

ABSOLUTELY CONTINUOUS INVARIANT MEASURES FOR NON-AUTONOMOUS DYNAMICAL SYSTEMS.

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ABSTRACT. We consider the non autonomous dynamical system $\{\tau_n\}$, where τ_n is a continuous map $X \rightarrow X$, and X is a compact metric space. We assume that $\{\tau_n\}$ converges uniformly to τ . The inheritance of chaotic properties as well as topological entropy by τ from the sequence $\{\tau_n\}$ has been studied in [4, 5, 10, 13, 17]. In [16] the generalization of SRB measures to non-autonomous systems has been considered. In this paper we study absolutely continuous invariant measures (acim) for non autonomous systems. After generalizing the Krylov-Bogoliubov Theorem [7] and Straube's Theorem [14] to the non autonomous setting, we prove that under certain conditions the limit map τ of a non autonomous sequence of maps $\{\tau_n\}$ with acims has an acim.

1. INTRODUCTION

Autonomous systems are rare in nature. A more realistic approach to modeling real life processes is to consider non autonomous models. In this note we consider a sequence of maps $\{\tau_n\}$ on a compact metric space $X \rightarrow X$. We assume that $\{\tau_n\}$ converges uniformly to τ . Let $\tau_{(0,n)} = \tau_n \circ \tau_{n-2} \circ \cdots \circ \tau_1 \circ \tau_0$. For an initial measure η we consider the sequence $\mu_n = (\tau_{(0,n)})_* \eta$. Since X is compact the space of probability measures on X is $*$ -weakly compact and hence we can assume that $\{\mu_n\}$ converges to a measure μ . In this note we study conditions under which the limit map τ preserves μ . In particular we are interested in the situation when μ_n and μ are absolutely continuous.

The behaviour of non autonomous sequences of piecewise expanding maps was studied before. In the paper [12] the authors consider a family \mathcal{E} of exact piecewise expanding maps with uniform expanding properties and show that for any two initial densities f_1, f_2 the iterates $P_{\tau_{(0,n)}} f_1$ and $P_{\tau_{(0,n)}} f_2$ get closer to each other with exponential speed. Using the notation of Section 2:

$$\int |P_{\tau_{(0,n)}} f_1 - P_{\tau_{(0,n)}} f_2| dm \leq C(f_1, f_2) \Lambda^n, \quad n \geq 1,$$

for some constants $C(f_1, f_2) > 0$, $0 < \Lambda < 1$ and any sequence of maps $\tau_n \in \mathcal{E}$. In this situation, in general, there is no limit map and the densities $P_{\tau_{(0,n)}} f$ do not converge. In this note we assume the uniform convergence $\tau_n \rightrightarrows \tau$. This allows us to prove that, under some assumptions, the densities $P_{\tau_{(0,n)}} f$ converge to a τ -invariant density.

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Another approach to dealing with compositions of different maps is to consider a random map. Maps from a family $\mathcal{E} = \{\tau_a\}_{a \in \mathcal{A}}$ are applied randomly according to a probability on \mathcal{A} , which might depend on the current position of the process. The literature on random maps is quite rich. A recent article is [1]. The authors study, in particular, random maps based on the set \mathcal{E} of the Liverani-Saussol-Vaienti maps

$$\tau_a(x) = \begin{cases} x(1 + 2^a x^a), & x \in [0, 1/2], \\ 2x - 1, & x \in (1/2, 1], \end{cases}$$

with parameters in $[a_0, a_1] \subset (0, 1)$ chosen independently with respect to a distribution ν on $[a_0, a_1]$. These maps have indifferent fixed points which makes them non-exponentially mixing. The authors study the fibre-wise (quenched) dynamics of the system. For this point of view a skew-product approach is convenient.

Let $(\mathcal{A}, \mathcal{F}, p)$ be a Borel probability space, let $\Omega = \mathcal{A}^{\mathbb{Z}}$ be equipped with the product measure $P := p^{\mathbb{Z}}$ and let $\sigma : \Omega \rightarrow \Omega$ denote the P -preserving two-sided shift map. Let (X, \mathcal{B}) be a measurable space. Suppose that $\tau_a : X \rightarrow X$ is a family of measurable maps defined for p -almost every $a \in \mathcal{A}$ such that the skew product

$$T : X \times \Omega \rightarrow X \times \Omega, \quad T(x, \omega) = (\tau_{[\omega]_0}, \sigma\omega),$$

is measurable with respect to $\mathcal{B} \times \mathcal{F}$. If $X_\omega = X \times \{\omega\}$ denotes the fiber over ω and

$$\tau_\omega^n = \tau_{\sigma^{n-1}\omega} \circ \cdots \circ \tau_\omega : X_\omega \rightarrow X_{\sigma^n\omega},$$

we have $T^n(x, \omega) = (\tau_\omega^n(x), \sigma^n\omega)$. If a probability measure μ is T -invariant and $\pi_*\mu = P$ (π is the projection onto Ω), then there exists a family of probability fiber measures μ_ω on X_ω such that $\mu(A) = \int \mu_\omega(A) dP(\omega)$ for any $A \in \mathcal{B} \times \mathcal{F}$. Since μ is T -invariant the measures $\{\mu_\omega\}$ form an equivariant family, i.e., $(\tau_\omega)_*\mu_\omega = \mu_{\sigma\omega}$ for almost all ω .

The authors study future and past quenched correlations: given $\phi, \psi : X \times \Omega \rightarrow \mathbb{R}$ the future and past fibre-wise correlations are defined as

$$\begin{aligned} Cor_{n,\omega}^{(f)} &= \int (\phi \circ \tau_\omega^n) \psi d\mu_\omega - \int \phi d\mu_{\sigma^n\omega} \int \psi d\mu_\omega, \\ Cor_{n,\omega}^{(p)} &= \int (\phi \circ \tau_{\sigma^{-n}\omega}^n) \psi d\mu_{\sigma^{-n}\omega} - \int \phi d\mu_\omega \int \psi d\mu_{\sigma^{-n}\omega}. \end{aligned}$$

They prove that for the random map based on family \mathcal{E} there exists an equivariant family of measures μ_ω which are absolutely continuous P -a.e., characterize their densities and show that both future and past quenched correlations are of order $\mathcal{O}(n^{1-1/a_0} + \delta)$ for bounded ϕ and Hölder continuous ψ and arbitrary $\delta > 0$. The system (T, μ) is mixing.

In this note we assume that $\tau_n \rightrightarrows \tau$ and consider the compositions $\tau_{(0,n)} = \tau_n \circ \tau_{n-2} \circ \cdots \circ \tau_1 \circ \tau_0$, so we can say that we study one fixed fiber under very special assumptions.

In Section 2 we give the definitions and introduce the notation. In Section 3 we generalize the Krylov-Bogoliubov Theorem [7] and Straube's Theorem [14] to the non autonomous setting. Section 4 is independent of the previous section. We make stronger assumptions on the τ_n 's and establish the existence of an acim for the limit map τ and show that any convergent subsequence of $\{P_{\tau_{(0,n)}} f\}_{n \geq 1}$ converges to an invariant density of the limit map, where $P_{\tau_{(0,n)}}$ is the Frobenius-Perron operator induced by $\tau_{(0,n)}$ and f is a density.

2. NOTATION AND DEFINITIONS

Let (X, ρ) be a compact metric space. Let $\{\tau_n\}$ be a sequence of maps $\tau_n : X \rightarrow X$ which converges uniformly to a continuous map τ . We shall consider the non-autonomous dynamical system defined by

$$x_{m+1} = \tau_m(x_m), \quad m = 0, 1, 2, \dots$$

where we assume that τ_0 is the identity and $x_0 \in I$.

We write

$$\tau_{(m,n)} = \tau_n \circ \tau_{n-2} \circ \dots \circ \tau_{m+1} \circ \tau_m, \quad n > m.$$

In particular,

$$\tau_{(0,n)} = \tau_n \circ \tau_{n-2} \circ \dots \circ \tau_1 \circ \tau_0.$$

Let $\mathcal{B}(X)$ be the σ -algebra of Borel subsets of X .

For a map $\tau : X \rightarrow X$ we define an operator on measures on $\mathcal{B}(X)$:

$$\tau_*\mu(A) = \mu(\tau^{-1}A),$$

for any measurable set A .

3. GENERALIZATION OF THE KRYLOV-BOGOLIUBOV THEOREM AND STRAUBE'S THEOREM

We will now prove a generalization of the Krylov-Bogoliubov Theorem:

Theorem 1. *Let $\{\tau_n\}$ be a sequence of transformations defining a nonautonomous dynamical system on the metric compact space X with a continuous limit τ . We assume that the τ_n 's converge uniformly to τ . Let η be a fixed probability measure on X . Define the measures $\mu_n = \frac{1}{n} \sum_{i=1}^n \nu_i$, where $\nu_i = (\tau_{(0,i)})_*(\eta)$. Let μ be a $*$ -weak limit point of the sequence $\{\mu_n\}_{n \geq 1}$. Then μ is a τ -invariant measure, i.e., $\tau_*\mu = \mu$.*

Proof. We follow the proof of the original Krylov-Bogoliubov Theorem. Let η be a probability measure on X . Then the sequence $\mu_n = \frac{1}{n} \sum_{i=1}^n \nu_i$, where $\nu_i = (\tau_{(0,i)})_*(\eta)$ is a sequence of probability measures and contains a convergent subsequence μ_{n_k} . Let $\mu = \lim_{k \rightarrow \infty} \mu_{n_k}$. We will prove that $\tau_*\mu = \mu$. To this end it is enough to show that for any $g \in C^0(X)$, $\mu(g) = \tau_*\mu(g) = \mu(g \circ \tau)$.

We estimate the difference

$$\begin{aligned} (1) \quad & |\mu_n(g) - \mu_n(g \circ \tau)| = \frac{1}{n} \left| \sum_{i=1}^n \nu_i(g) - \sum_{i=1}^n \nu_i(g \circ \tau) \right| \\ &= \frac{1}{n} \left| \eta(g \circ \tau_{(0,1)}) + \eta(g \circ \tau_{(0,2)}) + \dots + \eta(g \circ \tau_{(0,n-1)}) + \eta(g \circ \tau_{(0,n)}) \right. \\ &\quad \left. - \eta(g \circ \tau \circ \tau_{(0,1)}) - \eta(g \circ \tau \circ \tau_{(0,2)}) - \dots - \eta(g \circ \tau \circ \tau_{(0,n-1)}) - \eta(g \circ \tau \circ \tau_{(0,n)}) \right| \\ &= \frac{1}{n} \left| \eta(g \circ \tau_{(0,1)}) + \sum_{i=2}^n (\eta(g \circ \tau_{(0,i)}) - \eta(g \circ \tau \circ \tau_{(0,i-1)})) - \eta(g \circ \tau \circ \tau_{(0,n)}) \right|. \end{aligned}$$

Let ω_g be the modulus of continuity of g , i.e.,

$$\omega_g(\delta) = \sup_{\rho(x,y) < \delta} |g(x) - g(y)|.$$

For an arbitrary $\varepsilon > 0$ we can find a $\delta > 0$ such that $\omega_g(\delta) < \varepsilon$. Since $\tau_n \rightarrow \tau$ uniformly for this δ we can find an $N \geq 1$ such that $\sup_{x \in X} \rho(\tau_n(x), \tau(x)) < \delta$ for all $n > N$.

For $i > N$, we have

$$\begin{aligned} & \left| \eta(g \circ \tau_{(0,i)}) - \eta(g \circ \tau \circ \tau_{(0,i-1)}) \right| = \left| \eta(g \circ \tau_i \circ \tau_{(0,i-1)} - g \circ \tau \circ \tau_{(0,i-1)}) \right| \\ & = \left| \eta((g \circ \tau_i - g \circ \tau)(\tau_{(0,i-1)})) \right| \leq \omega_g(\delta) < \varepsilon. \end{aligned}$$

Thus, for $n > N$, we have

$$|\mu_n(g) - \mu_n(g \circ \tau)| \leq \frac{1}{n} (N \cdot 2 \cdot \sup |g| + (n - N)\varepsilon),$$

which becomes arbitrarily close to ε as $n \rightarrow \infty$. This shows that

$$\mu_{n_k}(g) - \mu_{n_k}(g \circ \tau) \rightarrow 0 \text{ as } k \rightarrow \infty.$$

We have $\mu_{n_k}(g) \rightarrow \mu(g)$ and since τ is continuous $\mu_{n_k}(g \circ \tau) \rightarrow \mu(g \circ \tau) = \tau_*\mu(g)$. Thus, μ is a τ -invariant measure. \square

Remark: The only place where we needed the continuity of τ is the last line of the proof: since τ is continuous $g \circ \tau$ is continuous for any continuous g and then the $*$ -weak convergence of μ_{n_k} implies $\mu_{n_k}(g \circ \tau) \rightarrow \mu(g \circ \tau)$.

Theorem 1 does not yield any more information about the τ -invariant measure μ . The next result is a generalization of a theorem by Straube [14], which provides a sufficient condition for μ to be absolutely continuous.

Theorem 2. *Let (X, \mathcal{B}, ν) be a normalized measure space and let $\{\tau_n\}$ be a sequence of non-singular transformations defining a non-autonomous dynamical system on X . We do not assume that the limit τ is continuous. Assume there exists $\delta > 0$ and $0 < \alpha < 1$ such that*

$$\nu(E) < \delta \implies \sup_{k \geq 1} \nu(\tau_{(0,k)}^{-1}(E)) < \alpha,$$

for all $E \in \mathcal{B}$. Then there exists a τ -invariant normalized measure μ which is absolutely continuous with respect to ν .

(The proof uses a number of facts from the theory of finitely additive measures which are collected in the Appendix. The proof is similar to the proof in [14] but is modified to allow the use of the estimates from the proof of Theorem 1.)

Proof. Let us define the measures

$$\nu_n(E) = \frac{1}{n} \sum_{k=0}^{n-1} \nu(\tau_{(0,k)}^{-1}(E)), \quad E \in \mathcal{B}.$$

Then, for all n ,

- (a) $\nu_n(X) = 1$;
- (b) $\nu_n \ll \nu$ (τ_n is non-singular for every n);
- (c) $\nu_n(\cdot) \geq 0$.

Thus, $\{\nu_n\}$ is a sequence of positive, normalized, absolutely continuous measures and can be treated as a sequence in the unit ball of $L_\infty^*(X)$ with the $*$ -weak topology. Thus, it contains a convergent subsequence $\nu_{n_k} \rightarrow z$ and z can be identified with a finitely additive measure on X . The measure z is finitely additive, positive, normalized and absolutely continuous with respect to ν .

By Lemma 7 in the Appendix we can uniquely decompose z into

$$z = z_c + z_p,$$

where z_c is countably additive and z_p is purely finitely additive. Now, we claim that $z_c \neq 0$. Otherwise, by Lemma 6, there exists a decreasing sequence $\{E_n\} \subset \mathcal{B}$ such that $\lim_{n \rightarrow \infty} \nu(E_n) = 0$ and $z(E_n) = z(X) = 1$ for all $n \geq 1$. Since $\nu(E_n) \rightarrow 0$, for any $\delta > 0$, there exists an n_0 such that $n > n_0 \implies \nu(E_n) < \delta$. Now, by our assumptions, there is an $\alpha < 1$ such that,

$$\sup_k \nu(\tau_{(0,k)}^{-1}(E_n)) < \alpha < 1.$$

Thus, $\nu(\tau_{(0,k)}^{-1}(E_n)) < \alpha$ for all k . So,

$$z(E_n) < \alpha < 1,$$

which is a contradiction. We have demonstrated that $z_c \neq 0$.

Now we will prove that z_c is τ -invariant. Consider the finitely additive measure

$$\kappa = z - z \circ \tau^{-1} = z_c - z_c \circ \tau^{-1} + z_p - z_p \circ \tau^{-1}.$$

In the proof of Theorem 1 we showed that for any continuous function g on X we have

$$\mu_{n_k}(g) - \mu_{n_k}(\tau^{-1}(g)) \rightarrow 0, \quad k \rightarrow \infty.$$

This means that for any continuous function g (which is bounded since X is compact) we have

$$\kappa(g) = z(g) - z \circ \tau^{-1}(g) = 0.$$

We do not need continuity of τ here as $\mu_{n_k}(h) \rightarrow z(h)$ for all bounded h . By Lemma 9 in the Appendix the countably additive component of κ is 0, which means

$$z_c - z_c \circ \tau^{-1} = 0,$$

or that z_c is τ -invariant. \square

In the following example we show that, unlike in the case of one transformation, the converse implication in Theorem 2 may not hold. We will construct a sequence of maps $\tau_n \rightarrow \tau$, such that τ admits an acim and

$$(2) \quad \forall \delta > 0 \exists E \in \mathcal{B} \sup_{k \geq 1} \nu(\tau_{(2,k)}^{-1}(E)) = 1.$$

Example 3. Let us consider maps $\tau_n : [0, 1] \rightarrow [0, 1]$, $n = 2, 3, \dots$, defined as follows

$$\tau_n(x) = \begin{cases} (1 - \frac{1}{n})x, & \text{for } x \in [0, \frac{1}{2}); \\ 2x - 1, & \text{for } x \in [\frac{1}{2}, 1]. \end{cases}$$

The limit map $\tau(x) = x\chi_{[0, \frac{1}{2})}(x) + (2x + 1)\chi_{[\frac{1}{2}, 1]}(x)$ admits an acim and condition (2) holds.

Proof. Let $\rho_n = \tau_n|_{[0, \frac{1}{2})}$ be the first branch of τ_n . The slope of $\rho_n = \frac{n-1}{n}$ so the slope of $\rho_{m,n} = \rho_n \circ \rho_{n-1} \circ \rho_{n-2} \circ \dots \circ \rho_m$, $n > m$, is $\frac{n-1}{n} \cdot \frac{n-2}{n-1} \cdot \frac{n-3}{n-2} \dots \frac{m-1}{m} = \frac{m}{n} < 1$. Then, the interval $\rho_{m,n}^{-1}([0, \delta])$ is the interval from 0 to the minimum of $\delta \cdot \frac{n}{m}$ and $\frac{1}{2}$. Note, that for any k , we have

$$(3) \quad \rho_k^{-1}([0, \frac{1}{2}]) = [0, \frac{1}{2}].$$

Letting $\varrho = \varrho_n = \tau_n|_{[\frac{1}{2}, 1]}$ be the second branch of τ_n , we have

$$\begin{aligned} \varrho^{-1}\left(\left[0, \frac{1}{2}\right]\right) &= \left[\frac{1}{2}, \frac{1}{2} + \frac{1}{4}\right]; \\ \varrho^{-1}\left(\left[\frac{1}{2}, \frac{1}{2} + \frac{1}{4}\right]\right) &= \left[\frac{1}{2} + \frac{1}{4}, \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right]; \\ &\vdots \\ \varrho^{-1}\left(\left[\sum_{i=1}^k \frac{1}{2^i}, \sum_{i=1}^{k+1} \frac{1}{2^i}\right]\right) &= \left[\sum_{i=1}^{k+1} \frac{1}{2^i}, \sum_{i=1}^{k+2} \frac{1}{2^i}\right]. \end{aligned}$$

This and (3) imply that

$$\tau_{(2, m-1)}^{-1}\left(\left[0, \frac{1}{2}\right]\right) = \left[0, \sum_{i=1}^{m-1} \frac{1}{2^i}\right].$$

Let $\varepsilon > 0$ and m such that $1 - \sum_{i=1}^{m-1} \frac{1}{2^i} < \varepsilon$. Let n satisfy $\delta \cdot \frac{n}{m} > \frac{1}{2}$. Then the Lebesgue measure of $\tau_{2,n}^{-1}([0, \delta])$ is larger than $1 - \varepsilon$. \square

4. EXISTENCE OF AN ABSOLUTELY CONTINUOUS INVARIANT MEASURE FOR THE LIMIT MAP

In this section we will assume that all the maps τ_n are piecewise expanding maps of an interval. For the general theory of such maps we refer the reader to [3] or [8].

Let $I = [0, 1]$. The map $\tau : I \rightarrow I$ is called piecewise expanding iff there exists a partition $\mathcal{P} = \{I_i := [a_{i-1}, a_i], i = 1, \dots, q\}$ of I such that $\tau : I \rightarrow I$ satisfies the following conditions:

- (i) τ is monotonic on each interval I_i ;
- (ii) $\tau_i := \tau|_{I_i}$ is C^2 , i.e., C^2 in the interior and the one-sided limits of the derivatives are finite at endpoints;
- (iii) $|\tau'_i(x)| \geq s_i \geq s > 1$ for any i and for all $x \in (a_{i-1}, a_i)$.

The following Frobenius-Perron operator $P_\tau : L^1(I, m) \rightarrow L^1(I, m)$, where m is Lebesgue measure, is a basic tool in the theory of piecewise expanding maps. For a general non-singular map τ [$m(A) = 0 \implies m(\tau^{-1}(A)) = 0$], we define $P_\tau f$ as a Radon-Nikodym derivative $\frac{d(\tau_* m)}{dm}$. For piecewise expanding maps the operator can be written explicitly [3]:

$$P_\tau f(x) = \sum_{i=1}^q \frac{f(\tau_i^{-1}(x))}{|\tau'_i(\tau_i^{-1}(x))|}.$$

In particular $P_\tau f = f$ iff $f \cdot m$ is an acim of τ . Piecewise expanding maps of the interval satisfy the following Lasota-Yorke inequality [9]. For any bounded variation function $f \in BV(I)$ the variation $V(P_\tau f)$ satisfies

$$V(P_\tau f) \leq AV(f) + B \int_I |f| dm,$$

where the constants $A = \frac{2}{s}$, $B = \frac{\max_s |\tau''|}{s} + \frac{2}{h}$ and $h = \min_i \{m(I_i)\}$. In particular, we can assume that $A < 1$, considering an iterate τ^k , if necessary. We always assume that bounded variation functions are modified to satisfy $f(x_0) = \limsup_{x \rightarrow x_0} f(x)$ for all $x_0 \in I$.

We will prove the following:

Theorem 4. *Assume that τ_n , $n = 1, 2, \dots$ are piecewise expanding maps of an interval and satisfy the Lasota-Yorke inequality with common constants $A < 1$ and B . Then, for any density $f \in BV(I)$, the sequence $f_n = \frac{1}{n} \sum_{i=1}^n P_{\tau_{(1,i)}} f$ forms a precompact set in L^1 and any convergent subsequence converges to a density of an acim of the limit map τ .*

Remark: We do not assume that the maps τ_n are defined on a common partition. We assume that they all satisfy Lasota-Yorke inequality with the same constant B . In the following lemma we show that this implies that the limit map τ is defined on a finite partition and the partitions for maps τ_n are “asymptotically” the same as the partition for τ .

Lemma 5. *Under the assumptions of Theorem 4 the limit map τ is piecewise monotonic and there exists a constant K such that for any interval J we have $m(\tau^{-1}(J)) \leq Km(J)$. In particular, it follows that the limit map τ is non-singular.*

Proof. Since the constant B depends on the reciprocal of h , there is a universal bound q_u on the number of elements of the partition \mathcal{P} for τ_n . This places a restriction on the number k of iterates we can use to make $A < 1$. Thus, there exists a universal lower bound s_u for the modulus of the derivative τ'_n .

Now, we prove that τ is piecewise monotonic. Assume that the graph of τ contains p points forming a “zigzag”, i.e., there exist $x_1 < x_2 < x_3 < \dots < x_{p-1} < x_p$ such that $\tau(x_i) < \tau(x_{i+1})$ for odd i and $\tau(x_i) > \tau(x_{i+1})$ for even i (or other way around). Then, $p \leq 2q_u$. If not, then since $\tau_n \rightrightarrows \tau$ uniformly, for large n the graph of τ_n also contains a zigzag of length p . This is impossible as τ_n has at most q_u branches of monotonicity. Thus, τ is piecewise monotonic with at most q_u branches of monotonicity.

Let $[a, b] \subset I$ be an interval. Each line $y = a$, $y = b$ intersects the graph of τ in at most q_u points. Let points (x_1, a) , (x_2, b) be the points of intersection of these lines with one monotonic, say increasing, branch of τ . Then,

$$b - a = \lim_{n \rightarrow \infty} \tau_n(x_2) - \tau_n(x_1) \geq \lim_{n \rightarrow \infty} s_u \cdot (x_2 - x_1) = s_u \cdot (x_2 - x_1).$$

If one (or two) of the intersections is empty, we replace appropriate x_i by the endpoint of the interval of monotonicity. Thus, for any interval J we have

$$(4) \quad m(\tau^{-1}(J)) \leq \frac{q_u}{s_u} m(J).$$

□

We can now prove Theorem 4.

Proof of Theorem 4. Since f is a density and the Frobenius-Perron operator preserves the integral of positive functions, we have $\int |P_{\tau_n} f| dm = 1$ for all $n \geq 1$. Since $P_{\tau_{(1,i)}} = P_{\tau_i} \circ P_{\tau_{i-1}} \circ \dots \circ P_{\tau_2} \circ P_{\tau_1}$, we can apply the Lasota-Yorke inequality consecutively and obtain

$$V(P_{\tau_{(1,i)}} f) \leq A^i V(f) + B(A^{i-1} + A^{i-2} + \dots + A^2 + A + 1) \leq A^i V(f) + \frac{B}{1-A}, \quad i \geq 1.$$

Thus, the functions $P_{\tau_{(1,i)}} f$ and also the functions f_n , $i, n \geq 1$, have uniformly bounded variation. Since for a bounded variation density f , $\sup_{x \in I} f(x) \leq 1 + V(f)$, these functions are also uniformly bounded. The sequence $\{f_n\}_{n \geq 1}$, being both

uniformly bounded and of uniformly bounded variation contains a subsequence $\{f_{n_k}\}_{k \geq 1}$ convergent almost everywhere to a function f^* of bounded variation by Helly's Theorem [11]. Additionally, by the Lebesgue Dominated Convergence Theorem, $\int_I f^* dm = 1$. This means that, by Scheffe's Theorem [2], $f_{n_k} \rightarrow f^*$ in the L^1 -norm. Thus, the sequence $\{f_n\}_{n \geq 1}$ forms a pre-compact set in L^1 and in particular, contains a subsequence convergent in L^1 to a function of bounded variation.

Now, we will prove that for any density F , $(P_{\tau_n} F - P_\tau F) \rightarrow 0$ weakly in L^1 , as $n \rightarrow \infty$. Let $g \in L^\infty(I, m)$ be an arbitrary bounded function and let us fix an $\varepsilon > 0$. By Lusin's Theorem [6, Th. 7.10] for any $\eta > 0$ there exists an open set $U \subset I$, $m(U) < \eta$, and a continuous function $G \in C^0(I)$ such that $g = G$ on $I \setminus U$ and $\sup |G| \leq \|g\|_\infty$. The Frobenius-Perron operator is a conjugate of the Koopman operator, that is for any $f \in L^1$ and any $g \in L^\infty$, we have $\int_I P_\tau f \cdot g dm = \int_I f \cdot g \circ \tau dm$. Therefore, we can write

$$\begin{aligned} & \left| \int_I (P_\tau F \cdot g - P_{\tau_n} F \cdot g) dm \right| \leq \int_I F |g \circ \tau - g \circ \tau_n| dm \\ & = \int_I F |g \circ \tau - G \circ \tau + G \circ \tau - G \circ \tau_n + G \circ \tau_n - g \circ \tau_n| dm \\ & \leq \int_{\tau^{-1}(U)} F |g \circ \tau - G \circ \tau| dm + \int_I F |G \circ \tau_n + G \circ \tau_n| dm + \int_{\tau_n^{-1}(U)} F |g \circ \tau_n - G \circ \tau_n| dm. \end{aligned}$$

Let $\sup G \leq \|g\|_\infty = M_g$. Let $I_F(t) = \sup_{\{A: m(A) < t\}} \int_A |F| dm$. It is known that $I_F(t) \rightarrow 0$ as $t \rightarrow 0$. Let ω_G be the modulus of continuity of G : $\omega_G(t) = \sup_{|x-y| \leq t} |G(x) - G(y)|$. Again, $\omega_G(t) \rightarrow 0$ as $t \rightarrow 0$. Using estimate (4) we obtain

$$\begin{aligned} & \left| \int_I (P_\tau F \cdot g - P_{\tau_n} F \cdot g) dm \right| \\ (5) \quad & \leq 2M_g I_F\left(\frac{qu}{su}\eta\right) + \omega_G(\sup |\tau_n - \tau|) + 2M_g I_F\left(\frac{qu}{su}\eta\right) \\ & = \omega_G(\|\tau_n - \tau\|_\infty) + 4M_g I_F\left(\frac{qu}{su}\eta\right). \end{aligned}$$

Let us fix an $\varepsilon > 0$. Since $\|\tau_n - \tau\|_\infty \rightarrow 0$, as $n \rightarrow \infty$ we can find $N \geq 1$ such that for all $n \geq N$ we have $\omega_G(\|\tau_n - \tau\|_\infty) < \varepsilon$. We can also find an $\eta > 0$ such that $4M_g I_F\left(\frac{qu}{su}\eta\right) < \varepsilon$. This shows that $(P_{\tau_n} F - P_\tau F) \rightarrow 0$ weakly in L^1 , as $n \rightarrow \infty$. Note, that this convergence is uniform over precompact subsets of L^1 , since the estimate (5) can be made common for all F in such a set (the functions in a precompact set are uniformly integrable).

Let $\{f_{n_k}\}_{k \geq 1}$ be a subsequence of $\{f_n\}_{n \geq 1}$ convergent in L^1 to f^* . To simplify the notation we will skip the subindex k . We will show that f^* is the density of an acim of τ , i.e., $P_\tau f^* = f^*$. We have

$$P_\tau f^* = P_\tau \left(\lim_{n \rightarrow \infty} f_n \right) = \lim_{n \rightarrow \infty} P_\tau f_n.$$

We will show that $P_\tau f_n - f_n$ converges weakly in L^1 to 0. Let $\phi_i = P_{\tau_{(1,i)}} f$, $i = 1, 2, \dots$. Then, $f_n = \frac{1}{n} (\phi_1 + \phi_2 + \dots + \phi_{n-1} + \phi_n)$. We can write

$$\begin{aligned} P_\tau f_n - f_n &= \frac{1}{n} (P_\tau \phi_1 + P_\tau \phi_2 + P_\tau \dots + P_\tau \phi_{n-1} + P_\tau \phi_n) - \frac{1}{n} (\phi_1 + \phi_2 + \dots + \phi_{n-1} + \phi_n) \\ &= \frac{1}{n} (P_\tau \phi_n - \phi_1) + \frac{1}{n} \sum_{i=1}^{n-1} (P_\tau \phi_i - \phi_{i+1}) = \frac{1}{n} (P_\tau \phi_n - \phi_1) + \frac{1}{n} \sum_{i=1}^{n-1} (P_\tau \phi_i - P_{\tau_{i+1}} \phi_i). \end{aligned}$$

Let I_Φ be a common I_F function for all ϕ_i 's. Let N and η be chosen as above. Let $n \geq N + 2$. Then, using estimate (5), we have

$$\begin{aligned} &\left| \int_I (P_\tau f_n - f_n) g \, dm \right| \\ &\leq \frac{1}{n} \int_I |(P_\tau \phi_n - \phi_1) g| \, dm + \frac{1}{n} \sum_{i=1}^N \int_I |(P_\tau \phi_i - P_{\tau_{i+1}} \phi_i) g| \, dm \\ &\quad + \frac{1}{n} \sum_{i=N+1}^{n-1} \int_I |(P_\tau \phi_i - P_{\tau_{i+1}} \phi_i) g| \, dm \\ &\leq \frac{2}{n} M_g + \frac{2}{n} N M_g + \frac{n-1-N}{n} (2\varepsilon). \end{aligned}$$

As $n \rightarrow \infty$ the right hand side becomes smaller than say 3ε . Since $\varepsilon > 0$ is arbitrary this proves that $P_\tau f_n - f_n$ converges weakly in L^1 to 0 and $P_\tau f^* = f^*$. \square

5. APPENDIX

Here we collect the results about finitely additive measures necessary for the proof of Theorem 2

Lemma 6. [Theorem 1.22 of [15]] *Let (X, \mathcal{B}) be a compact measure space. Let the measure η be purely finitely additive and $\eta \geq 0$. Let κ be a countably additive measure defined on (X, \mathcal{B}) such that $\kappa \geq 0$. Then, there exists a decreasing sequence $\{E_n\} \subset \mathcal{B}$ such that $\lim_{n \rightarrow \infty} \kappa(E_n) = 0$ and $\eta(E_n) = \eta(X)$ for all $n \geq 1$. Conversely, if κ is a measure and the above conditions hold for all countably additive κ , then η is purely finitely additive.*

Lemma 7. [Theorems 1.23 and 1.24 of [15]] *Let η be a measure such that $\eta \geq 0$. Then there exist unique measures η_p and η_c such that $\eta_p \geq 0$, $\eta_c \geq 0$, η_p is purely finitely additive, η_c is countably additive and*

$$\eta = \eta_p + \eta_c.$$

Lemma 8. [Contained in the proof of Theorem 1.23 of [15]] *Let η be a measure decomposed as $\eta = \eta_p + \eta_c$. Then, η_c is the greatest of the measures κ , such that $0 \leq \kappa \leq \eta$.*

Lemma 9. *If η is a non-negative finitely additive measure and*

$$\int_X g d\eta = 0,$$

for any continuous function on X , then η is purely finitely additive measure.

Proof. According to the Definition 1.13 of [15] we have to show that any countably additive measure κ satisfying

$$(6) \quad 0 \leq \kappa \leq \eta$$

is a zero measure. Let κ satisfy (6). Then for any continuous function g , we have

$$0 \leq \kappa(g) \leq \eta(g) = 0.$$

Therefore $\kappa(g) = 0$ for all continuous functions g . Since κ is a countably additive measure, $\kappa = 0$. \square

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