ - i\epsilon)^{\Delta-d/2+1}} \xrightarrow{\ell+m=2}$$

$$= \frac{(2\pi)^2 i^{\Delta-d/2+1}}{\Gamma(\Delta-d/2+1)}\theta\left(\frac{P^+}{2}\right)\theta\left(-\frac{P^2}{2P^+}\right)\left(-\frac{P^2}{2P^+}\right)^{\Delta-d/2-1}\int_0^\infty du\, u^{\Delta-d/2}\delta^2(1-u), \tag{G21}$$

where we have isolated the divergent term in the dimensionless integral over u.

Gathering all the results, bearing in mind that Eq. (G9) can be written as

$$\langle P|T_{--}|P\rangle = \frac{a}{4}(I_{2,0} - 2I_{1,1} + 2I_{0,2}), \tag{G22}$$

leads to

$$\langle P|T_{--}|P\rangle = \tilde{a}\,\theta\left(\frac{P^+}{2}\right)\theta(-P^2)\left(-\frac{P^2}{4}\right)^{\Delta-d/2-1}\left(\frac{P^+}{2}\right)^2, \tag{G23}$$

where

$$\tilde{a} = -\frac{a}{4}(2\pi^{d+2})2\frac{d-1}{\Gamma(d/2+1)\Gamma(d/2)}\frac{1}{\Gamma(\Delta+1)\Gamma(\Delta-d/2+1)}\lim_{\epsilon\to 0}\delta(\epsilon). \tag{G24}$$

Given that $a < 0$, for any unitary theory where $\Delta - d/2 + 1 \geq 0$, $\tilde{a}$ is positive definite.

_____

[1] G. Mack, All unitary ray representations of the conformal group SU(2,2) with positive energy, Commun. Math. Phys. 55, 1 (1977).

[2] S. Minwalla, Restrictions imposed by superconformal invariance on quantum field theories, Adv. Theor. Math. Phys. 2, 783 (1998).

[3] B. Grinstein, K. A. Intriligator, and I. Z. Rothstein, Comments on unparticles, Phys. Lett. B 662, 367 (2008).

[4] D. M. Hofman and J. Maldacena, Conformal collider physics: Energy and charge correlations, J. High Energy Phys. 05 (2008) 012.

[5] O. Nachtmann, Positivity constraints for anomalous dimensions, Nucl. Phys. B63, 237 (1973).

[6] Z. Komargodski and A. Zhiboedov, Convexity and liberation at large spin, J. High Energy Phys. 11 (2013) 140.

[7] T. Hartman, S. Jain, and S. Kundu, Causality constraints in conformal field theory, J. High Energy Phys. 05 (2016) 099.

[8] D. M. Hofman, Higher derivative gravity, causality and positivity of energy in a UV complete QFT, Nucl. Phys. B823, 174 (2009).

[9] M. Kulaxizi and A. Parnachev, Energy Flux Positivity and Unitarity in CFTs, Phys. Rev. Lett. 106, 011601 (2011).

[10] A. M. Polyakov, Scale Invariance of Strong Interactions and its application to Lepton-Hadron Reactions, Lecture Notes in International School of High Energy Physics in Erevan, 1971 (Chernogolovka, 1972); Acad. Sci. USSR, Bull., Phys. Ser. (1972).

[11] M. S. Costa, J. Penedones, D. Poland, and S. Rychkov, Spinning conformal correlators, J. High Energy Phys. 11 (2011) 071.

[12] A. Buchel, J. Escobedo, R. C. Myers, M. F. Paulos, A. Sinha, and M. Smolkin, Holographic GB gravity in arbitrary dimensions, J. High Energy Phys. 03 (2010) 111.

[13] Unpublished.

[14] J. Maldacena, D. Simmons-Duffin, and A. Zhiboedov, Looking for a bulk point, J. High Energy Phys. 01 (2017) 013.

[15] A. Adams, N. Arkani-Hamed, S. Dubovsky, A. Nicolis, and R. Rattazzi, Causality, analyticity and an IR obstruction to UV completion, J. High Energy Phys. 10 (2006) 014.

[16] V. W. Hughes and J. Kuti, Internal spin structure of the nucleon, Annu. Rev. Nucl. Part. Sci. 33, 611 (1983).

[17] A. V. Manohar, An Introduction to Spin Dependent Deep Inelastic Scattering, arXiv:hep-ph/9204208.

[18] H. Epstein, V. Glaser, and A. Martin, Polynomial behaviour of scattering amplitudes at fixed momentum transfer in theories with local observables, Commun. Math. Phys. 13, 257 (1969).

[19] S. B. Giddings and R. A. Porto, The gravitational S-matrix, Phys. Rev. D 81, 025002 (2010).

[20] E. Elkhidir, D. Karateev, and M. Serone, General three-point functions in 4D CFT, J. High Energy Phys. 01 (2015) 133.

[21] S. Weinberg and E. Witten, Limits on massless particles, Phys. Lett. 96B, 59 (1980).

[22] A. Dymarsky, K. Farnsworth, Z. Komargodski, M. A. Luty, and V. Prilepina, Scale invariance, conformality, and generalized free fields, J. High Energy Phys. 02 (2016) 099.

[23] A. Bzowski and K. Skenderis, Comments on scale and conformal invariance, J. High Energy Phys. 08 (2014) 027.

[24] H. Osborn and A. C. Petkou, Implications of conformal invariance in field theories for general dimensions, Ann. Phys. (N.Y.) 231, 311 (1994).

[25] J. M. Maldacena and G. L. Pimentel, On graviton non-Gaussianities during inflation, J. High Energy Phys. 09 (2011) 045.

[26] S. Giombi, S. Prakash, and X. Yin, A note on CFT correlators in three dimensions, J. High Energy Phys. 07 (2013) 105.

[27] A. L. Fitzpatrick, J. Kaplan, D. Poland, and D. Simmons-Duffin, The analytic bootstrap and AdS superhorizon locality, J. High Energy Phys. 12 (2013) 004.

[28] D. Li, D. Meltzer, and D. Poland, Conformal collider physics from the lightcone bootstrap, J. High Energy Phys. 01 (2016) 143.

[29] A. Zhiboedov, On conformal field theories with extremal a/c values, J. High Energy Phys. 04 (2014) 038.

[30] A. V. Belitsky, S. Hohenegger, G. P. Korchemsky, E. Sokatchev, and A. Zhiboedov, From correlation functions to event shapes, arXiv:1309.0769.

[31] A. Zhiboedov, A note on three-point functions of conserved currents, arXiv:1206.6370.

[32] B. Basso, S. Komatsu, and P. Vieira, Structure constants and integrable bootstrap in planar $N = 4$ SYM Theory, arXiv:1505.06745.

[33] R. Rattazzi, V. S. Rychkov, E. Tonni, and A. Vichi, Bounding scalar operator dimensions in 4D CFT, J. High Energy Phys. 12 (2008) 031.

[34] L. H. Ford, Constraints on negative energy fluxes, Phys. Rev. D 43, 3972 (1991).

[35] L. H. Ford and T. A. Roman, The quantum interest conjecture, Phys. Rev. D 60, 104018 (1999).

[36] R. Bousso, Z. Fisher, S. Leichenauer, and A. C. Wall, Quantum focusing conjecture, Phys. Rev. D 93, 064044 (2016).

[37] R. Bousso, Z. Fisher, J. Koeller, S. Leichenauer, and A. C. Wall, Proof of the quantum null energy condition, Phys. Rev. D 93, 024017 (2016).

[38] K. Farnsworth, M. A. Luty, and V. Prilepina, Positive energy conditions in 4D conformal field theory, J. High Energy Phys. 10 (2016) 001.

[39] I. Heemskerk, J. Penedones, J. Polchinski, and J. Sully, Holography from conformal field theory, J. High Energy Phys. 10 (2009) 079.

[40] X. O. Camanho, J. D. Edelstein, J. Maldacena, and A. Zhiboedov, Causality constraints on corrections to the graviton three-point coupling, J. High Energy Phys. 01 (2016) 020.