# Signal Set Tissue Systems and Overlapping Localities

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**Abstract**

In signal set tissue systems with overlapping localities (ssolt-systems) evolution rules can be associated with several cells. Rules may influence each other through instant signalling and, moreover, they are synchronised when sharing activated cells. The latter is a new feature. We study the behaviour of ssolt-systems in the form of labelled step transition systems. This allows to compare ssolt-systems with and without signalling and overlapping localities. Next the synthesis problem is considered, i.e., the question when given a step transition system, how to effectively construct an ssolt-system exhibiting this behaviour. To this end, ssolt-systems are related to a new class of Petri nets, that are behaviourally equivalent to ssolt-systems. It is shown how certain region based synthesis techniques can be applied to these nets and hence are also available for ssolt-systems.

*Key words:* natural computing, set tissue system, instant signalling, locality, concurrency, step transition system, Petri net, synthesis, theory of regions.

## 1 Introduction

Tissue systems are a computational model inspired by the way biochemical reactions and interactions take place inside and among living cells ([21, 22]). Like membrane systems, tissue systems are defined in terms of evolution rules

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that specify how objects (molecules) can be combined to form new objects. In both types of systems reactions are localised, i.e., they take place in compartments where their reactants reside and their products can either remain in the originating compartment or be delivered to a neighbouring one. A membrane system has an associated hierarchical structure (a tree) that reflects the structure of a cell; the tree corresponds to the nesting of its compartments separated by membranes. The more general tissue systems use a graph to formalise neighbourhood relations between compartments, now like cells arranged in a tissue. Motivated by reaction systems [6, 8, 9] that model biochemical processes using a qualitative rather than a quantitative approach, [12] has introduced a variant of membrane systems — set membrane systems, independently introduced also in [1] — with evolution rules that no longer refer to multisets of objects (indicating how many instances of each object are involved), but rather to sets (merely indicating presence/absence) of objects. In [13] set tissue systems, i.e., tissue systems with such qualitative evolution rules, are enhanced with instant signalling that makes it possible to fast process certain objects even within the same computational step that has produced them. This extension is an abstraction of cell signalling, a complex system of communication employed in cellular biochemical processes. In a biological micro environment, signalling is a fundamental feature, governing and coordinating the activities of cells. This ability to pass on and respond immediately to the presence of signalling objects has been abstracted in [13] in the framework of set tissue systems through the concurrent execution of evolution rules based on a local maximality (eagerness) principle that reflects that everything that can happen (in an active compartment) does not wait and occurs as soon as possible. As said, the various models of membrane and tissue systems discussed so far are based on the idea that reactions take place locally, i.e., in individual compartments. Moreover, executions in different compartments are unrelated.

In this paper, we introduce evolution rules that are associated with conglomerates of compartments, i.e., each rule has a *locality* consisting of all compartments where it may occur. The underlying idea is that not only the reactions taking place within and among cells are influencing each other by the production of biochemical substances, but that there may also be external stimuli (mechanical, temperature, light etc.) that simultaneously affect several cells within a tissue. A rule that is associated with thus 'activated' compartments will then be simultaneously executed in each of the compartments with which it is associated. In this paper we are interested in the behaviour of the resulting *signal set tissue system with overlapping localities*, or SSOLT-systems. To capture this behaviour, labelled transition systems (so-called *concurrent reachability graphs*) are used that describe how starting from an initial configuration (state), a SSOLT-system evolves from configuration to configuration through a 'locally maximal' execution of evolution rules. Here, we distinguish between two types of semantics depending on the interpretation of local maximality, i.e., the interpretation of the eagerness underlying the locally maximal execution of the enabled rules. In the first interpretation, a rule is involved

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whenever *all* its compartments are activated (have resources). In the second semantics, a rule is involved whenever at least one (*any*) of its compartments is activated. Consequently, there will be two kinds of concurrent reachability graphs that can be defined by a SSOLT-system. As will be shown, in both these semantics, signalling and simultaneous activation of compartments have each their own effect on the expressiveness of the resulting models in terms of the generated reachability graphs.

On the other hand, we will also consider how to obtain from a labelled step transition system a SSOLT-system that behaves as specified, i.e., has a concurrent reachability graph (of type *all* or *any*) that is isomorphic to the given transition system — if at all possible. Algorithmic methods to synthesise concurrent systems from behavioural specifications (like step transition systems) provide an attractive way to construct systems that behave according to specification. This so-called synthesis problem has been extensively studied in the context of Petri nets, a well-established mathematical framework for the modelling of concurrent and distributed systems (see, e.g., [23, 24]). As we will demonstrate, some of the theory developed there can be extended and then transferred to SSOLT-systems.

Thus, we will relate the concurrent reachability graphs of SSOLT-systems to those of a suitable class of Petri nets. At a very basic level, the dynamics of membrane and tissue systems is similar to that of Petri nets (see [10]). Molecules in a compartment correspond to tokens in a specific place and evolution rules can be viewed as transitions related to input and output places in accordance with the objects consumed and produced by the rule. Once a similar mode of execution for both models has been agreed upon, there is a one-to-one correspondence between configurations and markings (token distributions) and between the concerted execution of multiple rules and the simultaneous firing of their related transitions, guaranteeing isomorphic reachability graphs. This correspondence has been proven to be robust in the sense that various concepts and methods could be transferred from one framework to the other. From the observation that the evolution rules of a membrane or tissue system are localised (belong to compartments) came the idea to also associate a *locality* with Petri net transitions and to define a locally maximal execution mode ([18, 17]). And membrane systems with qualitative evolution rules relate to set nets — that have a firing rule based on sets rather than multisets — with localities ([12]). Instant signalling on the other hand involves an application of the special so-called a/sync places that can be used for the instantaneous transfer of tokens from an input transition to an output transition ([11]). Overlapping localities of transitions, resembling the overlap of localities of the rules in a SSOLT-system, have been considered in [20].

Our solution to the synthesis problem will be based on the notion of *region*, introduced in the seminal paper [7] for the synthesis of Elementary Net systems with a sequential execution semantics, a basic class of Petri Nets. Over the

years, this idea has been further developed and extended in various directions, including synthesis modules of tool kits, various application areas of other net classes, other execution semantics, and different behavioural specification models. One of the key advances in the design of region based solutions has been the development in [3] of a general approach for dealing with region based synthesis for a range of synthesis problems. This approach is founded on so-called $\tau$-nets and $\tau$-regions. The parameter $\tau$ conveniently captures the marking information and different connections between places and transitions of varying classes of Petri nets, removing the need to re-state and re-prove the basic results every time a net model is modified. Once a class of Petri nets has been shown to be a class of $\tau$-nets, i.e., to correspond to a class of $\tau$-nets for some suitable $\tau$, this general method can be applied.

Based on the above considerations, we thus introduce a new class of Petri nets, so-called signal set nets with overlapping localities, or SSOL-nets. It is demonstrated that indeed SSOLT-systems and SSOL-nets closely correspond to each other both in structure and dynamics and hence for each of the two execution modes (*all* and *any*), the related concurrent reachability graphs are essentially the same. Finally, we show that SSOL-nets are indeed a class of $\tau$-nets. Hence it is possible to address the synthesis problem using $\tau$-regions. However, since SSOL-nets have two kinds of places (normal ones and signal places), in this case the parameter $\tau$ is based on two net-types to describe the connections between places and transitions, which is a new feature.

The paper is organized as follows. After a preliminary section, SSOLT-systems are introduced together with their *all* and *any* execution semantics in Section 3. In the same section, the expressive power of the different features is compared in terms of the concurrent reachability graphs. Section 4 introduces SSOL-nets and relates them to SSOLT-systems, whereas it is demonstrated in Section 5 that SSOL-nets are $\tau$-nets. The synthesis problem is then addressed in Section 6. In the concluding Section 7 we summarise what has been achieved.

## 2 Preliminaries

A *labelled transition system*, or LT-system, is a triple $LTS = (Q, A, \delta)$, where $Q$ is a finite set of *states*, $A$ is a finite set of *labels*, and $\delta : Q \times A \to Q$ is a partial function. For every state $q \in Q$, we define $enbld_{LTS}(q) = \{a \mid \delta(q, a) \text{ is defined}\}$ is the set of labels *enabled at* $q$. $LTS$ can be identified with a labelled directed graph, where $Q$ is the set of nodes, and $\delta$ defines the labelled arcs, i.e., $\delta = \{(q, a, q') \in Q \times A \times Q \mid a \in enbld_{LTS}(q) \wedge q' = \delta(q, a)\}$.

A *step transition system*, or ST-system, (over a finite set $T$) is a quadruple $STS = (Q, A, \delta, q_0)$ such that $LTS = (Q, A, \delta)$ is an LT-system, $q_0 \in Q$ is

the designated *initial state*, and $A = 2^T$ is the set of *steps*. For every $q \in Q$, we define $enbld_{STS}(q) = enbld_{LTS}(q)$. Moreover, we may also associate with $STS$ the labelled directed graph defined through $LTS$. We require that each $t \in T$ occurs in the label of at least one arc in this graph, and that for each state $q \neq q_0$, there is a directed path from $q_0$ to $q$. Finally, we require that $\delta(q, \varnothing) = q$ for all $q \in Q$. (Note that the three requirements above are needed when applying the theory of regions in order to synthesise nets from transition systems.)

Let $STS = (Q, 2^T, \delta, q_0)$ and $STS' = (Q', 2^{T'}, \delta', q_0')$ be two ST-systems, and $\phi : T \to T'$ be a bijection. Then $STS$ and $STS'$ are *$\phi$-isomorphic* if there is a bijection $\nu : Q \to Q'$ such that $\nu(q_0) = q_0'$ and, for all $q, q' \in Q$ and $\alpha \subseteq T$,

$$(q, \alpha, q') \in \delta \iff (\nu(q), \phi(\alpha), \nu(q')) \in \delta' \ ,$$

where $\phi(\alpha) = \{\phi(t) \mid t \in \alpha\}$. We denote this by $STS \cong_{\phi,\nu} STS'$ or $STS \cong_\phi STS'$. Moreover, two ST-systems, $STS$ and $STS'$, are *isomorphic* (denoted $STS \cong STS'$) if there exists a bijection $\phi$ such that $STS$ and $STS'$ are $\phi$-isomorphic. That is, the relation $\cong$ considers two ST-systems isomorphic if they are the same up to the renaming of their states and arc labels.

## 3 Signal Set Tissue Systems with Overlapping Localities

A *tissue structure* $\gamma$ (of degree $m \geq 1$) is an undirected graph with $m$ nodes identified with the integers $1, \ldots, m$. For all $i, j \in \{1, \ldots, m\}$, we write $(i, j) \in \gamma_{edges}$ to indicate that there is an edge between $i$ and $j$, and $i \in \gamma_{nodes}$ means that $i$ is a node of $\gamma$ (see Figure 1 for an example). We will refer to the nodes of a tissue structure as *compartments*. An edge between $i$ and $j$ indicates that direct communication between the compartments $i$ and $j$ is possible.
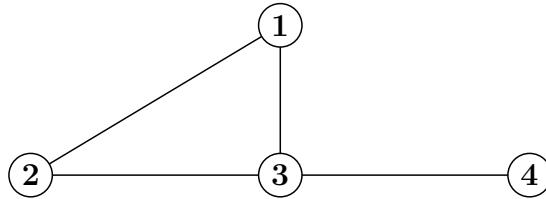


Fig. 1. A tissue structure $\gamma$ of degree 4.

Let $V$ be a finite set of *objects*, $V^{sgl} \subseteq V$ be a set of *signal objects*, and $\gamma$ be a tissue structure of degree $m$. A *signal set tissue system with overlapping localities*, or SSOLT-*system*, over $V$, $V^{sgl}$ and $\gamma$ is a tuple

$$\Sigma = (V, V^{sgl}, \gamma, w_1^0, \ldots, w_m^0, \mathcal{R})$$

such that, for every $i \in \gamma_{nodes}$, $w_i^0$ is a set of objects from $V$ present initially in the compartment $i$, and $\mathcal{R}$ is a set of (*evolution*) *rules*. A *configuration* of $\Sigma$ is a

tuple $C = (w_1, \ldots, w_m)$ of sets of objects, and $C_0 = (w_1^0, \ldots, w_m^0)$ is the *initial configuration*. Each evolution rule $r \in \mathcal{R}$ is of the form $r = (\mathfrak{c}^r : lhs^r \to rhs^r)$, where $\mathfrak{c}^r$, the *locality of* $r$, is a nonempty subset of $\{1, \ldots, m\}$; $lhs^r$, the *left hand side of* $r$, is a non-empty subset of $V$; and $rhs^r$, the *right hand side of* $r$, is a subset of

$$V \cup \{a_{\triangleright j} \mid a \in V \wedge j \in (\gamma_{nodes} \setminus \mathfrak{c}^r) \wedge \exists i \in \mathfrak{c}^r : (i, j) \in \gamma_{edges}\} .$$

A rule $r$, when executed, will consume some objects from the compartments belonging to its locality, and produce some objects in the compartments of its locality and possibly in some of the compartments linked to them according to $\gamma_{edges}$. Each $a \in V$ in the left hand side of $r$ represents an object to be consumed from each of the compartments of $\mathfrak{c}^r$. Each $a \in V$ in the right hand side of $r$ represents an object which is to be produced in each of the compartments of $\mathfrak{c}^r$. Therefore, one can understand the locality $\mathfrak{c}^r$ as the scope of $r$. The indexed symbols $a_{\triangleright j}$ represent an object $a$ that is sent to compartment $j$ not belonging to the locality of $r$ but linked in $\gamma$ with one of the compartments in the locality of $r$. Figure 2 shows a pictorial representation of a SSOLT-system

$$\Sigma_0 = (\{a, b, c\}, \{b\}, \gamma, \{a\}, \{a, c\}, \{a\}, \{a\}, \{r_1, r_2, r_3, r_4\}),$$

where:

$$
\begin{aligned}
r_1 &= (\{1, 2\} &&: \{a\} \to \{c\}) \\
r_2 &= (\{2\} &&: \{a, c\} \to \{b_{\triangleright 1}\}) \\
r_3 &= (\{1, 3, 4\} &&: \{a\} \to \{b\}) \\
r_4 &= (\{4\} &&: \{b\} \to \{c_{\triangleright 3}\}) .
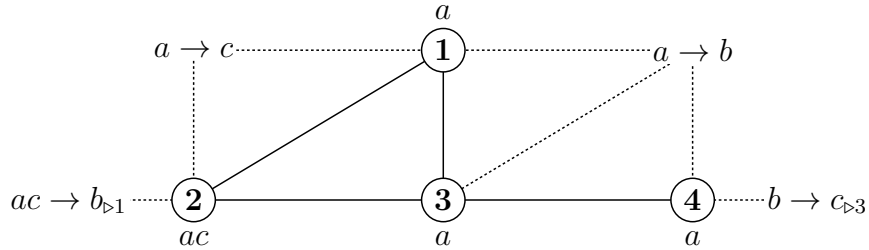\end{aligned}
$$



Fig. 2. An SSOLT-system over the tissue structure $\gamma$ shown in Figure 1. Note that the evolution rules are abbreviated (e.g., $ac$ denotes $\{a, c\}$).

A SSOLT-system evolves from configuration to configuration through the application (execution) of evolution rules based on the notion of a *set-rule* defined as a set of evolution rules.

For the rest of this section we let $\Sigma$ be a SSOLT-system as specified above.

Let $R \subseteq \mathcal{R}$ be a set-rule. Then $\mathfrak{c}^R = \bigcup_{r \in R} \mathfrak{c}^r$ denotes the set of compartments in the localities of the rules in $R$; we define tuples $lhs^R = (lhs_1^R, \ldots, lhs_m^R)$ and

$rhs^R = (rhs_1^R, \ldots, rhs_m^R)$ such that, for every $i \in \gamma_{nodes}$,[1]

$$lhs_i^R = \{a \in V \mid \exists r \in R : (i \in \mathfrak{c}^r \wedge a \in lhs^r)\}$$
$$rhs_i^R = \{a \in V \mid \exists r \in R : ((i \in \mathfrak{c}^r \wedge a \in rhs^r) \vee a_{\triangleright i} \in rhs^r)\} .$$

The tuple $lhs^R$ specifies which objects are needed per compartment for the simultaneous execution of all the evolution rules in $R$. In the case of non-signal objects, these must be already present, but any signal object required by $R$ may be created during the simultaneous execution of the evolution rules in $R$.[2] Thus, a set-rule $R \subseteq \mathcal{R}$ is *resource enabled*, or RES-enabled, at a configuration $C = (w_1, \ldots w_m)$ if, for all $i \in \gamma_{nodes}$, $lhs_i^R \setminus w_i \subseteq rhs_i^R \cap V^{sgl}$.
The set-rules that are RES-enabled at the initial configuration of the SSOLT-system depicted in Figure 2 are all subsets $R \subseteq \{r_1, r_2, r_3, r_4\}$ such that $r_4 \in R$ implies $r_3 \in R$. Also, $\mathfrak{c}^{\{r_1,r_2\}} = \{1,2\}$ and $\mathfrak{c}^{\{r_1,r_4\}} = \{1,2,4\}$. Moreover, for the singleton $R = \{r_3\}$, we have $lhs^R = (\{a\}, \varnothing, \{a\}, \{a\})$ and $rhs^R = (\{b\}, \varnothing, \{b\}, \{b\})$.

**Proposition 3.1** *If $R, R' \subseteq \mathcal{R}$ are set-rules RES-enabled at a configuration $C$, then $R \cup R'$ is also RES-enabled at $C$.*

**Proof.** Let $C = (w_1, \ldots w_m)$ and $i \in \gamma_{nodes}$. We first observe that

$$lhs_i^{R \cup R'} = lhs_i^R \cup lhs_i^{R'} \quad and \quad rhs_i^{R \cup R'} = rhs_i^R \cup rhs_i^{R'} .$$

Moreover, since $R$ and $R'$ are RES-enabled at a $C$,

$$lhs_i^R \setminus w_i \subseteq rhs_i^R \cap V^{sgl} \quad and \quad lhs_i^{R'} \setminus w_i \subseteq rhs_i^{R'} \cap V^{sgl} .$$

Hence we obtain:

$$
\begin{aligned}
lhs_i^{R \cup R'} \setminus w_i &= (lhs_i^R \cup lhs_i^{R'}) \setminus w_i \\
&= (lhs_i^R \setminus w_i) \cup (lhs_i^{R'} \setminus w_i) \\
&\subseteq (rhs_i^R \cap V^{sgl}) \cup (rhs_i^{R'} \cap V^{sgl}) \\
&= (rhs_i^R \cup rhs_i^{R'}) \cap V^{sgl} \\
&= rhs_i^{R \cup R'} \cap V^{sgl} .
\end{aligned}
$$

Thus $R \cup R'$ is RES-enabled at $C$. $\quad\square$

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[1] We do not define $lhs_i^r$ and $rhs_i^r$ for single evolution rules $r$. However, $R$ can be a singleton set containing just one rule.

[2] This is called instant signalling. In fact, it is possible that a single evolution rule produces some of its own input in the form of signal objects.

To be executable a set-rule must not only be RES-enabled, but also satisfy a maximality criterion that guarantees that evolution rules of 'active' compartments (belonging to the localities of rules already included in the set-rule) that could occur with the set-rule are also included in the set-rule. When formalising this, it appears that there are two levels of 'eagerness' justifying such inclusion: firstly, evolution rules of which the whole locality (all compartments) is active, need to be included; and secondly, evolution rules which have one or more active compartments, need to be included.

Let $R$ be a set-rule RES-enabled at a configuration $C$. Then $R$ is:

- *all-control enabled* (or $\text{CTRL}_{all}$-enabled) at $C$ if there is no set-rule $R' \supset R$ which is RES-enabled at $C$ with $\mathfrak{c}^{R'} = \mathfrak{c}^R$, and
- *any-control enabled* (or $\text{CTRL}_{any}$-enabled) at $C$ if there is no set-rule $R' \supset R$ which is RES-enabled at $C$ with $\mathfrak{c}^r \cap \mathfrak{c}^R \neq \varnothing$, for every $r \in R' \setminus R$.

There are two set-rules $\text{CTRL}_{any}$-enabled at the initial configuration of the SSOLT-system depicted Figure 2: $\varnothing$ and $\{r_1, r_2, r_3, r_4\}$. Moreover, there are five set-rules which are $\text{CTRL}_{all}$-enabled at the same configuration: $\varnothing$, $\{r_2\}$, $\{r_1, r_2\}$, $\{r_3, r_4\}$, and $\{r_1, r_2, r_3, r_4\}$.

Let $\mu \in \{all, any\}$ be an *execution mode*. If $R$ is $\text{CTRL}_\mu$-enabled at configuration $C$, then it can be *$\mu$-executed* leading to $C' = (w'_1, \ldots w'_m)$ such that, for all $i \in \gamma_{nodes}$,

$$w'_i = (w_i \setminus lhs^R_i) \cup (rhs^R_i \setminus (lhs^R_i \setminus w_i))$$
$$= ((w_i \cup rhs^R_i) \setminus lhs^R_i) \cup (w_i \cap rhs^R_i \cap lhs^R_i) .$$

We denote this by $C \xrightarrow{R}_\mu C'$. Note that $R = \varnothing$ is $\text{CTRL}_\mu$-enabled at any $C$, and that $C \xrightarrow{\varnothing}_\mu C$ always holds. Note also that the index $\mu$ refers to the type of enabledness, but that the result of executing a set-rule does not depend on whether it stands for *all* or *any*.

For the initial configuration of the SSOLT-system depicted Figure 2, we have:

$$(\{a\}, \{a, c\}, \{a\}, \{a\}) \xrightarrow{\{r_1, r_2, r_3, r_4\}}_{any} (\{b, c\}, \{c\}, \{b, c\}, \varnothing)$$
$$(\{a\}, \{a, c\}, \{a\}, \{a\}) \xrightarrow{\{r_3, r_4\}}_{all} (\{b\}, \{a, c\}, \{b, c\}, \varnothing)$$

**Proposition 3.2** *Let $R$ be a set-rule RES-enabled at a configuration $C$, and $\mu \in \{all, any\}$. Then there is a set-rule $R' \supseteq R$ which is $\text{CTRL}_\mu$-enabled at $C$.*

**Proof.** Follows from the fact that there are no infinite $\subset$-increasing chains of set-rules in $\Sigma$. $\square$

A *$\mu$-evolution* is a finite sequence of $\mu$-executions starting from $C_0$, and any configuration which can be obtained through such an evolution is *$\mu$-reachable*.

Moreover, the *μ-concurrent reachability graph of* $\Sigma$ is the step transition system $CRG_\mu(\Sigma) = ([C_0\rangle_\mu, 2^{\mathcal{R}}, \delta, C_0)$, where $[C_0\rangle_\mu$ is the set of all $\mu$-reachable configurations which are the states of the graph, $C_0$ is the initial state, and

$$\delta = \{(C, R, C') \mid C, C' \in [C_0\rangle_\mu \land R \subseteq \mathcal{R} \land C \xrightarrow{R}_\mu C'\} \ .$$

Note that we can assume without loss of generality that each evolution rule of $\Sigma$ occurs in at least one set-rule labelling an arc in $\delta$. Indeed, let $r \in \mathcal{R}$, and $R_C$ be the set of all set-rules RES-enabled at a configuration $C \in [C_0\rangle_\mu$. If there is $C \in [C_0\rangle_\mu$ and $R \in R_C$ such that $r \in R$ then, according to Proposition 3.2, there is a set-rule $R' \supseteq R$ which is CTRL$_\mu$-enabled at $C$, and so $r$ occurs in the label of an arc of $CRG_\mu(\Sigma)$. Otherwise, for each $C \in [C_0\rangle_\mu$, the sets of RES-enabled set-rules remains the same if we take $\mathcal{R} \setminus \{r\}$ instead of $\mathcal{R}$. Hence the set of set-rules CTRL$_\mu$-enabled at $C$ remains the same. As a result, $r$ can be deleted from $\mathcal{R}$ without any influence on the behaviour of $\Sigma$.

### 3.1 Overlapping localities vs. instant signalling

To assess the effect of the new feature of overlapping localities, we focus on the dynamics of the computational systems represented. Since the operational semantics of SSOLT-systems is fully captured by their concurrent reachability graphs, we will use the latter in order to compare the relative expressive power of the different sub-models. To carry out a meaningful comparison, we now single out three syntactical sub-models of SSOLT-systems.

A SSOLT-system $\Sigma = (V, V^{sgl}, \gamma, w_1^0, \ldots, w_m^0, \mathcal{R})$ is called :

- *signal set tissue system*, or SST-system, if $|\mathfrak{c}^r| = 1$, for every $r \in \mathcal{R}$ [13].
- *set tissue system with overlapping localities*, or SOLT-system, if $V^{sgl} = \varnothing$.
- *basic set tissue system*, or BST-system, if it is an SST-system and a SOLT-system [12].

Our first observation is that for an SST-system $\Sigma$, CTRL$_{all}$-enabledness and CTRL$_{any}$-enabledness are identical notions, and so $CRG_{all}(\Sigma) = CRG_{any}(\Sigma)$. We will in this case omit the reference to *all* and *any* and simply write e.g., $CRG(\Sigma)$ for the *concurrent reachability graph of* $\Sigma$.

We thus have the following families of concurrent reachability graphs:

$CRG_{ssolt}^{all}$    are the    *all*-concurrent    reachability graphs of SSOLT-systems.

$CRG_{ssolt}^{any}$    are the    *any*-concurrent    reachability graphs of SSOLT-systems.

$CRG_{solt}^{all}$    are the    *all*-concurrent    reachability graphs of SOLT-systems.

$CRG_{solt}^{any}$    are the    *any*-concurrent    reachability graphs of SOLT-systems.

$CRG_{sst}$    are the    concurrent    reachability graphs of SST-systems.

$CRG_{bst}$    are the    concurrent    reachability graphs of BST-systems.

The families of isomorphism classes (i.e., equivalence classes of the relation $\cong$) of the members of these six families will be respectively denoted by $\mathcal{CRG}_{ssolt}^{all}$, $\mathcal{CRG}_{ssolt}^{any}$, $\mathcal{CRG}_{solt}^{all}$, $\mathcal{CRG}_{solt}^{any}$, $\mathcal{CRG}_{sst}$, and $\mathcal{CRG}_{bst}$.

The relationships between the four classes of tissue systems described above are clear. What is not clear, however, is the relationship between the families of concurrent reachability graphs they generate. Directly from the definitions, we obtain:

**Proposition 3.3** $CRG_{bst} \subseteq CRG_{sst}$

$$CRG_{bst} \subseteq CRG_{solt}^{all} \quad CRG_{sst} \subseteq CRG_{ssolt}^{all} \quad CRG_{solt}^{all} \subseteq CRG_{ssolt}^{all}$$

$$CRG_{bst} \subseteq CRG_{solt}^{any} \quad CRG_{sst} \subseteq CRG_{ssolt}^{any} \quad CRG_{solt}^{any} \subseteq CRG_{ssolt}^{any} .$$

The above result, in turn, immediately yields (for the families of isomorphism classes):

**Proposition 3.4** $\mathcal{CRG}_{bst} \subseteq \mathcal{CRG}_{sst}$

$$\mathcal{CRG}_{bst} \subseteq \mathcal{CRG}_{solt}^{all} \quad \mathcal{CRG}_{sst} \subseteq \mathcal{CRG}_{ssolt}^{all} \quad \mathcal{CRG}_{solt}^{all} \subseteq \mathcal{CRG}_{ssolt}^{all}$$

$$\mathcal{CRG}_{bst} \subseteq \mathcal{CRG}_{solt}^{any} \quad \mathcal{CRG}_{sst} \subseteq \mathcal{CRG}_{ssolt}^{any} \quad \mathcal{CRG}_{solt}^{any} \subseteq \mathcal{CRG}_{ssolt}^{any} .$$

We will now strengthen the above results. The next three propositions will be used to demonstrate that all the inclusions in Proposition 3.4 are strict.

**Proposition 3.5** $\mathcal{CRG}_{sst} \setminus \mathcal{CRG}_{solt}^{all} \neq \varnothing$ *and* $\mathcal{CRG}_{sst} \setminus \mathcal{CRG}_{solt}^{any} \neq \varnothing$.

**Proof.** Let $\Sigma = (\{a\}, \{a\}, \gamma, \{a\}, \{a\}, \{r, r'\})$ be an SST-system, where $\gamma$ consists of nodes $\{1, 2\}$ with an edge between 1 and 2, and:

$$r = (\{1\} : \{a\} \to \varnothing)$$
$$r' = (\{2\} : \{a\} \to a_{\triangleright 1}) .$$

Then: (i) $enbld_{CRG(\Sigma)}(C_0) = 2^{\{r,r'\}}$, (ii) $C_0 \xrightarrow{\{r\}} C_1$, and (iii) $enbld_{CRG(\Sigma)}(C_1) = \{\varnothing, \{r'\}, \{r,r'\}\}$, where $C_0 = (\{a\}, \{a\})$ and $C_1 = (\varnothing, \{a\})$.

We show that there is no SOLT-system $\Sigma'$ such that $CRG(\Sigma) \cong CRG_{all}(\Sigma')$ or $CRG(\Sigma) \cong CRG_{any}(\Sigma')$. Suppose that such a $\Sigma'$ exists (in both cases). Thus, $\Sigma'$ has two rules $r$ and $r'$ (perhaps differently defined than their counterparts in $\Sigma$) and no others. Let $C_0'$ and $C_1'$ be the two configurations of $\Sigma'$ corresponding to $C_0$ and $C_1$, respectively.

We first observe that $\{r\}$ is RES-enabled at $C_1'$, because $\{r, r'\} \in enbld_{CRG(\Sigma)}(C_1)$ and $\Sigma'$ has no signal objects. We then consider two cases.

Case 1: $\mathfrak{c}^r \cap \mathfrak{c}^{r'} = \varnothing$. Then, since $\{r\}$ is RES-enabled at $C_1'$ and there are no other rules apart from $r'$, it is also CTRL$_{all}$-enabled and CTRL$_{any}$-enabled at $C_1'$, contradicting (iii).

Case 2: $\mathfrak{c}^r \cap \mathfrak{c}^{r'} \neq \varnothing$. Then, since $\{r, r'\} \in enbld_{CRG_{any}(\Sigma')}(C_1')$, it follows that $\{r, r'\}$ and $\{r'\}$ are RES-enabled at $C_1'$. So $\{r'\}$ is not CTRL$_{any}$-enabled at $C_1'$, a contradiction. In the case of CTRL$_{all}$-enabledness, it would have to be the case, by (iii), that $\mathfrak{c}^{r'} \subset \mathfrak{c}^r$. However, this would produce a contradiction with $\{r\}, \{r, r'\} \in enbld_{CRG_{all}(\Sigma')}(C_0')$ (see (i)).

Hence $\Sigma'$ does not exist. $\square$

**Proposition 3.6** $\mathcal{CRG}_{solt}^{all} \setminus \mathcal{CRG}_{sst} \neq \varnothing$.

**Proof.** Let $\Sigma = (\{a\}, \varnothing, \gamma, \{a\}, \{a\}, \{a\}, \{a\}, \{r, r', r''\})$ be an SOLT-system, where $\gamma$ has $\{1, 2, 3, 4\}$ as its set of nodes and no edges, and:

$$r = (\{1, 3\} : \{a\} \to \varnothing)$$
$$r' = (\{2, 4\} : \{a\} \to \varnothing)$$
$$r'' = (\{1, 2\} : \{a\} \to \varnothing).$$

Then $enbld_{CRG_{all}(\Sigma)}(C_0) = 2^{\{r,r',r''\}} \setminus \{r, r'\}$, where $C_0 = (\{a\}, \{a\}, \{a\}, \{a\})$ is the initial configuration of $\Sigma$.

We need to show is that there is no SST-system $\Sigma'$ such that $CRG_{all}(\Sigma) \cong CRG(\Sigma')$. Suppose that such a $\Sigma'$ exists.

Thus, the rules of $\Sigma'$ are $r$, $r'$, and $r''$ (perhaps differently defined than their counterparts in $\Sigma$). Their localities in $\Sigma'$ are given by $\mathfrak{c}^r = \{i\}$, $\mathfrak{c}^{r'} = \{j\}$, and $\mathfrak{c}^{r''} = \{k\}$. We observe that the following hold:

(1) $\{r\}$, $\{r'\}$, $\{r''\}$, $\{r, r''\}$, $\{r', r''\}$, and $\{r, r', r''\}$ are all RES-enabled at $C_0'$ (the initial configuration of $\Sigma'$).

(2) $i \neq j \neq k \neq i$. This follows from (1) and the definition of CTRL$_{all}$-enabledness $(CRG(\Sigma') = CRG_{all}(\Sigma') = CRG_{all}(\Sigma))$.

(3) $\{r, r'\}$ is RES-enabled at $C_0'$. This follows from (1) and Proposition 3.1.

Hence, by (2), (3) and the definition of CTRL$_{all}$-enabledness, $\{r, r'\}$ is CTRL$_{all}$-enabled at $C_0'$, contradicting $\{r, r'\} \notin enbld_{CRG_{all}(\Sigma)}(C_0) = enbld_{CRG_{all}(\Sigma')}(C_0')$. $\quad\square$

**Proposition 3.7** $\mathcal{CRG}_{solt}^{any} \setminus \mathcal{CRG}_{sst} \neq \varnothing$.

**Proof.** Let $\Sigma = (\{a, b\}, \varnothing, \gamma, \{a, b\}, \{a, b\}, \{r, r', r''\})$ be an SOLT-system, where $\gamma$ has $\{1, 2\}$ as its set of nodes and no edges, and:

$$
\begin{aligned}
r &= (\{1\} &: \{a\} \rightarrow \{a\}) \\
r' &= (\{2\} &: \{a\} \rightarrow \{a\}) \\
r'' &= (\{1, 2\} : \{b\} \rightarrow \varnothing) \, .
\end{aligned}
$$

Then: (i) $enbld_{CRG_{any}(\Sigma)}(C_0) = \{\varnothing, \{r, r', r''\}\}$, where $C_0 = (\{a, b\}, \{a, b\})$ is the initial configuration; (ii) $C_0 \xrightarrow{\{r,r',r''\}}_{any} C_1$, where $C_1 = (\{a\}, \{a\})$; and (iii) $enbld_{CRG_{any}(\Sigma)}(C_1) = \{\varnothing, \{r\}, \{r'\}, \{r, r'\}\}$.

We need to show that there is no SST-system $\Sigma'$ such that (iv) $CRG_{any}(\Sigma) \cong CRG(\Sigma')$.

Suppose that such a $\Sigma'$ exists. Let $C_0' = (w_1, \ldots, w_m)$ be its initial configuration. The rules of $\Sigma'$ are $r$, $r'$, and $r''$ (perhaps differently defined than their counterparts in $\Sigma$). Their localities in $\Sigma'$ are given by $\mathfrak{c}^r = \{i\}$, $\mathfrak{c}^{r'} = \{j\}$, and $\mathfrak{c}^{r''} = \{k\}$. Moreover, let $C_0' \xrightarrow{\{r,r',r''\}} C_1'$ where $C_1' = (u_1, \ldots, u_m)$. Recall that for $\Sigma'$, CTRL$_{any}$-enabledness and CTRL$_{all}$-enabledness are the same notion.

We first show that $\{r\}$ is RES-enabled at $C_0'$ in $\Sigma'$. Suppose that this does not hold. Then, by (i) and (iv), there is (in $\Sigma'$) a signal object $v$ such that $v \notin w_i$, $v \in lhs^r \setminus rhs^r$ and $v \in rhs^{r'} \cup rhs^{r''}$. By the definition of $any$-executability of $R = \{r, r', r''\}$ at $C_0'$, we have $v \notin u_i$. Hence $\{r\}$ is not RES-enabled at $C_1'$, contradicting (iii).

We then observe that, by (iii) and (iv), $enbld_{CRG_{any}(\Sigma')}(C_1') = \{\varnothing, \{r\}, \{r'\}, \{r, r'\}\}$. Hence, by the definition of CTRL$_{any}$-enabledness, we have $i \neq j$.

Suppose now that $i \neq k$. Since, as we just argued $\{r\}$ is RES-enabled at $C_0'$, we have, by the definition of CTRL$_{any}$-enabledness and $j \neq i \neq k$, that $\{r\}$ is CTRL$_{any}$-enabled at $C_0'$, contradicting (i).

Hence $i = k$ and, by symmetry, $j = k$. Thus $i = j$, a contradiction. $\quad\square$

From the above propositions we obtain a complete characterisation of the

relative expressive power of the various kinds of tissue systems considered here separately for the two different modes of execution.

**Theorem 3.8** *The graphs in Figure 3 depict strict inclusions between the domains forming their respective nodes. Moreover, no other inclusion holds.*

**Proof.** The weak $\subseteq$ inclusions all follow from Proposition 3.4. By Proposition 3.5 and Proposition 3.6, $\mathcal{CRG}_{solt}^{all}$ and $\mathcal{CRG}_{sst}$ are incomparable (not included in one another). Similarly, by Proposition 3.5 and Proposition 3.7, $\mathcal{CRG}_{solt}^{any}$ and $\mathcal{CRG}_{sst}$ are incomparable. As an immediate conseqence of the above, all inclusions depicted are strict. $\square$
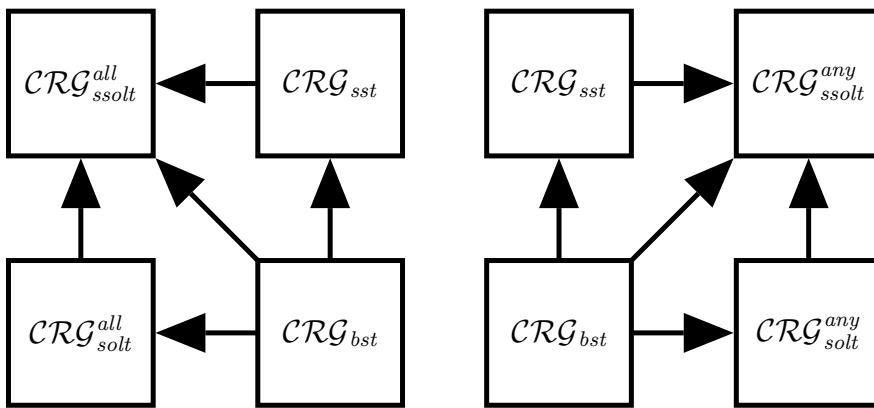


Fig. 3. Expressiveness of different classes of tissue systems.

## 4   Systems and nets

In this section we first add the concept of overlapping localities to the signal set nets of [13]. Next we compare the resulting net class with SSOLT-systems.

### 4.1   Signal set nets with overlapping localities

A *signal set net with overlapping localities*, or SSOL-net, is a tuple $N = (P, P^{sgl}, T, F, \ell, M_0)$, where $P$ and $T$ are finite disjoint sets of respectively *places* and *transitions*, $P^{sgl} \subseteq P$ are *signal places*, $F \subseteq (P \times T) \cup (T \times P)$ is the *flow relation*, $\ell : P \cup T \to 2^{\{1,2,\ldots\}} \setminus \varnothing$ is the *locality mapping*, and $M_0 \subseteq P$ is the *initial* marking. It is assumed that $\ell(t)$ is finite, for every transition $t \in T$, and that $|\ell(p)| = 1$, for every $p \in P$. For a transition $t$, we refer to $\ell(t)$ as the *locality of* $t$ and to the elements of $\ell(t)$ as *locations*. Thus a transition

may have several locations which can be shared with other transitions (in case of overlapping localities). Places have only one location, but different places may have the same location.

If $|\ell(t)| = 1$, for every $t \in T$, then $N$ is basically a signal set net from [13], and if, additionally, $P^{sgl} = \varnothing$ then we obtain a model as in [12]. The latter was derived from SET-nets [19], originally developed as a Petri net model for reaction systems [6, 8, 9]. In [20], overlapping localities are introduced, but the formalisation is different: a transition can belong to several designated subsets of transitions, referred to as localities; and there are no explicit localities assigned to places.

Let $N$ be a SSOL-net as specified above.

A set of places is a *marking*, and a set of transitions is a *step*. If $t$ is a transition, then we denote by $^\bullet t = \{p \in P \mid \exists (p,t) \in F\}$ its set of *input places* and by $t^\bullet = \{p \in P \mid (t,p) \in F\}$ its set of *output places*. Moreover, if $U$ is a step, then we define $^\bullet U = \bigcup_{t \in U} {}^\bullet t$ and $U^\bullet = \bigcup_{t \in U} t^\bullet$; and $\ell_U = \bigcup_{t \in U} \ell(t)$ are the locations associated with the transitions of $U$. Moreover, for $t \in T$: $^\bullet t = {}^\bullet \{t\}$ and $t^\bullet = \{t\}^\bullet$.

A step $U$ is *resource enabled*, or RES-enabled, at a marking $M$ if $^\bullet U \setminus M \subseteq U^\bullet \cap P^{sgl}$, i.e., if all input non-signal places of its transitions belong to $M$, and all signal input places that are not in $M$ are outputs of its transitions.

Markings indicate the presence of resources, but without any quantification (hence 'set net'). Consequently, if a place is marked (belongs to the current marking), it can provide input (resources) to any number of simultaneously executed transitions. Executing a transition empties all its input places and each of its output places will belong to the resulting marking. Moreover, resources produced as output for signal places can be consumed in the same step by another (or even the same) transition having that place as an input place.

To be executable at a marking, a RES-enabled step must also be 'maximal' in the sense that it cannot be extended with additional transitions with a locality that is already involved in that step. We distinguish between two criteria. According to the first one, there are no transitions left out which could be added to yield a new RES-enabled step with the same set of localities as the original step; whereas the second one requires that there are no transitions left out that have an active locality (their localities overlap with some of the localities of the transitions forming the step) and could be added to yield a new RES-enabled step.

Let $U$ be a step RES-enabled at a marking $M$. Then $U$ is:

- *all-control enabled* (or CTRL$_{all}$-enabled) at $M$ if there is no step $U' \supset U$ which is RES-enabled at $M$ with $\ell_{U'} = \ell_U$, and

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- *any-control enabled* (or $\text{CTRL}_{any}$-enabled) at $M$ if there is no step $U' \supset U$ which is RES-enabled at $M$ with $\ell(t) \cap \ell_U \neq \varnothing$, for every $t \in U' \setminus U$.

A step $U$ which is $\text{CTRL}_\mu$-enabled w.r.t. an execution mode $\mu \in \{all, any\}$ at $M$ can be *$\mu$-executed* leading to the marking $M'$ given by

$$M' = (M \setminus {}^\bullet U) \cup (U^\bullet \setminus ({}^\bullet U \setminus M)) = ((M \cup U^\bullet) \setminus {}^\bullet U) \cup (M \cap U^\bullet \cap {}^\bullet U) .$$

We denote this by $M[U\rangle_\mu M'$. It should be observed here that the subscript $\mu$ refers to the enabling condition; the effect of executing a step is in both modes the same. A *$\mu$-computation* of $N$ is a finite sequence of $\mu$-executions starting from $M_0$, and any marking which can be obtained through such a computation is *$\mu$-reachable*. Moreover, the *$\mu$-concurrent reachability graph of* $N$ is the step transition system $CRG_\mu(N) = ([M_0\rangle_\mu, 2^T, \delta, M_0)$, where $[M_0\rangle_\mu$ is the set of all $\mu$-reachable markings which are the states of the graph, $M_0$ is the initial state, and

$$\delta = \{(M, U, M') \mid M, M' \in [M_0\rangle_\mu \wedge U \subseteq T \wedge M[U\rangle_\mu M'\} .$$

Note that we can assume here - without loss of generality - that all transitions of $N$ are useful, i.e., they occur in at least one $\mu$-execution in a $\mu$-computation of $N$).

### 4.2   Relating SSOLT-*systems and* SSOL-*nets*

In this sub-section, we show that the two models of concurrent systems introduced in this paper are closely related in a behavioural sense. More precisely, we will identify for each SSOLT-system an SSOL-net that generates essentially the same concurrent reachability graph. As before in, e.g., [13, 14], the evolution rules become transitions and the objects will be places - in this case preserving their (non-)signalling properties; the flow relation preserves the input and output of the evolution rules; and the locality mapping corresponds to compartments and localities in the SSOLT-system. This leads to a natural partition of the constructed places with a subset for each different object and each place in that subset corresponding to one of the compartments where the object may reside. Conversely, given a tissue structure and an SSOL-net that has input and output relations between transitions and places 'consistent' with their localities, we will be able to construct an SSOLT-system with the same concurrent reachability graph.

Let $\gamma$ be a tissue structure, $N = (P, P^{sgl}, T, F, \ell, M_0)$ be an SSOL-net, and $\Pi$ be a partition of the set of places $P$ such that, for every $\pi \in \Pi$, $\ell$ is injective on $\pi$, and $\pi \subseteq P \setminus P^{sgl}$ or $\pi \subseteq P_{STS}^{sgl}$. Then $N$ is *spanned* over $\gamma$ w.r.t. $\Pi$ if, for all $p \in P$, $t \in T$, and $\pi \in \Pi$ such that $p \in \pi$, the following hold:

- $\ell(p), \ell(t) \subseteq \gamma_{nodes}$,
- $(p, t) \in F$ implies $\ell(p) \subseteq \ell(t)$ and, for every $i \in \ell(t)$, there is $p' \in \pi$ such that $(p', t) \in F$ and $\ell(p') = \{i\}$,
- $(t, p) \in F$ implies one of the following:
  - $\ell(p) \subseteq \ell(t)$ and, for every $i \in \ell(t)$, there is $p' \in \pi$ such that $(t, p') \in F$ and $\ell(p') = \{i\}$, or
  - $\ell(p) \cap \ell(t) = \varnothing$ and $(\ell(t) \times \ell(p)) \cap \gamma_{edges} \neq \varnothing$ (i.e., there is an edge in $\gamma$ between some $i$ and $j$, where $(i, j) \in \ell(t) \times \ell(p)$).

### 4.2.1 SSOL-*nets corresponding to* SSOLT-*systems*

Let $\Sigma = (V, V^{sgl}, \gamma, w_1^0, \ldots, w_m^0, \mathcal{R})$ be an arbitrary, but fixed, SSOLT-system. The SSOL-net *corresponding to* $\Sigma$ is $\mathsf{ts2pn}(\Sigma) = (P, P^{sgl}, T, F, \ell, M_0)$, where:

- $P = \{p_i^a \mid i \in \gamma_{nodes} \land a \in V\}$, $P^{sgl} = \{p_i^a \mid i \in \gamma_{nodes} \land a \in V^{sgl}\}$, $T = \{t^r \mid r \in \mathcal{R}\}$.
- $\ell(p) = \{i\}$ and $\ell(t) = \mathfrak{c}^r$, for all $p = p_i^a \in P$ and $t = t^r \in T$; moreover:

$$ {}^\bullet t = \{p_i^a \mid i \in \mathfrak{c}^r \land a \in lhs^r\} $$

$$ t^\bullet = \{p_i^a \mid i \in \mathfrak{c}^r \land a \in rhs^r\} \cup \{p_i^a \mid a_{\triangleright i} \in rhs^r\} \,. $$

- $M_0 = \{p_i^a \mid i \in \gamma_{nodes} \land a \in w_i^0\}$.

**Proposition 4.1** $\mathsf{ts2pn}(\Sigma)$ *is an* SSOL-*net spanned over the tissue structure* $\gamma$ *w.r.t. the partition* $\Pi = \{\{p_i^a \mid i \in \gamma_{nodes}\} \mid a \in V\}$ *of the places of* $\mathsf{ts2pn}(\Sigma)$.

**Proof.** Follows directly from the definitions. $\square$

The configurations of $\Sigma$ and the markings of $\mathsf{ts2pn}(\Sigma)$ are related by a mapping $\nu$ which, for every configuration $C = (w_1, \ldots, w_m)$ of $\Sigma$, yields a marking $\nu(C) = \{p_i^a \mid i \in \gamma_{nodes} \land a \in w_i\}$ of $\mathsf{ts2pn}(\Sigma)$. Moreover, $\phi(r) = t^r$ is the transition of $\mathsf{ts2pn}(\Sigma)$ corresponding to rule $r \in \mathcal{R}$.

**Proposition 4.2** *The mappings* $\nu$ *and* $\phi$ *are bijections, and*

$$ \nu^{-1}(M) = (\{a \in V \mid p_1^a \in M\}, \ldots, \{a \in V \mid p_m^a \in M\}) \,, $$

*for every marking* $M$ *of* $\mathsf{ts2pn}(\Sigma)$.

**Proof.** Follows directly from the definitions. $\square$

**Proposition 4.3** $\nu(C_0) = M_0$, *where* $C_0$ *is the initial configuration of* $\Sigma$.

**Proof.** Follows directly from the definitions. $\square$

The strong behavioural relationship between the operation of SSOLT-systems and the corresponding SSOL-nets is established in the next result.

**Proposition 4.4** *Let $C$ and $C'$ be configurations, $R \subseteq \mathcal{R}$ be a set-rule of $\Sigma$, and $\mu$ be an execution mode. Then:*

$$C \xrightarrow{R}_\mu C' \qquad \textit{if and only if} \qquad \nu(C) \, [\phi(R)\rangle_\mu \, \nu(C')$$
$$\textit{in } \Sigma \qquad\qquad\qquad\qquad \textit{in } \textsf{ts2pn}(\Sigma) \, .$$

**Proof.** Below, for a configuration $C = (w_1, \ldots, w_m)$ of $\Sigma$ and a set-rule $R$ of $\Sigma$, we denote:

$$\textsf{set}(C) = \{a@i \mid i \in \gamma_{nodes} \wedge a \in w_i\}$$
$$\textsf{in}(R) = \{a@i \mid i \in \mathfrak{c}^R \wedge a \in lhs_i^R\}$$
$$\textsf{out}(R) = \{a@i \mid i \in \mathfrak{c}^R \wedge a \in rhs_i^R\} \, ,$$

where $a@i$ denotes an object $a$ in compartment $i$ (an extended name).

Then $R$ is RES-enabled at a configuration $C$ iff

$$\textsf{in}(R) \setminus \textsf{set}(C) \subseteq \textsf{out}(R) \cap \{a@i \mid i \in \gamma_{nodes} \wedge a \in V^{sgl}\} \, .$$

The latter is equivalent to ${}^\bullet\phi(R) \setminus \nu(C) \subseteq \phi(R)^\bullet \cap P^{sgl}$ since each $p_i^a$ can be identified with $a@i$, and, for each $R \subseteq \mathcal{R}$, the set of input places of $\phi(R)$ can be identified with $\textsf{in}(R)$ and the set of output places of $\phi(R)$ with $\textsf{out}(R)$.

Hence a set-rule $R$ is RES-enabled at $C$ iff $\phi(R)$ is RES-enabled at $\nu(C)$. Moreover, $\mathfrak{c}^r = \ell(t^r)$, for every $r \in \mathcal{R}$, and so we have $\mathfrak{c}^R = \ell_{\phi(R)}$. Hence $R$ is CTRL$_\mu$-enabled at $C$ iff $\phi(R)$ is CTRL$_\mu$-enabled at $\nu(C)$. It therefore remains to show that the results of the executions of $R$ at $C$ and of $\phi(R)$ at $\nu(C)$ correspond to each other.

Suppose that $C \xrightarrow{R}_\mu C'$. Then

$$\textsf{set}(C') = (\textsf{set}(C) \setminus \textsf{in}(R)) \cup (\textsf{out}(R) \setminus (\textsf{in}(R) \setminus \textsf{set}(C))) \, ,$$

and so $\nu(C') = (\nu(C) \setminus {}^\bullet\phi(R)) \cup (\phi(R)^\bullet \setminus ({}^\bullet\phi(R) \setminus \nu(C)))$. Hence we obtain $\nu(C) \, [\phi(R)\rangle_\mu \, \nu(C')$ which completes the proof. $\quad\square$

Together with Propositions 4.2 and 4.3, this means that the concurrent reachability graphs of $\Sigma$ and $\textsf{ts2pn}(\Sigma)$ are isomorphic if we identify each rule with the corresponding transition. Hence we obtain

**Theorem 4.5** *Let $\mu$ be an execution mode. Then $CRG_\mu(\Sigma)$ and $CRG_\mu(\textsf{ts2pn}(\Sigma))$ are $\phi$-isomorphic step transition systems.*

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*4.2.2* SSOLT-*systems corresponding to* SSOL-*nets*

Let $N = (P, P^{sgl}, T, F, \ell, M_0)$ be an arbitrary, fixed, SSOL-net spanned over the tissue structure $\gamma$ of degree $m$ w.r.t. a partition $\Pi$ of $P$. The SSOLT-system *corresponding to* $N$ w.r.t. $\Pi$ is $\mathsf{pn2ts}(N, \Pi) = (V, V^{sgl}, \gamma, w_1^0, \ldots, w_m^0, \mathcal{R})$, where:

- $V = \{a^\pi \mid \pi \in \Pi\}$ and $V^{sgl} = \{a^\pi \mid \pi \in \Pi \wedge \pi \subseteq P^{sgl}\}$.
- $w_i^0 = \{a^\pi \mid \pi \in \Pi \wedge M_0 \cap \pi \cap \ell^{-1}(\{i\}) \neq \varnothing\}$, for every $i \leq m$. We then define $\nu'(M) = (w_1, \ldots, w_m)$, where $w_i = \{a^\pi \mid \pi \in \Pi \wedge M \cap \pi \cap \ell^{-1}(\{i\}) \neq \varnothing\}$, for every marking $M$ of $N$.
- $\mathcal{R} = \{r^t \mid t \in T\}$, where each $r^t = \phi'(t) = (\ell(t) : lhs \to rhs)$ is such that:

$$
\begin{aligned}
lhs \;&=\; \{a^\pi \mid \pi \in \Pi \wedge \pi \cap {}^\bullet t \neq \varnothing\} \\
rhs \;&=\; \{a^\pi \mid \pi \in \Pi \wedge \exists p \in \pi \cap t^\bullet : \ell(p) \subseteq \ell(t)\} \;\cup \\
&\quad\; \{a_{\rhd i}^\pi \mid \pi \in \Pi \wedge \exists p \in \pi \cap t^\bullet : \ell(p) \cap \ell(t) = \varnothing \wedge \ell(p) = \{i\}\} \;.
\end{aligned}
$$

**Proposition 4.6** $\mathsf{pn2ts}(N, \Pi)$ *is an* SSOLT-*system over the tissue structure* $\gamma$.

**Proof.** Follows from the definitions. $\square$

The translation results in a very close behavioural correspondence between the chosen classes of Petri nets and tissue systems. By proceeding similarly as in the case of the previous translation, we obtain

**Theorem 4.7** *Let* $\mu$ *be an execution mode. Then the step transition systems* $CRG_\mu(N)$ *and* $CRG_\mu(\mathsf{pn2ts}(N, \Pi))$ *are* $\phi'$-*isomorphic.*
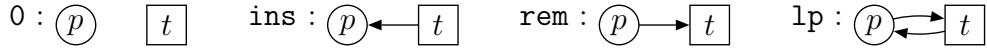
We can therefore conclude, by Theorems 4.5 and 4.7, that SSOLT-systems and SSOL-nets spanned over tissue structures are essentially equivalent semantical models. This means, in particular, that the synthesis of SSOLT-systems from step transition systems can be approached as the synthesis of SSOL-nets spanned over tissue structures.

# 5 Net-types

As discussed in, e.g., [3, 15], several Boolean Petri net classes (i.e., those types of nets where markings are subsets of places) can be distinguished on the basis of individual connections between places and transitions. Moreover, the effect of the simultaneous execution of a certain combination of transitions on a place can be calculated using a commutative monoid which returns the composite connection between the place and the given step. The resulting formalism, based on the generic notions of net-type $\tau$ and $\tau$-net, provides an alternative

presentation of net theory which is particularly suitable for dealing with the net synthesis problem. In this section we demonstrate that SSOL-nets are a class of $\tau$-nets.

A *net-type* is supposed to capture the behaviour of a place of a net of a particular kind. It is an LT-system $\tau = (\mathbb{Q}, \mathbb{S}, \delta)$ such that: $\mathbb{Q}$ is a set of states; $\mathbb{S}$ is a *connection monoid* (a set of *connections* equipped with a commutative and associative binary composition operation $\oplus$ and a neutral (identity) element 0); and $\delta$ is a partial function satisfying $\delta(q, 0) = q$, for all $q \in \mathbb{Q}$. In the case of SSOL-nets considered here, $\mathbb{Q} = \{\mathbf{0}, \mathbf{1}\}$ indicating whether a place is marked or not, and $\mathbb{S} = \{0, \mathtt{ins}, \mathtt{rem}, \mathtt{lp}\}$, i.e., there are four kinds of possible connections between places and transitions: 'no connection',' insert', 'remove' and 'loop'.

$$0 : \; \textcircled{p} \quad \boxed{t} \qquad \mathtt{ins} : \; \textcircled{p} \longleftarrow \boxed{t} \qquad \mathtt{rem} : \; \textcircled{p} \longrightarrow \boxed{t} \qquad \mathtt{lp} : \; \textcircled{p} \rightleftarrows \boxed{t}$$

The above connections give rise to two different interpretations in terms of net-types, depending on the intended 'signalling' status of a place. We therefore consider two net-types, $\tau^{nsgl} = (\mathbb{Q}, \mathbb{S}, \delta^{nsgl})$ and $\tau^{sgl} = (\mathbb{Q}, \mathbb{S}, \delta^{sgl})$, with $\delta^{nsgl}$ and $\delta^{sgl}$ given respectively in Figure 4(a) and Figure 4(c), and the common table, shown in Figure 4(b), defining the composition operation used.

**Proposition 5.1** $(\mathbb{S}, \oplus)$, *where* $\mathbb{S} = \{0, \mathtt{ins}, \mathtt{rem}, \mathtt{lp}\}$ *and* $\oplus$ *is defined in Figure 4(b), is a commutative monoid with* 0 *as neutral element,* $\mathtt{lp}$ *an absorbing element, and* $\mathtt{ins}$ *and* $\mathtt{rem}$ *idempotent elements.*

**Proof.** $\oplus$ is commutative as the table in Figure 4(b) is symmetric, and the listed properties of individual elements clearly hold. Hence, to show the associativity of $\oplus$, we only need to observe that $\mathtt{ins} \oplus (\mathtt{ins} \oplus \mathtt{rem}) = \mathtt{lp} = (\mathtt{ins} \oplus \mathtt{ins}) \oplus \mathtt{rem}$, $\mathtt{ins} \oplus (\mathtt{rem} \oplus \mathtt{ins}) = \mathtt{lp} = (\mathtt{ins} \oplus \mathtt{rem}) \oplus \mathtt{ins}$, and $\mathtt{rem} \oplus (\mathtt{ins} \oplus \mathtt{ins}) = \mathtt{lp} = (\mathtt{rem} \oplus \mathtt{ins}) \oplus \mathtt{ins}$. $\square$

The only difference between the net-types $\tau^{nsgl}$ and $\tau^{sgl}$ is in the 'loop' connection which in the signalling case is non-blocking (enabled) also for $\mathbf{0}$. However, it is not the case that $0 = \mathtt{lp}$ for signal places since they behave differently in compositions as 0 is the neutral element whereas $\mathtt{lp}$ is an absorbing element.

A class of nets can be rendered as a class of $\tau$-nets if we can find a suitable net-type $\tau$ to describe the behaviour of its places. Unlike in the standard definitions of $\tau$-nets, the class of $\tau$-nets corresponding to SSOL-nets, will employ two net-types, $\tau^{nsgl}$ and $\tau^{sgl}$, to accommodate both signal and non-signal places as they have different semantical properties.

As before, we let $\mathbb{Q} = \{\mathbf{0}, \mathbf{1}\}$ and $\mathbb{S} = \{0, \mathtt{ins}, \mathtt{rem}, \mathtt{lp}\}$.
A *signal set* $\tau$-*net with overlapping localities* (or SSOL-$\tau$-net) is a tuple $Z = (P, P^{sgl}, T, G, \ell, M_0)$, where $P$, $P^{sgl}$, $T$ and $\ell$ are as in the definition of an
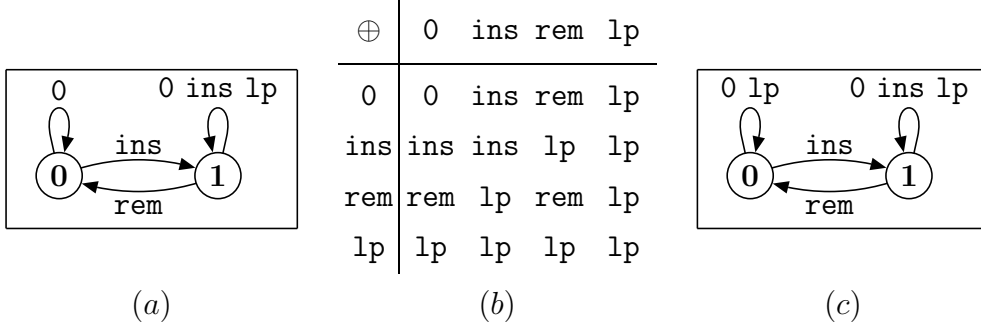
Fig. 4. (*a* & *b*) define the net-type $\tau^{nsgl}$; and (*b* & *c*) define the net-type $\tau^{sgl}$.

SSOL-net, $G : (P \times T) \to \mathbb{S}$ is the connection mapping, and $M_0 : P \to \mathbb{Q}$ is the initial marking (in general, any mapping $M : P \to \mathbb{Q}$ is a marking).

Below, for $p \in P$ and $U \subseteq T$, we denote $G(p, U) = \bigoplus_{t \in U} G(p, t)$ with $G(p, \varnothing) = 0$ and $\bigoplus_{t \in \{t_1, \ldots, t_n\}} G(p, t) = G(p, t_1) \oplus \cdots \oplus G(p, t_n)$, where $\oplus$ is as defined in Figure 4(*b*).

A step $U \subseteq T$ of $Z$ is *resource enabled* (or RES-enabled) at a marking $M$ if, for every $p \in P \setminus P^{sgl}$, $G(p, U) \in enbld_{\tau^{nsgl}}(M(p))$, and, for every $p \in P^{sgl}$, $G(p, U) \in enbld_{\tau^{sgl}}(M(p))$.

Let $U$ be a step RES-enabled at a marking $M$. Then $U$ is:

- *all-control enabled* (or CTRL$_{all}$-enabled) at $M$ if there is no step $U' \supset U$ which is RES-enabled at $M$ with $\ell_{U'} = \ell_U$, and
- *any-control enabled* (or CTRL$_{any}$-enabled) at $M$ if there is no step $U' \supset U$ which is RES-enabled at $M$ with $\ell(t) \cap \ell_U \neq \varnothing$, for every $t \in U' \setminus U$.

A step $U$ which is CTRL$_\mu$-enabled at $M$ can be *$\mu$-executed* leading to the marking $M'$ such that, for every $p \in P \setminus P^{sgl}$, $M'(p) = \delta^{nsgl}(M(p), G(p, U))$, and, for every $p \in P^{sgl}$, $M'(p) = \delta^{sgl}(M(p), G(p, U))$. We denote this by $M[U\rangle_\mu M'$. A *$\mu$-computation* of $Z$ is then a finite sequence of $\mu$-executions starting from $M_0$, and any marking which can be obtained through such a computation is *$\mu$-reachable*. Moreover, the *$\mu$-concurrent reachability graph of $Z$* is the step transition system $CRG_\mu(Z) = ([M_0\rangle_\mu, 2^T, \delta, M_0)$, where $[M_0\rangle_\mu$ is the set of all $\mu$-reachable markings which are the states of the graph, $M_0$ is the initial state, and

$$\delta = \{(M, U, M') \mid M, M' \in [M_0\rangle_\mu \wedge U \subseteq T \wedge M[U\rangle_\mu M'\}.$$

Again, we can assume that all transitions are useful.

Let $\gamma$ be a tissue structure and $\Pi$ be a partition of the set of places $P$ such that, for very $\pi \in \Pi$, $\ell$ is injective on $\pi$, and $\pi \subseteq P \setminus P^{sgl}$ or $\pi \subseteq P^{sgl}_{STS}$. Then $Z$ is *spanned* over $\gamma$ w.r.t. $\Pi$ if, for all $p \in P$, $t \in T$, and $\pi \in \Pi$ such that $p \in \pi$, the following hold:

- $\ell(p), \ell(t) \subseteq \gamma_{nodes}$;
- $G(p,t) \in \{\texttt{rem}, \texttt{lp}\}$ implies $\ell(p) \subseteq \ell(t)$ and, for every $i \in \ell(t)$, there is $p' \in \pi$ such that $G(p',t) = G(p,t)$ and $\ell(p') = \{i\}$,
- $G(p,t) = \texttt{ins}$ implies one of the following:
  - $\ell(p) \subseteq \ell(t)$ and, for every $i \in \ell(t)$, there is $p' \in \pi$ such that $G(p',t) = \texttt{ins}$ and $\ell(p') = \{i\}$, or
  - $\ell(p) \cap \ell(t) = \varnothing$ and $(\ell(t) \times \ell(p)) \cap \gamma_{edges} \neq \varnothing$.

**Theorem 5.2** *There is a bijection $\psi$ from* SSOL-*nets to* SSOL-$\tau$-*nets such that* $CRG_\mu(N) = CRG_\mu(\psi(N))$, *for every* SSOL-*net $N$ and every execution mode $\mu$. Moreover, the sets of places of $N$ and $\psi(N)$ are the same, and $N$ is spanned over $\gamma$ w.r.t. $\Pi$ iff $\psi(N)$ is spanned over $\gamma$ w.r.t. $\Pi$, for every tissue structure $\gamma$ and every partition $\Pi$ of the common set of places of $N$ and $\psi(N)$.*

**Proof.** Let $N = (P, P^{sgl}, T, F, \ell, M_0)$ be an SSOL-net. We then define the SSOL-$\tau$-net $\psi(N) = (P, P^{sgl}, T, G, \ell, M_0)$, where, for all $p \in P$ and $t \in T$:

$$
G(p,t) = \begin{cases}
\texttt{0} & if \ (p,t) \notin F \wedge (t,p) \notin F \\
\texttt{ins} & if \ (p,t) \notin F \wedge (t,p) \in F \\
\texttt{rem} & if \ (p,t) \in F \wedge (t,p) \notin F \\
\texttt{lp} & if \ (p,t) \in F \wedge (t,p) \in F \, .
\end{cases}
$$

It is easy to check that, for all $M \subseteq P$ and $U \subseteq T$, $U$ is RES-enabled at $M$ in $N$ iff $U$ is RES-enabled at $M$ in $\psi(N)$. Hence, for every execution mode $\mu$, $U$ is CTRL$_\mu$-enabled at $M$ in $N$ iff $U$ is CTRL$_\mu$-enabled at $M$ in $\psi(N)$. As a result, $CRG_\mu(N) = CRG_\mu(\psi(N))$. Moreover, the second part of the theorem is obvious. $\square$

Thanks to Theorems 4.7 and 5.2, the problem of synthesising an SSOLT-system from a given ST-system can be replaced by the problem of synthesising an SSOL-$\tau$-net from a given ST-system.


## 6  Synthesis


To specify the behaviour of a net to be synthesised, we use ST-systems. As the semantics of the nets we intend to synthesise, SSOL-nets, can be expressed by two net-types, it is possible to adapt the approach developed for the general theory of Petri net synthesis investigated, e.g., in [2, 3, 4, 16, 15].

We consider the following problem:

SYNTHESIS

Let $STS = (Q, 2^T, \delta, q_0)$ be an ST-system, $\gamma$ be a tissue structure, $\ell : T \to 2^{\gamma_{nodes}} \setminus \varnothing$ be a locality mapping for $T$, and $\mu$ be an execution mode. Provide necessary and sufficient conditions for $STS$ to be $\mu$-realisable by some SSOL-$\tau$-net $Z$, i.e., $STS \cong CRG_\mu(Z)$, so that $Z$ is spanned over $\gamma$ w.r.t. some partition of its places, and the locality mapping of $Z$ extends $\ell$. Moreover, construct a suitable $Z$ and a partition of its places if $STS$ is $\mu$-realisable.

In what follows, $STS$, $\gamma$, $\ell$, and $\mu$ are fixed.

Existing solutions to similar net synthesis problems tend to use some notion of a *region* of $STS$ whose definition depends on the net-type of the nets for which it is defined. Each region represents a place in the net to be constructed and contains all the information about its marking (local state) at every $q \in Q$, and its connection to every $t \in T$ given by an element of the connection monoid of the net-type. Therefore, in the definition of a region, we will have two mappings $\sigma$ and $\eta$ to calculate such markings and connections. Moreover, the specific notion of a region defined below will comprise the locality of the place it represents, and we will need to ensure that this locality is consistent with the fact that the net being constructed has to be spanned over the tissue structure $\gamma$. Finally, we will also need to define two kinds of regions to reflect the difference in behaviour of signal and non-signal places.

Let $\xi \in \{nsgl, sgl\}$ and $\tau^\xi = (\mathbb{Q}, \mathbb{S}, \delta^\xi)$. A $\tau^\xi$-*region* of $STS$ in the SYNTHESIS problem is a triple $\rho = (k \in \gamma_{nodes}, \sigma : Q \to \mathbb{Q}, \eta : T \to \mathbb{S})$ such that, for all $t \in T$, $q \in Q$, and $U \in enbld_{STS}(q)$:

- $\eta(t) \in \{\mathtt{rem}, \mathtt{lp}\}$ implies $k \in \ell(t)$,
- $\eta(t) = \mathtt{ins}$ and $k \notin \ell(t)$ implies $(\ell(t) \times \{k\}) \cap \gamma_{edges} \neq \varnothing$, and
- $\bigoplus \eta(U) \in enbld_{\tau^\xi}(\sigma(q))$ and $\delta^\xi(\sigma(q), \bigoplus \eta(U)) = \sigma(\delta(q, U))$,

where $\bigoplus \eta(\varnothing) = \mathtt{0}$ and $\bigoplus \eta(U) = \eta(t_1) \oplus \cdots \oplus \eta(t_n)$, for $U = \{t_1, \ldots, t_n\}$.

Below we denote $\rho = (k_\rho, \sigma_\rho, \eta_\rho)$. The set of $\tau^\xi$-regions is denoted by $Reg^\xi_{STS}$, and $Reg_{STS} = Reg^{nsgl}_{STS} \cup Reg^{sgl}_{STS}$ denotes the set of $\tau$-*regions* of $STS$.

By comparing the diagrams of net-types $\tau^{nsgl}$ and $\tau^{sgl}$ in Figure 4, we immediately obtain

**Proposition 6.1** $Reg^{nsgl}_{STS} \subseteq Reg^{sgl}_{STS}$.

Hence, checking whether a triple $(k, \sigma, \eta)$ is a $\tau$-region means checking it first for the non-signalling case, and only if this fails, checking the signalling case.

Having introduced $\tau$-regions, one may now introduce a derived notion of step enabledness. For every $q \in Q$, $enbld_{STS,\tau}(q)$ is the set of all $\tau$-enabled steps $U \subseteq T$ meaning that $\bigoplus \eta_\rho(U) \in enbld_\tau(\sigma_\rho(q))$, for each $\tau$-region $\rho$ of $STS$. Directly from the definitions, we obtain

**Proposition 6.2** $enbld_{STS}(q) \subseteq enbld_{STS,\tau}(q)$, for every $q \in Q$.

We then obtain a complete characterisation of the positive instances of the SYNTHESIS problem.

**Theorem 6.3** *The* SYNTHESIS *problem has a solution iff there exists a non-empty family $\Omega$ of non-empty sets of $\tau$-regions of STS such that the following hold, for all $\omega \in \Omega$, $\rho \in \omega$, $t \in T$, $q, q' \in Q$, and $U \subseteq T$:*

(1) *$\ell$ is injective on $\omega$, and $\omega \subseteq Reg_{STS}^{nsgl}$ or $\omega \subseteq Reg_{STS}^{sgl}$.*
(2) *If $\eta_\rho(t) \in \{\mathtt{rem}, \mathtt{ins}, \mathtt{lp}\}$ and $k_\rho \in \ell(t)$, then, for every $i \in \ell(t)$, there is $\rho' \in \omega$ with $k_{\rho'} = i$ and $\eta_{\rho'}(t) = \eta_\rho(t)$.*
(3) AXIOM I: STATE SEPARATION
   *If $q \neq q'$ then there is $\rho \in \bigcup \Omega$ with $\sigma_\rho(q) \neq \sigma_\rho(q')$.*
(4) AXIOM II: FORWARD CLOSURE
   *If $U \notin enbld_{STS}(q) \cup cds_\mu(enbld_{STS,\tau}(q))$ then there is $\rho \in \bigcup \Omega$ with $\bigoplus \eta_\rho(U) \notin enbld_\tau(\sigma_\rho(q))$, where, for every $\mathcal{X} \subseteq 2^T$:[3]*

$$cds_{all}(\mathcal{X}) = \{U \in \mathcal{X} \mid \exists U' \in \mathcal{X} : U \subset U' \wedge \ell_{U'} = \ell_U\}$$

$$cds_{any}(\mathcal{X}) = \{U \in \mathcal{X} \mid \exists U' \in \mathcal{X} : U \subset U' \wedge (\forall t \in U' \setminus U : \ell(t) \cap \ell_U \neq \varnothing)\} .$$

*Moreover, if (1)–(4) above hold, then $Z = (P, P^{sgl}, T, G, \ell, M_0)$, where:*

- *$P = \{p_\rho^\omega \mid \rho \in \omega \in \Omega\}$ and $P^{sgl} = \{p_\rho^\omega \mid \rho \in \omega \in \Omega \wedge \omega \subseteq Reg_{STS}^{sgl}\}$,*
- *$G(p_\rho^\omega, t) = \eta_\rho(t)$, for all $p_\rho^\omega \in P$ and $t \in T$, and*
- *$\ell(p_\rho^\omega) = \{k_\rho\}$ and $M_0(p_\rho^\omega) = \sigma_\rho(q_0)$, for every $p_\rho^\omega \in P$,*

*is a* SSOL-$\tau$-*net solving the* SYNTHESIS *problem. In addition, a suitable partition of $P$ is given by $\Pi = \{\{p_\rho^\omega \mid \rho \in \omega\} \mid \omega \in \Omega\}$.*

**Proof.** Note that $\Pi$ is a partition of $\bigcup \Pi$ even if $\Omega$ was not a partition of $\bigcup \Omega$. Moreover, $Z$ is spanned over $\gamma$ w.r.t. $\Pi$ which follows from (1) and (2) above and the definition of a $\tau$-region.

The theorem follows by a straightforward adaptation of the results presented in [4] and [5]. The only property we need to verify is that $cds_\mu$ defines a *step firing policy* in the sense of [4]. This means that we need to show that, for all $\mathcal{X} \subseteq 2^T$ and $\mathcal{Y} \subseteq \mathcal{X}$:

---

[3] It is always the case that $\varnothing \notin cds_{all}(\mathcal{X})$ and $\varnothing \notin cds_{any}(\mathcal{X})$. The notation '$cds$' comes from [4] and is meant to indicate 'control disabled steps'.

(i) $cds_\mu(\mathcal{X}) \subseteq \mathcal{X}$.
(ii) $cds_\mu(\mathcal{Y}) \subseteq cds_\mu(\mathcal{X})$.
(iii) $\mathcal{X} \subseteq 2^T$ and $\mathcal{X} \setminus cds_\mu(\mathcal{X}) \subseteq \mathcal{Y}$ imply $cds_\mu(\mathcal{X}) \cap \mathcal{Y} \subseteq cds_\mu(\mathcal{Y})$.

Clearly, (i) and (ii) hold. To show (iii), let $\mu = all$, $\mathcal{X} \setminus cds_{all}(\mathcal{X}) \subseteq \mathcal{Y} \subseteq \mathcal{X} \subseteq 2^T$ and $U \in cds_{all}(\mathcal{X}) \cap \mathcal{Y}$. Since $U \in cds_{all}(\mathcal{X})$ and $T$ is finite, there is a maximal (w.r.t. $\subseteq$) set $U' \in \mathcal{X}$ such that $U \subset U'$ and $\ell_{U'} = \ell_U$. Clearly, $U' \notin cds_{all}(\mathcal{X})$, and so $U' \in \mathcal{X} \setminus cds_{all}(\mathcal{X}) \subseteq \mathcal{Y}$. As $U \in \mathcal{Y}$ by assumption, we have $U \in cds_{all}(\mathcal{Y})$.

For $\mu = any$ the proof is similar. $\square$

Note that we only need to consider a minimal $\Omega$, i.e., such that deleting any of its components would make at least one instance of AXIOMS I or II unsatisfied.

It is interesting to observe the relationship between signal and non-signal places in the constructed net. By Proposition 6.1, $Reg^{nsgl}_{STS} \subseteq Reg^{sgl}_{STS}$. If $\rho \in Reg^{nsgl}_{STS}$ then its 'witnessing power' w.r.t. AXIOM I is the same as for $\rho \in Reg^{sgl}_{STS}$. However, considering AXIOM II, the same is no longer the case as $\rho$ will have a strictly greater witnessing power when interpreted as included in $Reg^{nsgl}_{STS}$ than when interpreted as included in $Reg^{sgl}_{STS}$ (this follows from the fact that the LT-system in Figure 4(c) enables at least as many connections as that in Figure 4(a) at the corresponding states). Hence, whenever we have a witnessing $\rho \in Reg^{nsgl}_{STS}$ there is no need to add its counterpart $\rho \in Reg^{sgl}_{STS}$ to the solution being constructed.

## 7    Conclusion

In this paper we have introduced a new class of set tissue systems by formalising the idea that the computational process progresses in synchronised overlapping localities. This feature has been shown to properly extend the expressive power of the original systems. Currently, Theorem 3.8 provides a full picture of relative expressiveness of tissue systems for each of the *all* and *any* modes separately. We plan to investigate the relationships between the two modes.

Next SSOLT-systems have been related to SSOL-nets with the same behaviour. Since SSOL-nets can be rendered as $\tau$-nets, their synthesis problem is solvable and thus also that of SSOLT-systems. In fact, to establish the class of $\tau$-nets that corresponds to SSOL-nets it was necessary to introduce two net-types rather than one as is standard. We consider this further evidence for the solidity and flexibility of the general region-based approach to synthesis of Petri nets based on $\tau$-nets.

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