ON THE BIFURCATION SET OF UNIQUE EXPANSIONS

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Abstract. Given a positive integer M, for $q\in(1,M+1]$ let $\mathcal{U}_q$ be the set of $x\in[0,M/(q-1)]$ having a unique q-expansion with the digit set $\{0,1,\ldots,M\}$, and let $\mathbf{U}_q$ be the set of corresponding q-expansions. Recently, Komornik et al. showed in [23] that the topological entropy function $H:q\mapsto h_{top}(\mathbf{U}_q)$ is a Devil's staircase in $(1,M+1]$.

Let $\mathscr{B}$ be the bifurcation set of H defined by

$$\mathscr{B}=\{q\in(1,M+1]:H(p)\neq H(q)\quad\text{for any}\quad p\neq q\}.$$

In this paper we analyze the fractal properties of $\mathscr{B}$, and show that for any $q\in\mathscr{B}$,

$$\lim_{\delta\to0}\dim_H(\mathscr{B}\cap(q-\delta,q+\delta))=\dim_H\mathcal{U}_q,$$

where $\dim_H$ denotes the Hausdorff dimension. Moreover, when $q\in\mathscr{B}$ the univoque set $\mathcal{U}_q$ is dimensionally homogeneous, i.e., $\dim_H(\mathcal{U}_q\cap V)=\dim_H\mathcal{U}_q$ for any open set V that intersect $\mathcal{U}_q$.

As an application we obtain a dimensional spectrum result for the set $\mathscr{U}$ containing all bases $q\in(1,M+1]$ such that 1 admits a unique q-expansion. In particular, we prove that for any $t>1$ we have

$$\dim_H(\mathscr{U}\cap(1,t])=\max_{q\le t}\dim_H\mathcal{U}_q.$$

We also consider the variations of the sets $\mathscr{U}=\mathscr{U}(M)$ when M changes.

1. Introduction

Fix a positive integer M. For any $q\in(1,M+1]$ each $x\in I_{q,M}:=[0,M/(q-1)]$ has a q-expansion, i.e., there exists a sequence $(x_i)=x_1x_2\ldots$ with each $x_i\in\{0,1,\ldots,M\}$ such that

$$x=\sum_{i=1}^{\infty}\frac{x_i}{q^i}=:\pi_q((x_i)). \tag{1.1}$$

The sequence (x_i) is called a *q-expansion* of x. If no confusion arises the *alphabet* is always assumed to be $\{0,1,\ldots,M\}$.

Non-integer base expansions have received a lot of attention since the pioneering papers of Rényi [34] and Parry [33]. It is well known that for any $q\in(1,M+1)$ Lebesgue almost every $x\in I_{q,M}$ has a continuum of q-expansions (cf. [35, 11]). Moreover, for any $k\in\mathbb{N}\cup\{\aleph_0\}$

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there exist $q \in (1, M+1]$ and $x \in I_{q,M}$ such that x has precisely k different q-expansions (see e.g., [19, 37]). For more information on non-integer base expansions we refer the reader to the survey paper [22] and the references therein.

In this paper we focus on studying unique q-expansions. For $q \in (1, M+1]$ let

$$\mathcal{U}_q := \{x \in I_{q,M} : x \text{ has a unique } q\text{-expansion}\},$$

and let $\mathbf{U}_q = \pi_q^{-1}(\mathcal{U}_q)$ be the set of corresponding q-expansions. These sets have been the object of study in many articles and have a very rich topological structure (see for example [25, 15]). Komornik et al. studied in [23] the Hausdorff dimension of $\mathcal{U}_q$, and showed that the dimension function $D : q \mapsto \dim_H \mathcal{U}_q$ has a Devil's staircase behavior (see also [3]). Moreover, they showed that the entropy function

$$H : (1, M+1] \to [0, \log(M+1)]; \qquad q \mapsto h_{top}(\mathbf{U}_q)$$

is a Devil's staircase (see Lemma 2.4 below). Recently, Alcaraz Barrera et al. investigated in [1] the dynamical properties of $\mathcal{U}_q$, and determined the maximal intervals on which the entropy function H is constant.

Let $\mathscr{B}$ be the *bifurcation set* of the function H defined by

$$\mathscr{B} = \{q \in (1, M+1] : H(p) \neq H(q) \text{ for any } p \neq q\}.$$

Then $\mathscr{B}$ is the set of bases where the entropy function H is not locally constant. In [1] Alcaraz Barrera et al. gave a characterization of $\mathscr{B}$ and showed that $\mathscr{B}$ has full Hausdorff dimension. In particular, we have

$$\mathscr{B} = (q_{KL}, M+1] \setminus \bigcup [p_L, p_R], \tag{1.2}$$

where q_{KL} is the *Komornik-Loreti constant* (cf. [24]) and the union on the right hand side is countable and pairwise disjoint (see Section 2 below for more explanation).

From [15] we know that the univoque set $\mathcal{U}_q$ has a fractal structure and might have isolated points. Our first result states that for $q \in \mathscr{B}$ the univoque set $\mathcal{U}_q$ is *dimensionally homogeneous*, i.e., the local Hausdorff dimension of $\mathcal{U}_q$ equals the full dimension of $\mathcal{U}_q$.

Theorem 1. *Let $q \in (q_{KL}, M+1] \setminus \bigcup (p_L, p_R]$. Then for any open set $V \subseteq \mathbb{R}$ with $\mathcal{U}_q \cap V \neq \emptyset$ we have*

$$\dim_H(\mathcal{U}_q \cap V) = \dim_H \mathcal{U}_q.$$

Remark 1.1.

(1) Note by (1.2) that $\mathscr{B} \subset (q_{KL}, M+1] \setminus \bigcup (p_L, p_R]$. So Theorem 1 implies that the univoque set $\mathcal{U}_q$ is dimensionally homogeneous for any $q \in \mathscr{B}$.

(2) In Theorem 3.6 we give a complete characterization of the set

$$\{q \in (1, M+1] : \mathcal{U}_q \text{ is dimensionally homogeneous}\}.$$

It turns out that the Lebesgue measure of this set is positive and strictly smaller than M.

Throughout the paper we will use $\overline{A}$ to denote the topological closure of a set $A \subset \mathbb{R}$. Our second result presents a close relationship between the bifurcation set $\overline{\mathscr{B}}$ and the univoque sets $\mathcal{U}_q$.

Theorem 2. *For any $q \in \overline{\mathscr{B}}$ we have*

$$\lim_{\delta \to 0} \dim_H(\overline{\mathscr{B}} \cap (q-\delta, q+\delta)) = \dim_H \mathcal{U}_q.$$

Remark 1.2.

(1) Since by (1.2) and (2.5) the difference between $\mathscr{B}$ and $\overline{\mathscr{B}}$ is countable, Theorem 2 also holds if we replace $\overline{\mathscr{B}}$ by $\mathscr{B}$.

(2) Note that $\dim_H \mathcal{U}_q > 0$ for any $q > q_{KL}$ (see Lemma 2.4 below). As a consequence of Theorem 2 it follows that

$$q \in \overline{\mathscr{B}} \setminus \{q_{KL}\} \quad \Longleftrightarrow \quad \lim_{\delta \to 0} \dim_H(\overline{\mathscr{B}} \cap (q-\delta, q+\delta)) = \dim_H \mathcal{U}_q > 0.$$

Recently, Allaart et al. [2, Corollary 3] gave another characterization of $\overline{\mathscr{B}}$, and showed that

$$q \in \overline{\mathscr{B}} \setminus \{q_{KL}\} \quad \Longleftrightarrow \quad \lim_{\delta \to 0} \dim_H(\mathscr{U} \cap (q-\delta, q+\delta)) = \dim_H \mathcal{U}_q > 0,$$

where $\mathscr{U} := \{q \in (1, M+1] : 1 \in \mathcal{U}_q\}$.

It is well-known that the univoque set $\mathcal{U}_q$ has a close connection with the set $\mathscr{U} = \mathscr{U}(M)$ of *univoque bases* $q \in (1, M+1]$ for which 1 has a unique q-expansion with alphabet $\{0, 1, \ldots, M\}$. For example, in [15] De Vries and Komornik showed that $\mathcal{U}_q$ is closed if and only if $q \notin \overline{\mathscr{U}}$. The set $\mathscr{U}$ has many interesting properties itself. Erdős et al. showed in [18] that $\mathscr{U}$ is an uncountable set of zero Lebesgue measure. Daróczy and Kátai proved in [14] that the Hausdorff dimension of $\mathscr{U}$ is 1 (see also [23]). Komornik and Loreti showed in [24] that the smallest element of $\mathscr{U}$ is q_{KL}. In [25] the same authors studied the topological properties of $\mathscr{U}$, and showed that its closure $\overline{\mathscr{U}}$ is a Cantor set. Recently, Kong et al. proved in [28] that for any $q \in \overline{\mathscr{U}}$ we have

$$\dim_H(\overline{\mathscr{U}} \cap (q-\delta, q+\delta)) > 0 \qquad \text{for any } \delta > 0. \tag{1.3}$$

On a different note, in [9] Bonanno et al. introduced a set

$$\Lambda = \{x \in [0,1] \,:\, S^k x \le x \text{ for all } n \ge 0\}, \tag{1.4}$$

where S is the tent map defined by $S : x \mapsto \min\{2x, 2-2x\}$ and showed that there is a one to one correspondence between the set $\mathscr{U}(1)$ and the set $\Lambda \backslash \mathbb{Q}_1$, where $\mathbb{Q}_1$ is the set of all rationals with odd denominator. This link is based on work by Allouche and Cosnard (see [4, 6, 7]), who related the set $\mathscr{U}(1)$ to kneading sequences of unimodal maps. The authors of [9] also explored a relationship between these sets and the real slice of the boundary of the Mandelbrot set.

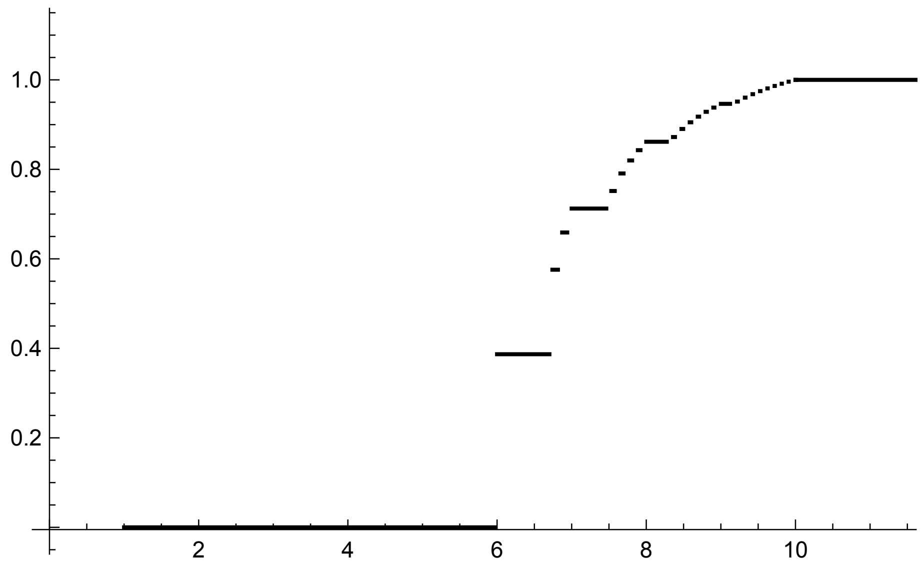

FIGURE 1. The asymptotic graph of the function $\phi(t) = \dim_H(\mathscr{U} \cap (1, t])$ for $t \in [4, 11.5]$ with $M = 9$ and $q_{KL} = q_{KL}(9) \approx 5.97592$.

By using Theorem 2 we investigate the dimensional spectrum of $\mathscr{U}$. Our next result strengthens the relationship between $\mathcal{U}_q$ and $\mathscr{U}$.

Theorem 3. *For any $t > 1$ we have*

$$\dim_H(\mathscr{U} \cap (1, t]) = \max_{q \le t} \dim_H \mathcal{U}_q.$$

Moreover, the function $\phi(t) := \dim_H(\mathscr{U} \cap (1, t])$ is a Devil's staircase on $(1, \infty)$.

Remark 1.3.

(1) In [25] it was shown that $\overline{\mathscr{U}} \setminus \mathscr{U}$ is a countable set. Hence, Theorem 3 still holds if we replace $\mathscr{U}$ by $\overline{\mathscr{U}}$.

(2) Results from [23] (see Lemma 2.4 below) give that $\dim_H \mathcal{U}_q = 1$ if and only if $q = M+1$. In view of Theorem 3 we obtain that $\dim_H(\mathscr{U} \cap (1, t]) < 1$ for any $t < M+1$.

This implies that the Hausdorff dimension of $\mathscr{U}$ is concentrated on the neighborhood of $M+1$.

As an application of Theorem 3 we investigate the variations of $\mathscr{U} = \mathscr{U}(M)$ when the parameter M changes. For $K \in \{1, 2, \ldots, M\}$, let $\mathscr{U}(K)$ be the set of bases $q \in (1, K+1]$ such that 1 has a unique q-expansion with respect to the alphabet $\{0, 1, \ldots, K\}$. Theorem 4 characterizes the Hausdorff dimensions of the intersection $\mathscr{U}(M) \cap \mathscr{U}(K)$ and the difference $\mathscr{U}(M) \setminus \mathscr{U}(K)$. Indeed, we prove the following stronger result.

Theorem 4.

(i) *Let $K \in \{1, 2, \ldots, M\}$. Then*

$$\dim_H \left(\bigcap_{J=K}^{M} \mathscr{U}(J) \right) = \max_{q \le K+1} \dim_H \mathcal{U}_q.$$

(ii) *For any positive integer L we have*

$$\dim_H \left(\mathscr{U}(L) \setminus \bigcup_{J \ne L} \mathscr{U}(J) \right) = 1.$$

Remark 1.4. By the proof of Theorem 4 it follows that for $K < M$ the intersection

$$\bigcap_{J=K}^{M} \mathscr{U}(J) = \mathscr{U}(M) \cap (1, K+1]$$

is a proper subset of $\mathscr{U}(K)$. This, together with (1.3), implies that for $K < M$ neither the intersection $\bigcap_{J=K}^{M} \mathscr{U}(J)$ nor the difference set $\mathscr{U}(M) \setminus \bigcap_{J=K}^{M} \mathscr{U}(J)$ contains isolated points.

We emphasize that for each $q \in (1, M+1]$ the univoque set $\mathcal{U}_q$ is related to the dynamical system

$$T_{q,j} : \left[0, \frac{M}{q-1}\right] \to \left[0, \frac{M}{q-1}\right]; \qquad x \mapsto qx - j$$

for $j \in \{0, 1, \ldots, M\}$. On the other hand, the set $\mathscr{U}$ contains all parameters $q \in (1, M+1]$ such that 1 has a unique q-expansion, and thus $\mathscr{U}$ is related to infinitely many dynamical systems. A similar set up involving a bifurcation set for infinitely many dynamical systems is considered in [9] (see also [10]). They considered the bifurcation set of an entropy map for a family of maps $\{T_\alpha : [\alpha - 1, \alpha] \to [\alpha - 1, \alpha]\}_{\alpha \in [0,1]}$, called α-continued fraction transformations [32], where for each $\alpha \in [0, 1]$ the map T_α is defined by

$$T_\alpha(x) = \begin{cases} \frac{1}{|x|} - \lfloor \frac{1}{|x|} + 1 - \alpha \rfloor & \text{if} \quad x \ne 0; \\ 0 & \text{if} \quad x = 0. \end{cases} \tag{1.5}$$

Each map T_α has a unique invariant measure μ_α that is absolutely continuous with respect to the Lebesgue measure. They showed that the map

$$\psi : \alpha \mapsto h_{\mu_\alpha}(T_\alpha),$$

assigning to each α the measure theoretic entropy $h_{\mu_\alpha}(T_\alpha)$, has countably many intervals on which it is monotonic. The complement of the union of these intervals in $[0,1]$, i.e., the bifurcation set of ψ denoted by F, has Lebesgue measure 0 (see [29] and [10]) and Hausdorff dimension 1 (see [9]). Moreover, in [9] the authors identified a homeomorphism between the set F and the set $\Lambda \setminus \{0\}$ from (1.4), giving also a relation to the set $\mathscr{U}(1)$. In [9], however, no information is given on the local structure of F. Recently, Dajani and the first author identified in [12] another set E that is linked to the sets $\mathscr{U}(1)$, Λ and F. They investigated a family of symmetric doubling maps $S_\gamma : [-1,1] \to [-1,1]$, given by

$$S_\gamma(x) = 2x - \gamma\lfloor 2x \rfloor,$$

where $\lfloor x \rfloor$ denotes the integer part of x, and showd that the set E of parameters $\gamma \in [1,2]$ for which the map S_γ does not have a piecewise smooth invariant density is homeomorphic to $\Lambda \setminus \{0\}$. Therefore, the results obtained in this paper about the set $\mathscr{U}(1)$ can be used to investigate the bifurcation sets E, F and the set Λ.

The rest of the paper is arranged in the following way. In Section 2 we fix the notation and recall some properties of unique q-expansions. Moreover, we recall from [1] some important properties of the bifurcation set $\mathscr{B}$. In Section 3 we give the proof of Theorem 1 for the dimensional homogeneousness of $\mathcal{U}_q$. In Section 4 we prove an auxiliary proposition that will be used to prove Theorem 2 in Section 5. The proof of Theorems 3 and 4 will be given in Sections 6 and 7, respectively. We end the paper with some remarks.

2. Unique expansions and bifurcation set

In this section we recall some properties of unique q-expansions and of the bifurcation set $\mathscr{B}$ as well. First we need some terminology from symbolic dynamics (cf. [30]).

2.1. **Symbolic dynamics.** Given a positive integer M, let $\{0,1,\dots,M\}^*$ denote the set of all finite strings of symbols from $\{0,1,\dots,M\}$, called *words*, together with the empty word denoted by ϵ. Let $\{0,1,\dots,M\}^{\mathbb{N}}$ be the set of sequences $(d_i) = d_1d_2\dots$ with each element $d_i \in \{0,1,\dots,M\}$. Let σ be the left shift on $\{0,1,\dots,M\}^{\mathbb{N}}$ defined by $\sigma((d_i)) = (d_{i+1})$. Then $(\{0,1,\dots,M\}^{\mathbb{N}}, \sigma)$ is a full shift. For a word $\mathbf{c} = c_1\dots c_n \in \{0,1,\dots,M\}^*$ we denote by $\mathbf{c}^k = (c_1\dots c_n)^k$ the k-fold concatenation of $\mathbf{c}$ to itself and by $\mathbf{c}^\infty = (c_1\dots c_n)^\infty$ the periodic sequence with period block $\mathbf{c}$. Moreover, for a word $\mathbf{c} = c_1\dots c_n$ with $c_n < M$ we denote by $\mathbf{c}^+$ the word

$$\mathbf{c}^+ = c_1 \dots c_{n-1}(c_n + 1).$$

Similarly, for a word $\mathbf{c} = c_1 \dots c_n$ with $c_n > 0$ we write $\mathbf{c}^- = c_1 \dots c_{n-1}(c_n - 1)$. For a sequence $(d_i) \in \{0, 1, \dots, M\}^{\mathbb{N}}$ we denote its *reflection* by

$$\overline{(d_i)} = (M - d_1)(M - d_2) \cdots .$$

Accordingly, for a word $\mathbf{c} = c_1 \dots c_n$ we denote its reflection by $\overline{\mathbf{c}} = (M - c_1) \cdots (M - c_n)$.

On words and sequences we consider the lexicographical ordering $\prec, \preccurlyeq, \succ$ or $\succcurlyeq$ which is defined as follows. For two sequences $(c_i), (d_i) \in \{0, 1, \dots, M\}^{\mathbb{N}}$ we say that $(c_i) \prec (d_i)$ if there exists $n \in \mathbb{N}$ such that $c_1 \dots c_{n-1} = d_1 \dots d_{n-1}$ and $c_n < d_n$. Moreover, we write $(c_i) \preccurlyeq (d_i)$ if $(c_i) \prec (d_i)$ or $(c_i) = (d_i)$. Similarly, we write $(c_i) \succ (d_i)$ if $(d_i) \prec (c_i)$, and $(c_i) \succcurlyeq (d_i)$ if $(d_i) \preccurlyeq (c_i)$. We extend this definition to words in the following way. For two words $\omega, \nu \in \{0, 1, \dots, M\}^*$ we write $\omega \prec \nu$ if $\omega 0^\infty \prec \nu 0^\infty$. Accordingly, for a sequence $(d_i) \in \{0, 1, \dots, M\}^{\mathbb{N}}$ and a word $\mathbf{c} = c_1 \dots c_m$ we say $(d_i) \prec \mathbf{c}$ if $(d_i) \prec \mathbf{c}0^\infty$.

Let $\mathcal{F} \subseteq \{0, 1, \dots, M\}^*$ and let $X = X_{\mathcal{F}} \subseteq \{0, 1, \dots, M\}^{\mathbb{N}}$ be the set of those sequences that do not contain any word from $\mathcal{F}$. We call the pair (X, σ) a *subshift.* If $\mathcal{F}$ can be chosen to be a finite set, then (X, σ) is called a *subshift of finite type.* For $n \in \mathbb{N} \cup \{0\}$ we denote by $\mathcal{L}_n(X)$ the set of words of length n occurring in sequences of X. In particular, for $n = 0$ we set $\mathcal{L}_0(X) = \{\epsilon\}$. The *languange* of (X, σ) is then defined by

$$\mathcal{L}(X) = \bigcup_{n=0}^{\infty} \mathcal{L}_n(X).$$

So, $\mathcal{L}(X)$ is the set of all finite words occurring in sequences from X.

For a subshift (X, σ) and a word $\omega \in \mathcal{L}(X)$ let $F_X(\omega)$ be the *follower set* of ω in X defined by

$$F_X(\omega) := \left\{(d_i) \in X : d_1 \dots d_{|\omega|} = \omega\right\}, \tag{2.1}$$

where $|\mathbf{c}|$ denotes the length of a word $\mathbf{c} \in \{0, 1, \dots, M\}^*$.

A subshift (X, σ) is called *topologically transitive* (or simply *transitive*) if for any two words $\omega, \nu \in \mathcal{L}(X)$ there exists a word γ such that $\omega\gamma\nu \in \mathcal{L}(X)$. In other words, in a transitive subshift (X, σ) any two words can be "connected" in $\mathcal{L}(X)$.

The *topological entropy* $h_{top}(X)$ of a subshift (X, σ) is a quantity that indicates its complexity. It is defined by

$$h_{top}(X) = \lim_{n \to \infty} \frac{\log \# \mathcal{L}_n(X)}{n} = \inf_{n \geq 1} \frac{\log \# \mathcal{L}_n(X)}{n}, \tag{2.2}$$

where $\#A$ denotes the cardinality of a set A. Accordingly, we define the topological entropy of a follower set $F_X(\omega)$ by changing X to $F_X(\omega)$ in (2.2) if the corresponding limit exists. Clearly, if X is a transitive subshift, then $h_{top}(F_X(\omega)) = h_{top}(X)$ for any $\omega \in \mathcal{L}(X)$.

2.2. **Unique expansions.** In this subsection we recall some results about unique expansions. For more information on this topic we refer the reader to the survey papers [36, 22] or the book chapter [16]. For $q \in (1, M+1]$, let

$$\alpha(q) = \alpha_1(q)\alpha_2(q)\ldots$$

be the *quasi-greedy* q-expansion of 1 (cf. [13]), i.e., the lexicographically largest q-expansion of 1 not ending with a string of zeros. The following characterization of quasi-greedy expansions was given in [8, Theorem 2.2].

Lemma 2.1. *The map $q \mapsto \alpha(q)$ is a strictly increasing bijection from $(1, M+1]$ onto the set of all sequences $(a_i) \in \{0, 1, \ldots, M\}^{\mathbb{N}}$ not ending with 0^∞ and satisfying*

$$a_{n+1}a_{n+2}\ldots \preceq a_1a_2\ldots \qquad \textit{whenever} \quad a_n < M.$$

Recall from (1.1) the definition of the projection map π_q for $q \in (1, M+1]$ mapping $\{0, 1, \ldots, M\}^{\mathbb{N}}$ onto the interval $I_{q,M} = [0, M/(q-1)]$. In general, π_q is not bijective. However, π_q is a bijection between $\mathbf{U}_q = \pi_q^{-1}(\mathcal{U}_q)$ and $\mathcal{U}_q$. The following lexicographical characterization of $\mathbf{U}_q$, or equivalently $\mathcal{U}_q$, was essentially due to Parry [33] (see also [8]).

Lemma 2.2. *Let $q \in (1, M+1]$. Then $(x_i) \in \mathbf{U}_q$ if and only if*

$$x_{n+1}x_{n+2}\ldots \prec \alpha(q) \qquad \textit{whenever} \quad x_n < M,$$
$$\overline{x_{n+1}x_{n+2}\ldots} \prec \alpha(q) \qquad \textit{whenever} \quad x_n > 0.$$

Observe that $\mathscr{U} = \{q \in (1, M+1] : \alpha(q) \in \mathbf{U}_q\}$. As a corollary of Lemma 2.2 we have the following characterizations of $\mathscr{U}$ and $\overline{\mathscr{U}}$.

Lemma 2.3.

(i) $q \in \mathscr{U} \setminus \{M+1\}$ *if and only if the quasi-greedy expansion $\alpha(q)$ satisfies*

$$\overline{\alpha(q)} \prec \sigma^n(\alpha(q)) \prec \alpha(q) \qquad \textit{for any } n \geq 1.$$

(ii) $q \in \overline{\mathscr{U}}$ *if and only if the quasi-greedy expansion $\alpha(q)$ satisfies*

$$\overline{\alpha(q)} \prec \sigma^n(\alpha(q)) \preccurlyeq \alpha(q) \qquad \textit{for any } n \geq 1.$$

Proof. Part (i) was shown in [17, Theorem 2.5] and Part (ii) was proven in [17, Theorem 3.9]. $\square$

In [15] it was shown that $(\mathbf{U}_q, \sigma)$ is not necessarily a subshift. Inspired by [23] we consider the set $\mathbf{V}_q$ which contains all sequences $(x_i) \in \{0, 1, \ldots, M\}^{\mathbb{N}}$ satisfying

$$\overline{\alpha(q)} \preccurlyeq \sigma^n((x_i)) \preccurlyeq \alpha(q) \qquad \text{for all} \quad n \geq 0.$$

Then $(\mathbf{V}_q, \sigma)$ is a subshift (cf. [23, Lemma 2.6]). Furthermore, Lemma 2.1 implies that the set-valued map $q \mapsto \mathbf{V}_q$ is increasing, i.e., $\mathbf{V}_p \subseteq \mathbf{V}_q$ whenever $p < q$.

Recall that the Komornik-Loreti constant q_{KL} is the smallest element of $\mathscr{U}$, which is defined in terms of the classical *Thue-Morse sequence* $(\tau_i)_{i=0}^{\infty} = 01101001\ldots$. Here the sequence $(\tau_i)_{i=0}^{\infty}$ is defined as follows (cf. [5]): $\tau_0 = 0$, and if $\tau_0 \ldots \tau_{2^n-1}$ has already been defined for some $n \geq 0$, then $\tau_{2^n} \ldots \tau_{2^{n+1}-1} = \overline{\tau_0 \ldots \tau_{2^n-1}}$. Then the Komornik-Loreti constant $q_{KL} = q_{KL}(M) \in (1, M+1]$ is the unique base satisfying

$$\alpha(q_{KL}) = \lambda_1 \lambda_2 \ldots, \tag{2.3}$$

where

$$\lambda_i = \begin{cases} k + \tau_i - \tau_{i-1} & \text{if} \quad M = 2k, \\ k + \tau_i & \text{if} \quad M = 2k+1, \end{cases}$$

for each $i \geq 1$. We emphasize that the sequence (λ_i) depends on M. By the definition of the Thue-Morse sequence $(\tau_i)_{i=0}^{\infty}$ it follows that (cf. [1])

$$\lambda_{2^n+1} \ldots \lambda_{2^{n+1}} = \overline{\lambda_1 \ldots \lambda_{2^n}}^{\,+} \qquad \text{for any} \quad n \geq 0. \tag{2.4}$$

Recall that a function $f : [a, b] \to \mathbb{R}$ is called a *Devil's staircase* (or *Cantor function*) if f is a continuous and non-decreasing function with $f(a) < f(b)$, and f is locally constant almost everywhere. The next lemma summarizes some results from [23] on the Hausdorff dimension of $\mathcal{U}_q$.

Lemma 2.4.

(i) *For any $q \in (1, M+1]$ we have*

$$\dim_H \mathcal{U}_q = \frac{h_{top}(\mathbf{V}_q)}{\log q}.$$

(ii) *The entropy function $H : q \mapsto h_{top}(\mathbf{V}_q)$ is a Devil's staircase in $(1, M+1]$:*

- *H is increasing and continuous in $(1, M+1]$;*
- *H is locally constant almost everywhere in $(1, M+1]$;*
- *$H(q) = 0$ if and only if $1 < q \leq q_{KL}$. Moreover, $H(q) = \log(M+1)$ if and only if $q = M+1$.*

Remark 2.5.

(1) Lemma 2.4 implies that the dimensional function $D : q \mapsto \dim_H \mathcal{U}_q$ has a Devil's staircase behavior: (i) D is continous in $(1, M+1]$; (ii) $D' < 0$ almost everywhere in $(1, M+1]$; (iii) $D(q) = 0$ for any $q \in (1, q_{KL}]$ and $D(q) = 1$ for $q = M+1$.
(2) In [23, Lemma 2.11] the authors showed that H is locally constant on the complement of $\overline{\mathscr{U}}$, i.e., $H'(q) = 0$ for any $q \in (1, M+1] \setminus \overline{\mathscr{U}}$.

2.3. **Bifurcation set.** In this subsection we recall some recent results obtained in [1], where the authors investigated the maximal intervals on which H is locally constant, called *entropy plateaus* (or simply called *plateaus*). For convenience of the reader we adopt much of the notation from [1]. We hope that this helps the interested reader who wants to access the relevant background information. Let $\mathscr{B}$ be the complement of these plateaus. From Lemma 2.4 (ii) we have

$$\mathscr{B} = \{q \in (1, M+1] : H(p) \neq H(q) \quad \text{for any } p \neq q\}.$$

Note by (1.2) that $\mathscr{B}$ is not closed. For the closure $\overline{\mathscr{B}}$ we have

$$\overline{\mathscr{B}} = \{q \in (1, M+1] : \forall \delta > 0, \exists p \in (q-\delta, q+\delta) \text{ such that } H(p) \neq H(q)\}.$$

In [1] $\overline{\mathscr{B}}$ was denoted by $\mathscr{E}$. The following lemma for $\overline{\mathscr{B}}$, the first part of which follows from Remark 2.5 (2), was established in [1, Theorem 3].

Lemma 2.6. *$\overline{\mathscr{B}} \subset \overline{\mathscr{U}}$, and $\overline{\mathscr{B}}$ is a Cantor set of full Hausdorff dimension.*

By Lemma 2.4 it follows that $\min \overline{\mathscr{B}} = q_{KL}$ and $\max \overline{\mathscr{B}} = M+1$. Since $\overline{\mathscr{B}}$ is a Cantor set, we can write

$$(q_{KL}, M+1] \setminus \overline{\mathscr{B}} = \bigcup (p_L, p_R), \tag{2.5}$$

where the union is pairwise disjoint and countable. By the definition of $\overline{\mathscr{B}}$ it follows that the intervals $[p_L, p_R]$ are the plateaus of H. In particular, since H is increasing, these intervals have the property that $H(q) = H(p_L)$ if and only if $q \in [p_L, p_R]$. This implies that the bifurcation set $\mathscr{B}$ can be rewritten as in (1.2), i.e.,

$$\mathscr{B} = (q_{KL}, M+1] \setminus \bigcup [p_L, p_R].$$

By (2.5) and (1.2) it follows that $\overline{\mathscr{B}} \setminus \mathscr{B}$ is countable. The fact that $\overline{\mathscr{B}}$ does not have isolated points gives the following lemma (see also [1]).

Lemma 2.7.

(i) *For any $q \in (q_{KL}, M+1] \setminus \bigcup (p_L, p_R]$ there exists a sequence of plateaus $\{[p_L(n), p_R(n)]\}$ such that $p_L(n) \nearrow q$ as $n \to \infty$.*

(ii) *For any* $q \in [q_{KL}, M+1) \setminus \bigcup [p_L, p_R)$ *there exists a sequence of plateaus* $\{[q_L(n), q_R(n)]\}$ *such that* $q_L(n) \searrow q$ *as* $n \to \infty$.

So, by (2.5), (1.2) and Lemma 2.7 it follows that $\overline{\mathscr{B}} \setminus \mathscr{B}$ is a countable and dense subset of $\overline{\mathscr{B}}$. In particular, the set of left endpoints of all plateaus of H is dense in $\overline{\mathscr{B}}$.

In [1] more detailed information on the structure of the plateaus of H is given. Before we can give the necessary details, we have to recall some notation from [1]. Let $\mathbf{V}$ be the set of sequences $(a_i) \in \{0, 1, \ldots, M\}^{\mathbb{N}}$ satisfying the inequalities

$$\overline{(a_i)} \preccurlyeq \sigma^n((a_i)) \preccurlyeq (a_i) \qquad \text{for all} \quad n \geq 0.$$

In [1, Lemma 3.3] it is proved that the subshift $(\mathbf{V}_q, \sigma)$ is not transitive for any $q \in (q_{KL}, q_T)$, where $q_T \in (1, M+1) \cap \mathscr{B}$ is the unique base such that

$$\alpha(q_T) = \begin{cases} (k+1)k^\infty & \text{if} \quad M = 2k, \\ (k+1)((k+1)k)^\infty & \text{if} \quad M = 2k+1. \end{cases} \tag{2.6}$$

The plateaus of H are characterized separately for the cases (A) $q \in [q_T, M+1]$ and (B) $q \in (q_{KL}, q_T)$.

(A). First we recall from [1] the following definition.

Definition 2.8. A sequence $(a_i) \in \mathbf{V}$ is called *irreducible* if

$$a_1 \ldots a_j (\overline{a_1 \ldots a_j}^{\,+})^\infty \prec (a_i) \qquad \text{whenever} \quad (a_1 \ldots a_j^-)^\infty \in \mathbf{V}.$$

Lemma 2.9. *Let* $[p_L, p_R] \subset [q_T, M+1]$ *be a plateau of* H.

(i) *There exists a word* $a_1 \ldots a_m \in \mathcal{L}(\mathbf{V}_{p_L})$ *such that*

$$\alpha(p_L) = (a_1 \ldots a_m)^\infty \text{ is irreducible,} \quad \text{and} \quad \alpha(p_R) = a_1 \ldots a_m^+ (\overline{a_1 \ldots a_m})^\infty.$$

(ii) $(\mathbf{V}_{p_L}, \sigma)$ *is a transitive subshift of finite type.*

(iii) *There exists a periodic sequence* $\nu^\infty \in \mathbf{V}_{p_L}$ *such that for any word* $\eta \in \mathcal{L}(\mathbf{V}_{p_L})$ *we can find a large integer* N *and a word* ω *satisfying*

$$\overline{\alpha_1(p_L) \ldots \alpha_N(p_L)} \prec \sigma^j(\eta\omega\nu^\infty) \prec \alpha_1(p_L) \ldots \alpha_N(p_L) \qquad \text{for any} \quad j \geq 0.$$

Proof. Part (i) follows by [1, Proposition 5.2], and Part (ii) follows by [1, Lemma 5.1 (1)].

For (iii) we take

$$\nu = \begin{cases} k & \text{if} \quad M = 2k, \\ (k+1)k & \text{if} \quad M = 2k+1. \end{cases}$$

Since $p_L \geq q_T$, by Lemma 2.1 we have $\alpha(p_L) \succcurlyeq \alpha(q_T)$. Then (2.6) gives that

$$\overline{\alpha_1(p_L)\alpha_2(p_L)} \preccurlyeq \overline{\alpha_1(q_T)\alpha_2(q_T)} \prec \sigma^j(\nu^\infty) \prec \alpha_1(q_T)\alpha_2(q_T) \preccurlyeq \alpha_1(p_L)\alpha_2(p_L) \tag{2.7}$$

for all $j \geq 0$. Note by (i) that $\alpha(p_L)$ is irreducible. By the proof of [1, Proposition 3.17] it follows that for any word $\eta \in \mathcal{L}(\mathbf{V}_{p_L})$ there exist a large integer $N \geq 2$ and a word ω satisfying

$$\overline{\alpha_1(p_L)\dots\alpha_N(p_L)} \prec \sigma^j(\eta\omega\nu^\infty) \prec \alpha_1(p_L)\dots\alpha_N(p_L) \quad \text{for any} \quad 0 \leq j < |\eta| + |\omega|.$$

This together with (2.7) proves (iii). □

(B). Now we consider plateaus of H in (q_{KL}, q_T). Let (λ_i) be the quasi-greedy q_{KL}-expansion of 1 as given in (2.3). Note that (λ_i) depends on M. For $n \geq 1$ let

$$\xi(n) = \begin{cases} \lambda_1\dots\lambda_{2^{n-1}}(\overline{\lambda_1\dots\lambda_{2^{n-1}}}^{\,+})^\infty & \text{if} \quad M = 2k, \\ \lambda_1\dots\lambda_{2^n}(\overline{\lambda_1\dots\lambda_{2^n}}^{\,+})^\infty & \text{if} \quad M = 2k+1. \end{cases} \tag{2.8}$$

Then $\xi(1) = \alpha(q_T)$, and $\xi(n)$ is strictly decreasing to $(\lambda_i) = \alpha(q_{KL})$ as $n \to \infty$. Moreover, [1, Lemma 4.2] gives that $\xi(n) \in \mathbf{V}$ for all $n \geq 1$. We recall from [1] the following definition.

Definition 2.10. A sequence $(a_i) \in \mathbf{V}$ is said to be $*$-*irreducible* if there exists $n \in \mathbb{N}$ such that $\xi(n+1) \preccurlyeq (a_i) \prec \xi(n)$, and

$$a_1\dots a_j(\overline{a_1\dots a_j}^{\,+})^\infty \prec (a_i)$$

whenever

$$(a_1\dots a_j^-)^\infty \in \mathbf{V} \quad \text{and} \quad j > \begin{cases} 2^n & \text{if} \quad M = 2k, \\ 2^{n+1} & \text{if} \quad M = 2k+1. \end{cases}$$

Lemma 2.11. *Let $[p_L, p_R] \subseteq (q_{KL}, q_T)$ be a plateau of H.*

(i) *There exists a word $a_1\dots a_m \in \mathcal{L}(\mathbf{V}_{p_L})$ such that*

$$\alpha(p_L) = (a_1\dots a_m)^\infty \text{ is } *\text{-irreducible}, \quad \text{and} \quad \alpha(p_R) = a_1\dots a_m^+(\overline{a_1\dots a_m})^\infty.$$

(ii) *$(\mathbf{V}_{p_L}, \sigma)$ is a subshift of finite type, and it contains a unique transitive subshift of finite type (X_{p_L}, σ) satisfying $h_{top}(X_{p_L}) = h_{top}(\mathbf{V}_{p_L})$.*

(iii) *There exists a periodic sequence $\nu^\infty \in X_{p_L}$ such that for any word $\eta \in \mathcal{L}(\mathbf{V}_{p_L})$ we can find a large integer N and a word ω satisfying*

$$\overline{\alpha_1(p_L)\dots\alpha_N(p_L)} \prec \sigma^j(\eta\omega\nu^\infty) \prec \alpha_1(p_L)\dots\alpha_N(p_L) \quad \text{for any} \quad j \geq 0.$$

Proof. Part (i) follows from [1, Proposition 5.11], and Part (ii) follows from [1, Lemma 5.9]. Then it remains to prove (iii).

By (i) we know that $\alpha(p_L)$ is a $*$-irreducible sequence. Then there exists $n \in \mathbb{N}$ such that $\xi(n+1) \preccurlyeq \alpha(p_L) \prec \xi(n)$. Note by (i) and (2.8) that $\alpha(p_L)$ is purely periodic while $\xi(n+1)$ is eventually periodic. Then $\alpha(p_L) \succ \xi(n+1)$. Let

$$\nu = \begin{cases} \lambda_1\dots\lambda_{2^n}^- & \text{if} \quad M = 2k, \\ \lambda_1\dots\lambda_{2^{n+1}}^- & \text{if} \quad M = 2k+1. \end{cases}$$

Then by the proof of [1, Lemma 5.9] we have $\nu^\infty \in X_{p_L}$. Observe by (2.4) and (2.8) that $\xi(n+1) = \nu^+(\overline{\nu})^\infty \in \mathbf{V}$. Then by using $\alpha(p_L) \succ \xi(n+1)$ it follows that there exists a large integer N such that

$$\overline{\alpha_1(p_L)\dots\alpha_N(p_L)} \prec \sigma^j(\nu^\infty) \prec \alpha_1(p_L)\dots\alpha_N(p_L) \quad \text{for any} \quad j \geq 0.$$

The remaining part of (iii) follows by the proof of [1, Lemma 5.8]. □

Finally, the following characterization of $\overline{\mathscr{B}}$ was established in [1, Theorem 3].

Lemma 2.12. $\overline{\mathscr{B}} = \overline{\{q \in (q_{KL}, M+1] : \alpha(q) \text{ is irreducible or } *-\text{irreducible}\}}$.

3. Dimensional homogeneity of $\mathcal{U}_q$

In this section we will prove Theorem 1. Instead of proving Theorem 1 we prove the following equivalent statement.

Theorem 3.1. *Let $q \in (1, q_{KL}] \cup ((q_{KL}, M+1] \setminus \bigcup (p_L, p_R])$. Then for any $x \in \mathcal{U}_q$ we have*

$$\dim_H(\mathcal{U}_q \cap (x-\delta, x+\delta)) = \dim_H \mathcal{U}_q \qquad \text{for any } \delta > 0.$$

Before giving the proof of Theorem 3.1 we first explain why Theorem 3.1 is equivalent to Theorem 1. Clearly, Theorem 1 implies Theorem 3.1. On the other hand, take $q \in \mathscr{B}$. Let $V \subseteq \mathbb{R}$ be an open set with $\mathcal{U}_q \cap V \neq \emptyset$. Then there exist $x \in \mathcal{U}_q \cap V$ and $\delta > 0$ such that

$$\mathcal{U}_q \cap V \supset \mathcal{U}_q \cap (x-\delta, x+\delta).$$

By Theorem 3.1 it follows that $\dim_H(\mathcal{U}_q \cap V) \geq \dim_H \mathcal{U}_q$, which gives Theorem 1.

Note that for $q \in (1, q_{KL}]$ the statement of Theorem 3.1 follows immediately from the fact that $\dim_H \mathcal{U}_q = 0$. For $q \in (q_{KL}, M+1]$ recall that $\mathbf{V}_q$ is the set of sequences $(x_i) \in \{0, 1, \dots, M\}^{\mathbb{N}}$ satisfying

$$\overline{\alpha(q)} \preccurlyeq \sigma^n((x_i)) \preccurlyeq \alpha(q) \quad \text{for all } n \geq 0.$$

Accordingly, let

$$\mathcal{V}_q := \{\pi_q((x_i)) : (x_i) \in \mathbf{V}_q\},$$

where π_q is the projection map defined in (1.1). For a set $A \subset \mathbb{R}$ and $r \in \mathbb{R}$ we denote by $rA := \{r \cdot a : a \in A\}$ and $r + A := \{r + a : a \in A\}$.

The following lemma for a relationship between $\mathcal{U}_q$ and $\mathcal{V}_q$ follows from Lemma 2.2 and the definition of $\mathcal{V}_q$.

Lemma 3.2. *Let $q \in (q_{KL}, M+1]$. Then $\mathcal{U}_q$ is a countable union of affine copies of $\mathcal{V}_q$ up to a countable set, i.e.,*

$$\mathcal{U}_q \cup \mathcal{N} = \left\{0, \frac{M}{q-1}\right\} \cup \bigcup_{c_1=1}^{M-1} \left(\frac{c_1}{q} + \frac{\mathcal{V}_q}{q}\right) \cup \bigcup_{m=2}^{\infty} \bigcup_{c_m=1}^{M} \left(\frac{c_m}{q^m} + \frac{\mathcal{V}_q}{q^m}\right)$$
$$\cup \bigcup_{m=2}^{\infty} \bigcup_{c_m=0}^{M-1} \left(\sum_{i=1}^{m-1} \frac{M}{q^i} + \frac{c_m}{q^m} + \frac{\mathcal{V}_q}{q^m}\right),$$

where the set $\mathcal{N}$ is at most countable.

Proof. For $q \in (q_{KL}, M+1]$ let $\mathbf{W}_q$ be the set of sequences (x_i) satisfying

$$\overline{\alpha(q)} \prec \sigma^n((x_i)) \prec \alpha(q) \quad \text{for any} \quad n \geq 0,$$

and let $\mathcal{W}_q = \pi_q(\mathbf{W}_q)$. Then $\mathcal{V}_q \setminus \mathcal{W}_q$ is at most countable (cf. [15]). By [23, Lemma 2.5] it follows that

$$\mathcal{U}_q = \left\{0, \frac{M}{q-1}\right\} \cup \bigcup_{c_1=1}^{M-1} \left(\frac{c_1}{q} + \frac{\mathcal{W}_q}{q}\right) \cup \bigcup_{m=2}^{\infty} \bigcup_{c_m=1}^{M} \left(\frac{c_m}{q^m} + \frac{\mathcal{W}_q}{q^m}\right)$$
$$\cup \bigcup_{m=2}^{\infty} \bigcup_{c_m=0}^{M-1} \left(\sum_{i=1}^{m-1} \frac{M}{q^i} + \frac{c_m}{q^m} + \frac{\mathcal{W}_q}{q^m}\right).$$

This establishes the lemma since $\mathcal{W}_q \subseteq \mathcal{V}_q$ and $\mathcal{V}_q \setminus \mathcal{W}_q$ is at most countable. $\square$

It immediately follows from Lemma 3.2 that

$$\dim_H \mathcal{U}_q = \dim_H \mathcal{V}_q \qquad \text{for any } q \in (q_{KL}, M+1].$$

Hence, it suffices to prove Theorem 3.1 for $\mathcal{V}_q$ instead of $\mathcal{U}_q$. We first prove it for q being the left endpoint of an entropy plateau.

Lemma 3.3. *Let $[p_L, p_R] \subset (q_{KL}, M+1)$ be a plateau of H. Then for any $x \in \mathcal{V}_{p_L}$ we have*

$$\dim_H(\mathcal{V}_{p_L} \cap (x-\delta, x+\delta)) = \dim_H \mathcal{V}_{p_L} \qquad \textit{for any} \quad \delta > 0.$$

Proof. Obviously, $\dim_H(\mathcal{V}_{p_L} \cap (x-\delta, x+\delta)) \leq \dim_H \mathcal{V}_{p_L}$. So, it suffices to the prove the reverse inequality.

Fix $\delta > 0$ and $x \in \mathcal{V}_{p_L}$. Suppose that $(x_i) \in \mathbf{V}_{p_L}$ is a p_L-expansion of x. Then there exists a large integer N such that

$$\pi_{p_L}(F_{\mathbf{V}_{p_L}}(x_1 \ldots x_N)) \subseteq \mathcal{V}_{p_L} \cap (x-\delta, x+\delta), \tag{3.1}$$

where the follower set $F_{\mathbf{V}_{p_L}}(x_1 \ldots x_N) = \{(y_i) \in \mathbf{V}_{p_L} : y_1 \ldots y_N = x_1 \ldots x_N\}$ is as defined in (2.1). We split the proof into the following two cases.

Case I. $[p_L, p_R] \subset [q_T, M+1]$. Then by Lemma 2.9 (ii) it follows that $(\mathbf{V}_{p_L}, \sigma)$ is a transitive subshift of finite type. This implies that

$$h_{top}(F_{\mathbf{V}_{p_L}}(x_1 \dots x_N)) = h_{top}(\mathbf{V}_{p_L}).$$

Then, by (3.1), Lemma 2.4 (i) and Lemma 3.2 it follows that

$$\begin{aligned}\dim_H(\mathcal{V}_{p_L} \cap (x-\delta, x+\delta)) &\geq \dim_H \pi_{p_L}(F_{\mathbf{V}_{p_L}}(x_1 \dots x_N)) \\ &= \frac{h_{top}(F_{\mathbf{V}_{p_L}}(x_1 \dots x_N))}{\log p_L} \\ &= \frac{h_{top}(\mathbf{V}_{p_L})}{\log p_L} = \dim_H \mathcal{U}_{p_L} = \dim_H \mathcal{V}_{p_L}.\end{aligned}$$

Case II. $[p_L, p_R] \subset (q_{KL}, q_T)$. Then by Lemma 2.11 (ii) it follows that $(\mathbf{V}_{p_L}, \sigma)$ is a subshift of finite type that contains a unique transitive subshift of finite type X_{p_L} such that

$$h_{top}(X_{p_L}) = h_{top}(\mathbf{V}_{p_L}). \tag{3.2}$$

Furthermore, by Lemma 2.11 (iii) there exist a sequence $\nu^\infty \in X_{p_L}$ and a word ω such that

$$x_1 \dots x_N \omega \nu^\infty \in F_{\mathbf{V}_{p_L}}(x_1 \dots x_N). \tag{3.3}$$

From [30, Proposition 2.1.7] there exists $m \geq 0$ such that $(\mathbf{V}_{p_L}, \sigma)$ is an m-step subshift of finite type. Note by (3.3) that the word $x_1 \dots x_N \omega \nu^m \in \mathcal{L}(\mathbf{V}_{p_L})$. Then by [30, Theorem 2.1.8] it follows that for any sequence $(d_i) \in F_{X_{p_L}}(\nu^m) \subseteq F_{\mathbf{V}_{p_L}}(\nu^m)$ we have $x_1 \dots x_N \omega d_1 d_2 \dots \in F_{\mathbf{V}_{p_L}}(x_1 \dots x_N)$. In other words,

$$\left\{x_1 \dots x_N \omega d_1 d_2 \dots : (d_i) \in F_{X_{p_L}}(\nu^m)\right\} \subseteq F_{\mathbf{V}_{p_L}}(x_1 \dots x_N).$$

Therefore, by (3.1) it follows that

$$\begin{aligned}\dim_H(\mathcal{V}_{p_L} \cap (x-\delta, x+\delta)) &\geq \dim_H \pi_{p_L}(F_{\mathbf{V}_{p_L}}(x_1 \dots x_N)) \\ &\geq \dim_H \pi_{p_L}(F_{X_{p_L}}(\nu^m)) = \dim_H \pi_{p_L}(X_{p_L}),\end{aligned} \tag{3.4}$$

where the last equality holds by the transitivity of (X_{p_L}, σ). Observe that $\pi_{p_L}(X_{p_L})$ is a graph-directed set satisfying the open set condition (cf. [31]). Then the Hausdorff dimension of $\pi_{p_L}(X_{p_L})$ is given by

$$\dim_H \pi_{p_L}(X_{p_L}) = \frac{h_{top}(X_{p_L})}{\log p_L}. \tag{3.5}$$

By (3.2), (3.4), (3.5) and Lemma 2.4 (i) we conclude that

$$\begin{aligned}\dim_H(\mathcal{V}_{p_L} \cap (x-\delta, x+\delta)) &\geq \dim_H \pi_{p_L}(X_{p_L}) \\ &= \frac{h_{top}(X_{p_L})}{\log p_L} = \frac{h_{top}(\mathbf{V}_{p_L})}{\log p_L} \\ &= \dim_H \mathcal{U}_{p_L} = \dim_H \mathcal{V}_{p_L}.\end{aligned}$$

□

Now we consider $q \in \mathscr{B}$. We need the following lemma.

Lemma 3.4. *Let $q \in (q_{KL}, M+1]$ and $x_1 \dots x_N \in \mathcal{L}(\mathbf{V}_q)$. Let $\{p_n\} \subset (1, M+1]$ be a sequence such that $\alpha(p_n) \in \mathbf{V}$ for each $n \geq 1$, and $p_n \nearrow q$ as $n \to \infty$. Then*

$$x_1 \dots x_N \in \mathcal{L}(\mathbf{V}_{p_n}) \qquad \text{for all sufficiently large } n.$$

Proof. Since $x_1 \dots x_N \in \mathcal{L}(\mathbf{V}_q)$, we have

$$\overline{\alpha_1(q) \dots \alpha_{N-i}(q)} \preccurlyeq x_{i+1} \dots x_N \preccurlyeq \alpha_1(q) \dots \alpha_{N-i}(q) \qquad \text{for any} \quad 0 \leq i < N.$$

Let $s \in \{0, 1, \dots, N-1\}$ be the smallest integer such that

$$x_{s+1} \dots x_N = \overline{\alpha_1(q) \dots \alpha_{N-s}(q)} \quad \text{or} \quad x_{s+1} \dots x_N = \alpha_1(q) \dots \alpha_{N-s}(q). \tag{3.6}$$

If there is no $s \in \{0, 1, \dots, N-1\}$ for which (3.6) holds, then we set $s = N$. By our choice of s it follows that

$$\overline{\alpha_1(q) \dots \alpha_{N-i}(q)} \prec x_{i+1} \dots x_N \prec \alpha_1(q) \dots \alpha_{N-i}(q) \qquad \text{for all} \quad 0 \leq i < s. \tag{3.7}$$

In terms of (3.6) we may assume by symmetry that

$$x_{s+1} \dots x_N = \alpha_1(q) \dots \alpha_{N-s}(q). \tag{3.8}$$

Since $p_n \nearrow q$ as $n \to \infty$, by Lemma 2.1 there exists $K \in \mathbb{N}$ such that

$$\alpha_1(p_n) \dots \alpha_N(p_n) = \alpha_1(q) \dots \alpha_N(q) \qquad \text{for any} \quad n \geq K.$$

By the assumption that $\alpha(p_n) \in \mathbf{V}$ for any $n \geq 1$, it follows from (3.7) and (3.8) that

$$x_1 \dots x_N \alpha_{N-s+1}(p_n) \alpha_{N-s+2}(p_n) \dots = x_1 \dots x_s \alpha_1(p_n) \alpha_2(p_n) \dots \in \mathbf{V}_{p_n}$$

for any $n \geq K$. So, $x_1 \dots x_N \in \mathcal{L}(\mathbf{V}_{p_n})$ for all $n \geq K$. □

Lemma 3.5. *Let $q \in \mathscr{B}$. Then for any $x \in \mathcal{V}_q$ we have*

$$\dim_H(\mathcal{V}_q \cap (x - \delta, x + \delta)) = \dim_H \mathcal{V}_q \qquad \text{for any} \ \ \delta > 0.$$

Proof. Take $q \in \mathscr{B}$. Since $\mathscr{B} \subset (q_{KL}, M+1] \setminus \bigcup (p_L, p_R]$, by Lemma 2.7 (i) there exists a sequence of plateaus $\{[p_L(n), p_R(n)]\}_{n=1}^\infty$ such that $p_L(n) \nearrow q$ as $n \to \infty$.

Now we fix $\delta > 0$ and $x \in \mathcal{V}_q$. Suppose $(x_i) \in \mathbf{V}_q$ is a q-expansion of x. Then there exists a large integer N such that

$$\pi_q(F_{\mathbf{V}_q}(x_1 \dots x_N)) \subseteq \mathcal{V}_q \cap (x - \delta, x + \delta). \tag{3.9}$$

By Lemmas 2.9 (i) and 2.11 (i) we have $\alpha(p_L(n)) \in \mathbf{V}$ for all $n \geq 1$. Then applying Lemma 3.4 to the sequence $\{p_L(n)\}$ gives a large integer K such that

$$x_1 \dots x_N \in \mathcal{L}(\mathbf{V}_{p_L(n)}) \qquad \text{for all} \ \ n \geq K.$$

Since $\mathbf{V}_{p_L(n)} \subset \mathbf{V}_q$ for any $n \geq 1$, it follows from (3.9) that

$$\pi_q(F_{\mathbf{V}_{p_L(n)}}(x_1 \dots x_N)) \subset \mathcal{V}_q \cap (x-\delta, x+\delta) \qquad \text{for all} \quad n \geq K. \tag{3.10}$$

By (3.10) and the proof of Lemma 3.3 it follows that for any $n \geq K$,

$$\dim_H(\mathcal{V}_q \cap (x-\delta, x+\delta)) \geq \dim_H \pi_q(F_{\mathbf{V}_{p_L(n)}}(x_1 \dots x_N)) \geq \frac{h_{top}(\mathbf{V}_{p_L(n)})}{\log q}.$$

Letting $n \to \infty$ we have $p_L(n) \nearrow q$, and then we conclude by the continuity of the function $q \mapsto h_{top}(\mathbf{V}_q)$ (see Lemma 2.4 (ii)) that

$$\dim_H(\mathcal{V}_q \cap (x-\delta, x+\delta)) \geq \frac{h_{top}(\mathbf{V}_q)}{\log q} = \dim_H \mathcal{U}_q = \dim_H \mathcal{V}_q. \qquad \square$$

Proof of Theorem 3.1. Take $q \in (1, q_{KL}] \cup ((q_{KL}, M+1] \setminus \bigcup(p_L, p_R])$. If $q \in (1, q_{KL}]$, then the result follows from the fact that $\dim_H \mathcal{U}_q = 0$ (see Lemma 2.4).

Assume $q \in (q_{KL}, M+1] \setminus \bigcup(p_L, p_R]$ where the union is taken over all plateaus $[p_L, p_R]$ of H. Take $x \in \mathcal{U}_q$. If $x \notin \{0, M/(q-1)\}$, then by Lemma 3.2 x belongs to an affine copy of $\mathcal{V}_q$. Since the Hausdorff dimension is invariant under affine transformations (cf. [20]), the statement follows from Lemmas 3.3 and 3.5.

So, it remains to consider $x = 0$ and $x = M/(q-1)$. By symmetry we may assume $x = 0$. Take $\delta > 0$. Then by Lemma 3.2 there exists a sufficiently large integer m such that

$$\frac{1}{q^m} + \frac{\mathcal{V}_q}{q^m} \subseteq (\mathcal{U}_q \cup \mathcal{N}) \cap (-\delta, \delta),$$

where $\mathcal{N}$ is at most countable. This proves the statement for $x = 0$. $\square$

At the end of this section we strengthen Theorem 3.1 and give a complete characterization of the set

$$\{q \in (1, M+1] : \mathcal{U}_q \text{ is dimensionally homogeneous}\}.$$

Let $[p_L, p_R] \subset (q_{KL}, M+1]$ be a plateau of H. Note that $p_L \in \overline{\mathscr{B}} \setminus \mathscr{B} \subset \overline{\mathscr{U}} \setminus \mathscr{U}$. Then by [15, Theorem 1.7] there exists a largest $\hat{p}_L \in (p_L, p_R)$ such that the set-valued map $q \mapsto \mathbf{V}_q$ is constant in $[p_L, \hat{p}_L)$. Furthermore, for $q = \hat{p}_L$ any sequence in the difference set $\mathbf{V}_{\hat{p}_L} \setminus \mathbf{V}_{p_L}$ is not contained in $\mathbf{U}_{\hat{p}_L}$. Then by the same argument as in the proof of Lemma 3.3 it follows that Theorem 3.1 also holds for any $q \in [p_L, \hat{p}_L]$. Clearly, $\mathcal{U}_q$ is dimensionally homogeneous for $q \leq q_{KL}$. So, the univoque set $\mathcal{U}_q$ is dimensionally homogeneous for any $q \in (1, q_{KL}] \cup ((q_{KL}, M+1] \setminus \bigcup(\hat{p}_L, p_R])$. This, combined with some recent progress obtained by Allaart et al. [2], implies the following.

Theorem 3.6.

(i) *If $M = 1$ or M is even, then $\mathcal{U}_q$ is dimensionally homogeneous if, and only if, $q \in (1, q_{KL}] \cup ((q_{KL}, M+1] \setminus \bigcup(\hat{p}_L, p_R])$.*

(ii) *If $M = 2k+1 \geq 3$, then $\mathcal{U}_q$ is dimensionally homogeneous if, and only if,* $q \in (1, q_{KL}] \cup ((q_{KL}, M+1] \setminus \bigcup(\hat{p}_L, p_R])$ *or* $q = \frac{k+3+\sqrt{k^2+6k+1}}{2}$.

Proof. By Theorem 3.1 and the above arguments it follows that $\mathcal{U}_q$ is dimensionally homogeneous for any $q \in (1, q_{KL}] \cup ((q_{KL}, M+1] \setminus \bigcup(\hat{p}_L, p_R])$. Then to prove the sufficiency it remains to prove the dimensional homogeneity of $\mathcal{U}_q$ for $q = \frac{k+3+\sqrt{k^2+6k+1}}{2} =: q_\star$ with $M = 2k+1 \geq 3$. Note that $q_\star$ is the right endpoint of the entropy plateau generated by $k+1$, i.e., $[p_\star, q_\star]$ is an entropy plateau with $\alpha(p_\star) = (k+1)^\infty$ and $\alpha(q_\star) = (k+2)k^\infty$. Then by [2, Corollary 3.10] it follows that

$$h_{top}(\mathbf{V}_{q_\star} \setminus \mathbf{V}_{p_\star}) = h_{top}(\mathbf{V}_{p_\star}) = \log 2, \tag{3.11}$$

where the second equality follows from that $\mathbf{V}_{p_\star} = \{k, k+1\}^{\mathbb{N}}$. Furthermore, any sequence in $\mathbf{V}_{q_\star} \setminus \mathbf{V}_{p_\star}$ eventually ends in a transitive sub-shift of finite type (X, σ) with states $\{k-1, k, k+1, k+2\}$ and adjacency matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}. \tag{3.12}$$

Observe that $h_{top}(X) = \log 2$. Using (3.11) and by a similar argument as in the proof of Lemma 3.3 it follows that $\mathcal{U}_{q_\star}$ is dimensionally homogeneous.

Now we prove the necessity. Without loss of generality we assume that $M = 1$ or M is even. Let $[p_L, p_R] \subset (q_{KL}, M+1]$ be an entropy plateau generated by $a_1 \ldots a_m$, and let $\hat{p}_L \in (p_L, p_R)$ be the largest point such that the map $q \mapsto \mathbf{V}_q$ is constant in $[p_L, \hat{p}_L)$. In fact, we have $\alpha(\hat{p}_L) = (a_1 \ldots a_m^+ \overline{a_1 \ldots a_m^+})^\infty$ (cf. [15]). Take $q \in (\hat{p}_L, p_R]$. Then $\mathbf{W}_q \setminus \mathbf{V}_{p_L} \neq \emptyset$, where $\mathbf{W}_q$ is the set of sequences (x_i) satisfying

$$\overline{\alpha(q)} \prec \sigma^n((x_i)) \prec \alpha(q) \quad \text{for any} \quad n \geq 0.$$

Furthermore, any sequence in $\mathbf{W}_q \setminus \mathbf{V}_{p_L}$ must end in the sub-shift of finite type (Y, σ) with states $\left\{\overline{a_1 \ldots a_m^+},\ \overline{a_1 \ldots a_m},\ a_1 \ldots a_m,\ a_1 \ldots a_m^+\right\}$ and adjacency matrix A defined in (3.12). In particular,

$$h_{top}(Y) = \frac{\log 2}{m} = h_{top}(\mathbf{V}_{p_R} \setminus \mathbf{V}_{p_L}) < h_{top}(\mathbf{V}_{p_L}), \tag{3.13}$$

where the inequality follows from [2, Corollary 3.10]. Observe that $\mathbf{W}_q \subseteq \mathbf{U}_q$. Therefore, by (3.13) and the same argument as in the proof of Lemma 3.3 it follows that for any $x \in \pi_q(\mathbf{W}_q \setminus \mathbf{V}_{p_L}) \subset \mathcal{U}_q$ there exists $\delta > 0$ such that

$$\dim_H(\mathcal{U}_q \cap (x - \delta, x + \delta)) \leq \frac{h_{top}(Y)}{\log q} < \frac{h_{top}(\mathbf{V}_{p_L})}{\log q} = \dim_H \mathcal{U}_q.$$

This completes the proof. □

4. Auxiliary Proposition

In this section we prove an auxiliary proposition that will be used to prove Theorem 2 in the next section.

Proposition 4.1. *Let $q \in \overline{\mathscr{B}} \setminus \{M+1\}$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$(1-\varepsilon)\dim_H \pi_q(\mathbf{B}_\delta(q)) \le \dim_H(\overline{\mathscr{B}} \cap (q-\delta, q+\delta)) \le (1+\varepsilon)\dim_H \pi_{q+\delta}(\mathbf{B}_\delta(q)),$$

where

$$\mathbf{B}_\delta(q) := \left\{\alpha(p) : p \in \overline{\mathscr{B}} \cap (q-\delta, q+\delta)\right\}.$$

The proof of Proposition 4.1 is based on the following lemma for the Hausdorff dimension under Hölder continuous maps (cf. [20]).

Lemma 4.2. *Let $f : (X, \rho_1) \to (Y, \rho_2)$ be a Hölder map between two metric spaces, i.e., there exist two constants $C > 0$ and $\lambda > 0$ such that*

$$\rho_2(f(x), f(y)) \le C\rho_1(x, y)^\lambda$$

for any $x, y \in X$ with $\rho_1(x, y) \le c$ (here c is a small constant). Then $\dim_H f(X) \le \frac{1}{\lambda}\dim_H X$.

First we prove the second inequality in Proposition 4.1.

Lemma 4.3. *Let $q \in \overline{\mathscr{B}} \setminus \{M+1\}$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\dim_H(\overline{\mathscr{B}} \cap (q-\delta, q+\delta)) \le (1+\varepsilon)\dim_H \pi_{q+\delta}(\mathbf{B}_\delta(q)).$$

Proof. Fix $\varepsilon > 0$ and $q \in \overline{\mathscr{B}} \setminus \{M+1\}$. Then there exists $\delta > 0$ such that

$$q-\delta > 1, \quad q+\delta < M+1 \quad \text{and} \quad \frac{\log(q+\delta)}{\log(q-\delta)} \le 1+\varepsilon. \tag{4.1}$$

Since $\overline{\mathscr{B}} \subseteq \overline{\mathscr{U}}$, by Lemmas 2.1 and 2.3 (ii) it follows that for each $p \in \overline{\mathscr{B}} \cap (q-\delta, q+\delta)$ we have

$$\overline{\alpha(q+\delta)} \prec \overline{\alpha(p)} \prec \sigma^i(\alpha(p)) \preccurlyeq \alpha(p) \prec \alpha(q+\delta) \qquad \text{for all} \quad i \ge 0.$$

So, by Lemma 2.2 $\alpha(p) \in \mathbf{U}_{q+\delta}$ for any $p \in \overline{\mathscr{B}} \cap (q-\delta, q+\delta)$. This implies that the map

$$g : \overline{\mathscr{B}} \cap (q-\delta, q+\delta) \to \pi_{q+\delta}(\mathbf{B}_\delta(q)); \qquad p \mapsto \pi_{q+\delta}(\alpha(p))$$

is bijective. By Lemma 4.2 it suffices to prove that there exists a constant $C > 0$ such that

$$\left|\pi_{q+\delta}(\alpha(p_2)) - \pi_{q+\delta}(\alpha(p_1))\right| \ge C\,|p_2 - p_1|^{1+\varepsilon}$$

for any $p_1, p_2 \in \overline{\mathscr{B}} \cap (q-\delta, q+\delta)$.

Take $p_1, p_2 \in \overline{\mathscr{B}} \cap (q-\delta, q+\delta)$ with $p_1 < p_2$. Then by Lemma 2.1 we have $\alpha(p_1) \prec \alpha(p_2)$. So, there exists $n \geq 1$ such that

$$\alpha_1(p_1)\ldots\alpha_{n-1}(p_1) = \alpha_1(p_2)\ldots\alpha_{n-1}(p_2) \quad \text{and} \quad \alpha_n(p_1) < \alpha_n(p_2).$$

Then

$$\begin{aligned} 0 < p_2 - p_1 &= \sum_{i=1}^{\infty}\frac{\alpha_i(p_2)}{p_2^{i-1}} - \sum_{i=1}^{\infty}\frac{\alpha_i(p_1)}{p_1^{i-1}} \\ &\leq \sum_{i=1}^{n-1}\left(\frac{\alpha_i(p_2)}{p_2^{i-1}} - \frac{\alpha_i(p_1)}{p_1^{i-1}}\right) + \sum_{i=n}^{\infty}\frac{\alpha_i(p_2)}{p_2^{i-1}} \; \leq p_2^{2-n}, \end{aligned} \tag{4.2}$$

where the last inequality follows from the property of quasi-greedy expansion $\alpha(p_2)$ that $\sum_{i=1}^{\infty}\alpha_{k+i}(p_2)/p_2^i \leq 1$ for any $k \geq 1$.

On the other hand, by (4.1) we have $\alpha(p_2) \preccurlyeq \alpha(q+\delta) \prec \alpha(M+1) = M^\infty$. Then there exists a large integer N (depending on $q+\delta$) such that

$$\alpha_1(p_2)\ldots\alpha_N(p_2) \preccurlyeq M^{N-1}(M-1). \tag{4.3}$$

Note that $p_2 \in \overline{\mathscr{B}} \subseteq \overline{\mathscr{U}}$. Then by Lemma 2.3 (ii) and (4.3) it follows that

$$\alpha_{m+1}(p_2)\alpha_{m+2}(p_2)\ldots \succ \overline{\alpha(p_2)} \succcurlyeq 0^{N-1}10^\infty \qquad \text{for any} \quad m \geq 1.$$

This implies that

$$\begin{aligned} &\pi_{q+\delta}(\alpha(p_2)) - \pi_{q+\delta}(\alpha(p_1)) \\ &= \sum_{i=1}^{\infty}\frac{\alpha_i(p_2)-\alpha_i(p_1)}{(q+\delta)^i} \\ &= \frac{\alpha_n(p_2)-\alpha_n(p_1)}{(q+\delta)^n} - \frac{1}{(q+\delta)^n}\sum_{i=1}^{\infty}\frac{\alpha_{n+i}(p_1)}{(q+\delta)^i} + \sum_{i=n+1}^{\infty}\frac{\alpha_i(p_2)}{(q+\delta)^i} \\ &\geq \frac{1}{(q+\delta)^n} - \frac{1}{(q+\delta)^n}\sum_{i=1}^{\infty}\frac{\alpha_{n+i}(p_1)}{p_1^i} + \sum_{i=n+1}^{\infty}\frac{\alpha_i(p_2)}{(q+\delta)^i} \\ &\geq \sum_{i=n+1}^{\infty}\frac{\alpha_i(p_2)}{(q+\delta)^i} \; \geq \frac{1}{(q+\delta)^{n+N}}, \end{aligned}$$

where the second inequality follows from the same property of the quasi-greedy expansion $\alpha(p_1)$ that was used before.

Therefore, by (4.1) and (4.2) it follows that

$$\begin{aligned} \pi_{q+\delta}(\alpha(p_2)) - \pi_{q+\delta}(\alpha(p_1)) &\geq \left((q+\delta)^{-\frac{n+N}{1+\varepsilon}}\right)^{1+\varepsilon} \\ &\geq \left((q-\delta)^{-n-N}\right)^{1+\varepsilon} \\ &\geq (q-\delta)^{-N(1+\varepsilon)}(p_2^{-n})^{1+\varepsilon} \geq C(p_2-p_1)^{1+\varepsilon}, \end{aligned}$$

where the constant $C = (q-\delta)^{-N(1+\varepsilon)}(q+\delta)^{-2(1+\varepsilon)}$. This completes the proof. $\square$

Now we turn to prove the first inequality of Proposition 4.1.

Lemma 4.4. *Let $q \in \overline{\mathscr{B}} \setminus \{M+1\}$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\dim_H(\overline{\mathscr{B}} \cap (q-\delta, q+\delta)) \geq (1-\varepsilon) \dim_H \pi_q(\mathbf{B}_\delta(q)).$$

Proof. The proof is similar to that of Lemma 4.3. Fix $\varepsilon > 0$ and take $q \in \overline{\mathscr{B}} \setminus \{M+1\}$. Then there exists $\delta > 0$ such that

$$q - \delta > 1, \quad q + \delta < M+1 \quad \text{and} \quad \frac{\log(q+\delta)}{\log q} \leq \frac{1}{1-\varepsilon}. \tag{4.4}$$

Take $p_1, p_2 \in \overline{\mathscr{B}} \cap (q-\delta, q+\delta)$ with $p_1 < p_2$. Then by Lemma 2.1 we have $\alpha(p_1) \prec \alpha(p_2)$, and therefore there exists a smallest integer $n \geq 1$ such that $\alpha_n(p_1) < \alpha_n(p_2)$. This implies that

$$\left|\pi_q(\alpha(p_2)) - \pi_q(\alpha(p_1))\right| = \left|\sum_{i=1}^{\infty} \frac{\alpha_i(p_2) - \alpha_i(p_1)}{q^i}\right| \leq \sum_{i=n}^{\infty} \frac{M}{q^i} = \frac{Mq}{q-1} q^{-n}. \tag{4.5}$$

On the other hand, observe that $q+\delta < M+1$. Then $\alpha(p_2) \preccurlyeq \alpha(q+\delta) \prec \alpha(M+1) = M^\infty$. So, there exists $N \geq 1$ such that

$$\alpha_1(p_2) \dots \alpha_N(p_2) \preccurlyeq M^{N-1}(M-1).$$

Since $p_2 \in \overline{\mathscr{B}} \subseteq \overline{\mathscr{U}}$, Lemma 2.3 (ii) gives

$$1 = \sum_{i=1}^{\infty} \frac{\alpha_i(p_2)}{p_2^i} > \sum_{i=1}^{n} \frac{\alpha_i(p_2)}{p_2^i} + \frac{1}{p_2^{n+N}},$$

which implies that

$$\begin{aligned} \frac{1}{p_2^{n+N}} &< 1 - \sum_{i=1}^{n} \frac{\alpha_i(p_2)}{p_2^i} = \sum_{i=1}^{\infty} \frac{\alpha_i(p_1)}{p_1^i} - \sum_{i=1}^{n} \frac{\alpha_i(p_2)}{p_2^i} \\ &\leq \sum_{i=1}^{n} \left(\frac{\alpha_i(p_2)}{p_1^i} - \frac{\alpha_i(p_2)}{p_2^i}\right) \\ &\leq \sum_{i=1}^{\infty} \left(\frac{M}{p_1^i} - \frac{M}{p_2^i}\right) = \frac{M}{(p_1-1)(p_2-1)}(p_2 - p_1). \end{aligned} \tag{4.6}$$

Here the second inequality holds since

$$\begin{aligned} &\alpha_1(p_1) \dots \alpha_{n-1}(p_1) = \alpha_1(p_2) \dots \alpha_{n-1}(p_2), \\ &\alpha_n(p_1) < \alpha_n(p_2) \quad \text{and} \quad \sum_{i=1}^{\infty} \alpha_{n+i}(p_1)/p_1^i \leq 1. \end{aligned}$$

Therefore, by (4.4)–(4.6) we conclude that

$$\begin{aligned}\left|\pi_q(\alpha(p_2)) - \pi_q(\alpha(p_1))\right| &\le \frac{Mq^{N+1}}{q-1}\big(q^{-\frac{n+N}{1-\varepsilon}}\big)^{1-\varepsilon}\\ &\le \frac{Mq^{N+1}}{q-1}(q+\delta)^{-(n+N)(1-\varepsilon)}\\ &\le \frac{Mq^{N+1}}{q-1}p_2^{-(n+N)(1-\varepsilon)} < C(p_2-p_1)^{1-\varepsilon},\end{aligned}$$

where

$$C = \frac{M^{2-\varepsilon}q^{N+1}}{(q-1)(q-\delta-1)^{2(1-\varepsilon)}}.$$

Note by Lemma 2.1 that the map $p \mapsto \alpha(p)$ is bijective from $\overline{\mathscr{B}} \cap (q-\delta, q+\delta)$ onto $\mathbf{B}_\delta(q)$. Hence, the lemma follows by letting $f = \pi_q \circ \alpha$ in Lemma 4.2. □

Proof of Proposition 4.1. The proposition follows from Lemmas 4.3 and 4.4. □

5. Local dimension of $\mathscr{B}$

In this section we will prove Theorem 2, which states that for any $q \in \overline{\mathscr{B}}$ we have

$$\lim_{\delta\to 0}\dim_H(\overline{\mathscr{B}} \cap (q-\delta, q+\delta)) = \dim_H \mathcal{U}_q.$$

First we prove the upper bound.

Proposition 5.1. *For any $q \in \overline{\mathscr{B}}$ we have*

$$\lim_{\delta\to 0}\dim_H(\overline{\mathscr{B}} \cap (q-\delta, q+\delta)) \le \dim_H \mathcal{U}_q.$$

Proof. Take $q \in \overline{\mathscr{B}}$. By Lemma 2.4 and Proposition 4.1 it follows that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\begin{aligned} \dim_H \mathcal{U}_{q+\delta} &\le \dim_H \mathcal{U}_q + \varepsilon,\\ \dim_H(\overline{\mathscr{B}} \cap (q-\delta, q+\delta)) &\le (1+\varepsilon)\dim_H \pi_{q+\delta}(\mathbf{B}_\delta(q)),\end{aligned} \tag{5.1}$$

where $\mathbf{B}_\delta(q) = \big\{\alpha(p) : p \in (q-\delta, q+\delta) \cap \overline{\mathscr{B}}\big\}$.

Since $\overline{\mathscr{B}} \subseteq \overline{\mathscr{U}}$, Lemmas 2.1 and 2.3 (ii) give that any sequence $\alpha(p) \in \mathbf{B}_\delta(q)$ satisfies

$$\overline{\alpha(q+\delta)} \prec \overline{\alpha(p)} \prec \sigma^n(\alpha(p)) \preccurlyeq \alpha(p) \prec \alpha(q+\delta) \qquad \text{for all} \quad n \ge 0.$$

By Lemma 2.2 this implies that $\mathbf{B}_\delta(q) \subseteq \mathbf{U}_{q+\delta}$. Therefore, by (5.1) it follows that

$$\begin{aligned}\dim_H(\overline{\mathscr{B}} \cap (q-\delta, q+\delta)) &\le (1+\varepsilon)\dim_H \pi_{q+\delta}(\mathbf{B}_\delta(q))\\ &\le (1+\varepsilon)\dim_H \mathcal{U}_{q+\delta} \le (1+\varepsilon)(\dim_H \mathcal{U}_q + \varepsilon).\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this completes the proof. □

The proof of the lower bound of Theorem 2 is tedious. We will prove this in several steps. First we need the following lemma.

Lemma 5.2. *Let $[p_L, p_R] \subseteq (q_{KL}, M+1)$ be a plateau of H such that $\alpha(p_L) = (\alpha_1 \dots \alpha_m)^\infty$ with period m. Then*

$$\alpha_{i+1} \dots \alpha_m \prec \alpha_1 \dots \alpha_{m-i} \qquad \text{for all} \quad 0 < i < m,$$
$$\alpha_{i+1} \dots \alpha_m \alpha_1 \dots \alpha_i \succ \overline{\alpha_1 \dots \alpha_m} \qquad \text{for all} \quad 0 \le i < m.$$

Proof. Since $(\alpha_1 \dots \alpha_m)^\infty$ is the quasi-greedy p_L-expansion of 1 with period m, the greedy p_L-expansion of 1 is $\alpha_1 \dots \alpha_m^+ 0^\infty$. So, by [17, Propostion 2.2] it follows that $\sigma^n(\alpha_1 \dots \alpha_m^+ 0^\infty) \prec \alpha_1 \dots \alpha_m^+ 0^\infty$ for any $n \ge 1$. This implies

$$\alpha_{i+1} \dots \alpha_m \prec \alpha_{i+1} \dots \alpha_m^+ \preccurlyeq \alpha_1 \dots \alpha_{m-i} \qquad \text{for any} \quad 0 < i < m.$$

Lemma 2.6 states that $p_L \in \overline{\mathscr{B}} \subset \overline{\mathscr{U}}$. Then by Lemma 2.3 (ii) we have that

$$(\alpha_{i+1} \dots \alpha_m \alpha_1 \dots \alpha_i)^\infty = \sigma^i((\alpha_1 \dots \alpha_m)^\infty) \succ (\overline{\alpha_1 \dots \alpha_m})^\infty$$

for any $0 \le i < m$. This implies that

$$\alpha_{i+1} \dots \alpha_m \alpha_1 \dots \alpha_i \succ \overline{\alpha_1 \dots \alpha_m} \qquad \text{for any } 0 \le i < m.$$ □

Let $[p_L, p_R] \subset (q_{KL}, M+1)$ be a plateau of H. For any $N \ge 1$ let $(\mathbf{W}_{p_L,N}, \sigma)$ be a subshift of finite type in $\{0, 1, \dots, M\}^{\mathbb{N}}$ with the set of forbidden blocks $c_1 \dots c_N$ satisfying

$$c_1 \dots c_N \preccurlyeq \overline{\alpha_1(p_L) \dots \alpha_N(p_L)} \qquad \text{or} \qquad c_1 \dots c_N \succcurlyeq \alpha_1(p_L) \dots \alpha_N(p_L).$$

Then any sequence $(x_i) \in \mathbf{W}_{p_L,N}$ satisfies

$$\overline{\alpha_1(p_L) \dots \alpha_N(p_L)} \prec \sigma^n((x_i)) \prec \alpha_1(p_L) \dots \alpha_N(p_L) \qquad \text{for all} \quad n \ge 0.$$

If $\alpha_N(p_L) > 0$, then $\mathbf{W}_{p_L,N}$ is indeed the set of sequences $(x_i) \in \{0, 1, \dots, M\}^{\mathbb{N}}$ satisfying

$$(\overline{\alpha_1(p_L) \dots \alpha_N(p_L)}^+)^\infty \preccurlyeq \sigma^n((x_i)) \preccurlyeq (\alpha_1(p_L) \dots \alpha_N(p_L)^-)^\infty$$

for all $n \ge 0$. By the definition of $\mathbf{W}_{p_L,N}$ it gives that

$$\mathbf{W}_{p_L,1} \subseteq \mathbf{W}_{p_L,2} \subseteq \dots \subseteq \mathbf{V}_{p_L}.$$

We emphasize that $\mathbf{W}_{p_L,1}$ can be an empty set, and the inclusions in the above equation are not necessarily strict.

Observe that $(\mathbf{V}_{p_L}, \sigma)$ is a subshift of finite type with positive topological entropy. The following asymptotic result was established in [23, Proposition 2.8].

Lemma 5.3. *Let $[p_L, p_R] \subseteq [q_T, M+1]$ be a plateau of H. Then*

$$\lim_{N \to \infty} h_{top}(\mathbf{W}_{p_L,N}) = h_{top}(\mathbf{V}_{p_L}).$$

Recall from (2.8) that

$$\xi(n) = \lambda_1 \ldots \lambda_{2^{n-1}}(\overline{\lambda_1 \ldots \lambda_{2^{n-1}}}^{\,+})^\infty \qquad \text{if} \quad M = 2k,$$
$$\xi(n) = \lambda_1 \ldots \lambda_{2^n}(\overline{\lambda_1 \ldots \lambda_{2^n}}^{\,+})^\infty \qquad \text{if} \quad M = 2k+1.$$

Note that the sequence (λ_i) in the definition of $\xi(n)$ depends on M. In the following lemma we show that the entropy of $(\mathbf{W}_{p_L,N},\sigma)$ is equal to the entropy of the follower set $F_{\mathbf{W}_{p_L,N}}(\nu)$ for all sufficiently large integers N, where ν is the word defined in Lemma 2.9 (iii) or Lemma 2.11 (iii).

Lemma 5.4.

(i) *Let $[p_L,p_R] \subset [q_T, M+1]$ be a plateau of H, and let*

$$\nu = \begin{cases} k & \text{if} \quad M = 2k, \\ (k+1)k & \text{if} \quad M = 2k+1. \end{cases}$$

Then for all sufficiently large integers N we have

$$h_{top}(F_{\mathbf{W}_{p_L,N}}(\nu^\ell)) = h_{top}(\mathbf{W}_{p_L,N}) \qquad \text{for any} \quad \ell \geq 1.$$

(ii) *Let $[p_L,p_R] \subset (q_{KL}, q_T)$ be a plateau of H with $\xi(n+1) \preccurlyeq \alpha(p_L) \prec \xi(n)$. Set*

$$\nu = \begin{cases} \lambda_1 \ldots \lambda_{2^n}^- & \text{if} \quad M = 2k, \\ \lambda_1 \ldots \lambda_{2^{n+1}}^- & \text{if} \quad M = 2k+1. \end{cases}$$

Then for all sufficiently large integers N we have

$$h_{top}(F_{\mathbf{W}_{p_L,N}}(\nu^\ell)) = h_{top}(\mathbf{W}_{p_L,N}) \qquad \text{for any} \quad \ell \geq 1.$$

Proof. Take $\ell \geq 1$. First we prove (i). By Lemma 2.9 (iii) there exists a large integer $N \geq 2$ such that $\nu^\ell \in \mathcal{L}(\mathbf{W}_{p_L,N})$. Since $(\mathbf{W}_{p_L,N},\sigma)$ is a subshift of finite type, to prove (i) it suffices to prove that for any word $\rho \in \mathcal{L}(\mathbf{W}_{p_L,N})$ there exists a word γ of uniformly bounded length for which $\nu^\ell\gamma\rho \in \mathcal{L}(\mathbf{W}_{p_L,N})$.

Take $\rho = \rho_1 \ldots \rho_m \in \mathcal{L}(\mathbf{W}_{p_L,N})$. If $M = 2k$, then $\nu = k$. Since $\alpha(p_L) \succcurlyeq \alpha(q_T) = (k+1)k^\infty$, we have

$$\overline{\alpha_1(p_L)} \leq k-1 < \nu < k+1 \leq \alpha_1(p_L).$$

So, $\nu^\ell\gamma\rho \in \mathcal{L}(\mathbf{W}_{p_L,N})$ by taking $\gamma = \epsilon$ the empty word. Similarly, if $M = 2k+1$ then $\nu = (k+1)k$. Observe that $\alpha(p_L) \succcurlyeq \alpha(q_T) = (k+1)((k+1)k)^\infty$. This implies that $\nu^\ell\gamma\rho \in \mathcal{L}(\mathbf{W}_{p_L,N})$ by taking $\gamma = \epsilon$ if the initial word $\rho_1 \geq k+1$, and by taking $\gamma = k+1$ if $\rho_1 \leq k$.

Now we turn to prove (ii). We only give the proof for $M = 2k$, since the proof for $M = 2k+1$ is similar. Then $\nu = \lambda_1 \ldots \lambda_{2^n}^-$. By Lemma 2.11 (iii) there exists a large integer $N \geq 2^{n+1}$ such that $\nu^\infty = (\lambda_1 \ldots \lambda_{2^n}^-)^\infty \in \mathbf{W}_{p_L,N}$. Since $h_{top}(\mathbf{V}_{p_L}) > 0$, by Lemma 5.3 we can choose N sufficiently large such that $h_{top}(\mathbf{W}_{p_L,N}) > 0$. Since $\mathbf{W}_{p_L,N}$ is a subshift of finite type, there exists a transitive subshift of finite type $X_N \subset \mathbf{W}_{p_L,N}$ for which $h_{top}(X_N) = h_{top}(\mathbf{W}_{p_L,N})$ (cf. [30, Theorem 4.4.4]). We claim that the word $\lambda_1 \dots \lambda_{2^n}$ or $\overline{\lambda_1 \dots \lambda_{2^n}}$ belongs to $\mathcal{L}(X_N)$.

By (2.8) and (2.4) it follows that

$$\xi(n) = \lambda_1 \dots \lambda_{2^{n-1}}(\overline{\lambda_1 \dots \lambda_{2^{n-1}}}^+)^\infty = \lambda_1 \dots \lambda_{2^n}(\overline{\lambda_1 \dots \lambda_{2^{n-1}}}^+)^\infty.$$

Then the assumption $\xi(n+1) \preccurlyeq \alpha(p_L) \prec \xi(n)$ gives that

$$\alpha_1(p_L) \dots \alpha_{2^n}(p_L) = \lambda_1 \dots \lambda_{2^n} = \alpha_1(q_{KL}) \dots \alpha_{2^n}(q_{KL}). \tag{5.2}$$

Suppose that the words $\lambda_1 \dots \lambda_{2^n}$ and $\overline{\lambda_1 \dots \lambda_{2^n}}$ do not belong to $\mathcal{L}(X_N)$. Then by (5.2) we have

$$X_N \subset \mathbf{W}_{p_L,2^n} = \mathbf{W}_{q_{KL},2^n} \subset \mathbf{V}_{q_{KL}}.$$

So, by Lemma 2.4 it follows that X_N has zero topological entropy, leading to a contradiction with $h_{top}(X_N) = h_{top}(\mathbf{W}_{p_L,N}) > 0$.

By the claim, to finish the proof of (ii) it suffices to prove that for any word $\rho \in \mathcal{L}(X_N)$ with a prefix $\lambda_1 \dots \lambda_{2^n}$ or $\overline{\lambda_1 \dots \lambda_{2^n}}$ there exists a word γ of uniformly bounded length such that $\nu^\ell \gamma\rho \in \mathcal{L}(\mathbf{W}_{p_L,N})$. In [26, Lemma 4.2] (see also, [1, Lemma 4.2]) it was shown that for any $n \ge 1$ we have

$$\overline{\lambda_1 \dots \lambda_{2^n - i}} \prec \lambda_{i+1} \dots \lambda_{2^n} \preccurlyeq \lambda_1 \dots \lambda_{2^n - i} \qquad \text{for any} \quad 0 \le i < 2^n.$$

This implies that for any $0 \le i < 2^n$ we have

$$\lambda_{i+1} \dots \lambda_{2^n}^- \prec \lambda_1 \dots \lambda_{2^n - i} \quad \text{and} \quad \lambda_{i+1} \dots \lambda_{2^n}^- \lambda_1 \dots \lambda_i \succ \overline{\lambda_1 \dots \lambda_{2^n}}. \tag{5.3}$$

Observe that

$$\nu = \lambda_1 \dots \lambda_{2^n}^- = \lambda_1 \dots \lambda_{2^{n-1}} \overline{\lambda_1 \dots \lambda_{2^{n-1}}}.$$

Then by (5.2) and (5.3) it follows that if $\lambda_1 \dots \lambda_{2^n}$ is a prefix of ρ, then $\nu^\ell \gamma\rho \in \mathcal{L}(\mathbf{W}_{p_L,N})$ by taking $\gamma = \epsilon$ the empty word, and if $\overline{\lambda_1 \dots \lambda_{2^n}}$ is a prefix of ρ then $\nu^\ell \gamma\rho \in \mathcal{L}(\mathbf{W}_{p_L,N})$ by taking $\gamma = \lambda_1 \dots \lambda_{2^{n-1}}$. ☐

In the following lemma we prove the lower bound of Theorem 2 for $q \in [q_T, M+1]$ being the left endpoint of an entropy plateau.

Lemma 5.5. *Let $[p_L, p_R] \subseteq [q_T, M+1]$ be a plateau of H. Then for any $\delta > 0$ we have*

$$\dim_H(\overline{\mathscr{B}} \cap (p_L - \delta, p_L + \delta)) \ge \dim_H \mathcal{U}_{p_L}.$$

Proof. By Lemma 2.9 (i) it follows that $\alpha(p_L) = (\alpha_i) = (\alpha_1 \dots \alpha_m)^\infty$ is an irreducible sequence, where m is the minimal period of $\alpha(p_L)$. Then, there exists a large integer $N_1 > m$ such that

$$\alpha_1 \cdots \alpha_j(\overline{\alpha_1 \dots \alpha_j}^+)^\infty \prec \alpha_1 \dots \alpha_{N_1} \qquad \text{if} \quad (\alpha_1 \dots \alpha_j^-)^\infty \in \mathbf{V} \text{ and } 1 \le j \le m. \tag{5.4}$$

Let ν be the word defined in Lemma 5.4 (i). Then by Lemma 2.9 (iii) there exist a large integer $N > N_1$ and a word ω such that

$$\overline{\alpha_1 \dots \alpha_N} \prec \sigma^n(\alpha_1 \dots \alpha_m \omega \nu^\infty) \prec \alpha_1 \dots \alpha_N \quad \text{for any} \quad n \geq 0. \tag{5.5}$$

Observe that $(\mathbf{W}_{p_L,N}, \sigma)$ is an N-step subshift of finite type. Note by (5.5) that $\alpha_1 \dots \alpha_m \omega \nu^N \in \mathcal{L}(\mathbf{W}_{p_L,N})$. Then by [30, Theorem 2.1.8] it follows that for any sequence $(d_i) \in F_{\mathbf{W}_{p_L,N}}(\nu^N)$ we have $\alpha_1 \dots \alpha_m \omega d_1 d_2 \dots \in F_{\mathbf{W}_{p_L,N}}(\alpha_1 \dots \alpha_m)$. In other words,

$$\left\{\alpha_1 \dots \alpha_m \omega d_1 d_2 \dots : (d_i) \in F_{\mathbf{W}_{p_L,N}}(\nu^N)\right\} \subseteq F_{\mathbf{W}_{p_L,N}}(\alpha_1 \dots \alpha_m) \subseteq \mathbf{W}_{p_L,N}.$$

So,

$$h_{top}(F_{\mathbf{W}_{p_L,N}}(\nu^N)) \leq h_{top}(F_{\mathbf{W}_{p_L,N}}(\alpha_1 \dots \alpha_m)) \leq h_{top}(\mathbf{W}_{p_L,N}).$$

Therefore, by Lemma 5.4 (i) we obtain

$$h_{top}(F_{\mathbf{W}_{p_L,N}}(\alpha_1 \dots \alpha_m)) = h_{top}(\mathbf{W}_{p_L,N}). \tag{5.6}$$

Let Λ_N be the set of sequences $(a_i) \in \{0, 1, \dots, M\}^\infty$ satisfying

$$a_1 \dots a_{mN} = (\alpha_1 \dots \alpha_m)^N \quad \text{and} \quad a_{mN+1} a_{mN+2} \dots \in F_{\mathbf{W}_{p_L,N}}(\alpha_1 \dots \alpha_m).$$

Fix $\delta > 0$. We claim that

$$\Lambda_N \subseteq \mathbf{B}_\delta(p_L) = \left\{\alpha(q) : q \in \overline{\mathscr{B}} \cap (p_L - \delta, p_L + \delta)\right\}$$

for all sufficiently large integers $N > N_1$.

Clearly, when N increases the length of the common prefix of sequences in Λ_N grows, and it coincides with a prefix of $\alpha(p_L) = (\alpha_1 \dots \alpha_m)^\infty$. So, by Lemmas 2.1 and 2.12 it suffices to show that for all $N > N_1$ any sequence $(a_i) \in \Lambda_N$ is irreducible.

Take $N > N_1$ and $(a_i) \in \Lambda_N$. First we claim that

$$\overline{\alpha_1 \dots \alpha_N} \prec \sigma^n((a_i)) \prec \alpha_1 \dots \alpha_N \qquad \text{for any } n \geq 1. \tag{5.7}$$

Observe that $a_1 \dots a_{mN} = (\alpha_1 \dots \alpha_m)^N$ and the tails $a_{mN+1} a_{mN+2} \dots \in F_{\mathbf{W}_{p_L,N}}(\alpha_1 \dots \alpha_m)$. Since $N > N_1 > m$, (5.7) follows directly from Lemma 5.2.

Note by the definition of Λ_N that $a_1 \dots a_N = \alpha_1 \dots \alpha_N$. By (5.7) it follows that $(a_i) \in \mathbf{V}$. So, by Definition 2.8 it remains to prove that

$$a_1 \dots a_j (\overline{a_1 \dots a_j}^{\,+})^\infty \prec (a_i) \qquad \text{whenever} \quad (a_1 \dots a_j^-)^\infty \in \mathbf{V}. \tag{5.8}$$

We split the proof of (5.8) into the following three cases.

- For $1 \leq j \leq m$, (5.8) follows from (5.4).

- For $m < j \le N$, let $j = j_1 m + r_1$ with $j_1 \ge 1$ and $r_1 \in \{1, 2, \ldots, m\}$. Since $(\alpha_1 \ldots \alpha_j^-)^\infty = ((\alpha_1 \ldots \alpha_m)^{j_1} \alpha_1 \ldots \alpha_{r_1}^-)^\infty \in \mathbf{V}$, we have

$$\alpha_{r_1+1} \ldots \alpha_m \alpha_1 \ldots \alpha_{r_1} \succ \alpha_{r_1+1} \ldots \alpha_m \alpha_1 \ldots \alpha_{r_1}^- \succcurlyeq \overline{\alpha_1 \ldots \alpha_m}.$$

  This implies that

$$\begin{aligned} a_1 \ldots a_j (\overline{a_1 \ldots a_j}^{\,+})^\infty &= (\alpha_1 \ldots \alpha_m)^{j_1} \alpha_1 \ldots \alpha_{r_1} \overline{\alpha_1 \ldots \alpha_m} \ldots \\ &\prec (\alpha_1 \ldots \alpha_m)^{j_1} \alpha_1 \ldots \alpha_{r_1} \alpha_{r_1+1} \ldots \alpha_m \alpha_1 \ldots \alpha_{r_1} 0^\infty \\ &\preccurlyeq (a_i). \end{aligned}$$

- For $j > N$, by (5.7) it follows that

$$(\overline{a_1 \ldots a_j}^{\,+})^\infty = (\overline{\alpha_1 \ldots \alpha_N a_{N+1} \ldots a_j}^{\,+})^\infty \prec a_{j+1} a_{j+2} \ldots,$$

  which implies that (5.8) also holds in this case.

Therefore, (a_i) is an irreducible sequence, and thus $(a_i) \in \mathbf{B}_\delta(p_L)$. So, $\Lambda_N \subseteq \mathbf{B}_\delta(p_L)$ for all $N > N_1$.

Note that $\pi_{p_L}(\Lambda_N)$ is a scaling copy of $\pi_{p_L}(F_{\mathbf{W}_{p_L,N}}(\alpha_1 \ldots \alpha_m))$ which is related to a graph-directed set satisfying the open set condition (cf. [23, Lemma 3.2]). By Proposition 4.1 and (5.6) it follows that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\begin{aligned} \dim_H(\overline{\mathscr{B}} \cap (p_L - \delta, p_L + \delta)) &\ge (1 - \varepsilon) \dim_H \pi_{p_L}(\mathbf{B}_\delta(p_L)) \\ &\ge (1 - \varepsilon) \dim_H \pi_{p_L}(\Lambda_N) \\ &= (1 - \varepsilon) \frac{h_{top}(F_{\mathbf{W}_{p_L,N}}(\alpha_1 \ldots \alpha_m))}{\log p_L} \\ &= (1 - \varepsilon) \frac{h_{top}(\mathbf{W}_{p_L,N})}{\log p_L} \end{aligned}$$

for all sufficiently large integers $N > N_1$. Letting $N \to \infty$ we conclude by Lemmas 5.3 and 2.4 that

$$\dim_H(\overline{\mathscr{B}} \cap (p_L - \delta, p_L + \delta)) \ge (1 - \varepsilon) \frac{h_{top}(\mathbf{V}_{p_L})}{\log p_L} = (1 - \varepsilon) \dim_H \mathcal{U}_{p_L}.$$

Since $\varepsilon > 0$ was taken arbitrarily, this establishes the lemma. □

Now we prove the lower bound of Theorem 2 for $q \in (q_{KL}, q_T)$ being the left endpoint of an entropy plateau.

Lemma 5.6. *Let $[p_L, p_R] \subset (q_{KL}, q_T)$ be a plateau of H. Then for any $\delta > 0$ we have*

$$\dim_H \overline{\mathscr{B}} \cap (p_L - \delta, p_L + \delta) \ge \dim_H \mathcal{U}_{p_L}.$$

Proof. The proof is similar to that of Lemma 5.5. We only give the proof for $M = 2k$, since the proof for $M = 2k+1$ is similar.

By Lemma 2.11 (i) it follows that $\alpha(p_L) = (\alpha_i) = (\alpha_1 \dots \alpha_m)^\infty$ is a $*$-irreducible sequence, where m is the minimal period of $\alpha(p_L)$. Then there exists $n \geq 1$ such that $\xi(n+1) \preccurlyeq \alpha(p_L) \prec \xi(n)$, where $\xi(n) = \lambda_1 \dots \lambda_{2^{n-1}}(\overline{\lambda_1 \dots \lambda_{2^{n-1}}}^+)^\infty$. By (2.4) this implies that $m > 2^n$. Since $\alpha(p_L) = (\alpha_i)$ is periodic while $\xi(n+1)$ is eventually periodic, we have $\xi(n+1) \prec \alpha(p_L) \prec \xi(n)$. So there exists a large integer N_0 such that

$$\xi(n+1) \prec \alpha_1 \dots \alpha_{N_0} \prec \xi(n). \tag{5.9}$$

Since $\alpha(p_L) = (\alpha_i)$ is $*$-irreducible, by Definition 2.10 there exists an integer $N_1 > N_0$ such that

$$\alpha_1 \dots \alpha_j(\overline{\alpha_1 \dots \alpha_j}^+)^\infty \prec \alpha_1 \dots \alpha_{N_1} \qquad \text{if} \quad (\alpha_1 \dots \alpha_j^-)^\infty \in \mathbf{V} \text{ and } 2^n < j \leq m. \tag{5.10}$$

Let $\nu = \lambda_1 \dots \lambda_{2^n}^-$ be the word defined as in Lemma 5.4 (ii). Then by Lemma 2.11 (iii) there exist a large integer $N \geq N_1$ and a word ω such that

$$\overline{\alpha_1 \dots \alpha_N} \prec \sigma^j(\alpha_1 \dots \alpha_m \omega \nu^\infty) \prec \alpha_1 \dots \alpha_N \quad \text{for any} \quad j \geq 0. \tag{5.11}$$

Observe that $(\mathbf{W}_{p_L,N}, \sigma)$ is an N-step subshift of finite type. Note by (5.11) that $\alpha_1 \dots \alpha_m \omega \nu^N \in \mathcal{L}(\mathbf{W}_{p_L,N})$. Then by [30, Theorem 2.1.8] it follows that for any sequence $(d_i) \in F_{\mathbf{W}_{p_L,N}}(\nu^N)$ we have $\alpha_1 \dots \alpha_m \omega d_1 d_2 \dots \in F_{\mathbf{W}_{p_L,N}}(\alpha_1 \dots \alpha_m)$. This implies

$$\left\{\alpha_1 \dots \alpha_m \omega d_1 d_2 \dots : (d_i) \in F_{\mathbf{W}_{p_L,N}}(\nu^N)\right\} \subseteq F_{\mathbf{W}_{p_L,N}}(\alpha_1 \dots \alpha_m) \subseteq \mathbf{W}_{p_L,N}.$$

So, by Lemma 5.4 (ii) we obtain

$$h_{top}(F_{\mathbf{W}_{p_L,N}}(\alpha_1 \dots \alpha_m)) = h_{top}(\mathbf{W}_{p_L,N}). \tag{5.12}$$

Let Δ_N be the set of sequences (a_i) satisfying

$$a_1 \dots a_{mN} = (\alpha_1 \dots \alpha_m)^N \quad \text{and} \quad a_{mN+1} a_{mN+2} \dots \in F_{\mathbf{W}_{p_L,N}}(\alpha_1 \dots \alpha_m).$$

Fix $\delta > 0$. Then we claim that

$$\Delta_N \subset \mathbf{B}_\delta(p_L) = \left\{\alpha(q) : q \in \overline{\mathscr{B}} \cap (p_L - \delta, p_L + \delta)\right\}$$

for all sufficiently large integers $N > N_1$. Observe that the common prefix of sequences in Δ_N has length at least $m(N+1)$ and it coincides with a prefix of $\alpha(p_L) = (\alpha_1 \dots \alpha_m)^\infty$. So, by Lemmas 2.1 and 2.12 it suffices to show that for all integers $N > N_1$ any sequence in Δ_N is $*$-irreducible.

Take $N > N_1$ sufficiently large and take $(a_i) \in \Delta_N$. Then by (5.9) we have $\xi(n+1) \prec (a_i) \prec \xi(n)$. Furthermore, by Lemma 5.2 and the definition of Δ_N it follows that

$$\overline{a_1 \dots a_N} \prec \sigma^j((a_i)) \prec a_1 \dots a_N \quad \text{for any} \quad j \geq 1. \tag{5.13}$$

This implies that $(a_i) \in \mathbf{V}$. Furthermore, by (5.10), (5.13) and arguments similar to those in the proof of Lemma 5.5 we can prove that

$$a_1 \dots a_j (\overline{a_1 \dots a_j}^{\,+})^\infty \prec (a_i)$$

whenever $j > 2^n$ and $(a_1 \dots a_j^-)^\infty \in \mathbf{V}$. Therefore, by Definition 2.10 the sequence (a_i) is $*$-irreducible, and then $\Delta_N \subset \mathbf{B}_\delta(p_L)$ for all $N > N_1$, proving the claim.

Hence, by Proposition 4.1 and (5.12) it follows that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\begin{aligned} \dim_H(\overline{\mathscr{B}} \cap (p_L - \delta, p_L + \delta)) &\ge (1-\varepsilon) \dim_H \pi_{p_L}(\mathbf{B}_\delta(p_L)) \\ &\ge (1-\varepsilon) \dim_H \pi_{p_L}(\Delta_N) \\ &= (1-\varepsilon) \frac{h_{top}(F_{\mathbf{W}_{p_L,N}}(\alpha_1 \dots \alpha_m))}{\log p_L} \\ &= (1-\varepsilon) \frac{h_{top}(\mathbf{W}_{p_L,N})}{\log p_L} \end{aligned}$$

for all sufficiently large integers $N > N_1$. Letting $N \to \infty$ we obtain by Lemmas 5.3 and 2.4 that

$$\dim_H(\overline{\mathscr{B}} \cap (p_L - \delta, p_L + \delta)) \ge (1-\varepsilon) \frac{h_{top}(\mathbf{V}_{p_L})}{\log p_L} = (1-\varepsilon) \dim_H \mathcal{U}_{p_L}.$$

Since $\varepsilon > 0$ was arbitrary, we complete the proof by letting $\varepsilon \to 0$. □

Proof of Theorem 2. Take $q \in \overline{\mathscr{B}}$ and $\delta > 0$. By Lemma 2.7 there exists a sequence of plateaus $\{[p_L(n), p_R(n)]\}$ such that $p_L(n)$ converges to q as $n \to \infty$. By Lemmas 5.5 and 5.6 it follows that

$$\dim_H(\overline{\mathscr{B}} \cap (q - \delta, q + \delta)) \ge \dim_H \mathcal{U}_{p_L(n)}$$

for all sufficiently large n. Letting $n \to \infty$ and by Lemma 2.4 we obtain that

$$\dim_H(\overline{\mathscr{B}} \cap (q - \delta, q + \delta)) \ge \dim_H \mathcal{U}_q. \tag{5.14}$$

Therefore, the theorem follows from (5.14) and Proposition 5.1. □

6. Dimensional spectrum of $\mathscr{U}$

Recall that $\mathscr{U}$ is the set of univoque bases $q \in (1, M+1]$ for which 1 has a unique q-expansion. In this section we will use Theorem 2 to prove Theorem 3 for the dimensional spectrum of $\mathscr{U}$, which states that

$$\dim_H(\mathscr{U} \cap (1, t]) = \max_{q \le t} \dim_H \mathcal{U}_q \quad \text{ for all } \ t > 1.$$

We focus on $t \in (q_{KL}, M+1)$, since by Lemma 2.4 the other cases are trivial.

Since the proof of Lemma 4.3 above only uses properties of $\overline{\mathscr{U}}$ instead of $\overline{\mathscr{B}}$, the proof also gives the following lemma for the set $\overline{\mathscr{U}}$.

Lemma 6.1. *Let $q \in \overline{\mathscr{U}} \setminus \{M+1\}$. Then for any $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\dim_H(\overline{\mathscr{U}} \cap (q-\delta, q+\delta)) \leq (1+\varepsilon) \dim_H \pi_{q+\delta}(\mathbf{U}_\delta(q)),$$

where $\mathbf{U}_\delta(q) = \big\{\alpha(p) : p \in \overline{\mathscr{U}} \cap (q-\delta, q+\delta)\big\}$.

To prove Theorem 3 we first consider the upper bound.

Lemma 6.2. *For any $t \in (q_{KL}, M+1)$ we have*

$$\dim_H(\overline{\mathscr{U}} \cap (1,t]) \leq \max_{q \leq t} \dim_H \mathcal{U}_q.$$

Proof. Fix $\varepsilon > 0$, and take $t \in (q_{KL}, M+1)$. Then it suffices to prove

$$\dim_H(\overline{\mathscr{U}} \cap (1,t]) \leq (1+\varepsilon)(\max_{q \leq t} \dim_H \mathcal{U}_q + \varepsilon). \tag{6.1}$$

By Lemmas 2.4 and 6.1 it follows that for each $q \in \overline{\mathscr{U}} \cap (1,t]$ there exists a sufficiently small $\delta = \delta(q,\varepsilon) > 0$ such that

$$\begin{aligned} \dim_H \mathcal{U}_{q+\delta} &\leq \dim_H \mathcal{U}_q + \varepsilon, \\ \dim_H(\overline{\mathscr{U}} \cap (q-\delta, q+\delta)) &\leq (1+\varepsilon) \dim_H \pi_{q+\delta}(\mathbf{U}_\delta(q)). \end{aligned} \tag{6.2}$$

Observe that $\big\{(q-\delta, q+\delta) : q \in \overline{\mathscr{U}} \cap (1,t]\big\}$ is an open cover of $\overline{\mathscr{U}} \cap (1,t]$, and that $\overline{\mathscr{U}} \cap (1,t] = \overline{\mathscr{U}} \cap [q_{KL}, t]$ is a compact set. Hence, there exist $q_1, q_2, \ldots, q_N$ in $\overline{\mathscr{U}} \cap (1,t]$ such that

$$\overline{\mathscr{U}} \cap (1,t] \subseteq \bigcup_{i=1}^{N} \big(\overline{\mathscr{U}} \cap (q_i - \delta_i, q_i + \delta_i)\big), \tag{6.3}$$

where $\delta_i = \delta(q_i, \varepsilon)$ for $1 \leq i \leq N$.

Note by Lemmas 2.2 and 2.3 that for each $i \in \{1, 2, \ldots, N\}$ we have

$$\pi_{q_i+\delta_i}(\mathbf{U}_{\delta_i}(q_i)) = \pi_{q_i+\delta_i}(\big\{\alpha(p) : p \in \overline{\mathscr{U}} \cap (q_i - \delta_i, q_i + \delta_i)\big\}) \subseteq \mathcal{U}_{q_i+\delta_i}.$$

Then by (6.2) and (6.3) it follows that

$$\begin{aligned} \dim_H(\overline{\mathscr{U}} \cap (1,t]) &\leq \dim_H \left(\bigcup_{i=1}^{N} \big(\overline{\mathscr{U}} \cap (q_i - \delta_i, q_i + \delta_i)\big) \right) \\ &= \max_{1 \leq i \leq N} \dim_H(\overline{\mathscr{U}} \cap (q_i - \delta_i, q_i + \delta_i)) \\ &\leq (1+\varepsilon) \max_{1 \leq i \leq N} \dim_H \pi_{q_i+\delta_i}(\mathbf{U}_{\delta_i}(q_i)) \\ &\leq (1+\varepsilon) \max_{1 \leq i \leq N} \dim_H \mathcal{U}_{q_i+\delta_i} \\ &\leq (1+\varepsilon) \max_{1 \leq i \leq N} (\dim_H \mathcal{U}_{q_i} + \varepsilon) \\ &\leq (1+\varepsilon)(\max_{q \leq t} \dim_H \mathcal{U}_q + \varepsilon). \end{aligned}$$

This proves (6.1), and completes the proof. $\square$

The next lemma gives the lower bound of Theorem 3.

Lemma 6.3. *For any* $t \in (q_{KL}, M+1)$ *we have*

$$\dim_H(\overline{\mathscr{U}} \cap (1,t]) \geq \max_{q \leq t} \dim_H \mathcal{U}_q.$$

Proof. Take $t \in (q_{KL}, M+1)$. Note by Lemma 2.4 that the dimension function $D : q \mapsto \dim_H \mathcal{U}_q$ is continuous. Then there exists $q_* \in [q_{KL}, t]$ such that

$$\dim_H \mathcal{U}_{q_*} = \max_{q \leq t} \dim_H \mathcal{U}_q.$$

Since the entropy function H is locally constant on the complement of $\mathscr{B}$, it follows by Lemma 2.4 that

$$q_* \in (q_{KL}, t] \setminus \bigcup (p_L, p_R] \subseteq (q_{KL}, t] \cap \overline{\mathscr{B}}.$$

If $q_* \in (q_{KL}, t) \cap \overline{\mathscr{B}}$, then the lemma follows by $\overline{\mathscr{B}} \subset \overline{\mathscr{U}}$ and Theorem 2. If $q_* = t$, then by Lemma 2.7 (i) there exists a sequence of plateaus $\{[p_L(n), p_R(n)]\}$ such that $p_L(n) \in (q_{KL}, t) \cap \overline{\mathscr{B}}$ and $p_L(n) \nearrow q_*$ as $n \to \infty$. Therefore, by Lemma 2.4 and Theorem 2 we also have

$$\dim_H(\overline{\mathscr{U}} \cap (1,t]) \geq \dim_H(\overline{\mathscr{B}} \cap (q_{KL}, t]) \geq \dim_H \mathcal{U}_{p_L(n)} \to \dim_H \mathcal{U}_{q_*}$$

as $n \to \infty$. This establishes the lemma. □

Proof of Theorem 3. For $1 < t \leq q_{KL}$ we have $\mathscr{U} \cap (1,t] \subseteq \{q_{KL}\}$ and thus by Lemma 2.4 (i) it follows that

$$\dim_H(\mathscr{U} \cap (1,t]) = 0 = \max_{q \leq t} \dim_H \mathcal{U}_q.$$

For $t \geq M+1$ we have $\mathscr{U} = \mathscr{U} \cap (1,t]$ and the result also follows from Lemma 2.4. For the remaining t the result follows from Lemmas 6.2 and 6.3, since $\overline{\mathscr{U}} \setminus \mathscr{U}$ is countable.

From Lemma 2.4 it follows that the dimension function $D : q \mapsto \dim_H \mathcal{U}_q$ has a Devil's staircase behavior (see also Remark 2.5 (1)). This implies that $\phi(t) := \max_{q \leq t} \dim_H \mathcal{U}_q$ is a Devil's staircase in $(1, \infty)$: (i) ϕ is non-decreasing and continuous in $(1, \infty)$; (ii) ϕ is locally constant almost everywhere in $(1, \infty)$; and (iii) $\phi(q_{KL}) = 0$, and $\phi(t) > 0$ for any $t > q_{KL}$. □

7. Variations of $\mathscr{U}(M)$

For any $K \in \{0, 1, \ldots, M\}$, let $\mathscr{U}(K)$ denote the set of bases $q > 1$ such that 1 has a unique q-expansion over the alphabet $\{0, 1, \ldots, K\}$. Then $\mathscr{U}(K) \subset (1, K+1]$. In this section we investigate the Hausdorff dimension of the intersection $\bigcap_{J=K}^{M} \mathscr{U}(J)$, and prove Theorem 4. Note that $q_{KL} = q_{KL}(M)$ is the smallest element of $\mathscr{U}(M)$, and $K+1$ is the largest element of $\mathscr{U}(K)$. So, if $K+1 < q_{KL}$ then $\mathscr{U}(M) \cap \mathscr{U}(K) = \emptyset$. Therefore, in the following we assume $K \in [q_{KL} - 1, M]$.

Lemma 7.1. *Let $K \in [q_{KL}-1, M]$ be an integer. Then for each $q \in \mathscr{U}(M) \cap (1, K+1]$ the unique expansion $\alpha(q) = (\alpha_i(q))$ satisfies*

$$M - K \le \alpha_i(q) \le K \qquad \text{for any} \quad i \ge 1.$$

Proof. Clearly, the lemma holds if $K = M$. So we assume $K < M$. Take $q \in \mathscr{U}(M) \cap (1, K+1] \subseteq [q_{KL}, K+1]$. Then

$$\alpha(q_{KL}) \preceq \alpha(q) \preceq \alpha(K+1) = K^\infty.$$

This, together with $\alpha_1(q_{KL}) \ge M - \alpha_1(q_{KL})$, implies that

$$M - K \le \alpha_1(q_{KL}) \le \alpha_1(q) \le K.$$

Since $M > K$ and $q \in \mathscr{U}(M)$, it follows from Lemma 2.3 (i) that

$$M - K \le M - \alpha_1(q) \le \alpha_i(q) \le \alpha_1(q) \le K \qquad \text{for any} \quad i \ge 1.$$

This completes the proof. □

Lemma 7.2. *Let $K \in [q_{KL}-1, M]$ be an integer. Then*

$$\mathscr{U}(M) \cap \mathscr{U}(K) = (1, K+1] \cap \mathscr{U}(M).$$

Proof. Since $\mathscr{U}(K) \subseteq (1, K+1]$, it suffices to prove that $\mathscr{U}(M) \cap (1, K+1] \subseteq \mathscr{U}(K)$. Take $q \in \mathscr{U}(M) \cap (1, K+1]$. Then by Lemma 2.3 it follows that $\alpha(q) = (\alpha_i(q))$ satisfies

$$(K - \alpha_i(q)) \preceq (M - \alpha_i(q)) \prec \alpha_{i+1}(q)\alpha_{i+2}(q) \cdots \prec \alpha(q) \quad \text{for all} \quad i \ge 1. \tag{7.1}$$

Note by Lemma 7.1 that $0 \le \alpha_i(q) \le K$ for all $i \ge 1$. Hence, by (7.1) and Lemma 2.3 we conclude that $q \in \mathscr{U}(K)$. □

Proof of Theorem 4. First we prove (i). Clearly, if $K < q_{KL} - 1$ then $\bigcap_{J=K}^{M} \mathscr{U}(J) = \emptyset$, and therefore (i) holds by Lemma 2.4 (i). If $q_{KL} - 1 \le K \le M$, then by repeatedly using Lemma

7.2 we conclude that

$$\begin{aligned}
\bigcap_{J=K}^{M} \mathscr{U}(J) &= \big(\mathscr{U}(M) \cap \mathscr{U}(M-1)\big) \cap \bigcap_{J=K}^{M-2} \mathscr{U}(J) \\
&= (1, M] \cap \mathscr{U}(M) \cap \bigcap_{J=K}^{M-2} \mathscr{U}(J) \\
&= (1, M] \cap \big(\mathscr{U}(M) \cap \mathscr{U}(M-2)\big) \cap \bigcap_{J=K}^{M-3} \mathscr{U}(J) \\
&= (1, M-1] \cap \mathscr{U}(M) \cap \bigcap_{J=K}^{M-3} \mathscr{U}(J) \\
&\quad \ldots \\
&= (1, K+1] \cap \mathscr{U}(M).
\end{aligned}$$

Therefore, by Theorem 3 we establish (i).

As for (ii), we observe that for any $L \geq 1$,

$$\mathscr{U}(L) = \Big(\mathscr{U}(L) \setminus \bigcup_{J \neq L} \mathscr{U}(J)\Big) \cup \bigcup_{J \neq L} \big(\mathscr{U}(L) \cap \mathscr{U}(J)\big). \tag{7.2}$$

From (i) and Lemma 2.4 (i) it follows that $\dim_H(\mathscr{U}(L) \cap \mathscr{U}(J)) < 1$ for any $J \neq L$. Furthermore, by Lemma 2.6 we have $\dim_H \mathscr{U}(L) = 1$ (see also, [23, Theorem 1.6]). Therefore, (ii) immediately follows from (7.2). $\square$

8. Final remarks

It was shown in Theorem 3 that the function $\phi(t) = \dim_H(\mathscr{U} \cap (1, t])$ is a Devil's staircase in $(1, \infty)$ (see Figure 1 for the sketch plot of ϕ). Then a natural question is to ask about the presence and position of plateaus for ϕ, i.e., maximal intervals on which ϕ is constant. By Lemma 2.4 (i) and Theorem 3 it follows that $\phi(t) = 0$ if and only if $t \leq q_{KL}$, and $\phi(t) = 1$ if and only if $t \geq M+1$. Hence, the first plateau of ϕ is $(1, q_{KL}]$, and the last plateau is $[M+1, \infty)$.

Since $\phi(t) = \max_{q \leq t} \dim_H \mathcal{U}_q$, an interval $[q_L, q_R]$ is a plateau of ϕ if and only if

$$\begin{aligned}
&\dim_H \mathcal{U}_p < \dim_H \mathcal{U}_{q_L} && \text{for any} && p < q_L, \\
&\dim_H \mathcal{U}_q \leq \dim_H \mathcal{U}_{q_L} && \text{for any} && q_L \leq q \leq q_R, \\
&\dim_H \mathcal{U}_r > \dim_H \mathcal{U}_{q_L} && \text{for any} && r > q_R.
\end{aligned}$$

By Lemma 2.4 it follows that for each plateau $[q_L, q_R]$ of ϕ we have $\dim_H \mathcal{U}_{q_L} = \dim_H \mathcal{U}_{q_R}$.

Question 1. Can we describe the plateaus of ϕ in $(q_{KL}, M+1)$?

Theorem 3 tells us that the set $\mathscr{U}$ gets heavier towards the right, but does not say anything about the local weight.

Question 2. For any $t_2 > t_1 > 1$, what is the local dimension $\dim_H(\mathscr{U} \cap [t_1, t_2])$?

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