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Torsion points and height jumping in higher-dimensional families of abelian varieties

David Holmes
Mathematisch Instituut, Universiteit Leiden, Postbus 9512
2300 RA Leiden, The Netherlands
holmesdst@math.leidenuniv.nl

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In 1983, Silverman and Tate showed that the set of points in a 1-dimensional family of abelian varieties where a section of infinite order has "small height" is finite. We conjecture a generalization to higher-dimensional families, where we replace "finite" by "not Zariski dense." We show that this conjecture would imply the uniform boundedness conjecture for torsion points on abelian varieties. We then prove a few special cases of this new conjecture.

Keywords: Heights; torsion points; abelian varieties.

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1. Introduction

The uniform boundedness conjecture predicts that the number of rational torsion points on an abelian variety over $\mathbb{Q}$ is bounded uniformly in the dimension of the abelian variety. It is a theorem of Mazur for elliptic curves (with generalizations to number fields by Kamienny, Merel and others), but is wide open for higher dimensional abelian varieties.

An easy consequence of this conjecture is that, given a family of abelian varieties over $\mathbb{Q}$ and a section of infinite order, the set of rational points in the base where the section becomes torsion is not Zariski dense. This is known unconditionally for families of elliptic curves by Mazur's theorem, but not in general. However, it is also known unconditionally for abelian varieties of any dimension if the base of the family has dimension 1 — this is a consequence of a theorem of Silverman [32] and Tate [35] (with refinements by a number of authors — [1, 5, 10, 19]).

The first main result of this paper is that the uniform boundedness conjecture is in fact *equivalent* to showing that the set of points where a section of infinite order becomes torsion is not Zariski dense (Theorem 2.4). In particular, if we could generalize the theorem of Silverman and Tate to families of arbitrary dimension (cf. Theorem 2.2), this would imply uniform boundedness. The proof of the equivalence is not difficult, it mainly uses induction on dimension and the fact that the stack of principally polarized abelian varieties of fixed dimension is a noetherian Deligne–Mumford stack.

By a recent result of Cadoret and Tamagawa, the uniform boundedness conjecture for abelian varieties is equivalent to the same conjecture for Jacobians of curves. Using this, we can also show that the restriction of Theorem 2.2 to Jacobians implies the full uniform boundedness conjecture.

In fact Silverman and Tate ([32, 35]) show not only the finiteness of the number of points where the section becomes torsion, but (stronger) the finiteness of the set of points where the section has bounded Néron–Tate height. Motivated by this, we wonder whether the same may be true for families of higher dimension. Fix a global field K. Writing $\hat{\mathrm{h}}$ for the Néron–Tate height, we define

Definition 1.1. Given a variety S/K, an abelian scheme A/S, and a section $\sigma \in A(S)$, we say $(A/S, \sigma)$ has *sparse small points* if for all $d \in \mathbb{Z}_{\geq 1}$ there exists $\epsilon > 0$ such that the set

$$\mathbf{T}_\epsilon(d) = \{s \in S(\bar{K}) | [\kappa(s) : K] \leq d \text{ and } \hat{\mathrm{h}}(\sigma(s)) \leq \epsilon\}$$

is not Zariski dense in S.

We tentatively propose

Conjecture 1.2. *Every pair $(A/S, \sigma)$ with σ of infinite order has sparse small points.*

This is a theorem of Silverman and Tate when S has dimension 1. Since torsion points have height zero the above conjecture would imply that sets of points where a section of infinite order becomes torsion are not Zariski dense, and hence (by Theorem 2.4 mentioned above) the uniform boundedness conjecture. The conjecture is true for families of elliptic curves, subject to a conjecture of Lang on lower bounds for heights in families (Theorem 3.2).

In the second half of this paper we prove some special cases of Theorem 1.2. In [15] we proved that torsion points are sparse for families of Jacobians which admit Néron models. Here, we consider again families admitting Néron models, but the proof is far more involved, and we are forced to impose some additional assumptions in order for our methods to work, see Theorem 3.17. Most important is a condition on the base S; a precise statement can be found in Theorem 3.17, but here we note that it is satisfied whenever $\dim_{\mathbb{Q}} \mathrm{Pic}(S) \otimes_{\mathbb{Z}} \mathbb{Q} = 1$, yielding a slight simplification of Theorem 3.17:

Theorem 1.3. *Let $S/\mathbb{Q}$ be a projective variety with* $\mathrm{Pic}(S) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$, *and let $U \subseteq S$ be a dense open subscheme. Let C/S be a family of nodal curves, smooth* *over U, with C regular. Write J for the Jacobian of C_U/U, and let $\sigma \in J(S)$ be a section of infinite order corresponding to the restriction to C_U of a divisor on C supported on some sections of C/S. Assume that S has a proper regular model over $\mathbb{Z}$ over which C has a proper nodal model. Assume that J admits a Néron model over S. Then $(J/U, \sigma)$ has sparse small points.*

For the sake of those readers not familiar with the theory of Néron models over higher dimensional bases developed in [14] we note that the assumption that J admit a Néron model over S can be replaced by the assumption that $S\backslash U$ (with reduced induced scheme structure) is smooth over $\mathbb{Q}$, or that the fibers of $C \to S$ have tree-like dual graphs.

Unfortunately it seems at present difficult to give interesting new examples of families to which the theorem applies. The family of nodal genus g curves C/S induces a proper map $s: S \to \overline{\mathcal{M}}_g$ to the moduli space of stable curves of genus g, and if for example s has image contained within the locus of tree-like curves and the pullback of every boundary divisor in $\overline{\mathcal{M}}_g$ is locally analytically irreducible in S, then J admits a Néron model over S. If $g \leq 2$ then the whole of $\overline{\mathcal{M}}_g$ is tree-like, so we only require that the pullback of every boundary divisor in $\overline{\mathcal{M}}_g$ is locally analytically irreducible in S, but constructing such families of dimension > 1 whose Jacobain admits a section of infinite order does not seem straightforward.

We conclude the introduction by giving an outline of the proof of Theorem 3.17. In [1] we gave a new proof of the theorem of Silverman and Tate over curves, and in this paper we follow essentially the same strategy of proof in higher dimensions. Namely, given a family of Jacobians J/S and a section $\sigma \in J(S)$ of infinite order, we consider the function

$$\hat{\mathrm{h}}_\sigma : S(\bar{\mathbb{Q}}) \to \mathbb{R}_{\geq 0}; \;\; s \mapsto \hat{\mathrm{h}}(\sigma(s)).$$

We decompose $\hat{\mathrm{h}}_\sigma$ as a sum of two functions,

$$\hat{\mathrm{h}}_\sigma = \mathrm{h}_{\mathscr{L}} + j$$

where $\mathrm{h}_{\mathscr{L}}$ is a Weil height on S with respect to a certain line bundle, and j is an "error term", called the *height jump*. When S is a curve, we were able to show that the bundle $\mathscr{L}$ is ample and the function j is bounded, from which the theorem of Silverman and Tate easily follows.

Recent work of DeMarco and Mavraki in the case where $\dim S = 1$ and the curve is of genus 1 shows that, by making a careful choice of metrics on the line bundle $\mathcal{L}$, the function j can even be made to vanish in that case. A key step in the construction of their metrics is an application of Silverman's [33, Theorem II.0.1], which in our terminology can be said to hold exactly because the local height jumps are bounded.

Unfortunately, in extending from curves to arbitrary varieties we encounter two substantial technical difficulties; the line bundle $\mathscr{L}$ is not in general ample, and the function j does not seem to be bounded[a]. The assumptions in Theorem 3.17 allow us to get around these problems. We will show that the jump is bounded if the Jacobian J has a Néron model over a suitable compactification of S, and that $\mathscr{L}$ is ample if S has sufficiently simple geometry (cf. our condition that $\dim_{\mathbb{Q}} \operatorname{Pic}(\bar{S}) \otimes_{\mathbb{Z}} \mathbb{Q} = 1$).

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1.1. *Conventions*

Given a field k, by a *variety over* k we mean an integral separated k-scheme of finite type. We write $\kappa(p)$ for the residue field of a point p.

2. Reformulating the Uniform Boundedness Conjecture

In this section, we work exclusively in characteristic zero, to avoid problems with inseparable Weil restrictions (cf. [7, Example A.3.8] for an example of an abelian variety whose Weil restriction is not proper). We begin by recalling the uniform boundedness conjecture:

Conjecture 2.1 (The strong uniform boundedness conjecture for abelian varieties). *Fix an integer $g \geq 0$. There exists a constant $B = B(g) \in \mathbb{Z}_{>0}$ such that for every g-dimensional abelian variety $A/\mathbb{Q}$ and point $p \in A(\mathbb{Q})$, we have that either p is of infinite order, or that the order of p is less than B.*

Note that this is equivalent to the usual formulation of the strong uniform boundedness conjecture (see for example [31, Conjecture 2.3.2]); Zarhin's trick reduces the general case to that of principally polarized abelian varieties, by Weil restriction we can obtain a bound for abelian varieties over finite extensions of $\mathbb{Q}$ which is uniform in the dimension of the variety and the degree of the field extension, and finally a bound on the order of torsion points and on the dimension implies a bound on the size of the torsion subgroup.

Our first main result (Theorem 2.4) is that Theorem 2.1 is equivalent to the following conjecture:

Conjecture 2.2. *Let $S/\mathbb{Q}$ be a variety and let A/S be an abelian scheme. Let $d \geq 1$ be an integer. Let $\sigma \in A(S)$ be a section of infinite order. Define*

$$\mathbf{T}(d) = \{s \in S(\bar{\mathbb{Q}}) \mid [\kappa(s) : \mathbb{Q}] \leq d \text{ and } \sigma(s) \text{ is torsion in } A_s(\bar{\mathbb{Q}})\}.$$

Then $\mathbf{T}(d)$ is not Zariski dense in S.

[a]More precisely, we can write $j = \sum_l j_l$ as a sum of local terms indexed by primes l of $\mathbb{Q}$, and each j_l is unbounded on $\mathbb{Q}_l$-valued points. We may hope that for $\mathbb{Q}$-valued points all these local jumps are bounded — indeed for torsion points on elliptic curves this should follow from Mazur's theorem — but we cannot prove this at present.

In the case when the base scheme S has dimension 1, this conjecture follows from a theorem of Silverman [32], see also [1, 5, 10, 19, 35] for various strengthened versions. In Sec. 3, we will discuss a variant which looks not only at torsion points but at all points of "small height".

Remark 2.3. This conjecture is reminiscent of conjectures of Pink [30] and Masser-Zannier [24, p. 1668], concerning the distribution of torsion points in the Zariski topology. But note that we always bound the degrees of the field extensions we consider, in contrast to these other conjectures, and there does not seem to be a more direct connection between them.

Theorem 2.4. *Theorem* 2.1 *is equivalent to Theorem* 2.2.

It is easy to see that Theorem 2.1 implies Theorem 2.2; we must show the converse. Before giving the proof we need a lemma and a definition.

Lemma 2.5. *Assume Theorem* 2.2. *Let $S/\mathbb{Q}$ be a variety, and let A/S be an abelian scheme with a section $\sigma \in A(S)$. Fix $d \in \mathbb{Z}_{>0}$. Then there exists an integer $c = c(d) > 0$ such that for every $L/\mathbb{Q}$ of degree at most d and every point $s \in S(L)$, either $\sigma(s)$ has infinite order, or $\sigma(s)$ is torsion of order less than c.*

Proof. We proceed by induction on the dimension of S. If $\dim S = 0$ then S has only finitely many $\bar{\mathbb{Q}}$-points, so the result is immediate.

In general, we fix an integer $\delta > 0$ and assume the lemma holds for every variety S of dimension less than δ. Now let S have dimension δ. If σ is torsion (say of order c_0) then for every $s \in S(\overline{\mathbb{Q}})$ it holds that $c_0\sigma(s) = 0$, and we are done. As such, we may and do assume that σ has infinite order. We apply Theorem 2.2 to obtain a proper closed subscheme $Z \subsetneq S$ such that

$$\{p \in S(L) \mid [L : \mathbb{Q}] \leq d \text{ and } \sigma(p) \text{ is torsion in } A_p(L)\} \subseteq Z(\overline{\mathbb{Q}}).$$

Now Z has only finitely many irreducible components, and each has dimension less than δ. We are done by the induction hypothesis. □

Definition 2.6. Fix an integer $g > 0$. Let $\mathscr{A}_g$ denote the moduli stack of PPAV of dimension g. Then $\mathscr{A}_g$ is a separated Deligne–Mumford stack of finite type over $\mathbb{Z}$. Given also an integer $n \geq 0$, let $\mathscr{A}_{g,n}$ denote the moduli stack of PPAV of dimension g together with a collection of n ordered marked sections (not assumed distinct).

We have natural maps $\phi_n \colon \mathscr{A}_{g,n+1} \to \mathscr{A}_{g,n}$ given by forgetting the last section. The map ϕ_n has n natural sections τ_i, given by "doubling up" the sections σ_i. Then $(\phi_n \colon \mathscr{A}_{g,n+1} \to \mathscr{A}_{g,n}, \tau_1, \dots, \tau_n)$ is the "universal PPAV with n marked sections". In particular, each $\mathscr{A}_{g,n}$ is a Deligne–Mumford stack, separated and of finite type over $\mathbb{Z}$.

Proof of Theorem 2.4. Consider the universal map $\phi_2 \colon \mathscr{A}_{g,2} \to \mathscr{A}_{g,1}$ with its tautological section σ_1. The basic idea is to apply Theorem 2.5 to this family, but we must be careful since $\mathscr{A}_{g,1}$ is a stack and not a scheme. However, since $\mathscr{A}_{g,1}$ is noetherian we can apply [21, Theorem 16.6] (every noetherian Deligne–Mumford stack admits a finite, surjective and generically étale morphism from a scheme) to construct a (separated) scheme S of finite type over K, a map $S \to \mathscr{A}_{g,1}$ and an integer $d \geq 1$ such that for every field-valued point $s\colon \operatorname{Spec} L \to \mathscr{A}_{g,1}$ there is an extension M/L of degree at most d and an M-valued point of S lying over s. We are then done by Theorem 2.5 applied to the pullback of $\mathscr{A}_{g,2}$ and σ_1 to (the underlying reduced subscheme of each irreducible component of) S. □

Remark 2.7. In the argument above, we make use of a general existence result of suitable finite covers of stacks by schemes, but in our setting one can make this much more concrete. Namely, we can take $S \to \mathscr{A}_{g,1}$ simply to be the moduli of pointed abelian varieties with sufficiently high level structure so that S is actually a scheme.

In fact, the reader more comfortable with level structures than with stacks can entirely avoid the use of the latter in the above argument. Let $\mathscr{A}'_{g,n}$ be the moduli space of n-pointed abelian varieties with sufficiently high level structure, so $\mathscr{A}'_{g,0}$ is the moduli of PPAV with level structure, $\mathscr{A}'_{g,1}$ is the universal abelian variety over it, $\mathscr{A}'_{g,2} = \mathscr{A}'_{g,1} \times_{\mathscr{A}'_{g,0}} \mathscr{A}'_{g,1}$, and $\sigma_1\colon \mathscr{A}'_{g,1} \to \mathscr{A}'_{g,1} \times_{\mathscr{A}'_{g,0}} \mathscr{A}'_{g,1}$ is the diagonal. We apply Theorem 2.5 directly to this family to conclude the proof.

2.1. *Extensions and generalizations*

Remark 2.8. Using a recent result of Cadoret and Tamagawa [4], we can reduce further to the case of families of curves:

Conjecture 2.9. *Let $S/\mathbb{Q}$ be a variety and let C/S be a proper smooth curve with Jacobian J/S. Let $d \geq 1$ be an integer. Let $\sigma \in J(S)$ be a section of infinite order. Define*

$$\mathbf{T}(d) = \{s \in S(\bar{\mathbb{Q}}) \,|\, [\kappa(s) : \mathbb{Q}] \leq d \text{ and } \sigma(s) \text{ is torsion in } J_s(\bar{\mathbb{Q}})\}.$$

Then $\mathbf{T}(d)$ is not Zariski dense in S.

The equivalence of Theorem 2.1 with Theorem 2.9 may be proven in an almost identical fashion to the equivalence of Theorem 2.1 with Theorem 2.2, after first appealing to the main result of [4] to reduce Theorem 2.1 to the case of curves. We omit the details.

Remark 2.10. It is possible to simultaneously "specialise" both conjectures to again obtain equivalent statements; for example:

- Theorem 2.1 for Weil restrictions of elliptic curves is equivalent to Theorem 2.9 for elliptic curves over base schemes of dimension at most $2 = \dim \overline{\mathcal{M}}_{1,2}$;
- Theorem 2.1 for Weil restrictions of principally polarized abelian surfaces is equivalent to Theorem 2.9 for genus 2 curves over base schemes of dimension at most $5 = \dim \overline{\mathcal{M}}_{2,2}$;

- Theorem 2.1 for Weil restrictions of principally polarized abelian varieties of dimension g is equivalent to Theorem 2.9 for curves of genus $G := 1 + 6^g(g-1)!\frac{g(g-1)}{2}$ over base schemes of dimension at most $\dim \overline{\mathcal{M}}_{G,2\lceil \frac{G}{2} \rceil}$ (see [4, Theorem 1.2] for the origin of this expression for G).

The proofs are identical and the possible variations numerous, so we do not give further details. One can also reduce further to the case of families of curves admitting compactifications with at-worst nodal singularities, etc.

Remark 2.11. Let A/S be an abelian scheme, σ a section of infinite order, and $f\colon S' \to S$ an alteration (a proper surjective generically finite map of varieties). Then Theorem 2.2 holds for σ in A/S if and only if it holds for $f^*\sigma$ in $A \times_S S'/S'$. A corresponding statement holds for Theorem 2.9. This has several convenient corollaries; for example, it is enough to prove Theorem 2.9 for families of curves C/S admitting stable models over some compactification of S. In Sec. 3, we will study Theorem 2.9 in detail in the case of such families.

3. Conjecture on Sparsity of Small Points

In the introduction we proposed (Theorem 1.2) that the sets of points where a section of infinite order has small height is not Zariski dense. It is clear that this conjecture implies Theorem 2.2. The remainder of this paper will be devoted to prove special cases of Theorem 1.2, first for elliptic curves and then for certain special families of abelian varieties of higher dimension (Theorem 3.17).

3.1. *Theorem 1.2 for elliptic curves*

To suggest that Theorem 1.2 is not completely unreasonable, we show that a conjecture of Lang implies Theorem 1.2 for elliptic curves and $d = 1$ (see Theorem 3.2 for a precise statement). We begin by recalling Lang's conjecture:

Conjecture 3.1 ([18] or [12, Conjecture F.3.4(a)]). *Fix a finite field extension $k/\mathbb{Q}$, there exists a constant $c = c(k) > 0$ such that for all elliptic curves E/k and all non-torsion points $a \in E(k)$, we have*

$$\hat{\mathrm{h}}(a) \geq c \cdot \log|N_{k/\mathbb{Q}}\Delta_{E/k}|.$$

Here $\Delta_{E/k}$ is the discriminant, and $N_{k/\mathbb{Q}}$ denotes the norm down to $\mathbb{Q}$.

Lemma 3.2. *Assume Theorem* 3.1 *holds. Fix a finite extension $k/\mathbb{Q}$ and a variety S/k. Then for every family of elliptic curves E/S together with a section σ of infinite order, there exists $\epsilon > 0$ such that the set $T_\epsilon(1)$ is not Zariski dense in S* (*cf. Theorem* 1.1)

Proof. Let Σ denote the finite set of elliptic curves over k with everywhere good reduction, and let $b > 0$ denote the smallest height of a non-torsion k-point appearing on any curve in Σ, or set $b = 1$ if no such exists. Let $c = c(k)$ be the constant from Lang's conjecture, and let m denote the infimum of the values taken by the expression $c \cdot \log|\Delta_{E/\mathbb{Q}}|$ as E runs over all elliptic curves over k with at least one place of bad reduction; this infimum is achieved (since there are only finitely many elliptic curves of bounded discriminant) and is positive (by our bad-reduction assumption).

Set $\delta = \min(b, m)$. For all $\delta > \epsilon \geq 0$ we have

$$\mathbf{T}_\epsilon(1) = \mathbf{T}(1).$$

Now Theorem 2.1 is known for elliptic curves over k by work of Merel Theorem [25] so we know that Theorem 2.2 holds in our situation, hence $\mathbf{T}(1)$ is not Zariski dense in S. ◻

Pazuki has proposed a refined form of the Lang–Silverman conjecture in [28, Conjecture 5.1], and remarked that it implies the strong torsion conjecture. In a similar fashion to the proof of Theorem 3.2 one can show that his conjecture implies the full form of Theorem 1.2. The author is grateful to an anonymous referee for pointing out this reference.

3.2. *Strategy for proving more special cases of Theorem* 1.2

The remainder of the paper will be devoted to proving Theorem 1.2 for families of curves over "simple" base schemes and assuming that the Jacobian admits a Néron model over that base (see Theorem 3.17 for the precise statement). As discussed in the introduction we will use a number of tools from the theory of Néron models and height jumps developed in the papers [1, 3, 13, 14]. In an attempt to keep the present work reasonably self-contained we will briefly recall the main definitions and results we use as we go along.

The strategy of the proof was briefly discussed in the introduction, but we will give a slightly more detailed outline here before proceeding. We begin with a smooth projective variety $S/\mathbb{Q}$ and a nodal curve C/S; by a *nodal curve* we mean a proper flat finitely presented morphism all of whose geometric fibers are reduced, connected, of dimension 1, and have at worst ordinary double point singularities. Write $U \subseteq S$ for the locus where C/S is smooth; we assume U is dense in S. Write J for the Jacobian of C_U/U, and let $\sigma \in J(S)$ be a section of infinite order. We consider the function

$$\hat{\mathrm{h}}_\sigma : U(\bar{\mathbb{Q}}) \to \mathbb{R}_{\geq 0}; \quad s \mapsto \hat{\mathrm{h}}(\sigma(s)).$$

We decompose $\hat{\mathrm{h}}_\sigma$ as a sum of two functions,

$$\hat{\mathrm{h}}_\sigma = \mathrm{h}_{\mathscr{L}} + j,$$

where $\mathrm{h}_{\mathscr{L}}$ is a Weil height on U with respect to a certain line bundle on S, and j is an "error term," the *height jump*.

The line bundle $\mathscr{L}$ will be the admissible extension of the Deligne pairing of σ with itself, cf. Sec. 3.3. In Sec. 3.3, we will also define the height jump j. In

Sec. 3.4, we will make the jump explicit in certain situations, following closely the presentation in [1]. This will require a discussion of Green's functions on resistive networks. In Sec. 3.5, we will show that the jump vanishes if and only if J admits a Néron model over a suitable $\mathbb{Z}$-model of S; this is not essential for our other results, but demonstrates the close link between Néron models and the height jump. In Sec. 3.6, we return to our main aim of proving Theorem 3.17, showing that the jump is bounded for curves the Jacobians of whose generic fibers admit Néron models. In Sec. 3.7, we compare ways of associating heights to line bundles which are not ample, giving a key inequality needed for the proof of Theorem 3.17. Finally, in Sec. 3.8, we put these ingredients together to prove our main result Theorem 3.17.

3.3. *Defining the algebraic height jump*

Let S be a regular scheme, $C \to S$ a generically smooth nodal curve, and $U \subseteq S$ the largest open over which C is smooth. Write J for the Jacobian of C_U/U. Let σ, $\tau \in J(U)$ be two sections. Write $\mathcal{P}$ for the rigidified Poincaré bundle on $J \times_U J$ (cf. [26]). Write $\langle \sigma, \tau \rangle$ for the dual of the pullback $(\sigma, \tau)^*\mathcal{P}$ — it is a line bundle on U called the *Deligne pairing* of σ and τ, cf. [1].

We know from [14] that there exists a largest open subset $U \subseteq V \subseteq S$ such that the complement of V in S has codimension 2 and such that J admits a Néron model over V. The Néron model is of finite type by [15], so there exists $n > 0$ such that $n\sigma$ and $n\tau$ pass through the identity component N^0 of the Néron model; choose such an n. By [27, Definition II.1.2.7 and Theorem II.3.6] the Poincaré bundle admits a unique rigidified extension to $N^0 \times_S N^0$, which we pull back to V along $(n\sigma, n\tau)$. Since S is regular and the complement of V has codimension 2, this line bundle on V has a unique extension to a line bundle on S. Raising this line bundle to the power $1/n^2$ we obtain a $\mathbb{Q}$-line bundle on S which we call the *admissible extension of the Deligne pairing* (or just *the admissible pairing*), and write as $\langle \sigma, \tau \rangle_a$ — it is independent of the choice of n. This is also known as the "Lear extension," as in the setting of complex geometry (or more generally Hodge theory) it can also be defined by requiring that certain metrics extend continuously outside some codimension 2 subset of the boundary, see for example [3, 11, 22].

Suppose T is another regular scheme, and $f\colon T \to S$ is a morphism. We say f is *non-degenerate* if $f^{-1}U$ is dense in T. Given a non-degenerate morphism $f\colon T \to S$, we have a canonical isomorphism of line bundles on $f^{-1}U$

$$f^*\langle \sigma, \tau \rangle \,|_{f^{-1}U} = \langle f^*\sigma, f^*\tau \rangle \,|_{f^{-1}U} = \langle f^*\sigma, f^*\tau \rangle_a \,|_{f^{-1}U}.$$

Thus the $\mathbb{Q}$-line bundle

$$f^*\langle \sigma, \tau \rangle_a^\vee \otimes \langle f^*\sigma, f^*\tau \rangle_a$$

is canonically trivial over $f^{-1}U$, and so the section "1" over U gives a canonical rational section over T. We define the *height jump* $j(\sigma, \tau, f)$ associated to σ, τ and f to be the corresponding $\mathbb{Q}$-divisor on T.

Lemma 3.3. *Assume $f\colon T \to S$ is a non-degenerate morphism such that $f^{-1}(S\backslash V)$ has codimension at least 2 in T. Then the formation of $\langle D, E\rangle_a$ is compatible with base change along f.*

This is a slight generalization of [1, Proposition 3.4], and the proof is essentially the same.

Proof. Recall that f being non-degenerate implies that T is regular. The formation of $\langle D, E\rangle_a$ is compatible with base change along f if and only if the jump divisor $j(f; D, E)$ is trivial. The triviality of a divisor can be checked on the complement of a codimension 2 subscheme, so we may assume that the image of f is contained in V.

Now the sections mD and nE of the Néron model $N(J_U)$ are contained in the fiberwise connected component of identity $N^0(J_U)$, by assumption. The pullback of $N^0(J_U)$ along f is again semiabelian, and prolongs the pullback of the Jacobian. Moreover, the sections mD and nE pull back to sections of $f^*N^0(J_U)$, and the rigidified extension of the Poincaré bundle pulls back to a rigidified extension of the Poincaré bundle. The result then follows from the uniqueness of semiabelian prolongations, cf. Theorem 1.2 in [9]. □

Corollary 3.4. *Assume that J_U extends into a Néron model over S. Then for all non-degenerate morphisms $f\colon T \to S$, the height jump divisor on T is trivial.*

In Theorem 3.10, we shall see that the sufficient condition of this corollary is also essentially a necessary condition, even if we restrict our test objects to traits.

Remark 3.5. The jump is defined for arbitrary T, but it is generally enough to be able to compute it when T is a trait, since the jump is stable under flat base-change by Theorem 3.3, in particular under localization at generic points of prime divisors in T.

3.4. *Computing the algebraic height jump*

In this section, we summarize some results on the height jump from [1]. In particular, Eq. (3.1) shows how to compute the jump explicitly in certain situations, which will be important for our later results. We will relate the height jump to Green's functions on graphs. For some of this we will use the language of electrical networks, but we only use it as a language and as a guide to intuition; all our proofs are still (intended to be) rigorous. More details of what follows in the remainder of this section can be found in [1, §6 and §7].

3.4.1. *Resistive networks and Green's functions*

For us graphs have finitely many edges and vertices, are allowed loops and multiple edges, and are not directed. A *resistive network* is a graph Γ together with a labeling

$\mu\colon \operatorname{Edges}(\Gamma) \to \mathbb{R}_{\geq 0}$ of the edges by non-negative real numbers, which we can think of as resistances (if the labels are all positive, it is called a *proper* resistive network, otherwise it is *improper*, which is an important distinction in what follows).

The *Laplacian* of a proper resistive network (Γ, μ) is the linear map $L\colon \mathbb{R}^{\operatorname{Vert}\Gamma} \to \mathbb{R}^{\operatorname{Vert}\Gamma}$ which sends a vector $(x_v : v \in \operatorname{Vert}\Gamma)$ to the vector whose component at a vertex u is

$$\sum_{v\in\operatorname{Vert}\Gamma}\sum_{e}\frac{x_u - x_v}{\mu(e)},$$

where the second sum is over all edges with one endpoint at v and the other at u. Note that the properness gives us that the $\mu(e)$ are nonzero.

Remark 3.6. If we think of a vector $X = (x_v : v \in \operatorname{Vert}\Gamma)$ as an assignment of a voltage to every vertex in Γ, then for a directed edge e from u to v, the term $(u-v)/e$ can be thought of as the current flowing along the edge e. The component of the vector LX at a vertex v is then the total current flowing out of the vertex v into the rest of the network.

Definition 3.7. Assume that (Γ, μ) is a proper resistive network with exactly one connected component. Let X, $Y \in \mathbb{R}^{\operatorname{Vert}\Gamma}$. We define the *Green's function* at X, Y to be

$$\operatorname{gr}(\Gamma, \mu; X, Y) = XL^{+}Y,$$

where L^{+} is the Moore–Penrose pseudo inverse (see [29]) of L.

If we fix X and Y and allow μ to vary over $\mathbb{R}_{>0}^{\operatorname{Edges}\Gamma}$ then it is easy to see that the function sending μ to $\operatorname{gr}(\Gamma, \mu; X, Y)$ is continuous. We can also extend the definition of the Green's function to improper networks:

Definition 3.8. Let (Γ, μ) be a resistive network with exactly one connected component and let X, $Y \in \mathbb{R}^{\operatorname{Vert}\Gamma}$. Let Γ' be the graph obtained from Γ by contracting every edge with resistance 0, and write μ', X' and Y' for the corresponding resistance function and vertex weightings on Γ'. Then we define the *Green's function* at X, Y to be

$$\operatorname{gr}(\Gamma, \mu; X, Y) = X'L'^{+}Y',$$

where L'^{+} is the Moore–Penrose pseudo inverse of the Laplacian L' on Γ'.

Now for fixed X and Y we get a function $\mathbb{R}_{\geq 0}^{\operatorname{Edges}\Gamma} \to \mathbb{R}$ sending μ to $\operatorname{gr}(\Gamma, \mu; X, Y)$. It is no longer obvious that this function should be continuous, but it is in fact continuous, see [1, Proposition 6.6].

3.4.2. *Some terminology for families of nodal curves*

In what follows we will need some precise descriptions of the local structure of families of nodal curves, which we will collect here. Let S be a scheme, and C/S a nodal curve. If $s \in S$ is a point then a *non-degenerate trait through* s is a morphism $f\colon T \to S$ from the spectrum T of a discrete valuation ring, sending the closed point of T to s, and such that f^*C is smooth over the generic point of T.

We say a nodal curve C/S is *quasisplit* if the morphism $\mathrm{Sing}(C/S) \to S$ is an immersion Zariski-locally on the source (for example, a disjoint union of closed immersions), and if for every field-valued fiber C_k of C/S, every irreducible component of C_k is geometrically irreducible.

Suppose we are given C/S a quasisplit nodal curve and $s \in S$ a point. Then we write Γ_s for the *dual graph* of C/S in the sense of [23, Definition 10.3.17] — this makes sense because C/S is quasisplit and so all the singular points are rational points, and all the irreducible components are geometrically irreducible. Assume that C/S is smooth over a schematically dense open of S. If we are also given a non-smooth point c in the fiber over s, then there exists an element $\alpha \in \mathcal{O}_{S,s}$ and an isomorphism of completed étale local rings (after choosing compatible geometric points lying over c and s)

$$\widehat{\mathcal{O}^{et}}_{C,c} \xrightarrow{\sim} \frac{\widehat{\mathcal{O}^{et}}_{S,s}[[x,y]]}{(xy-\alpha)}.$$

This element α is not unique, but the ideal it generates in $\mathcal{O}_{S,s}$ is unique. We label the edge of the graph Γ_s corresponding to c with the ideal $\alpha\mathcal{O}_{S,s}$. In this way the edges of Γ_s can be labeled by principal ideals of $\mathcal{O}_{S,s}$.

If η is another point of S with $s \in \overline{\{\eta\}}$ then we get a *specialization map*

$$\mathrm{sp}\colon \Gamma_s \to \Gamma_\eta$$

on the dual graphs, which contracts exactly those edges in Γ_s whose labels generate the unit ideal in $\mathcal{O}_{S,\eta}$. If an edge e of Γ_s has label ℓ, then the label on the corresponding edge of Γ_η is given by $\ell\mathcal{O}_{S,\eta}$.

3.4.3. *The jump in terms of Green's functions*

Now we will apply our discussion of Green's functions above to compute the height jump. Let S be a regular noetherian scheme, C/S a generically-smooth quasisplit nodal curve, and $U \subseteq S$ the largest open over which C is smooth. Let $s \in S$ be a point, and write (Γ, ℓ) for the labeled graph of C/S at s, where the labels take values in the monoid of principal ideals of $\mathcal{O}_{S,s}$, cf. Sec. 3.4.2. Let $Z_1, \ldots, Z_r$ be prime divisors in S forming the boundary of U, and for each i let z_i be a local equation of Z_i in the local ring $\mathcal{O}_{S,s}$. Now if e is an edge of Γ, the label $\ell(e)$ is generated by $z_1^{a_1} \cdots z_r^{a_r}$ for some unique $(a_1, \ldots, a_r) \in \mathbb{Z}^r_{\geq 0}$. Given $I \subseteq \{1, \ldots, r\}$ we define a new labeled graph (Γ, ℓ_I) by requiring that, if $\ell(e) = (z_1^{a_1} \cdots z_r^{a_r})$, then $\ell_I(e) = (\prod_{i \in I} z_i^{a_i})$. In particular, $\ell_{\{1,\ldots,r\}} = \ell$. We write ℓ_i for $\ell_{\{i\}}$.

Given a horizontal divisor D on C/S supported on sections through the smooth locus and of relative degree 0, we can define an associated combinatorial divisor $\mathcal{D}$

in $\mathbb{R}^{\operatorname{Vert}\Gamma}$ by setting

$$\mathcal{D}(y) = \deg D|_Y$$

for each vertex y of Γ with corresponding irreducible component Y of C_s.

Suppose now that we have a non-degenerate trait $f\colon T \to S$ through s. Then we have a pullback map $f^{\#}\colon \mathcal{O}_{S,s} \to \mathcal{O}_T(T)$. We obtain a labeling of the edges of Γ with values in $\mathbb{Z}_{\geq 0}$ by sending an edge e to $\operatorname{ord}_T f^{\#}\ell(e)$, and we write $\operatorname{ord}_T f^{\#}\ell$ for this labeling. We make corresponding definitions for the ℓ_I.

Theorem 3.9 ([1], Theorem 7.8). *Suppose we are given two horizontal divisors D and E on C/S of relative degree* 0 *and supported on sections through the smooth locus of C/S, with associated combinatorial divisors $\mathcal{D}$ and $\mathcal{E}$. Write σ (respectively, τ) for the section of the Jacobian J of C_U induced by D (respectively, E). Then the height jump associated to σ, τ and f is given by the formula $j(\sigma, \tau, f) = j \cdot [t]$ where t is the closed point of T, and the rational number j is given by*

$$j = \operatorname{gr}(\Gamma, \operatorname{ord}_T f^{\#}\ell; \mathcal{D}, \mathcal{E}) - \sum_{i=1}^{r} \operatorname{gr}(\Gamma, \operatorname{ord}_T f^{\#}\ell_i; \mathcal{D}, \mathcal{E}). \qquad (3.1)$$

3.5. *Vanishing of the jump is equivalent to the existence of a Néron model*

Let S be an integral, noetherian, regular scheme, and let $C \to S$ be a generically smooth nodal curve. Let $U \subset S$ be the largest open subscheme of S over which C is smooth. We have seen that if the Jacobian of the generic fiber of $C \to S$ has a Néron model over S, then for each non-degenerate morphism $f\colon T \to S$ the height jump divisor is trivial. In this section, we give a partial converse to this result, using Eq. (3.1). This result is not needed to prove our main Theorem 3.17, but is intended to demonstrate the close connection between the vanishing of the jump and the existence of Néron models.

We recall the following definition of alignment of labeled graphs and curves from [14]:

(1) given a graph Γ with an edge-labeling by a (multiplicatively written) monoid, we say Γ is *aligned* if for all cycles γ in Γ, and for all pairs of edges e_1, e_2 on γ, there exist positive integers n_1, n_2 such that

$$(\operatorname{label}(e_1))^{n_1} = (\operatorname{label}(e_2))^{n_2};$$

(2) given a geometric point $s \in S$, we say $C \to S$ is *aligned at* s if the labeled reduction graph Γ_s of C above s is aligned (where the monoid is the monoid of principal ideals in $\mathcal{O}_{S,s}$);
(3) we say $C \to S$ *is aligned* if $C \to S$ is aligned at s for all geometric points s in S.

Proposition 3.10. *Assume C admits a regular nodal model over S. The following conditions are equivalent*:

(1) *The Jacobian of $C_U \to U$ has a Néron model over S;*

(2) *For all nodal models* $\tilde{C} \to S$ *of* C, *for all non-degenerate morphisms* $f: T \to S$ *with* T *a trait and for all relative degree zero divisors* D, E *with support contained in the smooth locus* $\mathrm{Sm}(\tilde{C}/S)$ *of* $\tilde{C} \to S$, *the height jump divisor* $j(f; D, E)$ *is trivial*;
(3) *There exists a nodal model* $\tilde{C} \to S$ *of* C *with the same property as in item* (2);
(4) *There exists an aligned regular model of* $C \to S$;
(5) *Every nodal model of* C *over* S *is aligned.*

We note that following [8, Proposition 3.6], a sufficient condition for the existence of a regular nodal model of C is that $C \to S$ be split nodal (in the sense of [8]) and smooth over the complement of a strict normal crossings divisor in S.

Proof. The equivalence of (1), (4) and (5) follows from the main results of [14]. It is clear that (2) implies (3). The implication (1) $\Rightarrow$ (2) follows from Theorem 3.3. We will now show (3) implies (4). In fact we will show the contrapositive, that ($\neg$4) $\Rightarrow$ ($\neg$3). Assume ($\neg$4), and consider any regular nodal model of C over S. To simplify notation, we will assume this model is C itself. By ($\neg$4) we have that $C \to S$ is not aligned. Both questions are étale-local on the base, and so we may assume that S is the spectrum of a strictly henselian local ring, with closed point s, and that $C \to S$ is not aligned at s. Note that $C \to S$ is quasisplit since S is strictly henselian. Let $z_1, \dots, z_r$ be distinct height 1 prime ideals in $O_{S,s}$ such that C is smooth over $U := S\backslash V(\prod_{i=1}^r z_i)$. Write Γ_s for the labeled graph of C over s. Since C is regular, every label on Γ_s is equal to one of the z_i; non-trivial powers and products cannot occur.

Let γ be a cycle in Γ_s which contains two distinct labels — such a cycle exists exactly because $C \to S$ is not aligned at s. After possibly reordering the z_i, we may and do assume that γ contains two adjacent edges e_1, e_2 with labels z_1, z_2, respectively. Let p, $q \in C(S)$ be sections through the smooth loci of the two irreducible components of C_s corresponding to the (distinct) endpoints of e_1. Let $D = E = p - q$, degree-zero divisors on $C \to S$ supported on the smooth locus. Applying Eq. (3.1), we find that for any non-degenerate test curve $f: T \to S$ the height jump is given by

$$g(\Gamma_s, \mathrm{ord}_t\, f^{\#}\ell_s; \mathcal{D}, \mathcal{E}) - \sum_{i=1}^{r} g(\Gamma_s, \mathrm{ord}_t\, f^{\#}\ell_{s,i}; \mathcal{D}, \mathcal{E}).$$

We find that all the terms $g(\Gamma_s, \mathrm{ord}_t\, f^{\#}\ell_{s,i}; \mathcal{D}, \mathcal{E})$ vanish for $i \neq 1$. Indeed, if $i \neq 1$ then all the edges e with $\ell_s(e) = (z_1)$ have $\ell_{s,i}(e) = (1)$, in particular $\ell_{s,i}(e_1) = (1)$, so $\mathrm{ord}_t\, f^{\#}\ell_{s,i}(e_1) = 0$. It is then immediate from Theorem 3.8 that $g(\Gamma_s, \mathrm{ord}_t\, f^{\#}\ell_{s,i}; \mathcal{D}, \mathcal{E}) = 0$, since the edge e_1 is contracted in the computation of the Green's function, and so the vertex weightings are both zero.

To prove ($\neg$4), it suffices to show that the equality

$$g(\Gamma_s, \mathrm{ord}_t\, f^{\#}\ell_s; \mathcal{D}, \mathcal{E}) = g(\Gamma_s, \mathrm{ord}_t\, f^{\#}\ell_{s,1}; \mathcal{D}, \mathcal{E}) \tag{3.2}$$

fails for some non-degenerate test curve $f: T \to S$. Choose any non-degenerate test curve $f: T \to S$ such that the image of the closed point t of T is the closed point s of S. Then we find that every label of $(\Gamma_s, \mathrm{ord}_t\, f^{\#}\ell_s)$ is strictly positive.

Equation (3.2) is equivalent to the statement that the effective resistance between p and q on the graph Γ_s with edges given resistance equal to the $\mathrm{ord}_t\, f^{\#}\ell_s$ is equal to the resistance between p and q on the same graph but with all the edges *not* labeled by z_1 being contracted (equivalently, having their resistances set to zero). Then by [1, Corollary A.5], in the graph with "non-contracted" edges, no current flows through any edge not labeled by z_1 when 1 unit of current flows in at p and out at q. But since there is a path (extendable to a spanning tree) from p to q that either starts or ends with e_2, [1, Eq. (A.2)] tells us that the current along the z_2-labeled edge e_2 must be nonzero, so we have reached a contradiction. □

Remark 3.11. We point out that the regularity assumption in part (4) of the theorem is essential: a nodal curve over $S = \operatorname{Spec} \mathbb{C}[[u, v]]$ with labeled graph over the closed point a 1-gon with label (uv) — which is then necessarily not regular — is aligned, but its Jacobian does not admit a Néron model over S.

3.6. *Bounding the height jump for generically aligned curves*

Recall from Theorem 3.10 that the height jump vanishes whenever the Jacobian of a family of curves admits a Néron model. The aim of this section is to show that if the base-change of our family to $\mathbb{Q}$ admits a Néron model, then the height jump is bounded.

In this section, we work over an arbitrary global field K, since our proofs naturally work in that generality. We write Λ for the spectrum of the ring of integers of K if K is a number field, or for a smooth proper geometrically integral curve over a finite field with field of rational functions K if K is a global function field. If L/K is a finite extension then we define Λ_L similarly.

Theorem 3.12. *Let S be a regular scheme proper and flat over Λ, and let $C \to S$ be a nodal curve smooth over a dense open subscheme $U \hookrightarrow S$ and with C regular. Write J for the Jacobian of C_U/U. Suppose we are given integers $d_1, \ldots, d_n, e_1, \ldots, e_n$ with $\sum_i d_i = \sum_i e_i = 0$ and sections $\sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_n \in C^{sm}(S)$, and set $\sigma = [\sum_i d_i(\sigma_i)_U]$, $\tau = [\sum_i e_i(\tau_i)_U] \in J(U)$.*

Suppose also that J_K has a Néron model over S_K. Then there exists a constant B such that for all finite extensions L/K and for all $x \in U(L)$ we have

$$\left| \frac{\widehat{\deg} j(\sigma, \tau, \bar{x})}{[L : K]} \right| \leq B,$$

where $j(\sigma, \tau, \bar{x})$ denotes the height jump associated to σ, τ and $\bar{x}\colon \Lambda_L \to S$.

Proof. We proceed in several steps.

Step 1: Translate the problem into a statement which is local on Λ.

Recall from [14] that the existence of a Néron model of the Jacobian is equivalent to alignment of the family of curves. Using that C_K/S_K is aligned and the finite presentation of C/S, we see that there is a nonempty Zariski open $\Lambda^0 \hookrightarrow \Lambda$ such that the Jacobian of $C_{U_{\Lambda^0}}/U_{\Lambda^0}$ admits a Néron model over S_{Λ^0}. Then by Theorem 3.4 we deduce that there is a finite set of primes of Λ such that for every (finite flat quasi-)section, the height jump vanishes outside that finite set. Because the claim is stable under unramified base-change on Λ we may replace Λ by its strict henselization and completion $\Lambda_\mathfrak{p}$ at some prime $\mathfrak{p}$. Write $K_\mathfrak{p}$ for the field of rational functions on $\Lambda_\mathfrak{p}$.

Step 2: Translate the problem into a statement which is local on S.

Let $S' \to S$ be an étale cover, and let $L/K_\mathfrak{p}$ be a finite extension. Write Λ_L for the normalization of $\Lambda_\mathfrak{p}$ in L. Then Λ_L is finite over $\Lambda_\mathfrak{p}$ of degree $[L : K]$, and is strictly henselian. By properness of S/Λ the restriction map $S(\Lambda_L) \to S(L)$ is a bijection, and since Λ_L is strictly henselian we know that $S'(\Lambda_L) \to S(\Lambda_L)$ is surjective. By quasi-compactness of S, it therefore suffices to bound the height jump on every connected component of some étale cover of S.

Step 3: Make some reductions using the étale local nature of the statement.

To keep the notation concise we will write Λ in place of $\Lambda_\mathfrak{p}$, and (using step 2 and [13, Lemma 6.3]) we will replace S by a flat finite-type integral Λ-scheme such that for some $s \in S$ (which we will refer to as a "controlling point"), the induced specialization map $\Gamma_s \to \Gamma_t$ is surjective for all points $t \in S$.

Since S is noetherian, it suffices to consider quasi-sections in $S(\Lambda_L)$ (with L/K finite) such that the induced graph over the closed point of Λ is the same as the graph of the controlling point s. If $Z_1, \ldots, Z_n$ are prime Weil divisors on S such that every label on Γ can be written in terms of the Z_i (and the Z_i are minimal with respect to this) then we can restrict further to quasi-sections $x \in S(\Lambda_L)$ such that for all i, the divisor x^*Z_i on Λ is non-trivial. Write Σ for the set of such quasi-sections. If $x\colon \Lambda_L \to S$ is a quasi-section, we write $L = L_x$ and $\Lambda_x = \Lambda_L$.

Step 4: Conclude the argument by an appeal to a theorem about resistive networks.

We are now in the situation of Sec. 3.4, and we adopt the notation of that section. Further, we know that C_K/S_K is aligned, where we write K for the generic point of Λ. Some of the Z_i may be trivial after pullback to S_K, and some may not. Re-ordering we assume that $Z_1, \ldots, Z_r$ are not trivial on S_K and $Z_{r+1}, \ldots, Z_n$ are. As such, for each $r+1 \le i \le n$ the function

$$\Sigma \to \mathbb{R}_{>0}; \quad x \mapsto \frac{\operatorname{ord}_{\Lambda_x} x^*Z_i}{[L_x : K]}$$

is bounded.

Define divisors on C/S by $D = \sum_i d_i\sigma_i$, $E = \sum_i e_i\tau_i$, so $\sigma = [D_U]$ and $\tau = [E_U]$. Write $\mathcal{D}$, $\mathcal{E}$ for the combinatorial divisors associated to D and E. Then applying

[1, Proposition 6.8] we find that

$$\left| \operatorname{gr}\left(\Gamma, \frac{\operatorname{ord}_{\Lambda_x} x^{\#}\ell}{[L_x : K]}; \mathcal{D}, \mathcal{E}\right) - \operatorname{gr}\left(\Gamma, \frac{\operatorname{ord}_{\Lambda_x} x^{\#}\ell_{\{1,\ldots,r\}}}{[L_x : K]}; \mathcal{D}, \mathcal{E}\right) - \sum_{i=1}^{r} \operatorname{gr}\left(\Gamma, \frac{\operatorname{ord}_{\Lambda_x} x^{\#}\ell_i}{[L_x : K]}; \mathcal{D}, \mathcal{E}\right) \right|$$

is bounded independent of $x \in \Sigma$. To complete the proof it suffices to bound the absolute value of

$$B(x) := \operatorname{gr}\left(\Gamma, \frac{\operatorname{ord}_{\Lambda_x} x^{\#}\ell_{\{1,\ldots,r\}}}{[L_x : K]}; \mathcal{D}, \mathcal{E}\right) - \sum_{i=1}^{r} \operatorname{gr}\left(\Gamma, \frac{\operatorname{ord}_{\Lambda_x} x^{\#}\ell_i}{[L_x : K]}; \mathcal{D}, \mathcal{E}\right) \tag{3.3}$$

for which we will use that C_K/S_K is aligned. In fact, we will show that $B(x) = 0$ for all $x \in \Sigma$.

Let Γ_K be the graph obtained from Γ by contracting every edge whose label does not contain at least one of $Z_1, \ldots, Z_r$ with non-zero (i.e. positive) coefficient. It is clear that all terms in Eq. (3.3) can be computed on Γ_K just as well as on Γ. Moreover, Γ_K is the graph of C over a controlling point of S_K, so we can see easily what alignment of C_K/S_K means on Γ_K; namely, that all labels on edges in 2-vertex-connected components are multiplicatively related over S_K. Define a labeling ℓ' on the edges of Γ_K by composing ℓ with the map sending $\sum_{i=1}^{n} a_i Z_i$ to $\sum_{i=1}^{r} a_i Z_i$.

We will show $B(x) = 0$ in three steps:

Step 4.1: The case where Γ_K is 2-vertex connected.

By alignment there exists a divisor $\delta = \sum_{i=1}^{r} a_i Z_i$ and for each edge e of Γ_K a constant $\lambda_e \in \mathbb{Q}_{\geq 0}$ such that $\ell'(e) = \lambda_e \delta$. Let $\rho\colon \mathbb{R}_{\geq 0}^{\operatorname{Edges}\Gamma_K} \to \mathbb{R}_{\geq 0}$ be the map sending a labeling μ to the Green's function $\operatorname{gr}(\Gamma_K, \mu; \mathcal{D}, \mathcal{E})$. This function is homogenous of degree 1 by [1, Proposition 6.7(a)]. For $1 \leq i \leq r$ define

$$g_i\colon \Sigma \to \mathbb{R}_{\geq 0}^{\operatorname{Edges}\Gamma_K}; \quad x \mapsto \frac{\operatorname{ord}_{\Lambda_x} x^{\#}\ell_i}{[L_x : K]}$$

and define $g = \sum_{i=1}^{r} g_i$. Then by definition we have

$$\operatorname{gr}\left(\Gamma_K, \frac{\operatorname{ord}_{\Lambda_x} x^{\#}\ell_{\{1,\ldots,r\}}}{[L_x : K]}; \mathcal{D}, \mathcal{E}\right) = \rho \circ g$$

and

$$\operatorname{gr}\left(\Gamma_K, \frac{\operatorname{ord}_{\Lambda_x} x^{\#}\ell_i}{[L_x : K]}; \mathcal{D}, \mathcal{E}\right) = \rho \circ g_i$$

for $1 \leq i \leq r$. Define

$$f\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}^{\operatorname{Edges}\Gamma_K}; \quad t \mapsto (\lambda_e t)_{e \in \operatorname{Edges}\Gamma_K}$$

and define $\rho_0 := \rho \circ f \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, which is a "linear" map since ρ is homogeneous of weight 1 and the source has dimension 1. Setting

$$h_i \colon \Sigma \to \mathbb{R}_{\geq 0}; \quad x \mapsto a_i \frac{\mathrm{ord}_{\Lambda_x} x^{\#} Z_i}{[L_x : K]}$$

and $h = \sum_{i=1}^{r} h_i$, we find $g_i = f \circ h_i$ and $g = f \circ h$. Thus $\rho \circ g = \rho_0 \circ h$, and $\rho \circ g_i = \rho_0 \circ h_i$ for each i. Then

$$\begin{aligned} B &= \rho_0 \circ h - \sum_{i=1}^{r} \rho_0 \circ h_i \\ &= \rho_0 \circ \left(h - \sum_{i=1}^{r} h_i \right) \quad \text{(by linearity of } \rho_0\text{)} \\ &= \rho_0 \circ (\text{zero map}) = (\text{zero map}). \end{aligned}$$

Step 4.2: The case where $\mathcal{D} = \mathcal{E} = u - v$ for some vertices u and v.

We may assume Γ_K has no self-loops, since they do not contribute to the Green's functions. Write Γ' for the graph obtained from Γ_K by contracting to a point every 2-vertex-connected component which is not a single edge (so Γ' is a tree). Thus there is a unique path γ from the image of u to the image of v in Γ'.

Let $H_1, \ldots, H_n$ be the vertices appearing along γ, so the image of u lies in H_1 and the image of u lies in H_n. Let $b_1, \ldots, b_{n-1}$ be the edges along γ. Each H_i corresponds to either a vertex or a 2-vertex-connected subgraph of Γ_K, which we will denote by the same symbol H_i. For $1 \leq i \leq n-1$ let $t_i \in H_i$ be the vertex which the edge of γ out of H_i lifts to, and for $2 \leq i \leq n$ let $s_i \in H_i$ be the vertex of H_i where the edge of γ into H_i lifts to. The result might look something like Fig. 1. Then by additivity of Green's functions in trees we find that

$$\begin{aligned} B(x) = B_x(H_1; u, t_1) &+ \sum_{i=2}^{n-1} B_x(H_i; s_i, t_i) + \sum_{i=1}^{n-1} B_x(b_i; t_i, s_i) \\ &+ B_x(H_n; t_n, v), \end{aligned} \tag{3.4}$$

where for a subgraph $G \subseteq \Gamma_K$ and two vertices a and b we write

$$\begin{aligned} B_x(G; a, b) := \mathrm{gr}\left(G, \left. \frac{\mathrm{ord}_{\Lambda_x} x^{\#} \ell_{\{1,\ldots,r\}}}{[L_x : K]} \right|_G ; a - b, a - b \right) \\ - \sum_{i=1}^{r} \mathrm{gr}\left(G, \left. \frac{\mathrm{ord}_{\Lambda_x} x^{\#} \ell_i}{[L_x : K]} \right|_G ; a - b, a - b \right). \end{aligned}$$

But all the terms on the right-hand side of Eq. (3.4) vanish by step 4.1, so we are done.

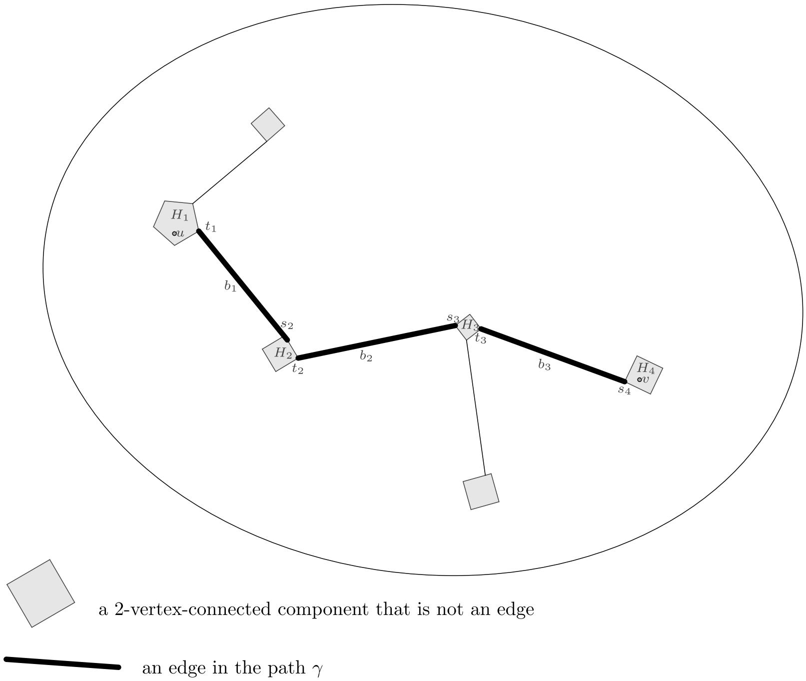

Fig. 1. An example of a graph Γ_K when $n = 4$.

Step 4.3: The general case.

To deduce the general case from the case where $\mathcal{D} = \mathcal{E} = u - v$, we fix x and consider B as a bilinear function on the space Div of combinatorial divisors of degree zero on Γ_K. Then B is positive semi-definite by [1, Corollary 6.7(b)] and it vanishes on a basis of Div by Step 4.2, so by the Cauchy–Schwarz inequality we find that B is zero. □

3.7. *Heights and rational maps*

In this section, we continue to work over a global field K. Let S/K be a projective scheme, and $\mathcal{L}$ on S a line bundle. To this data we can attach a Weil height

$$\mathrm{h}_{\mathcal{L}} \colon S(\bar{K}) \to \mathbb{R} \tag{3.5}$$

which is unique up to addition of a bounded function on $S(\bar{K})$. For example, this can be done via Arakelov theory, or by writing $\mathcal{L}$ as a difference of very ample line bundles and applying the usual height machinery on a variety embedded in projective space. In either case a number of choices must be made, but the resulting heights all differ by bounded amounts. We say $\mathrm{h}_{\mathcal{L}}$ (or $\mathcal{L}$) is *weakly non-degenerate* if the sets of points of bounded height with residue fields of bounded degree are not Zariski dense in S. For example, Northcott's theorem tells us that if $\mathcal{L}$ is ample then sets of points of bounded height and degree for $\mathrm{h}_{\mathcal{L}}$ are finite, so certainly not Zariski dense if S has positive dimension. The main goal of this section is to show that if $\mathrm{h}^0(S, \mathcal{L}) \geq 2$, then $\mathrm{h}_{\mathcal{L}}$ is weakly non-degenerate. We will start by developing a bit of theory about heights associated to rational maps. Since we are ultimately interested in weak non-degeneracy, we will work up to $O(1)$ everywhere.

In the above setup, let V be a non-zero sub-K-vector space of $\mathrm{H}^0(S, \mathcal{L})$. In the usual way we obtain a rational map

$$f_V : S \dashrightarrow \mathbb{P}(V)$$

(we think of $\mathbb{P}(V)$ as the space of rank-1 quotients of V, to avoid writing duals everywhere). Let U be an open subset of its domain of definition. By composing the map f_V with a standard height on $\mathbb{P}(V)$ (say after choosing a basis of V) we get a height

$$\mathrm{h}_V : U(\bar{K}) \to \mathbb{R}.$$

For example, the reader will easily verify that if $V \subseteq V' \subseteq \mathrm{H}^0(S, \mathcal{L})$ then $f_{V'}$ is defined on U and we have $\mathrm{h}_V \leq \mathrm{h}_{V'}$ on $U(\bar{K})$ (up to addition of a bounded function on $U(\bar{K})$). It is slightly harder to compare $\mathrm{h}_{\mathcal{L}}$ with h_V, but we have

Lemma 3.13. *On $U(\bar{K})$ we have $\mathrm{h}_V \leq \mathrm{h}_{\mathcal{L}}$, up to addition of a bounded function on $U(\bar{K})$.*

Note that $\mathrm{h}_{\mathcal{L}}$ is not in general equal to h_V. In the proof we will say that two functions on $U(\bar{K})$ are "equal up to $O(1)$" if their difference is bounded.

Proof. Let $\mathcal{S}$ be a proper flat reduced model of S over $\mathcal{O}_K$ such that $\mathcal{L}$ extends to a line bundle on $\mathcal{S}$; choose such an extension and denote it $\mathscr{L}$. Let $\mathcal{L}_V$ be the sub-$\mathcal{O}_S$-module of $\mathcal{L}$ generated by V, and let $\mathscr{L}_V$ be an extension to a coherent submodule of $\mathscr{L}$. Let

$$I_V := \mathscr{L}_V \otimes_{\mathcal{O}_{\mathcal{S}}} \mathscr{L}^\vee \to \mathcal{O}_{\mathcal{S}},$$

a coherent sheaf of ideals in $\mathcal{O}_{\mathcal{S}}$. Let $\pi : \tilde{\mathcal{S}} \to \mathcal{S}$ be the blowup of $\mathcal{S}$ in I_V.

Over each Archimedean place of K we choose a continuous metric on $\mathcal{O}_{\mathcal{S}}$ and on $\mathscr{L}$. The metric on $\mathcal{O}_{\mathcal{S}}$ induces a norm on each section of I_V. The choices of these metrics are not important, since any two will have bounded difference by properness of S, but it is important to make a choice so that the arithmetic degrees below make sense.

For a point $p \in U(K)$, we write $\bar{p}$ for the corresponding $\mathcal{O}_K$-point of $\mathcal{S}$, and $\tilde{p}$ for the corresponding $\mathcal{O}_K$-point of $\tilde{\mathcal{S}}$. Write $f = f_V : U \to \mathbb{P}(V)$, and define $\tilde{f} : \tilde{\mathcal{S}}_K \to \mathbb{P}(V)$ to be the natural map extending f. Then we have

$$\begin{aligned} h(f(p)) &= h(\tilde{f}(p)) \\ &\overset{(1)}{=} \widehat{\deg}\left(\tilde{p}^*(\pi^{-1} I_v \otimes_{\mathcal{O}_{\mathcal{S}}} \mathscr{L})\right) \end{aligned}$$

$$\stackrel{(2)}{=} \widehat{\deg}\,(\bar{p}^*(I_v \otimes_{\mathcal{O}_{\mathcal{S}}} \mathscr{L}))$$

$$\stackrel{(3)}{\leq} \widehat{\deg}\,(\bar{p}^*\mathscr{L}).$$

See for example [34, §1.2] or [20, IV, §3] for the definition of the arithmetic degree $\widehat{\deg}$; note that in the function field case it is just the usual degree of a line bundle on a proper smooth curve. We now justify the above (in)equalities:

(1) holds up to $O(1)$; it follows from the fact that on $\tilde{\mathcal{S}}_K$ we have $\left(\pi^{-1}I_v \otimes \pi^*\mathscr{L}\right)|_{\tilde{\mathcal{S}}_K} = \tilde{f}^*\mathcal{O}(1)$, and $h(\tilde{f}(p))$ is equal (up to $O(1)$) to the arithmetic degree of $\tilde{p}^*$ of any invertible model of $\tilde{f}^*\mathcal{O}(1)$;
(2) is where the content lies; because p is in U we know that I_V pulls back along $\bar{p}$ to a submodule of the line bundle $\bar{p}^*\mathscr{L}$ which is invertible on the generic point of $\mathcal{O}_K$, and hence is itself a line bundle. Then from the proof of [23, 8.1.5] we find that $\bar{p}^*I_V = \tilde{p}^*\pi^{-1}I_V$, from which the result follows;
(3) is because $I_V \subseteq \mathcal{O}_{\mathcal{S}}$, so $\bar{p}^*I_v \otimes_{\mathcal{O}_{\mathcal{S}}} \mathscr{L} \subseteq \bar{p}^*\mathscr{L}$.

Now up to $O(1)$ we have that $h_V(p) = h(f(p))$ and $h_{\mathcal{L}}(p) = \widehat{\deg}\,(\bar{p}^*\mathscr{L})$ and we are done. □

For the reader less comfortable with Arakelov theory, we provide an alternative proof of this lemma. By the remark just above the lemma we reduce to the case where $V = H^0(S, \mathcal{L})$. Write $\mathcal{L}_V$ for the coherent subsheaf of $\mathcal{L}$ generated by the elements of V, then tensoring the inclusion $\mathcal{L}_V \to \mathcal{L}$ with $\mathcal{L}^\vee$ yields an injection $I_V := \mathcal{L}_V \otimes \mathcal{L}^\vee \to \mathcal{O}_S$. Let $\pi\colon \tilde{S} \to S$ be the blowup of S along the ideal sheaf I_V and write $\pi^{-1}I_V$ for the inverse-image ideal sheaf, then $\mathcal{F} := \pi^{-1}I_v \otimes \pi^*\mathcal{L}$ is the subsheaf of $\pi^*\mathcal{L}$ generated by the pullbacks of the global sections of $\mathcal{L}$, and is an invertible sheaf by construction. Since $\mathcal{F}$ is invertible and globally generated it induces a morphism $\tilde{S} \to \mathbb{P}(V)$ (noting $H^0(\tilde{S}, \mathcal{F}) = V$), and for points in U the height induced by this map evidently coincides with h_V. So on $\tilde{S}$ we have an inclusion of line bundles $\mathcal{F} \subseteq \pi^*\mathcal{L}$, and (for points in U), $h_V = h_{\mathcal{F}}$ and $h_{\mathcal{L}} = h_{\pi^*\mathcal{L}}$. The inequality $h_{\mathcal{F}} \leq h_{\pi^*\mathcal{L}}$ is clear e.g., from the height machine.

Corollary 3.14. *Let S/K be a connected projective scheme, and $\mathcal{L}$ on S a line bundle with $h^0(S, \mathcal{L}^{\otimes n}) \geq 2$ for some $n > 0$. Then $\mathcal{L}$ is weakly non-degenerate.*

Proof. We may assume $n = 1$. In the above notation, let $V = \mathrm{H}^0(S, \mathcal{L})$. Then we can find a dominant rational map $f_V\colon S \dashrightarrow \mathbb{P}^1$ which is defined on some dense open $U \subseteq S$. From the Northcott property for $\mathbb{P}^1$ we see that h_V is weakly non-degenerate, and the result then follows from Theorem 3.13. □

3.8. *Proof of the second main Theorem* 3.17

Remark 3.15. In the proof of Theorem 3.17 we want to talk about two metrics on a line bundle having "bounded difference." If X is a finite-type scheme over $\mathbb{C}$

and L is a line bundle on X with metrics $\|-\|_1$ and $\|-\|_2$, we say $\|-\|_1$ and $\|-\|_2$ have *bounded difference* if there exists an open Zariski cover U_i of X and generating sections $\ell_i \in L(U_i)$ such that the function

$$|\|\ell_i\|_1 - \|\ell_i\|_2|$$

is bounded on U_i for every i.

Again, K is a global field and Λ its scheme of integers. Now let $\mathcal{X}$ be a proper scheme over Λ and $\mathcal{L}$ a line bundle on $\mathcal{X}$. Let $X = \mathcal{X}_\mathbb{C}$ and let $L = \mathcal{L}_\mathbb{C}$. To any metric $\|-\|$ on L we associate as usual a height function

$$\mathrm{h}_{\mathcal{L},\|-\|} \colon \mathcal{X}(\bar{K}) \to \mathbb{R}.$$

If $\|-\|_1$ and $\|-\|_2$ are two metrics on L having bounded difference in the above sense, then one checks without difficulty that the functions $\mathrm{h}_{\mathcal{L},\|-\|_1}$ and $\mathrm{h}_{\mathcal{L},\|-\|_2}$ have bounded difference.

Definition 3.16. Let S/K be a smooth projective connected scheme of dimension d. We say a line bundle L on S is *ample-positive* if for every ample H on S, we have that $L \cdot H^{d-1} > 0$.

Theorem 3.17. *Let $\mathcal{S}$ be a regular scheme, projective and flat over Λ. Let $C/\mathcal{S}$ be a nodal curve, smooth over a dense open of $\mathcal{S}$ and with $C_{\mathcal{S}_K}$ regular. Assume*:

(1) *$C_{\mathcal{S}_K}/\mathcal{S}_K$ is aligned*;
(2) *for every ample-positive L on $\mathcal{S}_K$ there exists $n > 0$ such that $h^0(\mathcal{S}_K, L^{\otimes n}) \geq 2$.*

Let $S \subseteq \mathcal{S}_K$ be any open over which C is smooth and write J for the Jacobian of C_S/S. Let $\sigma \in J(S)$ be a section of infinite order corresponding to a divisor supported on some sections of $C/\mathcal{S}_K$. Then given any $d \in \mathbb{Z}_{\geq 1}$ there exist $\epsilon \in \mathbb{R}_{>0}$ such that the set

$$\mathbf{T}_\epsilon(d) = \{p \in S(\bar{K}) \,|\, [\kappa(p) : K] \leq d \text{ and } \hat{\mathrm{h}}(\sigma(p)) \leq \epsilon\}.$$

is not Zariski dense in S.

Examples of varieties for which the second condition of the theorem holds include curves, and varieties X with $\dim_\mathbb{Q} \mathrm{Pic}(X) \otimes_\mathbb{Z} \mathbb{Q} = 1$.

Proof.

Step 1: Preliminary reductions.

We may assume S is integral; write L for its field of rational functions. Suppose first that the section σ lies in the image of the L/K trace of J_K (cf. [6]). Then after shrinking S we may assume there is an abelian variety B/K and a map $B_L \to J_L$ such that σ is the base-change to L of some section $\tau \in B(K)$. The section τ cannot have finite order since anything which kills τ also kills σ, and if τ is of infinite order then the result is clear since the set of points of sufficiently small height is empty. We may thus assume that σ does *not* lie in the image of the L/K trace of J_K.

We have already assumed that σ corresponds to a divisor supported on some sections of C/S but we may further assume that these sections extend $C/\mathcal{S}$ and (for example by looking at case I in the proof of Theorem 2.4 of [17]) are contained in the smooth locus of $C/\mathcal{S}$. This can be achieved by modifications which do not affect the fiber over K, since the sections already necessarily go through the smooth locus over K by our assumption that $C_{\mathcal{S}_K}$ be regular.

Step 2: Checking the admissible pairing has enough global sections.

Define a line bundle on $\mathcal{S}$ by $\mathcal{L} = \langle\sigma, \sigma\rangle_a$, the admissible pairing as defined in Sec. 3.3. We want to show that $\mathrm{h}^0(\mathcal{S}_K, \mathcal{L}^{\otimes n}) \geq 2$ for some $n > 0$. By Theorem 3.14 it is enough to show that $\mathcal{L}$ is ample-positive, which we will deduce from the proof of the Lang–Néron theorem (of which a modern exposition can be found in [6]). Let H be any ample line bundle on $\mathcal{S}_K$, then the pair $(\mathcal{S}_K, H)$ gives rise to a *generalized global field structure* on L, cf. [6, Example 8.4]. After quite some work unravelling the definitions, we find that the Néron–Tate height

$$\hat{\mathrm{h}}_H : J(\bar{L})/\operatorname{Tr}(J/\bar{K}) \to \mathbb{R}$$

differs by a bounded amount from the function

$$J(\bar{L})/\operatorname{Tr}(J/\bar{K}) \to \mathbb{R}; \quad \tau \mapsto c_1(\langle\tau, \tau\rangle_a) \cdot c_1(H)^{\dim \mathcal{S}_K - 1}.$$

By [6, Theorem 9.15] the function $\hat{\mathrm{h}}_H$ is positive definite, so (after possibly replacing σ by a positive multiple, which is harmless for the argument) we find that $c_1(\mathcal{L}) \cdot c_1(H)^{\dim \mathcal{S}_K - 1} > 0$ as required.

Step 3: Comparing metrics on $\mathcal{L}$.

This step is obvious trivial if our global field K has no Archimedean places, i.e. has positive characteristic.

Since $\mathcal{S}_K$ is projective there exists a continuous metric on the line bundle $\mathcal{L}$ (e.g., write $\mathcal{L}$ as a difference of very ample line bundles, to which we can pull back the Fubini–Study metric). This metric is far from unique, but (by compactness) any two such metrics have bounded difference. Write $\|-\|_c$ for one such metric. The bundle $\mathcal{L}$ also comes with a natural metric $\|-\|_{\mathcal{P}}$ over S given by pulling back the unique rigidified translation-invariant metric on the Poincaré bundle. We do not know[b] if this metric extends continuously to the whole of $\mathcal{S}_K$, but nonetheless we will see in the next paragraph how to use results from [3] to show that $\|-\|_{\mathcal{P}}$ has bounded difference from $\|-\|_c$.

[b]The metric does extend continuously if $\dim \mathcal{S}_K = 1$ by [16], it does not extend continuously in general in higher dimension (due to height-jumping), and in the present case (when we have a Néron model) we do not know whether it extends.

By [3, Theorem 1.1(1)] the logarithm of the norm in $\|-\|_{\mathcal{P}}$ of a local generating section of $\mathcal{L}$ differs by a bounded amount from a function of the form

$$q(\log(|z_1|)\ldots,\log(|z_n|)),$$

where $z_1,\ldots,z_n$ are suitably chosen local analytic coordinates, and q is some homogeneous rational function of degree 1. However, using that $C_{\mathcal{S}_K}/\mathcal{S}_K$ is aligned one can check (using test curves in $\mathcal{S}_K$) that this rational function q is in fact linear. Thus we see that $\|-\|_{\mathcal{P}}$ and $\|-\|_c$ have bounded difference.

Associated to the metrized line bundles $(\mathcal{L},\|-\|_c)$ and $(\mathcal{L},\|-\|_{\mathcal{P}})$ we get height functions $\mathrm{h}^c_{\mathcal{L}}$ and $\mathrm{h}^{\mathcal{P}}_{\mathcal{L}}\colon S(\bar{K})\to\mathbb{R}$ respectively. Then $\mathrm{h}^c_{\mathcal{L}}$ is weakly non-degenerate by Theorem 3.14 since $\mathrm{h}^0(\mathcal{S}_K,\mathcal{L}^{\otimes n})\geq 2$. Because $\mathrm{h}^c_{\mathcal{L}}$ and $\mathrm{h}^{\mathcal{P}}_{\mathcal{L}}$ have bounded difference (cf. Theorem 3.15) we deduce that $\mathrm{h}^{\mathcal{P}}_{\mathcal{L}}$ is also weakly non-degenerate.

Step 4: Concluding the proof using that the jump is bounded.

To conclude the proof of the theorem, we compare three functions from $S(\bar{K})$ to $\mathbb{R}$:

(a) $\hat{\mathrm{h}}_\sigma$ sending $s\in S(\bar{K})$ to the Néron–Tate height of $\sigma(s)$;
(b) $j\colon S(\bar{K})\to\mathbb{R}$ sending $s\in S(\bar{K})$ to $\frac{\deg j(\sigma,\sigma,\bar{s})}{[\kappa(s):K]}$ where $j(\sigma,\sigma,\bar{s})$ is the jump corresponding to σ and $\bar{s}$;
(c) the height $\mathrm{h}^{\mathcal{P}}_{\mathcal{L}}$,

and by construction these satisfy

$$\hat{\mathrm{h}}_\sigma=\mathrm{h}^{\mathcal{P}}_{\mathcal{L}}-j.$$

We have seen above that $\mathrm{h}^{\mathcal{P}}_{\mathcal{L}}$ is weakly non-degenerate, and $|j|$ is bounded by Theorem 3.12, so we see that $\hat{\mathrm{h}}_\sigma$ is also weakly non-degenerate as required. □

Note that the main results of [1–3] concern the nonnegativity of j (as conjectured in [11]), but for our purposes this does not seem very helpful, since we want to deduce positivity of $\hat{\mathrm{h}}_\sigma$ from positivity of $\mathrm{h}_{\mathcal{L}}$. In the presence of a Néron model on the generic fiber we can show that j is bounded (which is sufficient), but without this assumption it seems hard to control j.

It is interesting to compare our result to [36, Theorem 1.3.5] where Zhang proves a similar result where the section σ of the Jacobian is replaced by the Gross–Schoen cycle on the product of C with itself, and where the family of curves C/S is assumed to be smooth. It seems reasonable to speculate that it might be possible to generalize Zhang's result to the case where C/S is assumed only to be regular and aligned, rather than smooth.

References

[1] O. Biesel, D. Holmes and R. de Jong, Néron models and the height jump divisor, *Trans. Amer. Math. Soc.* **369** (2017) 8685–8723.

[2] J. Burgos Gil, D. Holmes and R. de Jong, Positivity of the height jump divisor, *Int. Math. Res. Not.* **2019**(7) (2019) 2044–2068; https://doi.org/10.1093/imrn/rnx169.

[3] J. Burgos Gil, D. Holmes and R. de Jong, Singularities of the biextension metric for families of abelian varieties, *Forum Math. Sigma* **6** (2018) e12.

[4] A. Cadoret and A. Tamagawa, Note on torsion conjecture, in *Geometric and Differential Galois Theories*, Séminaires et Congrès, Vol. 27 (Société Mathématique De France, Paris, 2013), pp. 57–68.

[5] G. S. Call, Local heights on families of abelian varieties, Ph.D. thesis, Harvard University (1986).

[6] B. Conrad, Chow's K/k-image and K/k-trace, and the Lang–Néron theorem, *Enseign. Math. (2)* **52**(1–2) (2006) 37–108.

[7] B. Conrad, O. Gabber and G. Prasad, *Pseudo-Reductive Groups*, Vol. 26 (Cambridge University Press, 2015).

[8] A. J. de Jong, Smoothness, semi-stability and alterations, *Inst. Hautes Études Sci. Publ. Math.* **83** (1996) 51–93.

[9] P. Deligne, Le lemme de Gabber, *Astérisque* **127** (1985) 131–150, Seminar on arithmetic bundles: The Mordell conjecture (Paris, 1983/84).

[10] W. Green, Heights in families of abelian varieties, *Duke Math. J.* **58**(3) (1989) 617–632.

[11] R. Hain, Normal functions and the geometry of moduli spaces of curves, in *Handbook of Moduli I*, G. Farkas and I. Morrison (eds.), Advanced Lectures in Mathematics, Vol. XXIV (International Press, Boston, 2013).

[12] M. Hindry and J. H. Silverman, *Diophantine Geometry: An Introduction*, Vol. 201 (Springer-Verlag, New York, 2000).

[13] D. Holmes, A Néron model of the universal Jacobian, preprint (2014), arXiv:1412.2243[Math.AG].

[14] D. Holmes, Néron models of Jacobians over base schemes of dimension greater than 1, preprint (2014), arXiv:1402.0647, to appear in *J. Reine Angew. Math.*

[15] D. Holmes, Quasi-compactness of Néron models, and an application to torsion points, *Manuscripta Math.* **153**(3) (2017) 323–330.

[16] D. Holmes and R. de Jong, Asymptotics of the Néron height pairing, *Math. Res. Lett.* **22**(5) (2015) 1337–1371.

[17] F. F. Knudsen, The projectivity of the moduli space of stable curves. II. The stacks $M_{g,n}$. *Math. Scand.* **52**(2) (1983) 161–199.

[18] S. Lang, *Elliptic Curves: Diophantine Analysis*, Vol. 231 (Springer, 1978).

[19] S. Lang, *Fundamentals of Diophantine Geometry* (Springer-Verlag, New York, 1983).

[20] S. Lang, *Introduction to Arakelov Theory* (Springer, 1988).

[21] G. Laumon and L. Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete, A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics] (Springer-Verlag, Berlin, Heidelberg, 2000).

[22] D. A. Lear, Extensions of normal functions and asymptotics of the height pairing, Ph.D. Thesis, University of Washington (1990)

[23] Q. Liu, *Algebraic Geometry and Arithmetic Curves*, Oxford Graduate Texts in Mathematics. (Oxford University Press, Oxford, 2002), translated from the French by Reinie Erné, Oxford Science Publications.

[24] D. Masser and U. Zannier, Torsion anomalous points and families of elliptic curves, *Amer. J. Math.* **132**(6) (2010) 1677–1691.

[25] L. Merel, Bornes pour la torsion des courbes elliptiques sur les corps de nombres, *Invent. Math.* **124**(1–3) (1996) 437–449.

[26] L. Moret-Bailly, Métriques permises, *Astérisque* **127** (1985) 29–87.
[27] L. Moret-Bailly, *Pinceaux de variétés abéliennes*, Vol. 129 (Société Mathématique de France, 1985).
[28] F. Pazuki, Heights, ranks and regulators of abelian varieties, preprint (2015), arXiv:1506.05165.
[29] R. Penrose, A generalized inverse for matrices, *Proc. Cambridge Philos. Soc.* **51** (1955) 406–413.
[30] R. Pink, A common generalization of the conjectures of André-Oort, Manin-Mumford, and Mordell-Lang, Unpublished.
[31] A. Silverberg, Open questions in arithmetic algebraic geometry, in *Arithmetic Algebraic Geometry*, IAS/Park City Mathematics Series, Vol. 9 (American Mathematical Society, Providence, RI, 2001), pp. 83–142.
[32] J. H. Silverman, Heights and the specialization map for families of abelian varieties, *J. Reine Angew. Math.* **342** (1983) 197–211.
[33] J. H. Silverman, Variation of the canonical height on elliptic-surfaces ii Local analyticity properties, *J. Number Theory* **48**(3) (1994) 291–329.
[34] L. Szpiro, Degrés, intersections, hauteurs, *Astérisque* **127** (1985) 11–28. Seminar on arithmetic bundles: The Mordell conjecture (Paris, 1983/84).
[35] J. Tate, Variation of the canonical height of a point depending on a parameter, *Amer. J. Math.* **105**(1) (1983) 287–294.
[36] S.-W. Zhang, Gross-Schoen cycles and dualising sheaves, *Invent. Math.* **179**(1) (2010) 1–73.