

DIFFERENTIABILITY IN PERTURBATION PARAMETER OF MEASURE SOLUTIONS TO PERTURBED TRANSPORT EQUATION

PIOTR GWIAZDA¹, SANDER C. HILLE², KAMILA ŁYCZEK³,
AND AGNIESZKA ŚWIERCZEWSKA-GWIAZDA³

ABSTRACT. We consider a linear perturbation in the velocity field of the transport equation. We investigate solutions in the space of bounded Radon measures and show that they are differentiable with respect to the perturbation parameter in a proper Banach space, which is predual to the Hölder space $\mathcal{C}^{1+\alpha}(\mathbb{R}^d)$. This result on differentiability is necessary for application in optimal control theory, which we also discuss.

1. INTRODUCTION

Analysis of perturbations in partial differential equation systems is an important issue. *Structured population models* [CnCC13, GLM10, GM10, CCGU12, CGR18], dynamics of system [CnCR11, BGSW13, DHL14, FLOS, AFM] and *vehicular traffic flow* [BDDR08, EHM16, GS16, GR17] were investigated for Lipschitz dependence on initial conditions in space of measures. However, the differentiability (not only Lipschitz dependence) is necessary for the application in optimal control theory or linearised stability. Previous considerations concerning the transport equation in the space of measures did not allow to analyse the differentiability of solutions with respect to a perturbation of the system [AGS08, Thi03, PF14].

In this paper we consider solutions to a perturbed transport equation in the space of bounded Radon measures, denoted by $\mathcal{M}(\mathbb{R}^d)$, where the perturbation is linear in the velocity field.

Consider the initial value problem for the transport equation in conservative form

$$\begin{cases} \partial_t \mu_t + \operatorname{div}_x(b\mu_t) = w\mu_t & \text{in } (\mathcal{C}_c^1([0, \infty) \times \mathbb{R}^d))^*, \\ \mu_{t=0} = \mu_0 \in \mathcal{P}(\mathbb{R}^d), \end{cases} \quad (1.1)$$

where the velocity field $(t \mapsto b(t, \cdot)) \in \mathcal{C}^0([0, +\infty) ; \mathcal{C}^{1+\alpha}(\mathbb{R}^d))$, the initial condition is a probability measure on \mathbb{R}^d and $w(t, x) \in \mathcal{C}^{1+\alpha}([0, \infty) \times \mathbb{R}^d)$. By $(\cdot)^*$ we denote the topological dual to (\cdot) , when the latter is equipped with a suitable locally convex or norm topology; \mathcal{C}_c^1 is

2010 Mathematics Subject Classification. Primary: 35Q93; Secondary: 28A33.

¹INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES, ŚNIADECKICH 8, 00-656 WARSZAWA, POLAND

²MATHEMATICAL INSTITUTE, LEIDEN UNIVERSITY P.O. BOX 9512, 2300 RA LEIDEN, THE NETHERLANDS

³INSTITUTE OF APPLIED MATHEMATICS AND MECHANICS, UNIVERSITY OF WARSAW, BANACHA 2, 02-097 WARSZAWA, POLAND

E-mail addresses: pgwiazda@mimuw.edu.pl, shille@math.leidenuniv.nl, k.lyczek@mimuw.edu.pl, aswiercz@mimuw.edu.pl.

the space of continuous functions with compact support and $\mathcal{C}^{1+\alpha}$ is the space of functions of which first order partial derivatives are Hölder continuous with exponent α , where $0 < \alpha \leq 1$.

Existence and uniqueness of solutions to equation (1.1) was proved in [Man07], see Lemma 2.1. The solution $\mu_t: [0, \infty) \rightarrow \mathcal{P}(\mathbb{R}^d)$ is a *narrowly continuous curve* (by [Man07], Lemma 3.2). Recall that a mapping $[0, \infty) \ni t \mapsto \mu_t \in \mathcal{P}(\mathbb{R}^d)$ is narrowly continuous if $t \mapsto \int_{\mathbb{R}^d} \eta d\mu_t$ is a continuous function for all η in the space of continuous and bounded functions defined on \mathbb{R}^d , $\mathcal{C}_b(\mathbb{R}^d)$.

We start by defining a weak solution to equation (1.1).

Definition 1.1. Let $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$ and $(t \mapsto b(t, \cdot)) \in \mathcal{C}^0([0, +\infty); \mathcal{C}^{1+\alpha}(\mathbb{R}^d))$.

We say that the narrowly continuous curve $t \mapsto \mu_t \in \mathcal{P}(\mathbb{R}^d)$ is a weak solution to (1.1) if

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^d} (\partial_t \varphi(t, x) + b \nabla_x \varphi(t, x)) d\mu_t(x) dt + \int_{\mathbb{R}^d} \varphi(0, \cdot) d\mu_0 \\ = \int_0^\infty \int_{\mathbb{R}^d} w(t, x) \varphi(t, x) d\mu_t(x) dt, \end{aligned} \quad (1.2)$$

holds for all test functions $\varphi \in \mathcal{C}_c^1([0, \infty) \times \mathbb{R}^d)$.

We introduce a perturbation to the velocity field b as follows

$$b^h(t, x) := b(t, x) + h \cdot b_1(t, x), \quad (1.3)$$

where $(t \mapsto b(t, \cdot)), (t \mapsto b_1(t, \cdot)) \in \mathcal{C}^0([0, +\infty); \mathcal{C}^{1+\alpha}(\mathbb{R}^d))$ and $h \in \mathbb{R}$, close to 0.

The perturbed problem corresponding to (1.1) has the form

$$\begin{cases} \partial_t \mu_t^h + \operatorname{div}_x (b^h(t, x) \mu_t^h) = w(t, x) \mu_t^h & \text{in } (\mathcal{C}_c^1([0, \infty) \times \mathbb{R}^d))^*, \\ \mu_{t=0}^h = \mu_0 \in \mathcal{P}(\mathbb{R}^d). \end{cases} \quad (1.4)$$

Notice that the initial conditions in (1.1) and (1.4) are the same ($\mu_{t=0} = \mu_{t=0}^h = \mu_0$). For the purpose of further considerations, without loss of generality, we may assume that $h \in [-\frac{1}{2}, \frac{1}{2}]$.

Before stating the main result, we need to define an appropriate Banach space. First recall that the Hölder space $\mathcal{C}^{1+\alpha}(\mathbb{R}^d)$ is a Banach space with the norm

$$\|f\|_{\mathcal{C}^{1+\alpha}(\mathbb{R}^d)} := \sup_{x \in \mathbb{R}^d} |f(x)| + \sup_{x \in \mathbb{R}^d} |\nabla_x f(x)| + \sup_{\substack{x_1 \neq x_2 \\ x_1, x_2 \in \mathbb{R}^d}} \frac{|\nabla_x f(x_1) - \nabla_x f(x_2)|}{|x_1 - x_2|^\alpha}. \quad (1.5)$$

The space of Radon measures $\mathcal{M}(\mathbb{R}^d)$ inherits the dual norm of $(\mathcal{C}^{1+\alpha}(\mathbb{R}^d))^*$ by means of embedding the former into the latter, where a measure is identified with the functional defined by integration against the measure. Throughout we identify the former with the subspace of $(\mathcal{C}^{1+\alpha}(\mathbb{R}^d))^*$. Let then

$$Z := \overline{\mathcal{M}(\mathbb{R}^d)}^{(\mathcal{C}^{1+\alpha}(\mathbb{R}^d))^*}, \quad (1.6)$$

which is a Banach space equipped with the dual norm $\|\cdot\|_{(\mathcal{C}^{1+\alpha}(\mathbb{R}^d))^*}$.

We show in Proposition 5.3 that such defined Z is a predual space of $\mathcal{C}^{1+\alpha}(\mathbb{R}^d)$: Z^* is linearly isomorphic to $\mathcal{C}^{1+\alpha}(\mathbb{R}^d)$.

The following theorem is the main result of this paper.

Theorem 1.1. *Assume that $(t \mapsto b(t, \cdot))$ and $(t \mapsto b_1(t, \cdot)) \in \mathcal{C}^0([0, +\infty); \mathcal{C}^{1+\alpha}(\mathbb{R}^d))$, and $w(t, x) \in \mathcal{C}^{1+\alpha}([0, \infty) \times \mathbb{R}^d)$. Let μ_t^h be the weak solution to problem (1.4) with velocity field defined by (1.3). Then the mapping*

$$\left[-\frac{1}{2}, \frac{1}{2}\right] \ni h \mapsto \mu_t^h \in \mathcal{P}(\mathbb{R}^d)$$

is differentiable in Z , i.e. $\partial_h \mu_t^h \in Z$.

Classically, the analysis of structured population models was carried out in Lipschitz setting [Web85, Thi03]. This approach is appropriate for considering the densities of populations. However, it does not allow to work with less regular distributions used in applications, like Dirac mass. Firstly, we would like to argue why this result cannot be obtained in the space $W^{1,\infty}$ with the *flat metric* (called also *bounded Lipschitz distance*) – what is a natural setting to consider transport equation in the space of bounded Radon measures [PR16, PFM, GJMU14, CLM13, GM10].

Recall that the flat metric is defined as follows

$$\rho_F(\mu, \nu) := \sup_{f \in W^{1,\infty}, \|f\|_{W^{1,\infty}} \leq 1} \left\{ \int_{\mathbb{R}^d} f d(\mu - \nu) \right\}.$$

It is worth recalling that the *generalized Wasserstein distance* coincides with the flat metric [PR16].

Now, we recall a counterexample presented in [Skr] for differentiability in the mentioned setting. Consider a perturbed transport equation for one dimensional x on \mathbb{R}

$$\begin{cases} \partial_t \mu_t^h + \partial_x((1+h)\mu_t^h) = 0, \\ \mu_0^h = \delta_0. \end{cases} \quad (1.7)$$

It can be easily checked that $\mu_t^h = \delta_{(1+h)t}$ is a measure solution to (1.7). Note that the map $h \mapsto \mu_t^h$ is Lipschitz continuous

$$\rho_F(\mu_t^h, \mu_t^{h'}) = \rho_F(\delta_{(1+h)t}, \delta_{(1+h')t}) \leq |h - h'|t.$$

However, it is not differentiable for the flat metric. If $\frac{\mu_t^h - \mu_t^0}{h}$ were convergent, it would satisfy Cauchy condition with respect to the flat metric. We compute

$$\int_{\mathbb{R}} f(x) \left(\frac{d\mu_t^{h_1}(x) - d\mu_t^0}{h_1} - \frac{d\mu_t^{h_2}(x) - d\mu_t^0}{h_2} \right) = \frac{f((1+h_1)t) - f(t)}{h_1} - \frac{f((1+h_2)t) - f(t)}{h_2}.$$

If we choose a test function from $W^{1,\infty}(\mathbb{R})$ such that

$$f(x) = \begin{cases} |x - t| - 1, & \text{if } |x - t| \leq 1, \\ 0, & \text{if } |x - t| > 1, \end{cases}$$

then for $h_1 > 0$ and $h_2 < 0$, we get $p_F \left(\frac{\mu_t^{h_1} - \mu_t^0}{h_1}, \frac{\mu_t^{h_2} - \mu_t^0}{h_2} \right) \geq 2t$. Thus $\frac{\mu_t^h - \mu_t^0}{h}$ does not converge. That is why we need a space with test functions a little bit more regular than $W^{1,\infty}(\mathbb{R})$.

Theorem 1.1, differentiability with respect to perturbing parameter, is required for various applications. One that we shall discuss in this paper is application to optimal control theory.

As additional results we have further characterizations of the Banach space Z , presented in Section 5. First, Z is separable as the span of Dirac measures at rational points is a dense countable subset of Z . Moreover we have that Z^* is linearly isomorphic to $\mathcal{C}^{1+\alpha}(\mathbb{R}^d)$.

The outline of the paper is as follows. Section 2 is devoted to preparing the necessary background in functional analysis. The proof of Theorem 1.1 is treated in Section 3. In Section 4 by discuss possible applications of the result of this paper. Characterization of the space Z is presented in Section 5.

2. PRELIMINARIES

The characteristic system associated to equation (1.1), has the following form

$$\begin{cases} \dot{X}_b(t, y) = b(t, X_b(t, y)), \\ X_b(t_0, y) = y \in \mathbb{R}^d, \end{cases} \quad (2.8)$$

where $(t \mapsto b(t, \cdot)) \in \mathcal{C}^0([0, +\infty); \mathcal{C}^{1+\alpha}(\mathbb{R}^d))$.

A solution to (2.8), X_b is called a *flow map*. Note that the flow maps are defined for all $t \in \mathbb{R}$ and thus $y \mapsto X_b(t, y)$ is a one-parameter group of diffeomorphisms on \mathbb{R}^d (dependent on the variable b).

Remark. The requirement $(t \mapsto b(t, \cdot)) \in \mathcal{C}^0([0, +\infty); \mathcal{C}^1(\mathbb{R}^d))$ is sufficient to conclude that $y \mapsto X_b(t, y)$ is a diffeomorphism. Higher regularity is needed when we estimate remainder terms of a Taylor expansion in the final proof of Theorem 1.1 (see e.g. equation (3.11)).

Now we define the *push-forward operator* [AGS08]. If Y_1, Y_2 are separable metric spaces, $\mu \in \mathcal{P}(Y_1)$, and $r: Y_1 \rightarrow Y_2$ is a μ -measurable map, we denote by $\mu \mapsto r\#\mu \in \mathcal{P}(Y_2)$ the push-forward of μ through r , defined by

$$r\#\mu(\mathbf{B}) := \mu(r^{-1}(\mathbf{B})), \quad \text{for all } \mathbf{B} \in \mathcal{B}(Y_2).$$

The following lemma guarantees that a weak solution μ_t is probability measure.

Lemma 2.1 (A representation formula for the non-homogenous continuity equation [Man07]). *Let $b(t, y)$ be a Borel velocity field in $L^1([0, T]; W^{1,\infty}(\mathbb{R}^d))$, $w(t, x)$ a Borel bounded and locally Lipschitz continuous (w.r.t. the space variable) scalar function and $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$. Then there exists a unique μ_t , narrowly continuous family of Borel finite positive measures solving (in the distributional sense) the initial value problem (1.1) and it is given by the explicit formula*

$$\mu_t = X_b(t, \cdot)\#(e^{\int_0^t w(s, X_b(s, \cdot)) ds} \cdot \mu_0), \quad \text{for all } t \in [0, T].$$

Remark. Since in our case $(t \mapsto b(t, \cdot)) \in \mathcal{C}^0([0, +\infty); \mathcal{C}^{1+\alpha}(\mathbb{R}^d))$ then b is globally Lipschitz and thus the solution X_t is global. Also $w(t, x)$ satisfies the assumption in Lemma 2.1. Thus we conclude that (1.1) has a unique weak solution $t \mapsto \mu_t$, that is defined for all t .

Since the representation formula could be generalized for the case when μ is a non-negative measure $\mathcal{M}^+(\mathbb{R}^d)$ we can also consider non-positive measures as an initial condition.

3. PROOF OF MAIN RESULT – THEOREM 1.1

By definition $Z = \overline{\text{span}\{\delta_x : x \in \mathbb{R}^d\}}^{(\mathcal{C}^{1+\alpha}(\mathbb{R}^d))^*}$ is a subspace of $(\mathcal{C}^{1+\alpha}(\mathbb{R}^d))^*$. The space Z inherits the norm of $(\mathcal{C}^{1+\alpha}(\mathbb{R}^d))^*$. Since Z is complete, it is enough to show that proper sequence of differential quotient is a Cauchy sequence.

The analogue of (2.8) for the system associated to perturbed equation (1.4) with velocity field defined by (1.3), where $(t \mapsto b(t, \cdot)), (t \mapsto b_1(t, \cdot)) \in \mathcal{C}^0([0, +\infty); \mathcal{C}^{1+\alpha}(\mathbb{R}^d))$, has the form

$$\begin{cases} \dot{X}_h(t, y) = (b + b_1 h)(t, X_h(t, y)), \\ X_h(t_0, y) = y \in \mathbb{R}^d. \end{cases} \quad (3.9)$$

As before, $y \mapsto X_h(t, y)$ is a diffeomorphism. To underline the dependence of $X_h(t, x)$ on the parameter h from now on we will use the notation $X(t, y; h) := X_h(t, y)$.

Lemma 3.1. *Let $(t \mapsto b(t, \cdot)), (t \mapsto b_1(t, \cdot)) \in \mathcal{C}^0([0, +\infty); \mathcal{C}^{1+\alpha}(\mathbb{R}^d))$. Then for all $(t, y) \in [0, +\infty) \times \mathbb{R}^d$ the mapping $(h \mapsto X(t, y; h)) \in \mathcal{C}^{1+\alpha}([-\frac{1}{2}, \frac{1}{2}])$.*

The proof goes in a similar way as the proof of higher order differentiability (\mathcal{C}^k , where $k \in \mathbb{N}$) of the solution with respect to parameters, which can be found in the book [Har02] p. 100.

We are in the position to prove the main result.

Proof of Theorem 1.1. Consider the weak solution μ_t to system (1.1) (where $h = 0$) and $\mu_t^{h_1}, \mu_t^{h_2}$ ($h_1 \neq h_2, h_{1,2} \neq 0$) to system defined by (1.4). They are unique and defined for all $t \in [0, \infty)$, according to Lemma 2.1.

Notice that for every $\lambda \in \mathbb{R}$, $\frac{\mu_t^{h+\lambda} - \mu_t^h}{\lambda} \in \mathcal{M}(\mathbb{R}^d) \subseteq Z$, which is a complete space. First we show differentiability at $h = 0$. Differentiability at other h follows from this result (see end of proof).

For the first part it suffices to show that

$$I_{h_1, h_2} := \left\| \frac{\mu_t^{h_1} - \mu_t}{h_1} - \frac{\mu_t^{h_2} - \mu_t}{h_2} \right\|_{(\mathcal{C}^{1+\alpha}(\mathbb{R}^d))^*}$$

can be made arbitrary small, when h_1 and h_2 are sufficiently close to 0. Then for any sequence $h_n \rightarrow 0$, $\frac{\mu_t^{h_n} - \mu_t}{h_n}$ is a Cauchy sequence in $(\mathcal{C}^{1+\alpha}(\mathbb{R}^d))^*$. Hence, converges to a limit that is the same for each sequence (h_n) such that $h_n \rightarrow 0$.

$$\begin{aligned} I_{h_1, h_2} &= \sup_{\|\psi\|_{\mathcal{C}^{1+\alpha}} \leq 1} \left| \int_{\mathbb{R}^d} \psi d \left(\frac{\mu_t^{h_1} - \mu_t}{h_1} - \frac{\mu_t^{h_2} - \mu_t}{h_2} \right) \right| \\ &= \sup_{\|\psi\|_{\mathcal{C}^{1+\alpha}} \leq 1} \left| \int_{\mathbb{R}^d} \psi \frac{d\mu_t^{h_1}}{h_1} - \int_{\mathbb{R}^d} \psi \frac{d\mu_t}{h_1} - \int_{\mathbb{R}^d} \psi \frac{d\mu_t^{h_2}}{h_2} + \int_{\mathbb{R}^d} \psi \frac{d\mu_t}{h_2} \right| \end{aligned} \quad (3.10)$$

First we use representation formula (Lemma 2.1) and the fact that $y \mapsto X(t, y; h)$ is a diffeomorphism. Introduce for convenience $\bar{w}(s, y; h) := w(s, X_b(s, y; h))$.

$$\begin{aligned}
I_{h_1, h_2} &= \sup_{\|\psi\|_{C^{1+\alpha}} \leq 1} \left| \int_{\mathbb{R}^d} \psi(X(t, y; h_1)) e^{\int_0^t \bar{w}(s, y; h_1) ds} \frac{d\mu_0}{h_1} - \int_{\mathbb{R}^d} \psi(X(t, y; 0)) e^{\int_0^t \bar{w}(s, y; 0) ds} \frac{d\mu_0}{h_1} \right. \\
&\quad \left. - \int_{\mathbb{R}^d} \psi(X(t, y; h_2)) e^{\int_0^t \bar{w}(s, y; h_2) ds} \frac{d\mu_0}{h_2} + \int_{\mathbb{R}^d} \psi(X(t, y; 0)) e^{\int_0^t \bar{w}(s, y; 0) ds} \frac{d\mu_0}{h_2} \right| \\
&= \sup_{\|\psi\|_{C^{1+\alpha}} \leq 1} \left| \underbrace{\int_{\mathbb{R}^d} (\psi(X(t, y; h_1)) - \psi(X(t, y; 0))) e^{\int_0^t \bar{w}(s, y; h_1) ds} \frac{d\mu_0}{h_1}}_{I_{h_1}^{(1)}} \right. \\
&\quad \left. - \underbrace{\int_{\mathbb{R}^d} (\psi(X(t, y; h_2)) - \psi(X(t, y; 0))) e^{\int_0^t \bar{w}(s, y; h_2) ds} \frac{d\mu_0}{h_2}}_{I_{h_2}^{(1)}} \right. \\
&\quad \left. - \underbrace{\int_{\mathbb{R}^d} \left(e^{\int_0^t \bar{w}(s, y; 0) ds} - e^{\int_0^t \bar{w}(s, y; h_1) ds} \right) \psi(X(t, y; 0)) \frac{d\mu_0}{h_1}}_{I_{h_1}^{(2)}} \right. \\
&\quad \left. + \underbrace{\int_{\mathbb{R}^d} \left(e^{\int_0^t \bar{w}(s, y; 0) ds} - e^{\int_0^t \bar{w}(s, y; h_2) ds} \right) \psi(X(t, y; 0)) \frac{d\mu_0}{h_2}}_{I_{h_2}^{(2)}} \right|
\end{aligned}$$

Let us consider $|I_{h_1}^{(1)} - I_{h_2}^{(1)}|$ and $|I_{h_1}^{(2)} - I_{h_2}^{(2)}|$ separately.

In $I_{h_1}^{(2)} - I_{h_2}^{(2)}$ expand $e^{\int_0^t \bar{w}(s, y; h_1) ds}$ and $e^{\int_0^t \bar{w}(s, y; h_2) ds}$ into Taylor series around $h = 0$

$$\begin{aligned}
|I_{h_1}^{(2)} - I_{h_2}^{(2)}| &= \left| \int_{\mathbb{R}^d} \psi(X(t, y; 0)) \left[e^{\int_0^t \bar{w}(s, y; 0) ds} - e^{\int_0^t \bar{w}(s, y; 0) ds} \right. \right. \\
&\quad \left. \left. - h_1 e^{\int_0^t \bar{w}(s, y; 0) ds} \partial_h \left(\int_0^t \bar{w}(s, y; h) ds \right) \Big|_{h=0} - \mathcal{O}(|h_1|^{1+\alpha}) \right] \frac{d\mu_0}{h_1} \right. \\
&\quad \left. - \int_{\mathbb{R}^d} \psi(X(t, y; 0)) \left[e^{\int_0^t \bar{w}(s, y; 0) ds} - e^{\int_0^t \bar{w}(s, y; 0) ds} \right. \right. \\
&\quad \left. \left. - h_2 e^{\int_0^t \bar{w}(s, y; 0) ds} \partial_h \left(\int_0^t \bar{w}(s, y; h) ds \right) \Big|_{h=0} - \mathcal{O}(|h_2|^{1+\alpha}) \right] \frac{d\mu_0}{h_2} \right| \tag{3.11} \\
&= \left| \psi(X(t, y; 0)) \int_{\mathbb{R}^d} \left[- e^{\int_0^t \bar{w}(s, y; 0) ds} \partial_h \left(\int_0^t \bar{w}(s, y; h) ds \right) \Big|_{h=0} \right. \right. \\
&\quad \left. \left. + e^{\int_0^t \bar{w}(s, y; 0) ds} \partial_h \left(\int_0^t \bar{w}(s, y; h) ds \right) \Big|_{h=0} - \frac{\mathcal{O}(|h_1|^{1+\alpha})}{h_1} + \frac{\mathcal{O}(|h_2|^{1+\alpha})}{h_2} \right] d\mu_0 \right|
\end{aligned}$$

$$\begin{aligned} &\leq \left| \psi(X(t, y; 0)) \int_{\mathbb{R}^d} \left(-\frac{\mathcal{O}(|h_1|^{1+\alpha})}{h_1} + \frac{\mathcal{O}(|h_2|^{1+\alpha})}{h_2} \right) d\mu_0 \right| \\ &\leq \left| c \int_{\mathbb{R}^d} \left(-\frac{\mathcal{O}(|h_1|^{1+\alpha})}{h_1} + \frac{\mathcal{O}(|h_2|^{1+\alpha})}{h_2} \right) d\mu_0 \right|. \end{aligned}$$

We now take into consideration $|I_{h_1}^{(1)} - I_{h_2}^{(1)}|$. Because $\psi \in \mathcal{C}^{1+\alpha}(\mathbb{R}^d)$, one has

$$\psi(x) = \psi(x_0) + \nabla_x \psi(x_0)(x - x_0) + R(x, x_0), \quad (\star)$$

with $|R(x, x_0)| \leq C|\nabla_x \psi|_\alpha \|x - x_0\|^{1+\alpha}$, where $|\nabla_x \psi|_\alpha$ is an α -Hölder constant. Thus, expand $\psi(X(t, y; h_1))$ and $\psi(X(t, y; h_2))$ into Taylor series around $X(t, y; 0)$

$$\begin{aligned} |I_{h_1}^{(1)} - I_{h_2}^{(1)}| &= \left| \int_{\mathbb{R}^d} [\psi(X(t, y; 0)) + \nabla_x \psi(X(t, y; h))|_{h=0} \cdot (X(t, y; h_1) - X(t, y; 0)) \right. \\ &\quad \left. + \mathcal{O}(|X(t, y; h_1) - X(t, y; 0)|^{1+\alpha}) - \psi(X(t, y; 0))] e^{\int_0^t \bar{w}(s, y; h_1) ds} \frac{d\mu_0}{h_1} \right. \\ &\quad \left. - \int_{\mathbb{R}^d} [\psi(X(t, y; 0)) + \nabla_x \psi(X(t, y; h))|_{h=0} (X(t, y; h_2) - X(t, y; 0)) \right. \\ &\quad \left. + \mathcal{O}(|X(t, y; h_2) - X(t, y; 0)|^{1+\alpha}) - \varphi(X(t, y; 0))] e^{\int_0^t \bar{w}(s, y; h_2) ds} \frac{d\mu_0}{h_2} \right|. \end{aligned}$$

Expanding $X(t, y; h_2)$ and $X(t, y; h_1)$ around $h = 0$, by Lemma 3.1 and expansion similar to (\star) for $h \mapsto X(t, y; h)$ we obtain

$$\begin{aligned} &|I_{h_1}^{(1)} - I_{h_2}^{(1)}| \\ &= \left| \int_{\mathbb{R}^d} \left[\nabla_x \psi(X(t, y; h))|_{h=0} \left(X(t, y; 0) + h_1 \partial_h X(t, y; h) \Big|_{h=0} + \mathcal{O}(|h_1|^{1+\alpha}) - X(t, y; 0) \right) \right. \right. \\ &\quad \left. \left. + \mathcal{O}(|X(t, y; h_1) - X(t, y; 0)|^{1+\alpha}) \right] e^{\int_0^t \bar{w}(s, y; h_1) ds} \frac{d\mu_0}{h_1} \right. \\ &\quad \left. - \int_{\mathbb{R}^d} \left[\nabla_x \psi(X(t, y; h))|_{h=0} \left(X(t, y; 0) + h_2 \partial_h X(t, y; h) \Big|_{h=0} + \mathcal{O}(|h_2|^{1+\alpha}) - X(t, y; 0) \right) \right. \right. \\ &\quad \left. \left. + \mathcal{O}(|X(t, y; h_2) - X(t, y; 0)|^{1+\alpha}) \right] e^{\int_0^t \bar{w}(s, y; h_2) ds} \frac{d\mu_0}{h_2} \right|. \end{aligned}$$

Since the remainder term $\mathcal{O}(|X(t, y; h) - X(t, y; 0)|^{1+\alpha}) \leq c|h|^{1+\alpha}$ for all $h \in [-\frac{1}{2}, \frac{1}{2}]$, we can further estimate

$$\begin{aligned} &|I_{h_1}^{(1)} - I_{h_2}^{(1)}| \\ &\leq \left| \int_{\mathbb{R}^d} \left[\nabla_x \psi(X(t, y; h))|_{h=0} \left(\partial_h X(t, y; h) \Big|_{h=0} + \mathcal{O}(|h_1|^\alpha) \right) + \mathcal{O}(|h_1|^\alpha) \right] \cdot e^{\int_0^t \bar{w}(s, y; h_1) ds} d\mu_0 \right. \\ &\quad \left. - \int_{\mathbb{R}^d} \left[\nabla_x \psi(X(t, y; h))|_{h=0} \left(\partial_h X(t, y; h) \Big|_{h=0} + \mathcal{O}(|h_2|^\alpha) \right) + \mathcal{O}(|h_2|^\alpha) \right] \cdot e^{\int_0^t \bar{w}(s, y; h_2) ds} d\mu_0 \right|. \end{aligned}$$

We consider function $\psi \in \mathcal{C}^{1+\alpha}$, with $\|\psi\|_{\mathcal{C}^{1+\alpha}} \leq 1$. Hence we can further estimate $\nabla_x \psi(X(t, y; h))|_{h=0} \leq 1$. This yields

$$|I_{h_1}^{(1)} - I_{h_2}^{(1)}| \leq \left| \int_{\mathbb{R}^d} \left[\partial_h X(t, y; h) \Big|_{h=0} \left(e^{\int_0^t \bar{w}(s, y; h_1) ds} - e^{\int_0^t \bar{w}(s, y; h_2) ds} \right) + \left(\mathcal{O}_1(|h_1|^\alpha) e^{\int_0^t \bar{w}(s, y; h_1) ds} - \mathcal{O}_1(|h_2|^\alpha) e^{\int_0^t \bar{w}(s, y; h_2) ds} \right) \right] d\mu_0 \right|.$$

To summarize estimations of $|I_{h_1}^{(1)} - I_{h_2}^{(1)}|$:

- $\partial_h X(t, y; h) \Big|_{h=0}$ is just finite number (Lemma 3.1),
- $\left(e^{\int_0^t \bar{w}(s, y; h_1) ds} - e^{\int_0^t \bar{w}(s, y; h_2) ds} \right)$ can be estimated by $c|h_1 - h_2|$ (argumentation is similar as in estimations of $|I_{h_1}^{(2)} - I_{h_2}^{(2)}|$),
- $\left(\mathcal{O}(|h_1|^\alpha) e^{\int_0^t \bar{w}(s, y; h_1) ds} - \mathcal{O}(|h_2|^\alpha) e^{\int_0^t \bar{w}(s, y; h_2) ds} \right)$ is going to zero when $h_1 \rightarrow 0$ and $h_2 \rightarrow 0$.

Thus I_{h_1, h_2} can be made arbitrarily small when h_1 and h_2 are sufficiently close to 0. Therefore we have shown that $\frac{\mu_t^{h+\lambda_n} - \mu_t^h}{\lambda_n}$ is a Cauchy sequence for every $\lambda_n \rightarrow 0$ in $(\mathcal{C}^{1+\alpha}(\mathbb{R}^d))^*$ for $h = 0$, with the same limit. Hence μ_t^h is differentiable with respect to parameter h at $h = 0$.

The same argumentation works for $h \neq 0$. Let us consider a sequence $\frac{\mu_t^{h+\lambda_n} - \mu_t^h}{\lambda_n}$, where $\lambda_n \rightarrow 0$ and $h \neq 0$. By definition of perturbation (1.3), i.e. $b^h := b + hb_1$, the solution $\mu_t^{h+\lambda_n}$ for velocity field

$$b^{h+\lambda_n} = b + hb_1 + \lambda_n b_1 =: \bar{b} + \lambda_n b_1$$

and initial condition $\mu_0^{h+\lambda_n} = \mu_0$ is equal (by Lemma 2.1) to the solution $\bar{\mu}_t^{\lambda_n}$ with velocity field $\bar{b} + \lambda_n b_1$ and initial condition μ_0 . A similar statement holds for the μ_t^h and the solution $\bar{\mu}_t^0$ of (1.1) with velocity field \bar{b} . Thus

$$\frac{\mu_t^{h+\lambda_n} - \mu_t^h}{\lambda_n} = \frac{\bar{\mu}_t^{\lambda_n} - \mu_0}{\lambda_n}$$

and the latter sequence converges in Z as $h \rightarrow \infty$, by the first part of the proof. \square

4. APPLICATION TO OPTIMAL CONTROL

The results discussed above can be applied in optimal control theory. The list of references on optimal problems concerning transport equation is steadily growing [BGSW13, BFRS17, ACFK17, BDT17, AHP, BR19].

There are two main approaches to solve optimal control problem when the solution is not differentiable with respect to the control parameter. The first one is just to use *non-smooth analysis*. The second one is to strengthen assumptions for the problem to provide the solution will be differentiable, and then use *smooth analysis* – for which there are developed significantly more tools, and which are less numerically complex than non-smooth methods.

In [BR19] the authors consider the following optimal control problem

$$\begin{cases} \max_{u \in \mathcal{U}} \left[\int_0^T L(\mu_t, u(t, \cdot)) dt + \psi(\mu_T) \right], \\ \begin{cases} \partial_t \mu_t + \operatorname{div}_x \left((v(t, \mu_t, \cdot) + u(t, \cdot)) \mu_t \right) = 0, \\ \mu_{t=0} = \mu_0 \in \mathcal{P}_c(\mathbb{R}^d), \end{cases} \end{cases} \quad (4.12)$$

where $\mathcal{P}_c(\mathbb{R}^d)$ is the subset of $\mathcal{P}(\mathbb{R}^d)$ of Borel probability measures with compact support. The function L can be interpreted as income dependent on the level of sales (which is described by measure μ_t) and a situation on the market, $u(t)$. In this optimal control problem, we want to maximize the total income in the period $[0, T]$. Function $\psi(\mu_T)$ describes the income in a terminal time T .

Notice that the period of time is finite and $v(t, \mu_t, x) + u(t, x)$ corresponds to the velocity field $b(t, x)$ in our transport equation. Contrary to the problem (1.1) the authors consider the term $v(t, \mu_t, x)$ which depends on the solution. The studies on such non-linear problems will appear in [GHŁ]. Nevertheless, briefly speaking, the assumptions for coefficients of (4.12) are weaker than the ones for (1.1). In particular the velocity field $v(t, \mu_t, x) + u(t, x)$ is not differentiable with respect to perturbation in $u(t, x)$, it just satisfies Lipschitz condition.

The authors formulated a new Pontryagin Maximum Principle in the language of subdifferential calculus in Wasserstein spaces.

Below we would like to present the second approach. We want to argue how differentiability of velocity field with respect to perturbing parameter in problem (1.4) can be applied in optimal control.

In control theory, the control is based on observation of the state of the system at each or some finite points: $u(t) := \phi(\mu_t^h)$. The state μ_t^h is in $\mathcal{M}(\mathbb{R}^d) \subset \mathcal{Z}$. Thus, a reasonable class of differentiable observation function ϕ is provided by the composition of a continuous linear functional on \mathcal{Z} and $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$. In Proposition 5.3 we show that every continuous linear functional on \mathcal{Z} is represented essentially by integration with respect to a $\mathcal{C}^{1+\alpha}(\mathbb{R}^d)$ -function – denote it here by K .

Thus, aiming at optimal control of the solution to (1.4), where h is a control parameter attaining values in \mathbb{R} , we start by considering functionals of the form

$$\gamma(\mu^h) := \widehat{\gamma} \left(\int_{\mathbb{R}^d} K(x) d\mu^h(x) \right), \quad (4.13)$$

where $\widehat{\gamma}$ is a \mathcal{C}^1 -function and $K \in \mathcal{C}^{1+\alpha}(\mathbb{R}^d)$.

The meaning is essentially the following: the integral operator $\int_{\mathbb{R}^d} K(x) d\mu(x)$ is well-defined for μ being a measure and necessary not every element from the space \mathcal{Z} is measure. Following lemma provides extension of the domain to whole space \mathcal{Z} .

Lemma 4.1 (Extension Theorem). [AE08, Theorem 2.1] *Suppose X and Y are metric spaces, and Y is complete. Also suppose X_1 is a dense subset of X , and $f: X_1 \rightarrow Y$ is uniformly*

continuous. Then f has a uniquely determined extension $\bar{f}: X \rightarrow Y$ given by

$$\bar{f}(x) = \lim_{x_1 \rightarrow x, x_1 \in X_1} f(x_1), \quad \text{for } x \in X,$$

and \bar{f} is also uniformly continuous.

In our case the operator $\int_{\mathbb{R}^d} K(x) d\mu(x)$ is of course well-defined for any $\mu \in \mathcal{M}(\mathbb{R}^d)$ and it can be uniquely extended to $Z = \overline{\mathcal{M}(\mathbb{R}^d)^{(\mathcal{C}^{1+\alpha}(\mathbb{R}^d))^*}} (\text{span}\{\delta_x: x \in \mathbb{R}^d\})$ is dense subset of Z , Proposition 5.1). Denote this uniquely determined extension by

$$\langle K(\cdot), \mu \rangle_{\mathcal{C}^{1+\alpha}(\mathbb{R}^d), Z},$$

where $\langle \cdot, \cdot \rangle$ is dual pair. Thus the functional corresponding to (4.13) has the form

$$\bar{\gamma}(\mu^h) = \hat{\gamma}(\langle K, \mu \rangle_{\mathcal{C}^{1+\alpha}(\mathbb{R}^d), Z}). \quad (4.14)$$

Now consider the problem

$$\min_{h \in \mathbb{R}} \bar{\gamma}(\mu^h). \quad (4.15)$$

That is, we wish to find an $h^* \in \mathbb{R}$ such that $\bar{\gamma}(\mu^{h^*}) \leq \bar{\gamma}(\mu^h)$ for all $h \in \mathbb{R}$.

A necessary condition for μ^{h^*} realizing a minimum is that the gradient of the function $\bar{\gamma}$ is zero at μ^{h^*}

$$\partial_h \bar{\gamma}(\mu^h)|_{h=h^*} = \hat{\gamma}'(\langle K, \mu^h \rangle) \cdot \langle K, \partial_h \mu^h|_{h=h^*} \rangle_{\mathcal{C}^{1+\alpha}(\mathbb{R}^d), Z} = 0. \quad (4.16)$$

For this condition to be satisfied it is necessary that $h \mapsto \bar{\gamma}(\mu^h) \in \mathcal{C}^1(Z, \mathbb{R})$. This is guaranteed by the following lemma when combined with the differentiability of μ^h with respect to h (Theorem 1.1).

Lemma 4.2. *If $K(x) \in \mathcal{C}^{1+\alpha}(\mathbb{R}^d)$ and $\hat{\gamma} \in \mathcal{C}^1(\mathbb{R})$ then $\bar{\gamma}$ defined by (4.14) is $\mathcal{C}^1(Z, \mathbb{R})$.*

Proof. What we want to show is that if $K \in \mathcal{C}^{1+\alpha}(\mathbb{R}^d)$ then the functional $\mu \mapsto \langle K, \mu \rangle_{\mathcal{C}^{1+\alpha}(\mathbb{R}^d), Z}$ is linear and bounded on Z . Then $\hat{\gamma}(\langle K, \mu \rangle_{\mathcal{C}^{1+\alpha}(\mathbb{R}^d), Z}) \in \mathcal{C}^1(Z, \mathbb{R})$, as a composition of \mathcal{C}^1 -function and a bounded linear functional.

Linearity of $\mu \mapsto \langle K, \mu \rangle_{\mathcal{C}^{1+\alpha}(\mathbb{R}^d), Z}$ is clear. Following holds

$$|\langle K, \mu \rangle_{\mathcal{C}^{1+\alpha}(\mathbb{R}^d), Z}| \leq \|K\|_{\mathcal{C}^{1+\alpha}(\mathbb{R}^d)} \cdot \|\mu\|_{(\mathcal{C}^{1+\alpha}(\mathbb{R}^d))^*} \leq C \|\mu\|_{(\mathcal{C}^{1+\alpha}(\mathbb{R}^d))^*},$$

where constant $C = \sup_{x \in \mathbb{R}^d} |K(x)| + \sup_{x \in \mathbb{R}^d} |\nabla K(x)| + \sup_{x, y \in \mathbb{R}^d, x \neq y} \frac{|\nabla K(x) - \nabla K(y)|}{|x - y|^\alpha}$.

Thus the functional $\mu \mapsto \langle K, \mu \rangle_{\mathcal{C}^{1+\alpha}(\mathbb{R}^d), Z}$ is bounded. We conclude that $\bar{\gamma} \in \mathcal{C}^1(Z, \mathbb{R})$. \square

Of course, there are many optimization methods which do not depend on finding derivative analytically and then setting it to zero. When a functional $\bar{\gamma}(\mu^h)$ is differentiable with respect to h , an optimization problem (4.15) can be solved with gradient-based analytical methods or through numerical methods such as the steepest descent. When $\bar{\gamma}(\mu^h)$ is not differentiable, the above-mentioned methods cannot be applied, the problem becomes more complex numerically. And for differentiability of $\bar{\gamma}(\mu^h)$ necessary is differentiability of μ^h , which is satisfied by Theorem 1.1.

Remark. If $\hat{\gamma}$ is convex then condition (4.16) is not only necessary but also sufficient for μ^{h^*} to realize a minimum.

4.1. Further application. In [CGR18] authors consider optimization in the structured population model defined by

$$\begin{cases} \partial_t \mu_t + \partial_x (b(t)(\mu_t, x)\mu_t) + w(t)(\mu_t, x)\mu_t = 0, \\ (b(t)(\mu_t, 0))D_\lambda \mu_t(0) = \int_0^\infty \beta(t)(\mu_t, x)d\mu_t, \\ \mu_{t=0} = \mu_0, \end{cases} \quad (4.17)$$

where $t \in [0, \infty)$ and $x \in \mathbb{R}_+$ is a biological parameter, typically age or size. The unknown μ_t is a time dependent, non-negative and finite Radon measure. The growth function b and the mortality rate w are strictly positive, while the birth function β is non-negative – b, w, β are Nemytskii operators. By $D_\lambda \mu_t$ we denote the Radon-Nikodym derivative of μ_t with respect to the Lebesgue measure λ computed at 0. The initial datum μ_0 is a non-negative Radon measure.

Remark. The reason for analyzing solutions to structured population models in the space of measures is as follows: typical experimental data are not continuous, they provide information on percentiles, i.e., the number of individuals in some intervals of the structural variable (like age). In the case of demography and epidemiology a number of births are typically used per years.

Aiming at the optimal control of the solution to (4.17), a control parameter h is introduced (possibly time and/or state dependent), attaining values in a given set \mathcal{H} . Therefore, we obtain:

$$\begin{cases} \partial_t \mu_t^h + \partial_x (b(t; h)(\mu_t^h, x)\mu_t^h) + w(t; h)(\mu_t^h, x)\mu_t^h = 0, \\ (b(t; h)(\mu_t^h, 0))D_\lambda \mu_t^h(0) = \int_0^\infty \beta(t; h)(\mu_t^h, x)d\mu_t, \\ \mu_{t=0}^h = \mu_0. \end{cases} \quad (4.18)$$

The goal is to find minimum of a given functional

$$\mathcal{J}(\mu_t^h) = \int_0^\infty j(t, \mu_t^h; h)dt, \quad (4.19)$$

within a suitable function space i.e. to find an $h^* \in \mathcal{H}$ such that $\mathcal{J}(\mu^{h^*}) \leq \mathcal{J}(\mu^h)$ for all $h \in \mathcal{H}$.

Aiming at the optimal control problem in [CGR18] the Escalator Boxcar Train (EBT) algorithm is adapted (defined in [GJMU14]), i.e. an appropriate ODE system is used approximating the original PDE model. Authors mention that solutions to conservation or balance laws typically depend in a Lipschitz continuous way on the initial datum as well as from the functions defining the equation. This does not allow the use of differential tools in the search for the optimal control.

Since solution to the transport equation is differentiable with respect to parameter, mathematical tools applied to (4.19) can be extended by e.g. gradient methods.

5. CHARACTERIZATION OF THE SPACE Z

In this section we establish some further properties of the space Z defined by (1.6). The identification of the dual space Z^* in Proposition 5.3 is particularly interesting eg. in view of the application to control theory, discussed in Section 4. By δ_x we denote the Dirac measure concentrated in x .

Proposition 5.1. *Let Z be given by (1.6). Then the set $\text{span}\{\delta_x : x \in \mathbb{Q}^d\}$ is dense in Z with respect to the $(C^{1+\alpha}(\mathbb{R}^d))^*$ -topology, i.e.*

$$Z = \overline{\text{span}\{\delta_x : x \in \mathbb{Q}^d\}}^{(C^{1+\alpha}(\mathbb{R}^d))^*}.$$

Consequently, Z is a separable space.

Proof. We want to show that for any measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ there exists a sequence $\{\mu_n\}_{n \in \mathbb{N}} \in \text{span}\{\delta_x : x \in \mathbb{Q}^d\}$ such that $\|\mu_n - \mu\|_{(C^{1+\alpha}(\mathbb{R}^d))^*} \rightarrow 0$ as $n \rightarrow \infty$.

We consider bounded Radon measures, thus for any $\mu \in \mathcal{M}(\mathbb{R}^d)$ and for any $\varepsilon > 0$ there exists R_ε such that $|\mu|(\mathbb{R}^d \setminus \mathcal{B}(0, R_\varepsilon)) \leq \frac{\varepsilon}{2}$. The closure of a ball $\mathcal{B}(0, R_\varepsilon)$ in \mathbb{R}^d as a compact set has finite cover $\{\mathcal{B}(g_i, \frac{\varepsilon}{4\|\mu\|_{\text{TV}}})\}_{i=1}^{n(\varepsilon)}$, where $g_i \in \mathbb{Q}^d$. Denote by $\mathcal{B}_i := \mathcal{B}(g_i, \frac{\varepsilon}{4\|\mu\|_{\text{TV}}})$. Then define

$$U_{i,\varepsilon} := (\mathcal{B}(0, R_\varepsilon) \cap \mathcal{B}_i) \setminus \bigcup_{j=1}^{i-1} \mathcal{B}_j \quad (5.20)$$

are disjoint Borel sets and $\bigcup_{i=1}^{n(\varepsilon)} U_{i,\varepsilon} = \mathcal{B}(0, R_\varepsilon)$. Notice that g_i (the center of \mathcal{B}_i) is not necessarily contained in $U_{i,\varepsilon}$. In case g_i is not contained in $U_{i,\varepsilon}$ we take any other point of the ball \mathcal{B}_i contained in $U_{i,\varepsilon}$, we will denote this point the same way, slightly abusing notation.

For any $\mu \in \mathcal{M}(\mathbb{R}^d)$ and any $\varepsilon > 0$ we consider $\mu^\varepsilon = \sum_{i=1}^{n(\varepsilon)} \mu(U_{i,\varepsilon}) \cdot \delta_{g_i}$ (linear combination of Dirac deltas concentrated at points $g_i \in \mathbb{Q}^d$). Denote by $\widehat{\mu} := \mu|_{\mathcal{B}(0, R_\varepsilon)}$ the measure restricted to $\mathcal{B}(0, R_\varepsilon)$. Then the following holds:

$$\|\mu^\varepsilon - \mu\|_{(C^{1+\alpha}(\mathbb{R}^d))^*} \leq \|\mu^\varepsilon - \widehat{\mu}\|_{(C^{1+\alpha}(\mathbb{R}^d))^*} + \|\widehat{\mu} - \mu\|_{(C^{1+\alpha}(\mathbb{R}^d))^*} \leq \|\mu^\varepsilon - \widehat{\mu}\|_{(C^{1+\alpha}(\mathbb{R}^d))^*} + \frac{\varepsilon}{2}.$$

We need to estimate the following

$$\begin{aligned} & \|\mu^\varepsilon - \widehat{\mu}\|_{(C^{1+\alpha}(\mathbb{R}^d))^*} \\ &= \sup \left\{ \int_{\mathbb{R}^d} f d(\mu^\varepsilon - \widehat{\mu}) : f \in C^{1+\alpha}(\mathbb{R}^d), \|f\|_{C^{1+\alpha}(\mathbb{R}^d)} \leq 1 \right\} \\ &\leq \sup \left\{ \int_{\mathbb{R}^d} f d(\mu^\varepsilon - \widehat{\mu}) : f \in \text{Lip}(\mathbb{R}^d), \|f\|_\infty + \|\nabla f\|_\infty \leq 1 \right\} \\ &= \sup \left\{ \int_{\mathbb{R}^d} f d \left(\sum_{i=1}^{n(\varepsilon)} (\mu(U_{i,\varepsilon}) \delta_{g_i} - \widehat{\mu}) \right) : f \in \text{Lip}(\mathbb{R}^d), \|f\|_\infty + \|\nabla f\|_\infty \leq 1 \right\} \end{aligned}$$

$$\begin{aligned}
 &= \sup \left\{ \sum_{i=1}^{n(\varepsilon)} \left(\int_{U_{i,\varepsilon}} f(g_i) d\mu - \int_{U_{i,\varepsilon}} f d\mu \right) : f \in \text{Lip}(\mathbb{R}^d), \|f\|_\infty + \|\nabla f\|_\infty \leq 1 \right\} \\
 &= \sup \left\{ \sum_{i=1}^{n(\varepsilon)} \int_{U_{i,\varepsilon}} (f(g_i) - f) d\mu : f \in \text{Lip}(\mathbb{R}^d), \|f\|_\infty + \|\nabla f\|_\infty \leq 1 \right\} \\
 &\leq \sup \left\{ \sum_{i=1}^{n(\varepsilon)} \int_{U_{i,\varepsilon}} |g_i - x| d|\mu| : f \in \text{Lip}(\mathbb{R}^d), \|f\|_\infty + \|\nabla f\|_\infty \leq 1 \right\} \\
 &= \sum_{i=1}^{n(\varepsilon)} \int_{U_{i,\varepsilon}} |g_i - x| d|\mu| \leq \sum_{i=1}^{n(\varepsilon)} \int_{U_{i,\varepsilon}} \frac{\varepsilon}{2\|\mu\|_{\text{TV}}} d|\mu| \\
 &= \frac{\varepsilon}{2\|\mu\|_{\text{TV}}} \int_{\mathcal{B}(0, R_\varepsilon)} d|\mu| = \frac{\varepsilon}{2\|\mu\|_{\text{TV}}} |\mu|(\mathcal{B}(0, R_\varepsilon)) \leq \frac{\varepsilon}{2}.
 \end{aligned}$$

And now we get that for any $\mu \in \mathcal{M}(\mathbb{R}^d)$ there exists an element $\mu^\varepsilon \in \text{span}\{\delta_x : x \in \mathbb{Q}^d\}$ such that $\|\mu^\varepsilon - \mu\|_{(\mathcal{C}^{1+\alpha}(\mathbb{R}^d))^*} \leq \varepsilon$.

Hence, $\text{span}\{\delta_x : x \in \mathbb{Q}^d\}$ is a dense subset of $\mathcal{M}(\mathbb{R}^d)$. Countability of $\text{span}\{\delta_x : x \in \mathbb{Q}^d\}$ is clear because of countability of \mathbb{Q}^d . This implies that the space Z is separable. \square

Moreover, we can characterize the dual space of Z , similar in spirit to [HW09, Theorem 3.6, Theorem 3.7]. This result may be of separate interest in other settings.

Before giving and proving this characterization, we need the following lemma.

Lemma 5.2. *The mapping defined by $\bar{\delta}(x) := \delta_x$ is $\mathcal{C}^{1+\alpha}(\mathbb{R}^d, Z)$.*

Proof. For $f \in \mathcal{C}^{1+\alpha}(\mathbb{R}^d)$, $\lambda \in \mathbb{R}^d$ and $x \in \mathbb{R}^d$ define $D\bar{\delta}(x) \in \mathcal{L}(\mathbb{R}^d, Z)$ by means of

$$\langle D\bar{\delta}(x)\lambda, f \rangle_{(\mathcal{C}^{1+\alpha}(\mathbb{R}^d))^*, \mathcal{C}^{1+\alpha}(\mathbb{R}^d)} := \lambda \bullet \nabla f(x).$$

By \bullet we denote an inner product on \mathbb{R}^d . Thus, $\lambda \bullet \nabla f(x)$ relates to the gradient of f in the direction given by λ . Then

$$\frac{1}{|\lambda|} [\delta_{x+\lambda} - \delta_x - D\bar{\delta}(x)\lambda] \rightarrow 0$$

in Z as $\lambda \rightarrow 0$. Thus $D\bar{\delta}(x)$ is the Fréchet derivative of $\bar{\delta}$ at x .

Of course, for $x, y \in \mathbb{R}^d$, $x \neq y$,

$$\|D\bar{\delta}(x) - D\bar{\delta}(y)\|_Z = \|D\bar{\delta}(x) - D\bar{\delta}(y)\|_{(\mathcal{C}^{1+\alpha}(\mathbb{R}^d))^*}$$

because Z is linear subspace of $(\mathcal{C}^{1+\alpha}(\mathbb{R}^d))^*$, thus $\|\cdot\|_Z = \|\cdot\|_{(\mathcal{C}^{1+\alpha}(\mathbb{R}^d))^*}$ coincides on Z . Now, we can estimate

$$\|D\bar{\delta}(x) - D\bar{\delta}(y)\|_{(\mathcal{C}^{1+\alpha}(\mathbb{R}^d))^*} = \sup_{\lambda \in \mathbb{R}^d, |\lambda| \leq 1} \|D\bar{\delta}(x)\lambda - D\bar{\delta}(y)\lambda\|_{(\mathcal{C}^{1+\alpha}(\mathbb{R}^d))^*}$$

$$\begin{aligned}
&= \sup_{\lambda \in \mathbb{R}^d, |\lambda| \leq 1} \sup_{\|f\|_{\mathcal{C}^{1+\alpha}} \leq 1} \left| \langle D\bar{\delta}(x)\lambda - D\bar{\delta}(y)\lambda, f \rangle_{(\mathcal{C}^{1+\alpha}(\mathbb{R}^d))^*, \mathcal{C}^{1+\alpha}(\mathbb{R}^d)} \right| \\
&= \sup_{\lambda \in \mathbb{R}^d, |\lambda| \leq 1} \sup_{\|f\|_{\mathcal{C}^{1+\alpha}} \leq 1} \left| \sum_{i=1}^d \lambda_i (\partial_{x_i} f(x) - \partial_{x_i} f(y)) \right| \\
&\leq \sup_{\lambda \in \mathbb{R}^d, |\lambda| \leq 1} \sup_{\|f\|_{\mathcal{C}^{1+\alpha}} \leq 1} |\lambda| \cdot \left(\sum_{i=1}^d |\partial_{x_i} f(x) - \partial_{x_i} f(y)|^2 \right)^{1/2} \\
&\leq \sup_{\|f\|_{\mathcal{C}^{1+\alpha}} \leq 1} \frac{|\nabla f(x) - \nabla f(y)|}{|x - y|^\alpha} \cdot |x - y|^\alpha \\
&\leq \sup_{\|f\|_{\mathcal{C}^{1+\alpha}} \leq 1} \|f\|_{(\mathcal{C}^{1+\alpha}(\mathbb{R}^d))^*} \cdot |x - y|^\alpha \leq |x - y|^\alpha.
\end{aligned}$$

This concludes that $\|D\bar{\delta}(x) - D\bar{\delta}(y)\|_Z \leq |x - y|^\alpha$, thus $\bar{\delta} \in \mathcal{C}^{1+\alpha}(\mathbb{R}^d, Z)$. \square

Proposition 5.3. *The space Z^* is isomorphic to $\mathcal{C}^{1+\alpha}(\mathbb{R}^d)$ under the map $\phi \mapsto T\phi$, where $T\phi(x) := \phi(\delta_x)$, $T: Z^* \rightarrow \mathcal{C}^{1+\alpha}(\mathbb{R}^d)$.*

Proof. We need to show that T is bijection from $(Z^*, \|\cdot\|_{Z^*})$ to $(\mathcal{C}^{1+\alpha}(\mathbb{R}^d), \|\cdot\|_{\mathcal{C}^{1+\alpha}})$ such that

$$T(\lambda_1 z_1^* + \lambda_2 z_2^*) = \lambda_1 T(z_1^*) + \lambda_2 T(z_2^*),$$

for $z_1^*, z_2^* \in Z^*$ and $\lambda_1, \lambda_2 \in \mathbb{R}^d$, where

$$\|z^*\|_{Z^*} = \sup_{z \in Z} \{|z^*(z)| : \|z\|_Z \leq 1\} = \sup_{z \in Z} \{z^*(z) : \|z\|_Z \leq 1\}.$$

In addition T is bounded. By Banach Isomorphism Theorem, T^{-1} is bounded.

Step 1. Obviously the mapping defined by $T\phi(x) = \phi(\delta_x)$ maps Z^* into $\mathbb{R}^{\mathbb{R}^d}$, where by $\mathbb{R}^{\mathbb{R}^d}$ we denote a function space from \mathbb{R}^d to \mathbb{R} . The mapping T is injective, because if $z_1^* \neq z_2^*$ then using density of $\text{span}\{\delta_x : x \in \mathbb{R}^d\}$ in Z (Proposition 5.1) there exists $x \in \mathbb{R}^d$ such that

$$z_1^*(\delta_x) \neq z_2^*(\delta_x) \Rightarrow (Tz_1^*)(x) \neq (Tz_2^*)(x).$$

Indeed,

$$z_1^* \neq z_2^* \Rightarrow \exists z \in Z \text{ such that } z_1^*(z) \neq z_2^*(z).$$

Since $\text{span}\{\delta_x : x \in \mathbb{R}^d\}$ is dense in Z , there exists $\{z_n\}_{n \in \mathbb{N}} \subset \text{span}\{\delta_x : x \in \mathbb{R}^d\}$ such that $z_n \rightarrow z$. Functionals z_1^*, z_2^* are continuous and thus there exists n such that $z_1^*(z_n) \neq z_2^*(z_n)$. Of course $z_n = \sum_{i=1}^{k(n)} \alpha_i \delta_{x_i}$ and z_1^*, z_2^* are linear

$$\sum_{i=1}^{k(n)} \alpha_i z_1^*(\delta_{x_i}) \neq \sum_{i=1}^{k(n)} \alpha_i z_2^*(\delta_{x_i}).$$

To show that the mapping T is linear we need to show

$$T(\lambda_1 z_1^* + \lambda_2 z_2^*) = \lambda_1 Tz_1^* + \lambda_2 Tz_2^*, \quad \text{for all } \lambda_1, \lambda_2 \in \mathbb{R}^d, z_1^*, z_2^* \in Z^*,$$

what means that $\forall x \in \mathbb{R}^d$, $T(\lambda_1 z_1^* + \lambda_2 z_2^*)(x) = \lambda_1 T z_1^*(x) + \lambda_2 T z_2^*(x)$. Indeed, $T(\lambda_1 z_1^* + \lambda_2 z_2^*)(x) = (\lambda_1 z_1^* + \lambda_2 z_2^*)(\delta_x) = \lambda_1 z_1^*(\delta_x) + \lambda_2 z_2^*(\delta_x) = \lambda_1 T(z_1^*)(x) + \lambda_2 T(z_2^*)(x)$.

Step 2. First we prove that $\text{im}(T(Z^*)) \subseteq \mathcal{C}^{1+\alpha}(\mathbb{R}^d)$. By Lemma 5.2 we know that $(x \mapsto \delta_x) \in \mathcal{C}^{1+\alpha}(\mathbb{R}^d, Z)$ and then $(x \mapsto z^*(\delta_x)) \in \mathcal{C}^{1+\alpha}(\mathbb{R}^d, \mathbb{R})$ – as a composition of two functions $(x \mapsto \delta_x) \in \mathcal{C}^{1+\alpha}(\mathbb{R}^d, Z)$ and $z^* \in \mathcal{L}(Z, \mathbb{R})$. Therefore $(x \mapsto T z^*(x)) \in \mathcal{C}^{1+\alpha}(\mathbb{R}^d, \mathbb{R})$.

Step 3. To prove the opposite inclusion $\mathcal{C}^{1+\alpha}(\mathbb{R}^d) \subseteq \text{im}(T(Z^*))$, let us consider an arbitrary $y \in \mathcal{C}^{1+\alpha}(\mathbb{R}^d)$. We want to show there exists z_y^* such that $y = T z_y^*$. Define a functional $z_y^*(\delta_x) := y(x)$. Our goal is to show that $z_y^* \in Z^*$. It is enough to consider only $z \in \text{span}\{\delta_x : x \in \mathbb{R}^d\}$ and then

$$|z_y^*(z)| = |z_y^*\left(\sum_{i=1}^n \alpha_i \delta_{x_i}\right)|,$$

functional z_y^* is linear thus above is equal to $|\sum_{i=1}^n \alpha_i \cdot z_y^*(\delta_{x_i})|$. Using the definition of z_y^* the following holds

$$\begin{aligned} \left| \sum_{i=1}^n \alpha_i z_y^*(\delta_{x_i}) \right| &= \left| \sum_{i=1}^n \alpha_i y(x_i) \right| = \left| \sum_{i=1}^n \alpha_i \int_{\mathbb{R}^d} y d\delta_{x_i} \right| \\ &= \left| \int_{\mathbb{R}^d} y d\left(\sum_{i=1}^n \alpha_i \delta_{x_i}\right) \right| = \left| \int_{\mathbb{R}^d} y dz \right| \leq \|y\|_{\mathcal{C}^{1+\alpha}} \|z\|_{(\mathcal{C}^{1+\alpha}(\mathbb{R}^d))^*}. \end{aligned}$$

Thus $\|z_y^*\|_{Z^*} = \sup\{z_y^*(z) : \|z\|_Z \leq 1\} \leq \|y\|_{\mathcal{C}^{1+\alpha}(\mathbb{R}^d)}$.

Step 4. To complete the proof we need continuity of the mapping T which is of course equivalent to boundedness. In fact it is easy to see that $T^{-1}y = z_y^*$ is bounded. Estimations in step 3 imply that $\|T^{-1}\| \leq 1$. By Banach Isomorphism Theorem $\|T\| \leq C$, what finishes the proof. \square

6. ACKNOWLEDGMENT

This work was partially supported by the Polish Government MNiSW: P.G and A.Ś-G received support from the National Science Centre, UMO-2015/18/M/ST1/00075; K.Ł acknowledges the support of from the National Science Centre, DEC-2012/05/E/ST1/02218.

REFERENCES

- [ACFK17] G. Albi, Y-P. Choi, M. Fornasier, and D. Kalise. Mean field control hierarchy. *Applied Mathematics and Optimization*, 76(1):93–135, 2017.
- [AE08] H. Amann and J. Escher. *Analysis II*. Birkhäuser Basel, 2008.
- [AFM] L. Ambrosio, M. Fornasier, and M. Morandotti. Spatially inhomogeneous evolutionary games. *arXiv:1805.04027v1*.
- [AGS08] L. Ambrosio, N. Gigli, and G. Savaré. *Gradient Flows In Metric Spaces and in the Space of Probability Measures*. Lectures in Mathematics. ETH Zürich, Birkhäuser, 2008.
- [AHP] G. Albi, M. Herty, and L. Pareschi. Linear multistep methods for optimal control problems and applications to hyperbolic relaxation systems. *arXiv:1807.08547v1*.
- [BDDR08] F. Berthelin, P. Degond, M. Delitala, and M. Rasche. A model for the formation and evolution of traffic jams. *Archive for Rational Mechanics and Analysis*, 187(2):185–220, 2008.
- [BDT17] N. Bellomo, P. Degond, and E. Tadmor. *Active Particles, Volume 1: Advances in Theory, Models, and Applications*. Springer, Birkhäuser, 2017.
- [BFRS17] M. Bongini, M. Fornasier, F. Rossi, and F. Solombrino. Mean-field Pontryagin maximum principle. *Journal of Optimization Theory and Applications*, 175(1):1–38, 2017.
- [BGSW13] Y. Brenier, W. Gangbo, G. Savaré, and M. Westdickenberg. Sticky particle dynamics with interactions. *Journal de Mathématiques Pures et Appliquées. Neuvième Série*, 99(5):577–617, 2013.
- [BR19] B. Bonnet and F. Rossi. The Pontryagin maximum principle in the Wasserstein space. *Calculus of Variations and Partial Differential Equations*, 58(1):Art. 11, 36, 2019.
- [CCGU12] J. A. Carrillo, R. M. Colombo, P. Gwiazda, and A. Ulikowska. Structured populations, cell growth and measure valued balance laws. *Journal of Differential Equation*, 252:3245–3277, 2012.
- [CGR18] R. M. Colombo, P. Gwiazda, and M. Rosińska. Optimization in structure population models through the Escalator Boxcar Train. *ESAIM: Control, Optimisation and Calculus of Variations*, 24(1):377–399, 2018.
- [CLM13] G. Crippa and M. Lécureux-Mercier. Existence and uniqueness of measure solutions for a system of continuity equations with non-local flow. *NoDEA. Nonlinear Differential Equations and Applications*, 20(3):523–537, 2013.
- [CnCC13] J. A. Cañizo, J. A. Carrillo, and S. Cuadrado. Measure solutions for some models in population dynamics. *Acta Applicandae Mathematicae*, 123:141–156, 2013.
- [CnCR11] J. A. Cañizo, J. A. Carrillo, and J. Rosado. A well-posedness theory in measures for some kinetic models of collective motion. *Mathematical Models and Methods in Applied Sciences*, 21(3):515–539, 2011.
- [DHL14] P. Degond, M. Herty, and J.-G. Liu. Flow on sweeping networks. *Multiscale Modeling & Simulation. A SIAM Interdisciplinary Journal*, 12(2):538–565, 2014.
- [EHM16] J. H. M. Evers, S. C. Hille, and A. Muntean. Measure-valued mass evolution problems with flux boundary conditions and solution-dependent velocities. *SIAM J. Math. Analysis*, 48(3):1929–1953, 2016.
- [FLOS] M. Fornasier, L. Lisini, C. Orrieri, and G. Savaré. Mean-field optimal control as Gamma-limit of finite agent controls. *arXiv:1803.04689v1*.
- [GHŁ] P. Gwiazda, S. C. Hille, and K. Łyczek. Differentiability in perturbation parameter of measure solution to non-linear perturbed transport equation. *In preparation*.
- [GJMU14] P. Gwiazda, J. Jabłoński, A. Marciniak-Czochra, and A. Ulikowska. Analysis of particle methods for structured population models with nonlocal boundary term in the framework of bounded Lipschitz distance. *Numerical Methods for Partial Differential Equations*, 30(6):1797–1820, 2014.
- [GLM10] P. Gwiazda, T. Lorenz, and A. Marciniak-Czochra. A non-linear structured population model: Lipschitz continuity of measure-valued solutions with respect to model ingredients. *Journal of Differential Equations*, 248:2703–2735, 2010.
- [GM10] P. Gwiazda and A. Marciniak-Czochra. Structured population equations in metric spaces. *Journal of Hyperbolic Differential Equations*, 7:733–773, 2010.

- [GR17] P. Goatin and F. Rossi. A traffic flow model with non-smooth metric interaction: well-posedness and micro-macro limit. *Communications in Mathematical Sciences*, 15(1):261–287, 2017.
- [GS16] P. Goatin and S. Scialanga. Well-posedness and finite volume approximations of the LWR traffic flow model with non-local velocity. *Networks and Heterogeneous Media*, 11(1):107–121, 2016.
- [Har02] P. Hartman. *Ordinary Differential Equations*. Society for Industrial and Applied Mathematics, 2002.
- [HW09] S. C. Hille and D. H. T. Worm. Embedding of semigroups of Lipschitz maps into positive linear semigroups on ordered Banach spaces generated by measures. *Integral Equations and Operator Theory*, 63(3):351371, 2009.
- [Man07] S. Maniglia. Probabilistic representation and uniqueness results for measure-valued solutions of transport equations. *Journal de Mathématiques Pures et Appliquées*, 87(6):601–626, 2007.
- [PF14] B. Piccoli and R. Francesco. Generalized Wasserstein distance and its application to transport equations with source. *Archive for Rational Mechanics and Analysis*, 211(6):335358, 2014.
- [PFM] B. Piccoli, M. Fornasier, and M. Morandotti. Measure differential equations. *arXiv:1708.09738v1*.
- [PR16] B. Piccoli and F. Rossi. On properties of the generalized Wasserstein distance. *Archive for Rational Mechanics and Analysis*, 222(3):1339–1365, 2016.
- [Skr] J. Skrzeczkowski. Measure solutions to perturbed structured population models – differentiability with respect to perturbation parameter. *arXiv:1812.01747v3*.
- [Thi03] H. R. Thieme. *Mathematics in Population Biology*. Princeton University Press, 2003.
- [Web85] G. F. Webb. *Theory of nonlinear age-dependent population dynamics*. M. Dekker New York, 1985.