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Into the Darkness: Forging a Stable Path Through the Gravitational Landscape

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The cover shows a greyscale image of a dock, with colours inverted. The unknown surroundings represent the vast gravitational landscape while the dock stands for the path of stable models within the landscape. The unfinished state of the path signifies the fact that our understanding of the set of stable gravitational models is still a work in progress.

The original image was made by Hoach Le Dinh and taken from the free license image repository unsplash.com.

In memory of Gonzalo

Contents

1 Introduction

1.1 Preface

More than twenty years ago, in 1998, two independent groups, the Supernova Cosmology Group [1] and the High-Z Supernova Search Team [2] made the astounding discovery that the universe is expanding in an accelerating fashion. This discovery heralded a new era in theoretical and obsvervational cosmology as the search for the true nature underlying this phenomenon commenced. This, combined with the discovery of Cold Dark Matter [3], led to the establishment of the highly succesful cosmological model, Λ-Cold Dark Matter (ΛCDM) [4, 5] where the cosmological constant Λ, interpreted as the energy density of the vacuum, sources cosmic acceleration.

While ΛCDM has been resistant to new tests up till now, it remains theoretically unsatisfying to many. The observed value of the cosmological constant, in terms of the Planck mass, is $\Lambda_{obs} \sim (10^{-30} M_{pl})^4$, about 60 orders of magnitude smaller than the theoretical prediction coming from the Standard Model. While such a small value can be reconciled with theory without any new ingredients it would imply an incredible amount of fine tuning. The *cosmological constant problem* has triggered a vast endeavour to find alternative sources of acceleration, leading to a landscape of theories which modify General Relativity (GR) in a variety of ways.

In this thesis we study the landscape of gravitational models which modify GR by introducing an additional scalar degree of freedom (d.o.f.) to source Cosmic Acceleration. In particular we answer the question "*What is the complete set of theoretical conditions a gravitational model must satisfy, in order to give a theoretically viable cosmology?*".

In Section 1.2 we present two typical extension of GR and briefly discuss the distinction. In Section 1.3 we introduce the Effective Field Theory of Dark Energy and Modified Gravity (EFToDE/MG), a unifying framework which allows us to study the landscape of gravitational models in a broad and model independent way. In Section 1.4 we discuss the notion of perturbative stability of a gravitational model. Stability will be the guiding principle in order to answer the main question of this thesis. Finally, in Section 1.5, we will present a summary of the following chapters.

1.2 Dark Energy versus Modified Gravity

Extensions of General Relativity typically fall into two categories, Dark Energy(DE) which introduces a fluid into the universe modifying the stress energy tensor, and Modified Gravity(MG) which directly modifies the gravitational sector leading to a modified Einstein tensor. We will briefly present two common candidates of cosmic acceleration: Quintessence[6–9], a typical Dark Energy model and $f(R)$[10, 11] which modifies the gravitational sector. Both introduce an additional scalar degree of freedom to General Relativity.

- **Quintessence**

  The simplest extension beyond the cosmological constant is a scalar field whose potential energy drives cosmic acceleration, in a fashion similar to cosmic inflation. Dubbed *quintessence*, this corresponds to the action of a scalar field, ϕ, minimally coupled to gravity in the presence of a potential:

$$S = \int d^4x\sqrt{-g}\Big(\frac{M_{pl}^2}{2}R - \frac{1}{2}(\partial\phi)^2 - V(\phi)\Big) + S_m \, , \qquad (1.1)$$

  where R is the usual Ricci tensor and S_m is the action for any matter field present. This action leads to the usual Einstein equations with an additional stress energy tensor sourced by the scalar field:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{M_{pl}^2}(T_{\mu\nu}^{\text{matter}} + T_{\mu\nu}^{\phi}) \qquad (1.2)$$

  where:

$$T_{\mu\nu}^{\phi} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}(\frac{1}{2}(\partial\phi)^2 + V(\phi)). \qquad (1.3)$$

  and $T_{\mu\nu}^{\text{matter}}$ is the matter stress energy tensor.

  When considering a cosmological scenario, one employs the cosmological principle which postulates that, on cosmological scales, the universe is homogeneous and isotropic. Various observations, such as the uniformity of the CMB at large scales, support the Cosmological principle to a very high degree. The metric for a homogeneous and isotropic universe is then of the Friedmann-Lemaître-Robertson-Walker (FLRW) form:

$$ds^2 = -dt^2 + a(t)^2 d\vec{x}^2, \qquad (1.4)$$

where $a(t)$ is dubbed the scale factor and encodes the time dependent change of the spatial volume.

On this background the scalar field has a spatially homogeneous profile, $\phi = \phi(t)$, and it behaves like a perfect fluid. The corresponding equation of state parameter, defined as the ratio of the pressure of the fluid and the density,$w = P/\rho$, has the following form:

$$w_\phi = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)}\,, \tag{1.5}$$

where the dot stands for the derivative with respect to cosmic time. The cosmological constant corresponds to an equation of state parameter value $w_\Lambda = -1$ hence, in order to fit the observed expansion, $w_\phi \simeq -1$. This leads to the, so called, slow-roll condition which corresponds to $\dot{\phi}^2 \ll V(\phi)$. As in the case of inflation sourced by a slow rolling field, quintessence exhibits a vast phenomenology due to the broad range of potentials one can construct and has been deeply explored over the years.

- **f(R)**

  In the case of $f(R)$, rather than explicitly introducing a field, one modifies the Einstein-Hilbert part of the action. This makes it a typical example of a modified gravity model. Its action takes on the following form:

$$S = \frac{M_{pl}^2}{2}\int d^4x\sqrt{-g}(R + f(R)) + S_m\,, \tag{1.6}$$

  where $f(R)$ is a function of the Ricci Scalar. The resulting modified Einstein tensor which yields the following Einstein Equation [12]

$$(1+f_R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R+f) + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_R = \frac{1}{M_{pl}^2}T^m_{\mu\nu}. \tag{1.7}$$

  with f_R being the derivative of the function $f(R)$ with respect to the Ricci scalar. The new Einstein equation is now higher order in derivatives as it contains derivatives of f_R which depends on the Ricci scalar. This will promote one constraint to a dynamical equation, hence introducing a scalar degree of freedom. We will elaborate more on this at the end of this section.

  It is now possible to isolate the new contributions in (1.7) and interpret them as an effective fluid with stress energy tensor:

$$\frac{1}{M_{pl}^2}T^{\rm eff}_{\mu\nu} \equiv f_R R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_R. \tag{1.8}$$

and an effective equation of state:

$$w_{\text{eff}} = -\frac{1}{3} - \frac{2}{3}\frac{H^2 f_R - f/6 - H\dot{f}_R - \ddot{f}_R/2}{-H^2 f_R - f/6 - H\dot{f}_R + f_R R/6}, \tag{1.9}$$

where, $H = \frac{1}{a}\frac{da}{dt}$ is the Hubble parameter.

Interpreting the modification to gravity as an effective fluid clarifies the way it can source acceleration as it provides a direct link to traditional Dark Energy models. The tradeoff is that the distinction between Dark Energy models and modified gravity models becomes obscure, yet one must keep in mind that, the effects come purely from the gravitational sector.

Let us now expose the presence of an additional scalar degree of freedom in this theory [12] by taking the trace of (1.7) which yields:

$$\Box f_R = \frac{1}{3}(R + 2f - f_R R + \frac{1}{M_{pl}^2}T) \equiv \frac{dV_{eff}}{df_R} \tag{1.10}$$

This is the equation of motion of a scalar degree of freedom f_R, called the *scalaron*, with an effective potential V_{eff}.

Concluding we would like to stress that the dark energy and modified gravity models presented here are two standard examples. There are a variety of ways to source acceleration, yet all of them introduce a scalar d.o.f. regardless of the type of modification. For a deeper discussion into the distinction between Dark Energy and Modified Gravity we refer the reader to [13], where the authors try to enhance the definition with the use of the Equivalence Principle.

1.3 The Effective Field Theory of DE/MG

The question of testing the validity of General Relativity has occupied the scientific community for over 100 years. This has lead to GR surviving a battery of tests ranging from Solar System to Galactic scales and more. Yet, at cosmological scales, GR still remains largely untested. As cosmology has entered the golden era of high accuracy data provided by ESA and NASA missions, this is bound to change in the near future. On the theoretical side, the quest to explain cosmic acceleration has led to a wealth of models modifying GR at cosmological scales. Thus it becomes crucial to develop methods which are capable of quantifying all possible deviations from GR in a structured way as well as providing an efficient framework to confront different extensions of gravity with observational data.

For gravitational models exhibiting an additional scalar degree of freedom, a framework addressing these demands was constructed in Ref [14–16] under the name of *"Effective Field Theory of Dark Energy and Modified Gravity"* (henceforth dubbed EFToDE/MG). At the core of this approach lies the notion that dynamical cosmological perturbations are the Goldstone modes of spontaneously broken time-translations, in a fashion reminiscent of inflation where the breaking of de-Sitter invariance introduces a Goldstone mode, the *inflaton*. Using techniques of Effective Field Theories in Quantum Field Theory it is then possible to construct the most general action describing linear perturbations around the symmetry-breaking background. This was initially done in the context of Inflation [17] and Quintessence [18], and subsequently applied to cosmic acceleration.

The major strength of the EFToDE/MG lies in the fact that, besides being able to parametrise all possible deviations from General Relativity in a complete set of operators, it also provides a "Unifying" framework. The latter implies that each individual model corresponds to a subset of operators and can be studied within the framework.

In order to construct the action one needs to make the following considerations.

- The Weak Equivalence Principle (WEP) is to hold. This implies that all the matter species are universally coupled to the same metric $g_{\mu\nu}$. In order to simplify the inclusion of matter we choose to work in the Jordan frame. This frame choice dictates that the matter fields are not coupled to the scalar field.

- Additionally, we choose a particular time slicing where each equal time hypersurface corresponds to a uniform field hypersurface. This sets the fluctuations of the scalar field to zero and sets the, so called, unitary gauge. This gauge is a familiar concept from the standard model where one can set it in order to absorb the Higgs d.o.f. into the gauge field. In this case the unitary gauge makes the breaking of time-translations manifest thus leaving the unbroken spatial diffeomorphisms as residual symmetries.

Having taken these considerations, one can now construct the most general action based on operators satisfying the residual gauge symmetries. In order to facilitate this procedure and identify the relevant geometrical terms, a 3+1 space and time decomposition will be employed[19]. This decomposition identifies two key quantities on the constant time hypersurfaces: the normal vector n_μ and the induced 3-dimensional metric

$h_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu$. This leads to the following general form:

$$S = \int d^4x\sqrt{-g}\left[g^{00}, K^\mu_\mu, K^\nu_\mu K^\mu_\nu, R, \mathcal{R}, \mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}, .., ; t\right] + S_m[g_{\mu\nu}, \psi^i]\,, \tag{1.11}$$

where $\mathcal{R}_{\mu\nu}$ and $K^\mu_\nu = h^\rho_\nu \nabla_\rho n^\mu$ are the intrinsic and extrinsic curvatures of the constant-time hypersurfaces respectively. The matter fields ψ^i are universally coupled to the same Jordan metric as dictated by the WEP. As the EFT strictly modifies the gravitational sector we will proceed to neglect the matter fields in the rest of this introductory Chapter. Their inclusion will be significant in Chapter 3 and we refer the reader to that Chapter for their complete treatment.

Out of the set of operators, three contribute to the background. The corresponding action is then, where the Planck Mass is denoted as m_0:

$$S = \int d^4x\sqrt{-g}\left[\frac{m_0}{2}\left(1+\Omega(t)\right)R + \Lambda(t) - c(t)\delta g^{00}\right] \tag{1.12}$$

Note the presence of explicit time dependent functions, multiplying the curvature terms, dubbed *"EFT functions"* and of $\delta g^{00} \equiv g^{(00)} + 1$. Both are now allowed due to the breaking of time-translation invariance, in contrast to regular GR. The functions $\Omega(t)$ and $\Lambda(t)$ are the, time dependent, conformal coupling to the Ricci Scalar and the Cosmological constant, respectively. This action leads to the following Einstein Equations, while neglecting matter, which determine the background:

$$\begin{aligned}
&3H\dot{\Omega}m_0^2 - 2c + 3H^2 m_0^2(1+\Omega) + \Lambda = 0\,,\\
&3H^2 m_0^2(1+\Omega) + 2\dot{H}m_0^2(1+\Omega) + 2m_0^2 H\dot{\Omega} + m_0^2\ddot{\Omega} + \Lambda = 0\,.
\end{aligned} \tag{1.13}$$

Finally, the complete action describing linear perturbations around the time-translation breaking background is the following

$$\begin{aligned}
\mathcal{S}^{(2)} = \int d^4x\sqrt{-g}\Bigg[&\frac{m_0^2}{2}(1+\Omega(t))R + \Lambda(t) - c(t)\delta g^{00} + \frac{M_2^4(t)}{2}(\delta g^{00})^2\\
&-\frac{\bar{M}_1^3(t)}{2}\delta g^{00}\delta K - \frac{\bar{M}_2^2(t)}{2}(\delta K)^2 - \frac{\bar{M}_3^2(t)}{2}\delta K^\mu_\nu \delta K^\nu_\mu + \frac{\hat{M}^2(t)}{2}\delta g^{00}\delta\mathcal{R}\\
&+m_2^2(t)h^{\mu\nu}\partial_\mu g^{00}\partial_\nu g^{00}\Bigg]\,.
\end{aligned} \tag{1.14}$$

The choice of the unitary gauge in the above action guarantees that the scalar d.o.f. has been absorbed by the metric, hence it does not appear explicitly in the action. One could make it manifest by applying the so called "Stückelberg technique" [14, 15]. In this thesis we will stick to the unitary gauge, yet the results obtained will be applicable in other gauges as well.

The "effective" and "unifying" aspects of the EFToDE/MG make it ideal in order to aid the endeavour to confront the gravitational landscape with cosmological data. In order to achieve this, the EFToDE/MG framework was recently implemented into the Einstein-Boltzmann solver CAMB [20, 21] via the *EFTCAMB* patch [22–24]. This then allows users to do an agnostic exploration in the *pure EFT* mode or study individual models through the *mapping EFT* mode. In this thesis we will show how both aspects are important when approaching the vast landscape of theories which extend GR.

As a final comment it is important to stress that it is possible to include derivatives of the geometrical objects. This has not been done in this initial setup as it will introduce higher-order time derivatives. These higher order time derivatives risk the introduction of ghosts through the Ostrogradsky instability [25]. In [26] the newly developed DHOST theories[27, 28], which are free of the Ostrogradsky ghost, have been incorporated in this setup by introducing an operator with higher order time derivatives. This case will not be taken into consideration in the remainder of this thesis.

1.4 Stability in the language of Effective Field Theories

1.4.1 The Ghost Instability

A common pathology encountered in EFTs is the presence of *ghosts*, fields with negative energy quanta or negative norm. This typically corresponds to a field with the wrong sign for the kinetic term. In a vacuum this does not pose a problem for the theory as the sign is purely a matter of convention. The sign of the kinetic term does matter when one couples the ghost field with another field, which has the opposite sign as a change of convention will not alleviate the issue. A simple example is the following action of two scalar fields:

$$\mathcal{L} = \frac{1}{2}(\partial\psi)^2 - \frac{m_\psi}{2} - \frac{1}{2}(\partial\phi)^2 - \frac{m_\phi}{2} + \lambda\psi^2\phi^2. \tag{1.15}$$

Here the field ψ has the wrong sign and is directly coupled with the field ϕ with interaction strength λ . This lead to an unstable vacuum as it is sensitive to the spontaneous decay $0 \to \phi\phi + \psi\psi$ which costs zero energy and has an infinite decay rate[29, 30].

1.4.2 The Gradient Instability

In a similar way as the ghost fields appear due to a wrong sign of the kinetic term, the gradient instability manifests itself when a field has the wrong sign for its gradient term, i.e. the term containing spatial derivatives of the field. This leads to modes that grow fast leading to instabilities in the theory as we shall show. Let us consider the following scalar field, in Fourier space, with a general speed of sound:

$$\mathcal{L} = \frac{1}{2}\dot{\chi}^2 - c_s^2 \frac{1}{2}(k\chi)^2. \tag{1.16}$$

Obviously the field is not Lorentz-Invariant when the speed of sound differs from 1. The solutions for the wave equation of this field are the following

$$\chi_k \sim e^{\pm i\omega t} \tag{1.17}$$

with $\omega = \sqrt{c_s^2 k^2}$. When the speed of sound is imaginary, the sign of the gradient term flips, resulting in the following unbounded solutions:

$$\chi_k \sim e^{\pm \omega t}. \tag{1.18}$$

with a typical timescale of $\tau \sim 1/(c_s k)$. Within the language of effective field theories this implies that, for modes below the energy cutoff Λ, an instability will arise if the system is allowed to evolve long enough. Additionally, the modes most sensitive are the ones with the highest-energy.

Within cosmology and in particular DE/MG the appearance of gradient instabilities are a common occurence and are thus one of the first tests a theory has to pass to be considered viable. In that case the typical timescale of the universe is taken to be the Hubble time and thus the inverse rate of instability is not allowed to exceed this timescale. As was shown in [18] it is possible, when considering a theory with higher order derivatives, to have a gradient instability for a finite range of modes which evolve over scales larger than the Hubble scale and thus do not create an unviable theory. In order to avoid unecessarily constraining such a theory we consider in the rest of this manuscript only the leading order term of the speed of sound and demand it to have the correct sign.

1.4.3 The Tachyonic Instability

Finally a, rather, underemphasized pathology in Effective Field Theories is the tachyonic instability. This instability appears at large scales and is sourced by a mass term with a wrong sign. Thus its behaviour is analogous to the gradient instability but on large scales, as will become clear below. We consider again a scalar field but now with a mass term and in the large scale limit, we ignore the gradient term:

$$\mathcal{L} = \frac{1}{2}\dot{\chi}^2 - m^2\frac{1}{2}(\chi)^2 \tag{1.19}$$

The solutions for the field are now of the same form as for the gradient instability:

$$\chi_k \sim e^{\pm i\omega t} \tag{1.20}$$

but now we have $\omega = \sqrt{m^2}$. When the mass of the field is imaginary, we have again unbounded solutions due to the appearance of the square root:

$$\chi_k \sim e^{\pm mt}. \tag{1.21}$$

As before this instability comes with a characteristic timescale $\tau \sim m^{-1}$. A clear distinction with the gradient instability is that here the timescale is not scale dependent. This implies that high-energy modes which satisfy $m \ll k \ll \Lambda$ are insensitive to the tachyonic instability and thus the theory in itself is not *a-priori* ill-defined. Rather, the tachyonic instability can be seen as a statement on the vacuum or, analogously, the cosmological background one is perturbing around. When one encounters this instability one has not chosen the true vacuum/background of the model under study. There is a well known example of this in the the Standard model, namely the the Higgs field which appears as a tachyon.

In the field of Dark Energy and Modified Gravity the study of the tachyonic instability has not been a high priority in the literature. While we argued that its appearance does not signify that the theory is ill-defined it is important to consider it for the following reasons. When one confronts a theory with cosmological data one requires initially the behaviour of the background and subsequently the perturbations to match our observed universe. Hence, when a tachyonic instability occurs on a cosmologically viable background, it is impossible to reconcile both the background and the perturbations with observations, rendering the model unviable from a cosmological rather than a stability perspective. In the remainder of this thesis the tachyonic instability will play an important role in completing the set of conditions, furthering the goal of answering the main question of this thesis .

1.5 Summary of this thesis

1.5.1 Chapter 2

In this Chapter we proceed to expand the EFToDE/MG to include Lorentz-violating theories of modified gravity. In particular we focus on the theory of high-energy Hořava gravity which has drawn much attention due to it being simultaneously a quantum gravity and a cosmic acceleration candidate. This leads to an action which encompasses a set of 6 additional operators , on top of the original construction, in order to be able to cover the additional signatures.

Having established the new, expanded, action we construct a comprehensive dictionary which provides the means to map a particular theory into the EFT. This is of importance as, besides being the most general action capturing deviations from GR, the EFT provides a unifying framework which allows models to be studied in its language. The dictionary covers models like $f(R)$, Horndeski, beyond Horndeski and the newly added Hořava gravity.

In the final part of this Chapter we start to address the main question of this thesis by doing a comprehensive stability analysis of the EFToDE/MG, while neglecting matter. In any realistic scenario matter is present during cosmic acceleration but the choice to neglect it simplifies the problem and is usually made due to the fact that the energy budget of our universe is DE dominated. Based on these assumptions, we obtain a set of conditions guaranteeing the absence of ghost, gradient and tachyonic instabilities. These conditions are not universal but were derived for all available subcases such as beyond Horndeski, Hořava gravity and so on.

This Chapter is based on [31]: *An Extended action for the effective field theory of dark energy: a stability analysis and a complete guide to the mapping at the basis of EFTCAMB* with N. Frusciante and A. Silvestri.

1.5.2 Chapter 3

In Chapter 2 the stability of the EFToDE/MG was studied in a vacuum, i.e. neglecting matter. In the present Chapter we present a generalisation of this result where we redo the calculation in the presence of radiation and Cold Dark Matter(CDM), i.e. presureless, fluids. This significantly complicates the problem at hand as the gravitational interaction couples the different degrees of freedom in a variety of ways, making the identification of the relevant quantities problematic.

Initially we focus on the Ghost and Gradient instabilities. We choose to model the fluids with the Sorkin-Schutz action, which comes with a number of advantages covered in the main text. As before, a number of subcases need to be considered and studied individually which are then compared with the previously established results. With the exception of the beyond Horndeski models, matter turns out not to significantly alter previously established results, partially vindicating the simplifying assumptions made in previous works.

Moving further, we tackle the main goal of the chapter, namely to study the tachyonic instability in the presence of matter. In order to achieve this we proceed to consider only a single matter field, as the multiple stages required in this calculation become increasingly untractable when including additional degrees of freedom. At the end we present the main novel result of this thesis, the two new conditions required to avoid the tachyonic instability when the EFToDE/MG includes the presence of the main matter component of our universe, CDM. These conditions will be the main topic of study in Chapter 5.

This Chapter is based on [32]: *On the stability conditions for theories of modified gravity in the presence of matter fields* with A. De Felice and N. Frusciante.

1.5.3 Chapter 4

The future of our universe, if cosmic acceleration keeps on acting without significant alterations, is expected to be a de-Sitter like end state. In this end state the Hubble parameter becomes a constant and matter has been diluted away. This motivation lies behind the main topic of this chapter, the EFToDE/MG in the de-Sitter limit.

Guided by the question lying at the basis of this thesis we perform a stability analysis of the EFToDE/MG in the de Sitter limit. As before, this leads to a set of conditions for different subcases such as beyond Horndeski. Additionally, we manage to solve the equations of motion analytically due to their simplicity. These results, while done in the context of DE/MG, also hold for the EFT of Inflation and can be freely applied.

Parallel to the study of the curvature perturbation described above, we construct a gauge-invariant quantity describing the DE/MG variable and derive the corresponding conditions. We do this as a test to check the validity of the original conditions, which were derived for a gauge dependent variable. This study lead us to conclude that once one set of conditions is satisfied the other one will be instantly satisfied as well, a result both expected and welcomed.

This Chapter is based on [33]: *de Sitter limit analysis for dark energy and modified gravity models* with A. De Felice and N. Frusciante.

1.5.4 Chapter 5

The final chapter of this thesis is distinct from the others as, rather than deriving viability conditions, we proceed to test them. In particular we aim to test the novel conditions forbidding tachyonic instabilities, derived in Chapter 3. The ghost and gradient conditions have been employed in previous work in the literature so, while included, we will not focus on their contribution.

In order to proceed we employ the code *EFTCAMB*, a patch adding the EFToDE/MG formalism to the Einstein Boltzmann solver CAMB, thus allowing us to study the cosmologies of different theories. Before this work, *EFTCAMB* already included the subset of ghost and gradient conditions yet lacked the tachyonic conditions. In order to deal with this deficiency and avoid diverging perturbations at large scales, it employs a set of ad-hoc mathematical conditions derived from the equation of motion of the scalar field. It is therefore these mathematical conditions to which the tachyonic conditions will be compared.

By studying a large ensemble of models we manage to achieve this comparison showing that the tachyonic conditions have an equivalent or stronger constraining impact than the ad-hoc math conditions. The parameters we took under consideration are the well known μ and Σ which encode the deviations from GR in the gravitational Poisson and lensing equation respectively. In some cases, such as the Brans-Dicke models a visible impact was seen. This led to the first work where it was possible to exclude models, at large scales, based on theoretical considerations without resorting to ad-hoc conditions which suffer from severe limitations.

This Chapter is based on [34]: *The role of the tachyonic instability in Horndeski gravity* with N. Frusciante, S. Peirone and A. Silvestri.

2 An Extended action for the effective field theory of dark energy: a stability analysis and a complete guide to the mapping at the basis of EFTCAMB

2.1 Introduction

In the present Chapter we propose an extension of the original EFT action for DE/MG [14, 15] by including extra operators with up to sixth order spatial derivatives acting on perturbations. This will allow us to cover a wider range of theories, e.g. Hořava gravity [35, 36], as shown in Refs. [37–39]. The latter model has recently gained attention in the cosmological context [39–58], as well as in the quantum gravity sector [35, 36, 59–61], since higher spatial derivatives have been shown to be relevant in building gravity models exhibiting powercounting and renormalizable behaviour in the ultra-violet regime (UV) [62–64].

We will work out a very general recipe that can be directly applied to any gravity theory with one extra scalar d.o.f. in order to efficiently map it into the EFT language, once the corresponding Lagrangian is written in the Arnowitt-Deser-Misner (ADM) formalism. We will pay particular attention to the different conventions by adapting all the calculations to the specific convention used in *EFTCAMB*, in order to provide a ready-to-use guide on the full mapping of models into this code. This method has already been used in Refs. [37, 65] and here we will further extend it by including the operators in our extended action. Additionally, we will revisit some of the already known mappings in order to accommodate the *EFTCAMB* conventions. Moreover, we will present for the first time the complete mapping of the covariant formulation of the GLPV theories [66, 67] into the EFT formalism. Subsequently, we will perform a detailed study of the stability conditions for the gravity sector of our extended

EFT action. For a restricted subset of EFT models such an analysis can already be found in the literature [14, 15, 65, 67, 68]. Doing this analysis will allow us to have a first glimpse at the viable parameter space of theories covered by the extended EFT framework and to obtain very general conditions to be implemented in *EFTCAMB*. In particular, we will compute the conditions necessary to avoid ghost instabilities and to avoid gradient instabilities, both for scalar and tensor modes. We will also present the condition to avoid tachyonic instabilities in the scalar sector. Finally, we will proceed to extend the ReParametrized Horndeski (RPH) basis, or α-basis, of Ref. [69] in order to include all the models of our generalized EFT action. This will require the introduction of new functions and we will proceed to comment on their impact on the kinetic terms and speeds of propagation of both scalar and tensor modes.

The work in this Chapter is based on [31]: *An Extended action for the effective field theory of dark energy: a stability analysis and a complete guide to the mapping at the basis of EFTCAMB* with N. Frusciante and A. Silvestri. In Section 2.2, we propose a generalization of the EFT action for DE/MG that includes all operators with up to six-th order spatial derivatives. In Section 2.3, we outline a general procedure to map any theory of gravity with one extra scalar d.o.f., and a well defined Jordan frame, into the EFT formalism. We achieve this through an interesting, intermediate step which consists of deriving an equivalent action in the ADM formalism, in Section 2.3.2, and work out the mapping between the EFT and ADM formalism, in Section 2.3.3. In order to illustrate the power of such method, in Section 2.4 we provide some mapping examples: minimally coupled quintessence, f(R)-theory, Horndeski/GG, GLPV and Hořava gravity. In Section 2.5, we work out the physical stability conditions for the extended EFT action, guaranteeing the avoidance of ghost and tachyonic instabilities and positive speeds of propagation for tensor and scalar modes. In Section 2.6, we extend the RPH basis to include the class of theories described by the generalized EFT action and we elaborate on the phenomenology associated to it. The last two sections are more or less independent, so the reader interested only in one of these can skip the other parts. Finally, in Section 2.7, we summarize and comment on our results.

2.2 An extended EFT action

The EFT framework for DE/MG models, introduced in Refs. [14, 15], provides a systematic and unified way to study the dynamics of linear perturbations in a wide range of DE/MG models characterized by an additional scalar d.o.f. and for which there exists a well defined Jordan frame [10, 11, 70–73]. The action is constructed in the unitary gauge as an expansion up to second order in perturbations around the FLRW background of all operators that are invariant under time-dependent spatial-diffeomorphisms. Each of the latter appear in the action accompanied by a time dependent coefficient. The choice of the unitary gauge implies that the scalar d.o.f. is "eaten" by the metric, thus it does not appear explicitly in the action. It can be made explicit by the Stükelberg technique which, by means of an infinitesimal time-coordinate transformation, allows one to restore the broken symmetry by introducing a new field describing the dynamic and evolution of the extra d.o.f.. For a detailed description of this formalism we refer the readers to Refs. [14, 15, 65, 74, 75]. In this Chapter we will always work in the unitary gauge.

The original EFT action introduced in Refs. [14, 15], and its follow ups in Refs. [65, 75–77], cover most of the theories of cosmological interest, such as Horndeski/GG [78, 79], GLPV [66] and low-energy Hořava [35, 36]. However, operators with higher order spatial derivatives are not included. On the other hand, theories which exhibit higher than second order spatial derivatives in the field equations have been gaining attention in the cosmological context [37, 38, 53, 64, 76], moreover, they appear to be interesting models for quantum gravity as well [35, 36, 59–62]. As long as one deals with scales that are sufficiently larger than the non-linear cutoff, the EFT formalism can be safely used to study these theories. In the following, we propose an extended EFT action that includes operators up to sixth order in spatial derivatives:

$$\begin{aligned}\mathcal{S}_{EFT} = \int d^4x\sqrt{-g}\Bigg[&\frac{m_0^2}{2}(1+\Omega(t))R + \Lambda(t) - c(t)\delta g^{00} + \frac{M_2^4(t)}{2}(\delta g^{00})^2\\
&-\frac{\bar{M}_1^3(t)}{2}\delta g^{00}\delta K - \frac{\bar{M}_2^2(t)}{2}(\delta K)^2 - \frac{\bar{M}_3^2(t)}{2}\delta K^\mu_\nu \delta K^\nu_\mu + \frac{\hat{M}^2(t)}{2}\delta g^{00}\delta\mathcal{R}\\
&+m_2^2(t)h^{\mu\nu}\partial_\mu g^{00}\partial_\nu g^{00} + \frac{\bar{m}_5(t)}{2}\delta\mathcal{R}\delta K + \lambda_1(t)(\delta\mathcal{R})^2 + \lambda_2(t)\delta\mathcal{R}^\mu_\nu\delta\mathcal{R}^\nu_\mu\\
&+\lambda_3(t)\delta\mathcal{R}h^{\mu\nu}\nabla_\mu\partial_\nu g^{00} + \lambda_4(t)h^{\mu\nu}\partial_\mu g^{00}\nabla^2\partial_\nu g^{00} + \lambda_5(t)h^{\mu\nu}\nabla_\mu\mathcal{R}\nabla_\nu\mathcal{R}\\
&+\lambda_6(t)h^{\mu\nu}\nabla_\mu\mathcal{R}_{ij}\nabla_\nu\mathcal{R}^{ij} + \lambda_7(t)h^{\mu\nu}\partial_\mu g^{00}\nabla^4\partial_\nu g^{00}\\
&+\lambda_8(t)h^{\mu\nu}\nabla^2\mathcal{R}\nabla_\mu\partial_\nu g^{00}\Bigg], \qquad (2.1)\end{aligned}$$

where m_0^2 is the Planck mass, g is the determinant of the four dimensional metric $g_{\mu\nu}$, $h^{\mu\nu} = (g^{\mu\nu} + n^\mu n^\nu)$ is the spatial metric on constant-time hypersurfaces, n_μ is the normal vector to the constant-time hypersurfaces, δg^{00} is the perturbation of the upper time-time component of the metric, R is the trace of the four dimensional Ricci scalar, $\mathcal{R}_{\mu\nu}$ is the three dimensional Ricci tensor and $\mathcal{R}$ is its trace, $K_{\mu\nu}$ is the extrinsic curvature and K is its trace and $\nabla^2 = \nabla_\mu\nabla^\mu$ with ∇_μ being the covariant derivative constructed with $g_{\mu\nu}$. The coefficients $\{\Omega, \Lambda, c, M_2^4, \bar{M}_1^3, \bar{M}_2^2, \bar{M}_3^2, \hat{M}^2, m_2^2, \bar{m}_5, \lambda_i\}$ (with $i = 1$ to 8) are free functions of time and hereafter we will refer to them as *EFT functions*. $\{\Omega, \Lambda, c\}$ are usually called background EFT functions as they are the only ones contributing to both the background and linear perturbation equations, while the others enter only at the level of perturbations. Let us notice that the operators corresponding to $\bar{m}_5, \lambda_{1,2}$ have already been considered in Ref. [65], while the remaining operators have been introduced by some of the authors of this paper in Ref. [39], where it is shown that they are necessary to map the high-energy Hořava gravity action [64] in the EFT formalism.

The EFT formalism offers a unifying approach to study large scale structure (LSS) in DE/MG models. Once implemented into an Einstein-Boltzmann solver like CAMB [20], it clearly provides a very powerful software with which to test gravity on cosmological scales. This has been achieved with the patches *EFTCAMB/EFTCosmoMC*, introduced in Refs. [22–24]. This software can be used in two main realizations: the *pure EFT* and the *mapping EFT*. The former corresponds to an agnostic exploration of dark energy, where the user can turn on and off different EFT functions and explore their effects on the LSS. In the latter case instead, one specializes to a model (or a class of models, e.g. $f(R)$ gravity), maps it into the EFT functions and proceed to study the corresponding dynamics of perturbations. We refer the reader to Ref. [80] for technical details of the code.

There are some key virtues of *EFTCAMB* which make it a very interesting tool to constrain gravity on cosmological scales. One is the possibility of imposing powerful yet general conditions of stability at the level of the EFT action, which makes the exploration of the parameter space very efficient [23]. We will elaborate on this in Section 2.5. Another, is the fact that a vast range of specific models of DE/MG can be implemented *exactly* and the corresponding dynamics of perturbations be evolved, in the same code, guaranteeing unprecedented accuracy and consistency.

In order to use *EFTCAMB* in the mapping mode it is necessary to determine the expressions of the EFT functions corresponding to the given model. Several models are already built-in in the currently public version of *EFTCAMB*. This Chapter offers a complete guide on how to map specific models and classes of models of DE/MG all the way into the EFT language at the basis of *EFTCAMB*, whether they are initially formulated in the ADM or covariant formalism; all this, without the need of going through the cumbersome expansion of the models to quadratic order in perturbations around the FLRW background.

2.3 From a General Lagrangian in ADM formalism to the EFToDE/MG

In this Section we use a general Lagrangian in the ADM formalism which covers the same class of theories described by the EFT action (2.1). This will allow us to make a parallel between the ADM and EFT formalisms, and to use the former as a convenient platform for a general mapping description of DE/MG theories into the EFT language. In particular, in Section 2.3.1 we will expand a general ADM action up to second order in perturbations, in Section 2.3.2 we will write the EFT action in ADM form and, finally, in Section 2.3.3 we will provide the mapping between the two.

2.3.1 A General Lagrangian in ADM formalism

Let us introduce the 3+1 decomposition of spacetime typical of the ADM formalism, for which the line element reads:

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)\,, \tag{2.2}$$

where $N(t, x^i)$ is the lapse function, $N^i(t, x^i)$ the shift and $h_{ij}(t, x^i)$ is the three dimensional spatial metric. We also adopt the following definition of the normal vector to the hypersurfaces of constant time and the corresponding extrinsic curvature:

$$n_\mu = N\delta_{\mu 0}, \qquad K_{\mu\nu} = h^\lambda_\mu \nabla_\lambda n_\nu. \tag{2.3}$$

The general Lagrangian we use in this Section has been proposed in Ref. [37] and can be written as follows:

$$L = L(N, \mathcal{R}, \mathcal{S}, K, \mathcal{Z}, \mathcal{U}, \mathcal{Z}_1, \mathcal{Z}_2, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5; t)\,, \tag{2.4}$$

where the above geometrical quantities are defined as follows:

$$\begin{aligned}
&\mathcal{S} = K_{\mu\nu}K^{\mu\nu}\,,\ \mathcal{Z} = \mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}\,,\ \mathcal{U} = \mathcal{R}_{\mu\nu}K^{\mu\nu}\,,\ \mathcal{Z}_1 = \nabla_i\mathcal{R}\nabla^i\mathcal{R}\,,\\
&\mathcal{Z}_2 = \nabla_i\mathcal{R}_{jk}\nabla^i\mathcal{R}^{jk}\,, \alpha_1 = a^i a_i\,,\ \alpha_2 = a^i\Delta a_i\,,\ \alpha_3 = \mathcal{R}\nabla_i a^i\,,\\
&\alpha_4 = a_i\Delta^2 a^i\,,\ \alpha_5 = \Delta\mathcal{R}\nabla_i a^i,
\end{aligned} \tag{2.5}$$

with $\Delta = \nabla_k\nabla^k$ and a^i is the acceleration of the normal vector, $n^\mu\nabla_\mu n_\nu$. ∇_μ and ∇_k are the covariant derivatives constructed respectively with the four dimensional metric, $g_{\mu\nu}$ and the three metric, h_{ij}.

The operators considered in the Lagrangian (2.4) allow to describe gravity theories with up to sixth order spatial derivatives, therefore the range of theories covered by such a Lagrangian is the same as the EFT action proposed in Section 2.2. The resulting general action, constructed with purely geometrical quantities, is sufficient to cover most of the candidate models of modified gravity [10, 11, 70–73].

We shall now proceed to work out the mapping of Lagrangian (2.4) into the EFT formalism. The procedure that we will implement in the following retraces that of Refs. [37, 65]. However, there are some tricky differences between the EFT language of Ref. [65] and the one at the basis of *EFTCAMB* [22, 23]. Most notably the different sign convention for the normal vector, n_μ, and the extrinsic curvature, $K_{\mu\nu}$ (see Eq. (2.3)), a different notation for the conformal coupling and the use of δg^{00} in the action instead of g^{00}, which changes the definition of some EFT functions. It is therefore important that we present all details of the calculation as well as derive a final result which is compatible with *EFTCAMB*. In particular, the results of this Section account for the different convention for the normal vector.

We shall now expand the quantities in the Lagrangian (2.4) in terms of perturbations by considering for the background a flat FLRW metric of the form:

$$ds^2 = -dt^2 + a(t)^2\delta_{ij}dx^i dx^j\,, \tag{2.6}$$

where $a(t)$ is the scale factor. Therefore, we can define:

$$\begin{aligned}
&\delta K_{\mu\nu} = Hh_{\mu\nu} + K_{\mu\nu}\,, \quad \delta\mathcal{S} = \mathcal{S} - 3H^2 = -2H\delta K + \delta K^\mu_\nu \delta K^\nu_\mu\,,\\
&\delta K = 3H + K\,, \quad \delta\mathcal{U} = -H\delta\mathcal{R} + \delta K^\mu_\nu\delta K^\nu_\mu\,, \quad \delta\alpha_1 = \partial_i\delta N\partial^i\delta N\,,\\
&\delta\alpha_2 = \partial_i\delta N\nabla_k\nabla^k\partial^i\delta N\,, \qquad \delta\alpha_3 = \mathcal{R}\nabla_i\partial^i\delta N\,, \quad \delta\alpha_4 = \partial_i\delta N\Delta^2\partial^i\delta N\,,\\
&\delta\alpha_5 = \Delta^2\mathcal{R}\nabla_i\partial^i\delta N\,, \quad \delta\mathcal{Z}_1 = \nabla_i\delta\mathcal{R}\nabla^i\delta\mathcal{R}\,, \quad \delta\mathcal{Z}_2 = \nabla_i\delta\mathcal{R}_{jk}\nabla^i\delta\mathcal{R}^{jk}
\end{aligned} \tag{2.7}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter and ∂_μ is the partial derivative w.r.t. the coordinate x^μ. The operators $\mathcal{R}, \mathcal{Z}$ and $\mathcal{U}$ vanish on a flat FLRW background, thus they contribute only to perturbations, and for convenience we can write $\mathcal{R} = \delta\mathcal{R} = \delta_1\mathcal{R} + \delta_2\mathcal{R}$, $\mathcal{Z} = \delta\mathcal{Z}$, $\mathcal{U} = \delta\mathcal{U}$, where $\delta_1\mathcal{R}$ and $\delta_2\mathcal{R}$ are the perturbations of the Ricci scalar respectively at first and second order. We now proceed with a simple expansion of the Lagrangian (2.4) up to second order:

$$\begin{aligned}
\delta L \;=\;& \bar{L} + L_N\delta N + L_K\delta K + L_\mathcal{S}\delta\mathcal{S} + L_\mathcal{R}\delta\mathcal{R} + L_\mathcal{U}\delta\mathcal{U} + L_\mathcal{Z}\delta\mathcal{Z} + \sum_{i=1}^{5} L_{\alpha_i}\delta\alpha_i\\
+\;& \sum_{i=1}^{2} L_{\mathcal{Z}_i}\delta\mathcal{Z}_i + \frac{1}{2}\left(\delta N\frac{\partial}{\partial N} + \delta K\frac{\partial}{\partial K} + \delta\mathcal{S}\frac{\partial}{\partial\mathcal{S}} + \delta\mathcal{R}\frac{\partial}{\partial\mathcal{R}} + \delta\mathcal{U}\frac{\partial}{\partial\mathcal{U}}\right)^2 L\\
+\;& \mathcal{O}(3),
\end{aligned} \tag{2.8}$$

where $\bar{L}$ is the Lagrangian evaluated on the background and $L_X = \partial L/\partial X$ is the derivative of the Lagrangian w.r.t the quantity X. It can be shown that by considering the perturbed quantities in (2.7) and, after some manipulations, it is possible to obtain the following expression for the action up to second order in perturbations:

$$\begin{aligned}
\mathcal{S}_{ADM} \quad &= \quad \int d^4x\sqrt{-g}\left[\bar{L} + \dot{\mathcal{F}} + 3H\mathcal{F} + (L_N - \dot{\mathcal{F}})\delta N + \left(\dot{\mathcal{F}} + \frac{1}{2}L_{NN}\right)(\delta N)^2\right. \\
&+ \quad L_{\mathcal{S}}\delta K^\nu_\mu \delta K^\mu_\nu + \frac{1}{2}\mathcal{A}(\delta K)^2 + \mathcal{B}\delta N\delta K\mathcal{C}\delta K\delta\mathcal{R} + \mathcal{D}\delta N\delta\mathcal{R} + \mathcal{E}\delta\mathcal{R} \\
&+ \quad \frac{1}{2}\mathcal{G}(\delta\mathcal{R})^2 + L_{\mathcal{Z}}\delta\mathcal{R}^\mu_\nu\mathcal{R}^\nu_\mu + L_{\alpha_1}\partial_i\delta N\partial^i\delta N + L_{\alpha_2}\partial_i\delta N\nabla_k\nabla^k\partial^i\delta N \\
&+ \quad L_{\alpha_3}\mathcal{R}\nabla_i\partial^i\delta N + L_{\alpha_4}\partial_i\delta N\Delta^2\partial^i\delta N + L_{\alpha_5}\Delta\mathcal{R}\nabla_i\partial^i\delta N \\
&+ \quad \left. L_{\mathcal{Z}_1}\nabla_i\delta\mathcal{R}\nabla^i\delta\mathcal{R} + L_{\mathcal{Z}_2}\nabla_i\delta\mathcal{R}_{jk}\nabla^i\delta\mathcal{R}^{jk}\right], \qquad (2.9)
\end{aligned}$$

where:

$$\begin{aligned}
&\mathcal{A} = L_{KK} + 4H^2L_{\mathcal{SS}} - 4HL_{\mathcal{S}K}, \\
&\mathcal{B} = L_{KN} - 2HL_{\mathcal{S}N}, \\
&\mathcal{C} = L_{KR} - 2HL_{\mathcal{S}R} + \frac{1}{2}L_{\mathcal{U}} - HL_{K\mathcal{U}} + 2H^2L_{\mathcal{SU}}, \\
&\mathcal{D} = L_{N\mathcal{R}} + \frac{1}{2}\dot{L}_{\mathcal{U}} - HL_{N\mathcal{U}}, \\
&\mathcal{E} = L_{\mathcal{R}} - \frac{3}{2}HL_{\mathcal{U}} - \frac{1}{2}\dot{L}_{\mathcal{U}}, \\
&\mathcal{F} = L_K - 2HL_{\mathcal{S}}, \\
&\mathcal{G} = L_{\mathcal{RR}} + H^2L_{\mathcal{UU}} - 2HL_{\mathcal{RU}}. \qquad (2.10)
\end{aligned}$$

Here and throughout the Chapter, unless stated otherwise, dots indicate derivatives w.r.t. cosmic time, t. The above quantities are general functions of time evaluated on the background. In order to obtain action (2.9), we have followed the same steps as in Refs. [37, 65], however, there are some differences in the results due to the different convention that we use for the normal vector (Eq. (2.3)). As a result the differences stem from the terms which contain K and $K_{\mu\nu}$. More details are in Appendix 2.8, where we derive the contribution of δK and $\delta\mathcal{S}$, and in Appendix 2.9, where we explicitly comment and derive the perturbations generated by $\mathcal{U}$.

Finally, we derive the modified Friedmann equations considering the first order action, which can be written as follows:

$$\mathcal{S}^{(1)}_{ADM} = \int d^4x\left[\delta\sqrt{h}(\bar{L} + 3H\mathcal{F} + \dot{\mathcal{F}}) + a^3(L_N + 3H\mathcal{F} + \bar{L})\delta N + a^3\mathcal{E}\delta_1\mathcal{R}\right], \qquad (2.11)$$

where $\delta_1\mathcal{R}$ is the contribution of the Ricci scalar at first order. Notice that we used $\sqrt{-g} = N\sqrt{h}$, where h is the determinant of the three dimensional metric. It is straightforward to show that by varying the above action w.r.t. δN and $\delta\sqrt{h}$, one finds the Friedmann equations:

$$\begin{aligned} &L_N + 3H\mathcal{F} + \bar{L} = 0\,, \\ &\bar{L} + 3H\mathcal{F} + \dot{\mathcal{F}} = 0. \end{aligned} \tag{2.12}$$

Hence, the homogeneous part of action (2.9) vanishes after applying the Friedmann equations.

2.3.2 The EFT action in ADM notation

We shall now go back to the EFT action (2.1) and rewrite it in the ADM notation. This will allow us to easily compare it with action (2.9) and obtain a general recipe to map an ADM action into the EFT language. To this purpose, an important step is to connect the δg^{00} used in this formalism with δN used in the ADM formalism:

$$g^{00} = -\frac{1}{N^2} = -1 + 2\delta N - 3(\delta N)^2 + ... \equiv -1 + \delta g^{00}\,, \tag{2.13}$$

from which follows that $(\delta g^{00})^2 = 4(\delta N)^2$ at second order. Considering the Eqs. (2.7) and (2.13), it is very easy to write the EFT action in terms of ADM quantities, the only term which requires a bit of manipulation is $(1 + \Omega(t))R$, which we will show in the following. First, let us use the Gauss-Codazzi relation [19] which allows one to express the four dimensional Ricci scalar in terms of three dimensional quantities typical of ADM formalism:

$$R = \mathcal{R} + K_{\mu\nu}K^{\mu\nu} - K^2 + 2\nabla_\nu(n^\nu\nabla_\mu n^\mu - n^\mu\nabla_\mu n^\nu)\,. \tag{2.14}$$

Then, we can write:

$$\begin{aligned} \int d^4x\sqrt{-g}\frac{m_0^2}{2}(1+\Omega)R = &\int d^4x\sqrt{-g}\frac{m_0^2}{2}(1+\Omega)\left[\mathcal{R} + K_{\mu\nu}K^{\mu\nu} - K^2\right. \\ &\left. +2\nabla_\nu\left(n^\nu\nabla_\mu n^\mu - n^\mu\nabla_\mu n^\nu\right)\right]\,, \\ = &\int d^4x\sqrt{-g}\frac{m_0^2}{2}(1+\Omega)\left[\mathcal{R} + \mathcal{S} - K^2 + 2\nabla_\nu\left(n^\nu K - a^\nu\right)\right]\,, \\ = &\int d^4x\sqrt{-g}\left[\frac{m_0^2}{2}(1+\Omega)\left(\mathcal{R} + \mathcal{S} - K^2\right) + m_0^2\dot{\Omega}\frac{K}{N}\right]\,, \end{aligned} \tag{2.15}$$

where in the last line we have used that $\nabla^\nu a_\nu = 0$. Proceeding as usual and employing the relation (2.139), we obtain:

$$\begin{aligned}\int d^4x\sqrt{-g}\frac{m_0^2}{2}(1+\Omega)R = \int d^4x\sqrt{-g}m_0^2\Big\{\frac{1}{2}(1+\Omega)\mathcal{R} + 3H^2(1+\Omega)\\ +2\dot{H}(1+\Omega) + 2H\dot{\Omega} + \ddot{\Omega} + \Big[H\dot{\Omega} - 2\dot{H}(1+\Omega) - \ddot{\Omega}\Big]\delta N\\ -\dot{\Omega}\delta K\delta N + \frac{(1+\Omega)}{2}\delta K^\mu_\nu \delta K^\nu_\mu - \frac{(1+\Omega)}{2}(\delta K)^2\\ + \Big[2\dot{H}(1+\Omega) + 2H\dot{\Omega} + \ddot{\Omega} - 3H\dot{\Omega}\Big](\delta N)^2\Big\}.\end{aligned} \tag{2.16}$$

Finally, after combining terms correctly, we obtain the final form of the EFT action in the ADM notation, up to second order in perturbations:

$$\begin{aligned}\mathcal{S}_{EFT} = \int d^4x\sqrt{-g}\Big\{\frac{m_0^2}{2}(1+\Omega)\mathcal{R} + 3H^2 m_0^2(1+\Omega) + 2\dot{H}m_0^2(1+\Omega)\\ +2m_0^2 H\dot{\Omega} + m_0^2\ddot{\Omega} + \Lambda + \Big[H\dot{\Omega}m_0^2 - 2\dot{H}m_0^2(1+\Omega) - \ddot{\Omega}m_0^2 - 2c\Big]\delta N\\ -(m_0^2\dot{\Omega} + \bar{M}_1^3)\delta K\delta N + \frac{1}{2}\left[m_0^2(1+\Omega) - \bar{M}_3^2\right]\delta K^\mu_\nu\delta K^\nu_\mu - \frac{1}{2}\left[m_0^2(1+\Omega)\right.\\ \left.+\bar{M}_2^2\right](\delta K)^2 + \hat{M}^2\delta N\delta\mathcal{R} + \Big[2\dot{H}m_0^2(1+\Omega) + \ddot{\Omega}m_0^2 - Hm_0^2\dot{\Omega} + 3c\\ +2M_2^4\Big](\delta N)^2 + 4m_2^2 h^{\mu\nu}\partial_\mu\delta N\partial_\nu\delta N + \frac{\bar{m}_5}{2}\delta\mathcal{R}\delta K + \lambda_1(\delta\mathcal{R})^2\\ +\lambda_2\delta\mathcal{R}^\mu_\nu\delta\mathcal{R}^\nu_\mu + 2\lambda_3\delta\mathcal{R}h^{\mu\nu}\nabla_\mu\partial_\nu\delta N + 4\lambda_4 h^{\mu\nu}\partial_\mu\delta N\nabla^2\partial_\nu\delta N\\ +\lambda_5 h^{\mu\nu}\nabla_\mu\mathcal{R}\nabla_\nu\mathcal{R} + \lambda_6 h^{\mu\nu}\nabla_\mu\mathcal{R}_{ij}\nabla_\nu\mathcal{R}^{ij} + 4\lambda_7 h^{\mu\nu}\partial_\mu\delta N\nabla^4\partial_\nu\delta N\\ +2\lambda_8 h^{\mu\nu}\nabla^2\mathcal{R}\nabla_\mu\partial_\nu\delta N\Big\}\,.\end{aligned} \tag{2.17}$$

This final form of the action will be the starting point from which we will construct a general mapping between the EFT and ADM formalisms.

2.3.3 The Mapping

We now proceed to explicitly work out the mapping between the EFT action (2.17) and the ADM one (2.9). The result will be a very convenient recipe in order to quickly map any model written in the ADM notation into the EFT formalism. In the next Section we will apply it to most of the interesting candidate models of DE/MG, providing a complete guide on how to go from covariant formulations all the way to the EFT formalism at the basis of the Einstein-Boltzmann solver *EFTCAMB* [22, 23].

A direct comparison between actions (2.9) and (2.17) allows us to straightforwardly identify the following:

$$\frac{m_0^2}{2}(1+\Omega)=\mathcal{E}\,,\qquad -2c+m_0^2\left[-2\dot{H}(1+\Omega)-\ddot{\Omega}+H\dot{\Omega}\right]=L_N-\dot{\mathcal{F}}\,,$$
$$\Lambda+m_0^2\left[3H^2(1+\Omega)+2\dot{H}(1+\Omega)+2H\dot{\Omega}+\ddot{\Omega}\right]=\bar{L}+3H\mathcal{F}+\dot{\mathcal{F}}\,,$$
$$m_0^2\left[2\dot{H}(1+\Omega)-H\dot{\Omega}+\ddot{\Omega}\right]+2M_2^4+3c=\dot{\mathcal{F}}+\frac{L_{NN}}{2},$$
$$-m_0^2(1+\Omega)-\bar{M}_2^2=\mathcal{A},\quad \lambda_1=\frac{\mathcal{G}}{2},\quad -m_0^2\dot{\Omega}-\bar{M}_1^3=\mathcal{B},\quad \frac{\bar{m}_5}{2}=\mathcal{C},$$
$$\hat{M}^2=\mathcal{D},\quad \frac{m_0^2}{2}(1+\Omega)-\frac{\bar{M}_3^2}{2}=L_{\mathcal{S}},\quad 4m_2^2=L_{\alpha_1},\quad \lambda_5=L_{\mathcal{Z}_1},\quad 4\lambda_4=L_{\alpha_2},$$
$$2\lambda_3=L_{\alpha_3},\qquad 4\lambda_7=L_{\alpha_4},\quad 2\lambda_8=L_{\alpha_5},\quad \lambda_2=L_{\mathcal{Z}},\quad \lambda_6=L_{\mathcal{Z}_2}. \tag{2.18}$$

It is now simply a matter of inverting these relations in order to obtain the desired general mapping results:

$$\Omega(t)=\frac{2}{m_0^2}\mathcal{E}-1,\qquad c(t)=\frac{1}{2}(\dot{\mathcal{F}}-L_N)+(H\dot{\mathcal{E}}-\ddot{\mathcal{E}}-2\mathcal{E}\dot{H}),$$
$$\Lambda(t)=\bar{L}+\dot{\mathcal{F}}+3H\mathcal{F}-(6H^2\mathcal{E}+2\ddot{\mathcal{E}}+4H\dot{\mathcal{E}}+4\dot{H}\mathcal{E})\,,\qquad \bar{M}_2^2(t)=-\mathcal{A}-2\mathcal{E},$$
$$M_2^4(t)=\frac{1}{2}\left(L_N+\frac{L_{NN}}{2}\right)-\frac{c}{2},\qquad \bar{M}_1^3(t)=-\mathcal{B}-2\dot{\mathcal{E}},\qquad \bar{M}_3^2(t)=-2L_{\mathcal{S}}+2\mathcal{E},$$
$$m_2^2(t)=\frac{L_{\alpha_1}}{4},\qquad \bar{m}_5(t)=2\mathcal{C},\qquad \hat{M}^2(t)=\mathcal{D},\qquad \lambda_1(t)=\frac{\mathcal{G}}{2},$$
$$\lambda_2(t)=L_{\mathcal{Z}},\qquad \lambda_3(t)=\frac{L_{\alpha_3}}{2},\qquad \lambda_4(t)=\frac{L_{\alpha_2}}{4},\qquad \lambda_5(t)=L_{\mathcal{Z}_1},$$
$$\lambda_6(t)=L_{\mathcal{Z}_2},\qquad \lambda_7(t)=\frac{L_{\alpha_4}}{4},\qquad \lambda_8(t)=\frac{L_{\alpha_5}}{2}. \tag{2.19}$$

Let us stress that the above definitions of the EFT functions are very useful if one is interested in writing a specific action in EFT language. Indeed the only step required before applying (2.19), is to write the action which specifies the chosen theory in ADM form, without the need of perturbing the theory and its action up to quadratic order.

The expressions of the EFT functions corresponding to a given model, and their time-dependence, are all that is needed in order to implement a specific model of DE/MG in *EFTCAMB* and have it solve for the dynamics of perturbations, outputting observable quantities of interest. Since *EFTCAMB* uses the scale factor as the time variable and the Hubble parameter expressed w.r.t conformal time, one needs to convert the cosmic time t in the argument of the functions in Eq. (2.19) into the scale factor, a, their time derivatives into derivatives w.r.t. the scale factor and transform the Hubble parameter into the one in conformal time τ, while considering it a function of a, see Ref. [80]. This is a straightforward step and we will give some examples in Appendix 2.10.

Let us conclude this Section looking at the equations for the background. Working with the EFT action, and expanding it to first order while using the ADM notation, one obtains:

$$\begin{aligned}
\mathcal{S}^{(1)}_{EFT} &= \int d^4x \left\{ a^3 \frac{m_0^2}{2} (1+\Omega)\, \delta_1 \mathcal{R} + \Big[3H^2 m_0^2 (1+\Omega) + 2\dot{H} m_0^2 (1+\Omega) \right. \\
&+ 2m_0^2 H\dot{\Omega} + m_0^2 \ddot{\Omega} + \Lambda \Big] \delta\sqrt{h} + a^3 \Big[3H\dot{\Omega} m_0^2 - 2c + 3H^2 m_0^2 (1+\Omega) \\
&+ \left. \Lambda \right] \delta N \Big\} \,, \qquad (2.20)
\end{aligned}$$

therefore the variation w.r.t. δN and $\delta\sqrt{h}$ yields:

$$\begin{aligned}
& 3H\dot{\Omega} m_0^2 - 2c + 3H^2 m_0^2 (1+\Omega) + \Lambda = 0 \,, \\
& 3H^2 m_0^2 (1+\Omega) + 2\dot{H} m_0^2 (1+\Omega) + 2m_0^2 H\dot{\Omega} + m_0^2 \ddot{\Omega} + \Lambda = 0 \,.
\end{aligned} \qquad (2.21)$$

Using the mapping (2.19), it is easy to verify that these equations correspond to those in the ADM formalism (2.12). Once the mapping (2.19) has been worked out, it is straightforward to obtain the Friedmann equations without having to vary the action for each specific model.

2.4 Model mapping examples

Having derived the precise mapping between the ADM formalism and the EFT approach in Section 2.3.3, we proceed to apply it to some specific cases which are of cosmological interest, i.e. minimally coupled quintessence [71], $f(R)$ theory [11], Horndeski/GG [78, 79], GLPV [66] and Hořava gravity [64]. The mapping of some of these theories is already present in the literature (see Refs. [14, 15, 39, 65, 74, 75] for more details). However, since one of the main purposes of this work is to provide a self-contained and general recipe that can be used to easily implement a specific theory in *EFTCAMB*, we will present all the mapping of interest, including those that are already in the literature due to the aforementioned differences in the definition of the normal vector and some of the EFT functions. Let us notice that the mapping of the GLPV Lagrangians in particular, is one of the new results obtained in this work.

2.4.1 Minimally coupled quintessence

As illustrated in Refs. [14, 15, 75], the mapping of minimally coupled quintessence [71] into EFT functions is very straightforward. The typical action for such a model is of the following form:

$$\mathcal{S}_\phi = \int d^4x\sqrt{-g}\left[\frac{m_0^2}{2}R - \frac{1}{2}\partial^\nu\phi\partial_\nu\phi - V(\phi)\right], \qquad (2.22)$$

where $\phi(t, x^i)$ is a scalar field and $V(\phi)$ is its potential. Let us proceed by rewriting the second term in unitary gauge and in ADM quantities:

$$-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \;\rightarrow\; -\frac{\dot\phi_0^2(t)}{2}g^{00} \equiv \frac{\dot\phi_0^2(t)}{2N^2}, \qquad (2.23)$$

where $\phi_0(t)$ is the field background value. Substituting back into the action we get, in the ADM formalism, the following action:

$$\mathcal{S}_\phi = \int d^4x\sqrt{-g}\left\{\frac{m_0^2}{2}\left[\mathcal{R} + \mathcal{S} - K^2\right] + \frac{1}{N^2}\frac{\dot\phi_0^2(t)}{2} - V(\phi_0)\right\}, \quad (2.24)$$

where we have used the Gauss-Codazzi relation (2.14) to express the four dimensional Ricci scalar in terms of three dimensional quantities. Now, since the initial covariant action has been written in terms of ADM quantities, we can finally apply the results in Eqs. (2.19) to get the EFT functions:

$$\Omega(t) = 0, \qquad c(t) = \frac{\dot\phi_0^2}{2}, \qquad \Lambda(t) = \frac{\dot\phi_0^2}{2} - V(\phi_0). \qquad (2.25)$$

Notice that the other EFT functions are zero. In Refs. [14, 15] the above mapping has been obtained directly from the covariant action while our approach follows more strictly the one adopted in Ref. [75]. However, let us notice that w.r.t. it, our results differ due to a different definition of the background EFT functions.*

*The background EFT functions adopted here are related to the ones in Ref. [75], by the following relations:

$$1 + \Omega(t) = f(t), \qquad \Lambda(t) = -\tilde\Lambda(t) + c(t), \qquad c(t) = \tilde c(t). \qquad (2.26)$$

where f and tildes quantities correspond to the EFT functions in Ref. [75]. These differences are due to the fact that in our formalism we have in the EFT action the term $-c\delta g^{00}$ while in the other formalism the authors use $-\tilde c g^{00}$, therefore an extra contribution to $\tilde\Lambda$ from this operator comes when using $g^{00} = -1 + \delta g^{00}$. Instead the different definition of the conformal coupling function, Ω, is due to numerical reasons related to the implementation of the EFT approach in CAMB.

Moreover, in order to use them in *EFTCAMB* one need to convert them in conformal time τ, therefore one has:

$$c(\tau) = \mathcal{H}^2 \frac{\phi_0'^2}{2}\,, \qquad \Lambda(\tau) = \mathcal{H}^2 \frac{\phi_0'^2}{2} - V(\phi_0)\,, \tag{2.27}$$

where the prime indicates the derivative w.r.t. the scale factor, $a(\tau)$, and $\mathcal{H} \equiv \frac{1}{a}\frac{da}{d\tau}$ is the Hubble parameter in conformal time. Minimally coupled quintessence models are already implemented in the public versions of *EFTCAMB* [80].

2.4.2 $f(R)$ gravity

The second example we shall illustrate is that of $f(R)$ gravity [10, 11]. The mapping of the latter into the EFT language was derived in Refs. [14, 75]. Here, we present an analogous approach which uses the ADM formalism. Let us start with the action :

$$\mathcal{S}_f = \int d^4x\sqrt{-g}\frac{m_0^2}{2}\left[R + f(R)\right], \tag{2.28}$$

where $f(R)$ is a general function of the four dimensional Ricci scalar.

In order to map it into our EFT approach, we will proceed to expand this action around the background value of the Ricci scalar, $R^{(0)}$. Therefore, we choose a specific time slicing where the constant time hypersurfaces coincide with uniform R hypersurfaces. This allows us to truncate the expansion at the linear order because higher orders will always contribute one power or more of δR to the equations of motion, which vanishes. For a more complete analysis we refer the reader to Ref. [14] . After the expansion we obtain the following Lagrangian:

$$\mathcal{S}_f = \int d^4x\sqrt{-g}\frac{m_0^2}{2}\left\{\left[1 + f_R(R^{(0)})\right]R + f(R^{(0)}) - R^{(0)}f_R(R^{(0)})\right\}, \tag{2.29}$$

where $f_R \equiv \frac{df}{dR}$. In the ADM formalism the above action reads:

$$\begin{aligned}\mathcal{S}_f &= \int d^4x\sqrt{-g}\frac{m_0^2}{2}\left\{\left[1 + f_R(R^{(0)})\right]\left[\mathcal{R} + \mathcal{S} - K^2\right] + \frac{2}{N}\dot{f}_R K\right.\\ &+ \left. f(R^{(0)}) - R^{(0)}f_R(R^{(0)})\right\},\end{aligned} \tag{2.30}$$

where we have used as usual the Gauss Codazzi relation (2.14). Using Eqs. (2.19), it is easy to calculate that the only non zero EFT functions for $f(R)$ gravity are:

$$\Omega(t) = f_R(R^{(0)})\,, \qquad \Lambda(t) = \frac{m_0^2}{2}f(R^{(0)}) - R^{(0)}f_R(R^{(0)})\,. \tag{2.31}$$

The public version of *EFTCAMB* already contains the designer $f(R)$ models [12, 80, 81], while the specific Hu-Sawicki model has been implemented through the full mapping procedure [82].

2.4.3 The Galileon Lagrangians

The Galileon class of theories were derived in Ref. [83], by studying the decoupling limit of the five dimensional model of modified gravity known as DGP [84]. In this limit, the dynamics of the scalar d.o.f., corresponding to the longitudinal mode of the massive graviton, decouple from gravity and enjoy a galilean shift symmetry around Minkowski background, as a remnant of the five dimensional Poincare' invariance [85]. Requiring the scalar field to obey this symmetry and to have second order equations of motion allows one to identify a finite amount of terms that can enter the action. These terms are typically organized into a set of Lagrangians which, subsequently, have been covariantized [86] and the final form is what is known as the Generalized Galileon (GG) model [79]. This set of models represent the most general theory of gravity with a scalar d.o.f. and second order field equations in four dimensions and has been shown to coincide with the class of theories derived by Horndeski in Ref. [78]. It is therefore common to refer to these models with the terms GG and Horndeski gravity, alternatively. GG models have been deeply investigated in the cosmological context, since they display self accelerated solutions which can be used to realize both a single field inflationary scenario at early times [87–96] and a late time accelerated expansion [97–101]. Moreover, on small scales these models naturally display the Vainshtein screening mechanism [102, 103], which can efficiently hide the extra d.o.f. from local tests of gravity [83, 85, 104–108].

GG models include most of the interesting and viable theories of DE/MG that we aim to test against cosmological data. To this extent, the Einstein-Boltzmann solver *EFTCAMB* can be readily used to explore these theories both in a model-independent way, through a subset of the EFT functions, and in a model-specific way [22, 80]. In the latter case, the first step consists of mapping a given GG model into the EFT language. In the following we derive the general mapping between GG and EFT functions, in order to provide an instructive and self-consistent compendium to easily map any given GG model into the formalism at the basis of *EFTCAMB*.

Let us introduce the GG action:

$$\mathcal{S}_{GG} = \int d^4x\sqrt{-g}\left(L_2 + L_3 + L_4 + L_5\right), \tag{2.32}$$

where the Lagrangians have the following structure:

$$
\begin{aligned}
L_2 &= \mathcal{K}(\phi, X)\,, \\
L_3 &= G_3(\phi, X)\Box\phi\,, \\
L_4 &= G_4(\phi, X)R - 2G_{4X}(\phi, X)\left[(\Box\phi)^2 - \phi^{;\mu\nu}\phi_{;\mu\nu}\right]\,, \\
L_5 &= G_5(\phi, X)G_{\mu\nu}\phi^{;\mu\nu} + \frac{1}{3}G_{5X}(\phi, X)\left[(\Box\phi)^3 - 3\Box\phi\phi^{;\mu\nu}\phi_{;\mu\nu}\right. \\
&\quad \left. + 2\phi_{;\mu\nu}\phi^{;\mu\sigma}\phi_{;\sigma}^{;\nu}\right]\,,
\end{aligned} \tag{2.33}
$$

here $G_{\mu\nu}$ is the Einstein tensor, $X \equiv \phi^{;\mu}\phi_{;\mu}$ is the kinetic term and $\{\mathcal{K}, G_i\}$ $(i = 3, 4, 5)$ are general functions of the scalar field ϕ and X, and $G_{iX} \equiv \partial G_i/\partial X$. Moreover, $\Box = \nabla^2$ and ; stand for the covariant derivative w.r.t. the metric $g_{\mu\nu}$. The mapping of GG is already present in the literature. For instance in Ref. [74] the mapping is obtained directly from the covariant Lagrangians, while in Refs. [65, 75] the authors start from the ADM version of the action. In this Chapter we present in details all the steps from the covariant Lagrangians (2.33) to their expressions in ADM quantities; we then use the mapping (2.19) to obtain the EFT functions corresponding to GG. This allows us to give an instructive presentation of the method, while providing a final result consistent with the EFT conventions at the basis of *EFTCAMB*. Throughout these steps, we will highlight the differences w.r.t. Refs. [65, 74, 75] which arise because of different conventions. Finally, in Appendix 2.10 we rewrite the results of this Section with the scale factor as the independent variable and the Hubble parameter defined w.r.t. the conformal time, making them readily implementable in *EFTCAMB*.

Since the GG action is formulated in covariant form, we shall use the following relations to rewrite the GG Lagrangians in ADM form:

$$
n_\mu = \gamma\phi_{;\mu}, \qquad \gamma = \frac{1}{\sqrt{-X}}, \qquad \dot{n}_\mu = n^\nu n_{\mu;\nu}\,, \tag{2.34}
$$

where we have, as usual, assumed that constant time hypersurfaces correspond to uniform field ones. We notice that the acceleration, $\dot{n}_\mu$, and the extrinsic curvature $K^{\mu\nu}$ are orthogonal to the normal vector. This allows us to decompose the covariant derivative of the normal vector as follows:

$$
n_{\nu;\mu} = K_{\mu\nu} - n_\mu\dot{n}_\nu. \tag{2.35}
$$

With these definitions it can be easily verified that:

$$
\phi_{;\mu\nu} = \gamma^{-1}(K_{\mu\nu} - n_\mu\dot{n}_\nu - n_\nu\dot{n}_\mu) + \frac{\gamma^2}{2}\phi^{;\lambda}X_{;\lambda}n_\mu n_\nu, \tag{2.36}
$$

$$
\Box\phi = \gamma^{-1}K - \frac{\gamma^2}{2}\phi^{;\lambda}X_{;\lambda}. \tag{2.37}
$$

- **$\mathbf{L_2}$- Lagrangian**

Let us start with the simplest of the Lagrangians which can be Taylor expanded in the kinetic term X, around its background value X_0, as follows:

$$\mathcal{K}(\phi, X) = \mathcal{K}(\phi_0, X_0) + \mathcal{K}_X(\phi_0, X_0)(X - X_0) + \frac{1}{2}\mathcal{K}_{XX}(X - X_0)^2, \quad (2.38)$$

where in terms of ADM quantities we have:

$$X = -\frac{\dot{\phi}_0(t)^2}{N^2} = \frac{X_0}{N^2}\,. \quad (2.39)$$

Now by applying the results in Eqs. (2.19), the corresponding EFT functions can be written as:

$$\Lambda(t) = \mathcal{K}(\phi_0, X_0), \qquad c(t) = \mathcal{K}_X(\phi_0, X_0)X_0 \qquad M_2^4(t) = \mathcal{K}_{XX}(\phi_0, X_0)X_0^2. \quad (2.40)$$

The differences with previous works in this case are the ones listed in Eq. (2.26).

- **$\mathbf{L_3}$- Lagrangian**

In order to rewrite this Lagrangian into the desired form, which depends only on ADM quantities, we introduce an auxiliary function:

$$G_3 \equiv F_3 + 2XF_{3X}\,. \quad (2.41)$$

We proceed to plug this in the L_3-Lagrangian (2.33) and using Eq. (2.37) we obtain, up to a total derivative:

$$L_3 = -F_{3\phi}X - 2(-X)^{3/2}F_{3X}K\,. \quad (2.42)$$

Now going to unitary gauge and considering Eq. (2.39), we can directly use (2.19). Let us start with $c(t)$:

$$\begin{aligned} c(t) \;&= \tfrac{1}{2}(\mathcal{F} - L_N) = -3\dot{\phi}_0^2\ddot{\phi}_0F_{3X} + 2\ddot{\phi}_0F_{3XX}\dot{\phi}_0^4 - \dot{\phi}_0^4F_{3X\phi} + F_{3\phi}\dot{\phi}_0^2 \\ &\quad -F_{3\phi X}\dot{\phi}_0^4 - 6H\dot{\phi}_0^5F_{3XX} + 9HF_{3X}\dot{\phi}_0^3\,. \end{aligned} \quad (2.43)$$

Now we want to eliminate the dependence on the auxiliary function F_3. In order to do this, we need to recombine terms by using the following:

$$G_3 = F_3 + 2XF_{3X}, \quad G_{3\phi} = F_{3\phi} - 2\dot{\phi}_0^2F_{3X\phi}, \quad G_{3X} = 3F_{3X} - 2\dot{\phi}_0^2F_{3XX}\,,$$
$$G_{3XX} = 3F_{3XX} - 2\dot{\phi}_0^2F_{3XXX} + 2F_{3XX}, \quad G_{3\phi X} = 3F_{3X\phi} - 2\dot{\phi}_0^2F_{3\phi XX}\,, \quad (2.44)$$

which gives the final expression:

$$c(t) = \dot{\phi}_0^2 G_{3X}(3H\dot{\phi}_0 - \ddot{\phi}_0) + G_{3\phi}\dot{\phi}_0^2. \tag{2.45}$$

Now let us move on to the remaining non zero EFT functions corresponding to the L_3 Lagrangian:

$$\begin{aligned}
&\Lambda(t) = \bar{L} + \dot{\mathcal{F}} + 3H\mathcal{F} = G_{3\phi}\dot{\phi}_0^2 - 2\ddot{\phi}_0\dot{\phi}_0^2 G_{3X}\,, \\
&\bar{M}_1^3(t) = -L_{KN} = -2G_{3X}\dot{\phi}_0^3\,, \\
&M_2^4(t) = \frac{1}{2}\left(L_N + \frac{L_{NN}}{2}\right) - \frac{c}{2} = G_{3X}\frac{\dot{\phi}_0^2}{2}(\ddot{\phi}_0 + 3H\dot{\phi}_0) - 3HG_{3XX}\dot{\phi}_0^5 \\
&\qquad -G_{3\phi X}\frac{\dot{\phi}_0^4}{2}\,,
\end{aligned} \tag{2.46}$$

where we have used the relations (2.44). In the definitions of the EFT functions, G_3 and its derivatives are evaluated on the background. We suppressed the dependence on (ϕ_0, X_0) to simplify the final expressions. Before proceeding to map the remaining GG Lagrangians, let us comment on the differences w.r.t. the results in literature [65, 74, 75]. The results coincide up to two notable exceptions. The background functions are redefined as presented in Eq. (2.26) and $\bar{M}_1^3 = -\bar{m}_1^3$. In the latter term, the minus sign is not a simple redefinition but rather comes from the fact that our extrinsic curvature has an overall minus sign difference due to the definition of the normal vector. Therefore, the term proportional to $\delta K \delta g^{00}$ will always differ by a minus sign.

- **$\mathbf{L_4}$- Lagrangian**

Let us now consider the L_4 Lagrangian:

$$L_4 = G_4 R - 2G_{4X}\left[(\Box\phi)^2 - \phi^{;\mu\nu}\phi_{;\mu\nu}\right]. \tag{2.47}$$

After some preliminary manipulations of the Lagrangian, we get:

$$L_4 = G_4\mathcal{R} + 2G_{4X}(K^2 - K_{\mu\nu}K^{\mu\nu}) + 2G_{4X}X_{;\lambda}(Kn^\lambda - \dot{n}^\lambda)\,. \tag{2.48}$$

We proceed by using the relation:

$$\partial_\mu G_4 = G_{4X}X_{;\mu} + G_{4\phi}\phi_{;\mu}\,, \tag{2.49}$$

which we substitute in the last term of the Lagrangian (2.48) and, using integration by parts, we get:

$$L_4 = G_4\mathcal{R} + (2G_{4X}X - G_4)(K^2 - K_{\mu\nu}K^{\mu\nu}) + 2G_{4\phi}\sqrt{-X}K\,, \tag{2.50}$$

where we have used the Gauss-Codazzi relation (2.14). Let us recall that we can relate $\phi_{;\mu}$ to X by using Eq. (2.39).

Finally, in the same spirit as for L_3, we derive from the Lagrangian (2.50) the corresponding non zero EFT functions by using the results (2.19):

$$
\begin{aligned}
\Omega(t) &= -1 + \frac{2}{m_0^2} G_4\,, \\
c(t) &= -\frac{1}{2}\Big(-\dot{L}_K + 2\dot{H}L_{\mathcal{S}} + 2H\dot{L}_{\mathcal{S}}\Big) + H\dot{L}_{\mathcal{R}} - \ddot{L}_{\mathcal{R}} - 2\dot{H}L_{\mathcal{R}} \\
&= G_{4X}(2\ddot{\phi}_0^2 + 2\dot{\phi}_0\dddot{\phi}_0 + 4\dot{H}\dot{\phi}_0^2 + 2H\dot{\phi}_0\ddot{\phi}_0 - 6H^2\dot{\phi}_0^2) \\
&\quad + G_{4X\phi}(2\dot{\phi}_0^2\ddot{\phi}_0 + 10H\dot{\phi}_0^3) + G_{4XX}(12H^2\dot{\phi}_0^4 - 8H\dot{\phi}_0^3\ddot{\phi}_0 - 4\dot{\phi}_0^2\ddot{\phi}_0^2)\,, \\
\Lambda(t) &= \bar{L} + \dot{\mathcal{F}} + 3H\mathcal{F} - (6H^2L_{\mathcal{R}} + 2\ddot{L}_{\mathcal{R}} + 4H\dot{L}_{\mathcal{R}} + 4\dot{H}L_{\mathcal{R}}), \\
&= G_{4X}\left[12H^2\dot{\phi}_0^2 + 8\dot{H}\dot{\phi}_0^2 + 16H\dot{\phi}_0\ddot{\phi}_0 + 4(\ddot{\phi}_0^2 + \dot{\phi}_0\dddot{\phi}_0)\right] \\
&\quad - G_{4XX}\big(16H\dot{\phi}_0^3\ddot{\phi}_0 + 8\dot{\phi}_0^2\ddot{\phi}_0^2\big) + 8HG_{4X\phi}\dot{\phi}_0^3\,, \\
M_2^4(t) &= \frac{1}{2}(L_N + L_{NN}/2) - \frac{c}{2} = G_{4\phi X}\big(4H\dot{\phi}_0^3 - \ddot{\phi}_0\dot{\phi}_0^2\big) - 6H\dot{\phi}_0^5 G_{4\phi XX} \\
&\quad - G_{4X}\left(2\dot{H}\dot{\phi}_0^2 + H\dot{\phi}_0\ddot{\phi}_0 + \dot{\phi}_0\dddot{\phi}_0 + \ddot{\phi}_0^2\right) \\
&\quad + G_{4XX}\big(18H^2\dot{\phi}_0^4 + 2\dot{\phi}_0^2\ddot{\phi}_0^2 + 4H\ddot{\phi}_0\dot{\phi}_0^3\big) - 12H^2 G_{4XXX}\dot{\phi}_0^6\,, \\
\bar{M}_2^2(t) &= -L_{KK} - 2L_{\mathcal{R}} = 4G_{4X}\dot{\phi}_0^2\,, \\
\bar{M}_3^2(t) &= -2L_{\mathcal{S}} + 2L_{\mathcal{R}} = -4G_{4X}\dot{\phi}_0^2 \equiv -\bar{M}_2^2(t)\,, \\
\hat{M}^2(t) &= L_{N\mathcal{R}} = 2\dot{\phi}_0^2 G_{4X}\,, \\
\bar{M}_1^3(t) &= 2HL_{\mathcal{S}N} - 2\dot{L}_{\mathcal{R}} - L_{KN} = G_{4X}(4\dot{\phi}_0\ddot{\phi}_0 + 8H\dot{\phi}_0^2) \\
&\quad - 16HG_{4XX}\dot{\phi}_0^4 - 4G_{4\phi X}\dot{\phi}_0^3\,,
\end{aligned}
\tag{2.51}
$$

where also in this case G_4 and its derivative are evaluated on the background. Let us notice that the above relations satisfy the conditions which define Horndeski/GG theories, i.e.:

$$
\bar{M}_2^2 = -\bar{M}_3^2(t) = 2\hat{M}^2(t), \tag{2.52}
$$

as found in Refs. [65, 74]. Finally, besides the differences mentioned previously for the L_2 and L_3 Lagrangians which also apply here, we notice that $\hat{M}^2 = \mu_1^2$ when comparing with Ref. [65].

- **$\mathbf{L_5}$- Lagrangian**

Finally, let us conclude with the L_5 Lagrangian. This Lagrangian contains cubic terms which makes it more complicated to express it in the ADM form:

$$L_5 = G_5(\phi,X)G_{\mu\nu}\phi^{;\mu\nu}+\frac{1}{3}G_{5X}(\phi,X)\left[(\Box\phi)^3-3\Box\phi\phi^{;\mu\nu}\phi_{;\mu\nu}+2\phi_{;\mu\nu}\phi^{;\mu\sigma}\phi^{;\nu}_{;\sigma}\right]. \tag{2.53}$$

In order to rewrite L_5, we have to enlist once again the help of an auxiliary function, F_5, which is defined as follows:

$$G_{5X}\equiv F_{5X}+\frac{F_5}{2X}. \tag{2.54}$$

Then, using this definition, we get the following relation:

$$G_{5X}X_{;\rho}=\gamma\nabla_\rho(\gamma^{-1}F_5)-F_{5\phi}\gamma^{-1}n_\rho. \tag{2.55}$$

Let us start with the first term of the Lagrangian, which can be written as:

$$G_5G_{\mu\nu}\phi^{;\mu\nu}=F_5\phi^{;\mu\nu}G_{\mu\nu}-\frac{\gamma}{2}X^{;\nu}n^\mu G_{\mu\nu}F_5+(F_{5\phi}-G_{5\phi})\gamma^{-2}n^\mu n^\nu G_{\mu\nu}, \tag{2.56}$$

hence we need to rewrite $F_5\phi^{;\mu\nu}G_{\mu\nu}$ in terms of ADM quantities which can be achieved by employing the following relation:

$$\begin{aligned} K^{\mu\nu}G_{\mu\nu} &= KK^{\mu\nu}K_{\mu\nu}-K^3_{\mu\nu}+\mathcal{R}_{\mu\nu}K-K^{\mu\nu}n^\sigma n^\rho R_{\mu\sigma\nu\rho}-\frac{1}{2}K\big(\mathcal{R}-K^2\\ &+ K_{\mu\nu}K^{\mu\nu}-2R_{\mu\nu}n^\mu n^\nu\big). \end{aligned} \tag{2.57}$$

This leads to the following:

$$\begin{aligned} F_5\phi^{;\mu\nu}G_{\mu\nu} &= F_5(\gamma^{-1}(-2R_{\mu\nu}n^\mu\dot{n}^\nu)+\frac{\gamma^2}{2}n^\mu n^\nu\phi^{;\lambda}X_{;\lambda}G_{\mu\nu})\\ &+F_5\gamma^{-1}\big[KK^{\mu\nu}K_{\mu\nu}-K^3_{\mu\nu}+\mathcal{R}_{\mu\nu}K^{\mu\nu}-K^{\mu\nu}n^\sigma n^\rho R_{\mu\sigma\nu\rho}\\ &-\frac{1}{2}K\big(\mathcal{R}-K^2+K_{\mu\nu}K^{\mu\nu}-2R_{\mu\nu}n^\mu n^\nu\big)\big]. \end{aligned} \tag{2.58}$$

The second term of the Lagrangian can be computed by considering Eqs. (2.36)-(2.37), which yields:

$$\begin{aligned} &\frac{1}{3}G_{5X}\left[(\Box\phi)^3-3\Box\phi\phi^{;\mu\nu}\phi_{;\mu\nu}+2\phi_{;\mu\nu}\phi^{;\mu\sigma}\phi^{;\nu}_{;\sigma}\right]=\\ &=\frac{G_{5X}}{3}\gamma^{-3}\big(K^3-3KS+2K_{\mu\nu}K^{\mu\sigma}K^\nu_\sigma\big)+G_{5X}\big(-\frac{1}{2}K^2\phi_{;\lambda}X^{;\lambda}-2\dot{n}_\sigma\dot{n}_\nu K^{\nu\sigma}\\ &+\frac{S}{2}\phi_{;\lambda}X^{;\lambda}+2\gamma^{-3}K\dot{n}^\nu\dot{n}_\nu\big)\\ &=\frac{G_{5X}}{3}\gamma^{-3}\tilde{\mathcal{K}}+G_{5X}\mathcal{J}, \end{aligned} \tag{2.59}$$

where the definitions of $\tilde{\mathcal{K}}$ and $\mathcal{J}$ come directly from the second line of the above expression. In Appendix 2.11 we treat in detail the $G_{5X}\mathcal{J}$ term but for now we simply state the final result:

$$G_{5X}\mathcal{J} = F_5\gamma^{-1}\Big[\frac{\tilde{\mathcal{K}}}{2}+K^{\mu\nu}n^\sigma n^\rho R_{\mu\sigma\nu\rho}+\dot{n}^\sigma n^\rho R_{\sigma\rho}-Kn^\sigma n^\rho R_{\sigma\rho}\Big]-\frac{F_{5\phi}}{2}(K^2-S). \tag{2.60}$$

Hence, after collecting all the terms, we get:

$$L_5 = F_5\sqrt{-X}\Big(K^{\mu\nu}\mathcal{R}_{\mu\nu} - \frac{1}{2}K\mathcal{R}\Big) + (G_{5\phi} - F_{5\phi})X\frac{\mathcal{R}}{2} + \frac{(-X)^{3/2}}{3}G_{5X}\tilde{\mathcal{K}} + \frac{G_{5\phi}}{2}X(K^2 - K_{\mu\nu}K^{\mu\nu})\,. \tag{2.61}$$

Now, in order to proceed with the mapping, we need to analyse $\tilde{\mathcal{K}}$ and $\mathcal{U} = K^{\mu\nu}\mathcal{R}_{\mu\nu}$ terms. The latter will be treated as in Appendix 2.9, while the former can be written up to third order as follows:

$$\tilde{\mathcal{K}} = -6H^3 - 6H^2K - 3HK^2 + 3HK_{\mu\nu}K^{\mu\nu} + \mathcal{O}(3). \tag{2.62}$$

Finally, the ultimate Lagrangian is:

$$L_5 = F_5\sqrt{-X}\Big(\mathcal{U} - \frac{1}{2}K\mathcal{R}\Big) + (G_{5\phi} - F_{5\phi})X\frac{\mathcal{R}}{2} + \frac{(-X)^{3/2}}{3}G_{5X}(-6H^3 - 6H^2K - 3HK^2 + 3H\mathcal{S}) + \frac{G_{5\phi}}{2}X(K^2 - \mathcal{S})\,. \tag{2.63}$$

Although F_5 is present in the above Lagrangian, it will disappear when computing the EFT functions as was the case for L_3. At this point we can write down the non zero EFT functions as follows:

$$\begin{aligned}
\Omega(t) &= \frac{2}{m_0^2}\left(G_{5X}\ddot{\phi}_0\dot{\phi}_0^2 - G_{5\phi}\frac{\dot{\phi}_0^2}{2}\right) - 1\,,\\
c(t) &= \frac{1}{2}\dot{\tilde{\mathcal{F}}} + \frac{3}{2}Hm_0^2\dot{\Omega} - 3H^2\dot{\phi}_0^2G_{5\phi} + 3H^2\dot{\phi}_0^4G_{5\phi X} - 3H^3\dot{\phi}_0^3G_{5X}\\
&\quad + 2H^3\dot{\phi}_0^5G_{5XX}\,,\\
\Lambda(t) &= \dot{\tilde{\mathcal{F}}} - 3m_0^2H^2(1+\Omega) + 4G_{5X}H^3\dot{\phi}_0^3 + 3HG_{5\phi}\dot{\phi}_0^2\,,\\
M_2^4(t) &= -\frac{\dot{\tilde{\mathcal{F}}}}{4} - \frac{3}{4}Hm_0^2\dot{\Omega} - 2H^3G_{5XXX}\dot{\phi}_0^7 - 3H^2\dot{\phi}_0^6G_{5\phi XX} + 6G_{5XX}H^3\dot{\phi}_0^5\\
&\quad + 6H^2G_{5\phi X}\dot{\phi}_0^4 - \frac{3}{2}H^3G_{5X}\dot{\phi}_0^3\,,\\
\hat{M}^2(t) &= -G_{5X}\dot{\phi}_0^2\ddot{\phi}_0 + HG_{5X}\dot{\phi}_0^3 + G_{5\phi}\dot{\phi}_0^2\,,
\end{aligned}$$

$$\bar{M}_2^2(t) = -\bar{M}_3^2(t) = 2\hat{M}^2(t)\,,$$
$$\bar{M}_1^3(t) = -m_0^2\dot{\Omega} + 4H\dot{\phi}_0^2 G_{5\phi} - 4H\dot{\phi}_0^4 G_{5\phi X} - 4H^2\dot{\phi}_0^5 G_{5XX} + 6H^2\dot{\phi}_0^3 G_{5X}\,, \tag{2.64}$$

with $\tilde{\mathcal{F}} = \mathcal{F} - m_0^2\dot{\Omega} - 2Hm_0^2(1+\Omega) = 2H^2 G_{5X}\dot{\phi}_0^3 + 2HG_{5\phi}\dot{\phi}_0^2 - m_0^2\dot{\Omega} - 2Hm_0^2(1+\Omega)$. We have omitted, in the EFT functions, the dependence on the background quantities ϕ_0 and X_0 of G_5 and its derivatives. Finally we recover, as expected, the relation (2.52).

2.4.4 GLPV Lagrangians

We shall now move on to the beyond Hordenski models derived by Gleyzes *et al.* [66, 67], known as GLPV. These build on the premises of the Galileon models and include some extra terms in the Lagrangians that, while contributing higher order spatial derivatives in the field equations, maintain second order equations of motion for the true propagating d.o.f.. Specifically, the GLPV action assumes the following form:

$$\mathcal{S}_{\rm GLPV} = \int d^4x\sqrt{-g}\left[L_2^{GG} + L_3^{GG} + L_4^{GG} + L_5^{GG} + L_4^{\rm GLPV} + L_5^{\rm GLPV}\right]\,, \tag{2.65}$$

where L_i^{GG}(i=2,3,4,5) are the GG Lagrangians listed in Eq.(2.33) and the new terms to be added to the GG Lagrangians are the following:

$$L_4^{\rm GLPV} = \tilde{F}_4(\phi,X)\epsilon^{\mu\nu\rho}{}_{\sigma}\epsilon^{\mu'\nu'\rho'\sigma}\phi_{;\mu}\phi_{;\mu'}\phi_{;\nu\nu'}\phi_{\rho\rho'}\,,$$
$$L_5^{\rm GLPV} = \tilde{F}_5(\phi,X)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\mu'\nu'\rho'\sigma'}\phi_{;\mu}\phi_{;\mu'}\phi_{;\nu\nu'}\phi_{;\rho\rho'}\phi_{;\sigma\sigma'}\,, \tag{2.66}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric Levi-Civita tensor and $\tilde{F}_4, \tilde{F}_5$ are two new arbitrary functions of (ϕ, X).

As usual, we will first express the new Lagrangians in terms of ADM quantities using, among others, relations (2.36)-(2.37), and we get:

$$L_4^{\rm GLPV} = -X^2\tilde{F}_4(\phi,X)(K^2 - K_{ij}K^{ij})\,,$$
$$L_5^{\rm GLPV} = \tilde{F}_5(\phi,X)(-X)^{5/2}\tilde{\mathcal{K}}$$
$$= \tilde{F}_5(\phi,X)(-X)^{5/2}(-6H^3 - 6H^2K - 3HK^2 + 3HK_{\mu\nu}K^{\mu\nu})\,. \tag{2.67}$$

The last equality holds up to second order in perturbations. It is now easy to apply the familiar procedure. Moreover, since different Lagrangians contribute separately to the EFT functions, we can simply calculate the EFT functions corresponding to the new Lagrangians (2.67) and add those to the results previously derived for the GG Lagrangians.

- $\mathbf{L_4^{\mathrm{GLPV}}}$**- Lagrangian**

Let us start with the operators included in the L_4^{GLPV} Lagrangian:

$$L_4^{\mathrm{GLPV}} = -X^2\tilde{F}_4(K^2 - \mathcal{S}). \tag{2.68}$$

We can easily derive the following quantities that are useful for the mapping:

$$\begin{aligned}
L_K &= 6H\dot{\phi}_0^4\tilde{F}_4, \quad L_{\mathcal{S}} = \dot{\phi}_0^4\tilde{F}_4, \quad L_{KK} = -2\dot{\phi}_0^4\tilde{F}_4, \\
L_N &= 4\frac{\dot{\phi}_0^4}{N^5}\tilde{F}_4(K^2 - \mathcal{S}) = 24H^2\dot{\phi}_0^4\tilde{F}_4 \quad L_{NN} = -120\dot{\phi}_0^4\tilde{F}_4H^2, \\
L_{NK} &= -24H\dot{\phi}_0^4\tilde{F}_4, \quad L_{N\mathcal{S}} = -4\dot{\phi}_0^4\tilde{F}_4, \quad \mathcal{F} = 4H\dot{\phi}_0^4\tilde{F}_4\,, \\
\dot{\mathcal{F}} &= 4\dot{H}\dot{\phi}_0^4\tilde{F}_4 + 16H\tilde{F}_4\dot{\phi}_0^3\ddot{\phi}_0 - 8H\dot{\phi}_0^5\ddot{\phi}_0\tilde{F}_{4X} + 4H\dot{\phi}_0^5\tilde{F}_{4\phi}\,.
\end{aligned} \tag{2.69}$$

Using the relations (2.19), we obtain the non-zero EFT functions corresponding to L_4^{GLPV}:

$$\begin{aligned}
c(t) &= 2\dot{H}\dot{\phi}_0^4\tilde{F}_4 + 8H\dot{\phi}_0^3\ddot{\phi}_0\tilde{F}_4 - 4H\dot{\phi}_0^5\ddot{\phi}_0\tilde{F}_{4X} + 2H\tilde{F}_{4\phi}\dot{\phi}_0^5 - 12H^2\dot{\phi}_0^4\tilde{F}_4\,, \\
\Lambda(t) &= 6H^2\dot{\phi}_0^4\tilde{F}_4 + 4\dot{H}\dot{\phi}_0^4\tilde{F}_4 + 16H\dot{\phi}_0^3\ddot{\phi}_0\tilde{F}_4 + 4H\dot{\phi}_0^5\tilde{F}_{4\phi} - 8H\dot{\phi}_0^5\ddot{\phi}_0\tilde{F}_{4X}\,, \\
M_2^4(t) &= -18\dot{\phi}_0^4\tilde{F}_4H^2 - \dot{H}\dot{\phi}_0^4\tilde{F}_4 - 4H\dot{\phi}_0^3\ddot{\phi}_0\tilde{F}_4 + 2H\dot{\phi}_0^5\ddot{\phi}_0\tilde{F}_{4X} - H\tilde{F}_{4\phi}\dot{\phi}_0^5 + 6H^2\dot{\phi}_0^4\tilde{F}_4\,, \\
\bar{M}_2^2(t) &= 2\dot{\phi}_0^4\tilde{F}_4, \\
\bar{M}_1^3(t) &= 16H\dot{\phi}_0^4\tilde{F}_4, \\
\bar{M}_3^2(t) &= -\bar{M}_2^2(t)\,.
\end{aligned} \tag{2.70}$$

As before, $\tilde{F}_4$ and its derivatives are evaluated on the background, therefore they only depend on time.

- $\mathbf{L_5^{\mathrm{GLPV}}}$**- Lagrangian**

Let us now consider the last Lagrangian:

$$L_5^{\mathrm{GLPV}} = -(-X)^{5/2}\tilde{F}_5(-6H^3 - 6H^2K - 3HK^2 + 3H\mathcal{S})\,, \tag{2.71}$$

which gives the derivatives, w.r.t. ADM quantities, one needs to obtain the mapping:

$$\begin{aligned}
L_K &= -12H^2\dot{\phi}_0^5\tilde{F}_5, \quad L_{\mathcal{S}} = -3H\dot{\phi}_0^5\tilde{F}_5, \quad L_N = 5\frac{\dot{\phi}_0^5}{N^6}\tilde{F}_5\tilde{\mathcal{K}} = -30\dot{\phi}_0^5H^3\tilde{F}_5\,, \\
L_{KK} &= 6H\dot{\phi}_0^5\tilde{F}_5, \quad L_{NN} = 180H^3\dot{\phi}_0^5\tilde{F}_5, \quad L_{NK} = 60\dot{\phi}_0^5\tilde{F}_5H^2, \\
L_{N\mathcal{S}} &= 15H\dot{\phi}_0^5\tilde{F}_5, \quad \mathcal{F} = -6H^2\dot{\phi}_0^5\tilde{F}_5, \\
\dot{\mathcal{F}} &= 12H^2\dot{\phi}_0^6\tilde{F}_{5X}\ddot{\phi}_0 - 12H\dot{H}\dot{\phi}_0^5\tilde{F}_5 - 30H^2\dot{\phi}_0^4\tilde{F}_5\ddot{\phi}_0 - 6H^2\dot{\phi}_0^6\tilde{F}_{5\phi}\,.
\end{aligned} \tag{2.72}$$

Employing these, allows us to obtain the non-zero EFT functions:

$$\Lambda(t) = -3H^3\dot{\phi}_0^5\tilde{F}_5 - 12H\dot{H}\dot{\phi}_0^5\tilde{F}_5 - 30H^2\dot{\phi}_0^4\tilde{F}_5\ddot{\phi}_0 + 12H^2\dot{\phi}_0^6\tilde{F}_{5X}\ddot{\phi}_0 - 6H^2\dot{\phi}_0^6\tilde{F}_{5\phi}\,,$$
$$c(t) = 6H^2\dot{\phi}_0^6\ddot{\phi}_0\tilde{F}_{5X} - 6H\dot{H}\dot{\phi}_0^5\tilde{F}_5 - 15H^2\dot{\phi}_0^4\tilde{F}_5\ddot{\phi}_0 - 3H^2\dot{\phi}_0^6\tilde{F}_{5\phi} + 15\dot{\phi}_0^5H^3\tilde{F}_5\,,$$
$$M_2^4(t) = \frac{45}{2}\dot{\phi}_0^5H^3\tilde{F}_5 + 3H\dot{H}\dot{\phi}_0^5\tilde{F}_5 + \frac{15}{2}H^2\dot{\phi}_0^4\ddot{\phi}_0\tilde{F}_5 - 3H^2\dot{\phi}_0^6\ddot{\phi}_0\tilde{F}_{5X} + \frac{3}{2}H^2\dot{\phi}_0^6\tilde{F}_{5\phi}\,,$$
$$\bar{M}_2^2(t) = -6H\dot{\phi}_0^5\tilde{F}_5,$$
$$\bar{M}_1^3(t) = -30H^2\dot{\phi}_0^5\tilde{F}_5,$$
$$\bar{M}_3^2(t) = -\bar{M}_2^2(t)\,. \qquad (2.73)$$

As usual the functions $\tilde{F}_5$ and its derivatives are functions of time. Their expressions in terms of the scale factor and the Hubble parameter w.r.t. conformal time can be found in Appendix 2.10. Let us notice that GLPV models correspond to:

$$\bar{M}_2^2 = -\bar{M}_3^2, \qquad (2.74)$$

which is a less restrictive condition than the one defining GG theories (2.52) as $\bar{M}_2^2 \neq 2\hat{M}^2$.

Let us conclude this Section by working out the mapping between the EFT functions and a common way to write the GLPV action. This action is built directly in terms of geometrical quantities, hence guaranteeing the unitary gauge since the scalar d.o.f. has been eaten by the metric [66]. Therefore now we will consider the following GLPV Lagrangian instead of the one defined previously:

$$\begin{aligned} L_{\rm GLPV} \;=\;& A_2(t,N) + A_3(t,N)K + A_4(t,N)(K^2 - K_{ij}K^{ij}) + B_4(t,N)\mathcal{R} \\ +\;& A_5(t,N)\left(K^3 - 3KK_{ij}K^{ij} + 2K_{ij}K^{ik}K^j_k\right) \\ +\;& B_5(t,N)K^{ij}\left(\mathcal{R}_{ij} - h_{ij}\frac{\mathcal{R}}{2}\right)\,, \end{aligned} \qquad (2.75)$$

where A_i, B_i are general functions of t and N, and can be expressed in terms of the scalar field, ϕ, , as shown in Ref. [66], effectively creating the equivalence between the above Lagrangian and the one introduced in Eq. (2.65).

It is very easy to write the above Lagrangian in terms of the quantities introduced in Section 2.3.1:

$$\begin{aligned} L_{\rm GLPV} \;=\;& A_2(t,N) + A_3(t,N)K + A_4(t,N)(K^2 - \mathcal{S}) + B_4(t,N)\mathcal{R} \\ +\;& A_5(t,N)\left(-6H^3 - 6H^2K - 3HK^2 + 3H\mathcal{S}\right) \\ +\;& B_5(t,N)\left(\mathcal{U} - \frac{\mathcal{R}K}{2}\right)\,. \end{aligned} \qquad (2.76)$$

Now, we can compute the quantities that we need for the mapping (2.19):

$$\begin{aligned}
&\bar{L} = \bar{A}_2 - 3H\bar{A}_3 + 6H^2\bar{A}_4 - 6H^3\bar{A}_5\,, \quad \mathcal{E} = \bar{B}_4 - \frac{1}{2}\dot{\bar{B}}_5\,, \\
&\mathcal{F} = \bar{A}_3 - 4H\bar{A}_4 + 6H^2\bar{A}_5\,, \quad L_{\mathcal{S}} = -\bar{A}_4 + 3H\bar{A}_5\,, \\
&L_K = \bar{A}_3 - 6H\bar{A}_4 + 12H^2\bar{A}_5, \quad L_{KK} = 2\bar{A}_4 - 6H\bar{A}_5, \\
&L_N = \bar{A}_{2N} - 3H\bar{A}_{3N} + 6H^2\bar{A}_{4N} - 6H^3\bar{A}_{5N}\,, \quad L_{\mathcal{U}} = \bar{B}_5\,, \\
&L_{NN} = \bar{A}_{2NN} - 3H\bar{A}_{3N} + 6H^2\bar{A}_{4NN} - 6H^3\bar{A}_{5NN}\,, \\
&L_{\mathcal{S}N} = -\bar{A}_{4N} + 3H\bar{A}_{5N}, \quad L_{KN} = \bar{A}_{3N} - 6H\bar{A}_{4N} + 12H^2\bar{A}_{5N}\,, \\
&L_{K\mathcal{R}} = -\frac{1}{2}\bar{B}_5\,, \quad L_{N\mathcal{U}} = \bar{B}_{5N}\,, \quad L_{N\mathcal{R}} = \bar{B}_{4N} + \frac{3}{2}H\bar{B}_{5N}\,,
\end{aligned} \tag{2.77}$$

where the quantities with the bar are evaluated in the background and A_{iY} means derivative of A_i w.r.t. Y. Then the EFT functions follow from Eq. (2.19):

$$\begin{aligned}
\Omega(t) &= \frac{2}{m_0^2}\left(\bar{B}_4 - \frac{1}{2}\dot{\bar{B}}_5\right) - 1\,, \\
\Lambda(t) &= \bar{A}_2 - 6H^2\bar{A}_4 + 12H^3\bar{A}_5 + \dot{\bar{A}}_3 - 4\dot{H}\bar{A}_4 - 4H\dot{\bar{A}}_4 + 6H^2\dot{\bar{A}}_5 + 12H\dot{H}\bar{A}_5 \\
&\quad - \left[2(3H^2 + 2\dot{H})\left(\bar{B}_4 - \frac{1}{2}\dot{\bar{B}}_5\right) + 2\ddot{\bar{B}}_4 - \bar{B}_5^{(3)} + 4H\left(\dot{\bar{B}}_4 - \frac{1}{2}\ddot{\bar{B}}_5\right)\right], \\
c(t) &= \frac{1}{2}\left(\dot{\bar{A}}_3 - 4\dot{H}\bar{A}_4 - 4H\dot{\bar{A}}_4 + 6H^2\dot{\bar{A}}_5 + 12H\dot{H}\bar{A}_5 - \bar{A}_{2N} + 3H\bar{A}_3N\right) \\
&\quad -6H^2\bar{A}_{4N} + 6H^3\bar{A}_{5N} + H\left(\dot{\bar{B}}_4 - \frac{1}{2}\ddot{\bar{B}}_5\right) - \ddot{\bar{B}}_4 + \frac{1}{2}\bar{B}_5^{(3)} \\
&\quad -2\dot{H}\left(\bar{B}_4 - \frac{1}{2}\dot{\bar{B}}_5\right), \\
\bar{M}_2^2(t) &= = -2\bar{A}_4 + 6H\bar{A}_5 - 2\bar{B}_4 + \dot{\bar{B}}_5\,, \\
\bar{M}_1^3(t) &= -\bar{A}_{3N} + 4H\bar{A}_{4N} - 6H^2\bar{A}_{5N} - 2\dot{\bar{B}}_4 + \ddot{\bar{B}}_5\,, \\
\bar{M}_3^2(t) &\equiv -\bar{M}_2^2(t)\,, \\
M_2^4(t) &= \frac{1}{4}\left(\bar{A}_{2NN} - 3H\bar{A}_{3NN} + 6H^2\bar{A}_{4NN} - 6H^3\bar{A}_{5NN}\right) - \frac{1}{4}\left(\dot{\bar{A}}_3 - 4\dot{H}\bar{A}_4\right. \\
&\quad \left.-4H\dot{\bar{A}}_4 + 6H^2\dot{\bar{A}}_5 + 12H\dot{H}\bar{A}_5\right) + \frac{3}{4}\left(\bar{A}_{2N} - 3H\bar{A}_{3N} + 6H^2\bar{A}_{4N}\right. \\
&\quad \left.-6H^3\bar{A}_{5N}\right) - \frac{1}{2}\left[H\left(\dot{\bar{B}}_4 - \frac{1}{2}\ddot{\bar{B}}_5\right) - \ddot{\bar{B}}_4 + \frac{1}{2}\bar{B}_5^{(3)}\right. \\
&\quad \left.-2\dot{H}\left(\bar{B}_4 - \frac{1}{2}\dot{\bar{B}}_5\right)\right], \\
\hat{M}^2(t) &= \bar{B}_{4N} + \frac{1}{2}H\bar{B}_{5N} + \frac{1}{2}\dot{\bar{B}}_5\,.
\end{aligned} \tag{2.78}$$

The condition (2.74) is satisfied as desired and one can focus on the GG subset of theories by enforcing the condition $\bar{M}_2^2(t) = 2\hat{M}^2(t)$.

2.4.5 Hořava Gravity

One of the main aspects of our paper is the inclusion of operators with higher order spatial derivatives in the EFT action. Thus, it is natural to proceed with the mapping of the most popular theory containing such operators, i.e. Hořava gravity [35, 36]. This theory is a recent proposed candidate to describe the gravitational interaction in the ultra-violet regime (UV). This is done by breaking the Lorentz symmetry resulting in a modification of the graviton propagator. Practically, this amounts to adding higher-order spatial derivatives to the action while keeping the time derivatives at most second order, in order to avoid Ostrogradsky instabilities [25]. As a result, time and space are treated on a different footing, therefore the natural formulation in which to construct the action is the ADM one. It has been shown that, in order to obtain a power-counting renormalizable theory, the action needs to contain terms with up to sixth-order spatial derivatives [62–64]. The resulting action does not demonstrate full diffeomorphism invariance but is rather invariant under a restricted symmetry, the foliation preserving diffeomorphisms (for a review see [55, 59] and references therein). Besides the UV regime, Hořava gravity has taken hold on the cosmological side as well as it exhibits a rich phenomenology [40–47, 49–51, 53] and very recently it has started to be constrained in that context [39, 48, 52, 54, 56–58].

Here, we will consider the following action which contains up to six order spatial derivatives, (and is therefore included in the extended EFT action):

$$
\begin{aligned}
\mathcal{S}_H \;&=\; \frac{1}{16\pi G_H}\int d^4x\sqrt{-g}\,\big[K_{ij}K^{ij} - \lambda K^2 - 2\xi\bar{\Lambda} + \xi\mathcal{R} + \eta a_i a^i + g_1\mathcal{R}^2 \\
&+\; g_2\mathcal{R}_{ij}\mathcal{R}^{ij} + g_3\mathcal{R}\nabla_i a^i + g_4 a_i\Delta a^i + g_5\mathcal{R}\Delta\mathcal{R} + g_6\nabla_i\mathcal{R}_{jk}\nabla^i\mathcal{R}^{jk} \\
&+\; g_7 a_i\Delta^2 a^i + g_8\Delta\mathcal{R}\nabla_i a^i\big]\,, \qquad (2.79)
\end{aligned}
$$

where the coefficients λ, η, ξ and g_i are running coupling constants, $\bar{\Lambda}$ is the "bare" cosmological constant and G_H is the coupling constant [39, 64]:

$$
\frac{1}{16\pi G_H} = \frac{m_0^2}{(2\xi - \eta)}\,. \qquad (2.80)
$$

The above action is already in unitary gauge and ADM form, then we just need few steps to write it in terms of the quantities introduced in

Section 2.3.1:

$$\mathcal{S}_H = \frac{1}{16\pi G_H}\int d^4x\sqrt{-g}\left[S-\lambda K^2-2\xi\bar{\Lambda}+\xi\mathcal{R}+\eta\alpha_1+g_1\mathcal{R}^2+g_2\mathcal{Z}\right.$$
$$\left.+g_3\alpha_3+g_4\alpha_2-g_5\mathcal{Z}_1+g_6\mathcal{Z}_2+g_7\alpha_4+g_8\alpha_5\right], \quad (2.81)$$

then by using the results (2.19) it is easy to show that the EFT functions read:

$$m_0^2(1+\Omega)=\frac{2m_0^2\xi}{(2\xi-\eta)}, \qquad c(t)=-\frac{m_0^2}{(2\xi-\eta)}(1+2\xi-3\lambda)\dot{H},$$
$$\Lambda(t)=\frac{2m_0^2}{(2\xi-\eta)}\left[-\xi\bar{\Lambda}-(1-3\lambda+2\xi)\left(\frac{3}{2}H^2+\dot{H}\right)\right],$$
$$\bar{M}_3^2=-\frac{2m_0^2}{(2\xi-\eta)}(1-\xi), \qquad \bar{M}_2^2=-2\frac{m_0^2}{(2\xi-\eta)}(\xi-\lambda),$$
$$m_2^2=\frac{m_0^2}{4(2\xi-\eta)}\eta, \qquad M_2^4(t)=\frac{m_0^2}{2(2\xi-\eta)}(1+2\xi-3\lambda)\dot{H},$$
$$\lambda_1=g_1\frac{m_0^2}{(2\xi-\eta)}, \qquad \lambda_2=g_2\frac{m_0^2}{(2\xi-\eta)},$$
$$\lambda_3=g_3\frac{m_0^2}{2(2\xi-\eta)}, \qquad \lambda_4=g_4\frac{m_0^2}{4(2\xi-\eta)}, \qquad \lambda_5=-g_5\frac{m_0^2}{(2\xi-\eta)}$$
$$\lambda_6=g_6\frac{m_0^2}{(2\xi-\eta)}, \qquad \lambda_7=g_7\frac{m_0^2}{4(2\xi-\eta)}, \qquad \lambda_8=g_8\frac{m_0^2}{2(2\xi-\eta)}, \quad (2.82)$$

and the remaining EFT functions are zero. The mapping of Hořava gravity has been worked out in details in Ref. [39], by some of the authors of this paper. Subsequently, the low-energy part of Hořava action, which is described by $\{\Omega, c, \Lambda, \bar{M}_3^2, \bar{M}_2^2, M_2^4, m_2^2\}$, has been implemented in *EFTCAMB* [80] and constraints on the low-energy parameters $\{\xi, \eta, \lambda\}$ have been obtained in Ref. [39].

2.5 Stability

Along with its unifying aspect, a very important advantage of the EFToDE/MG formalism is that of being formulated at the level of the action in a model independent way. By inspecting the EFT action expanded to quadratic order in the perturbations, it is possible to impose conditions on the EFT functions to ensure that none of the undesired instabilities develop. It has been preliminary shown in Ref. [23], that the impact of such conditions can be quite significant as they can efficiently reduce the parameter space that one needs to explore when performing a fit to data. In some cases they have been shown to dominate over the constraining power of current data [23].

The study of the theoretical viability of the EFT action has already been performed to some extent in the literature [14, 15, 65, 67, 68], however here we will include in the analysis, for the first time, higher order operators and consider also the instabilities related to a negative squared mass of the scalar d.o.f.. Specifically, we will consider three possible instabilities: ghost and gradient instabilities both in the scalar and tensor sector, and tachyonic scalar modes (for a review see Ref. [109]). Starting from the general action (2.17), we expand it up to quadratic order in tensor and scalar perturbations of the metric around a flat FLRW background. Our focus is on the stability of the gravity sector, hence we will not consider matter fluids. The complete stability analysis of the general action (2.17) in the presence of a matter sector is the main topic of the next Chapter.

Let us consider the following metric perturbations for the scalar components:

$$ds^2 = -(1+2\delta N)dt^2 + 2\partial_i\psi dt dx^i + a^2(1+2\zeta)\delta_{ij}dx^i dx^j \,, \tag{2.83}$$

where as usual $\delta N(t,x^i)$ is the perturbation of the lapse function, $\partial_i\psi(t,x^i)$ and $\zeta(t,x^i)$ are the scalar perturbations respectively of the shift function and the three dimensional metric. Then, the scalar perturbations of the quantities involved in the action (2.17) are:

$$\begin{aligned}
\delta K &= -3\dot{\zeta} + 3H\delta N + \frac{1}{a^2}\partial^2\psi \,, \\
\delta K_{ij} &= a^2\delta_{ij}(H\delta N - 2H\zeta - \dot{\zeta}) + \partial_i\partial_j\psi \,, \\
\delta K^i_j &= (H\delta N - \dot{\zeta})\delta^i_j + \frac{1}{a^2}\partial^i\partial_j\psi \,, \\
\delta\mathcal{R}_{ij} &= -(\delta_{ij}\partial^2\zeta + \partial_i\partial_j\zeta) \,, \\
\delta_1\mathcal{R} &= -\frac{4}{a^2}\partial^2\zeta \,, \\
\delta_2\mathcal{R} &= -\frac{2}{a^2}[(\partial_i\zeta)^2 - 4\zeta\partial^2\zeta].
\end{aligned} \tag{2.84}$$

Now, we can expand action (2.17) to quadratic order in metric perturbations. In the following we will Fourier transform the spatial part* and

*More properly, in Fourier space we should write $(\zeta(t,k))^2 \to \zeta_{\mathbf{k}}\zeta_{-\mathbf{k}}$, however in the following we prefer to drop the indices in order to simplify the notation.

after regrouping terms, we obtain:

$$\begin{aligned}\mathcal{S}^{(2)}_{EFT} &= \frac{1}{(2\pi)^3}\int d^3k dt\, a^3 \Big\{ -\left(\mathcal{W}_0+\mathcal{W}_3 k^2+\mathcal{W}_2 k^4\right)k^2\zeta^2 - 3a^2\mathcal{W}_4\dot{\zeta}\delta N \\ &\quad -\frac{3}{2}a^2\mathcal{W}_5(\dot{\zeta})^2 - \left(\mathcal{W}_4\delta N+\mathcal{W}_5\dot{\zeta}-\mathcal{W}_7 k^2\psi+\frac{2}{a^4}\bar{m}_5 k^2\zeta\right)k^2\psi \\ &\quad +\left(\mathcal{W}_1+4m_2^2\frac{k^2}{a^2}-4\frac{\lambda_4}{a^4}k^4+4\frac{\lambda_7}{a^6}k^6\right)(\delta N)^2 \\ &\quad -\left(\mathcal{W}_6+8\lambda_3\frac{k^2}{a^4}+8\frac{\lambda_8}{a^6}k^4\right)\delta N k^2\zeta\Big\}\,,\end{aligned} \tag{2.85}$$

where:

$$\begin{aligned}\mathcal{W}_0 &= -\frac{1}{a^2}\left[m_0^2(1+\Omega)+3H\bar{m}_5+3\dot{\bar{m}}_5\right],\\ \mathcal{W}_1 &= c+2M_2^4-3m_0^2H^2(1+\Omega)-3m_0^2H\dot{\Omega}-\frac{3}{2}H^2\bar{M}_3^2-\frac{9}{2}H^2\bar{M}_2^2-3H\bar{M}_1^3\,,\\ \mathcal{W}_2 &= -16\frac{\lambda_5}{a^6}-6\frac{\lambda_6}{a^6}\,,\\ \mathcal{W}_3 &= -16\frac{\lambda_1}{a^4}-6\frac{\lambda_2}{a^4}\,,\\ \mathcal{W}_4 &= \frac{1}{a^2}\left(-2m_0^2H(1+\Omega)-m_0^2\dot{\Omega}-H\bar{M}_3^2-\bar{M}_1^3-3H\bar{M}_2^2\right),\\ \mathcal{W}_5 &= \frac{1}{a^2}\left(2m_0^2(1+\Omega)+\bar{M}_3^2+3\bar{M}_2^2\right),\\ \mathcal{W}_6 &= -\frac{4}{a^2}\left(\frac{1}{2}m_0^2(1+\Omega)+\hat{M}^2\right)-6H\frac{\bar{m}_5}{a^2}\,,\\ \mathcal{W}_7 &= -\frac{1}{2a^4}\left(\bar{M}_3^2+\bar{M}_2^2\right).\end{aligned} \tag{2.86}$$

In this action we have three d.o.f. $\{\zeta, \delta N, \psi\}$, but in reality only one, ζ, is dynamical, while the other two, $\{\delta N, \psi\}$, are auxiliary fields. This implies that they can be eliminated through the constraint equations obtained by varying the above action w.r.t. them. We will leave for the next Sections the details of such a calculation, here we want to outline the general procedure we are adopting. After replacing back in the action the general expression for δN and ψ , we end up with an action of the form:

$$\mathcal{S}^{(2)}_{EFT} = \frac{1}{(2\pi)^3}\int d^3k dt\, a^3\left\{\mathcal{L}_{\dot{\zeta}\dot{\zeta}}(t,k)\dot{\zeta}^2-\left[k^2G(t,k)+\bar{M}(t,k)\right]\zeta^2\right\}. \tag{2.87}$$

where $\bar{M}(t,k)$ depends on inverse powers of k. $\mathcal{L}_{\dot{\zeta}\dot{\zeta}}(t,k)$ is usually called the kinetic term and its positivity guarantees that the theory is free from ghost in the scalar sector. The variation of the above action w.r.t. ζ gives:

$$\ddot{\zeta} + \left(3H + \frac{\dot{\mathcal{L}}_{\dot{\zeta}\dot{\zeta}}}{\mathcal{L}_{\dot{\zeta}\dot{\zeta}}}\right)\dot{\zeta} + \left(k^2 \frac{G}{\mathcal{L}_{\dot{\zeta}\dot{\zeta}}} + \frac{\bar{M}}{\mathcal{L}_{\dot{\zeta}\dot{\zeta}}}\right)\zeta = 0\,, \tag{2.88}$$

where the coefficient of $\dot{\zeta}$ is called the friction term and its sign will damp or enhance the amplitude of the field fluctuations. $\bar{M}/\mathcal{L}_{\dot{\zeta}\dot{\zeta}}$ is called the dispersion coefficient which, in principle, can be both negative and positive. Finally, we define the propagation speed as:

$$c_s^2 \equiv \frac{G}{\mathcal{L}_{\dot{\zeta}\dot{\zeta}}}. \tag{2.89}$$

Let us note that the speed of propagation and the dispersion coefficient (or "mass" term) and their effective counterparts have non-local expressions. Therefore, their interpretation as the actual physical entities might be ambiguous at first glance because usually these quantities are defined in some specific limit, where they assume local expressions. In this work, we still retain the labeling of speed of propagation and mass term for the non-local expressions, because they reduce to the corresponding local and physical quantity when the proper limit is considered. Moreover, the non-local definitions are the ones which serve to our purpose, since they represent the proper quantities on which the stability conditions have to be imposed in order to guarantee a viable theory at all times and scales.

Now, let us perform a field redefinition in order to have a canonical action. This step is important in order to identify the correct conditions to avoid the gradient and tachyonic instabilities, in particular the last one which is related to the condition of boundedness from below of the corresponding canonical Hamiltonian. We will show that not only the mass is sensitive to this normalization, as it is known, but that in the general case in which the kinetic term is scale-dependent also the speed of propagation, is affected by the field redefinition. In general, we can use:

$$\zeta(t,k) = \frac{\phi(t,k)}{\sqrt{2\mathcal{L}_{\dot{\zeta}\dot{\zeta}}(t,k)}}, \tag{2.90}$$

which, once applied to the action (2.87), gives:

$$\mathcal{S}^{(2)}_{EFT} = \frac{1}{(2\pi)^3}\int d^3k dt\, a^3 \left[\frac{1}{2}\dot{\phi}^2 - c^2_{s,\text{eff}}(t,k)\frac{k^2}{2}\phi^2 - m^2_{\text{eff}}(t,k)\phi^2\right]\,, \tag{2.91}$$

where $m_{\rm eff}(t,k)$ is an effective mass and depends on inverse powers of k, while $c^2_{s,\rm eff}(t,k)$ is the effective speed of propagation.

When $\mathcal{L}_{\dot\zeta\dot\zeta}$ is only a function of time, the field redefinition (2.90) will give time-dependent contributions only to $\bar{M}$ thus generating $m^2_{\rm eff}$ and leaving G unaffected. In this case we have:

$$
\begin{aligned}
c^2_{s,\rm eff}(t,k) &= c^2_s(t,k)\,,\\
m^2_{\rm eff}(t) &= \frac{\mathcal{L}_{\dot\zeta\dot\zeta}\left(4\bar{M}(t)-2\ddot{\mathcal{L}}_{\dot\zeta\dot\zeta}\right)+\dot{\mathcal{L}}^2_{\dot\zeta\dot\zeta}-6H\mathcal{L}_{\dot\zeta\dot\zeta}\dot{\mathcal{L}}_{\dot\zeta\dot\zeta}}{8\mathcal{L}^2_{\dot\zeta\dot\zeta}}\,.
\end{aligned}
\tag{2.92}
$$

Let us notice that in case in which the kinetic term depends only on time, the term $\bar{M}$ usually turns out to be zero or at most a function of time.

On the contrary, when $\mathcal{L}_{\dot\zeta\dot\zeta}$ exhibits a k-dependence, the field redefinition will affect both $\bar{M}$ and G and in general $c^2_{s,\rm eff}\neq c^2_s$ and the above expression for the effective mass does not hold anymore. In Section 2.5.2 we will discuss the general expressions for these two quantities. In general, the GLPV class of theories belongs to the case in which $\mathcal{L}_{\dot\zeta\dot\zeta}$ is only a function of time. When one starts including operators like $\{m_2^2,\bar{m}_5,\lambda_i,\bar{M}_3^2\neq-\bar{M}_2^2\}$, k-dependence will be generated in the kinetic term. In the following sections we will analyse these cases in details.

Finally, in order to study the stability, one has to analyse the evolution of the field equation obtained by varying the action (2.91) w.r.t. ϕ, i.e.:

$$
\ddot\phi+3H\dot\phi+\left(k^2c^2_{s,\rm eff}+m^2_{\rm eff}\right)\phi=0, \tag{2.93}
$$

In this case H represents a friction term, which is always positive, and $m^2_{\rm eff}$ is the dispersion coefficient. A negative value of the effective mass squared generates a tachyonic instability, however requiring $m^2_{\rm eff}$ to be positive is a stringent condition. It is sufficient to guarantee that the time scale on which the instability evolves is longer then the time evolution of the system [109] in order to be free of said instability. Therefore, we can require that, when $m^2_{\rm eff}<0$, the typical evolution scale is of the same order as the Hubble time, H_0.

Continuing, in order to avoid gradient instabilities one enforces a positive value of the effective speed of propagation. In the simpler cases in which the kinetic term depends only on time (e.g. Horndeski and GLPV theories), the normalization of the field leaves the speed of sound unchanged, i.e. $c^2_s=c^2_{\rm eff}$, thus the condition to impose is $c^2_{\rm eff}=c^2_s>0$. For the more general case in which the kinetic term depends on scale in a non trivial way, $c^2_{\rm eff}=c^2_s+f(t,k)$ (see Section 2.5.2 for the full expression of $f(t,k)$); however, in the high k-limit, where the gradient instability shows up, $f(t,k)$ is maximally of order $\mathcal{O}(1/k^2)$ which can be neglected in this limit. Therefore, the condition on the effective speed of propagation reduces indeed to the original condition on the speed of propagation, i.e. $c_s^2 > 0$. In summary, in order to guarantee the stability of the scalar sector the combination of $c_{\rm eff}^2 > 0$ and $m_{\rm eff}^2 > 0$, along with the no-ghost condition, i.e. $\mathcal{L}_{\dot{\zeta}\dot{\zeta}} > 0$, provides the full set of stability conditions.

We conclude with the stability analysis on the tensor modes. The perturbed metric components which contribute to tensor modes are:

$$g_{ij}^T(t,x^i) = a^2 h_{ij}^T(t,x^i)\,, \tag{2.94}$$

therefore, the terms containing tensor perturbations in (2.17), are the following:

$$\delta\mathcal{R}_{ij} = -\frac{\delta^{lk}}{a^2}\partial_l\partial_k h_{ij}^T, \quad \delta_2\mathcal{R} = \frac{1}{a^2}\Big(\frac{3}{4}\partial_k h_{ij}^T\partial^k h^{ij\,T} + h_{ij}^T\partial^2 h^{ij\,T} -\frac{1}{2}\partial_k h_{ij}^T\partial^j h^{ik\,T}\Big), \quad \delta K_j^i = -\frac{\dot{h}_j^{i\,T}}{2} \tag{2.95}$$

where $\delta_2\mathcal{R}$ is the second order perturbation of the Ricci scalar, $\mathcal{R}$. Then, the EFT action for tensor perturbations up to second order reads:

$$\mathcal{S}_{EFT}^{T\,(2)} = \int d^4x\, a^3\left\{\frac{m_0^2}{2}(1+\Omega)\delta_2\mathcal{R} + \left(\frac{m_0^2}{2}(1+\Omega) - \frac{\bar{M}_3^2}{2}\right)\delta K_j^i\delta K_i^j + \lambda_2\delta\mathcal{R}_{ij}\delta\mathcal{R}^{ij} + \lambda_6\frac{\tilde{g}^{kl}}{a^2}\partial_k\mathcal{R}_{ij}\partial_l\mathcal{R}^{ij}\right\}, \tag{2.96}$$

from which we can notice that only four EFT functions describe the dynamics of tensors, i.e. $\{\Omega, \bar{M}_3^2, \lambda_2, \lambda_8\}$. Among the extra operators that we added in action (2.17), only two contribute to tensor modes $\{\lambda_2, \lambda_8\}$. Now, using (2.95), the action becomes:

$$\mathcal{S}_{EFT}^{T\,(2)} = \int d^4x\, a^3\left\{-\frac{m_0^2}{2}(1+\Omega)\frac{1}{4a^2}(\partial_k h_{ij}^T)^2 + \left(\frac{m_0^2}{2}(1+\Omega) - \frac{\bar{M}_3^2}{2}\right)\frac{(\dot{h}_{ij}^T)^2}{4} + \lambda_2\left(\frac{\delta^{lk}}{a^2}\partial_l\partial_k h_{ij}^T\right)^2 + \lambda_6\frac{1}{a^6}(\partial_k\partial_l\partial^l h_{ij}^T)^2\right\}. \tag{2.97}$$

It is clear that the additional operators associated to higher spatial derivatives do not affect the kinetic term. However, they affect the speed of propagation of the tensor modes, as we will show in the following. Indeed, action (2.97) can be written in the compact form:

$$\mathcal{S}_{EFT}^{T\,(2)} = \frac{1}{(2\pi)^3}\int d^3k dt\, a^3\frac{A_T(t)}{8}\left[(\dot{h}_{ij}^T)^2 - \frac{c_T^2(t,k)}{a^2}k^2(h_{ij}^T)^2\right], \tag{2.98}$$

with

$$
\begin{aligned}
A_T(t) &= m_0^2(1+\Omega) - \bar{M}_3^2\,, \\
c_T^2(t,k) &= \bar{c}_T^2(t) - 8\frac{\lambda_2\frac{k^2}{a^2} + \lambda_6\frac{k^4}{a^4}}{m_0^2(1+\Omega) - \bar{M}_3^2}\,, \\
\bar{c}_T^2(t) &= \frac{m_0^2(1+\Omega)}{m_0^2(1+\Omega) - \bar{M}_3^2}\,,
\end{aligned}
\tag{2.99}
$$

where we have Fourier transformed the spatial part. $\bar{c}_T^2$ is the tensor speed of propagation for all the theories belonging to the GLPV class, as shown in Refs. [67, 69]. However, GLPV theories are characterized by the condition $\bar{M}_3^2(t) = -\bar{M}_2^2(t)$, while the present definition of the tensor speed does not rely on this constraint as it holds for a wider class of theories. In order to avoid the development of instabilities in the tensorial sector, one generally demands the kinetic term to be positive, i.e. $A_T > 0$, and to have a positive speed of propagation $c_T^2 > 0$. From Eqs. (2.99) it is easy to identify the corresponding conditions on the EFT functions.

2.5.1 Stability conditions for the GLPV class of theories

Let us focus on the GLPV class of theories by considering the appropriate set of operators:

$$
\begin{aligned}
\mathcal{S}^{(2)}_{\rm GLPV} \;=\; & \frac{1}{(2\pi)^3}\int d^3k dt\, a^3\left[-\mathcal{W}_6\delta N k^2\zeta - \mathcal{W}_4\delta N k^2\psi - \mathcal{W}_5 k^2\psi\dot{\zeta}\right. \\
& \left. -\; \mathcal{W}_0 k^2\zeta^2 + \mathcal{W}_1(\delta N)^2 - 3a^2\mathcal{W}_4\delta N\dot{\zeta} - \frac{3}{2}a^2\mathcal{W}_5\dot{\zeta}^2\right]\,,
\end{aligned}
\tag{2.100}
$$

which is obtained from action (2.85) by imposing the following constraints:

$$
\mathcal{W}_7 = 0\,, \qquad \left\{m_2^2, \bar{m}_5, \lambda_i\right\} = 0. \tag{2.101}
$$

By varying the above action w.r.t. δN and ψ we get, respectively,:

$$
\begin{aligned}
&-\mathcal{W}_6 k^2\zeta - \mathcal{W}_4 k^2\psi + 2\mathcal{W}_1\delta N - 3a^2\mathcal{W}_4\dot{\zeta} = 0\,, \\
&-\mathcal{W}_4\delta N - \mathcal{W}_5\dot{\zeta} = 0.
\end{aligned}
\tag{2.102}
$$

Inverting these relations gives:

$$
\begin{aligned}
\delta N &= -\frac{\mathcal{W}_5}{\mathcal{W}_4}\dot{\zeta}\,, \\
k^2\psi &= -\frac{1}{\mathcal{W}_4^2}\left[\left(3a^2\mathcal{W}_4^2 + 2\mathcal{W}_1\mathcal{W}_5\right)\dot{\zeta} + \mathcal{W}_4\mathcal{W}_6 k^2\zeta\right]\,,
\end{aligned}
\tag{2.103}
$$

which, once substituted back in the action (2.100), yields:

$$\mathcal{S}^{(2)}_{\rm GLPV} = \frac{1}{(2\pi)^3}\int d^3k dt\, a^3\left\{\left(\frac{3}{2}a^2\mathcal{W}_5 + \frac{\mathcal{W}_1\mathcal{W}_5^2}{\mathcal{W}_4^2}\right)\dot\zeta^2 - k^2\left[\frac{3}{2}H\frac{\mathcal{W}_5\mathcal{W}_6}{\mathcal{W}_4} + \frac{1}{2}\frac{d}{dt}\left(\frac{\mathcal{W}_5\mathcal{W}_6}{\mathcal{W}_4}\right) + \mathcal{W}_0\right]\zeta^2\right\}. \tag{2.104}$$

This particular form has been obtained after integrating by parts the term containing $\dot\zeta\zeta$. The above action has the same form of (2.87), where $\bar M = 0$. Therefore, it is easy to read the no-ghost condition:

$$\mathcal{L}_{\dot\zeta\dot\zeta}(t) \equiv \frac{3}{2}a^2\mathcal{W}_5 + \frac{\mathcal{W}_1\mathcal{W}_5^2}{\mathcal{W}_4^2} > 0\,, \tag{2.105}$$

and the condition on the speed of propagation ($c_s^2 > 0$):

$$c_s^2(t) = \frac{3H\mathcal{W}_5\mathcal{W}_6\mathcal{W}_4 + \mathcal{W}_6\mathcal{W}_4\dot{\mathcal{W}}_5 + \mathcal{W}_5\mathcal{W}_4\dot{\mathcal{W}}_6 - \mathcal{W}_5\mathcal{W}_6\dot{\mathcal{W}}_4 + 2\mathcal{W}_0\mathcal{W}_4^2}{3a^2\mathcal{W}_5\mathcal{W}_4^2 + 2\mathcal{W}_1\mathcal{W}_5^2}\,. \tag{2.106}$$

The speed of propagation coincides with the phase velocity due to the lack of k-dependence in the kinetic term, as discussed at earlier stage. Additionally, this implies that only the mass term will be sensitive to the field redefinition which, in this case, reads:

$$\zeta(t,k) = \frac{\phi(t,k)}{\sqrt{2\left(\frac{3}{2}a^2\mathcal{W}_5 + \frac{\mathcal{W}_1\mathcal{W}_5^2}{\mathcal{W}_4^2}\right)}}\,. \tag{2.107}$$

After this transformation the effective mass follows directly form Eq. (2.92), i.e.:

$$m^2_{\rm eff}(t) = \frac{-2\mathcal{L}_{\dot\zeta\dot\zeta}\ddot{\mathcal{L}}_{\dot\zeta\dot\zeta} + \dot{\mathcal{L}}^2_{\dot\zeta\dot\zeta} - 6H\mathcal{L}_{\dot\zeta\dot\zeta}\dot{\mathcal{L}}_{\dot\zeta\dot\zeta}}{8\mathcal{L}^2_{\dot\zeta\dot\zeta}}\,, \tag{2.108}$$

where the kinetic term is given by Eq. (2.105).

2.5.2 Stability conditions for the class of theories beyond GLPV

To go beyond the GLPV class of theories we start by naively considering the general action (2.85) with all the higher order operators. We proceed to integrate out the auxiliary fields δN and ψ by solving the following field equations:

$$
\begin{aligned}
&-2\bar{m}_5\frac{k^2}{a^4}\zeta+2\mathcal{W}_7k^2\psi-\mathcal{W}_4\delta N-\mathcal{W}_5\dot{\zeta}=0\,,\\
&8\left(m_2^2-\frac{\lambda_4}{a^2}k^2+\frac{\lambda_7}{a^4}k^4\right)\frac{k^2}{a^2}\delta N-\left(\mathcal{W}_6+8\lambda_3\frac{k^2}{a^4}+8\frac{\lambda_8}{a^6}k^4\right)k^2\zeta\\
&-\mathcal{W}_4k^2\psi+2\mathcal{W}_1\delta N-3a^2\mathcal{W}_4\dot{\zeta}=0\,,
\end{aligned}
\tag{2.109}
$$

and we finally end up with an action of the form:

$$
\mathcal{S}^{(2)}_{EFT}=\frac{1}{(2\pi)^3}\int d^3kdt\,a^3\left\{\mathcal{L}_{\dot{\zeta}\dot{\zeta}}(t,k)\dot{\zeta}^2-k^2\bar{\mathcal{B}}(t,k)\zeta^2-k^2\bar{\mathcal{V}}(t,k)\dot{\zeta}\zeta\right\}\,,
\tag{2.110}
$$

where:

$$
\begin{aligned}
&\mathcal{L}_{\dot{\zeta}\dot{\zeta}}(t,k)=\frac{\left(6a^2\mathcal{W}_7+\mathcal{W}_5\right)\left[3a^4{\mathcal{W}_4}^2+2a^2\mathcal{W}_1\mathcal{W}_5+8k^2\mathcal{W}_5\left(m_2^2-\frac{\lambda_4}{a^2}k^2+\frac{\lambda_7}{a^4}k^4\right)\right]}{2a^2\left({\mathcal{W}_4}^2-4\mathcal{W}_1\mathcal{W}_7\right)-32k^2\mathcal{W}_7\left(m_2^2-\frac{\lambda_4}{a^2}k^2+\frac{\lambda_7}{a^4}k^4\right)}\,,\\
&\bar{\mathcal{B}}(t,k)=\left\{a^2\mathcal{W}_0\left(\mathcal{W}_4^2-4\mathcal{W}_1\mathcal{W}_7\right)+k^2\left[\frac{1}{a^6}\left(-a^6\mathcal{W}_7\left(a^2\mathcal{W}_6^2+16m_2^2\mathcal{W}_0\right)\right.\right.\right.\\
&\left.-2a^4\bar{m}_5\mathcal{W}_4\mathcal{W}_6+a^8\left(\mathcal{W}_4^2-4\mathcal{W}_1\mathcal{W}_7\right)\mathcal{W}_3-4\bar{m}_5^2\mathcal{W}_1\right)\\
&+k^2\left(\frac{1}{a^8}\left(a^{10}\left(\mathcal{W}_4^2-4\mathcal{W}_1\mathcal{W}_7\right)\mathcal{W}_2-16\left(a^6\mathcal{W}_7\left(a^2m_2^2\mathcal{W}_3+\lambda_3\mathcal{W}_6-\lambda_4\mathcal{W}_0\right)\right.\right.\right.\\
&\left.\left.\left.+a^2\bar{m}_5\lambda_3\mathcal{W}_4+\bar{m}_5^2m_2^2\right)\right)\right)\\
&+k^4\left(-\frac{16}{a^{10}}\left(a^4\mathcal{W}_7\left(a^6m_2^2\mathcal{W}_2-a^4\lambda_4\mathcal{W}_3+a^2\lambda_7\mathcal{W}_0+4{\lambda_3}^2\right)\right.\right.\\
&\left.\left.+a^2\lambda_8\left(a^4\mathcal{W}_6\mathcal{W}_7+\bar{m}_5\mathcal{W}_4\right)-{\bar{m}_5}^2\lambda_4\right)\right)\\
&+k^6\left(\frac{16}{a^{12}}\left(a^4\mathcal{W}_7\left(a^6\lambda_4\mathcal{W}_2-a^4\lambda_7\mathcal{W}_3-8\lambda_3\lambda_8\right)-{\bar{m}_5}^2\lambda_7\right)\right)\\
&\left.\left.+k^8\left(-\frac{16}{a^{10}}\mathcal{W}_7\left(a^6\lambda_7\mathcal{W}_2+4{\lambda_8}^2\right)\right)\right]\right\}/\left\{a^2\left(\mathcal{W}_4^2-4\mathcal{W}_1\mathcal{W}_7\right)\right.\\
&\left.-16k^2\mathcal{W}_7\left(m_2^2-\frac{\lambda_4}{a^2}k^2+\frac{\lambda_7}{a^4}k^4\right)\right\},\\
&\bar{\mathcal{V}}(t,k)=-\left\{\frac{k^2}{a^2}\left[8\mathcal{W}_4\left(6a^2\mathcal{W}_7+\mathcal{W}_5\right)\left(\lambda_3+\lambda_8\frac{k^2}{a^2}\right)+16\frac{\bar{m}_5\mathcal{W}_5}{a^2}\left(m_2^2-\frac{\lambda_4}{a^2}k^2\right.\right.\right.\\
&\left.\left.\left.+\frac{\lambda_7}{a^4}k^4\right)\right]+6a^4\mathcal{W}_4\mathcal{W}_7\mathcal{W}_6+a^2\mathcal{W}_4\mathcal{W}_5\mathcal{W}_6+6\bar{m}_5\mathcal{W}_4^2+4\frac{\bar{m}_5}{a^2}\mathcal{W}_1\mathcal{W}_5\right\}/\left\{a^2\right.\\
&\left.\left(\mathcal{W}_4^2-4\mathcal{W}_1\mathcal{W}_7\right)-16k^2\mathcal{W}_7\left(m_2^2-\frac{\lambda_4}{a^2}k^2+\frac{\lambda_7}{a^4}k^4\right)\right\}.
\end{aligned}
\tag{2.111}
$$

It is easy to notice that the above expressions can be written in a more compact form as:

$$\begin{aligned}
\mathcal{L}_{\dot\zeta\dot\zeta}(t,k) &= \frac{k^2\mathcal{A}_4(t,k)+\mathcal{A}_1(t)}{k^2\mathcal{A}_2(t,k)+\mathcal{A}_3(t)}\,,\\
\bar{\mathcal{B}}(t,k) &= \frac{k^2\mathcal{B}_2(t,k)+\mathcal{B}_1(t)}{k^2\mathcal{A}_2(t,k)+\mathcal{A}_3(t)}\,,\\
\bar{\mathcal{V}}(t,k) &= \frac{k^2\mathcal{V}_2(t,k)+\mathcal{V}_1(t)}{k^2\mathcal{A}_2(t,k)+\mathcal{A}_3(t)}\,. \qquad (2.112)
\end{aligned}$$

By considering the above definitions the action can be written in the same form of (2.87), i.e.:

$$\mathcal{S}^{(2)}_{EFT} = \frac{1}{(2\pi)^3}\int d^3k dt\, a^3\left\{\mathcal{L}_{\dot\zeta\dot\zeta}(t,k)\dot\zeta^2 - k^2 G(t,k)\zeta^2\right\}\,, \qquad (2.113)$$

where we have identified the "gradient" term as:

$$\begin{aligned}
G(t,k) &= \Big\{k^2\Big[\mathcal{V}_2\left(k^2\dot{\mathcal{A}}_2+\dot{\mathcal{A}}_3-3H\left(k^2\mathcal{A}_2+\mathcal{A}_3\right)\right)+\mathcal{A}_2\mathcal{A}_3\,(2\mathcal{B}_1\\
&- \dot{\mathcal{V}}_1-k^2\dot{\mathcal{V}}_2+2k^2\mathcal{B}_2\Big)+\mathcal{V}_1\left(\dot{\mathcal{A}}_2-3H\mathcal{A}_2\right)\Big]\\
&+ \mathcal{V}_1\left(\dot{\mathcal{A}}_3-3H\mathcal{A}_3\right)\Big\}\,/\,\{2\left(k^2\mathcal{A}_2+\mathcal{A}_3\right)^2\}\\
&\equiv \frac{k^2\mathcal{G}_2(t,k)+\mathcal{G}_1(t)}{\left(k^2\mathcal{A}_2(t,k)+\mathcal{A}_3(t)\right)^2}\,. \qquad (2.114)
\end{aligned}$$

Then the speed of propagation is $c_s^2(t,k) = G/\mathcal{L}_{\dot\zeta\dot\zeta}$ and the friction term in the field equation of ζ turn out to be a function of both t and k. Let us notice that when considering the most general case, at least one of the functions $\{m_2^2, \lambda_i\}$ is not zero and none of the $\mathcal{A}_i$ functions are nil. Additionally the action does not contain the term $\bar{M}$. We will show in the next Section some particular cases of the action (2.85) for which such a term is present.

Let us now normalize the field by means of (2.90) with the kinetic term given by Eq. (2.111). Since the kinetic term is a function of k, the normalization will affect both the effective mass and speed of propagation. Thus we have:

$$\begin{aligned}
m^2_{\rm eff}(t,k) &= \Big(\mathcal{A}_1{}^2\left[2\mathcal{A}_3\left(3H\dot{\mathcal{A}}_3+\ddot{\mathcal{A}}_3\right)-3\dot{\mathcal{A}}_3^2\right]-2\mathcal{A}_3\mathcal{A}_1\Big[\mathcal{A}_3\left(3H\dot{\mathcal{A}}_1+\ddot{\mathcal{A}}_1\right)\\
&- \dot{\mathcal{A}}_1\dot{\mathcal{A}}_3\Big]+\mathcal{A}_3^2\dot{\mathcal{A}}_1^2\Big)\left(8\left(k^2\mathcal{A}_4+\mathcal{A}_1\right)^2\left(k^2\mathcal{A}_2+\mathcal{A}_3\right)^2\right)\,,\\
c^2_{s,\rm eff}(t,k) &= \Big\{6H\Big[\Big[k^2\left(\dot{\mathcal{A}}_4k^2+\dot{\mathcal{A}}_1\right)\mathcal{A}_2^2+2\Big[\mathcal{A}_3\left(\dot{\mathcal{A}}_4k^2+\dot{\mathcal{A}}_1\right)-k^2\mathcal{A}_4\Big(\dot{\mathcal{A}}_2k^2
\end{aligned}$$

$$
\begin{aligned}
+ & \quad \dot{\mathcal{A}}_3\Big)\Big]\,\mathcal{A}_2 + \mathcal{A}_3\left(\mathcal{A}_3\dot{\mathcal{A}}_4 2\mathcal{A}_4\left(\dot{\mathcal{A}}_2 k^2 + \dot{\mathcal{A}}_3\right)\right)\Big]\,\mathcal{A}_1 \\
- & \quad \mathcal{A}_1^2\left(\mathcal{A}_3\dot{\mathcal{A}}_2 + \mathcal{A}_2\left(\dot{\mathcal{A}}_2 k^2 + \dot{\mathcal{A}}_3\right)\right) + \left(\mathcal{A}_2 k^2 + \mathcal{A}_3\right)\mathcal{A}_4\Big[\mathcal{A}_2\left(\dot{\mathcal{A}}_4 k^2 + \dot{\mathcal{A}}_1\right)k^2 \\
- & \quad k^2\mathcal{A}_4\left(\dot{\mathcal{A}}_2 k^2 + \dot{\mathcal{A}}_3\right) + \mathcal{A}_3\left(\dot{\mathcal{A}}_4 k^2 + \dot{\mathcal{A}}_1\right)\Big]\Big] + \Big[3\mathcal{A}_4^2\dot{\mathcal{A}}_2^2 k^6 - 4\mathcal{A}_3\mathcal{A}_4\mathcal{G}_2 k^4 \\
+ & \quad 6\mathcal{A}_4^2\dot{\mathcal{A}}_2\dot{\mathcal{A}}_3 k^4 - 2\mathcal{A}_3\mathcal{A}_4\dot{\mathcal{A}}_2\dot{\mathcal{A}}_4 k^4 - 2\mathcal{A}_3\mathcal{A}_4^2\ddot{\mathcal{A}}_2 k^4 + 3\mathcal{A}_4^2\dot{\mathcal{A}}_3^2 k^2 - \mathcal{A}_3^2\dot{\mathcal{A}}_4^2 k^2 \\
- & \quad 4\mathcal{A}_3\mathcal{A}_4\mathcal{G}_1 k^2 - 2\mathcal{A}_3\mathcal{A}_4\dot{\mathcal{A}}_1\dot{\mathcal{A}}_2 k^2 - 2\mathcal{A}_3\mathcal{A}_4\dot{\mathcal{A}}_3\dot{\mathcal{A}}_4 k^2 - 2\mathcal{A}_3\mathcal{A}_4^2\ddot{\mathcal{A}}_3 k^2 \\
+ & \quad 2\mathcal{A}_3^2\mathcal{A}_4\ddot{\mathcal{A}}_4 k^2 - \mathcal{A}_2^2\left(\dot{\mathcal{A}}_4^2 k^4 + 2\dot{\mathcal{A}}_1\dot{\mathcal{A}}_4 k^2 - 2\mathcal{A}_4\left(\ddot{\mathcal{A}}_4 k^2 + \ddot{\mathcal{A}}_1\right)k^2 + \dot{\mathcal{A}}_1^2\right)k^2 \\
- & \quad 2\mathcal{A}_3\mathcal{A}_4\dot{\mathcal{A}}_1\dot{\mathcal{A}}_3 - 2{\mathcal{A}_3}^2\dot{\mathcal{A}}_1\dot{\mathcal{A}}_4 + 2\mathcal{A}_3^2\mathcal{A}_4\ddot{\mathcal{A}}_1 + \mathcal{A}_1^2\Big[3k^2{\dot{\mathcal{A}}_2}^2 + 6\dot{\mathcal{A}}_3\dot{\mathcal{A}}_2 \\
- & \quad 2\left(\mathcal{A}_3\ddot{\mathcal{A}}_2 + \mathcal{A}_2\left(\ddot{\mathcal{A}}_2 k^2 + \ddot{\mathcal{A}}_3\right)\right)\Big] - 2\mathcal{A}_2\Big[{\mathcal{A}_4}^2\left(\ddot{\mathcal{A}}_2 k^2 + \ddot{\mathcal{A}}_3\right)k^4 \\
+ & \quad \mathcal{A}_4\Big(2\mathcal{G}_2 k^4 + \dot{\mathcal{A}}_2\dot{\mathcal{A}}_4 k^4 + 2\mathcal{G}_1 k^2 + \dot{\mathcal{A}}_1\dot{\mathcal{A}}_2 k^2 + \dot{\mathcal{A}}_3\dot{\mathcal{A}}_4 k^2 - 2\mathcal{A}_3\ddot{\mathcal{A}}_4 k^2 \\
+ & \quad +\dot{\mathcal{A}}_1\dot{\mathcal{A}}_3 - 2\mathcal{A}_3\ddot{\mathcal{A}}_1\Big)k^2 + \mathcal{A}_3\left(\dot{\mathcal{A}}_4 k^2 + \dot{\mathcal{A}}_1\right)^2\Big] + 2\mathcal{A}_1\Big[k^2\left(\ddot{\mathcal{A}}_4 k^2 + \ddot{\mathcal{A}}_1\right)\mathcal{A}_2^2 \\
- & \quad \Big(2\mathcal{G}_2 k^4 + \dot{\mathcal{A}}_2\dot{\mathcal{A}}_4 k^4 + 2\mathcal{A}_4\ddot{\mathcal{A}}_2 k^4 + 2\mathcal{G}_1 k^2 + \dot{\mathcal{A}}_1\dot{\mathcal{A}}_2 k^2 \\
+ & \quad \dot{\mathcal{A}}_3\dot{\mathcal{A}}_4 k^2 + 2\mathcal{A}_4\ddot{\mathcal{A}}_3 k^2 - 2\mathcal{A}_3\ddot{\mathcal{A}}_4 k^2 + \dot{\mathcal{A}}_1\dot{\mathcal{A}}_3 - 2\mathcal{A}_3\dot{\mathcal{A}}_1\Big)\,\mathcal{A}_2 \\
+ & \quad 3\mathcal{A}_4\left(\dot{\mathcal{A}}_2 k^2 + \dot{\mathcal{A}}_3\right)^2 - \mathcal{A}_3\,\big(2\mathcal{G}_2 k^2 \\
+ & \quad \dot{\mathcal{A}}_2\dot{\mathcal{A}}_4 k^2 + 2\mathcal{A}_4\ddot{\mathcal{A}}_2 k^2 + 2\mathcal{G}_1 + \dot{\mathcal{A}}_1\dot{\mathcal{A}}_2 + \dot{\mathcal{A}}_3\dot{\mathcal{A}}_4 + 2\mathcal{A}_4\ddot{\mathcal{A}}_3\big) \\
+ & \quad \mathcal{A}_3^2\ddot{\mathcal{A}}_4\Big]\Big]\Big\}\,/[8\left(\mathcal{A}_2 k^2 + \mathcal{A}_3\right)^2\left(\mathcal{A}_4 k^2 + \mathcal{A}_1\right)^2] \\
\equiv & \quad c_s^2 + f(t,k).
\end{aligned}
\tag{2.115}
$$

As said before the effective mass is a function of inverse powers of k. For sufficiently high k, the effective mass is negligible while in the low k limit, which is the one of interest in linear cosmology, it is solely a function of time. Let us notice that the effective mass in this case has been obtained directly from action (2.113), not from Eq. (2.92) which is valid only for cases when the kinetic term does not depend on k.

2.5.3 Special cases

Although the subset of theories with higher than second order spatial derivatives treated in the previous Section is very general, there are some special cases for which the action assumes some particular forms due to specific combinations of the EFT functions in the kinetic term. In order to illustrate said cases, we will consider the following action for practical examples:

$$\begin{aligned}\mathcal{S}^{(2)}_{EFT} &= \frac{1}{(2\pi)^3}\int d^3k dt\, a^3\Big[4m_2^2\frac{k^2}{a^2}(\delta N)^2 - \mathcal{W}_6\delta N k^2\zeta - \mathcal{W}_4\delta N k^2\psi - \mathcal{W}_5 k^2\psi\dot{\zeta} \\ & -\mathcal{W}_0 k^2\zeta^2 + \mathcal{W}_7(k^2\psi)^2 + \mathcal{W}_1(\delta N)^2 - 3a^2\mathcal{W}_4\delta N\dot{\zeta} - \frac{3}{2}a^2\mathcal{W}_5\dot{\zeta}^2\Big] \end{aligned} \tag{2.116}$$

for which the following conditions hold:

$$\mathcal{W}_7 \neq 0 \qquad \{\bar{m}_5, \lambda_i\} = 0\,. \tag{2.117}$$

By solving the Eqs. (2.109) for δN and ψ we get:

$$\begin{aligned}\delta N &= \frac{\mathcal{W}_4\left(6a^2\mathcal{W}_7+\mathcal{W}_5\right)\dot{\zeta} + 2\mathcal{W}_6\mathcal{W}_7 k^2\zeta}{16m_2^2\mathcal{W}_7\frac{k^2}{a^2} - \mathcal{W}_4^2 + 4\mathcal{W}_1\mathcal{W}_7}\,, \\ k^2\psi &= \frac{\mathcal{W}_4\mathcal{W}_6 k^2\zeta + \left(2\mathcal{W}_1\mathcal{W}_5 + 3a^2\mathcal{W}_4^2 + 8m_2^2\mathcal{W}_5\frac{k^2}{a^2}\right)\dot{\zeta}}{16m_2^2\mathcal{W}_7\frac{k^2}{a^2} - \mathcal{W}_4^2 + 4\mathcal{W}_1\mathcal{W}_7}\end{aligned} \tag{2.118}$$

which allow us to eliminate the two auxiliary fields in the action. Substituting back in the action we get:

$$\begin{aligned}\mathcal{S}^{(2)}_{EFT} &= \frac{1}{(2\pi)^3}\int d^3k dt\, a^3\left\{\left[\frac{\left(6a^2\mathcal{W}_7+\mathcal{W}_5\right)\left(3a^4\mathcal{W}_4^2+2a^2\mathcal{W}_1\mathcal{W}_5+8m_2^2\mathcal{W}_5k^2\right)}{2a^2\left(\mathcal{W}_4^2-4\mathcal{W}_1\mathcal{W}_7\right)-32m_2^2\mathcal{W}_7k^2}\right]\dot{\zeta}^2\right. \\ &+ k^2\left[\left(\left(a^2\left(\mathcal{W}_0\left(\mathcal{W}_4^2-4\mathcal{W}_1\mathcal{W}_7\right)-k^2\mathcal{W}_6^2\mathcal{W}_7\right)-16m_2^2\mathcal{W}_0\mathcal{W}_7k^2\right)\zeta^2\right.\right. \\ &- \left.\left.\left.\left(a^2\mathcal{W}_4\mathcal{W}_6\left(6a^2\mathcal{W}_7+\mathcal{W}_5\right)\right)\dot{\zeta}\zeta\right)/\left(16m_2^2\mathcal{W}_7k^2-a^2\left(\mathcal{W}_4^2-4\mathcal{W}_1\mathcal{W}_7\right)\right)\right]\right\}\,,\end{aligned} \tag{2.119}$$

where the kinetic term reads:

$$\mathcal{L}_{\dot{\zeta}\dot{\zeta}}(t,k) \equiv \frac{\left(6a^2\mathcal{W}_7+\mathcal{W}_5\right)\left(3a^4\mathcal{W}_4^2+2a^2\mathcal{W}_1\mathcal{W}_5+8k^2m_2^2\mathcal{W}_5\right)}{2a^2\left(\mathcal{W}_4^2-4\mathcal{W}_1\mathcal{W}_7\right)-32k^2m_2^2\mathcal{W}_7}\,. \tag{2.120}$$

In the following we will consider two special cases in which 1) the kinetic term depends only on time; 2) the kinetic term has a particular k-dependence, which needs to be studied carefully in order to correctly identify the speed of propagation.

- First case: $3a^2\mathcal{W}_4^2 + 2\mathcal{W}_1\mathcal{W}_5 \neq 0$ and $m_2^2 = 0$. The kinetic term is only a function of time:

$$\mathcal{L}_{\dot{\zeta}\dot{\zeta}}(t) = \frac{\left(6a^2\mathcal{W}_7+\mathcal{W}_5\right)\left(3a^4\mathcal{W}_4^2+2a^2\mathcal{W}_1\mathcal{W}_5\right)}{2a^2\left(\mathcal{W}_4^2-4\mathcal{W}_1\mathcal{W}_7\right)}\,, \tag{2.121}$$

which corresponds to the case $\mathcal{A}_2 = \mathcal{A}_4 = 0$. The above expression must be positive in order to guarantee that the theory does not exhibit ghost instabilities. Then, the speed of propagation can be easily obtained from action (2.119) once the terms proportional to $\dot{\zeta}\zeta$ have been integrated by parts and it reads:

$$
\begin{aligned}
c_s^2(t,k) &= \frac{1}{(\mathcal{W}_4^2 - 4\mathcal{W}_1\mathcal{W}_7)(3a^2\mathcal{W}_4{}^2 + 2\mathcal{W}_1\mathcal{W}_5)(6a^2\mathcal{W}_7 + \mathcal{W}_5)} \\
&\times \Big\{30a^2\mathcal{W}_4\mathcal{W}_6\mathcal{W}_7\left(\mathcal{W}_4{}^2 - 4\mathcal{W}_1\mathcal{W}_7\right)H + 3\mathcal{W}_4\mathcal{W}_5\mathcal{W}_6\left(\mathcal{W}_4^2\right. \\
&- \left.4\mathcal{W}_1\mathcal{W}_7\right)H - \mathcal{W}_6\mathcal{W}_4^2\mathcal{W}_5\dot{\mathcal{W}}_4 - 4\mathcal{W}_1\mathcal{W}_6\mathcal{W}_7\mathcal{W}_5\dot{\mathcal{W}}_4 \\
&+ \mathcal{W}_4^3\left(\mathcal{W}_6\dot{\mathcal{W}}_5 + \mathcal{W}_5\dot{\mathcal{W}}_6\right) + 4\mathcal{W}_4\Big[\mathcal{W}_5\Big(\mathcal{W}_6\Big(\mathcal{W}_7\dot{\mathcal{W}}_1 \\
&+ \mathcal{W}_1\dot{\mathcal{W}}_7\Big) - \mathcal{W}_1\mathcal{W}_7\dot{\mathcal{W}}_6\Big) - \mathcal{W}_1\mathcal{W}_6\mathcal{W}_7\dot{\mathcal{W}}_5\Big] \\
&+ 2\mathcal{W}_0\left(\mathcal{W}_4^2 - 4\mathcal{W}_1\mathcal{W}_7\right)^2 + 6a^2\Big[\mathcal{W}_4^3\left(\mathcal{W}_7\dot{\mathcal{W}}_6 + \mathcal{W}_6\dot{\mathcal{W}}_7\right) \\
&+ 4\mathcal{W}_7^2\mathcal{W}_4\left(\mathcal{W}_6\dot{\mathcal{W}}_1 - \mathcal{W}_1\dot{\mathcal{W}}_6\right) - 4\mathcal{W}_1\mathcal{W}_6\mathcal{W}_7^2\dot{\mathcal{W}}_4 \\
&- \mathcal{W}_4^2\mathcal{W}_6\mathcal{W}_7\dot{\mathcal{W}}_4\Big] - 2k^2a\mathcal{W}_6^2\mathcal{W}_7(\mathcal{W}_4^2 - 4\mathcal{W}_1\mathcal{W}_7)\Big\}\,, \qquad (2.122)
\end{aligned}
$$

where the k-dependence of the speed is due to $W_7 \neq 0$. Moreover, in this case, the final action is of the form (2.87) with $\bar{M} = 0$. Since the kinetic terms is free from any k-dependence there is no ambiguity in defining the mass term which, after the normalization (2.90), ends up being of the same form as in Eq. (2.92) where, in this case, $\mathcal{L}_{\dot{\zeta}\dot{\zeta}}$ is given by Eq. (2.121). Finally, the effective speed of propagation remains invariant under the field redefinition.

- Second case: $3a^2\mathcal{W}_4^2 + 2\mathcal{W}_1\mathcal{W}_5 = 0$ and $m_2^2 \neq 0$. In this case the kinetic term reduces to:

$$
\mathcal{L}_{\dot{\zeta}\dot{\zeta}}(t,k) = \frac{4m_2^2\mathcal{W}_5^2\left(6a^2\mathcal{W}_7 + \mathcal{W}_5\right)\frac{k^2}{a^2}}{\mathcal{W}_4^2\left(6a^2\mathcal{W}_7 + \mathcal{W}_5\right) - 16\frac{k^2}{a^2}m_2^2\mathcal{W}_5\mathcal{W}_7}\,, \qquad (2.123)
$$

which corresponds to $\mathcal{A}_1 = 0$ and $\mathcal{A}_2(t)$, $\mathcal{A}_4(t)$ both being functions of time. From the action (2.119) it follows that there is an overall factor k^2 in front of the Lagrangian which can be reabsorbed by redefining the field as $\tilde{\zeta} = k\zeta$. As a result we obtain an action of the form (2.110). Let us notice that, in this case, $\mathcal{V}_2 = 0$. After integrating by parts the term $\sim \dot{\zeta}\zeta$, we end up with an action as in (2.87) where $\bar{M} \neq 0$, and both the friction and dispersive coefficients in the field equation are functions of time and k. Now we can compute the speed of propagation which is:

$$c_s^2(t,k) = \frac{\mathcal{V}_1\dot{\mathcal{A}}_2 + \mathcal{A}_2(2k^2\mathcal{B}_2 - \dot{\mathcal{V}}_1 + 2\mathcal{B}_1) + 2\mathcal{A}_3\mathcal{B}_2 - 3H\mathcal{A}_2\mathcal{V}_1}{2\mathcal{A}_4\left(k^2\mathcal{A}_2 + \mathcal{A}_3\right)} \,. \tag{2.124}$$

In conclusion, we give the expressions for the effective mass and speed of propagation:

$$\begin{aligned}
m_{\rm eff}^2(t,k) &= \Big\{6\mathcal{A}_2\mathcal{A}_3H(\mathcal{A}_2\dot{\mathcal{A}}_3 - \mathcal{A}_3\dot{\mathcal{A}}_2) + \mathcal{A}_3\mathcal{A}_2(2\dot{\mathcal{A}}_2\dot{\mathcal{A}}_3 - 2\mathcal{A}_3\ddot{\mathcal{A}}_2 \\
&+ \mathcal{G}_1) + {\mathcal{A}_2}^2(2\mathcal{A}_3\ddot{\mathcal{A}}_3 - 3{\dot{\mathcal{A}}_3}^2) + {\mathcal{A}_3}^2{\dot{\mathcal{A}}_2}^2\Big\}\left\{8{\mathcal{A}_4}^2\left(k^2\mathcal{A}_2 + \mathcal{A}_3\right)^2\right\} \\
c_{s,\rm eff}^2(t,k) &= \Big\{6\mathcal{A}_4H\Big[\mathcal{A}_2\Big(\mathcal{A}_4\Big(k^2\dot{\mathcal{A}}_2 + \dot{\mathcal{A}}_3\Big) - 2\mathcal{A}_3\dot{\mathcal{A}}_4\Big) + \mathcal{A}_3\mathcal{A}_4\dot{\mathcal{A}}_2 \\
&- k^2\mathcal{A}_2^2\dot{\mathcal{A}}_4\Big] + 2\mathcal{A}_2\Big[\mathcal{A}_4\Big(k^2\dot{\mathcal{A}}_2\dot{\mathcal{A}}_4 + 2k^2\mathcal{G}_2 + \dot{\mathcal{A}}_3\dot{\mathcal{A}}_4 \\
&- 2\mathcal{A}_3\ddot{\mathcal{A}}_4 + 2\mathcal{G}_1\Big) + {\mathcal{A}_4}^2\Big(k^2\ddot{\mathcal{A}}_2 + \ddot{\mathcal{A}}_3\Big) + \mathcal{A}_3\dot{\mathcal{A}}_4^2\Big] \\
&+ \mathcal{A}_4\Big[2\mathcal{A}_3\Big(\dot{\mathcal{A}}_2\dot{\mathcal{A}}_4 + \mathcal{A}_4\ddot{\mathcal{A}}_2 + 2\mathcal{G}_2\Big) \\
&- 3\mathcal{A}_4\dot{\mathcal{A}}_2\Big(k^2\dot{\mathcal{A}}_2 + 2\dot{\mathcal{A}}_3\Big)\Big] + k^2\mathcal{A}_2^2\Big(\dot{\mathcal{A}}_4^2 - 2\mathcal{A}_4\ddot{\mathcal{A}}_4\Big)\Big\} \\
&/ \Big[8\mathcal{A}_4^2\left(k^2\mathcal{A}_2 + \mathcal{A}_3\right)^2\Big] \,,
\end{aligned} \tag{2.125}$$

where the function $\mathcal{G}_i(i = 1, 2)$ can be read from:

$$G(t,k) = \frac{\mathcal{V}_1\dot{\mathcal{A}}_2 + \mathcal{A}_2(-\dot{\mathcal{V}}_1 + 2\mathcal{B}_1) + 2\mathcal{A}_3\mathcal{B}_2 - 3H\mathcal{A}_2\mathcal{V}_1 + 2k^2\mathcal{A}_2\mathcal{B}_2}{2\left(k^2\mathcal{A}_2 + \mathcal{A}_3\right)^2}. \tag{2.126}$$

Finally, let us notice that in the case $\bar{M} \neq 0$, one may wonder if the conservation of the curvature perturbation is preserved on super-horizon scales. It is not so trivial to draw a general conclusion about the behaviour of ζ in such limit, because the EFT functions involved in the $\bar{M}$ term are all unknown functions of time. Therefore, we can conclude that in the general field equation for ζ on super-horizon scales such term might be non zero, possibly leading to a non conserved curvature perturbation. However, we expect that well behaved DE/MG models will have either $\bar{M} = 0$ or that such term will contribute a decaying mode, thus leaving the conservation of ζ unaffected. In this regard, we will argue our last statement by using an explicit example, which is not conclusive but can give an insight on how $\bar{M}$ can behave in the low k regime when theoretical models are considered. Considering the mapping (2.82), it is easy to verify that the low energy Hořava gravity falls in the special case under analysis and that the corresponding $\bar{M} \neq 0$. However, when considering the super-horizon limit the $\bar{M}$ term goes to zero and the equation for ζ reduces to

$$\ddot{\zeta} + H\dot{\zeta} = 0, \tag{2.127}$$

which solution is $\zeta \to \zeta_c - \frac{c_1 e^{-\sqrt{2}t\sqrt{\frac{\xi\Lambda}{9\lambda-3}}}}{\sqrt{2}\sqrt{\frac{\xi\Lambda}{9\lambda-3}}}$. ζ_c, c_1 are constant and the second term is a decaying mode. Hence, the conservation of ζ is preserved.

Let us conclude by saying that the cases treated in this Section are only few examples of "special" cancellations that might happen.

2.6 An extended basis for theories with higher spatial derivatives

In Ref. [69], the authors proposed a new basis to describe Horndeski theories, in terms of four free functions of time which parametrize the departure from GR. Specifically, these functions are: $\{\alpha_B, \alpha_M, \alpha_K, \alpha_T\}$, hereafter referred to as ReParametrized Horndeski (RPH). They are equivalent and an alternative to the EFT functions needed to describe the dynamics of perturbations in the Horndeski class, i.e. $\{\Omega, M_2^4, \bar{M}_2^2, \bar{M}_1^3\}$. In both cases one needs to supply also the Hubble parameter, $H(a)$. The latest publicly released version of *EFTCAMB* contains also the RPH basis as a built-in alternative [80]. RPH is also the building block at the basis of HiCLASS [110].

The RPH basis was constructed in order to encode departures from GR in terms of some key properties of the (effective) DE component. As discussed in details in Ref. [69], the braiding function α_B is connected to the clustering of DE, α_M parametrizes the time-dependence of the Planck mass and, along with α_T, is related to the anisotropic stress while large values of the kinetic function, α_K correspond to suppressed values of the speed of propagation of the scalar mode. In Ref. [67], the RPH basis has been extended to include the GLPV class of theories by adding the function α_H, which parametrizes the deviation from the Horndeski class.

In this Section we introduce an extended version of the RPH basis which generalizes the original one [69], as well as its extension to GLPV [67], by encompassing the higher order spatial derivatives terms appearing in action (2.1). We also present the explicit mapping between this new basis and the EFT functions in the extended action (2.1), in order to facilitate the link between phenomenological properties and the theory which is responsible for them.

Let us start with tensor perturbations of the EFT action (2.17) analysed in Section 2.5. Here, for completeness we rewrite its compact form:

$$\mathcal{S}^{T\,(2)}_{EFT} = \frac{1}{(2\pi)^3}\int d^3k dt\, a^3 \frac{A_T(t)}{8}\left[(\dot{h}^T_{ij})^2 - \frac{c_T^2(t,k)}{a^2}k^2(h^T_{ij})^2\right]. \quad (2.128)$$

Now, following Ref. [69], we define the deviation from GR of the tensor speed of propagation as:

$$c_T^2(t,k) = 1 + \tilde{\alpha}_T(t,k), \quad (2.129)$$

where:

$$\tilde{\alpha}_T(t,k) = \alpha_T(t) + \alpha_{T_2}(t)\frac{k^2}{a^2} + \alpha_{T_6}(t)\frac{k^4}{a^4}\,, \quad (2.130)$$

with:

$$\alpha_T(t) = \tfrac{\bar{M}_3^2}{m_0^2(1+\Omega)-\bar{M}_3^2} \equiv \bar{c}_T^2 - 1\,, \quad \alpha_{T_2}(t) = -8\tfrac{\lambda_2}{m_0^2(1+\Omega)-\bar{M}_3^2}\,,$$
$$\alpha_{T_6}(t) = -8\tfrac{\lambda_6}{m_0^2(1+\Omega)-\bar{M}_3^2}. \quad (2.131)$$

As expected, the additional higher order operators will contribute by adding a k-dependence in the original definition of the α_T function introduced in Ref. [69]. Moreover, we can define the rate of evolution of the mass function $M^2(t) \equiv A_T(t)$ (defined in Eq. (2.99)) as:

$$\alpha_M(t) = \frac{1}{H(t)}\frac{d}{dt}\left(\ln M^2(t)\right). \quad (2.132)$$

It is clear that α_T and α_M differ from the ones in Ref. [69] since, in general, $\bar{M}_3^2(t) \neq -\bar{M}_2^2(t)$ for theories with higher spatial derivatives. It is important to notice that the EFT functions which are involved in the definition of α_M and α_T are $\{\Omega, \bar{M}_3^2\}$. Therefore, the class of theories which can contribute to a time dependent Planck mass and modify the tensor speed of propagation, are the ones which are non-minimally coupled with gravity and/or contain the $\mathcal{S}$-term in the action; specifically, Horndeski models with non zero L_4^{GG}, L_5^{GG}, GLPV models with non zero L_4^{GLPV}, L_5^{GLPV} and Hořava gravity. Moreover, the k-dependence in the speed of propagation is related to the α_{T2}, α_{T6} functions which are present in Hořava gravity. Finally, let us notice that, since M^2 appears in the denominator of c_T^2, high values of M^2 will generally suppress the speed of propagation and in case only background EFT functions are at play or theories for which $\{\bar{M}_3^2(t), \lambda_{2,6}\} = 0$ are considered, c_T^2 is identically one. Therefore, it would be not possible to discriminate between minimally and non-minimally coupled models.

Let us now focus on the scalar perturbations. Collecting terms with the same perturbations, the second order action (2.17) can be written as follows:

$$\begin{aligned}
\mathcal{S}^{(2)}_{EFT} &= \frac{1}{(2\pi)^3}\int d^3k dt\, a^3\, \frac{M^2}{2}\Big\{(1+\tilde{\alpha}_H)\,\delta N\delta_1\tilde{\mathcal{R}} - 4H\alpha_B\delta N\delta\tilde{K} \\
&+ \delta\tilde{K}^\mu_\nu\delta\tilde{K}^\nu_\mu - (\alpha_B^{GLPV}+1)(\delta\tilde{K})^2 + \tilde{\alpha}_K H^2(\delta N)^2 \\
&- \frac{1}{4}\left(\alpha_{T_2}+\alpha_{T_6}\frac{k^2}{a^2}\right)\delta\tilde{\mathcal{R}}_{ij}\delta\tilde{\mathcal{R}}^{ij} + (1+\alpha_T)\delta_2\tilde{\mathcal{R}} + (1+\alpha_T)\delta_1\tilde{\mathcal{R}}\delta(\tilde{\sqrt{h}}) \\
&+ \left(\alpha_1+\alpha_5\frac{k^2}{a^2}\right)(\delta\tilde{\mathcal{R}})^2 + \bar{\alpha}_5\delta_1\tilde{\mathcal{R}}\delta\tilde{K}\Big\}\,,
\end{aligned} \tag{2.133}$$

where the geometrical quantities with tildes are the Fourier transform of the corresponding quantities in Eq. (2.84), moreover we have identified the following functions:

$$\begin{aligned}
&\alpha_B(t) = \frac{m_0^2\dot{\Omega}+\bar{M}_1^3}{2HM^2}\,, \qquad \alpha_B^{GLPV}(t) = \frac{\bar{M}_3^2+\bar{M}_2^2}{M^2}\,, \\
&\tilde{\alpha}_K(t,k) = \alpha_K(t) + \alpha_{K_2}(t)\frac{k^2}{a^2} + \alpha_{K_4}(t)\frac{k^4}{a^4} + \alpha_{K_7}(t)\frac{k^6}{a^6}\,, \\
&\text{where} \quad \alpha_K(t) = \frac{2c+4M_2^4}{H^2M^2}\,, \quad \alpha_{K_2}(t) = \frac{8m_2^2}{M^2H^2}\,, \quad \alpha_{K_4}(t) = -\frac{8\lambda_4}{M^2H^2}\,, \\
&\alpha_{K_7}(t) = \frac{8\lambda_7}{H^2M^2}\,, \quad \tilde{\alpha}_H(t,K) = \alpha_H(t) + \alpha_{H_3}(t)\frac{k^2}{a^2} + \alpha_{H_8}(t)\frac{k^4}{a^4}\,, \\
&\text{where} \quad \alpha_H(t) = \frac{2\hat{M}^2+\bar{M}_3^2}{M^2}\,, \quad \alpha_{H_3}(t) = -\frac{4\lambda_3}{M^2}\,, \quad \alpha_{H_8}(t) = \frac{4\lambda_8}{M^2}\,, \\
&\alpha_1(t) = \frac{2\lambda_1}{M^2}\,, \qquad \alpha_5(t) = \frac{2\lambda_5}{M^2}\,, \qquad \bar{\alpha}_5(t) = \frac{\bar{m}_5}{M^2}.
\end{aligned} \tag{2.134}$$

The relations between the $\mathcal{W}$-functions introduced in Section 2.5 and the above α-functions are the following:

$$\begin{aligned}
&\mathcal{W}_0 \equiv -\frac{M^2}{a^2}\left(\alpha_T + 1 + 3H\bar{\alpha}_5 + 3\dot{\bar{\alpha}}_5 + 3\bar{\alpha}_5 H\alpha_M\right)\,, \\
&\mathcal{W}_1 \equiv \frac{M^2H^2}{2}\alpha_K + \frac{3}{2}a^2H\mathcal{W}_4 - 3H^2M^2\alpha_B\,, \\
&\mathcal{W}_2 \equiv \frac{M^2}{a^6}\left(-8\alpha_5 + \frac{3}{4}\alpha_{T_6}\right)\,, \qquad \mathcal{W}_3 \equiv \frac{M^2}{a^4}\left(-8\alpha_1 + \frac{3}{4}\alpha_{T_2}\right)\,, \\
&\mathcal{W}_4 \equiv -\frac{HM^2}{a^2}\left(2+2\alpha_B+3\alpha_B^{GLPV}\right)\,, \quad \mathcal{W}_5 \equiv \frac{M^2}{a^2}\left(2+3\alpha_B^{GLPV}\right)\,, \\
&\mathcal{W}_6 \equiv -\frac{2M^2}{a^2}\left(1+\alpha_H+3H\bar{\alpha}_5\right)\,, \quad \mathcal{W}_7 \equiv -\frac{M^2}{2a^4}\alpha_B^{GLPV}\,.
\end{aligned} \tag{2.135}$$

Before discussing in details the meaning of the α-functions and how they contribute to the evolution of the propagating d.o.f., we introduce the perturbed linear equations which will help us in the discussion. The variation of the action (2.133) w.r.t to ψ and δN gives:

$$H\left[2(1+\alpha_B)+3\alpha_B^{GLPV}\right]\delta N-(2+3\alpha_B^{GLPV})\dot{\zeta}-\alpha_B^{GLPV}\frac{k^2\psi}{a^2}-2\bar{\alpha}_5\frac{k^2\zeta}{a^2}=0\,,$$
$$\Big[3H^2\left(2-4\alpha_B-3\alpha_B^{GLPV}\right)+H^2\tilde{\alpha}_K\Big]\delta N+2H\Big[2\alpha_B+3\alpha_B^{GLPV}+2\Big]\frac{k^2}{a^2}\psi$$
$$+\Big[3H\left(2+2\alpha_B+3\alpha_B^{GLPV}\right)\Big]\dot{\zeta}+2\Big[1+H\bar{\alpha}_5+\tilde{\alpha}_H\Big]\frac{k^2}{a^2}\zeta=0. \qquad (2.136)$$

These equations allow us to eliminate the auxiliary fields δN and ψ from the action, yielding an action solely in terms of the dynamical field ζ. A detailed description of how to eliminate the auxiliary fields was the subject of the previous Section 2.5, indeed the above equations are equivalent to Eqs. (2.109), once the relations (2.135) have been considered. At this point, we can describe the meaning of the different α-functions in terms of the phenomenology of ζ.

Let us now focus on the definition of the α-functions which characterize the new basis, $\{\alpha_M,\tilde{\alpha_T},\alpha_B,\alpha_B^{GLPV},\tilde{\alpha_H},\tilde{\alpha}_K,\bar{\alpha}_5,\alpha_1,\alpha_5\}$, extending and generalizing the RPH one. A first difference that can be noticed w.r.t. the RPH parametrization, is the presence of $\{\tilde{\alpha}_H,\tilde{\alpha}_K\}$ which are now functions of k, since they contain the contributions from operators with higher spatial derivatives. Let us now describe the new basis in details with the help of the definitions (2.134) and Eqs. (2.136):

- $\{\alpha_B,\alpha_B^{GLPV}\}$: α_B is the braiding function as defined in Ref. [69]. * Its role is clear by looking at Eqs. (2.136), indeed α_B regulates the relation between the auxiliary field δN and the dynamical d.o.f. ζ. Analogously, we define α_B^{GLPV}, which contributes to the braiding since it mediates the relationship of ψ and δN with ζ. The effects of these braiding coefficients on the kinetic term and the speed of propagation is more involved. Indeed, by looking at the action (2.133) we can notice that α_B^{GLPV} has a direct contribution to the kinetic term since it is the pre-factor of $(\delta K)^2$, which contains $\dot{\zeta}^2$. Moreover, both α_B and α_B^{GLPV} affect indirectly the kinetic term: the δN term in Eq. (2.136), whose pre-factor contains the braiding functions, turns out to be proportional to $\dot{\zeta}$, then substituting it back to action (2.133), the term in $(\delta N)^2$ will generate a contribution to the kinetic term. Furthermore, their involvement in the speed of propagation of the scalar d.o.f. comes in two ways: 1) from the kinetic term as previously mentioned. Indeed through

*The definition of α_B presented here differs from the one in Ref. [69] by a minus sign and a factor 2.

Eq. (2.89) they enter in the denominator of the definition of the propagating speed; 2) because they multiply both the δN and ψ terms in Eq. (2.136) which result to be proportional to $k^2\zeta$ which contributes to G in Eq. (2.89). Moreover, analogously to the definition of α_H, which parametrizes the deviation w.r.t. Horndeski/GG theories, α_B^{GLPV} is defined such as to parametrize the deviation from GLPV theories; indeed the latter are characterized by the condition $\alpha_B^{GLPV} = 0$, hence the name. If $\alpha_B^{GLPV} \neq 0$, higher spatial derivatives appear in the ζ equation. Finally, α_B is different from zero for all the theories showing non-minimal coupling to gravity and/or possessing the $\delta N\delta K$ operator in the action, i.e. $f(R), L_3^{GG}, L_4^{GG}, L_5^{GG}, L_4^{GLPV}, L_5^{GLPV}$. This operator does not appear when one considers quintessence and k-essence models (L_2^{GG}) and Hořava gravity. α_B^{GLPV} is non zero for the low-energy Hořava gravity action.

- $\tilde{\alpha}_K(t,k)$: it is the generalization of the purely kinetic function $\alpha_K(t)$ and it describes the extension of the kinetic term to higher order spatial derivatives in the case of non zero $\{\alpha_{K2}, \alpha_{K4}, \alpha_{K7}\}$. It is easy to see that $\tilde{\alpha}_K(t,k)$ is related to the kinetic term of the scalar d.o.f. since it appears in action (2.133) as a coefficient of the operator $(\delta N)^2$ and, through the linear perturbed equations (2.136), $\delta N \sim \dot{\zeta}$. Since it describes the kinetic term, it will affect the speed of propagation of ζ as well as the condition for the absence of a scalar ghost. The last point is easy to understand because as we extensively discussed in Section 2.5 the kinetic terms goes in the denominator of the speed of propagation of scalar perturbation (see Eq. (2.89)). The α_K function is characteristic of theories belonging to GLPV, while for Hořava gravity it is identically zero. On the other hand, Hořava gravity contributes non zero $\{\alpha_{K2}, \alpha_{K4}, \alpha_{K7}\}$. Finally, let us note that when considering theories beyond GLPV the braiding coefficient discussed in the previous point, α_B^{GLPV}, gives a direct contribution to the kinetic term through the operator $(\delta K)^2$.

- $\{\alpha_1, \alpha_5, \bar{\alpha}_5, \tilde{\alpha}_H\}$: from the constraint equations (2.136), it can be noticed that $\tilde{\alpha}_H$ and $\bar{\alpha}_5$ contribute to the speed of propagation of the scalar d.o.f. since they multiply the term $k^2\zeta$. In particular, if $\bar{\alpha}_5$ and the k-dependent parts of $\tilde{\alpha}_H$ are different from zero, the dispersion relation of ζ will be modified and the speed of propagation will depend on k. The functions $\{\alpha_1, \alpha_5\}$ have a similar impact since they are the pre-factors of $\delta_1\mathcal{R}$ in the action which, once expressed in terms of the perturbations of the metric, gives a term proportional to $k^2\zeta$. In this case by looking at Eq. (2.89) these functions will enter in the definition of G. The theories where these functions are present are GLPV and Hořava gravity models. In particular, in the case of Hořava gravity the functions associated with higher order spatial derivatives terms are present.

The above represents an interesting extension and generalization of the original RPH parametrization [69], carefully built while considering the different phenomenological aspects of the dark energy fluid. However, let us notice that the desired correspondence between the α-functions and actual observables becomes weaker as we go beyond the Horndeski class. Indeed, due to the high number of α-functions involved, their dependence on many EFT functions and the way they enter in the actual physical quantities, such as the speed of sound and the kinetic term, identifying exactly the underlying theory of gravity responsible for a specific effect is a hard task.

2.7 Conclusions

We started this Chapter by generalizing the original EFToDE/MG action for DE/MG by including operators up to sixth order in spatial derivatives. This was motivated by the recent rise of theories containing a (sub)set of these operators with higher-order spatial derivatives, like Hořava gravity. As such, these theories were not covered by the operators included in the first proposal of the EFToDE/MG action as presented in Refs. [37, 39]. From there on, the extended Lagrangian (2.1) became the basis of the rest of the Chapter as the new operators play a central role.

Starting from the extended Lagrangian (2.1) we proceeded to show an efficient method to map theories of gravity, expressed in terms of geometrical quantities, into the EFToDE/MG language. This led to a general mapping between the ADM and the EFToDE/MG formalism for our new extended Lagrangian. Subsequently, we illustrated this procedure by mapping models of DE/MG, with an additional scalar d.o.f., into the EFToDE/MG formalism, resulting in a vast set of worked out examples. These include minimally coupled quintessence, f(R), Hornedski/GG, GLPV and Hořava gravity. The preliminary step of writing the theories in the ADM formalism has also been presented as it is an integral part of the procedure. Therefore we created a very useful guide for the theoretical steps necessary in order to implement a given model of DE/MG into *EFTCAMB* and a "dictionary" for many of the existing DE/MG models. To this extent, we have been very careful and explicit about the conventions which lie at the basis of the EFToDE/MG formalism and, by extension, *EFTCAMB*. These become obvious when comparing with the equivalent approaches in the literature as there are some clear differences.

Thus the take-home message is that the user should be careful with the conventions when implementing a given model into *EFTCAMB*.

An ongoing field of research regarding the EFToDE/MG is the determination of the parameter space corresponding to physically healthy theories, as we introduced at the beginning of this thesis. This is vital from a theoretical as well as from a numerical point of view. As such it was natural to subject our extended Lagrangian to a thorough stability analysis while considering only the gravity sector. In fact, since the EFToDE/MG formalism is based on an action, we were able to determine general conditions of theoretical viability which are model independent and can, a priori, greatly reduce the parameter space. The most common criteria would be the absence of ghosts and gradient instabilities in the scalar and tensor sector and the exclusion of tachyonic instabilities. Regarding the first two criteria, one can find results in the literature either with or without the inclusion of a matter sector [14, 15, 37, 38, 65, 67, 68, 111]. In this work the study of the physical stability is particularly interesting due to the appearance of operators with higher order spatial derivatives. We proceeded, without including a matter sector, to study the stability of different sets of theories, leaving the analysis of the matter backreactions to the next Chapter. After integrating out the auxiliary fields, we obtained an EFToDE/MG action describing only the dynamics of the propagating d.o.f.. From this action, we identified the kinetic term and the speed of propagation which have now become functions of scale and time, due to the presence of higher derivative operators. We required both to be positive in order to guarantee a viable theory free from ghost and gradient instabilities. Subsequently we identified, at the level of the equations of motion, the friction and dispersive coefficients. We did this both for the scalar and tensor d.o.f.. Finally, we normalized the scalar d.o.f. in order to obtain an action in the canonical form. This form allowed us to identify the effective mass term on which we imposed conditions in order to avoid the appearance of tachyonic instabilities in the scalar sector. As a result, we obtained a set of very general stability conditions which must be imposed in order to ensure theoretical viability of models with operators containing up to sixth order in spatial derivatives, in absence of matter. It is worth noting that due to the complicated nature of some classes of theories, when written in the EFToDE/MG formalism, we had to divide the treatment and the resulting conditions in different subsets.

In the final part of this Chapter, we have built an extended and generalized version of the phenomenological parametrization in terms of α functions introduced in Ref. [69], to which we refer as ReParametrized Horndeski (RPH). This parametrization was originally built to include all models in the Horndeski class, and was afterwards extended to encompass beyond Horndeski models known as GLPV, in Ref. [67]. This was achieved by introducing an additional function which parametrizes the deviation from Horndeski theories. From this point we proceeded to introduce new functions and generalize the definition of the original ones, in order to account for all the beyond GLPV models described by the higher order operators that we have included in our extended EFToDE/MG action (2.1). In particular, we have found a new function parametrizing the braiding, which also contributes to the kinetic term; we have generalized the definitions of the kinetic and tensor speed excess functions, the latter one now being both time and scale dependent; finally, we have identified four extra functions entering in the definition of the speed of propagation of the scalar d.o.f.. It is important to notice that the structure of this extended phenomenological basis in terms of α functions becomes quite cumbersome when higher order operators are considered and the correspondence between the different functions and cosmological observables becomes weaker.

2.8 Appendix A: On δK and δS perturbations

In this Section we explicitly work out the perturbations associated to δK and δS used in Section 2.3.1 and show the difference with previous approaches [37, 65]. For this purpose, we consider the following terms of the Lagrangian (2.8):

$$\delta L \supset L_K\delta K + L_S\delta S = \mathcal{F}\delta K + L_S\delta K^\mu_\nu \delta K^\nu_\mu \equiv \mathcal{F}(K+3H) + L_S\delta K^\mu_\nu \delta K^\nu_\mu\,, \tag{2.137}$$

where we have defined:

$$\mathcal{F} \equiv L_K - 2HL_S. \tag{2.138}$$

Now, let us prove a relation which is useful in order to obtain action (2.9):

$$\int d^4x\sqrt{-g}\mathcal{F}K = \int d^4x\sqrt{-g}\mathcal{F}\nabla_\mu n^\mu = -\int d^4x\sqrt{-g}\nabla_\mu\mathcal{F}n^\mu = \int d^4x\sqrt{-g}\frac{\dot{\mathcal{F}}}{N}\,. \tag{2.139}$$

Using the above relation and the expansion of the lapse function:

$$N = 1 + \delta N + \delta N^2 + \mathcal{O}(3), \tag{2.140}$$

finally, we obtain:

$$L_K\delta K + L_S\delta S = 3H\mathcal{F} + \dot{\mathcal{F}}\left(1 - \delta N + (\delta N)^2\right) + L_S\delta K^\mu_\nu\delta K^\nu_\mu. \tag{2.141}$$

The differences with previous works are due to the different convention on the normal vector, n^μ (see Eq. (2.3)), which is responsible of the different sign in Eq. (2.139) w.r.t. the definition used in Refs. [37, 65] and then in the final results (2.141). Moreover, the difference in the definition of the extrinsic curvature, see Eq. (2.3), which is a consequence of the convention adopted for the normal vector, leads to the minus sign in Eq. (2.138) because its background value is $K^{i(0)}_j = -H\delta^i_j$.

2.9 Appendix B: On $\delta\mathcal{U}$ perturbation

Due to the different convention for n^μ we adopted here (see Eq. (2.3)), the result obtained in Refs. [37, 65] concerning the perturbation associated to $\mathcal{U} = \mathcal{R}_{\mu\nu}K^{\mu\nu}$, can not be directly applied to our Lagrangian (2.8). Therefore, we need to derive again such result, which is crucial in order to obtain the coefficients of the action (2.9). Then, let us prove the following relation:

$$\int d^4x\sqrt{g}\lambda(t)\mathcal{R}_{\mu\nu}K^{\mu\nu} = \int d^4x\sqrt{g}\left(\frac{\lambda(t)}{2}\mathcal{R}K - \frac{\dot{\lambda}(t)}{2N}\mathcal{R}\right), \tag{2.142}$$

where $\lambda(t)$ is a generic function of time. We notice that in Ref. [65] the above relation is defined with a plus in front of the second term in the last expression. Using the relation $K = \nabla^\mu n_\mu$ we obtain:

$$\int d^4x\sqrt{-g}\left(\lambda(t)\mathcal{R}_{\mu\nu}K^{\mu\nu} - \frac{\lambda(t)}{2}\mathcal{R}\nabla_\mu n^\mu + \frac{\dot{\lambda}(t)}{2N}\mathcal{R}\right) = 0\,. \tag{2.143}$$

Now, after integration by parts of the second term and using $n^\mu = \big(-1/N, N^i/N\big)$, the last term cancels and we are left with:

$$\int d^4x\sqrt{-g}\left(\lambda(t)\mathcal{R}_{\mu\nu}K^{\mu\nu} + \frac{\lambda(t)}{2}n^\mu\nabla_\mu\mathcal{R}\right) = 0\,. \tag{2.144}$$

The first term can be rewritten using the expression for the extrinsic curvature in the ADM formalism:

$$K_{ij} = -\frac{1}{2N}\big[\partial_t h_{ij} - \nabla_i N_j - \nabla_j N_i\big], \tag{2.145}$$

where covariant derivative is w.r.t. the spatial metric h_{ij}. The overall minus sign which appears in the above definition makes the expression to differ from the one usually encountered that follows from the definition of n^μ we employed. After substituting this expression into Eq. (2.144) we get:

$$\int d^4x\sqrt{h}\lambda(t)\left[-\frac{1}{2}\left(\mathcal{R}_{ij}h^{il}h^{jk}\dot{h}_{lk}+\dot{\mathcal{R}}\right)+\nabla^i N^j\mathcal{R}_{ij}+\frac{1}{2}N^i\nabla_i\mathcal{R}\right]=0\,. \tag{2.146}$$

From here on the subsequent steps follows Ref. [65], indeed the last two terms vanish due to the Bianchi identity and the first two can be combined as a total divergence. Hence, the relation (2.142) holds.

Finally, using the above relation we can now compute the perturbations coming from $\mathcal{U}=\mathcal{R}_{\mu\nu}K^{\mu\nu}$. Indeed we have:

$$\begin{aligned}\int d^4x\sqrt{-g}L_{\mathcal{U}}\mathcal{R}_{\mu\nu}K^{\mu\nu} &= \int d^4x\sqrt{-g}\left[\frac{1}{2}L_{\mathcal{U}}\mathcal{R}K-\frac{1}{2N}\dot{L}_{\mathcal{U}}\mathcal{R}\right]\\ &= \int d^4x\sqrt{-g}\left[\frac{1}{2}L_{\mathcal{U}}\left(K^{(0)}\delta\mathcal{R}+\delta K\delta\mathcal{R}\right)\right.\\ &- \left.\frac{1}{2}\dot{L}_{\mathcal{U}}\mathcal{R}\left(1-\delta N\right)\right]\,,\end{aligned} \tag{2.147}$$

then we get:

$$L_{\mathcal{U}}\delta\mathcal{U}=-\frac{1}{2}\left(3L_{\mathcal{U}}+\frac{1}{2}\dot{L}_{\mathcal{U}}\right)\delta\mathcal{R}+\left(\frac{1}{2}L_{\mathcal{U}}\delta K+\frac{1}{2}\dot{L}_{\mathcal{U}}\delta N\right)\delta\mathcal{R}\,. \tag{2.148}$$

2.10 Appendix C: Conformal EFT functions for Generalized Galileon and GLPV

In this Appendix we collect the results of Sections 2.4.3 and 2.4.4, and convert them to functions of the scale factor; the Hubble parameter and its time derivative are defined in terms of the conformal time, still they need to be considered functions of the scale factor. This further step is important for a direct implementation in *EFTCAMB* of Horndeski/GG and GLPV theories. *In this Section only*, primes indicate derivatives w.r.t. the scale factor. Furthermore, $\mathcal{H}\equiv d\ln a/d\tau$ and $\dot{\mathcal{H}}\equiv d\mathcal{H}/d\tau$, where τ is the conformal time. In order to get the correct results $\{\mathcal{K},G_i,\tilde{F}_i\}$ have to be considered functions of the scale factor.

First, we consider the EFToDE/MG functions derived in Section 2.4.3 for Horndeski/GG theories:

- L_2-Lagrangian

$$\Lambda(a) = \mathcal{K},$$
$$c(a) = \mathcal{K}_X X_0\,,$$
$$M_2^4(a) = \mathcal{K}_{XX} X_0^2\,, \tag{2.149}$$

where X_0 is:

$$X_0 = -\mathcal{H}^2 \phi_0'^2\,. \tag{2.150}$$

- L_3-Lagrangian

$$\begin{aligned}
\Lambda(a) &= \mathcal{H}^2\phi_0'^2\left[G_{3\phi} - 2G_{3X}\left(\frac{\dot{\mathcal{H}}}{a}\phi_0' + \mathcal{H}^2\phi_0''\right)\right]\,,\\
c(a) &= \mathcal{H}^2\phi_0'^2\left[G_{3X}\left(\left(3\mathcal{H}^2 - \dot{\mathcal{H}}\right)\frac{\phi_0'}{a} - \mathcal{H}^2\phi_0''\right) + G_{3\phi}\right]\,,\\
M_2^4(a) &= \frac{G_{3X}}{2}\mathcal{H}^2\phi_0'^2\left(\left(3\mathcal{H}^2 + \dot{\mathcal{H}}\right)\frac{\phi_0'}{a} + \mathcal{H}^2\phi_0''\right) - 3\frac{\mathcal{H}^6}{a}G_{3XX}\phi_0'^5 - \frac{G_{3\phi X}}{2}\mathcal{H}^4\phi_0'^4\,,\\
\bar{M}_1^3(a) &= -2\mathcal{H}^3 G_{3X}\phi_0'^3\,.
\end{aligned} \tag{2.151}$$

- L_4-Lagrangian

$$\begin{aligned}
\Omega(a) &= -1 + \frac{2}{m_0^2}G_4\,,\\
c(a) &= G_{4X}\left[2\left(\dot{\mathcal{H}}^2 + \mathcal{H}\ddot{\mathcal{H}} + 2\mathcal{H}^2\dot{\mathcal{H}} - 5\mathcal{H}^4\right)\frac{\phi_0'^{\,2}}{a^2} + 2\left(5\mathcal{H}^2\dot{\mathcal{H}} + \mathcal{H}^4\right)\frac{\phi_0'}{a}\phi_0'' + 2\mathcal{H}^4\phi_0''^{\,2}\right.\\
&\quad \left. + 2\mathcal{H}^4\phi_0'\phi_0'''\right] + G_{4X\phi}\left[2\mathcal{H}^2\phi_0'^2\left(\frac{\dot{\mathcal{H}}}{a}\phi_0' + \mathcal{H}^2\phi_0''\right) + 10\frac{\mathcal{H}^4}{a}\phi_0'^3\right]\\
&\quad + G_{4XX}\left[12\frac{\mathcal{H}^6}{a^2}\phi_0'^4 - 8\frac{\mathcal{H}^4}{a}\phi_0'^3\left(\frac{\dot{\mathcal{H}}}{a}\phi_0' + \mathcal{H}^2\phi_0''\right)\right.\\
&\quad \left. - 4\mathcal{H}^2\phi_0'^2\left(\frac{\dot{\mathcal{H}}^2}{a^2}\phi_0'^2 + 2\frac{\dot{\mathcal{H}}\mathcal{H}^2}{a}\phi_0'\phi_0'' + \mathcal{H}^4\phi_0''^2\right)\right]\,,\\
\Lambda(a) &= G_{4X}\left[4\left(\mathcal{H}^4 + 5\mathcal{H}^2\dot{\mathcal{H}} + \dot{\mathcal{H}}^2 + \mathcal{H}\ddot{\mathcal{H}}\right)\frac{\phi_0'^{\,2}}{a^2} + 4\left(4\mathcal{H}^4 + 5\mathcal{H}^2\dot{\mathcal{H}}\right)\frac{\phi_0'}{a}\phi_0'' + 4\mathcal{H}^4\phi_0''^{\,2}\right]\\
&\quad + 4\mathcal{H}^4\phi_0'\phi_0''' + 8\frac{\mathcal{H}^4}{a}G_{4X\phi}\phi_0'^3 - 8G_{4XX}\mathcal{H}^2\phi_0'^{\,2}\left(\dot{\mathcal{H}}\frac{\phi_0'}{a} + \mathcal{H}^2\phi_0''\right)\left(2\mathcal{H}^2\frac{\phi_0'}{a} + \dot{\mathcal{H}}\frac{\phi_0'}{a}\right.\\
&\quad \left. + \mathcal{H}^2\phi_0''\right)\,,\\
M_2^4(a) &= G_{4X\phi}\left[4\frac{\mathcal{H}^4}{a}\phi_0'^3 - \mathcal{H}^2\phi_0'^{\,2}\left(\frac{\dot{\mathcal{H}}}{a}\phi_0' + \mathcal{H}^2\phi_0''\right)\right] - 6\frac{\mathcal{H}^6}{a}\phi_0'^5 G_{4\phi XX} - 12\frac{\mathcal{H}^8}{a^2}G_{4XXX}\phi_0'^6
\end{aligned}$$

$$
\begin{aligned}
&+ G_{4XX}\mathcal{H}^2\phi_0'^{\,2}\left[2\left(9\mathcal{H}^4+\dot{\mathcal{H}}^2+2\mathcal{H}^2\dot{\mathcal{H}}\right)\frac{\phi_0'^{\,2}}{a^2}+2\left(2\mathcal{H}^2\dot{\mathcal{H}}+2\mathcal{H}^4\right)\frac{\phi_0'}{a}\phi_0''+2\mathcal{H}^4\phi''^{\,2}\right]\\
&+ G_{4X}\left[\left(-2\dot{\mathcal{H}}\mathcal{H}^2+2\mathcal{H}^4-\dot{\mathcal{H}}^2-\mathcal{H}\ddot{\mathcal{H}}\right)\frac{\phi_0'^{\,2}}{a^2}-\left(\mathcal{H}^4+5\mathcal{H}^2\dot{\mathcal{H}}\right)\frac{\phi_0'}{a}\phi_0''-\mathcal{H}^4\phi''^{\,2}\right.\\
&\left.-\,\mathcal{H}^4\phi_0'\phi_0'''\right],\\
\bar{M}_1^3(a) &= 4G_{4X}\mathcal{H}\phi_0'\left[\left(\dot{\mathcal{H}}+2\mathcal{H}^2\right)\frac{\phi_0'}{a}+\mathcal{H}^2\phi_0''\right]-16G_{4XX}\frac{\mathcal{H}^5}{a}\phi_0'^4-4G_{4X\phi}\mathcal{H}^3\phi_0'^3\,,\\
\bar{M}_2^2(a) &= 4\mathcal{H}^2G_{4X}\phi_0'^2=-\bar{M}_3^2(a)=2\hat{M}^2(a)\,. \qquad (2.152)
\end{aligned}
$$

- L_5-Lagrangian

$$
\begin{aligned}
\Omega(a) &= \frac{2\mathcal{H}^2}{m_0^2}\phi_0'^2\left[G_{5X}\left(\frac{\dot{\mathcal{H}}}{a}\phi_0'+\mathcal{H}^2\phi_0''\right)-\frac{G_{5\phi}}{2}\right]-1\,,\\
c(a) &= \frac{\mathcal{H}}{2}\tilde{\mathcal{F}}'+\frac{3}{2}\frac{\mathcal{H}^2}{a}m_0^2\Omega'-3\frac{\mathcal{H}^4}{a^2}\phi_0'^2G_{5\phi}+\frac{3\mathcal{H}^6}{a^2}\phi_0'^4G_{5\phi X}-3\frac{\mathcal{H}^6}{a^3}\phi_0'^3G_{5X}\\
&\quad+2\frac{\mathcal{H}^8}{a^3}\phi_0'^5G_{5XX}\,,\\
\Lambda(a) &= \tilde{\mathcal{F}}-3m_0^2\frac{\mathcal{H}^2}{a^2}(1+\Omega)+4G_{5X}\frac{\mathcal{H}^6}{a^3}\phi_0'^3+3\frac{\mathcal{H}^3}{a}G_{5\phi}\phi_0'^2\,,\\
M_4^2(a) &= -\mathcal{H}\frac{\tilde{\mathcal{F}}'}{4}-\frac{3}{4}\frac{\mathcal{H}^2}{a}m_0^2\Omega'-2\frac{\mathcal{H}^{10}}{a^3}\phi_0'^7G_{5XXX}-3\frac{\mathcal{H}^8}{a^2}\phi_0'^6G_{5\phi XX}+6G_{5XX}\frac{\mathcal{H}^8}{a^3}\phi_0'^5\\
&\quad+6\frac{\mathcal{H}^6}{a^2}\phi_0'^4G_{5\phi X}-\frac{3}{2}\frac{\mathcal{H}^6}{a^3}\phi_0'^3G_{5X}\,,\\
\bar{M}_2^2(a) &= 2\left[\mathcal{H}^2\phi_0'^2G_{5\phi}-G_{5X}\left[-\frac{\mathcal{H}^4}{a}\phi_0'^3+\mathcal{H}^2\phi_0'^2\left(\frac{\dot{\mathcal{H}}}{a}\phi_0'+\mathcal{H}^2\phi_0''\right)\right]\right]\\
&= -\bar{M}_3^2(a)=2\hat{M}^2(a)\,,\\
\bar{M}_1^3(a) &= -\mathcal{H}m_0^2\Omega'+4\frac{\mathcal{H}^3}{a}\phi_0'^2G_{5\phi}-4\frac{\mathcal{H}^5}{a}\phi_0'^4G_{5\phi X}-4\frac{\mathcal{H}^7}{a^2}\phi_0'^5G_{5XX}\\
&\quad+6\frac{\mathcal{H}^5}{a^2}\phi_0'^3G_{5X}\,, \qquad (2.153)
\end{aligned}
$$

where $\tilde{\mathcal{F}}(a)=\mathcal{F}-m_0^2\mathcal{H}\Omega'-2\frac{\mathcal{H}}{a}m_0^2(1+\Omega)$ and $\mathcal{F}(\tau)=2\frac{\mathcal{H}^5}{a^2}G_{5X}\phi_0'^3+2\frac{\mathcal{H}^3}{a}G_{5\phi}\phi_0'^2$.

Let us now consider the two Lagrangians which extend the Horndeski/GG theories to the GLPV ones introduced in Section 2.4.4:

- L_4^{GLPV}-Lagrangian

$$
\begin{aligned}
c(a) &= 2\frac{\mathcal{H}^4}{a^2}\phi_0'^4(\dot{\mathcal{H}}-\mathcal{H}^2)\tilde{F}_4 + 8\frac{\mathcal{H}^4}{a}\phi_0'^3\tilde{F}_4\left(\frac{\dot{\mathcal{H}}}{a}\phi_0' + \mathcal{H}^2\phi_0''\right) \\
&\quad - 4\frac{\mathcal{H}^6}{a}\tilde{F}_{4X}\left(\frac{\dot{\mathcal{H}}}{a}\phi_0' + \mathcal{H}^2\phi_0''\right)\phi_0'^5 + 2\mathcal{H}^6\frac{\tilde{F}_{4\phi}}{a}\phi_0'^5 - 12\frac{\mathcal{H}^6}{a^2}\phi_0'^4\tilde{F}_4\,, \\
\Lambda(a) &= 6\frac{\mathcal{H}^6}{a^2}\tilde{F}_4\phi_0'^4 + 4\frac{\mathcal{H}^4}{a^2}(\dot{\mathcal{H}}-\mathcal{H}^2)\phi_0'^4\tilde{F}_4 + 16\frac{\mathcal{H}^4}{a}\phi_0'^3\tilde{F}_4\left(\frac{\dot{\mathcal{H}}}{a}\phi_0' + \mathcal{H}^2\phi_0''\right) \\
&\quad - 8\frac{\mathcal{H}^6}{a}\tilde{F}_{4X}\left(\frac{\dot{\mathcal{H}}}{a}\phi_0' + \mathcal{H}^2\phi_0''\right)\phi_0'^5 + 4\frac{\mathcal{H}^6}{a}\tilde{F}_{4\phi}\phi_0'^5\,, \\
M_2^4(a) &= -18\frac{\mathcal{H}^6}{a^2}\phi_0'^4\tilde{F}_4 - \frac{\mathcal{H}^4}{a^2}\phi_0'^4(\dot{\mathcal{H}}-\mathcal{H}^2)\tilde{F}_4 - 4\frac{\mathcal{H}^4}{a}\phi_0'^3\tilde{F}_4\left(\frac{\dot{\mathcal{H}}}{a}\phi_0' + \mathcal{H}^2\phi_0''\right) \\
&\quad + 2\frac{\mathcal{H}^6}{a}\phi_0'^5\tilde{F}_{4X}\left(\frac{\dot{\mathcal{H}}}{a}\phi_0' + \mathcal{H}\phi_0''\right) - \frac{\mathcal{H}^6}{a}\phi_0'^5\tilde{F}_{4\phi} + 6\frac{\mathcal{H}^6}{a^2}\phi_0'^4\tilde{F}_4\,, \\
\bar{M}_2^2(a) &= 2\mathcal{H}^4\phi_0'^4\tilde{F}_4 = -\bar{M}_3^2(a)\,, \\
\bar{M}_1^3(a) &= 16\frac{\mathcal{H}^5}{a}\phi_0'^4\tilde{F}_4\,. \qquad (2.154)
\end{aligned}
$$

- L_5^{GLPV}-Lagrangian

$$
\begin{aligned}
\Lambda(a) &= -3\frac{\mathcal{H}^8}{a^3}\phi_0'^5\tilde{F}_5 - 12\frac{\mathcal{H}^6}{a^3}\phi_0'^5(\dot{\mathcal{H}}-\mathcal{H}^2)\tilde{F}_5 - 30\frac{\mathcal{H}^6}{a^2}\tilde{F}_5\left(\frac{\dot{\mathcal{H}}}{a}\phi_0' + \mathcal{H}^2\phi_0''\right)\phi_0'^4 \\
&\quad + 12\frac{\mathcal{H}^8}{a^2}\tilde{F}_{5X}\left(\frac{\dot{\mathcal{H}}}{a}\phi_0' + \mathcal{H}^2\phi_0''\right)\phi_0'^6 - 6\frac{\mathcal{H}^8}{a^2}\tilde{F}_{5\phi}\phi_0'^6\,, \\
c(a) &= 6\frac{\mathcal{H}^8}{a^2}\phi_0'^6\tilde{F}_{5X}\left(\frac{\dot{\mathcal{H}}}{a}\phi_0' + \mathcal{H}^2\phi_0''\right) - 6\frac{\mathcal{H}^6}{a^3}(\dot{\mathcal{H}}-\mathcal{H}^2)\phi_0'^5\tilde{F}_5 \\
&\quad - 15\frac{\mathcal{H}^6}{a^2}\phi_0'^4\left(\frac{\dot{\mathcal{H}}}{a}\phi_0' + \mathcal{H}^2\phi_0''\right) - 3\frac{\mathcal{H}^8}{a^2}\phi_0'^6\tilde{F}_{5\phi} + 15\frac{\mathcal{H}^8}{a^3}\tilde{F}_5\phi_0'^5\,, \\
M_2^4(a) &= \frac{45}{2}\frac{\mathcal{H}^8}{a^3}\phi_0'^5\tilde{F}_5 + 3\frac{\mathcal{H}^6}{a^3}(\dot{\mathcal{H}}-\mathcal{H}^2)\phi_0'^5\tilde{F}_5 + \frac{15}{2}\frac{\mathcal{H}^6}{a^2}\phi_0'^4\tilde{F}_5\left(\frac{\dot{\mathcal{H}}}{a}\phi_0' + \mathcal{H}^2\phi_0''\right) \\
&\quad - 3\frac{\mathcal{H}^8}{a^2}\phi_0'^6\left(\frac{\dot{\mathcal{H}}}{a}\phi_0' + \mathcal{H}^2\phi_0''\right)\tilde{F}_{5X} + \frac{3}{2}\frac{\mathcal{H}^8}{a^2}\phi_0'^6\tilde{F}_{5\phi}\,, \\
\bar{M}_2^2(a) &= -6\frac{\mathcal{H}^6}{a}\phi_0'^5\tilde{F}_5 = -\bar{M}_3^2(a)\,,
\end{aligned}
$$

$$\bar{M}_1^3(a) = -30\frac{\mathcal{H}^7}{a^2}\tilde{F}_5\phi_0'^5 \,. \tag{2.155}$$

Finally, we write the EFT functions obtained from the GLPV action (2.76) in Section 2.4.4 in the appropriate form adopted in *EFTCAMB* :

$$\begin{aligned}
\Omega(a) &= \frac{2}{m_0^2}\Big(\bar{B}_4 - \frac{\mathcal{H}}{2}\bar{B}_5'\Big) - 1\,, \\
\Lambda(a) &= \bar{A}_2 - 6\frac{\mathcal{H}^2}{a}\bar{A}_4 + 12\frac{\mathcal{H}^3}{a^3}\bar{A}_5 + \mathcal{H}\bar{A}_3' - \frac{4}{a^2}(\dot{\mathcal{H}} - \mathcal{H}^2)\bar{A}_4 - 4\frac{\mathcal{H}^2}{a}\bar{A}_4' \\
&+ 6\frac{\mathcal{H}^3}{a^2}\bar{A}_5' + 12\frac{\mathcal{H}}{a^3}(\dot{\mathcal{H}} - \mathcal{H}^2)\bar{A}_5 - \bigg[\frac{2}{a^2}\left(\mathcal{H}^2 + 2\dot{\mathcal{H}}\right)\bar{B}_4 + \frac{2}{a}\left(\dot{\mathcal{H}} + 2\mathcal{H}^2\right)\bar{B}_4' \\
&+2\mathcal{H}^2\bar{B}_4'' - \frac{\mathcal{H}}{a^2}\left(\mathcal{H}^2 + 3\dot{\mathcal{H}} + \frac{\ddot{\mathcal{H}}}{\mathcal{H}}\right)\bar{B}_5' - \frac{\mathcal{H}}{a}\left(3\dot{\mathcal{H}} + 2\mathcal{H}^2\right)\bar{B}_5'' - \mathcal{H}^3\bar{B}_5'''\bigg]\,, \\
c(a) &= \frac{1}{2}\Big(\mathcal{H}\bar{A}_3' - \frac{4}{a^2}(\dot{\mathcal{H}} - \mathcal{H}^2)\bar{A}_4 - 4\frac{\mathcal{H}^2}{a}\bar{A}_4' + 6\frac{\mathcal{H}^3}{a^2}\bar{A}_5' + 12\frac{\mathcal{H}}{a^3}(\dot{\mathcal{H}} - \mathcal{H}^2)\bar{A}_5 \\
&- \bar{A}_{2N} + 3\mathcal{H}\bar{A}_{3N} - 6\frac{\mathcal{H}^2}{a^2}\bar{A}_{4N} + 6\frac{\mathcal{H}^3}{a^3}\bar{A}_{5N}\Big) \\
&+ \frac{1}{a}\left(\mathcal{H}^2 - \dot{\mathcal{H}}\right)\bar{B}_4' + \frac{\mathcal{H}}{2a}\left(3\dot{\mathcal{H}} - \mathcal{H}^2\right)\bar{B}_5'' - \mathcal{H}^2\bar{B}_4'' + \frac{\mathcal{H}^3}{2}\bar{B}_5''' \\
&+ \frac{1}{2a^2}\left(\ddot{\mathcal{H}} - 2\mathcal{H}^3\right)\bar{B}_5' - \frac{2}{a^2}(\dot{\mathcal{H}} - \mathcal{H}^2)\bar{B}_4\,, \\
M_2^4(a) &= \frac{1}{4}\Big(\bar{A}_{2NN} - 3\frac{\mathcal{H}}{a}\bar{A}_{3NN} + 6\frac{\mathcal{H}^2}{a^2}\bar{A}_{4NN} - 6\frac{\mathcal{H}^3}{a^3}\bar{A}_{5NN}\Big) \\
&- \frac{1}{4}\bigg[\mathcal{H}\bar{A}_3' - 4\frac{\bar{A}_4}{a^2}(\dot{\mathcal{H}} - \mathcal{H}^2) - 4\frac{\mathcal{H}^2}{a}\bar{A}_4' + 6\frac{\mathcal{H}^3}{a^2}\bar{A}_5' \\
&+12\bar{A}_5\frac{\mathcal{H}}{a^3}(\dot{\mathcal{H}} - \mathcal{H}^2)\bigg] + \frac{3}{4}\Big(\bar{A}_{2N} - 3\frac{\mathcal{H}}{a}\bar{A}_{3N} + 6\frac{\mathcal{H}^2}{a^2}\bar{A}_{4N} - 6\frac{\mathcal{H}^3}{a^3}\bar{A}_{5N}\Big) \\
&- \frac{1}{2}\bigg[-\frac{2}{a^2}(\dot{\mathcal{H}} - \mathcal{H}^2)\bar{B}_4 + \frac{1}{a}\left(\mathcal{H}^2 - \dot{\mathcal{H}}\right)\bar{B}_4' - \mathcal{H}^2\bar{B}_4'' \\
&+\frac{1}{a^2}\left(\ddot{\mathcal{H}} - \mathcal{H}^3\right)\bar{B}_5' + \frac{\mathcal{H}}{2a}\left(3\dot{\mathcal{H}} - \mathcal{H}^2\right)\bar{B}_5'' + \frac{\mathcal{H}^3}{2}\bar{B}_5'''\bigg]\,, \\
\bar{M}_2^2(a) &= -2\bar{A}_4 + 6\frac{\mathcal{H}}{a}\bar{A}_5 - 2\bar{B}_4 + \mathcal{H}\bar{B}_5' = -\bar{M}_3^2(a)\,, \\
\bar{M}_1^3(a) &= -\bar{A}_{3N} + 4\frac{\mathcal{H}}{a}\bar{A}_{4N} - 6\frac{\mathcal{H}^2}{a^2}\bar{A}_{5N} - 2\bar{B}_4'\mathcal{H} + \frac{\dot{\mathcal{H}}}{a}\bar{B}_5' + \mathcal{H}^2\bar{B}_5''\,, \\
\hat{M}^2(a) &= \bar{B}_{4N} + \frac{\mathcal{H}}{2a}\bar{B}_{5N} + \frac{\mathcal{H}}{2}\bar{B}_5'\,.
\end{aligned} \tag{2.156}$$

2.11 Appendix D: On the $\mathcal{J}$ coefficient in the L_5 Lagrangian

In this Appendix we will show the details of the calculation regarding the J coefficient in the L_5 Lagrangian (2.53). Let us consider the following term:

$$\begin{aligned}G_{5X}\mathcal{J} &= G_{5X}\Big(-\frac{1}{2}\phi^{;\rho}X_{;\rho}(K^2-S)+2\gamma^{-3}(\gamma^2\frac{h^\rho_\mu}{2}X_{;\rho})(K\dot{n}_\mu-K^{\mu\nu}\dot{n}_\nu)\Big)\\ &= -\frac{1}{2}\Big(\gamma\nabla_\rho(\gamma^{-1}F_5)-F_{5\phi}\gamma^{-1}n_\rho\Big)(K^2-S)\phi^{;\rho}\\ &+\gamma^{-1}(K\dot{n}^\mu-K^{\mu\nu}\dot{n}_\nu)h^\mu_\rho\Big(\gamma\nabla_\rho(\gamma^{-1}F_5)+F_{5\phi}\gamma^{-1}n^\rho\Big)\,. \qquad (2.157)\end{aligned}$$

The last parenthesis contains a quantity which is orthogonal to the quantities that multiply it, hence it vanishes. Therefore, we have:

$$\begin{aligned}G_{5X}\mathcal{J} &= \frac{F_{5\phi}}{2}n_\rho n^\rho(K^2-S)-\frac{1}{2}n^\rho\nabla_\rho(\gamma^{-1}F_5)(K^2-S)+h^\rho_\mu\nabla_\rho(\gamma^{-1}F_5)(K\dot{n}^\mu-K^{\mu\nu}\dot{n}_\nu)\\ &= -\frac{F_{5\phi}}{2}(K^2-S)+\frac{F_5}{\gamma}\left[\frac{1}{2}\nabla_\rho(n^\rho K^2-n^\rho K_{\mu\nu}K^{\mu\nu})-(K\dot{n}^\mu-K^{\mu\nu}\dot{n}_\nu)_{;\mu}\right]\\ &= \frac{F_5}{\gamma}\Bigg(\frac{K^3}{2}+n^\rho K\nabla_\rho K-\frac{K}{2}K_{\mu\nu}K^{\mu\nu}-n^\rho K^{\mu\nu}\nabla_\rho K_{\mu\nu}-\dot{n}^\rho\nabla_\rho K\\ &-K\nabla_\rho\dot{n}^\rho+\dot{n}_\nu\nabla_\rho K^{\rho\nu}+K^{\rho\nu}\nabla_\rho\dot{n}_\nu)-\frac{F_{5\phi}}{2}(K^2-S), \qquad (2.158)\end{aligned}$$

where in the second line we have used the fact that n_μ is orthogonal to $\dot{n}_\mu$ and $K^{\mu\nu}$. Now, employing the following geometrical quantities:

$$\begin{aligned}R_{\mu\nu}n^\mu n^\nu &= -n^\mu\nabla_\mu K+\nabla_\mu\dot{n}^\mu+n^\mu\nabla^\nu K_{\mu\nu}\,,\\ R_{\mu\nu}n^\nu\dot{n}^\mu &= \dot{n}^\mu\nabla^\nu K_{\mu\nu}-\dot{n}^\mu\dot{n}_\nu\nabla^\nu n_\mu-\dot{n}^\mu\nabla_\mu K\,,\\ K^{\mu\nu}n^\rho n^\sigma R_{\mu\sigma\nu\rho} &= K^{\gamma\alpha}n^\beta(\nabla_\alpha K_{\beta\gamma})-K^{\gamma\alpha}n^\beta(\nabla_\beta K_{\alpha\gamma})+K^{\gamma\alpha}(\nabla_\alpha\dot{n}_\gamma)+K^{\gamma\alpha}\dot{n}_\gamma\dot{n}_\alpha\,,\end{aligned} \qquad (2.159)$$

we obtain:

$$\begin{aligned}G_{5X}\mathcal{J} &= \frac{F_5}{\gamma}\Big(\frac{K^3}{2}+n^\rho K\nabla_\rho K-\frac{K}{2}K_{\mu\nu}K^{\mu\nu}-n^\rho K^{\mu\nu}\nabla_\rho K_{\mu\nu}-\dot{n}^\rho\nabla_\rho K\\ &-K\nabla_\rho\dot{n}^\rho+\dot{n}_\nu\nabla_\rho K^{\rho\nu}+K^{\rho\nu}\nabla_\rho\dot{n}_\nu\Big)-\frac{F_{5\phi}}{2}(K^2-S)\\ &= \frac{F_5}{\gamma}\Big(\frac{K^3}{2}-\frac{K}{2}K_{\mu\nu}K^{\mu\nu}-KR_{\mu\nu}n^\mu n^\nu+n^\mu K(\nabla^\nu K_{\mu\nu})+K^{\mu\nu}n^\rho n^\sigma\\ &+K^{\mu\nu}n^\sigma n^\rho R_{\mu\sigma\nu\rho}-K^{\gamma\alpha}n^\beta(\nabla_\alpha K_{\beta\gamma})\end{aligned}$$

$$
\begin{aligned}
&- K^{\gamma\alpha}\dot{n}_\gamma \dot{n}_\alpha + R_{\mu\nu}n^\mu \dot{n}^\nu + \dot{n}^\mu \dot{n}^\nu \nabla_\nu n_\mu \Big) - \frac{F_{5\phi}}{2}(K^2 - S) \\
&= \frac{F_5}{\gamma}\Big(\frac{K^3}{2} - \frac{K}{2}K_{\mu\nu}K^{\mu\nu} - KR_{\mu\nu}n^\mu n^\nu - KK^{\mu\nu}K_{\mu\nu} + K^{\mu\nu}n^\rho n^\sigma \\
&+ K^{\mu\nu}n^\sigma n^\rho R_{\mu\sigma\nu\rho} + K^{\gamma\alpha}K^\beta_\alpha K_{\beta\gamma} \\
&- K^{\gamma\alpha}\dot{n}_\gamma \dot{n}_\alpha + R_{\mu\nu}n^\mu \dot{n}^\nu + \dot{n}^\mu \dot{n}^\nu \nabla_\nu n_\mu \Big) - \frac{F_{5\phi}}{2}(K^2 - S)\,,
\end{aligned}
\tag{2.160}
$$

where we have dropped a total derivative term. Finally, we use the definition $\tilde{\mathcal{K}}$ in Eq. (2.59) and we obtain the final result used in Section 2.4.3:

$$
G_{5X}\mathcal{J} = F_5\gamma^{-1}\Big[\frac{\tilde{\mathcal{K}}}{2} + K^{\mu\nu}n^\sigma n^\rho R_{\mu\sigma\nu\rho} + \dot{n}^\sigma n^\rho R_{\sigma\rho} - Kn^\sigma n^\rho R_{\sigma\rho}\Big] - \frac{F_{5\phi}}{2}(K^2 - S)\,.
\tag{2.161}
$$

3 On the stability conditions for theories of modified gravity in the presence of matter fields

3.1 Introduction

In the previous Chapter we started to study the set of conditions an extension of gravity must satisfy in order to guarantee a viable cosmological scenario. This was done by considering the model in a vacuum and as such does not represent our universe as it contains matter fields which are gravitationally coupled to the new d.o.f.. In fact, the stability conditions might be altered by the presence of the additional matter fields, thus changing the viability space of the theory [37, 67, 112–115]. Identifying the correct viability requirements is important when testing MG theories with cosmological data by using statistical tools [22, 23, 116, 117], as they can reduce the viability space one needs to explore. Additionally they can even dominate over the constraining power of observational data as recently shown in the case of designer f(R)-theory on wCDM background [23]. Therefore, in this Chapter we proceed to fix this deficiency and to quantify the modifications induced by the new fields.

With the aim to obtain general results, we will employ once more the Effective Field Theory of Dark Energy and Modified Gravity [14, 15]. This EFToDE/MG approach has been implemented into the Einstein Boltzmann solver, CAMB/CosmoMC [20, 21, 118], creating *EFTCAMB/EFTCosmoMC* [22, 23, 39, 80, 82, 119] (`http://www.eftcamb.org/`), providing a perfect tool to test gravity models through comparison with observational data. *EFTCAMB* comes with a built-in module to explore the viability space of the underlying theory of gravity, which then can be used as *priors*. The results of the present work have a direct application as they can be employed to improve the current *EFTCAMB* viability requirements but not limited to it as they can be easily mapped to other parametrizations [80].

For the matter sector we chose to employ the Sorkin-Schutz action, which allows one to treat general matter fluids [120, 121]. Among many models used to describe matter Lagrangians and which have been extensively used and investigated in the past years [37, 67, 112–115], we choose to follow the recent arguments in ref. [122]. Indeed, it has been shown that such an action, along with an appropriate choice for the matter field, describes the dynamics of all matter fluids avoiding some problems which might arise when including pressure-less matter fluids, like dust or cold dark matter (CDM), which are relevant in the evolution of the Universe.

Previously, a stability analysis of the EFToDE/MG in the presence of matter had been done in [37]. However, in our work we present also the conditions which allow to avoid tachyonic instabilities and we analyse all possible sub-cases concerning the stability conditions. Furthermore, due to the different choice of matter action, modifications can be seen when one includes presureless fluids.

With this machinery, we proceed to derive the viability constraints one needs to impose on the free parameters of the theory by focusing on three sources of possible instabilities, ghost, gradient and tachyonic instabilities. We will proceed while retaining the full generality of the EFToDE/MG approach, i.e. without limiting to specific models. However, where relevant, we will make connections to specific theories, such as low-energy Hořava gravity [35, 36, 55] and beyond Horndeski models [66] and we will analyse the results within the context of these models.

This Chapter is based on the work in [32]: *On the stability conditions for theories of modified gravity in the presence of matter fields* with A. De Felice and N. Frusciante. In section 3.2, we briefly recap the EFToDE/MG formalism we use to parametrize the DE/MG models with one extra scalar d.o.f.. In section 3.3, we introduce the Sorkin-Schutz action to describe the dynamics of matter fluids and we discuss the advantage of using this action with respect to previous approaches. We also work out the corresponding continuity equation and second order perturbed action. In section 3.4, we work out the action for both gravity and matter fields up to second order in perturbations. Then, we calculate and discuss the stability requirements to avoid ghost instabilities (section 3.4.1), to guarantee positive speeds of propagation (section 3.4.2) and to prevent tachyonic instabilities (section 3.4.3). Finally, we conclude in section 4.6.

3.2 The Effective Field Theory of Dark Energy and Modified Gravity

The EFToDE/MG has been proposed as a unifying framework to study the dynamic and evolution of linear order perturbations of a broad class of DE/MG theories [14, 15]. This approach encompasses all the theories of gravity exhibiting one extra scalar and dynamical d.o.f. and admits a well-defined Jordan frame. As discussed in the Introduction, the EFToDE/MG is constructed in the unitary gauge by expanding around the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric. Each operator si accompanied by a time dependent function dubbed *EFT function*. The explicit form of the perturbed EFToDE/MG action is the following:

$$\begin{aligned}\mathcal{S}_{EFT} = \int d^4x\sqrt{-g}\Bigg[&\frac{m_0^2}{2}(1+\Omega(t))R^{(4)}+\Lambda(t)-c(t)\delta g^{00}+\frac{M_2^4(t)}{2}(\delta g^{00})^2\\ &-\frac{\bar{M}_1^3(t)}{2}\delta g^{00}\delta K-\frac{\bar{M}_2^2(t)}{2}(\delta K)^2-\frac{\bar{M}_3^2(t)}{2}\delta K^\mu_\nu\delta K^\nu_\mu\\ &+\frac{\hat{M}^2(t)}{2}\delta g^{00}\delta R^{(3)}+m_2^2(t)\left(g^{\mu\nu}+n^\mu n^\nu\right)\partial_\mu g^{00}\partial_\nu g^{00}\Bigg],\end{aligned} \tag{3.1}$$

where m_0^2 is the Planck mass, $g_{\mu\nu}$ is the four dimensional metric and g is its determinant, δg^{00} is the perturbation of the upper time-time component of the metric, n_μ is the normal vector to the constant-time hypersurfaces, $R^{(4)}$ and $R^{(3)}$ are respectively the trace of the four dimensional and three dimensional Ricci scalar, $K_{\mu\nu}$ is the extrinsic curvature and K is its trace. Finally, with $\delta A = A - A^{(0)}$ we indicate the linear perturbation of the quantity A and $A^{(0)}$ is the corresponding background value.

Moreover, it has been shown that appropriate combinations of the EFT functions in action (5.1) allows one to describe specific classes of DE/MG models. We group such combinations as follows:

- $M_2^2 = -\bar{M}_3^2 = 2\hat{M}^2$ and $m_2^2 = 0$: Horndeski [78] or Generalized Galileon class of models [79] (and all the models belonging to them);
- $M_2^2 + \bar{M}_3^2 = 0$ and $m_2^2 = 0$: Beyond Horndeski class of models [66];
- $m_2^2 \neq 0$: Lorentz violating theories (e.g. low-energy Hořava gravity [35, 36, 55]).

For a detailed guide to map a specific theory into the EFToDE/MG language we refer the reader to the previous Chapter as well as refs. [14, 15, 37, 65]. Finally, an extended version of the above EFToDE/MG action has been presented in the previous Chapter which includes operators with higher than second order spatial derivatives.

In the following we will briefly recap the construction of the EFToDE/MG action up to second order in terms of the scalar metric perturbations as it will be the starting point for the stability analysis.

Because of the unitary gauge in action (3.1), it is natural to choose the Arnowitt-Deser-Misner (ADM) formalism [19] to write the line element, which reads:

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)\,, \tag{3.2}$$

where $N(t, x^i)$ is the lapse function, $N^i(t, x^i)$ the shift and $h_{ij}(t, x^i)$ is the metric tensor of the three dimensional spatial slices. Proceeding with the expansion around a flat FLRW background, the metric can be written as:

$$ds^2 = -(1 + 2\delta N)dt^2 + 2\partial_i\psi dt dx^i + a^2(1 + 2\zeta)\delta_{ij}dx^i dx^j\,, \tag{3.3}$$

where as usual $\delta N(t, x^i)$ is the perturbation of the lapse function, $\partial_i\psi(t, x^i)$ and $\zeta(t, x^i)$ are the scalar perturbations respectively of N_i and h_{ij} and a is the scale factor. Then, the scalar perturbations of the quantities involved in the action (3.1) are:

$$\begin{aligned}
\delta g^{00} &= 2\delta N\,, \\
\delta K &= -3\dot{\zeta} + 3H\delta N + \frac{1}{a^2}\partial^2\psi\,, \\
\delta K_{ij} &= a^2\delta_{ij}(H\delta N - 2H\zeta - \dot{\zeta}) + \partial_i\partial_j\psi\,, \\
\delta K^i_j &= (H\delta N - \dot{\zeta})\delta^i_j + \frac{1}{a^2}\partial^i\partial_j\psi\,, \\
\delta R^{(3)} &= -\frac{4}{a^2}\partial^2\zeta\,,
\end{aligned} \tag{3.4}$$

where we have made use of the following definitions of the normal vector and extrinsic curvature:

$$n_\mu = N\delta^0_\mu, \qquad K_{\mu\nu} = h^\lambda_\mu \nabla_\lambda n_\nu, \tag{3.5}$$

with $h^{\mu\nu} = g^{\mu\nu} + n^\mu n^\nu$, $H \equiv \frac{1}{a}\frac{da}{dt}$ is the Hubble function and dots are the derivatives with respect to time. Then, the action (3.1) can be explicitly expanded in terms of metric scalar perturbations up to second order and after some manipulations, we obtain the following final form:

$$\begin{aligned}
S^{(2)}_{EFT} &= \int dt d^3x a^3 \Bigg\{ -\frac{F_4(\partial^2\psi)^2}{2a^4} - \frac{3}{2}\,F_1\dot{\zeta}^2 + m_0^2(\Omega + 1)\,\frac{(\partial\zeta)^2}{a^2} \\
&- \frac{\partial^2\psi}{a^2}\left(F_2\delta N - F_1\dot{\zeta}\right) + 4m_2^2\frac{[\partial(\delta N)]^2}{a^2} + \frac{F_3}{2}\,\delta N^2 \\
&+ \left[3F_2\dot{\zeta} - 2\left(m_0^2(\Omega + 1) + 2\hat{M}^2\right)\frac{\partial^2\zeta}{a^2}\right]\delta N \Bigg\},
\end{aligned} \tag{3.6}$$

where we have defined

$$\begin{aligned} F_1 &= 2m_0^2(\Omega+1)+3\bar{M}_2^2+\bar{M}_3^2\,, \\ F_2 &= HF_1+m_0^2\dot{\Omega}+\bar{M}_1^3\,, \\ F_3 &= 4M_2^4+2c-3H^2F_1-6m_0^2H\dot{\Omega}-6H\bar{M}_1^3\,, \\ F_4 &= \bar{M}_2^2+\bar{M}_3^2\,, \end{aligned} \tag{3.7}$$

and other terms vanish because of the background equations of motion. This result will be considered along with the matter sector which will be presented in the next section in order to facilitate the complete study of conditions that guarantee that a gravity theory, in presence of matter fields, is free from instabilities.

3.3 The Matter Sector

The goal of the present work is to investigate the emergence of instabilities in modified theories of gravity under the influence of matter fluids and subsequently set appropriate stability conditions. Therefore, a crucial step is to make the appropriate choice for the matter action, S_m. Moreover, the generality of the EFToDE/MG approach in describing the gravity sector makes that even for the matter action there is an equally general treatment. It is common in literature to choose for the matter Lagrangian a k-essence like form, $P(\mathcal{X})$ [37, 111–114, 123, 124], to model the matter d.o.f. where $\mathcal{X}\equiv\chi_{;\mu}\chi^{;\mu}$ is the kinetic term of the field χ. However, this choice displays problematic behaviours which motivates us to decide for a different action. The easiest way of identifying those issues is to consider the corresponding action for $P(\mathcal{X})$ when it has been specialized to a dust fluid. In that case it can be easily shown that the action diverges. Subsequently, in ref. [122], it has been shown that the real problem arising in the K-essence like matter Lagrangian lies in the choice of the canonical field one uses to describe the d.o.f. of the fluid. The usual choice for the fluid variable, the velocity v_m, satisfies a closed first order equation of motion, which requires only one independent initial condition. Then, the dust fluid would have only one d.o.f. (rather than two) and for that field the action tends to blow up as the speed of propagation goes to zero ($c_{s,d}^2\to 0$). Instead, the appropriate variable for the fluid is the matter density perturbation, δ_m.

In order to avoid the issues described above we choose the Sorkin-Schutz action, see refs. [120, 121] which is well defined for a dust component and can describe in full generality perfect fluids. As observed above the appropriate fluid variable is the density perturbation which is exactly the one employed by this action and thus satisfies a second order equation of motion as will be evident in the following. The Sorkin-Schutz matter action reads:

$$S_{\rm m} = -\int d^4x[\sqrt{-g}\,\rho(n) + J^\nu \partial_\nu \ell]\,, \tag{3.8}$$

where ρ is the energy density, which depends on the number density n, ℓ is a scalar field, whereas J^ν is a vector with weight one. Additionally, we define n as

$$n = \sqrt{\frac{J^\alpha J^\beta g_{\alpha\beta}}{g}}\,. \tag{3.9}$$

Then, the four velocity vector u^α is defined as

$$u^\alpha = \frac{J^\alpha}{n\sqrt{-g}}\,, \tag{3.10}$$

and satisfies the usual relation $u^\alpha u_\alpha = -1$. Variation of the matter Lagrangian with respect to J^α leads to

$$u_\alpha = \frac{1}{\partial\rho/\partial n}\,\partial_\alpha \ell\,, \tag{3.11}$$

while taking its variation with respect to the metric we find that the stress energy tensor can be defined as

$$T_{\alpha\beta} \equiv \frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\alpha\beta}} = n\,\frac{\partial\rho}{\partial n}\,u_\alpha u_\beta + \left(n\,\frac{\partial\rho}{\partial n} - \rho\right) g_{\alpha\beta}\,, \tag{3.12}$$

which is a barotropic perfect fluid with pressure given by

$$p \equiv n\,\frac{\partial\rho}{\partial n} - \rho\,. \tag{3.13}$$

Let us notice that a particular choice for the density, i.e. $\rho \propto n^{1+w}$, allows to have the usual relation $p = w\rho$, where w is the barotropic coefficient. Finally, by varying the matter action with respect to ℓ, one gets the conservation constraint

$$\partial_\alpha J^\alpha = 0. \tag{3.14}$$

On a flat FLRW background the above relation gives $J^0 = \mathcal{N}_0$, where $\mathcal{N}_0$ is the total particle number and from Eq. (3.9) we have $n = \mathcal{N}_0/a^3$.

Let us now proceed to write the matter action (3.8) up to second order in the scalar fields by using the metric scalar perturbations in Eq. (3.3). For the fluid variables we proceed to expand them as follows

$$\begin{aligned} J^0 &= \mathcal{N}_0 + \delta J\,, \\ J^i &= \frac{1}{a^2}\,\partial^i \delta j\,, \\ \ell &= -\int^t \frac{\partial\rho}{\partial n}\,dt' - \frac{\partial\rho}{\partial n}\,v_m\,, \end{aligned} \tag{3.15}$$

where v_m is the velocity of the matter species. Furthermore, we note that since

$$\rho = \bar{\rho} + \frac{\partial \rho}{\partial n}\left(-\frac{3\mathcal{N}_0}{a^3}\zeta + \frac{\delta J}{a^3}\right) \equiv \bar{\rho} + \delta\rho\,, \tag{3.16}$$

where $\bar{\rho}$ is the density at the background, one can obtain

$$\delta J = \frac{a^3 \bar{\rho}\, \delta_m}{\partial\rho/\partial n} + 3\mathcal{N}_0\,\zeta\,, \tag{3.17}$$

where, as usual, $\delta_m = \delta\rho/\bar{\rho}$. We can thus rewrite δJ in terms of δ_m in the perturbed matter action. Finally, we can use the equation of motion for δj

$$\delta j = -\mathcal{N}_0(\psi + v_m) \tag{3.18}$$

in order to eliminate it in favour of v_m and ψ.

Combining the above results and after some integrations by parts, we obtain the action for the scalar perturbations up to second order:

$$\begin{aligned} S_m^{(2)} &= \int dt d^3x a^3 \Bigg[-\frac{n\rho_{,n}(\partial v)^2}{2a^2} + \Bigg(\frac{3H\left(n{\rho_{,n}}^2 - n\bar{\rho}\,\rho_{,nn} - \bar{\rho}\,\rho_{,n}\right)\delta_m}{\rho_{,n}} \\ &+ \frac{n\rho_{,n}\partial^2\psi}{a^2} - 3\,n\rho_{,n}\dot{\zeta} - \bar{\rho}\,\dot{\delta}_m\Bigg) v_m - \frac{\rho_{,nn}\bar{\rho}^2\delta_m^2}{2\rho_{,n}^2} - \bar{\rho}\,\delta N \delta_m\Bigg]. \end{aligned} \tag{3.19}$$

Notice that the velocity v_m can always be integrated out, as $n\rho_{,n} = \bar{\rho} + p \neq 0$.

3.4 Study of Stability conditions

In this section we present the main bulk of our work, i.e. the study of the general conditions that a gravity theory has to satisfy in order to be free from instabilities when additional matter fields are considered. These set of requirements include: no ghost instabilities, positive speeds of propagation (squared) and no tachyonic instabilities as presented in the Introduction. Recently, it has been shown that physical stability plays an important role when testing specific gravity models with cosmological data [23, 125]. In particular, the *EFTCAMB* patch [22, 23] includes a specific module with the task to identify the viable parameter space of a selected theory. The results of the present work can be used to improve such modules and improve on the efficiency of the selection process.

To achieve this goal we consider the general EFToDE/MG parametrization presented in section 3.2 in the presence of two different matter fluids, described by the action (3.19), for which we made the following, realistic, choices: a pressure-less fluid, i.e. cold dark matter/dust (d) and radiation (r). A treatment which includes two general fluids complicates the process substantially and we do not expect to learn much more in such a case. So the relevant action required in order to proceed is of the following form:

$$
\begin{aligned}
S^{(2)} &= \frac{1}{(2\pi)^3}\int dtd^3ka^3\left\{\bar{\rho}_d\left(-\frac{k^2\psi}{a^2}-3\dot{\zeta}-\dot{\delta}_d\right)v_d+\bar{\rho}_r\left(-\frac{4}{3}\frac{k^2\psi}{a^2}\right.\right.\\
&- \left.4\dot{\zeta}-\dot{\delta}_r\right)v_r-\bar{\rho}_d\frac{(kv_d)^2}{2a^2}-\frac{2}{3}\bar{\rho}_r\frac{(kv_r)^2}{a^2}-\frac{F_4}{2a^4}\left(k^2\psi\right)^2\\
&+ \left(\frac{2k^2\zeta\left(2\hat{M}^2+m_0^2(\Omega+1)\right)}{a^2}+3F_2\dot{\zeta}\right)\delta N+\left(\delta NF_2-F_1\dot{\zeta}\right)\frac{k^2\psi}{a^2}+\\
&+ \frac{4m_2^2(k\delta N)^2}{a^2}+\frac{m_0^2(\Omega+1)(k\zeta)^2}{a^2}-\frac{4}{3}\bar{\rho}_r\frac{(kv_r)^2}{2a^2}-\bar{\rho}_d\delta N\delta_d\\
&+ \left.\frac{1}{2}F_3\delta N^2-\frac{3}{2}F_1\dot{\zeta}^2-\frac{\bar{\rho}_r}{8}\delta_r^2-\bar{\rho}_r\delta N\delta_r\right\}, \qquad (3.20)
\end{aligned}
$$

where we have Fourier transformed the spatial coordinates and we have considered the following relations for the number densities:

$$
n_d=\bar{\rho}_d\,,\qquad n_r=(\bar{\rho}_r)^{\frac{3}{4}}, \qquad (3.21)
$$

being $\bar{\rho}_d, \bar{\rho}_r$ respectively the density of dust and radiation at background.

An action constructed in such a way admits only three d.o.f. described by $\{\zeta, \delta_d, \delta_r\}$. Therefore in the above action we notice the presence of four Lagrange multipliers δN, ψ, v_d and v_r. Consequently, we proceed with the removal of the latter by using the constraint equations obtained after the variations of the action with respect to the Lagrange multipliers. The resulting set of constraint equations is:

$$
\begin{aligned}
&\bar{\rho}_r\left(-\frac{4}{3}\frac{k^2\psi}{a^2}-4\dot{\zeta}-\dot{\delta}_r\right)-\frac{4}{3}\bar{\rho}_r\frac{k^2v_r}{a^2}=0\,,\\
&\bar{\rho}_d\left(-\frac{k^2\psi}{a^2}-3\dot{\zeta}-\dot{\delta}_d\right)-\bar{\rho}_d\frac{k^2v_d}{a^2}=0\,,\\
&\frac{2k^2\zeta\left(2\hat{M}^2+m_0^2(\Omega+1)\right)}{a^2}+3F_2\dot{\zeta}+\frac{8m_2^2k^2\delta N}{a^2}+F_2\frac{k^2\psi}{a^2}-\bar{\rho}_d\delta_d\\
&+F_3\delta N-\bar{\rho}_r\delta_r=0\,,\\
&-\bar{\rho}_dv_d-\frac{4}{3}\bar{\rho}_rv_r+\delta NF_2-F_1\dot{\zeta}-\frac{F_4}{a^2}k^2\psi=0\,. \qquad (3.22)
\end{aligned}
$$

After solving for the auxiliary fields and substituting the results back into action (3.20), we get an action containing only the three dynamical d.o.f. $\{\zeta, \delta_d, \delta_r\}$:

$$S^{(2)} = \frac{1}{(2\pi)^3}\int d^3k dt a^3 \left(\dot{\vec{\chi}}^t \mathbf{A}\dot{\vec{\chi}} - k^2\vec{\chi}^t\mathbf{G}\vec{\chi} - \dot{\vec{\chi}}^t\mathbf{B}\vec{\chi} - \vec{\chi}^t\mathbf{M}\vec{\chi}\right) , \quad (3.23)$$

where we have defined the dimensionless vector:

$$\vec{\chi}^t = (\zeta, \delta_d, \delta_r), \quad (3.24)$$

and the matrix components are listed in Appendix 3.6. In the next sections we will derive the stability conditions one needs to impose on the above action in order to guarantee the viability of the underlying theory of gravity.

Before proceeding with this in-depth analysis of the final action we present the background equations corresponding to our set-up:

$$\begin{aligned}
E_1 &\equiv 3m_0^2\left[1+\Omega+a\frac{d\Omega}{da}\right]H^2+\Lambda-2\,c-\sum\bar{\rho}_i=0\,,\\
E_2 &\equiv m_0^2(1+\Omega)(3H^2+2\dot{H})+2m_0^2\,H\dot{\Omega}+m_0^2\ddot{\Omega}+\sum_i p_i+\Lambda=0\,,\\
E_i &\equiv \dot{\bar{\rho}}_i+3H(\bar{\rho}_i+p_i)=0\,. \quad (3.25)
\end{aligned}$$

where the Friedmann equations have been supplemented by the continuity equations for the fluids. Finally, in order to close the system of equations, one needs to use the well-known equations of state for dust and radiation. As a side comment, from the background equations it is not possible to define in general a modified gravitational constant because c and Λ can be functions of H^2. The latter statement is clear when looking at the mapping of specific theories in the EFToDE/MG language as done in Chapter 2.

3.4.1 The presence of ghosts

A negative kinetic term of a field is usually considered as a pathology of the theory, since the high energy vacuum is unstable to the spontaneous production of particles [29]. Such a pathology must be constrained demanding for a positive kinetic term.

Recently in ref. [126], it has been shown that such a constraint has to be imposed only in the high energy regime, in other words, an infrared ghost does not lead to a catastrophic vacuum collapse. On the contrary it was shown that it corresponds to a well known physical phenomenon, the Jeans instability.

In fact, expanding the ghost conditions in high-k one can show that, when using appropriate field re-definitions, the sub-leading terms can be recast into the form of a Jeans mass instability, and viceversa. For example, the Hamiltonian $\mathcal{H} = -P^2 + Q^2$ (where a ghost is present), can be recast into $\mathcal{H} = p^2 - q^2$ (with negative squared speed of propagation and/or tachyonic mass), upon using the trivial canonical transformation $Q = p$, $P = -q$. Therefore, we will consider only the constraints coming from the high-k behaviour for the ghost conditions as only in this regime they correspond to a true theoretical instability and not to a hidden physical phenomenon. As for the tachyonic squared mass (i.e. negative mass), it is problematic only when the time of evolution of the instability is much larger than H^2. We will elaborate on the latter in section 3.4.3.

Although the EFToDE/MG approach has been discussed in the context of energies smaller than the cut-off of the theory, $\Lambda_{\rm cut-off}$, here and in the following we will assume that we can still perform a high-k expansion, namely we assume that in this regime we have $H \ll k/a \ll \Lambda_{\rm cut-off}$. This assumption is assumed to be valid at least for medium-low redshifts, those for which we can apply all the known cosmological-data constraints, namely BBN, CMB, BAO, etc.

In action (3.23) we have a non-diagonal kinetic matrix for the three fields, i.e. $\mathcal{L} \ni A_{ij}\dot{\chi}_i\dot{\chi}_j$. As previously mentioned, in order to guarantee the absence of ghosts, one needs to demand the high-k limit of the kinetic matrix to be positive definite. It is clear that one case encompassing all viable theories does not exist as a result of the wide range of operators which depend differently on the momentum. In particular, one has to pay attention to the operators accompanying $\bar{M}_2^3, \bar{M}_2^2$ and m_2^2, which exhibit a higher order dependence on k. Therefore, we will present a number of clear sub-cases which we consider relevant

We can identify a few cases:

1. In this case all the functions in the Lagrangian are present, in particular $m_2^2 \neq 0$ and $F_4 \neq 0$. As a reference we note that the low-energy Hořava gravity belongs to this general case. Expanding at high-k, we find

$$\mathcal{G}_1 = \frac{(F_1 - 3\,F_4)\,a^3 F_1}{2F_4} > 0\,, \qquad (3.26)$$

$$\mathcal{G}_l = \frac{a^5\bar{\rho}_l^2}{2k^2(\bar{\rho}_l + p_l)} > 0\,, \qquad (3.27)$$

   where the index l indicates the matter components, i.e. dust and radiation, $\mathcal{G}_1 \equiv {\rm Det}(\mathbf{A})/(A_{22}A_{33} - A_{23}^2)$, $\mathcal{G}_r = A_{33}$ and $\mathcal{G}_d = A_{22} - A_{23}^2/A_{33}$. The $\mathcal{G}_l$ conditions represent the standard matter no-ghost conditions, which are trivially satisfied.

2. $F_4 = 0 = m_2^2$. This case corresponds to the well known class of

beyond Horndeski theories. We find:

$$\mathcal{G}_1 = \frac{\left(F_1 F_3 + 3\,{F_2}^2\right) F_1 a^3}{2{F_2}^2} > 0\,, \tag{3.28}$$

$$\mathcal{G}_l = \frac{a^5 \bar{\rho}_l^2}{2k^2(\bar{\rho}_l + p_l)} > 0\,. \tag{3.29}$$

3. $F_4 = 0$ and $m_2^2 \neq 0$. The ghost conditions change into

$$\mathcal{G}_1 = \frac{4F_1^2 m_2^2 k^2 a}{F_2^2} > 0\,, \tag{3.30}$$

$$\mathcal{G}_d = \frac{F_2^2 a^5 \bar{\rho}_d}{2k^2(F_2^2 - 8\,m_2^2 \bar{\rho}_d)} > 0\,, \tag{3.31}$$

$$\mathcal{G}_r = \frac{9\,a^5\left(F_2^2 - 8\,m_2^2 \bar{\rho}_d\right)\bar{\rho}_r}{8\,k^2[3\,F_2^2 - 8\,m_2^2(3\bar{\rho}_d + 4\bar{\rho}_r)]} > 0\,, \tag{3.32}$$

and in this case the matter no-ghost conditions get non-trivially modified. In particular, we find $0 < m_2^2 < F_2^2/[8(\bar{\rho}_d + 4\bar{\rho}_r/3)]$. Such condition prevents m_2^2 to be arbitrarily large ensuring the stability of the theory. One might wondering about the role of spatial gradients of the lapse in the stability of matter, since in the action (3.20) there is no direct coupling between matter and gravity. However, the spatial gradient of the lapse turns out to be proportional to $\dot{\delta}_d^2$, $\dot{\delta}_r^2$ and $\dot{\zeta}^2$ through eqs. (3.22), then it is directly involved in the above ghost conditions. In this sense there is a "coupling" between gravity and matter fields.

4. $m_2^2 = 0$ and $F_4 \neq 0$. In this case we have

$$\mathcal{G}_1 = \frac{a^3\,(F_1 - 3\,F_4)\left(F_1 F_3 + 3\,F_2^2\right)}{2({F_2}^2 + F_4 F_3)} > 0\,, \tag{3.33}$$

$$\mathcal{G}_l = \frac{a^5 \bar{\rho}_l^2}{2k^2(\bar{\rho}_l + p_l)} > 0\,. \tag{3.34}$$

5. $F_1 = 0$. In this case the no-ghost conditions can be written as:

$$\mathcal{G}_1 = -\frac{9\,F_2^2 a^5}{16\,m_2^2 k^2} > 0\,, \tag{3.35}$$

$$\mathcal{G}_l = \frac{a^5 \bar{\rho}_l^2}{2k^2(\bar{\rho}_l + p_l)} > 0\,, \tag{3.36}$$

so that $m_2^2 < 0$.

6. Cases: $F_1 = 3F_4$, or $F_1 = 0 = F_2$, or $m_2^2 = 0 = F_1F_3 + 3F_2^2$. In this cases the determinant of the kinetic matrix identically vanishes. This behaviour, in general, leads to strong coupling, so that this class of theories cannot be considered as a valid EFT.

A final remark on the first two cases, which are the most noticeable since they are strictly related to well known models: the presence of matter fluids does not affect the form of the ghost conditions, indeed, we recover the same results as in the previous Chapter where no matter fluids were included, once the high-k limit has been taken. However, let us note that the parameters space identified by these conditions can change because of the evolution of the scale factor, which in turns is the solution of different Friedmann equations. Moreover, no-ghost conditions have been previously obtained in presence of matter fields described by a $P(\mathcal{X})$ action as in refs. [37, 111, 123, 124] (and references therein). Such results are obtained for the variable v_m and they can be safely applied for all matter fluids but not for dust. Indeed, in the specific case of pressureless fluids ($w \to 0$) the ghost condition turns out to be ill defined. This can be explained by the fact that the no-ghost conditions need to be derived at the level of the action, which diverges in this limit. From a physical point of view this is related to a "bad" choice of physical variable which has to describe the matter d.o.f. as we discussed in section 3.3. However, in some cases they can be extended to non relativistic matter species as for eg. in ref. [111], where the authors use for the barotropic coefficient of these species the case $w = 0^+$ which implies a small yet non-negligible pressure and speed of propagation. In conclusion, by using appropriate precautions in some cases present in literature one can find some of the above results, mostly related to case 2. In this sense our results are more general and robust.

3.4.2 The speeds of propagation

We will now proceed with the study of the speeds of propagation associated to the scalar d.o.f. in action (3.23). As usual, their positivity guarantees the avoidance of any potential gradient instabilities at high-k. Hereafter, we will consider the action purely in the high-k limit. This is a necessary step in order to obtain the physical speeds of propagation. Indeed, if one does not assume the high-k limit the resulting "speeds of propagation" would be complicated and non-local expressions due to the complex dependence on the momentum of the action (3.23) and the interaction between the three fields. Of course, in order to study the gradient instability in full generality one needs to work out such expressions. However, let us say that in such case the fields do not decouple from each other and it turns out to be very difficult to obtain analytical expressions for the speeds of propagation. Moreover, the regime in which the gradient instability manifests itself faster and thus becomes potentially dangerous within the lifetime of the universe is in the high-k limit, thus justifying our restriction to such a regime.

In order to achieve this, it is necessary to diagonalise the kinetic matrix, therefore we will proceed with the following field redefinition:

$$\zeta = \Psi_1, \qquad \delta_d = \Psi_2 k - \frac{A_{12}A_{33} - A_{13}A_{23}}{A_{22}A_{33} - A_{23}^2}\,\Psi_1,$$
$$\delta_r = k\Psi_3 + \frac{A_{12}A_{23} - A_{13}A_{22}}{A_{22}A_{33} - A_{23}^2}\,\Psi_1 - \frac{A_{23}}{A_{33}}\,\Psi_2 k\,. \tag{3.37}$$

The k dependence of the transformation is a convenient choice in order to obtain, in the high-k limit, a scale invariant kinetic matrix and the new kinetic matrix, $\mathcal{L} \ni a^3 K_{ij}\dot{\Psi}_i\dot{\Psi}_j$, is now diagonal without approximations. Finally, we get a Lagrangian of the form:

$$\begin{aligned}\mathcal{L}^{(2)} &= K_{11}\dot{\Psi}_1^2 + K_{22}\dot{\Psi}_2^2 + K_{33}\dot{\Psi}_3^2 + Q_{12}(\dot{\Psi}_1\Psi_2 - \dot{\Psi}_2\Psi_1)\\ &+ Q_{13}(\dot{\Psi}_1\Psi_3 - \dot{\Psi}_3\Psi_1) + Q_{23}(\dot{\Psi}_2\Psi_3 - \dot{\Psi}_3\Psi_2) - \mathcal{M}_{ij}\Psi_i\Psi_j,\end{aligned} \tag{3.38}$$

where the kinetic matrix coefficients are:

$$K_{11} = \frac{A_{33}A_{12}{}^2 - 2A_{13}A_{23}A_{12} + A_{13}^2A_{22} + A_{11}\left(A_{23}{}^2 - A_{22}A_{33}\right)}{A_{23}{}^2 - A_{22}A_{33}}\,,$$
$$K_{22} = k^2\left(A_{22} - \frac{A_{23}{}^2}{A_{33}}\right),$$
$$K_{33} = k^2A_{33} \qquad K_{ij} = 0 \ \text{ with } \ i \neq j\,, \tag{3.39}$$

and the Q_{ij} and $\mathcal{M}_{ij}$ matrix coefficients will be specified in the following case by case.

Due to the different scaling with k of the operators in action (3.23), it is necessary to analyse the sub-cases identified before separately. As it will become clear every sub-case exhibits a different behaviour, as expected.

1. General case ($m_2^2 \neq 0$ and $F_4 \neq 0$). The kinetic matrix elements at high-k read

$$K_{11} = \frac{F_1\,(F_1 - 3F_4)}{2F_4} + \mathcal{O}(k^{-2})\,, \quad K_{22} = \frac{1}{2}a^2\bar{\rho}_d + \mathcal{O}(k^{-2})\,,$$
$$K_{33} = \frac{3}{8}a^2\bar{\rho}_r + \mathcal{O}(k^{-2})\,, \tag{3.40}$$

which are scale invariant. In its full generality, the action reduces in such a limit to a system of three decoupled fields:

$$
\begin{aligned}
S^{(2)} &= \frac{1}{(2\pi)^3}\int dk^3 dt \frac{a^3}{8}\left\{4a^2\bar{\rho}_d\dot{\Psi}_2^2 + 3a^2\bar{\rho}_r\dot{\Psi}_3^2 + \frac{4F_1\left(F_1-3F_4\right)}{F_4}\dot{\Psi}_1^2\right.\\
&- \frac{k^2}{a^2}\left[\frac{2\left(-4m_0^2(\Omega+1)\left(m_2^2-\hat{M}^2\right)+4\hat{M}^4+m_0^4(\Omega+1)^2\right)}{m_2^2}\Psi_1^2\right.\\
&+ \left.\left. a^2\bar{\rho}_r\Psi_3^2\right] + \mathcal{O}(k^{-1})\right\}, \qquad (3.41)
\end{aligned}
$$

from which it is easy to read off the Q_{ij} and $\mathcal{M}_{ij}$ coefficients. Then, for high-k, the elements Q_{ij} are corrections and the matrix $\mathcal{M}_{ij}$ becomes diagonal. This decoupling is very helpful when obtaining the speeds of propagation from the Euler-Lagrange equations:

$$
\begin{aligned}
&\frac{F_1\left(F_1-3F_4\right)}{F_4}\ddot{\Psi}_1 + \frac{k^2}{a^2}\left(\frac{-4m_0^2(\Omega+1)\left(m_2^2-\hat{M}^2\right)+4\hat{M}^4+m_0^4(\Omega+1)^2}{2m_2^2}\right)\Psi_1\\
&+\left(\frac{F_1^2\left(3F_4H-\dot{F}_4\right)+F_1F_4\left(2\dot{F}_1-9F_4H\right)}{F_4^2}-3\dot{F}_1\right)\dot{\Psi}_1 \approx 0\,,\\
&\ddot{\Psi}_2 + 2H\dot{\Psi}_2 \approx 0\,,\\
&3\ddot{\Psi}_3 + 3H\dot{\Psi}_3 + \frac{k^2}{a^2}\Psi_3 \approx 0. \qquad (3.42)
\end{aligned}
$$

It is now straightforward to isolate the three speeds of propagation and look at their functional dependence:

$$
\begin{aligned}
c_{s,g}^2 &= \frac{F_4\left(-4m_0^2(\Omega+1)\left(m_2^2-\hat{M}^2\right)+4\hat{M}^4+m_0^4(\Omega+1)^2\right)}{2F_1m_2^2\left(F_1-3F_4\right)}\,,\\
c_{s,d}^2 &= 0\,,\quad c_{s,r}^2=\frac{1}{3}, \qquad (3.43)
\end{aligned}
$$

where we have used the suffix $''g''$ to indicate the speed of propagation associated to the d.o.f. of the gravity sector. It is clear that when we consider all the operators active, including the higher order in spatial derivative operators, one gets a completely decoupled system where the fields do not influence each other and evolve separately.

2. Case $F_4 = 0 = m_2^2$. After applying the fields re-definitions (3.37),

we get in the large k-limit the following action:

$$
\begin{aligned}
S^{(2)} &= \frac{1}{(2\pi)^3}\int dk^3 dt a^3 \left\{\frac{1}{2}F_1\left(\frac{F_1F_3}{F_2^2}+3\right)\dot{\Psi}_1^2+\frac{1}{2}a^2\bar{\rho}_d\dot{\Psi}_2^2\right.\\
&+ \frac{3}{8}a^2\bar{\rho}_r\dot{\Psi}_3^2-k^2\frac{\bar{\rho}_r}{8}\Psi_3^2+k\frac{1}{2F_2}\left(4\hat{M}^2+2m_0^2(\Omega+1)-F_1\right)\\
&\times \left[\bar{\rho}_d\left(\Psi_2\dot{\Psi}_1-\Psi_1\dot{\Psi}_2\right)+\bar{\rho}_r\left(\Psi_3\dot{\Psi}_1-\Psi_1\dot{\Psi}_3\right)\right]\\
&+ \frac{k^2}{3a^2F_2^2}\left[-3F_1F_2H\left(2\hat{M}^2+m_0^2(\Omega+1)\right)+(6\bar{\rho}_d+8\bar{\rho}_r)\right.\\
&\times \left(2\hat{M}^2+m_0^2(\Omega+1)\right)^2+3m_0^2\left[F_1(\Omega+1)\dot{F}_2\right.\\
&- F_2\left(F_1\dot{\Omega}+(\Omega+1)\dot{F}_1\right)+F_2^2(\Omega+1)\Big]-6\left(F_2\left(\dot{F}_1\hat{M}^2\right.\right.\\
&+ \left.\left.\left.\left.2F_1\hat{M}\dot{\hat{M}}\right)-F_1\dot{F}_2\hat{M}^2\right)\right]\Psi_1^2\right\}+\mathcal{O}(k^{-2}).
\end{aligned}
\tag{3.44}
$$

As it is clear, the resulting action in the high-k limit exhibits some substantial deviations from the previous case. The complication arises due to the fact that now the fields are coupled in antisymmetric configurations. This will force us to change approach when obtaining the speeds of propagation. Namely, we will choose firstly to Fourier transform the time component in the Lagrangian by using ($\partial_t \to -i\omega$) and then proceed to obtain the dispersion relations. This will yield the following:

$$
\mathcal{L}^{(2)} \sim \vec{\Psi}^T \begin{pmatrix} \frac{1}{2}F_1\left(\frac{F_3F_1}{F_2^2}+3\right)\omega^2-\frac{k^2}{a^2}\mathcal{G}_{11} & -i\omega k\mathcal{B}_{12} & -i\omega k\mathcal{B}_{13} \\ i\omega k\mathcal{B}_{12} & \frac{1}{2}a^2\bar{\rho}_d\omega^2 & 0 \\ i\omega k\mathcal{B}_{13} & 0 & \frac{3}{8}a^2\bar{\rho}_r\omega^2-k^2\frac{\bar{\rho}_r}{8} \end{pmatrix}\cdot\vec{\Psi}\,,
\tag{3.45}
$$

where $\mathcal{G}_{11}$ and $\mathcal{B}_{ij}$ can be read off from the action and we have defined the field vector $\vec{\Psi}$. Now, setting the determinant of the above matrix to zero and considering that $\omega^2=\frac{k^2}{a^2}c_s^2$ in the high-k limit, we obtain the following results:

$$
\begin{aligned}
&c_{s,d}^2 = 0\,,\\
&(3c_s^2-1)\bar{\rho}_r\left[\bar{\rho}_d\left(c_s^2(F_3F_1^2+3F_2^2F_1)-2a^2F_2^2\mathcal{G}_{11}\right)-4\mathcal{B}_{12}^2{F_2}^2\right]\\
&-16c_s^2\mathcal{B}_{13}^2F_2^2\bar{\rho}_d = 0
\end{aligned}
\tag{3.46}
$$

with $F_2\neq 0$ and where c_s^2 is the double solution of the dispersion relation obtained after observing that the dust speed of propagation,

$c^2_{s,d}$ is zero. It is clear that, while the speed of propagation of the dust component remains unaffected by the presence of radiation and gravity, the last dispersion relation manifests the clear interaction between radiation and gravity, which modifies their speeds of propagation. Hence, this result shows us clearly that the interaction with matter can affect the gravity sector in a very deep way.

The only case in which the gravity sector and the radiation one completely decouple is when the following condition applies:

$$4\hat{M}^2 + 2m_0^2(\Omega + 1) - F_1 = 0. \tag{3.47}$$

In this case from (3.46) the standard speed of propagation for the radiation is recovered and the speed of gravity is

$$c^2_{s,g} = \frac{2F_2^2\mathcal{G}_{11}}{F_3F_1^2 + 3F_2^2F_1}\,. \tag{3.48}$$

Let us notice that the condition (3.47) is trivially satisfied for the Horndeski class of models. In Eq. (3.48), $\mathcal{G}_{11}$ depends on the background densities of dust and radiation, then one can use the background equations (3.25) to eliminate the dependence from the densities of the matter fluids, thus obtaining

$$\begin{aligned} c^2_{s,g} &= \frac{2}{F_1\left(3F_2^2 + F_1F_3\right)} \times \Big(2cF_1^2 + 2m_0^2F_1^2\dot{H}(\Omega + 1) \\ &+ F_1^2H\left(F_2 - m_0^2\dot{\Omega}\right) + m_0^2F_1^2\ddot{\Omega} - 2m_0^2F_2^2(\Omega + 1) - F_1^2\dot{F}_2 \\ &+ 2F_2F_1\dot{F}_1\Big)\,. \end{aligned} \tag{3.49}$$

Even though the radiation and the dust sector appear unaltered there is some interplay between gravity and the matter sector. Although the above expression for the speed of propagation of the gravity mode holds both in the vacuum and matter case, the parameters space defined through Eq. (3.49) changes drastically in the two cases. Indeed, firstly one has to consider a different evolution for the scale factor, $a(t)$ accordingly to the corresponding Friedmann equations, secondly in the vacuum case Eq. (3.49) simplifies because a combination of terms turns to be zero due to the Friedmann equations. Instead, such combination of terms when matter is included gives a non zero contribution.

The same result for this sub-case has been obtained in ref. [37, 124], starting from a $P(\mathcal{X})$ action for the matter sector and the v_m variable. It is important to note that, in contrast to the no-ghost conditions, the results also agree for the case of dust. This can be explained by the fact that the speed of propagation can be obtained at the level of the equations of motion, hence avoiding the issues plaguing the action, described in the previous sections.

3. Case $F_4 = 0$ and $m_2^2 \neq 0$. The action at high-k reads

$$
\begin{aligned}
S^{(2)} &= \frac{1}{(2\pi)^3}\int dk^3 dta^3 \left\{ \frac{4k^2F_1^2m_2^2}{a^2F_2^2}\dot{\Psi}_1^2 + \frac{a^2\bar{\rho}_dF_2^2}{2F_2^2-16\bar{\rho}_dm_2^2}\dot{\Psi}_2^2 \right.\\
&+ \frac{9a^2\bar{\rho}_r\left(F_2^2-8\bar{\rho}_dm_2^2\right)}{8\left(-24\bar{\rho}_dm_2^2+3F_2^2-32m_2^2\bar{\rho}_r\right)}\dot{\Psi}_3^2 - \frac{k^2\bar{\rho}_r}{8}\Psi_3^2\\
&- \frac{128k^2\bar{\rho}_d^2m_2^4\bar{\rho}_r}{9\left(F_2^2-8\bar{\rho}_dm_2^2\right)^2}\Psi_2^2 - \frac{128k^4F_1^2m_2^4\bar{\rho}_r}{9a^4F_2^4}\Psi_1^2 - \frac{8k^3F_1m_2^2\bar{\rho}_r}{3a^2F_2^2}\Psi_1\Psi_3\\
&+ \frac{256k^3a\bar{\rho}_dF_1m_2^4\bar{\rho}_r}{9a^3F_2^4-72\bar{\rho}_dF_2^2m_2^2}\Psi_1\Psi_2 + k\left(16F_1F_2m_2^2H\right.\\
&+ \left(4F_2\left(F_2\hat{M}^2-2m_2^2\dot{F}_1\right)+2m_0^2F_2^2(\Omega+1)\right.\\
&- \left.\left.F_1\left(16F_2m_2\dot{m}_2-16m_2^2\dot{F}_2+F_2^2\right)\right)\right)\\
&\times \left[\frac{\bar{\rho}_d}{2\left(F_2^3-8\bar{\rho}_dF_2m_2^2\right)}\left(\Psi_2\dot{\Psi}_1-\Psi_1\dot{\Psi}_2\right)\right.\\
&+ \left.\frac{3\bar{\rho}_r}{2F_2\left(-24\bar{\rho}_dm_2^2+3F_2^2-32m_2^2\bar{\rho}_r\right)}\left(\Psi_3\dot{\Psi}_1-\Psi_1\dot{\Psi}_3\right)\right]\\
&+ \left.\frac{8k^2\bar{\rho}_dm_2^2\bar{\rho}_r}{3F_2^2-24\bar{\rho}_dm_2^2}\Psi_2\Psi_3\right\} + O(k^{-2})\,. \qquad (3.50)
\end{aligned}
$$

We find that the solutions of the discriminant equation

$$
\det\left(\frac{c_2^2k^2}{a^2}K_{ij}-\mathcal{M}_{ij}\right) = \frac{a^5{c_s}^4\bar{\rho}_m\left({c_s}^2\rho_{r,n}-n_r\rho_{r,nn}\right)\bar{\rho}_r^2{m_2}^2{F_1}^2k^6}{\left(-8\,{m_2}^2n_r\rho_{r,n}-8\,{m_2}^2\bar{\rho}_m+{F_2}^2\right)n_r{\rho_{r,n}}^2}\,, \qquad (3.51)
$$

reduce to

$$
c_{s,g}^2 = 0\,, \qquad c_{s,d}^2 = 0\,, \qquad c_{s,r}^2 = \frac{1}{3}\,. \qquad (3.52)
$$

The results for this case can be found in the limit $F_4 \to 0$ for the general case discussed above.

4. Case $F_4 \neq 0$ and $m_2^2 = 0$. The action for this sub-case at high-k

reads

$$
\begin{aligned}
S^{(2)} &= \frac{1}{(2\pi)^3}\int dk^3 dt a^3 \left\{ \frac{\left(3F_2^2+F_1F_3\right)\left(F_1-3F_4\right)}{2\left(F_2^2+F_3F_4\right)}\dot{\Psi}_1^2 \right.\\
&+ \frac{1}{2}a^2\bar{\rho}_d\dot{\Psi}_2^2+\frac{3}{8}a^2\bar{\rho}_r\dot{\Psi}_3^2-k^2\frac{\bar{\rho}_r\left(F_2^2+F_3F_4+4F_4\bar{\rho}_r\right)}{8\left(F_2^2+F_3F_4\right)}\Psi_3^2\\
&- \frac{k^2\bar{\rho}_d^2F_4}{2\left(F_2^2+F_3F_4\right)}\Psi_2^2-k^4\frac{2F_4\left(2\hat{M}^2+m_0^2(\Omega+1)\right)^2}{a^4\left(F_2^2+F_3F_4\right)}\Psi_1^2\\
&+ k^3\frac{2F_4\left(2\hat{M}^2+m_0^2(\Omega+1)\right)}{a^2\left(F_2^2+F_3F_4\right)}\left(\bar{\rho}_r\Psi_3\Psi_1+\bar{\rho}_d\Psi_1\Psi_2\right)\\
&+ k\frac{F_2\left(-F_1+3F_4+4\hat{M}^2+2m_0^2(\Omega+1)\right)}{2\left(F_2^2+F_3F_4\right)}\left[\bar{\rho}_d\left(\Psi_2\dot{\Psi}_1-\Psi_1\dot{\Psi}_2\right)\right.\\
&+ \left.\left.\bar{\rho}_r\left(\Psi_3\dot{\Psi}_1-\Psi_1\dot{\Psi}_3\right)\right]-\frac{k^2\bar{\rho}_dF_4\bar{\rho}_r}{\left(F_2^2+F_3F_4\right)}\Psi_2\Psi_3\right\}+\mathcal{O}(k^{-2})\,,
\end{aligned}
\tag{3.53}
$$

where the kinetic terms K_{11}, K_{22}, K_{33} are of order $\mathcal{O}(k^0)$ for high values of k and the elements Q_{12} and Q_{13} are of order k and cannot be neglected. Furthermore, the leading component of $\mathcal{M}_{11}$ is of order k^4. Therefore now we need to consider the discriminant equation as

$$
\mathrm{Det}(\omega^2 K_{ij} - i\,\omega\,Q_{ij} - \mathcal{M}_{ij}) = 0\,, \tag{3.54}
$$

this equation can be recast as

$$
\omega^6 + \left(\mathcal{A}\,\frac{k^4}{a^4}+\mathcal{O}(k^2)\right)\omega^4+\left(\mathcal{B}\,\frac{k^6}{a^6}+\mathcal{O}(k^4)\right)\omega^2 = 0\,, \tag{3.55}
$$

with

$$
\mathcal{A} = 4\,\frac{\left(\left(\Omega+1\right)m_0{}^2+2\,\hat{M}^2\right)^2 F_4}{\left(3\,F_4-F_1\right)\left(F_1F_3+3\,F_2{}^2\right)}\,,\qquad \mathcal{B}=-\frac{1}{3}\,\mathcal{A}\,. \tag{3.56}
$$

For high-k, we find the following solutions:

- One solution can be found by assuming $\omega^2 = W\,k^4/a^4$. In this case we find

$$
\left(W+\mathcal{A}\right)W^2\,\frac{k^{12}}{a^{12}}+\mathcal{O}(k^{10}) = 0\,, \tag{3.57}
$$

which is verified by $W = -\mathcal{A}$, so that

$$\omega^2 = -\mathcal{A}\,\frac{k^4}{a^4}\,, \qquad c_{s,g}^2 = -4\mathcal{A}\,\frac{k^2}{a^2}\,, \tag{3.58}$$

or

$$c_{s,g}^2 = \frac{16F_4\left((\Omega+1)\,m_0^2 + 2\,\hat{M}^2\right)^2}{(F_1F_3 + 3\,F_2^2)\,(F_1 - 3F_4)}\,\frac{k^2}{a^2}\,, \tag{3.59}$$

which tends to large values.

- The other two solutions of Eq. (3.55) can be found by assuming $\omega^2 = Wk^2/a^2$, so that

$$\left(\mathcal{A}\,W^2 + \mathcal{B}W\right)\,\frac{k^8}{a^8} + \mathcal{O}(k^6) = 0\,, \tag{3.60}$$

which implies the following standard results

$$c_{s,d}^2 = 0\,, \qquad c_{s,r}^2 = \frac{1}{3}\,. \tag{3.61}$$

5. Case $F_1 = 0$. The action reads:

$$\begin{aligned} S^{(2)} &= \frac{1}{(2\pi)^3}\int dk^3 dta^3\left[-\frac{9a^2F_2^2}{16m_2^2}\dot{\Psi}_1^2 + \frac{\bar{\rho}_d}{2}a^2\dot{\Psi}_2^2 + \frac{3}{8}\bar{\rho}_r a^2\dot{\Psi}_3^2\right.\\ &- \frac{k^2\bar{\rho}_r}{8}\psi_3^2 + k^2\frac{\bar{\rho}_d\left(2\hat{M}^2 + m_0^2(\Omega+1)\right)}{4m_2^2}\psi_1\Psi_2\\ &+ k^2\frac{\bar{\rho}_r\left(4\hat{M}^2 + 2m_0^2(\Omega+1) + 8m_2^2\right)}{8m_2^2}\Psi_1\Psi_3\\ &- \left. k^4\frac{4\hat{M}^4 - 4m_0^2(\Omega+1)\left(m_2^2 - \hat{M}^2\right) + m_0^4(\Omega+1)^2}{4a^2m_2^2}\Psi_1^2\right]\\ &+ \mathcal{O}(k^{-2}). \end{aligned} \tag{3.62}$$

In this case, the matrix Q_{ij} can be neglected, but the $\mathcal{M}_{11}$ coefficient has a term in k^4, therefore we have the discriminant equation

$$\mathrm{Det}(\omega^2 K_{ij} - \mathcal{M}_{ij}) = 0\,, \tag{3.63}$$

which leads to

$$\omega^6 + \left(\mathcal{A}\,\frac{k^4}{a^4} + \mathcal{O}(k^2)\right)\omega^4 + \left(-\frac{1}{3}\mathcal{A}\,\frac{k^6}{a^6} + \mathcal{O}(k^4)\right)\omega^2 + \mathcal{O}(k^6) = 0\,. \tag{3.64}$$

with

$$\mathcal{A} = \frac{4\left(-4m_0^2(\Omega+1)\left(m_2^2-\hat{M}^2\right)+4\hat{M}^4+m_0^4(\Omega+1)^2\right)}{9F_2^2}\,. \tag{3.65}$$

Once more we have a solution

$$\omega^2 = -\mathcal{A}\,\frac{k^4}{a^4}\,, \tag{3.66}$$

which leads to

$$c_{s,g}^2 = \frac{16}{9}\,\frac{4\,(m_2^2-\hat{M}^2)\,(1+\Omega)\,m_0^2-(1+\Omega)^2\,m_0^4-4\,\hat{M}^4}{F_2^2}\,\frac{k^2}{a^2}\,, \tag{3.67}$$

whereas the other two solutions are found to be

$$c_{s,d}^2 = 0\,, \qquad c_{s,r}^2 = \frac{1}{3}\,. \tag{3.68}$$

In summary, in this section we have derived the speeds of propagation for the three dynamical fields describing our system. In general for all the sub-cases analysed we have found that in the high-k regime the three d.o.f. decouple and the resulting speeds of propagation are unaltered with respect to the vacuum case. This can be easily verified by considering the high-k limit of the results in Chapter 2. Only one case stands aside, the beyond Horndeski case. In this case the dust field completely decouples from the other fields, while the radiation and gravity fields are coupled and their speeds result to be modified. We also recall that even in the cases the expressions for the speeds of propagation do not differ from the respective cases in vacuum, the parameters space may change accordingly to a different evolution of the scale factor, which in turns is the solution of different background equations. In conclusion, for all the cases analysed we demand a positive speed of propagation in order to guarantee the viability of the underlying theory of gravity.

3.4.3 Tachyonic and Jeans instabilities

The final aspect of our work which tends to be the one least studied in the literature in the context of MG theories and especially in the EFToDE/MG framework, is the study of the canonical mass of the fields and consequently the boundedness of the Hamiltonian at low momenta. These results are related to the usual tachyonic and Jeans instabilities, the latter being characteristic of the fluids components.

In this section we will restrict the analysis to the EFToDE/MG action in the presence of only one matter fluid. We choose the dust over radiation because we know that the dust component clusters and hence for our purpose it might show an interesting behaviour related to the Jeans instability. Thus, the results presented here will be applicable during the dust-dominated era and onwards when the dynamics of the two d.o.f. starts to play a role. A second fluid can be straightforwardly added, but it makes the procedure substantially more difficult.

One can obtain the action for the EFToDE/MG with a dust component by setting $\delta_r = 0$ in the action (3.20) and $\bar{\rho}_r = 0$ in the remaining functions. Now, let us assume the no-ghost conditions hold, and proceed to rewrite the action in its canonical form. The first step is to diagonalise the (2x2) kinetic matrix as in the previous section by making the following field redefinitions

$$\begin{aligned} \zeta &= \Psi_1 \, , \\ \delta_d &= k\Psi_2 - \frac{A_{12}\Psi_1}{A_{22}} \, , \end{aligned} \tag{3.69}$$

with the following diagonal terms:

$$\bar{K}_{11} = A_{11} - \frac{A_{12}^2}{A_{22}}, \qquad \bar{K}_{22} = k^2 A_{22} \, , \tag{3.70}$$

where A_{ij} are the ones defined in Appendix 3.6 after setting $\bar{\rho}_r = 0$. Next, the canonical form is obtained by normalising the fields accordingly to:

$$\begin{aligned} \Psi_1 &= \frac{1}{\sqrt{2\bar{K}_{11}}} \bar{\Psi}_1 \, , \\ \Psi_2 &= \frac{1}{\sqrt{2\bar{K}_{22}}} \bar{\Psi}_2 \, . \end{aligned} \tag{3.71}$$

After grouping the different terms and performing a number of integrations by parts, we obtain the Lagrangian as:

$$\mathcal{L}^{(2)} = \frac{a^3}{2} \left[\dot{\bar{\Psi}}_1^2 + \dot{\bar{\Psi}}_2^2 + \bar{B}(t,k) \, (\dot{\bar{\Psi}}_1 \bar{\Psi}_2 - \dot{\bar{\Psi}}_2 \bar{\Psi}_1) - \bar{C}_{ij}(t,k) \bar{\Psi}_i \bar{\Psi}_j \right] , \tag{3.72}$$

where we refer the reader to the Appendix 3.6 for the functional forms of the $\bar{B}, C_{ij}$ coefficients.

In order to obtain the mass eigenvalues we need to proceed with the diagonalization of the mass matrix C_{ij} while keeping the canonical form of the action. For this purpose we consider a field rotation via an orthogonal matrix, in the following way:

$$\begin{aligned} \bar{\Psi}_1 &= \cos\alpha \, \Phi_1 + \sin\alpha \, \Phi_2 \, , \\ \bar{\Psi}_2 &= -\sin\alpha \, \Phi_1 + \cos\alpha \, \Phi_2 \, . \end{aligned} \tag{3.73}$$

Now, it is possible to choose α in a very specific way in order to diagonalize the mass matrix. This leads to the following relation:

$$\tan(2\alpha) = \beta \equiv -\frac{2\bar{C}_{12}}{\bar{C}_{11} - \bar{C}_{22}}\,, \tag{3.74}$$

accompanied by:

$$\frac{d[\tan(2\alpha)]}{dt} = 2[1+\tan^2(2\alpha)]\dot{\alpha} \qquad \Longleftrightarrow \qquad \dot{\alpha} = \frac{\dot{\beta}}{2(1+\beta^2)}\,. \tag{3.75}$$

Then, the Lagrangian becomes

$$\mathcal{L}^{(2)} = \frac{a^3}{2}\left[\dot{\Phi}_1^2 + \dot{\Phi}_2^2 + B(t,k)\,(\dot{\Phi}_1\Phi_2 - \dot{\Phi}_2\Phi_1) - \mu_1(t,k)\Phi_1^2 - \mu_2(t,k)\Phi_2^2\right], \tag{3.76}$$

with the following definitions:

$$\begin{aligned} B &= \bar{B} + 2\dot{\alpha}\,, \\ \mu_1 &= -\dot{\alpha}^2 - \bar{B}\dot{\alpha} + \frac{(\bar{C}_{11}-\bar{C}_{22})^2 + 4C_{12}^2}{\bar{C}_{11}-\bar{C}_{22}}\cos^2\alpha + \frac{\bar{C}_{11}\bar{C}_{22} - 2\bar{C}_{12}^2 - \bar{C}_{22}^2}{\bar{C}_{11}-\bar{C}_{22}}\,, \\ \mu_2 &= -\dot{\alpha}^2 - \bar{B}\dot{\alpha} - \frac{(\bar{C}_{11}-\bar{C}_{22})^2 + 4C_{12}^2}{\bar{C}_{11}-\bar{C}_{22}}\cos^2\alpha + \frac{\bar{C}_{11}^2 - \bar{C}_{11}\bar{C}_{22} + 2\bar{C}_{12}^2}{\bar{C}_{11}-\bar{C}_{22}}\,. \end{aligned} \tag{3.77}$$

It is straightforward to obtain the energy function (which is equal in value to the Hamiltonian, see e.g. [127] for details) which reduces to a formally simple form (see Appendix 3.7), namely

$$H(\Phi_i, \dot{\Phi}_i) = \frac{a^3}{2}\left[\dot{\Phi}_1^2 + \dot{\Phi}_2^2 + \mu_1(t,k)\,\Phi_1^2 + \mu_2(t,k)\,\Phi_2^2\right], \tag{3.78}$$

so that the Hamiltonian will be unbounded from below if the eigenvalues satisfy $\mu_1 < 0$ or $\mu_2 < 0$, for example on the line ($\dot{\Phi}_i = 0$). Naively one could then proceed and constrain the mass eigenvalues to be non-negative. From a physical point of view this condition must be considered too stringent as there is a very well known phenomenon related to a negative mass, the Jeans Instability. As this corresponds to a negative mass but with a slow enough evolution rate in order to be countered by gravity we can reformulate the condition avoiding a catastrophical tachyon instability. For cosmological purposes, in order for a theory to be viable we will proceed to demand the eigenvalues, if negative, to satisfy the condition $|\mu_i(t,0)| \lesssim H^2$, so that the evolution rate of the instability will not affect the whole stability of the system for time-intervals much shorter than the Hubble time (see also ref. [126]). Finally, one would expect that the μ_2 eigenvalue can become negative as the dust sector must exhibit a Jeans instability in order for structure to form in our universe.

- **Minimally coupled quintessence model in presence of a dust fluid**

We will now proceed to exemplify the previous, rather abstract, approach by studying a specific model in the presence of dust: minimally coupled quintessence, which has the following action [71]

$$S_\phi = \int d^4x\sqrt{-g}\left[\frac{m_0^2}{2}R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right] + S_m\,, \tag{3.79}$$

where ϕ is the scalar field and V the corresponding potential. The above can be mapped in the EFToDE/MG formalism by making the following correspondence as was done in Chapter 2 and the following refs. [14, 15, 75]

$$c = \frac{1}{2}\dot{\phi}_0^2\,, \qquad \Lambda = \frac{1}{2}\dot{\phi}_0^2 - V\left(\phi_0\right)\,, \qquad \left\{\Omega, \hat{M}^2, \bar{M}_2^2, \bar{M}_3^2, M_1^3, M_2^4\right\} = 0, \tag{3.80}$$

where $\phi_0(t)$ is the background value of the scalar field.

Let us now consider that the minimally coupled quintessence model can be also parametrized by assuming that the modification to the gravity sector can be recast as a DE perfect fluid by introducing the following:

$$w_{\rm DE}(a) \equiv \frac{P_{\rm DE}}{\bar{\rho}_{\rm DE}} = \frac{\dot{\phi}^2/2 - V(\phi)}{\dot{\phi}^2/2 + V(\phi)}\,, \tag{3.81}$$

and assuming that the DE density has the standard perfect fluid form

$$\bar{\rho}_{\rm DE} = 3m_0^2 H_0^2\,\Omega_{\rm DE}^0\,\exp\left[-3\int_1^a \frac{(1+w_{\rm DE}(a))}{a}\,da\right]\,, \tag{3.82}$$

with H_0, Ω_{DE}^0 be the present day values of the Hubble and density parameter respectively. Then, the Friedmann equation simply reads:

$$3m_0^2H^2 = \bar{\rho}_d + \bar{\rho}_{\rm DE}. \tag{3.83}$$

and the EFT functions can be written accordingly as

$$c = \frac{1}{2}\bar{\rho}_{\rm DE}(1+w_{\rm DE})\,, \qquad \Lambda = w_{\rm DE}\bar{\rho}_{\rm DE}. \tag{3.84}$$

This choice for the parametrization makes the whole treatment of the minimally coupled quintessence case more handy. Indeed, this will allow

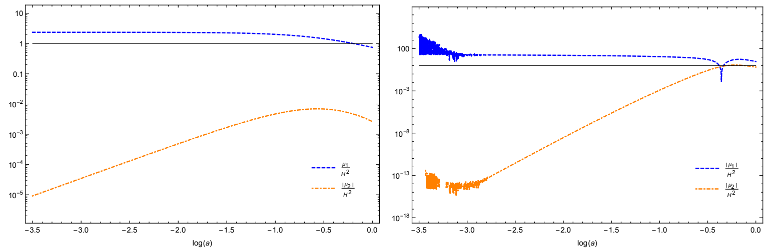

Figure 3.1: The figures show the behaviours of the mass eigenvalues μ_1/H^2 (blue dashed line), μ_2/H^2 (orange dot dashed line) for minimally coupled quintessence on a CPL background. *Left panel*: stable tachyonic configuration with $w_0 = -0.9$ and $w_a = 0.009$. *Right panel*: unstable tachyonic configuration with $w_0 = -2.9$ and $w_a = -2$. For this figure the cosmological parameters are chosen to be: $\Omega^0_{\rm DE} = 0.69, \Omega^0_d = 0.31, H_0 = 67.74$ [128]. See section 3.4.3 for the whole discussion.

us to rewrite the mass eigenvalues (3.95), presented in appendix **??**, purely in terms of the fluid parameter, i.e. $\mu_i(w_{\rm DE})$. As a general remark, from the expressions (3.95) it becomes clear that in general the mass eigenvalues tend to be quite complicated. More complicated theories, especially the non minimally coupled ones, will be substantially harder to treat, yet not impossible. Having the explicit results of the mass eigenvalues for the minimally coupled quintessence model, we want now to proceed and gain some intuition regarding their behaviour compared to H^2.

We will make a specific choice of the DE equation of state, out of the many options, which will help in illustrating different behaviours:

- The CPL parametrization [129, 130]: $w_{\rm DE}(a) = w_0 + w_a(1-a)$, where w_0 and w_a are constant and indicate, respectively, the value and the derivative of $w_{\rm DE}$ today;

as illustrative examples, for the values of $\{w_0, w_a\}$ in the DE equation of state we choose two sets, one for which the system is free from tachyonic instability and one where the gravity sector shows an unstable configuration since a tachyonic instability is manifest. The results are illustrated in figure 3.1. In the left panel, for the choice of the parameters $w_0 = -0.9$ and $w_a = 0.009$, we notice that the eigenvalue associated to the gravity sector (i.e. μ_1) is always positive and approximately of the same order as H^2. On the contrary, the eigenvalue associated with the dust sector (μ_2), here plotted in its absolute value, is negative and $|\mu_2| \ll H^2$. This, of course, has to be expected as it is a manifestation of the well known Jeans instability which allows structure to form. In the right panel, we chose a rather unrealistic set of the parameters in order to show a tachyonic instability, namely $w_0 = -2.9$ and $w_a = -2$. Both the eigenvalues oscillate and switch the sign very fast at early time, then μ_2 becomes positive around $\log(a) > -2.9$ and μ_1 turns to be firstly negative and finally positive at very late time. In this case since the eigenvalue associated to gravity is negative (for most of the time) and additionally $|\mu_1| \gg H^2$, this implies that the tachyonic instability evolves very fast, resulting in an unstable system. The dust eigenvalue on the other hand is positive during the matter dominated era which means that matter does not cluster.

The above discussion concerns only the tachyonic and Jeans instabilities, thus can not be considered exhaustive. In order to complete the set of stability conditions for minimally coupled quintessence one needs to study the ghost conditions and the speeds of propagation, as presented in the previous sections, which in the case of minimally coupled quintessence simply reduce to $w_{\rm DE}(a) > -1$.

3.5 Conclusion

In this Chapter we presented a thorough analysis of the viability conditions which guarantee the stability of the scalar d.o.f. in the presence of matter fields. These conditions guarantee the avoidance of ghosts and tachyonic instabilities, supplemented with a positive speed of propagation. The study of the viability of specific gravity theories in vacuum or in the presence of matter fields has already yielded an extensive literature. However, our results are more general and directly applicable to most of the well known models which are of cosmological interest.

For the gravity sector, we employed the general EFT approach for DE/MG, which has the advantage of being a model independent parametrization of gravity theories with one extra scalar d.o.f. while at the same time preserving a direct link with a wide class of theoretical models which can be explicitly mapped into this formalism. In order to describe the standard perfect fluids we chose the Sorkin-Schutz action which has been shown to be well behaved in contrast to other choices made in the past when considering presureless fluids. In detail, we specialised to the case where the matter fluids are dust (or CDM) and radiation. From these starting blocks we proceeded to derive the action up to second order in scalar perturbations accompanied by the background equations. Finally, we moved to the study of the viability requirements which we will summarise and discuss in the following.

After constructing the Lagrangian for the perturbations one can straightforwardly guarantee the absence of ghosts by imposing the positivity of the kinetic term, or matrix in case more than one field is considered as in the present chapter. In deriving such conditions we have considered only the Lagrangian in the high-k regime, following the recent results in ref. [126] where it was shown that only the high energy terms can turn out in catastrophic instabilities. Sub-leading terms can be recast in mass-like terms through appropriate field redefinitions. Because the EFToDE/MG approach encompasses a variety of DE/MG models, which in some cases show different and non trivial k-dependence, it is not possible to obtain one general result applicable to all possible theories. Therefore, we have identified five relevant sub-cases for which we have worked out the corresponding no-ghost conditions. In particular, two of the aforementioned sub-cases correspond to well known theoretical models, i.e. low-energy Hořava gravity and beyond Horndeski, while the remaining three do not correspond to any specific class of theories but can be useful in a model independent study. In general, we found three no-ghost conditions for each of the sub-cases. Out of those, the conditions corresponding to matter fields were trivially satisfied. Finally, we have also identified conditions which lead to strong coupling regimes, thus excluding these theories from an effective description.

The next step was to identify the speeds of propagation of the three d.o.f., in the high-k limit, for the sub-cases mentioned before. In order to avoid gradient instabilities it was then required that they have a positive sign. Depending on the sub-case the results change drastically because the momentum dependence of various terms differs in each sub-case. In general, the gravity speed of propagation does not depend on fluid variables once one consider theories with higher (than second) order spatial derivatives. In particular, for the sub-case to which low-energy Hořava gravity belongs, we find that at high-k the d.o.f. are completely decoupled and the speeds of propagation of dust and radiation components are unaffected by the coupling to gravity, leading to the standard results. On the contrary, the sub-case corresponding to beyond Horndeski shows non trivial modifications as only the dust speed of propagation stays unaltered. Furthermore, when specifying to Horndeski, the dust and radiation reduce to the standard results while the speed of propagation associated to the gravity mode still is modified with respect to the vacuum case. The three remaining sub-cases exhibit unaltered matter speeds and the gravity one is not influenced by the matter sector. A surprising result is the sub-case corresponding to $F_4 = 0, m_2^2 \neq 0$, for which the gravitational sector has a vanishing speed of sound.

In the last part of the Chapter we considered the tachyonic instability. This instability is tightly related with the Hamiltonian being unbounded as was shown in the extensive calculation in Section 3.4.3 . Only for this case we have simplified the approach by assuming only the dominant matter fluid, Cold Dark Matter. From this starting point, we have identified the two eigenvalues (μ_i) of the system which need to be constrained in the limit $k \to 0$ in order to guarantee the Hamiltonian of the system is bounded from below. These conditions then correspond to the ones one needs to apply in order to avoid tachyonic instabilities. A stringent condition is then to demand both the eigenvalues to be positive definite. On the other hand, it is well known that the dust fluid exhibits a Jeans instability, which is necessary in order to allow structures formation. Therefore, it is more realistic to allow them to become negative in sign but with a evolution rate which sufficiently slow in order to avoid the system to become unviable. In other words this translates to demanding that, in case $\mu_i < 0$, we have $|\mu_i| \ll H^2$. Due to the complexity of these results, we have chosen to exemplify our findings by studying the minimally coupled quintessence case. We have parametrized the gravity modification in terms of the equation of state for DE, i.e. $w_{\rm DE}(a)$ and then through the appropriate mapping, we were able to write $\mu_i(w_{\rm DE})$. By choosing the CPL parametrization for the DE fluid and two sets of values for the DE parameters we then presented, figure 3.1, we two typical situations, i.e. the case in which the tachyonic instability shows up and the theory becomes pathological and a stable case exhibiting a Jeans instability for the dust sector.

Concluding we would like to stress that, in the case of ghost and gradient instabilities, we presented the results in a model independent way. This implies that it covers the results existing in the literature, typically derived in the context of a specific theory, and generalises them to cover cases left unexplored. We proved this by comparing our results with the ones in the literature. Additionally we considered in depth the tachyonic instability, in the presence of matter, a result completely novel and of severe importance as the usual two conditions are not enough to produce a stable theory. This will be discussed more in depth in Chapter 5 where we will proceed to implement these results in *EFTCAMB* and test them. That will be the final step of this line of work as then they can be employed in depth in studying models of DE/MG.

3.6 Appendix A: Matrix coefficients

For completeness in this Appendix we will explicitly list the matrix coefficients used in sections 3.4,3.4.2 and 3.4.3. Let us start by considering the action (3.23) introduced in section 3.4:

$$S^{(2)} = \frac{1}{(2\pi)^3}\int d^3kdta^3\left(\dot{\vec{\chi}}^t\mathbf{A}\dot{\vec{\chi}} - k^2\vec{\chi}^t\mathbf{G}\vec{\chi} - \dot{\vec{\chi}}^t\mathbf{B}\vec{\chi} - \vec{\chi}^t\mathbf{M}\vec{\chi}\right)\,, \quad (3.85)$$

where the coefficients are

$$A_{11} = \frac{3(F_1-3F_4)\left(k^2a^2\left(-8m_2^2(3\bar{\rho}_d+4\bar{\rho}_r)+3F_2^2+F_1F_3\right)-a^4F_3(3\bar{\rho}_d+4\bar{\rho}_r)+8k^4F_1m_2^2\right)}{2k^2a^2\left(-8m_2^2(3\bar{\rho}_d+4\bar{\rho}_r)+3F_3F_4+3{F_2}^2\right)-2a^4F_3(3\bar{\rho}_d+4\bar{\rho}_r)+48k^4m_2^2F_4}\,,$$

$$
\begin{aligned}
A_{12} = A_{21} &= -\frac{3a^2\bar{\rho}_d(F_1-3F_4)\left(a^2F_3+8k^2m_2^2\right)}{2k^2a^2\left(-8m_2^2(3\bar{\rho}_d+4\bar{\rho}_r)+3F_3F_4+3F_2^2\right)-2a^4F_3(3\bar{\rho}_d+4\bar{\rho}_r)+48k^4m_2^2F_4}\,,\\
A_{13} = A_{31} &= -\frac{3a^2(F_1-3F_4)\bar{\rho}_r\left(a^2F_3+8k^2m_2^2\right)}{2k^2a^2\left(-8m_2^2(3\bar{\rho}_d+4\bar{\rho}_r)+3F_3F_4+3F_2^2\right)-2a^4F_3(3\bar{\rho}_d+4\bar{\rho}_r)+48k^4m_2^2F_4}\,,\\
A_{22} &= \frac{a^2\bar{\rho}_d\left(k^2a^2\left(3F_3F_4+3F_2^2-32m_2^2\bar{\rho}_r\right)-4a^4F_3\bar{\rho}_r+24k^4m_2^2F_4\right)}{2k^2\left(k^2a^2\left(-8m_2^2(3\bar{\rho}_d+4\bar{\rho}_r)+3F_3F_4+3F_2^2\right)-a^4F_3(3\bar{\rho}_d+4\bar{\rho}_r)+24k^4m_2^2F_4\right)}\,,\\
A_{23} = A_{32} &= \frac{3a^4\bar{\rho}_d\bar{\rho}_r\left(a^2F_3+8k^2m_2^2\right)}{2k^2\left(k^2a^2\left(-8m_2^2(3\bar{\rho}_d+4\bar{\rho}_r)+3F_3F_4+3F_2^2\right)-a^4F_3(3\bar{\rho}_d+4\bar{\rho}_r)+24k^4m_2^2F_4\right)}\,,\\
A_{33} &= \frac{9a^2\bar{\rho}_r\left(k^2a^2\left(-8m_2^2\bar{\rho}_d+F_3F_4+F_2^2\right)-a^4F_3\bar{\rho}_d+8k^4m_2^2F_4\right)}{8k^2\left(k^2a^2\left(-8m_2^2(3\bar{\rho}_d+4\bar{\rho}_r)+3F_3F_4+3F_2^2\right)-a^4F_3(3\bar{\rho}_d+4\bar{\rho}_r)+24k^4m_2^2F_4\right)}\,,\\
B_{11} &= -\frac{6k^4F_2(F_1-3F_4)\left(2\hat{M}^2+m_0^2(\Omega+1)\right)}{k^2a^2\left(-8m_2^2(3\bar{\rho}_d+4\bar{\rho}_r)+3F_3F_4+3{F_2}^2\right)-a^4F_3(3\bar{\rho}_d+4\bar{\rho}_r)+24k^4m_2^2F_4}\,,\\
B_{12} &= \frac{3k^2a^2F_2\bar{\rho}_d(F_1-3F_4)}{k^2a^2\left(-8m_2^2(3\bar{\rho}_d+4\bar{\rho}_r)+3F_3F_4+3{F_2}^2\right)-a^4F_3(3\bar{\rho}_d+4\bar{\rho}_r)+24k^4m_2^2F_4}\,,\\
B_{13} &= \frac{3k^2a^2F_2(F_1-3F_4)\bar{\rho}_r}{k^2a^2\left(-8m_2^2(3\bar{\rho}_d+4\bar{\rho}_r)+3F_3F_4+3{F_2}^2\right)-a^4F_3(3\bar{\rho}_d+4\bar{\rho}_r)+24k^4m_2^2F_4}\,,\\
B_{22} &= \frac{3a^4F_2{\bar{\rho}_d}^2}{k^2a^2\left(8m_2^2(3\bar{\rho}_d+4\bar{\rho}_r)-3F_3F_4-3{F_2}^2\right)+a^4F_3(3\bar{\rho}_d+4\bar{\rho}_r)-24k^4m_2^2F_4}\,,\\
B_{21} &= \frac{6k^2a^2F_2\bar{\rho}_d\left(2\hat{M}^2+m_0^2(\Omega+1)\right)}{k^2a^2\left(-8m_2^2(3\bar{\rho}_d+4\bar{\rho}_r)+3F_3F_4+3{F_2}^2\right)-a^4F_3(3\bar{\rho}_d+4\bar{\rho}_r)+24k^4m_2^2F_4}\,,\\
B_{23} = B_{32} &= \frac{3a^4F_2\bar{\rho}_d\bar{\rho}_r}{k^2a^2\left(8m_2^2(3\bar{\rho}_d+4\bar{\rho}_r)-3F_3F_4-3{F_2}^2\right)+a^4F_3(3\bar{\rho}_d+4\bar{\rho}_r)-24k^4m_2^2F_4}\,,\\
B_{33} &= \frac{3a^4F_2{\bar{\rho}_r}^2}{k^2a^2\left(8m_2^2(3\bar{\rho}_d+4\bar{\rho}_r)-3F_3F_4-3{F_2}^2\right)+a^4F_3(3\bar{\rho}_d+4\bar{\rho}_r)-24k^4m_2^2F_4}\,,\\
B_{31} &= \frac{6k^2a^2F_2\bar{\rho}_r\left(2\hat{M}^2+m_0^2(\Omega+1)\right)}{k^2a^2\left(-8m_2^2(3\bar{\rho}_d+4\bar{\rho}_r)+3F_3F_4+3F_2^2\right)-a^4F_3(3\bar{\rho}_d+4\bar{\rho}_r)+24k^4m_2^2F_4}\,,
\end{aligned}
\tag{3.86}
$$

$$
\begin{aligned}
G_{11} \;&=\; \Big\{k^2a^2\left(m_0^2(\Omega+1)\left(-8\left(3\bar{\rho}_d+4\bar{\rho}_r\right)\left(m_2^2-\hat{M}^2\right)+3F_3F_4\right.\right.\\
&+\; \left.\left. 3{F_2}^2\right)+8\hat{M}^4\left(3\bar{\rho}_d+4\bar{\rho}_r\right)+2m_0^4(\Omega+1)^2\left(3\bar{\rho}_d+4\bar{\rho}_r\right)\right)\\
&-\; m_0^2a^4F_3(\Omega+1)\left(3\bar{\rho}_d+4\bar{\rho}_r\right)-6k^4F_4\left(-4m_0^2(\Omega+1)\left(m_2^2-\hat{M}^2\right)\right.\\
&+\; \left.4(\hat{M}^2)^2+m_0^4(\Omega+1)^2\right)\Big\}\Big/\left\{a^2\left(k^2a^2\left(8m_2^2\left(3\bar{\rho}_d+4\bar{\rho}_r\right)\right.\right.\right.\\
&-\; \left.\left.\left.3F_3F_4-3{F_2}^2\right)+a^4F_3\left(3\bar{\rho}_d+4\bar{\rho}_r\right)-24k^4m_2^2F_4\right)\right\}\,,
\end{aligned}
\tag{3.87}
$$

$$
\begin{aligned}
G_{12} = G_{21} &= -\frac{\bar{\rho}_d\left(2\hat{M}^2+m_0^2(\Omega+1)\right)\left(-a^2(3\bar{\rho}_d+4\bar{\rho}_r)+3k^2\bar{M}_2^2+3k^2\bar{M}_3^2\right)}{k^2a^2\left(-8m_2^2(3\bar{\rho}_d+4\bar{\rho}_r)+3F_3F_4+3{F_2}^2\right)-a^4F_3(3\bar{\rho}_d+4\bar{\rho}_r)+24k^4m_2^2F_4}\,,\\
G_{13} = G_{31} &= -\frac{\bar{\rho}_r\left(2\hat{M}^2+m_0^2(\Omega+1)\right)\left(-a^2(3\bar{\rho}_d+4\bar{\rho}_r)+3k^2\bar{M}_2^2+3k^2\bar{M}_3^2\right)}{k^2a^2\left(-8m_2^2(3\bar{\rho}_d+4\bar{\rho}_r)+3F_3F_4+3{F_2}^2\right)-a^4F_3(3\bar{\rho}_d+4\bar{\rho}_r)+24k^4m_2^2F_4}\,,
\end{aligned}
$$

$$
\begin{aligned}
G_{22} &= \frac{3a^2F_4\bar{\rho}_d{}^2}{2k^2a^2\left(-8m_2^2(3\bar{\rho}_d+4\bar{\rho}_r)+3F_3F_4+3F_2{}^2\right)-2a^4F_3(3\bar{\rho}_d+4\bar{\rho}_r)+48k^4m_2^2F_4}\,,\\
G_{23} = G_{32} &= \frac{3a^2F_4\bar{\rho}_d\bar{\rho}_r}{2k^2a^2\left(-8m_2^2(3\bar{\rho}_d+4\bar{\rho}_r)+3F_3F_4+3F_2{}^2\right)-2a^4F_3(3\bar{\rho}_d+4\bar{\rho}_r)+48k^4m_2^2F_4}\,,\\
G_{33} &= \frac{\bar{\rho}_r\left(a^2\left(-4\left(m_2^2(6\bar{\rho}_d+8\bar{\rho}_r)-3F_4\bar{\rho}_r\right)+3F_3F_4+3F_2{}^2\right)+24k^2m_2^2F_4\right)}{8\left(k^2a^2\left(-8m_2^2(3\bar{\rho}_d+4\bar{\rho}_r)+3F_3F_4+3F_2{}^2\right)-a^4F_3(3\bar{\rho}_d+4\bar{\rho}_r)+24k^4m_2^2F_4\right)}\,,\\
&M_{11} = M_{12} = M_{21} = M_{13} = M_{31} = 0\,,\\
M_{22} &= -\frac{a^4\bar{\rho}_d{}^2(3\bar{\rho}_d+4\bar{\rho}_r)}{2k^2a^2\left(-8m_2^2(3\bar{\rho}_d+4\bar{\rho}_r)+3F_3F_4+3F_2{}^2\right)-2a^4F_3(3\bar{\rho}_d+4\bar{\rho}_r)+48k^4m_2^2F_4}\,,\\
M_{23} = M_{32} &= -\frac{a^4\bar{\rho}_d\bar{\rho}_r(3\bar{\rho}_d+4\bar{\rho}_r)}{2k^2a^2\left(-8m_2^2(3\bar{\rho}_d+4\bar{\rho}_r)+3F_3F_4+3F_2{}^2\right)-2a^4F_3(3\bar{\rho}_d+4\bar{\rho}_r)+48k^4m_2^2F_4}\,,\\
M_{33} &= -\frac{a^4\bar{\rho}_r(3\bar{\rho}_d+4\bar{\rho}_r)(F_3+4\bar{\rho}_r)}{8\left(k^2a^2\left(-8m_2^2(3\bar{\rho}_d+4\bar{\rho}_r)+3F_3F_4+3F_2{}^2\right)-a^4F_3(3\bar{\rho}_d+4\bar{\rho}_r)+24k^4m_2^2F_4\right)}\,.
\end{aligned}
\tag{3.88}
$$

Now, we write down the matrix coefficients of eq. (3.72) in section 3.4.3:

$$
\mathcal{L}^{(2)} = \frac{a^3}{2}\left[\dot{\bar{\Psi}}_1^2+\dot{\bar{\Psi}}_2^2+\bar{B}(t,k)\,(\dot{\bar{\Psi}}_1\bar{\Psi}_2-\dot{\bar{\Psi}}_2\bar{\Psi}_1)-\bar{C}_{ij}(t,k)\bar{\Psi}_i\bar{\Psi}_j\right],
\tag{3.89}
$$

where

$$
\begin{aligned}
\bar{B} &= -\frac{k\left(A_{22}\left(-2\dot{A}_{12}+B_{12}-B_{21}\right)+2A_{12}\dot{A}_{22}\right)}{4A_{22}\sqrt{\bar{K}_{11}}\sqrt{\bar{K}_{22}}}\,,\\
\bar{C}_{12} &= \bar{C}_{21} = \{k\left(a\left(A_{22}\left(A_{12}\left(\bar{K}_{11}\left(4\bar{K}_{22}\left(\ddot{A}_{22}+\dot{B}_{22}-2M_{22}\right)+2\dot{A}_{22}\dot{\bar{K}}_{22}\right)\right.\right.\right.\right.\right.\\
&- 2\bar{K}_{22}\dot{A}_{22}\dot{\bar{K}}_{11}-8k^2G_{22}\bar{K}_{11}\bar{K}_{22}\Big)\,4\bar{K}_{11}\bar{K}_{22}\dot{A}_{12}\dot{A}_{22}\Big)\\
&+ A_{22}^2\left(-\bar{K}_{11}\left(2\bar{K}_{22}\left(2\ddot{A}_{12}+\dot{B}_{12}+\dot{B}_{21}\right)+\dot{\bar{K}}_{22}\left(2\dot{A}_{12}-B_{12}+B_{21}\right)\right)\right.\\
&+ \bar{K}_{22}\dot{\bar{K}}_{11}\left(2\dot{A}_{12}-B_{12}+B_{21}\right)+8k^2G_{12}\bar{K}_{11}\bar{K}_{22}\Big)-4A_{12}\bar{K}_{11}\bar{K}_{22}\dot{A}_{22}^2\Big)\\
&- 6A_{22}\bar{K}_{11}\bar{K}_{22}aH\left(A_{22}\left(2\dot{A}_{12}+B_{12}+B_{21}\right)\right.\\
&- 2A_{12}\left(\dot{A}_{22}+B_{22}\right)\Big)\Big)\Big)\}/\{16aA_{22}^2\bar{K}_{11}^{-3/2}\bar{K}_{22}^{-3/2}\}\,,\\
\bar{C}_{11} &= \Big[6A_{22}\bar{K}_{11}H\left(A_{22}A_{12}\left(A_{12}\dot{\bar{K}}_{11}+B_{12}\bar{K}_{11}+B_{21}\bar{K}_{11}\right)\right.\\
&- A_{22}{}^2\left(A_{11}\dot{\bar{K}}_{11}+B_{11}\bar{K}_{11}\right)-A_{12}{}^2B_{22}\bar{K}_{11}\Big)\\
&+ 2A_{12}A_{22}\bar{K}_{11}\left(A_{12}\left(-\dot{A}_{22}\dot{\bar{K}}_{11}+\bar{K}_{11}\left(2M_{22}-\dot{B}_{22}\right)+2k^2G_{22}\bar{K}_{11}\right)\right.\\
&+ \bar{K}_{11}\dot{A}_{22}\left(4\dot{A}_{12}-B_{12}+B_{21}\right)\Big)+A_{22}{}^3\left(-2\bar{K}_{11}\left(\dot{A}_{11}\dot{\bar{K}}_{11}+A_{11}\ddot{\bar{K}}_{11}\right)\right.\\
&+ 3A_{11}\dot{\bar{K}}_{11}{}^2-2\bar{K}_{11}{}^2B'_{11}+4k^2G_{11}\bar{K}_{11}{}^2\Big)+A_{22}{}^2\left(2A_{12}\bar{K}_{11}\left(2\dot{A}_{12}\dot{\bar{K}}_{11}\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad barK_{11}\left(\dot{B}_{12}+\dot{B}_{21}\right)-4k^2G_{12}\bar{K}_{11}\Big)+2\bar{K}_{11}{}^2\dot{A}_{12}\left(-2\dot{A}_{12}+B_{12}-B_{21}\right)\\
&+ \quad A_{12}{}^2\left(2\bar{K}_{11}\dddot{\bar{K}}_{11}-3\dot{\bar{K}}_{11}{}^2\right)\Big)-4A_{12}{}^2\bar{K}_{11}{}^2\dot{A}_{22}{}^2\Big]/\left\{8A_{22}{}^3\bar{K}_{11}{}^3\right\}\,,\\
\bar{C}_{22} &= \frac{k^2}{8\bar{K}_{22}{}^3}\left[-2\bar{K}_{22}\left(\dot{A}_{22}\dot{\bar{K}}_{11}+A_{22}\ddot{\bar{K}}_{11}\right)+3A_{22}\dot{\bar{K}}_{11}{}^2+4k^2G_{22}\bar{K}_{22}{}^2\right.\\
&- \left.6\bar{K}_{22}H\left(A_{22}\dot{\bar{K}}_{11}+B_{22}\bar{K}_{22}\right)+\bar{K}_{22}{}^2\left(4M_{22}-2\dot{B}_{22}\right)\right]. \qquad (3.90)
\end{aligned}
$$

Note that the $A_{ij}, B_{ij}, G_{ij}, M_{ij}$ matrix components that appear in the last four coefficients have been obtained from the full expressions defined above by setting $\bar{\rho}_r = 0$.

3.7 Appendix B: Obtaining the Hamiltonian

In this appendix, we will present the derivation of the Hamiltonian used in section 3.4.3 and explain why the antisymmetric matrix B does not affect the unboundedness of the Hamiltonian. For this purpose we star from the Lagrangian (3.76):

$$
\mathcal{L}^{(2)} = \frac{a^3}{2}\left[\dot{\Phi}_1^2+\dot{\Phi}_2^2+B(t,k)\left(\dot{\Phi}_1\Phi_2-\dot{\Phi}_2\Phi_1\right)-\mu_1(t,k)\Phi_1^2-\mu_2(t,k)\Phi_2^2\right]. \qquad (3.91)
$$

Defining, the canonical momenta as:

$$
\begin{aligned}
p_1 &= a^3\left(\dot{\Phi}_1+B(t,k)\Phi_2\right),\\
p_2 &= a^3\left(\dot{\Phi}_2-B(t,k)\Phi_1\right), \qquad (3.92)
\end{aligned}
$$

the Hamiltonian can be written as follows

$$
\begin{aligned}
H &= \left\{p_1\left(\frac{p_1}{a^3}-B\Phi_2\right)+p_2\left(\frac{p_2}{a^3}+B\Phi_1\right)-\frac{a^3}{2}\left[\left(\frac{p_1}{a^3}-B\Phi_2\right)^2+\left(\frac{p_2}{a^3}+B\Phi_1\right)^2\right.\right.\\
&\quad \left.\left.+B\left[\left(\frac{p_1}{a^3}-B\Phi_2\right)\Phi_2-\left(\frac{p_1}{a^3}+B\Phi_1\right)\Phi_1\right]-\mu_1\Phi_1^2-\mu_2\Phi_2^2\right]\right\}\\
&= \frac{a^3}{2}\left[\left(\frac{p_1}{a^3}-B\Phi_2\right)^2+\left(\frac{p_2}{a^3}+B\Phi_1\right)^2+\mu_1\Phi_1^2+\mu_2\Phi_2^2\right]. \qquad (3.93)
\end{aligned}
$$

Now, in terms of $\{\dot{\Phi}_i, \Phi_i\}$ the above Hamiltonian becomes (3.78):

$$
H(\Phi_i,\dot{\Phi}_i) = \frac{a^3}{2}\left[\dot{\Phi}_1^2+\dot{\Phi}_2^2+\mu_1(t,k)\,\Phi_1^2+\mu_2(t,k)\,\Phi_2^2\right]. \qquad (3.94)
$$

From this expression it is clear that the antisymmetric matrix does not influence the unboundedness from below of the Hamiltonian, instead such issues are encoded within the functions μ_i.

3.8 Appendix C: Mass eigenvalues for beyond Horndeski case

In section 3.4.3 we studied the mass eigenvalues of the EFToDE/MG in the presence of a dust fluid. We have presented the procedure and the results in the case of minimally coupled quintessence but refrained from showing general expressions due to their complexity. Here we present the mass eigenvalues for the beyond Horndeski theories which, when using the appropriate mapping, will yield the previously discussed quintessence results. Then, the eigenvalues are:

$$
\begin{aligned}
\mu_1 \;=\;& \left(-\frac{4F_1F_3\left(F_2\left(F_3\dot{F}_1+F_1\dot{F}_3\right)-2F_1F_3\dot{F}_2\right)^2F_2{}^2}{3F_2{}^2+F_1F_3}\right.\\
+\;& \frac{4F_1F_3\left(F_2\left(F_3\dot{F}_1+F_1\dot{F}_3\right)-2F_1F_3\dot{F}_2\right)^2}{\frac{F_1F_3}{F_2{}^2}+3}+\left(F_1F_2{}^2\left(6F_1F_3\left(3F_2{}^2\right.\right.\right.\\
+\;& \left.F_1F_3\right)\dot{a}\left(3\dot{F}_3F_2{}^2+6F_3\left(\bar{\rho}_d-\dot{F}_2\right)F_2-2F_3{}^2\dot{F}_1\right)F_2{}^2\\
+\;& a\left(6F_3{}^3\dot{F}_1{}^2F_2{}^4+F_1{}^2F_3\left(\left(6F_3\ddot{F}_3-9\dot{F}_3{}^2\right)F_2{}^3+12F_3\left(\left(\dot{F}_2-\bar{\rho}_d\right)\dot{F}_3\right.\right.\right.\\
+\;& \left.\left.F_3\left(\dot{\bar{\rho}}_d-\ddot{F}_2\right)\right)F_2{}^2+4F_3{}^2\left(3\bar{\rho}_d\left(\dot{F}_2-\bar{\rho}_d\right)+\dot{F}_1\dot{F}_3-F_3\ddot{F}_1\right)F_2\\
-\;& \left.8F_3{}^3\dot{F}_1\dot{F}_2\right)F_2+2F_1{}^3F_3{}^2\left(F_2\dot{F}_3-2F_3\dot{F}_2\right)^2+F_1\left(9\left(2F_3\ddot{F}_3-3\dot{F}_3{}^2\right)F_2{}^6\right.\\
+\;& 36F_3\left(\left(\dot{F}_2-\bar{\rho}_d\right)\dot{F}_3+F_3\left(\dot{\bar{\rho}}_d-\ddot{F}_2\right)\right)F_2{}^5\\
-\;& \left.\left.\left.\left.12F_3{}^2\left(3\bar{\rho}_d\left(\bar{\rho}_d-\dot{F}_2\right)+F_3\ddot{F}_1\right)F_2{}^4+4F_3{}^4\dot{F}_1{}^2F_2{}^2\right)\right)\right)\right)/\left(a\left(3F_2{}^2\right.\right.\\
+\;& \left.\left.F_1F_3\right)\left(3F_2{}^2+2F_1F_3\right)\right)-\left(9F_2{}^6\left(1+\frac{1}{3\sqrt{\frac{F_2{}^4}{(3F_2{}^2+2F_1F_3)^2}}}\right)\right.\\
\times\;& \left(6F_1F_2F_3\left(3F_2{}^2+F_1F_3\right)\dot{a}\left(F_2F_3\dot{F}_1+F_1\left(2F_3\left(\bar{\rho}_d-\dot{F}_2\right)+F_2\dot{F}_3\right)\right)\right.\\
+\;& a\left(-3F_3{}^2\dot{F}_1{}^2F_2{}^4+2F_1F_3{}^2\left(3F_2{}^2\ddot{F}_1-F_3\dot{F}_1{}^2\right)F_2{}^2+F_1{}^2\left(\left(6F_3\ddot{F}_3\right.\right.\right.\\
-\;& \left.9\dot{F}_3{}^2\right)F_2{}^3+12F_3\left(\left(\dot{F}_2-\bar{\rho}_d\right)\dot{F}_3+F_3\left(\dot{\bar{\rho}}_d\right.\right.\\
-\;& \left.\left.\ddot{F}_2\right)\right)F_2{}^2+2F_3{}^2\left(6\bar{\rho}_d\left(\dot{F}_2-\bar{\rho}_d\right)-\dot{F}_1\dot{F}_3+F_3\ddot{F}_1\right)F_2+4F_3{}^3\dot{F}_1\dot{F}_2\right)F_2\\
+\;& 2F_1{}^3F_3\left(-2\left(\bar{\rho}_d{}^2-\dot{F}_2\bar{\rho}_d+\dot{F}_2{}^2+F_2\left(\ddot{F}_2-\dot{\bar{\rho}}_d\right)\right)F_3{}^2\right.\\
+\;& \left.\left.\left.\left.F_2\left(F_2\ddot{F}_3-2\left(\bar{\rho}_d-2\dot{F}_2\right)\dot{F}_3\right)F_3-2F_2{}^2\dot{F}_3{}^2\right)\right)\right)\right)/\left(a\left(3F_2{}^2+F_1F_3\right)^2\right.
\end{aligned}
$$

$$
\begin{aligned}
\times\quad & \left(3F_2{}^2+2F_1F_3\right)))\frac{1}{(16F_1{}^2F_2{}^4F_3{}^2)}\,,\\
\mu_2 \quad = \quad & \left(-\frac{4F_1F_3\left(F_2\left(F_3\dot{F}_1+F_1\dot{F}_3\right)-2F_1F_3\dot{F}_2\right)^2F_2{}^2}{3F_2{}^2+F_1F_3}\right.\\
+\quad & \frac{4F_1F_3(\left(F_2\left(F_3\dot{F}_1+F_1\dot{F}_3\right)-2F_1F_3\dot{F}_2\right)^2}{\frac{F_1F_3}{F_2{}^2}+3}+\left(9F_2{}^6\left(1+\frac{1}{3\sqrt{\frac{F_2{}^4}{(3F_2{}^2+2F_1F_3)^2}}}\right)\right.\\
\times\quad & \left(6F_1F_2F_3\left(3F_2{}^2+F_1F_3\right)\dot{a}\left(F_2F_3\dot{F}_1+F_1\left(2F_3\left(\bar{\rho}_d-\dot{F}_2\right)+F_2\dot{F}_3\right)\right)\right.\\
+\quad & a\ \left(-3F_3{}^2\dot{F}_1{}^2F_2{}^4+2F_1F_3{}^2\left(3F_2{}^2\ddot{F}_1-F_3\dot{F}_1{}^2\right)F_2{}^2\right.\\
+\quad & F_1{}^2\left(\left(\left(6F_3\ddot{F}_3-9\dot{F}_3{}^2\right)F_2{}^3+12F_3\left(\left(\dot{F}_2-\bar{\rho}_d\right)\dot{F}_3+F_3\left(\dot{\bar{\rho}}_d-\ddot{F}_2\right)\right)F_2{}^2\right.\right.\\
+\quad & 2F_3{}^2\left(6\bar{\rho}_d\left(\dot{F}_2-\bar{\rho}_d\right)-\dot{F}_1\dot{F}_3+F_3\ddot{F}_1\right)F_2+4F_3{}^3\dot{F}_1\dot{F}_2\Big)F_2\\
+\quad & 2F_1{}^3F_3\left(-2\left(\bar{\rho}_d{}^2-\dot{F}_2\bar{\rho}_d+\dot{F}_2{}^2+F_2\left(\ddot{F}_2-\dot{\bar{\rho}}_d\right)\right)F_3{}^2\right.\\
+\quad & F_2\left(F_2\ddot{F}_3-2\left(\bar{\rho}_d-2\dot{F}_2\right)\dot{F}_3\right)F_3-2F_2{}^2\dot{F}_3{}^2\Big)\Big)\Big)\Big)\Big)\Big/\left(a\left(3F_2{}^2+F_1F_3\right)^2\right.\\
\times\quad & \left(3F_2{}^2+2F_1F_3\right)\Big)-\left(2F_2{}^2F_3\left(6F_1F_3\left(3F_2{}^2+F_1F_3\right)\dot{a}\left(9\dot{F}_1F_2{}^4+6F_1F_3\dot{F}_1F_2{}^2\right.\right.\right.\\
+\quad & F_1{}^2\left(2F_3\left(F_3\dot{F}_1+3F_2\left(\dot{F}_2-\bar{\rho}_d\right)\right)-3F_2{}^2\dot{F}_3\right)\Big)F_2{}^2\\
+\quad & a\left(-27F_3\dot{F}_1{}^2F_2{}^8+18F_1F_3\left(3F_2{}^2\ddot{F}_1-2F_3\dot{F}_1{}^2\right)F_2{}^6+18F_1{}^2F_3\left(3F_2{}^2F_3\ddot{F}_1\right.\right.\\
-\quad & \dot{F}_1\left(\dot{F}_3F_2{}^2-2F_3\dot{F}_2F_2+F_3{}^2\dot{F}_1\right)\Big)F_2{}^4+2F_1{}^3\left(9\left(\dot{F}_3{}^2-F_3\ddot{F}_3\right)F_2{}^4\right.\\
+\quad & 18F_3\left(\bar{\rho}_d\dot{F}_3+F_3(t)\left(\ddot{F}_2-\dot{\bar{\rho}}_d\right)\right)F_2{}^3+6F_3{}^2\left(3\bar{\rho}_d{}^2\right.\\
-\quad & 3\dot{F}_2\bar{\rho}_d-3\dot{F}_2{}^2-\dot{F}_1\dot{F}_3+2F_3\ddot{F}_1\Big)F_2{}^2+12F_3{}^3\dot{F}_1\dot{F}_2F_2-2F_3{}^4\dot{F}_1{}^2\Big)F_2{}^2\\
+\quad & F_1{}^4F_3\left(3\left(\dot{F}_3{}^2-2F_3\ddot{F}_3\right)F_2{}^2+12F_3\left(\left(\bar{\rho}_d+\dot{F}_2\right)\dot{F}_3+F_3\left(\ddot{F}_2-\dot{\bar{\rho}}_d\right)\right)F_2{}^2\right.\\
+\quad & 4F_3{}^2\left(3\left(\bar{\rho}_d+\dot{F}_2\right)\left(\bar{\rho}_d-2\dot{F}_2\right)-\dot{F}_1\dot{F}_3+F_3\ddot{F}_1\right)F_2+8F_3{}^3\dot{F}_1\dot{F}_2\Big)F_2\\
-\quad & 2F_1{}^5F_3{}^2\left(F_2\dot{F}_3-2F_3\dot{F}_2\right)^2\Big)\Big)\Big/\left(a\left(3F_2{}^2+F_1F_3\right)^2\left(3F_2{}^2+2F_1F_3\right)\right)\Big)\\
/\quad & \left(16F_1{}^2F_2{}^4F_3{}^2\right)\,. && (3.95)
\end{aligned}
$$

4 de Sitter limit analysis for dark energy and modified gravity models

4.1 Introduction

In the previous Chapter we presented a choice of variables which allowed us to obtain a Hamiltonian of the form:

$$\mathbf{H}(\Phi_i, \dot{\Phi}_i) = \frac{a^3}{2}\left[\dot{\Phi}_1^2 + \dot{\Phi}_2^2 + \mu_1(t,k)\,\Phi_1^2 + \mu_2(t,k)\,\Phi_2^2\right], \tag{4.1}$$

where $\Phi_i(t,k)$ are two linear combinations of the physical fields $\zeta(t,k)$, the curvature perturbation, and $\delta\rho_d(t,k)$, the perturbation of the dust energy density. This Hamiltonian was then the basis from which we constructed the conditions guaranteeing the absence of tachyonic instabilities on large scales. As the final set of variables is a mix of the initial, gauge-dependent quantities, one might naturally wonder if a different choice of variables would yield different results. Thus we wished to study this question and compare the gauge-dependent variables with a choice of gauge-independent variables.

In order to get an insight into the dependence of the mass on the choice of variables we will choose a gauge invariant combination which will describe the perturbations. Then, we will make a change of coordinates to this new field, δ_ϕ, and proceed to study the mass. In order to simplify the comparison we will choose to study the final-state de Sitter (dS) background. This is a reasonable choice as our universe already experiences a dark-energy dominated phase. On doing this, we will employ the EFToDE/MG formalism which allows for late-time dS solutions citeCreminelli:2006xe. Since we restrict our attention to the dS background, we will have one, and only one, propagating scalar d.o.f., because matter fields are sub-dominant. Then, it is possible to exactly define the speed of propagation and the mass of this gauge-invariant field representing δ_ϕ. Even though the value for the mass of the dark energy field is exact only on the dS background, it is expected to be a reliable approximation for its value at late times, i.e. when $z \simeq 0$.

Besides the mass we will proceed to investigate, during the dS stage, the behaviour of the speed of propagation in a model independent fashion. On doing so we need to consider the limit $k/(aH) \gg 1$ as a potential gradient instability might manifest itself at those scales. However, on dS, as time progresses one needs to consider increasingly larger values for k, as a grows exponentially (whereas H remains constant). Subsequently, as the system evolves, the same modes will be rapidly stretched to cosmological scales. Now, in general, we find that the speed of propagation for the dark energy perturbation does not necessarily vanish, even for the Horndeski subclasses of theories. In fact, the numerical value of the speed of propagation is model dependent, and its non-negativity can be set as a constraint in order to have a final stable dS. If this constraint is not satisfied (i.e. $c_s^2 < 0$) then we will expect that the late time evolution cannot evolve towards a dS background even though at the level of the background the dS case is an attractor solution. On the other hand, for lower value of $k/(aH)$, the mass of the mode will play a more important role. In this case one needs to impose, in general, a constraint on the value of the mass for the dark energy perturbation field in order to obtain a stable dS.

A final source of instability might show up for those theories which exhibit a small or vanishing speed of propagation. In this case the sub-leading order term in the high $k/(aH)$ expansion becomes relevant and can potentially lead to unstable solutions. We will discuss this in depth and we will present the necessary constraints in order to avoid such instability.

The work in this Chapter is based on [33]: *de Sitter limit analysis for dark energy and modified gravity models* with A. De Felice and N. Frusciante. In Sec. 4.2 we give a general overview of the EFToDE/MG approach for dark energy and modified gravity and we introduce a gauge invariant quantity to describe the dark energy field. In Sec. 4.3 we show that the parameter space identified by imposing the no-ghost condition and a positive speed of propagation for scalar modes does not change when considering different quantities describing the dynamics of the extra d.o.f.. In Sec. 4.4, we discuss the dS limit by using the EFToDE/MG framework, we discuss the evolution of the extra scalar d.o.f. on different regime, i.e. low and large k, by deriving the speed of propagation and the mass term. In Sec. 4.5, in order to make our results concrete we apply them to specific well known models, such as K-essence, Galileons and low-energy Hořava gravity. Finally, in Sec. 4.6 we summarize and discuss potential future steps.

4.2 Modifying General Relativity

In the present analysis we will employ a general and unifying approach to parametrize any deviation from General Relativity obtained by including one extra scalar d.o.f. in the action, i.e. the effective field theory for Dark Energy and Modified Gravity [14, 15]. For the present purpose the EFToDE/MG approach has the advantage of keeping our results very general as all the well known theories of gravity with one extra scalar d.o.f. can be cast in the EFToDE/MG framework[14, 15, 37, 39, 65].

The EFToDE/MG is constructed in the unitary gauge, i.e. uniform time hypersurfaces correspond to uniform field hypersurfaces. This results in the scalar perturbation being absorbed by the metric. Let us now introduce the action which can be constructed by solely geometric quantities. The general form is:

$$\begin{aligned}\mathcal{S}^{(2)} = \int d^4x\sqrt{-g}\bigg[&\frac{m_0^2}{2}(1+\Omega(t))R^{(4)} + \Lambda(t) - c(t)\delta g^{00} + \frac{M_2^4(t)}{2}(\delta g^{00})^2 \\ &- \frac{\bar{M}_1^3(t)}{2}\delta g^{00}\delta K - \frac{\bar{M}_2^2(t)}{2}(\delta K)^2 - \frac{\bar{M}_3^2(t)}{2}\delta K^\mu_\nu \delta K^\nu_\mu \\ &+ \frac{\hat{M}^2(t)}{2}\delta g^{00}\delta R^{(3)} + m_2^2(t)\left(g^{\mu\nu}+n^\mu n^\nu\right)\partial_\mu g^{00}\partial_\nu g^{00}\bigg], \end{aligned} \tag{4.2}$$

where as usual m_0^2 is the Planck mass, $g_{\mu\nu}$ and g are respectively the four dimensional metric and its determinant, $\delta g^{00} = 1 + g^{00}$, whereas $R^{(4)}$ and $R^{(3)}$ are respectively the trace of the four dimensional and three dimensional Ricci scalar, n_μ is the normal vector, $K_{\mu\nu}$ and K are the extrinsic curvature and its trace. All the operators appearing in the action are invariant under the time dependent spatial-diffeomorphisms and they are expanded in perturbations up to second order around a flat Friedmann-Lemaître-Robertson-Walker (FLRW) background. The notation $\delta A = A - A^{(0)}$ indicates the linear perturbation of the operator A with $A^{(0)}$ its background value. The functions appearing in front of each operator are unknown functions of time and usually they are named EFT functions. In particular, $\{\Omega(t), c(t), \Lambda(t)\}$ are called background EFT functions because these are the only functions that appear in the background Friedmann equations. Finally one can opt to work directly with the field perturbation by restoring the full diffeomorphism invariance, through the Stückelberg technique. This step is useful either when the gauge is not well defined or when studying the evolution of the perturbations with numerical tool, such as *EFTCAMB/EFTCosmoMC* [**eftweb**, 22, 23, 80].

For the present purpose we adopt the action (4.2), which includes theories like Horndeski/Generalized Galileon [78, 79], beyond Horndeski

(GLPV) [66] and low-energy Hořava gravity [35, 36, 55], without the new operators presented in Chapter 2.

Now, let us use the Arnowitt-Deser-Misner (ADM) formalism [19] and expand the line element around the flat FLRW background. Keeping only the scalar part of the metric, we get

$$ds^2 = -(1+2\delta N)dt^2 + 2\partial_i\psi dt dx^i + [a^2(1+2\zeta)\delta_{ij} + 2\partial_i\partial_j\gamma]\, dx^i dx^j \,, \quad (4.3)$$

where as usual $\delta N(t,x^i)$ is the perturbation of the lapse function, $\partial_i\psi(t,x^i)$, $\zeta(t,x^i)$ and $\gamma(t,x^i)$ are the scalar perturbations respectively of N_i and of the metric tensor of the three dimensional spatial slices, h_{ij}, and $a(t)$ is the scale factor. In the following, since we choose the unitary gauge, we also set $\gamma(t,x^i) = 0$.

We have shown in Chapter 3 that the above EFToDE/MG action can be written as:

$$\begin{aligned} \mathcal{S}^{(2)} &= \int dt d^3x a^3 \Bigg\{ -\frac{F_4(\partial^2\psi)^2}{2a^4} - \frac{3}{2}\,F_1\dot{\zeta}^2 + m_0^2(\Omega+1)\,\frac{(\partial\zeta)^2}{a^2} - \frac{\partial^2\psi}{a^2}\left(F_2\delta N - F_1\dot{\zeta}\right) \\ &+ 4m_2^2\frac{[\partial(\delta N)]^2}{a^2} + \frac{F_3}{2}\,\delta N^2 + \left[3F_2\dot{\zeta} - 2\left(m_0^2(\Omega+1) + 2\hat{M}^2\right)\frac{\partial^2\zeta}{a^2}\right]\delta N \Bigg\}, \quad (4.4) \end{aligned}$$

where we have defined

$$\begin{aligned} F_1 &= 2m_0^2(\Omega+1) + 3\bar{M}_2^2 + \bar{M}_3^2 \,, \\ F_2 &= HF_1 + m_0^2\dot{\Omega} + \bar{M}_1^3 \,, \\ F_3 &= 4M_2^4 + 2c - 3H^2F_1 - 6m_0^2H\dot{\Omega} - 6H\bar{M}_1^3 \,, \\ F_4 &= \bar{M}_2^2 + \bar{M}_3^2 \,, \end{aligned} \quad (4.5)$$

and $H \equiv \dot{a}/a$ is the Hubble function and δN and ψ are auxiliary fields. Varying the action with respect to δN and ψ yields the Hamiltonian and momentum constraints:

$$\begin{aligned} &\frac{2k^2\zeta\left(2\hat{M}^2 + m_0^2(\Omega+1)\right)}{a^2} + 3F_2\dot{\zeta} + \frac{8m_2^2k^2\delta N}{a^2} + F_2\frac{k^2\psi}{a^2} + F_3\delta N = 0\,, \\ &\delta N F_2 - F_1\dot{\zeta} - \frac{F_4}{a^2}k^2\psi = 0\,. \end{aligned} \quad (4.6)$$

Finally, solving for the auxiliary fields one can eliminate them from the action, hence obtaining the following Lagrangian, written in compact form in 3D Fourier space:

$$\mathcal{S}^{(2)} = \int d^4x\, a^3 \left\{ \mathcal{L}_{\dot{\zeta}\dot{\zeta}}(t,k)\dot{\zeta}^2 - \frac{k^2}{a^2}G(t,k)\zeta^2 \right\} , \quad (4.7)$$

where

$$\mathcal{L}_{\dot{\zeta}\dot{\zeta}}(t,k) = \frac{\mathcal{A}_1(t) + \frac{k^2}{a^2}\mathcal{A}_4(t)}{\mathcal{A}_2(t) + \frac{k^2}{a^2}\mathcal{A}_3(t)}\,, \qquad G(t,k) = \frac{\mathcal{G}_1(t) + \frac{k^2}{a^2}\mathcal{G}_2(t) + \frac{k^4}{a^4}\mathcal{G}_3}{(\mathcal{A}_2(t) + \frac{k^2}{a^2}\mathcal{A}_3(t))^2}\,, \tag{4.8}$$

are respectively the kinetic and gradient term. The $A_i(t)$ and $G_i(t)$ coefficients are listed in Appendix 4.7 for a general FLRW background. In the next section they will be specified in the dS limit.

Besides the curvature perturbation $\zeta(t,k)$ one can choose to undo the unitary gauge and work directly with the Stückelberg field, namely π, by performing a broken time translation $t \to t - \pi(t,\vec{x})$. In order to obtain an unperturbed metric after the translation one needs to recognize that $\zeta = -H\pi$ [131]. However, these fields are not gauge invariant. In this work, we will define a gauge invariant quantity which will describe the evolution of the dark energy field at level of perturbations. Let us introduce the one-form

$$n_\mu = \frac{\partial_\mu \phi}{\sqrt{-g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi}} = \frac{\delta^0_\mu}{\sqrt{-g^{00}}}\,, \tag{4.9}$$

which would define the 4-velocity along the field-fluid. On the other hand, looking for deviation from General Relativity, when the matter fields are negligible we can can rewrite the Einstein equations as follows

$$m_0^2\, G_{\mu\nu} = T^\phi_{\mu\nu}\,. \tag{4.10}$$

This equation can always be written, and the modifications of gravity have been named in terms of its effective stress-energy tensor, $T^\phi_{\mu\nu}$, independently of the EFToDE/MG which we are considering. Therefore, we can define

$$\rho_\phi \equiv T^\phi_{\mu\nu}\, n^\mu n^\nu = m_0^2\, G_{\mu\nu} n^\mu n^\nu\,, \tag{4.11}$$

where the second part of this equation holds on-shell, that is, on implementing the equations of motion (at any order). Notice that the definition given in eq. (4.11) is covariant and, as such, valid even at non-linear order, and does not depend on the choice of the gauge. Since we want the results to match a more phenomenological approach we will define, at linear order the following gauge invariant combination to describe the dark energy field, namely

$$\delta_\phi \;\equiv\; \frac{\delta\rho_\phi}{\bar{\rho}_\phi} + \frac{\dot{\bar{\rho}}_\phi}{\bar{\rho}_\phi}\left[\psi - a^2\,\frac{d}{dt}\left(\frac{\gamma}{a^2}\right)\right], \tag{4.12}$$

where, using the background Friedmann equation from action (4.2) and assuming that no matter fields are present, on the background we can define

$$\bar{\rho}_\phi = 2c - \Lambda - 3m_0^2 H^2 \left(\Omega + a\Omega_{,a}\right), \tag{4.13}$$

and

$$\delta\rho_\phi \equiv \rho_\phi - \bar{\rho}_\phi \,. \tag{4.14}$$

We notice here that δ_ϕ reduces to $\delta\rho_\phi/\bar{\rho}_\phi$ in the Newtonian gauge. Comma a is the derivative with respect to the scale factor.

We will find the equation of motion for δ_ϕ which in general assumes the following from

$$\ddot{\delta}_\phi + \mu_3(t,k)\,\dot{\delta}_\phi + \mu_6(t,k)\,\delta_\phi = 0 \,. \tag{4.15}$$

The coefficient of $\dot{\delta}_\phi$ is the friction term and its sign will damp or enhance the amplitude of the field fluctuations. While μ_6 contains both the speed of propagation of the dark energy field and the information about of the mass which, in principle, can be both negative or positive. The above equation will allow us to define the mass of the dark energy perturbation field, which in the next section will be exact on the de Sitter background, and approximate at low redshifts, $z \simeq 0$.

4.3 The Ghost and Gradient instabilities

By studying the curvature perturbation field, one can immediately work out the stability conditions, namely the no-ghost condition, the positive speed of propagation and the tachyonic condition as weas done in Chapter 2 and 3. The first two conditions, i.e. the combination of no-ghost and positive-squared-speed conditions, give equivalents constraints for both the ζ and δ_ϕ fields, in the high-k regime [126]. We will show it in the following. Let us consider the action (4.7) and the field transformation

$$\delta_\phi = \alpha_3(t,k)\dot{\zeta} + \alpha_6(t,k)\zeta. \tag{4.16}$$

We will show in the following section that it is possible to derive this relation and find explicit expressions for $\{\alpha_3, \alpha_6\}$. For the moment we assume that such an expression exist, since we have only one independent d.o.f. (the curvature perturbation, ζ), so that any other field (for example δ_ϕ in this case) can be constructed out of a linear combination of ζ and its first time derivative $\dot{\zeta}$. Then, on introducing an arbitrary function, $E(t,k)$ (note, it is not a field), we can construct the action

$$\mathcal{S}^{(2)} = \int d^4x\, a^3 \left\{ \mathcal{L}_{\dot{\zeta}\dot{\zeta}}(t,k)\dot{\zeta}^2 - \frac{k^2}{a^2}\, G(t,k)\zeta^2 - E(t,k)\left(\delta_\phi - \alpha_3\dot{\zeta} - \alpha_6\zeta\right)^2 \right\}, \tag{4.17}$$

and it is clear that δ_ϕ is a Lagrange multiplier so that we can use its own equation of motion to remove it from the action. On performing this step we can see that eq. (4.17) reduces to eq. (4.16). This step may look like superfluous, but it allows us to change the dynamical field variable in the Lagrangian from ζ to δ_ϕ. Thus, since E is a free function, if $\alpha_3 \neq 0$, on choosing it to be $E = \mathcal{L}_{\dot\zeta\dot\zeta}/\alpha_3^2$, we immediately see that the kinetic quadratic term proportional to $\dot\zeta^2$ disappears and the action can be rewritten, after integrations by parts, as

$$\begin{aligned}\mathcal{S}^{(2)} &= \int d^4x\, a^3 \Bigg\{ \left[\frac{(H\,(\eta_\mathcal{L} - \eta_3 + \eta_6 + 3)\,\alpha_3 - \alpha_6)\,\alpha_6 \mathcal{L}_{\dot\zeta\dot\zeta}}{\alpha_3^2} - \frac{k^2}{a^2} G \right] \zeta^2 \\ &+ \left(-\frac{2\mathcal{L}_{\dot\zeta\dot\zeta}\dot\delta_\phi}{\alpha_3} + \frac{(-2\,H\,(\eta_\mathcal{L} - \eta_3 + 3)\,\alpha_3 + 2\,\alpha_6)\,\mathcal{L}_{\dot\zeta\dot\zeta}\delta_\phi}{{\alpha_3}^2} \right) \zeta \\ &- \frac{\delta_\phi^2 \mathcal{L}_{\dot\zeta\dot\zeta}}{\alpha_3^2} \Bigg\}, \end{aligned} \tag{4.18}$$

where we have defined

$$\eta_\mathcal{L} \equiv \frac{\dot{\mathcal{L}}_{\dot\zeta\dot\zeta}}{H\mathcal{L}_{\dot\zeta\dot\zeta}}, \qquad \eta_3 \equiv \frac{\dot\alpha_3}{H\alpha_3}, \qquad \eta_6 \equiv \frac{\dot\alpha_6}{H\alpha_6}. \tag{4.19}$$

Therefore, we have succeeded to make ζ become a Lagrange multiplier and, as such, in general, it can be integrated out (using its own equation of motion), leaving δ_ϕ as the propagating independent scalar d.o.f..

It should be noted, that integrating out ζ is only possible whenever the term proportional to ζ^2 in Eq. (4.18) does not vanish. If this case occurs, as we shall see later on happening in some theories for which both α_6 and G vanish, then the field δ_ϕ cannot be chosen as the independent field used to describe the system of scalar perturbations.

After removing the auxiliary field ζ, we can rewrite the action as

$$\mathcal{S}^{(2)} = \int d^4x\, a^3 \left[\frac{a^2}{k^2} \left(Q(t,k)\, \dot\delta_\phi^2 - \mathcal{G}(t,k)\, \frac{k^2}{a^2}\, \delta_\phi^2 \right) \right], \tag{4.20}$$

where the coefficients are listed in Appendix 4.7. Therefore, the no-ghost condition for the field δ_ϕ can be read as

$$\lim_{\frac{k}{aH}\to\infty} Q = \lim_{\frac{k}{aH}\to\infty} \frac{\mathcal{L}_{\dot\zeta\dot\zeta}^2}{G\alpha_3^2} = \frac{\mathcal{A}_3(t)^2}{\mathcal{G}_3(t)} \lim_{\frac{k}{aH}\to\infty} \frac{\mathcal{L}_{\dot\zeta\dot\zeta}^2}{\alpha_3^2} > 0\,, \tag{4.21}$$

which implies

$$\mathcal{G}_3(t) > 0\,, \tag{4.22}$$

and we have assumed that for any function $f(t,k)$ in the Lagrangian, we have, for large k's, that $f(t,k) = \bar{f}(t) + \mathcal{O}(k^{-2})$. If the previous assumption does not hold, then we need to discuss case by case what happens for the limit. On using again the above assumption, the speed of propagation can be defined as

$$c_s^2 = \lim_{\frac{k}{aH}\to\infty} \frac{\mathcal{G}}{Q} = \lim_{\frac{k}{aH}\to\infty} \frac{G}{\mathcal{L}_{\dot{\zeta}\dot{\zeta}}} = \frac{\mathcal{G}_3(t)}{\mathcal{A}_3(t)\mathcal{A}_4(t)}\,, \tag{4.23}$$

which we require to be positive defined. On combining both the constraints we find

$$\mathcal{A}_3(t)\,\mathcal{A}_4(t) > 0\,. \tag{4.24}$$

If we consider the stability conditions defined by the field ζ, we find the no-ghost condition

$$\lim_{\frac{k}{aH}\to\infty} \mathcal{L}_{\dot{\zeta}\dot{\zeta}} = \frac{\mathcal{A}_4(t)}{\mathcal{A}_3(t)} > 0\,, \tag{4.25}$$

which, together with

$$c_s^2 = \lim_{\frac{k}{aH}\to\infty} \frac{G}{\mathcal{L}_{\dot{\zeta}\dot{\zeta}}} = \frac{\mathcal{G}_3(t)}{\mathcal{A}_3(t)\mathcal{A}_4(t)} \geq 0\,, \tag{4.26}$$

imply $\mathcal{G}_3 > 0$. Thus, both fields propagate with the same speed. Note that these results apply on a general FLRW background.

This calculation shows that the no-ghost condition and the speed of propagation must be calculated in the high-k regime and in such a limit they become invariants, meaning that they do not change when we change the propagating scalar d.o.f.. It should be noticed that the no-ghost conditions do not coincide but the final set of conditions do for ζ and δ_ϕ.

Since the mass term is not a quantity which is sensitive to the high k regime, we should not expect, in general, it behaves as an invariant. Therefore, each propagating field will have its own mass. However, here we are considering physical fields, i.e. fields for which we can attach a clear physical meaning and both δ_ϕ and ζ need to remain less than unity for the background to be stable. Therefore, a mass instability for δ_ϕ, leading this field to reach unity, will imply in general some instability for the field ζ and viceversa. In order to find the mass of the field δ_ϕ we will investigate its equation of motion. We will perform this calculation in the following sections.

4.4 The de Sitter Limit

In this section we will consider the EFToDE/MG action (4.7) in the limit of a dS universe. Such a limit is a good approximation in those regimes in which the dark energy component is dominant over any matter fluids, e.g. very late time. In this case the background Friedmann equation simply reduces to

$$3m_0^2 H_0^2 = \bar{\rho}_\phi, \tag{4.27}$$

where the dark energy density, $\bar{\rho}_\phi$ has been defined in eq. (4.13). From the assumption of a dS universe, it follows that $H = \text{const} = H_0$ and the dark energy density is a constant as well. Therefore, eq. (4.13) is a constraint. As a result the dark energy density acts like a cosmological constant. As it is well known such a realization can be obtained, beside the cosmological constant itself, by considering a modified gravity theory with a scalar field whose solution can mimic such a behaviour. Then, eq. (4.27) can be integrated and one immediately gets

$$a(t) = a_0 e^{tH_0}, \tag{4.28}$$

where a_0 is an integration constant.

The EFToDE/MG approach preserves a direct link with those theories of modified gravity which show one extra scalar d.o.f. and they can be fully mapped in the EFToDE/MG language as in Chapter 2 and refs. [14, 15, 37, 39, 65]. Then, by using the mapping with specific theories and the solution in the dS limit for the chosen theories, we can deduce the behaviour of the EFT functions. In case of Horndeski [78] or Generalized Galileon [79] and beyond Horndeski/GLPV [66], when the shift symmetry is applied, the dS universe can be realized when the kinetic term is a constant, i.e. $X = -\dot{\phi}^2 = const$ [132, 133]. In this case all the EFT functions are constants and the constraint (4.13) is always satisfied. K-essence models [134] also admit a dS limit with $\dot{\phi} = const$, when the general function of the kinetic term, namely $\mathcal{K}(X)$, has a polynomial form. In this case the roots of the polynomial obtained by solving the equation $d\mathcal{K}/dX = 0$ are the constant values for the derivative of the field. A more general class of theories is the one with $m_2^2 \neq 0$, to which low-energy Hořava gravity [35, 36, 55] belongs. Such theory admits a dS solution [46, 135] and also in this case the EFT functions are constants. We will assume that the EFT functions on a dS background for all theory having $m_2^2 \neq 0$ are constant. In the following, assuming constant EFT functions will greatly simplify the whole treatment.

Moreover, by assuming $\Omega = const$ in the dS limit the EFToDE/MG background equations reduce to the following forms

$$
\begin{aligned}
&3m_0 H_0^2(1+\Omega)+\Lambda=0\,,\\
&3m_0 H_0^2(1+\Omega)+\Lambda-2c=0. \qquad (4.29)
\end{aligned}
$$

Then, it is easy to deduce the following relations

$$
c=0\,,\quad \bar{\rho}_\phi=-\frac{\Lambda}{1+\Omega}. \qquad (4.30)
$$

The generality of the EFToDE/MG approach in describing linear modifications of gravity due to an extra scalar d.o.f., allows us to perform a very general analysis in the dS limit for a wide range of theories. However, it is worth to notice that a unique treatment is not possible because subclasses of models, corresponding to specific choices of EFT functions are expected to show up. Therefore, in the following we will mainly consider three subclasses corresponding to

- General case: $\{F_4, m_2^2\}\neq 0$, to this class belong all models with higher then two spatial derivatives;
- Beyond Horndeski (or GLPV) models: $\{F_4, m_2^2\}=0$;
- Hořava gravity-like models: $m_2^2\neq 0$ and $3F_2^2+F_3F_1=0$.

For all of them we will study the behaviours of the curvature perturbation, $\zeta(t,k)$ as well as of the gauge independent quantity describing the dark energy field $\delta_\phi(t,k)$.

4.4.1 The general case

We will now investigate the stability of the dS universe in the general case, i.e. by assuming all operators to be active. In contrast to the next cases this corresponds to the case $\{F_4, m_2^2\}\neq 0$. The kinetic and gradient terms for this case have the same form as in (4.8), where now the terms $\mathcal{A}_i$ and $\mathcal{G}_i$ are constants and they can be obtained from the time dependent expressions in the Appendix 4.7 by setting all the EFT functions to be constant. They are:

$$
\begin{aligned}
\mathcal{L}_{\dot\zeta\dot\zeta}(t,k) &= \frac{(F_1-3F_4)\left(\left(3F_2^2+F_1F_3\right)+8\frac{k^2}{a(t)^2}F_1m_2^2\right)}{2\left(\left(F_2^2+F_3F_4\right)+8\frac{k^2}{a(t)^2}F_4m_2^2\right)}\,, \qquad (4.31)\\
G(t,k) &= \left(16F_4^2m_2^2\left(-4m_0^2(\Omega+1)\left(m_2^2-\hat{M}^2\right)+4\hat{M}^4+m_0^4(\Omega+1)^2\right)\frac{k^4}{a^4}\right.\\
&+ 8F_4\left(4m_0^2(\Omega+1)\left(F_2^2\left(\hat{M}^2-2m_2^2\right)+F_3F_4\left(\hat{M}^2-2m_2^2\right)\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
+ &\quad 3\left(F_1-3F_4\right)F_2H_0m_2^2\big)+4\hat{M}^2\Big(F_2^2\hat{M}^2+F_3F_4\hat{M}^2\\
+ &\quad 6\left(F_1-3F_4\right)F_2H_0m_2^2\big)+\left(F_2^2+F_3F_4\right)m_0^4(\Omega+1)^2\big)\frac{k^2}{a^2}\\
+ &\quad F_2\left(F_1-3F_4\right)\left(F_2^2+F_3F_4\right)H_0\Big(2\hat{M}+m_0^2(\Omega+1)\Big)\\
- &\quad \left(F_2^2+F_3F_4\right){}^2m_0^2(\Omega+1)\big)\Big/\left(\left(\left(F_2^2+F_3F_4\right)+8F_4\frac{k^2}{a(t)^2}m_2^2\right){}^2\right). \qquad (4.32)
\end{aligned}
$$

We assume that $\{F_2,(F_1-3F_4),2\hat{M}^2+m_0^2(\Omega+1)\}\neq 0$, leaving the treatment of these special cases at the end of this section. Now from action (4.7), one can derive the field equation for the curvature perturbation, ζ, in the dS limit, which reads

$$
\ddot{\zeta}+\left(3H_0+\frac{\dot{\mathcal{L}}_{\dot{\zeta}\dot{\zeta}}}{\mathcal{L}_{\dot{\zeta}\dot{\zeta}}}\right)\dot{\zeta}+\frac{k^2}{a(t)^2}\frac{G}{\mathcal{L}_{\dot{\zeta}\dot{\zeta}}}\zeta=0\,. \qquad (4.33)
$$

We notice that in the above equation there is no dispersion coefficient.

Let us now analyse two limiting cases of the above equation. In the limiting case in which k^2/a^2 is small, the term proportional to ζ in the above equation is sub-dominant and it can be neglected, thus the curvature perturbation behaves as follows

$$
\zeta(t)=C_2-\frac{C_1e^{-3H_0t}}{3H_0}\,, \qquad (4.34)
$$

where C_i are integration constant. Because the second term is a decaying mode, we can deduce from the above result that the curvature perturbation is conserved. On the contrary, when k^2/a^2 really matters, the equation of ζ reduces to

$$
\ddot{\zeta}+3H_0\dot{\zeta}+\left(\frac{k^2}{a(t)^2}c_s^2+\tilde{\mu}_{un}\right)\zeta=0\,, \qquad (4.35)
$$

where we have defined the squared speed of propagation of the mode ζ at high-k as in eq. (4.26) and $\tilde{\mu}_{un}$ is the next to leading order term in the high-k expansion of $G/\mathcal{L}_{\dot{\zeta}\dot{\zeta}}$. We will refer to $\tilde{\mu}_{un}$ as the undamped effective mass of the mode. When considering a dS background these two terms assume the following constant form

$$
\begin{aligned}
c_s^2 &= \frac{\mathcal{G}_3}{\mathcal{A}_3\mathcal{A}_4}=\frac{F_4\left(-4m_0^2(\Omega+1)\left(m_2^2-\hat{M}^2\right)+4\hat{M}^4+m_0^4(\Omega+1)^2\right)}{2F_1\left(F_1-3F_4\right)m_2^2},\\
\tilde{\mu}_{un} &= -(-12F_1\left(F_1-3F_4\right)F_2H_0m_2^2\Big(2\hat{M}^2+m_0^2(\Omega+1)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad F_2^2 \left(3F_4 \left(-4m_0^2(\Omega+1)\left(m_2^2-\hat{M}^2\right)+4\hat{M}^4\right.\right. \\
&+ \quad \left. m_0^4(\Omega+1)^2\right) + 4F_1 m_2^2 m_0^2(\Omega+1)\Big) \\
&+ \quad F_1F_3F_4\left(2\hat{M}^2+m_0^2(\Omega+1)\right)^2\Big)/(16F_1^2\,(F_1-3F_4)\,m_2^4). \qquad (4.36)
\end{aligned}
$$

Now, let us consider (4.35) for a general friction coefficient, χ. Then for high-k, we choose an approximate plane wave solution of the form $\zeta \propto \exp(-i\,\omega\,t)$, which, after substituting in the previous equation we get the following algebraic equation:

$$
-\omega^2 - \chi i H_0 \omega + \left(\frac{c_s^2 k^2}{a(t)^2} + \tilde{\mu}_{un}\right) = 0\,, \tag{4.37}
$$

The equation has the following solution

$$
\omega = -\frac{\chi}{2}\,H_0\,i \pm \omega_0\,, \tag{4.38}
$$

where

$$
\omega_0 \equiv \sqrt{\frac{c_s^2 k^2}{a(t)^2} + \tilde{m}^2}\,, \qquad \tilde{m}^2 \equiv \tilde{\mu}_{un} - \frac{\chi^2}{4}\,H_0^2\,, \tag{4.39}
$$

and $\tilde{m}^2$ represents the damped mass of the oscillatory part of the solution. The imaginary part of ω corresponds instead to the decaying (damped) part of the solution. Since we are in the high-k regime, we expect that in general $\frac{c_s^2 k^2}{a^2} + \tilde{m}^2 > 0$. In this case we are in the presence of an underdamped oscillator, for which the solution reads

$$
\zeta(t) \approx e^{-\chi H_0 t/2}(C_1 \cos\omega_0 t + C_2 \sin\omega_0 t)\,,
$$

and no instability occurs.

Now, an example of where the next to leading order term becomes relevant for stability is when the speed of sound is small or vanishing, i.e. $c_s^2 \simeq 0$. Then, when $\tilde{m}^2 < 0$, one has $\frac{c_s^2 k^2}{a^2} + \tilde{m}^2 < 0$, yielding the following solution:

$$
\zeta(t) \approx e^{-\chi H_0 t/2}(C_1 e^{-|\omega_0| t} + C_2 e^{|\omega_0| t})\,, \tag{4.40}
$$

which represents overdamped solutions when $|\omega_0| < \chi H_0/2$. On the other hand, if the model has $\frac{c_s^2 k^2}{a^2} + \tilde{m}^2 < 0$ and $|\omega_0| > \chi H_0/2$, then the mode

$$
\zeta(t) \propto e^{(-\frac{\chi}{2}H_0 + |\omega_0|)t} \tag{4.41}
$$

is exponentially growing. For $c_s^2 \simeq 0$ and $\tilde{m}^2 < 0$ this implies a catastrophic instability when

$$\tilde{\mu}_{un} < 0 \qquad \text{and} \qquad |\tilde{\mu}_{un}| \gg H_0^2. \tag{4.42}$$

Besides the case described above, when $c_s^2 < 0$ and $|\tilde{\mu}_{un}| \simeq H_0$ another instability arises. This is then the usual gradient instability.

The above discussion is directly applicable to the eq. (4.35) presented in this section when $\chi = 3$. We will show that the above arguments will be still valid in the high-k-limit of the dark energy field for the general case as well as for the other sub-cases discussed in the following, for which one will only need to employ this analysis for different values of χ. In such instances we will refer back to this paragraph instead of repeating the whole discussion.

However, in general the speed of propagation is not vanishing, thus the extra d.o.f. propagates also in a dS universe and the solution, when $\tilde{\mu}_{un}$ is negligible reads:

$$\begin{aligned} \zeta(t,k) &= \frac{1}{8H_0}\left[\sin\left(\frac{k}{a(t)}\frac{c_s}{H_0}\right)\left(3C_2H_0 + 8C_1\frac{k}{a(t)}c_s\right)\right. \\ &+ \left.\cos\left(\frac{k}{a(t)}\frac{c_s}{H_0}\right)\left(8C_1H_0 - 3C_2\frac{k}{a(t)}c_s\right)\right], \end{aligned} \tag{4.43}$$

which can be approximated as

$$\zeta(t,k) \approx \frac{c_s}{8H_0}\frac{k}{a(t)}\left[8C_1\sin\left(\frac{k}{a(t)}\frac{c_s}{H_0}\right) - 3C_2\cos\left(\frac{k}{a(t)}\frac{c_s}{H_0}\right)\right]. \tag{4.44}$$

This solution decays, even for a very large k as the scale factor grows exponentially.

Finally, in order to ensure a stable dS universe one has to impose some stability requirements. Following the discussion in the previous section and the results in Chapters 2 and 3, we have respectively for the avoidance of scalar and tensor ghosts

$$\frac{F_1(F_1 - 3F_4)}{F_4} > 0\,, \qquad m_0^2(1+\Omega) - \bar{M}_3^2 > 0\,, \tag{4.45}$$

which need to be combined with the requirement of positive speeds of propagation for scalar and tensor modes

$$\begin{aligned} c_s^2 &= \frac{F_4\left(-4m_0^2(\Omega+1)\left(m_2^2 - \hat{M}^2\right) + 4\hat{M}^4 + m_0^4(\Omega+1)^2\right)}{8F_1\left(F_1 - 3F_4\right)m_2^2}\,, \\ c_T^2 &= 1 + \frac{\bar{M}_3^2}{m_0^2(1+\Omega) - \bar{M}_3^2}. \end{aligned} \tag{4.46}$$

At this point one may wonder if a tachyonic condition can be applied. In Chapter 2 it has been shown that by performing a field redefinition in order to obtain a canonical action, one can define an effective mass term, which in the small k limit gives the correct condition. If we apply such condition in the dS limit the effective mass associated to our general case is vanishing. Moreover as discussed before, in case the speed of propagation for the ζ field becomes very small at high-k, one has also to ensure that the following conditions do not apply: $\tilde{\mu}_{un} < 0$ and $|\tilde{\mu}_{un}| \gg H_0^2$. However, as already discussed the mass term is sensitive to a field redefinition, thus in order to impose a condition on the mass which holds regardless of the considered field but containing the real information about the mass of the dark energy field, we need to investigate the behaviour of the gauge invariant quantity δ_ϕ.

In the dS universe the gauge invariant quantity defined in eq. (4.12) reads

$$\delta_\phi = \frac{\delta\rho_\phi}{\bar{\rho}_\phi} = \frac{2\dot{\zeta}}{H} - 2\,\delta N - \frac{2}{3}\,\frac{\nabla^2\zeta + \nabla^2\psi}{a^2H^2}\,, \tag{4.47}$$

which can be easily obtained from the first line in eq. (4.6). Moreover, from the same equations we found that δ_ϕ can be written as in eq. (4.16) and it is then used to derive the eq. (4.15). In the dS universe the coefficients of the eqs. (4.15)-(4.16) are

$$\begin{aligned} \alpha_3(t,k) &= \frac{\tilde{\alpha}_3 + \frac{k^2}{a(t)^2}\frac{4}{F_1}\mathcal{A}_4}{3H_0(\mathcal{A}_2 + \frac{k^2}{a(t)^2}\mathcal{A}_3)}\,, \quad \alpha_6(t,k) = \frac{2k^2}{3H_0^2 a(t)^2}\left[\frac{\tilde{\alpha}_6}{(\mathcal{A}_2 + \frac{k^2}{a(t)^2}\mathcal{A}_3)} + 1\right]\,, \\ \mu_3(t,k) &= H_0\frac{\sum_{m=0}^{7} b_m \frac{k^{2m}}{a^{2m}}}{\sum_{m=0}^{7} c_m \frac{k^{2m}}{a^{2m}}}\,, \quad \mu_6(t,k) = \frac{\sum_{n=0}^{10} d_n \frac{k^{2n}}{a^{2n}}}{\sum_{n=0}^{9} f_n \frac{k^{2n}}{a^{2n}}}\,, \end{aligned} \tag{4.48}$$

where here the $\{b_i, c_i, d_i, f_i, \tilde{\alpha}_i\}$ are constants. Note that the above results might have some limiting cases when the determinants of the above relations go to zero. In what follows we are assuming a non-vanishing denominator.

For the dark energy field in the regime in which k^2/a^2 is negligible, we have

$$\mu_3 = 5H_0 + \mathcal{O}(k^2)\,, \qquad \mu_6 = 6H_0^2 + \mathcal{O}(k^2)\,, \tag{4.49}$$

where $\mu_6 \equiv m^2$ can be read as a mass term, which in this case is positive and of the same order of H_0^2, thus no instability takes place. Moreover, because the value of the mass is fixed (i.e. does not depend on the specific value of the EFT functions one can assume), this result is quite general. We also stress that such results can be also safely applicable at low redshifts, as we know at those z the universe is mostly dark energy dominated and thus approaching a dS universe. Finally, the dark energy field evolves following

$$\delta_\phi(t) = C_1 e^{-3H_0 t} + C_2 e^{-2H_0 t}, \tag{4.50}$$

and because the friction term is positive, its effect will be to damp the amplitude of the field. Then, in this regime the δ_ϕ field effectively has a mass, while the ζ field has not. This is one of the main differences which characterize the gauge invariant field δ_ϕ.

In the opposite regime, we have

$$\mu_3 = 7H_0 + \mathcal{O}(k^{-2})\,, \qquad \mu_6 = \left(c_s^2 \frac{k^2}{a(t)^2} + \mu_{un}\right) + \mathcal{O}(k^{-2})\,, \tag{4.51}$$

where also in this case we have defined a speed of propagation of the mode δ_ϕ at high-k, which coincides with the speed of propagation for the field ζ as discussed in Sec. 4.3 and we have defined, in analogy with the previous case, μ_{un} as the effective undamped mass for the dark energy filed, which in this case assumes the following form:

$$\mu_{un} = 10H_0^2 + \frac{\mathcal{A}_3(\mathcal{A}_4\mathcal{G}_2 - \mathcal{A}_1\mathcal{G}_3) + \mathcal{A}_4\mathcal{G}_3(4\mathcal{A}_4H_0^2 - \mathcal{A}_2)}{\mathcal{A}_3^2\mathcal{A}_4^2}\,, \tag{4.52}$$

which is the next to leading order term in μ_6. From (4.51) we see that the equation of motion has the form

$$\ddot{\delta}_\phi + 7H_0\dot{\delta}_\phi + \left(\frac{c_s^2 k^2}{a^2} + \mu_{un}\right)\delta_\phi = 0\,, \tag{4.53}$$

which is exactly the same form of the equation of the ζ field at high-k. Thus the discussion presented earlier is also applicable here, for $\chi = 7$ and μ_{un} given by eq. (4.52). Finally, the the damped mass of the oscillatory mode is

$$\hat{m}^2 \equiv \mu_{un} - \frac{49}{4}\,H_0^2\,. \tag{4.54}$$

Therefore, an instability might manifest itself when $c_s^2 \simeq 0$ and $\hat{m}^2 < 0$. To be precise, when one has $\frac{c_s^2 k^2}{a^2} + \hat{m}^2 < 0$, one must impose $\mu_{un} < 0$ and $|\mu_{un}| \gg H_0^2$ in order to avoid said instability.

Finally we present the solution at leading order and when μ_{un} is negligible:

$$\delta_\phi(t,k) \approx \frac{k^3}{a(t)^3}\frac{c_s^3}{1920H_0^3}\left(1575c_2\cos\left(\frac{k}{a(t)}\frac{c_s}{H_0}\right) - 128c_1\sin\left(\frac{k}{a(t)}\frac{c_s}{H_0}\right)\right)\,, \tag{4.55}$$

which is decaying for an exponentially growing scale factor.

When one considers the case where all the operators are active it is necessary to highlight a number of limiting cases where a different behaviour emerges:

- case $F_2 = 0$. In this case, one is still able to solve the constraint equation to write the action in the form (4.7), with the following coefficients:

$$
\begin{aligned}
\mathcal{L}_{\dot\zeta\dot\zeta} &= \frac{1}{2}F_1\left(\frac{F_1}{F_4}-3\right), \\
G(t,k) &= \frac{1}{8\frac{k^2}{a(t)^2}m_2^2+F_3}\left[2\frac{k^2}{a(t)^2}\left(-4m_0^2(\Omega+1)\left(m_2^2-\hat{M}^2\right)\right.\right. \\
&+ \left.\left. 4\hat{M}^4+m_0^4(\Omega+1)^2\right)-m_0^2F_3(\Omega+1)\right], \qquad (4.56)
\end{aligned}
$$

  the speed of propagation of the curvature perturbation in the high-k limit (k^2/a^2) is

$$
c_s^2 = \frac{F_4\left(-4m_0^2(\Omega+1)\left(m_2^2-\hat{M}^2\right)+4\hat{M}^4+m_0^4(\Omega+1)^2\right)}{2F_1\left(F_1-3F_4\right)m_2^2}. \qquad (4.57)
$$

  The results and the discussion we had in the general case work also in this case, one has just to replace the correct speed of propagation.

- case $F_1 - 3F_4 = 0$. In this case the kinetic term in action (4.7) is vanishing, thus follows that the curvature perturbation $\zeta = 0$ as well as the dark energy field. These theories lead to strong coupling thus they cannot be considered in the EFT context.

- case $2\hat{M}^2 + m_0^2(1+\Omega) = 0$. After computing the kinetic and gradient terms, it is straightforward to verify that the gradient term is negative. Indeed, it has the form $G = -m_0^2(\Omega+1)$, and the stability condition to avoid ghost in tensor modes imposes that $1 + \Omega > 0$. Now, considering that the kinetic terms is positive as well, to guarantee that the scalar modes have no-ghosts, we can conclude that the speed of propagation in negative, thus this subclass of theories in the dS limit shows an instability.

In summary, we have analysed the evolution and stability of the curvature perturbation and the gauge invariant dark energy field for a quite general case. We have found that the curvature perturbation is conserved at large scale, as expected, and at small scale it evolves with a non zero speed of propagation, which finally decays as the scale factor grows with time (eq. (4.55)). The δ_ϕ field at large scale appears to have mass which results to be positive and of same order of H_0^2, thus avoiding the tachyonic instability and along with the fact that at these scale it decays, these are the two characteristics that makes the two fields analysed to be different. We conclude this section saying that in order to have a stable dS universe the conditions which need to be satisfied are the requirements on the kinetic terms and speeds of propagations for scalar and tensor modes (see eqs. (4.45)-(4.46)) since the condition on the avoidance of tachyonic instability at large scale is always satisfied. However, one has to make sure that at high-k, in case $c_s^2 \simeq 0$ the mass associated to these modes do not show an instability, i.e. $\tilde{m}^2 < 0$ when $\tilde{\mu}_{un} < 0$ and $|\tilde{\mu}_{un}| \gg H_0^2$ for the ζ field and $\hat{m}^2 < 0$ when $\mu_{un} < 0$ and $|\mu_{un}| \gg H_0^2$ for the dark energy field.

4.4.2 Beyond Horndeski class of theories

In this section we will consider the EFToDE/MG action restricted to the beyond Horndeski class of theories, which corresponds to set $m_2^2 = 0$, $F_4 = 0$ in action (4.2). For such case in general we have both $\mathcal{L}_{\dot{\zeta}\dot{\zeta}}$ and G to be functions of time as in Chapter 2, but in the dS limit the kinetic and the gradient terms reduce to constants with the following expressions:

$$\mathcal{L}_{\dot{\zeta}\dot{\zeta}} = \frac{1}{2} F_1 \left(\frac{F_1 F_3}{F_2^2} + 3 \right) , \quad G = \frac{F_1 H_0 \left(2\hat{M}^2 + m_0^2(\Omega + 1) \right) - F_2 m_0^2(\Omega + 1)}{F_2} , \tag{4.58}$$

and because they are constant we can define the speed of propagation from the beginning without requiring any limit, and it reads

$$c_s^2 = \frac{2F_2 \left(F_1 H_0 \left(2\hat{M}^2 + m_0^2(\Omega + 1) \right) - F_2 m_0^2(\Omega + 1) \right)}{F_1 \left(3F_2^2 + F_1 F_3 \right)} . \tag{4.59}$$

In the following we will consider $\{F_2, F_1, \left(3F_2^2 + F_1 F_3\right) \neq 0\}$. The requirement $F_1 \neq 0$ is ensured by the assumption that our theory reduces to GR, while the others cases will be considered at the end of this section. The stability conditions requires $\mathcal{L}_{\dot{\zeta}\dot{\zeta}} > 0$ and $c_s^2 > 0$ to guarantee the theory to be free from ghost in the scalar sector and to prevent gradient instabilities. To complete the set of stability conditions one has to include the conditions from the tensor modes, i.e. the no-ghost condition which reads $F_1/2 > 0$ and a positive tensor speed of propagation, that is $c_T^2 = 2m_0^2(1 + \Omega)/F_1 > 0$. For the ζ field we can perform a filed redefinition and construct a canonical action as was done in Chapter 2, from which we can read the effective mass. In the dS universe, such term is identically zero at all scale.

In the dS limit the analysis of the dynamical equation for ζ is straightforward:

$$\ddot{\zeta} + 3H_0\dot{\zeta} + \frac{k^2}{a(t)^2}c_s^2\zeta = 0\,, \tag{4.60}$$

which has the same form of the equation for ζ in the general case (see eq. (4.33)), thus it has the same solutions in both the regimes, but the speed is now given by eq. (4.59). In summary, the curvature perturbation is conserved in the limit in which $k^2/a(t)^2$ is heavily suppressed and it slowly decays at high-k (see eq. (4.55)).

Now, let us consider the dark energy field, δ_ϕ defined in eq. (4.16). For the beyond Horndeski sub-case, the coefficients of eqs. (4.16)-(4.15) reduce as follows

$$\begin{aligned}
\alpha_3 &= -\frac{2F_1\left(2F_2H_0+F_3\right)}{3F_2^2H_0} \equiv \alpha_3^0\,,\\
\alpha_6(t,k) &= \frac{2k^2\left(-2F_2H_0\left(2\hat{M}^2+m_0^2(\Omega+1)\right)+F_2^2\right)}{3F_2^2H_0^2a(t)^2} \equiv \frac{k^2}{a(t)^2}\alpha_6^0\,,\\
\mu_3(t,k) &= -\frac{H_0\left(5\alpha_3^0\left(\alpha_6^0H_0-\alpha_3^0c_s^2\right)-7(\alpha_6^0)^2\frac{k^2}{a(t)^2}\right)}{\alpha_3^0\left(\alpha_3^0c_s^2-\alpha_6^0H_0\right)+(\alpha_6^0)^2\frac{k^2}{a(t)^2}}\,,\\
\mu_6(t,k) &= \frac{1}{\alpha_3^0\left(\alpha_3^0c_s^2-\alpha_6^0H_0\right)+(\alpha_6^0)^2\frac{k^2}{a(t)^2}}\left[6\alpha_3^0H_0^2\left(\alpha_3c_s^2-\alpha_6^0H_0\right)\right.\\
&+ \left.\frac{k^2}{a(t)^2}\left[\alpha_6^0\alpha_3^0H_0c_s^2+(\alpha_3^0)^2(c_s^2)^2+10(\alpha_6^0)^2H_0^2\right]+(\alpha_6^0)^2\frac{k^4}{a(t)^4}c_s^2\right]\,,
\end{aligned} \tag{4.61}$$

where α_3^0 and α_6^0 are constants. These relations have been obtained from eqs. (4.48), and from them it is easy to identify the b_i, c_i, d_i coefficients. The above expressions hold for $F_2 \neq 0$ and $\alpha_3^0\left(\alpha_3^0c_s^2-\alpha_6^0H_0\right)+(\alpha_6^0)^2\frac{k^2}{a^2} \neq 0$. Let us note that in the latter, in order to realize $\alpha_3^0\left(\alpha_3^0c_s^2-\alpha_6^0H_0\right)+(\alpha_6^0)^2\frac{k^2}{a^2} \to 0$, we have to consider that since all the coefficients are k-independent we need to have $\alpha_6^0 = 0$, then the remaining option is $c_s^2 = 0$. That is because $\alpha_3^0 \neq 0$ otherwise the dark energy field disappears. Therefore, the only configuration is with $\{c_s^2, \alpha_6^0\} = 0$. We will consider the case $F_2 = c_s = 0$ at the end of this section.

In the limit in which k^2/a^2 is suppressed, these coefficients reduce to

$$\mu_3 = 5H_0 + \mathcal{O}(k)\,, \qquad \mu_6 = 6H_0^2 + \mathcal{O}(k^2)\,. \tag{4.62}$$

Then, the friction term μ_3 will dump the amplitude of the dark energy field, while $\mu_6 = m^2$ will act as positive dispersive coefficient or a "mass" one. These results are independent on the specific theory one may consider and the mass of the dark energy field is positive. This is a general result, which allows us to conclude that all the theories belonging to this subclass do not experience tachyonic instability in a dS universe, and it is quite safe to assume that this results holds also at $z \approx 0$. Moreover, the solution of eq. (4.15) reads

$$\delta_\phi(t,0) = D_1 e^{-3H_0 t} + D_2 e^{-2H_0 t}, \tag{4.63}$$

where D_i are integration constant. Therefore, we can conclude that the dark energy field is damped.

On the other hand, for large k^2/a^2 we get

$$\mu_3 = 7H_0 + \mathcal{O}(k^{-2})\,, \qquad \mu_6(t,k) = 2H_0\left(\frac{\alpha_3^0 c_s^2}{\alpha_6^0} + 5H_0\right) + \frac{k^2}{a(t)^2}c_s^2 + \mathcal{O}(k^{-2})\,, \tag{4.64}$$

with $\alpha_6^0 \neq 0$. Also in this limit the μ_3 coefficient will dump the amplitude of the dark energy field, while the second coefficient assumes the form

$$\mu_6(t,k) \equiv \left(\frac{k^2}{a(t)^2}c_s^2 + \mu_{un}\right)\,, \tag{4.65}$$

where the speed of the dark energy field in this regime is the same of the original ζ field and μ_{un} follows directly from the previous expression. The analysis done in the previous section for the high-k limit of the dark energy field is directly applicable to this case. Let us just recall that an instability might occurs when at high-k the speed of propagation is very small, as it can happen that $\hat{m}^2 < 0$ when $\mu_{un} < 0$ and $|\mu_{un}| \gg H_0^2$. When, μ_{un} is negligible as in the previous case, we can solve the equation and we find the same behaviour of the general case (eq. (4.55)).

As before we now separately consider some special cases:

- $\left\{c_s^2, \alpha_6\right\}$=0. In case $c_s^2 = 0$ the ζ field has the solution

  $$\zeta(t) = \tilde{C}_1 - \frac{\tilde{C}_2}{3H_0}e^{-3H_0 t}\,, \tag{4.66}$$

  which predicts the conservation of the curvature perturbation at any scale.

  When going to the dark energy field, δ_ϕ, which is related to the ζ field through the eq. (4.16), one can notice two main aspects. Firstly, because $\alpha_6 = 0$, the dark energy field is identified as $\dot{\zeta}$ up to a constant (α_3) and hence it requires one boundary condition less. Additionally, when carefully studying the Lagrangian after changing the field, eq. (4.20), it is clear that the kinetic term for the dark energy field diverges for high-k. This is due to the fact that the speed is vanishing which translates to the gradient term being zero. Hence, it must be concluded that, for this particular case, the choice for the dark energy field is inappropriate and should not be considered.

- $F_2 = 0$: Considering the action (4.4), by varying with respect to ψ immediately follows that $\dot{\zeta} = 0$. Thus the extra scalar d.o.f. does not propagate.
- $3F_2^2 + F_1F_3 = 0$: in this case the kinetic term is zero and the curvature perturbation is vanishing. These theories show strong coupling thus they cannot be considered in the EFT approach.

We conclude by saying that the results of the previous section also apply to the beyond Horndeski class of theories considered in the present section. Moreover, the main result here is also that the speed of propagation of the scalar mode in general does not vanish as instead previously found in literature. We will show some practical examples in Sec. 4.5.

4.4.3 Hořava gravity like models

Let us now consider a special case in which $m_2^2 \neq 0$ and $3F_2^2 + F_3F_1 = 0$. This subclass of models includes the low-energy Hořava gravity model. The action can be written as

$$\begin{aligned}\mathcal{S}^{(2)} \quad &= \int d^4x a(t)^3 \frac{k^2}{a(t)^2}\left\{\frac{\mathcal{A}_4}{\mathcal{A}_2 + \frac{k^2}{a(t)^2}\mathcal{A}_3}\dot{\zeta}^2 - \left(\frac{k^2}{a(t)^2}\frac{\mathcal{G}_2 + \frac{k^2}{a(t)^2}\mathcal{G}_3}{(\mathcal{A}_2 + \frac{k^2}{a(t)^2}\mathcal{A}_3)^2}\right.\right.\\ &\left.\left. + \frac{\mathcal{G}_1}{(\mathcal{A}_2 + \frac{k^2}{a(t)^2}\mathcal{A}_3)^2}\right)\zeta^2\right\} \qquad (4.67)\end{aligned}$$

with an overall factor $k^2/a(t)^2$. For this case in the dS limit the no-ghost and positive speed conditions read

$$\frac{\mathcal{A}_4}{\mathcal{A}_3} > 0, \qquad c_s^2 = \frac{\mathcal{G}_3}{\mathcal{A}_3\mathcal{A}_4} > 0\,, \qquad (4.68)$$

along with the usual conditions for the stability of tensor modes

$$m_0^2(1+\Omega) - \bar{M}_3^2 > 0\,, \qquad c_T^2(t) = 1 + \frac{\bar{M}_3^2}{m_0^2(1+\Omega) - \bar{M}_3^2} > 0\,. \quad (4.69)$$

The conditions on the speeds reduce to $\mathcal{G}_3 > 0$ and $1 + \Omega > 0$. Just for simplicity, let us rewrite the above action as follows

$$\mathcal{S}^{(2)} = \int d^4 x a^3 \frac{k^2}{a(t)^2} \left\{ \tilde{\mathcal{L}}_{\dot{\zeta}\dot{\zeta}}(t,k)\dot{\zeta}^2 - \left(\frac{k^2}{a^2}\tilde{G}(t,k) + \tilde{M}(t,k) \right) \zeta^2 \right\} , \tag{4.70}$$

where the definitions of the above coefficients immediately follows from the action (4.67). The field equation for the curvature perturbation can then be written in a compact form as

$$\ddot{\zeta} + \left(3H_0 + \frac{\dot{\tilde{\mathcal{L}}}_{\dot{\zeta}\dot{\zeta}}}{\tilde{\mathcal{L}}_{\dot{\zeta}\dot{\zeta}}} \right) \dot{\zeta} + \left(\frac{k^2}{a^2} \frac{\tilde{G}}{\tilde{\mathcal{L}}_{\dot{\zeta}\dot{\zeta}}} + \frac{\tilde{M}}{\tilde{\mathcal{L}}_{\dot{\zeta}\dot{\zeta}}} \right) \zeta = 0 \, , \tag{4.71}$$

where in this case a dispersion coefficient for the field ζ appears in the evolution equation. Let us now analyse the two limit as in the previous cases.

In the case k^2/a^2 is sub-dominant $\tilde{M} \neq 0$ and we have

$$\ddot{\zeta} + 3H_0\dot{\zeta} + \bar{m}^2\zeta = 0 \, , \tag{4.72}$$

where we have defined the mass term at low k as

$$\bar{m}^2 = \lim_{\frac{k^2}{a^2}\to 0} \frac{\tilde{M}}{\tilde{\mathcal{L}}_{\dot{\zeta}\dot{\zeta}}} = \frac{\mathcal{G}_1}{\mathcal{A}_2\mathcal{A}_4} \, . \tag{4.73}$$

In order to avoid a instability coming from the mass term we require $|\bar{m}^2| << H_0^2$. The solution reads

$$\zeta(t) = C_1 e^{\frac{1}{2}t\left(-\sqrt{9H_0^2 - 4\bar{m}^2} - 3H_0\right)} + C_2 e^{\frac{1}{2}t\left(\sqrt{9H_0^2 - 4\bar{m}^2} - 3H_0\right)} . \tag{4.74}$$

When $9H_0^2 - 4\bar{m}^2 > 0$, both the exponentials are purely negative hence both modes are decaying. In the opposite case the solution is a decaying oscillator.

In the limit in which k^2/a^2 is dominant the above equation reduces to

$$\ddot{\zeta} + 5H_0\dot{\zeta} + \left(\frac{k^2}{a(t)^2}c_s^2 + \tilde{\mu}_{un} \right) \zeta = 0 \, , \tag{4.75}$$

where

$$\tilde{\mu}_{un} = \frac{(\mathcal{A}_3\mathcal{G}_2 - \mathcal{A}_2\mathcal{G}_3)}{\mathcal{A}_3^2\mathcal{A}_4} . \tag{4.76}$$

Let us note that in this limit $\tilde{M}$ is of $\mathcal{O}(k^{-2})$ and the above mass-like term comes from the 0th order expansion of the term $\tilde{G}/\tilde{\mathcal{L}}_{\dot{\zeta}\dot{\zeta}}$. Also in this case we can apply the analysis of Sec. 4.4.1, for $\chi = 5$, and conclude that, when $\frac{k^2}{a^2}c_s^2 + \tilde{m}^2 > 0$ no instability occurs, while when the speed is small or negligible some growing modes or instability might take place if $\bar{\mu}_{un} < 0$. In the case $\tilde{\mu}_{un} << \frac{k^2}{a^2}c_s^2$, the solution of the above equation at leading order is:

$$\zeta(t,k) \approx -\frac{k^2}{a(t)^2}\frac{c_s^2}{96H_0^2}\left(45c_2\sin\left(\frac{k}{a(t)}\frac{c_s}{H_0}\right) + 32c_1\cos\left(\frac{k}{a(t)}\frac{c_s}{H_0}\right)\right), \tag{4.77}$$

which decays in time.

Now, let us consider the dark energy field. The definitions of the α_i functions which enter in the relation between ζ and δ_ϕ can be found in Appendix 4.7 after applying the restriction to this subcase. In the regime in which k^2/a^2 is sub-dominant, the equation for the dark energy field has the following coefficients

$$\mu_3 = 3H_0 + \mathcal{O}(k^2)\,, \qquad \mu_6 = \frac{\mathcal{G}_1}{\mathcal{A}_2\mathcal{A}_4} + \mathcal{O}(k^2)\,. \tag{4.78}$$

As expected in this case the mass term is the dominant one and $\mu_6 \equiv \bar{m}^2$. Thus in this limit the solution is the same of the curvature perturbation.

In the opposite regime, we have

$$\mu_3 = 9H_0 + \mathcal{O}(k^{-2})\,, \qquad \mu_6 = \left(\mu_{un} + \frac{k^2}{a(t)^2}c_s^2\right) + \mathcal{O}(k^{-2})\,, \tag{4.79}$$

where

$$\mu_{un} = -\frac{-\mathcal{A}_3\left(6F_2^2\mathcal{G}_3H_0^2 + F_3F_4\left(6\mathcal{G}_3H_0^2 + \mathcal{G}_2\right)\right) + \mathcal{A}_2F_3F_4\mathcal{G}_3 - 14\mathcal{A}_3^2\mathcal{A}_4F_3F_4H_0^2}{\mathcal{A}_3^2\mathcal{A}_4F_3F_4}\,. \tag{4.80}$$

Again here we obtain a behaviour following the one of the ζ field but with a different dispersive coefficient. When μ_{un} is negligible the solution at leading order is again an oscillatory decaying mode

$$\delta_\phi \approx \frac{k^4}{a(t)^4}\frac{c_s^4}{53760H_0^4}\left(99225C_2\sin\left(\frac{k}{a(t)}\frac{c_s}{H_0}\right) + 512C_1\cos\left(\frac{k}{a(t)}\frac{c_s}{H_0}\right)\right). \tag{4.81}$$

In conclusion, along with the conditions discussed in the beginning of this section for avoiding ghosts and having positive squared speeds of propagations, we need to make sure that $|\bar{m}^2| << H_0^2$. Additionally, when the speed of propagation is small, one needs to guarantee that both μ_{un} and $\tilde{\mu}_{un}$ do not cause an instability. This set of conditions will ensure the system to be stable. We will provide a working example in Sec. 4.5, where the above results are applied for low-energy Hořava gravity.

4.5 Working examples

In this section we will apply the results we have derived in the previous sections to specific models, i.e. K-essence, Horndeski/Galileon models, low-energy Hořava gravity.

4.5.1 Galileons

We consider here the Generalised Galileon Lagrangians, and we will apply the stability conditions derived for the beyond Horndeski models (Sec. 4.4.2). The complete Galileon action is the following [79]:

$$\mathcal{S}_{GG} = \int d^4x\sqrt{-g}\,(L_2 + L_3 + L_4 + L_5)\,, \tag{4.82}$$

where the Lagrangians have the following structure:

$$\begin{aligned}
L_2 &= \mathcal{K}(\phi, X)\,,\\
L_3 &= G_3(\phi, X)\Box\phi\,,\\
L_4 &= G_4(\phi, X)R - 2G_{4X}(\phi, X)\left[(\Box\phi)^2 - \phi^{;\mu\nu}\phi_{;\mu\nu}\right]\,,\\
L_5 &= G_5(\phi, X)G_{\mu\nu}\phi^{;\mu\nu} + \frac{1}{3}G_{5X}(\phi, X)\left[(\Box\phi)^3 - 3\Box\phi\phi^{;\mu\nu}\phi_{;\mu\nu} + 2\phi_{;\mu\nu}\phi^{;\mu\sigma}\phi^{;\nu}_{;\sigma}\right]\,,
\end{aligned} \tag{4.83}$$

here $G_{\mu\nu}$ is the Einstein tensor, $X \equiv \phi^{;\mu}\phi_{;\mu}$ is the kinetic term and $\{\mathcal{K}, G_i\}$ $(i = 3, 4, 5)$ are general functions of the scalar field ϕ and X, and $G_{iX} \equiv \partial G_i/\partial X$.

The Cubic Galileon model

We start by specializing action (4.82) to a well known model, i.e. the Cubic Galileon, which corresponds to the following choice of the functions

$$K(X) = -\frac{g_2}{2}X\,, \quad G_3(X) = \frac{g_3}{M^3}X\,, \quad G_4 = \frac{m_0^2}{2}\,, \quad G_5 = 0\,, \tag{4.84}$$

where $\{g_2, g_3\}$ are constant and $M^3 = m_0 H_0^2$.

In a dS universe the background equations become

$$3m_0^2H_0^2 = 6\frac{g_3}{M^3}H_0\dot{\phi}^3 + \frac{1}{2}g_2\dot{\phi}^2\,, \qquad 3m_0^2H_0^2 = -\frac{1}{2}g_2\dot{\phi}^2\,. \tag{4.85}$$

From the first Friedmann equation one can define the density of the dark energy field at the background, that is

$$\bar{\rho}_\phi = 6\frac{g_3}{M^3}H_0\dot{\phi}^3 + \frac{1}{2}g_2\dot{\phi}^2\,, \tag{4.86}$$

and after manipulating the equations, one gets a constraint equation

$$6\frac{g_3}{M^3}H_0\dot{\phi}^3 + g_2\dot{\phi}^2 = 0\,, \tag{4.87}$$

which corresponds to $c = 0$, and from which follows [86, 132]

$$\dot{\phi} \equiv \dot{\phi}_0 = const\,, \qquad g_2 = -6\frac{g_3}{M^3}H_0\dot{\phi}_0. \tag{4.88}$$

Considering the above results, the EFT functions corresponding to this model in the dS limit read

$$\Lambda = -3\tfrac{g_3}{M_3}H_0\dot{\phi}_0^3 = -3m_0^2H_0^2\,, \qquad M_2^4 = \tfrac{3}{2}\tfrac{g_3}{M^3}H_0\dot{\phi}_0^3 = \tfrac{3}{2}m_0^2H_0^2\,,$$
$$M_1^3 = -2\tfrac{g_3}{M^3}\dot{\phi}_0^3 = -2m_0^2H_0\,, \tag{4.89}$$

while the others are vanishing.

Using the mapping and the results obtained in the previous section we obtain that the speed of propagation reduces to zero while the kinetic term diverges, implying that there is no scalar d.o.f. propagating in dS. This is an expected result as the cubic Galileon decouples from gravity in dS with a speed of sound of the form [14]

$$c_s^2 = \frac{c}{c + M_2^4}\,, \tag{4.90}$$

which is exactly zero on the background.

K-essence

Motivated by the result for the cubic Galileon, where no scalar d.o.f. propagates on dS, we proceed to check if this holds in more Horndeski class theories. According to ref. [136] the complete set of Horndeski models (with $c = 0$) does not possess a scalar d.o.f. on a dS background, a statement which we wish to confront with specific examples.

We start with a well studied and rather simple theory by considering a K-essence model with a general $\mathcal{K}(X)$ and a standard Einstein-Hilbert term. In this case, we see that the background equations of motion impose

$$\mathcal{K}_{,X}|_{X=X_0} = 0\,, \qquad \mathcal{K}(X_0) = -3m_0^2H_0^2\,, \tag{4.91}$$

where X_0 is the background value of X. The speed of propagation can be written, along with the no-ghost condition, as

$$c_s^2 = \frac{\mathcal{K}_{,X}}{2X\mathcal{K}_{,XX} + \mathcal{K}_{,X}}, \qquad \mathcal{L}_{\dot{\zeta}\dot{\zeta}} = 2X\mathcal{K}_{,XX} + \mathcal{K}_{,X}. \tag{4.92}$$

Now, if $\mathcal{K}$ is analytical, we can consider a Taylor expansion around the point $X = X_0$. In such a case the background equations of motion impose

$$\mathcal{K} = -3m_0^2 H_0^2 + \frac{\mathcal{K}_2}{2}\,(X - X_0)^2 + \frac{\mathcal{K}_3}{6}\,(X - X_0)^3 + \mathcal{O}[(X - X_0)^4]\,, \quad (4.93)$$

where $\mathcal{K}_2 \equiv \mathcal{K}_{,XX}(X_0)$, and $\mathcal{K}_3 \equiv \mathcal{K}_{,XXX}(X_0)$. Then, one finds that

$$c_s^2 = \frac{1}{2X_0}\,(X - X_0) - \frac{1}{4\mathcal{K}_2 X_0^2}\,(3\mathcal{K}_2 + X_0\,\mathcal{K}_3)\,(X - X_0)^2 + \mathcal{O}[(X - X_0)^3]\,, \quad (4.94)$$

and we have that, for an analytical function, $c_s^2 \to 0$ on dS. Hence, if one would want to design a K-essence model with a non zero speed of sound one has to resort to a non analytic form for $\mathcal{K}$. Therefore, this is an example for which in the class of Horndeski models it is still possible to have a propagating d.o.f. in the dS universe. In the following we will show more.

Covariant Galileons

Let us study the dS solution for the Covariant Galileon [86], defined by the following choice of the functions:

$$K(X) = \tfrac{c_2}{2}\,X\,, \quad G_3(X) = \tfrac{c_3\,X}{2M^3}\,, \quad G_4(X) = \tfrac{m_0^2}{2} - \tfrac{c_4 X^2}{4M^6}\,, \quad G_5(X) = \tfrac{3c_5\,X^2}{4M^9}\,, \quad (4.95)$$

We proceed by adopting the following definitions [99]

$$X = -x_{dS}^2\,m_0^2\,H_0^2\,, \quad \alpha \equiv c_4\,x_{dS}^4\,, \quad \beta \equiv c_5\,x_{dS}^5\,, \quad (4.96)$$

where $x_{dS} = \frac{\dot\phi_0}{m_0 H_0}|_{dS}$ being the dS solution and M has been defined before. Then, we find that the equations of motion for the background are fulfilled provided that

$$c_2\,x_{dS}^2 = 9\alpha - 12\beta + 6\,, \qquad c_3 x_{dS}^3 = 9\alpha - 9\beta + 2\,. \quad (4.97)$$

In this case the no-ghost condition for the scalar mode can be written as

$$\frac{\mathcal{L}_{\dot\zeta\dot\zeta}}{m_0^2} = -\frac{(3\,\alpha - 6\,\beta + 2)\,(3\,\alpha - 6\,\beta - 2)}{6\,(\alpha - 2\,\beta)^2} > 0\,, \quad (4.98)$$

and the speed of propagation reduces to

$$c_s^2 = \frac{(2\,\beta - \alpha)\left(15\,\alpha^2 - 48\,\alpha\,\beta + 36\,\beta^2 + 4\right)}{18\,\alpha^2 - 72\,\alpha\,\beta + 72\,\beta^2 - 8}\,, \quad (4.99)$$

which does not vanish in general. Finally, from the dark energy field sector, we obtain

$$\mu_{un} = -\tfrac{1}{(-3\alpha+6\beta+2)^2} 4H_0^2 \left(15\alpha^3 - 6\alpha^2(13\beta+5) + 2\alpha\left(66\beta^2 +57\beta+17\right) - 4\left(18\beta^3+27\beta^2+17\beta+3\right)\right), \qquad (4.100)$$

which must be constrained, as discussed before, in the case of a vanishing speed of propagation. Correspondingly, we obtain for the tensor sector the following:

$$\frac{A_T^2}{m_0^2} = \frac{1}{8}\,(3\,\alpha - 6\,\beta + 2) > 0, \qquad c_T^2 = \frac{\alpha - 2}{6\,\beta - 3\,\alpha - 2}\,. \qquad (4.101)$$

Considering the no-ghost condition and a positive speed of propagation it can be easily shown that a part of the parameter space allows for stable dS solutions with a non-vanishing speed of propagation. For example the choice $\alpha = -\frac{7}{5}$, and $\beta = -\frac{4}{5}$ achieves this. These values result in a relatively small speed of propagation for which $\mu_{un} > 0$. Thus no instability is present for these choice of parameters.

Models with $G_5(X) = 0$, and $G_4(X) = m_0^2/2$

Now, for the Covariant Galileon, setting $\alpha = 0 = \beta$, that is $G_4 = m_0^2/2$ and $G_5 = 0$, yields once more a vanishing c_s^2 while for the kinetic term implies $\mathcal{L}_{\dot\zeta\dot\zeta} \to +\infty$, i.e. weak coupling regime (see the Cubic Galileon case in the previous section). Therefore, the Covariant Galileon requires non-trivial G_4, G_5 in order to have a non-zero speed of propagation for the scalar modes.

It is possible to find models for which $G_4 = m_0^2/2$ and $G_5 = 0$, and, on dS, the speed of propagation does not vanish. We illustrate this by considering the model:

$$K(X) = -c_2\mu^4\left(\tfrac{-X}{2M^4}\right)^p, \quad G_3(X) = c_3\,\mu\left(\tfrac{-X}{2M^4}\right)^q, \quad G_4(X) = \tfrac{m_0^2}{2}, \quad G_5(X) = 0\,, \qquad (4.102)$$

where p and q are constants and μ is a typical length scale of the system. Using the same notation as for the Covariant Galileon we obtain from the background equations of motion the following:

$$c_2 = \frac{3m_0^2H^2}{\mu^4(-X/(2M^4))^p}\,, \qquad c_3 = -\frac{p\,m_0^2\,H}{\mu\,q\,(-X)^{1/2}\,(-X/(2M^4))^q}\,. \qquad (4.103)$$

and subsequently we obtain:

$$\frac{\mathcal{L}_{\dot\zeta\dot\zeta}}{m_0^2} = \tfrac{3p\,(1-p+2q)}{(1-p)^2}\,, \quad c_s^2 = \tfrac{1-p}{3(1-p)+6q}\,, \quad \mu_{un} = \tfrac{2H_0^2\left(21p^2-2p(18q+11)+1\right)}{3p(p-2q-1)}\,, \qquad (4.104)$$

whereas the tensor modes do not add any new constraint. It is possible to find a stable dS on choosing $0 < p < 1$, and $q > -\frac{1}{2}(1-p)$. Finally, in order for the μ_{un} term to create an instability, one needs to look at the case of a very small (or vanishing) speed of sound, i.e. $p \to 1$ or $q \to \infty$. In both cases it turns out that $\mu_{un} = 12H_0^2$ hence no issues arise. As an example for the choice of parameters, we choose $p = 1/2$ and $q = 2$, for which all the conditions are satisfied with a speed of propagation of $c_s^2 = 1/27$, and the undamped mass of the modes is not negligible, as $\mu_{un} = 326/27\, H_0^2$.

Therefore we have showed that, even in the absence of non-trivial G_4, G_5, it is still possible to find models for which c_s^2 does not vanish on dS. This concludes our demonstration of the fact that Horndeski models do not necessarily imply a vanishing d.o.f. on a dS background as suggested by ref. [136].

4.5.2 Low-energy Hořava gravity

One well known model which falls in the above sub-case is the low-energy Hořava gravity [35, 36, 55]. The action of this theory is

$$\mathcal{S}_H = \frac{1}{16\pi G_H}\int d^4x\sqrt{-g}\left(K_{ij}K^{ij} - \lambda K^2 - 2\xi\bar{\Lambda} + \xi\mathcal{R} + \eta a_i a^i\right), \tag{4.105}$$

$\{\lambda, \xi, \eta\}$ are dimensionless running coupling constants, $\bar{\Lambda}$ is the "bare" cosmological constant, G_H is the coupling constant which can be expressed as [64]

$$\frac{1}{16\pi G_H} = \frac{m_0^2}{(2\xi - \eta)}. \tag{4.106}$$

Expanding the above action in terms of the perturbed metric (4.3) and considering the mapping between this action and the EFToDE/MG framework, the action up to second order in perturbations can be recast in the the same form of action (4.67) and by using the redefinition (4.16) the action becomes the one in (4.70). In order to specify the coefficients for action (4.70) and then analyse the solutions for this specific model, let us consider the background equation which in the dS limit is

$$H_0^2 = \frac{2\xi\bar{\Lambda}}{3(3\lambda - 1)}, \tag{4.107}$$

from which follows

$$\bar{\rho}_\phi = m_0^2\frac{2\xi\bar{\Lambda}}{(3\lambda - 1)}. \tag{4.108}$$

Now, we can specify all the EFT functions [39],

$$(1+\Omega) = \frac{2\xi}{(2\xi-\eta)}, \quad \Lambda = -\frac{4m_0^2\xi^2\bar{\Lambda}}{(2\xi-\eta)(3\lambda-1)} \quad \bar{M}_3^2 = -\frac{2m_0^2}{(2\xi-\eta)}(1-\xi),$$
$$\bar{M}_2^2 = -2\frac{m_0^2}{(2\xi-\eta)}(\xi-\lambda), \quad m_2^2 = \frac{m_0^2\eta}{4(2\xi-\eta)}, \quad \bar{M}_1^3 = \hat{M}^2 = c = M_2^4 = 0. \tag{4.109}$$

Then, the no-ghost and gradient conditions at high-k read

$$\frac{2(1-3\lambda)}{(\lambda-1)(\eta-2\xi)} > 0, \qquad c_s^2 = \frac{(\lambda-1)\xi(2\xi-\eta)}{\eta(3\lambda-1)} > 0\,, \tag{4.110}$$

where the latter is different from zero even in the PPN limit ($\eta \to 2\xi - 2$). Additionally, when k/a is sub-dominant we obtain a vanishing mass term for the ζ field, i.e. $\bar{m}^2 = 0$. When k/a is dominant, we also need to consider the undamped mass for the ζ field, which is

$$\tilde{\mu}_{un} = \frac{4H_0^2\xi}{\eta}. \tag{4.111}$$

When studying the parameter space allowed further below it turns out that $\tilde{\mu}_{un}$ will remain manifestly positive, hence no instabilities will occur due to its presence. Now, when it comes to the gauge independent choice, the dark energy field, δ_ϕ adds no new conditions when demanding no-ghost and a positive speed of propagation as analysed in the previous section. The mass for this field at low k is vanishing as well. At high-k for the dark energy field we can define

$$\mu_{un} = \frac{2H_0^2(\eta(21\lambda+2\xi-7)+2\xi(3\lambda-2\xi-1))}{\eta(3\lambda-1)}\,, \tag{4.112}$$

which has to be constrained if the speed is very small. Further below we will comment on its effect on the parameter space. Finally the tensor sector add the following set of constraints to the model:

$$\frac{2}{2\xi-\eta} > 0, \qquad c_T^2 = \xi > 0. \tag{4.113}$$

Now it is possible to define a range of viability for the parameters of low-energy Hořava gravity based on this set of conditions, namely:

$$0 < \eta < 2\xi, \qquad \lambda > 1 \quad \text{or} \quad \lambda < \frac{1}{3}, \tag{4.114}$$

which is a very well known result. Keeping the above conditions in mind we turn our attention to the regime of a small speed, i.e. $\lambda \to 1$ or $\eta \to 2\xi$.

In both cases it is easy to see that (4.111) will always be positive. On the contrary, (4.112) does show different behaviours. For $\lambda \to 1$ it is clear that an increasing ξ pushes it more and more to the strongly negative regime while η does the opposite. Now, for $\eta \to 2\xi$, it reduces to a constant, $\mu_{un} = 16H_0^2$. On top of these theoretical considerations one may want to consider additional constraints coming from PPN, binary pulsar or CherenKov [137]. Such additional constraints are complementary to the ones obtained in our paper and in case these have to be imposed, our results ensure that the theory is stable.

Finally, we will note that the relation between the original field ζ and the dark energy one for this specific case can be obtained by using the relations in Appendix 4.7 after applying the mapping provided in this section. The functions α_i appearing in eq. (4.16) result to be both functions of k and time.

4.6 Conclusion

Until now, when considering the EFToDE/MG in the unitary gauge, the curvature perturbation ζ has been the main focus of investigation when considering the question of stability. However, this choice of variable is gauge dependent, hence one might question if going to a gauge independent one the viable parameter space of the model changes and, most importantly, if such a gauge invariant quantity can be defined as the one describing the dynamical dark energy field. This motivated us to look for and construct a gauge independent quantity and, consequently, to perform a comparison with the results for the original field, ζ.

In this Chapter, we first proceeded to define a gauge invariant quantity which describes the linear density perturbation of the dark energy field. Such a definition is very general and applicable both in the presence of matter fields and in the late time universe. Then, moving to the explicit stability study of the scalar d.o.f., we focused on avoiding the usual set of instabilities namely ghost, gradient and tachyonic instabilities for both scalar and tensor modes. These are related to the sign of the kinetic term, the speed of propagation at high-k and the mass term at low-k respectively. Additionally, we studied the effect of sub-leading term in the high-k expansion as it might become important when the speed of propagation is small. Dubbed the effective undamped mass it can become problematic when it is strongly negative as the corresponding modes are unstable. Moreover, we showed that, by doing a field redefinition in the second order action from the curvature perturbation ζ to the dark energy field, the constrains arising by imposing the absence of ghost and gradient instabilities do not change.

Moving on to the mass terms the agreement between the two perturbations does not seem to hold as the mass terms are substantially different. It is then important to also consider the mass of the dark energy field, when setting the proper condition for the avoidance of tachyonic instability, as it has a real physical interpretation. In order to have an idea of the behaviour of the mass term, we studied modifications of gravity on a dS background and then we set and discussed the proper conditions one has to impose in order to ensure a stable dS universe. The existence of stable dS solutions is of value as it is expected to be the late time stage of the universe. As we wished to achieve model independent results we employed the usual EFToDE/MG while neglecting any matter components due to their heavily sub-leading behaviour.

As we saw in the previous Chapters, the all-encompassing nature of the original EFToDE/MG action dictates that a unique approach is not feasible as sub-cases might show up which need to be treated separately, a behaviour appearing due to the higher spatial derivative operators. In this Chapter we identified three main cases that deserved our attention: the case with all operators active, the beyond Horndeski class of models and the case encompassing low-energy Hořava gravity.

Starting with these three subcases we proceeded to study their theoretical stability by deriving the kinetic term and the speed of propagation. By demanding them to be positive, one guarantees that the theory is free of ghosts and gradient instabilities. Additionally, we supplemented them with the same conditions guaranteeing a stable tensor sector. As already discussed we find that the parameter space identified by the no-ghost and gradient conditions is independent of the field chosen to describe the scalar d.o.f.. In the general case when considering the low k/a limit it becomes clear that the two fields satisfy a different equation of motion. The curvature perturbations is conserved at those scales as the equation does not contain any mass term. On the contrary, the equation for the gauge invariant dark energy field appears to have a mass term which is positive and of the same order of H_0^2, hence a tachyonic instability does not develop and the solution is an exponentially decaying mode. We can infer the same conclusion for the beyond Horndeski sub-case. On the other hand, we find that for the Hořava like class both the curvature perturbation and the gauge invariant dark energy field satisfy the same equation of motion with a mass term dependent on the theory. The non zero mass term for the curvature perturbation can be attributed to the Lorentz violating nature of Hořava gravity. Thus we have to require that $|\bar{m}^2| \leq H_0^2$ in order to guarantee a stable dS universe.

In the high-k limit usually only the leading order is considered, identified as the speed of propagation. Constraining this to be positive is usually considered to be enough to guarantee stability of the corresponding modes. We proceeded to expand this analysis by not neglecting the next to leading order contribution, a term we dubbed the effective undamped mass. This term turns out be relevant for theories with a very small speed of propagation as it can become the source of an instability. Thus, in such a case, one needs to impose an additional constraint.

As a final comment we would like to emphasize that the speed of propagation was never identically zero. This is an interesting results when considering the Horndeski class of models as it was claimed that they do not propagate a scalar d.o.f. in dS [136]. While this can happen for specific cases, such as the Cubic Galileon and any analytic K-essence model, the statement does not hold in its full generality. To name one, the very well known Covariant Galileon theory has been studied and shown to propagate a d.o.f.. To complete its study we presented a parameter choice which not only propagates a d.o.f. but also guarantees a stable dS background.

Finally we believe that our results can be apllied at present time ($z \sim 0$) as well, as the dark energy field is dominating. However, in order to guarantee that this field remains stable along all the whole evolution of the universe, one has to properly derive the mass coefficient when the matter fluids are considered, as was done in the previous Chapter. In order to provide the mass associated to the, gauge invariant, dark energy density field one should construct the Hamiltonian for all the fields (perturbed dark energy density+ fluid densities), then work out the associated eigenvalues and finally apply the . In light of the results of this work it might be important to investigate also the behaviour of the effective undamped mass term at high-k. Concluding, the gauge invariant quantity defined to describe dark energy is still valid when one includes the presence of matter. The difficulty will then lie in the disentangling of its dynamics from these new field and will require a separate investigation.

4.7 Appendix A: Notation

In this Appendix we will explicitly list all the coefficients used in the main text.

The kinetic term in action (4.7) reads

$$\mathcal{L}_{\dot{\zeta}\dot{\zeta}}(t,k) = \frac{\mathcal{A}_1(t) + \frac{k^2}{a^2}\mathcal{A}_4(t)}{\mathcal{A}_2(t) + \frac{k^2}{a^2}\mathcal{A}_3(t)}\,, \tag{4.115}$$

where

$$\mathcal{A}_1(t) = (F_1 - 3F_4)\left(3F_2^2 + F_1F_3\right)\,,$$

$$\begin{aligned}
\mathcal{A}_2(t) &= 2\left(F_2^2 + F_3F_4\right) ,\\
\mathcal{A}_3(t) &= 16F_4m_2^2 ,\\
\mathcal{A}_4(t) &= 8F_1m_2^2\left(F_1 - 3F_4\right) ,
\end{aligned} \tag{4.116}$$

and the gradient term is

$$G(t,k) = \frac{\mathcal{G}_1(t) + \frac{k^2}{a^2}\mathcal{G}_2(t) + \frac{k^4}{a^4}\mathcal{G}_3}{(\mathcal{A}_2(t) + \frac{k^2}{a^2}\mathcal{A}_3(t))^2} , \tag{4.117}$$

where

$$\begin{aligned}
\mathcal{G}_1(t) &= 4\Big[F_2\left(F_2^2 + F_3F_4\right)\left(F_1 - 3F_4\right)H\left(2\hat{M}^2 + m_0^2(\Omega+1)\right)\\
&+ 2\left({F_2}^3\left(\left(\dot{F}_1 - 3\dot{F}_4\right)\hat{M}^2 + \left(F_1 - 3F_4\right)2\hat{M}\dot{\hat{M}}\right)\right.\\
&+ F_2\left(F_4\left(3F_4 - F_1\right)\dot{F}_3\hat{M}^2 + F_3\left(-6{F_4}^2\hat{M}\dot{\hat{M}} + F_4\left(\dot{F}_1\hat{M}^2\right.\right.\right.\\
&+ \left.\left.\left. F_12\hat{M}\dot{\hat{M}}\right) - F_1\dot{F}_4\hat{M}^2\right)\right) - F_2^2\left(F_1 - 3F_4\right)\dot{F}_2\hat{M}^2\\
&+ \left. F_3F_4\left(F_1 - 3F_4\right)\dot{F}_2\hat{M}^2\right) + m_0^2\left(-\left(-{F_2}^3\left(F_1\dot{\Omega} - 3F_4\dot{\Omega}\right.\right.\right.\\
&+ \left.\left.(\Omega+1)\left(\dot{F}_1 - 3\dot{F}_4\right)\right) + F_2\left(F_4\left(F_3\left(3F_4\dot{\Omega} - (\Omega+1)\dot{F}_1\right)\right.\right.\right.\\
&- \left.\left. 3F_4(\Omega+1)\dot{F}_3\right) + F_1\left(F_3\left((\Omega+1)\dot{F}_4 - F_4\dot{\Omega}\right) + F_4(\Omega+1)\dot{F}_3\right)\right)\\
&+ {F_2}^2(\Omega+1)\left(\left(F_1 - 3F_4\right)\dot{F}_2 + 2F_3F_4\right)\\
&+ \left.\left. F_3F_4(\Omega+1)\left(F_3F_4 - \left(F_1 - 3F_4\right)\dot{F}_2\right) + {F_2}^4(\Omega+1)\right)\right)\Big] ,\\
\mathcal{G}_2(t) &= 8\left(4m_0^2\left(-F_4(\Omega+1)\left(F_3F_4\left(2m_2^2 - \hat{M}^2\right) - \left(F_1 - 3F_4\right)m_2^2\dot{F}_2\right)\right.\right.\\
&+ F_4F_2^2(-(\Omega+1))\left(2m_2^2 - \hat{M}^2\right) + F_4\left(F_1 - 3F_4\right)F_2Hm_2^2(\Omega+1)\\
&+ 3F_2\left(F_4\left(3F_4\left((\Omega+1)2m_2\dot{m}_2 - m_2^2\dot{\Omega}\right) + m_2^2(\Omega+1)\dot{F}_1\right) + F_1\left(F_4\left(m_2^2\dot{\Omega}\right.\right.\right.\\
&- \left.\left.\left.\left.(\Omega+1)2m_2\dot{m}_2\right) - m_2^2(\Omega+1)\dot{F}_4\right)\right)\right) + 4\left(6F_2F_4\left(F_1 - 3F_4\right)Hm_2^2\hat{M}^2\right.\\
&+ F_4\hat{M}^2\left(F_3F_4\hat{M}^2 + 2\left(F_1 - 3F_4\right)m_2^2\dot{F}_2\right)\\
&- 2F_2\left(F_4^2\left(6m_2^2\hat{M}\dot{\hat{M}} - 3\hat{M}^2m_2^2\right) - F_4\left(m_2^2\left(\dot{F}_1\hat{M}^2 + 2F_1\hat{M}\dot{\hat{M}}\right)\right.\right.\\
&- \left.\left.\left. 2F_1\hat{M}m_2\dot{m}_2\right)F_1m_2^2\dot{F}_4\hat{M}^2\right) + F_2^2F_4\hat{M}^4\right)m_0^4F_4\left(F_2^2 + F_3F_4\right)(\Omega+1)^2\Big) ,\\
\mathcal{G}_3(t) &= 64F_4^2m_2^2\left(-4m_0^2(\Omega+1)\left(m_2^2 - \hat{M}^2\right) + 4\hat{M}^4 + m_0^4(\Omega+1)^2\right)
\end{aligned} \tag{4.118}$$

The kinetic and Gradient coefficients here are in a FLRW universe.

Here, we define the coefficients of action (4.20)

$$S = \int d^4x\, a^3 \left[\frac{a^2}{k^2}\left(Q\,\dot\delta_\phi^2 - \mathcal{G}\,\frac{k^2}{a^2}\,\delta_\phi^2\right)\right], \tag{4.119}$$

with

$$\begin{aligned}
Q &\equiv \frac{\mathcal{L}_{\dot\zeta\dot\zeta}^2\left(G{\alpha_3}^2\frac{k^2}{a^2} - [H(\eta_{\mathcal{L}} - \eta_3 + \eta_6 + 3)\alpha_3 - \alpha_6]\alpha_6\mathcal{L}_{\dot\zeta\dot\zeta}\right)\frac{k^2}{a^2}}{\left(\alpha_6\left(H\left(\eta_3 - \eta_6 - \eta_{\mathcal{L}} - 3\right)\alpha_3 + \alpha_6\right)\mathcal{L}_{\dot\zeta\dot\zeta} + \frac{k^2}{a^2}G{\alpha_3}^2\right)^2}, \qquad (4.120)\\
\mathcal{G} &\equiv \frac{\mathcal{L}_{\dot\zeta\dot\zeta}}{\left[\alpha_6\left(H\left(\eta_3 - \eta_6 - \eta_{\mathcal{L}} - 3\right)\alpha_3 + \alpha_6\right)\mathcal{L}_{\dot\zeta\dot\zeta} + \frac{k^2}{a^2}G\alpha_3^2\right]^2}\left(G^2\alpha_3^2\frac{k^4}{a^4}\right.\\
&+ G\{[{\eta_{\mathcal{L}}}^2 + (5 - 2\,\eta_3 - \eta_G + s_{\mathcal{L}})\eta_{\mathcal{L}} + \eta_3^2 + (\eta_G - s_3 - 5)\eta_3 - 3\,\eta_G\\
&+ 6]H^2{\alpha_3}^2 + 3\,H(\eta_3 - \eta_6 + 1/3\,\eta_G - 2/3\,\eta_{\mathcal{L}} - 5/3)\alpha_6\alpha_3 + {\alpha_6}^2\}\frac{k^2}{a^2}\mathcal{L}_{\dot\zeta\dot\zeta}\\
&+ H^2\eta_6[H({\eta_3}^2\alpha_3 - \eta_3\eta_6\alpha_3 - 2\,\eta_{\mathcal{L}}\eta_3\alpha_3 + \eta_3\alpha_3 s_3 - \eta_3\alpha_3 s_6 + \eta_{\mathcal{L}}\eta_6\alpha_3\\
&+ {\eta_{\mathcal{L}}}^2\alpha_3 + \eta_{\mathcal{L}}\alpha_3 s_6 - \eta_{\mathcal{L}}\alpha_3 s_{\mathcal{L}} - 6\,\eta_3\alpha_3 + 3\,\alpha_3\eta_6 + 6\,\alpha_3\eta_{\mathcal{L}}\\
&+ \left.3\,\alpha_3 s_6 + 9\,\alpha_3) + \alpha_6\eta_6 - \alpha_6\eta_{\mathcal{L}} - \alpha_6 s_6 - 3\,\alpha_6]\alpha_6\mathcal{L}_{\dot\zeta\dot\zeta}^2\right), \qquad (4.121)
\end{aligned}$$

and

$$s_{\mathcal{L}} \equiv \frac{\dot\eta_{\mathcal{L}}}{H\eta_{\mathcal{L}}}, \quad s_3 \equiv \frac{\dot\eta_3}{H\eta_3}, \quad s_6 \equiv \frac{\dot\eta_6}{H\eta_6}, \quad \eta_G = \frac{\dot G}{H\,G}. \tag{4.122}$$

Moreover, the explicit expressions for the α_i and μ_i coefficients in the dS limit used in Sec. 4.4.1 are

$$\begin{aligned}
\alpha_3(t,k) &= -\frac{2\left(F_1 - 3F_4\right)\left(2F_2H_0 + F_3 + 8\frac{k^2}{a^2}m_2^2\right)}{3H_0\left(F_2^2 + F_3F_4 + 8F_4\frac{k^2}{a^2}m_2^2\right)}\\
\alpha_6(t,k) &= \frac{2k^2}{a(t)^2}\,\frac{H_0\left(F_4H_0 - 2F_2\right)\left(2\hat M^2 + m_0^2(\Omega+1)\right) + F_2^2 + F_3F_4 + 8F_4\frac{k^2}{a^2}m_2^2}{3H_0^2\left(F_2^2 + F_3F_4 + 8F_4\frac{k^2}{a^2}m_2^2\right)},
\end{aligned} \tag{4.123}$$

and

$$\begin{aligned}
\mu_3(t,k) &= \frac{1}{\mathcal{L}_{\dot\zeta\dot\zeta}\left(\mathcal{L}_{\dot\zeta\dot\zeta}\left({\alpha_6}^2 - 3H_0\alpha_3\alpha_6 + \alpha_6\dot\alpha_3 - \alpha_3\dot\alpha_6\right) - \alpha_6\alpha_3\dot{\mathcal{L}}_{\dot\zeta\dot\zeta} + \frac{k^2}{a^2}G{\alpha_3}^2\right)}\\
&\times \left\{\mathcal{L}_{\dot\zeta\dot\zeta}\left(-6H_0\alpha_6\alpha_3\dot{\mathcal{L}}_{\dot\zeta\dot\zeta} + \alpha_6\alpha_3\ddot{\mathcal{L}}_{\dot\zeta\dot\zeta} + {\alpha_6}^2\dot{\mathcal{L}}_{\dot\zeta\dot\zeta} + 2\alpha_6\dot{\mathcal{L}}_{\dot\zeta\dot\zeta}\dot\alpha_3\right)\right.\\
&+ \mathcal{L}_{\dot\zeta\dot\zeta}^2\left(3H_0\alpha_6\left(2\dot\alpha_3 + \alpha_6\right) - 9H_0^2\alpha_3\alpha_6\right.
\end{aligned}$$

$$
\begin{aligned}
- & \quad \alpha_6 \left(2\dot{\alpha}_6 + \ddot{\alpha}_3\right) + \alpha_3\ddot{\alpha}_6) + 2\alpha_3\dot{\mathcal{L}}_{\dot{\zeta}\dot{\zeta}} \left(-\alpha_6\dot{\mathcal{L}}_{\dot{\zeta}\dot{\zeta}}\right) - \frac{k^2}{a^2}\mathcal{L}_{\dot{\zeta}\dot{\zeta}}{\alpha_3}^2\dot{G} \\
+ & \quad \frac{k^2}{a^2} G\alpha_3 \left(\mathcal{L}_{\dot{\zeta}\dot{\zeta}} \left(5H_0\alpha_3 - 2\dot{\alpha}_3\right) + 2\alpha_3\dot{\mathcal{L}}_{\dot{\zeta}\dot{\zeta}}\right)\Bigg\}
\end{aligned}
\tag{4.124}
$$

$$
\begin{aligned}
\mu_6(t,k) \quad = & \quad \frac{1}{a^2\mathcal{L}_{\dot{\zeta}\dot{\zeta}}\left(a^2\left(\mathcal{L}_{\dot{\zeta}\dot{\zeta}}\left(-3H_0\alpha_3\alpha_6 + \alpha_6\dot{\alpha}_3 - \alpha_3\dot{\alpha}_6 + {\alpha_6}^2\right) - \alpha_6\alpha_3\dot{\mathcal{L}}_{\dot{\zeta}\dot{\zeta}}\right) + k^2G{\alpha_3}^2\right)} \\
\times & \quad \Big\{a^2\left(a^2\left(\mathcal{L}_{\dot{\zeta}\dot{\zeta}}\left(-3H_0\alpha_3\left(-2\dot{\mathcal{L}}_{\dot{\zeta}\dot{\zeta}}\dot{\alpha}_6\right) - \dot{\mathcal{L}}_{\dot{\zeta}\dot{\zeta}}\left(2\dot{\alpha}_3 + \alpha_6\right)\dot{\alpha}_6\right.\right.\right. \\
+ & \quad \alpha_3\left(-\ddot{\mathcal{L}}_{\dot{\zeta}\dot{\zeta}}\dot{\alpha}_6 + \dot{\mathcal{L}}_{\dot{\zeta}\dot{\zeta}}\ddot{\alpha}_6\right)\Big) + \mathcal{L}^2_{\dot{\zeta}\dot{\zeta}}\left(-3H_0\left(\alpha_6\dot{\alpha}_6 + 2\dot{\alpha}_3\dot{\alpha}_6 - \alpha_3\ddot{\alpha}_6\right) + 9H_0^2\alpha_3\dot{\alpha}_6\right. \\
+ & \quad \dot{\alpha}_6\ddot{\alpha}_3 - \left(\dot{\alpha}_3 + \alpha_6\right)\ddot{\alpha}_6 + 2{\dot{\alpha}_6}^2\big) + \alpha_3\dot{\mathcal{L}}_{\dot{\zeta}\dot{\zeta}}\left(2\dot{\mathcal{L}}_{\dot{\zeta}\dot{\zeta}}\dot{\alpha}_6\right)\Big) \\
+ & \quad k^2\alpha_3\dot{G}\left(\mathcal{L}_{\dot{\zeta}\dot{\zeta}}\left(-3H_0\alpha_3 + \dot{\alpha}_3 + \alpha_6\right) - \alpha_3\dot{\mathcal{L}}_{\dot{\zeta}\dot{\zeta}}\right)\Big) \\
+ & \quad k^2a^2G\left(\alpha_3\left(5H_0\alpha_3\dot{\mathcal{L}}_{\dot{\zeta}\dot{\zeta}} + \alpha_3\ddot{\mathcal{L}}_{\dot{\zeta}\dot{\zeta}} - 2\alpha_6\dot{\mathcal{L}}_{\dot{\zeta}\dot{\zeta}} - 2\dot{\mathcal{L}}_{\dot{\zeta}\dot{\zeta}}\dot{\alpha}_3\right)\right. \\
+ & \quad \mathcal{L}_{\dot{\zeta}\dot{\zeta}}\left(-5H_0\alpha_3\left(\dot{\alpha}_3 + \alpha_6\right) + 6H_0^2{\alpha_3}^2 - 3\alpha_3\dot{\alpha}_6 + 2{\dot{\alpha}_3}^2 + 3\alpha_6\dot{\alpha}_3\right. \\
- & \quad \alpha_3\ddot{\alpha}_3 + {\alpha_6}^2\big)\big) + k^4G^2{\alpha_3}^2\Big\} \\
. &
\end{aligned}
\tag{4.125}
$$

5 The role of the tachyonic instability in Horndeski gravity

5.1 Introduction

In this Chapter we implement the conditions guaranteeing the absence of tachyonic instabilities, as derived in Chapter 2 and presented in Appendix 3.8, into the stability module of the Einstein-Boltzmann solver *EFTCAMB* [22, 23, 80]. This completes the stability module as it already encompasses the no-ghost and no-gradient conditions and guarantees stability on the whole range of linear scales. The already existing no-ghost and no-gradient, are in fact inherently high-k statements, and as such cannot guarantee stability on the whole range of scales. The common practice in Einstein-Boltzmann solvers so far, was to include a set of mathematical conditions designed to eliminate models with exponential growth of the perturbations. The latter are ad-hoc conditions derived, under some simplifying assumptions, at the level of the equation of motion implemented in the numerical codes. Here we show that this will not be necessary anymore and one can rely on the rigorous, theoretically motivated, set of conditions to ensure stability both in the high-k and low-k regime.

After the implementation we proceeded to study the impact of the novel conditions on the parameter space of Horndeski [78], as well as of its subclasses $f(R)$ gravity and Generalized Brans Dicke theories, identifying several interesting features. Among other things, we confirm that the constraining power of the stability conditions is significant and will certainly give an important contribution towards physically informed cosmological tests of gravity.

The work in this Chapter is based on [34]: *The role of the tachyonic instability in Horndeski gravity* with N. Frusciante, S. Peirone and A. Silvestri. In Sec. 5.2 we review the EFT of DE/MG and the formulation of the stability conditions. In Sec. 5.3 we present the class of models used in this study and their implementation in the EFToDE/MG language. In Sec. 5.4 we illustrate the methodology that allows us to build large ensembles of models. We also describe the parameter spaces that we use for studying the impact of the different stability conditions. Finally, in Sec. 5.5 we present and discuss the results and in Sec. 5.6 we conclude.

5.2 Stability conditions in the Effective Field Theory of dark energy and Modified Gravity

In this section we review the general conditions that a theory of gravity needs to satisfy in order to be free from instabilities, i.e. no-ghost, no-gradient and no-tachyon conditions [109]. As discussed in the Introduction, we include matter fields in our derivation, focusing on CDM, which is the relevant one for late times.

We employ the EFT of DE/MG framework, which is at the basis of the numerical code, *EFTCAMB*, that we use for our analysis. This framework offers a unified language for a broad class of DE/MG models with one additional scalar degree of freedom [14, 15]. The corresponding action is constructed in the unitary gauge as a quadratic expansion in perturbations and their derivatives, around a flat Friedmann-Lemaître-Robertson-Walker (FLRW) background. The different operators that enter in the action are spatial-diffeomorphism curvature invariants. For the purpose of this Chapter we restrict to Horndeski gravity, for which the corresponding EFToDE/MG action reads:

$$\begin{aligned}\mathcal{S} = \int d^4x\sqrt{-g}\left\{\frac{m_0^2}{2}(1+\Omega(t))R^{(4)} + \Lambda(t) - c(t)\delta g^{00}\right.\\ +\frac{\bar{M}_3^2(t)}{2}\left[(\delta K)^2 - \delta K^\mu_\nu \delta K^\nu_\mu - \frac{1}{2}\delta g^{00}\delta R^{(3)}\right] + \frac{M_2^4(t)}{2}(\delta g^{00})^2\\ \left.-\frac{\bar{M}_1^3(t)}{2}\delta g^{00}\delta K + L_m[g_{\mu\nu},\chi_m]\right\},\end{aligned} \tag{5.1}$$

where m_0^2 is the Planck mass, g the determinant of the four dimensional metric $g_{\mu\nu}$, δg^{00} the perturbation of the upper time-time component of the metric, $R^{(4)}$ and $R^{(3)}$ are respectively the trace of the four dimensional and three dimensional Ricci scalar, $K_{\mu\nu}$ is the extrinsic curvature and K its trace. Ω, c, Λ, M_i are free functions of time dubbed EFT functions. Finally, L_m is the matter Lagrangian for all matter fields. In this chapter we strictly follow the approach of Chapter 3 where we adopt the Sorkin-Schutz Lagrangian

After decomposing the action (5.1) into the actual perturbations of the metric and matter fields, and removing spurious d.o.f., one obtains the following action for the propagating d.o.f.s in Fourier space:

$$\mathcal{S}^{(2)} = \int \frac{d^3k}{(2\pi)^3} dt a^3 \left(\dot{\vec{\chi}}^t \mathbf{A} \dot{\vec{\chi}} - k^2 \vec{\chi}^t \mathbf{G} \vec{\chi} - \dot{\vec{\chi}}^t \mathbf{B} \vec{\chi} - \vec{\chi}^t \mathbf{M} \vec{\chi} \right) , \qquad (5.2)$$

where $\vec{\chi}^t \equiv (\zeta, \delta_d)$ with ζ the scalar degree of freedom and $\delta_d = \delta\rho/\bar{\rho}$ the density perturbations of the matter component. **A**, **B**, **G**, **M** are 2×2 time and scale dependent matrices (see Chapter 3 for their expressions) and dots indicate time derivatives with respect to cosmic time.

The stability requirements can be now obtained from action (5.2). In the following we will list them, discussing their relevance and range of applicability:

- *no-ghost:* Requiring the absence of ghosts translates into A_{ij} being positive definite. This must be done in the high-k limit as a low-k instability does not lead to a catastrophic vacuum collapse and is rather relatable to the Jeans instability, as discussed in [126].

- *no-gradient:* In order to avoid diverging solutions at high-k one needs to demand the speed of propagation to be positive, i.e. $c_s^2 > 0$. The speed of propagation can be obtained, after a diagonalization of the kinetic matrix A_{ij}, from the dispersion relations coming from action (5.2).

- *no-tachyon*: In the case of a single scalar canonical field, this amounts to demanding that either the mass term in the Lagrangian is positive or, in case it is negative, the rate of instability is slower than the Hubble rate. The latter case corresponds to the Jeans instability for the scalar d.o.f.. When matter fields are involved, one needs to study the mass matrix of the Hamiltonian associated to the canonical fields as illustrated in Chapter 3. For more details about the nature of the tachyon instability we refer the reader to [85]. Here, we will focus on the practical condition one has to impose in order to avoid such instability.

  The Hamiltonian associated to the action (5.2) assumes the following general form for the canonical d.o.f.s:

  $$\mathcal{H} = \frac{a^3}{2} \left[\dot{\Phi}_1^2 + \dot{\Phi}_2^2 + \mu_1(t,k)\, \Phi_1^2 + \mu_2(t,k)\, \Phi_2^2 \right] , \qquad (5.3)$$

  where Φ_i are the canonical fields and, for $k \to 0$, μ_i are the mass eigenvalues.* As discussed in Chapter 3, the above Hamiltonian

*It is important to note that the canonical field is a result of a number of field redefinitions, hence is a mix of the scalar and matter d.o.f..

exhibits a tachyonic instability when at low momenta a mass eigenvalue μ_i becomes negative and evolves rapidly, i.e. $|\mu_i| \gg H^2$. Thus, for a theory to be viable, it must hold that $|\mu_i| \lesssim H^2$. Formulated in this way one also includes theories with instabilities which evolve over scales much larger than the Hubble scale. This particular case is typical of clustering fluids, where the instability corresponds to the well known Jeans instability, which is vital for structure formation.

While we focused on the scalar d.o.f.s in the EFToDE/MG action, one can repeat the same expansion for the tensorial part. Starting from the quadratic action for tensor perturbations, one can then work out the equivalent conditions to ensure that tensor modes are free from ghost and gradient instabilities (see e.g. [14, 15, 65, 67, 68]).

From the above discussion, we have five conditions (three for the scalar sector and two for the tensor one) which need to be imposed in order to guarantee the stability of a theory at any scale and time.

The relevance of these conditions reflects also in the choice of the parameter space one has to sample when performing a fit to data. This was established in [23, 39, 138], where it was shown that they might dominate over the constraining power of cosmological data.

In Einstein-Boltzmann solvers the no-ghost and no-gradient conditions are commonly employed while the no-tachyon ones are typically not included. The former two conditions guarantee the stability at high-k, thus in order to guarantee stability on the whole k-spectrum, the codes usually employ ad-hoc conditions that eliminate models with exponentially growing modes at low-k. In the EFToDE/MG framework, these additional requirements are typically worked out at the level of the dynamical equation for the perturbations of the scalar field. While in action (5.1) the scalar field is hidden inside the metric degrees of freedom, one can leave the unitary gauge by the Stückelberg trick and make explicit the perturbations associated to the scalar field. This amounts to an infinitesimal time coordinate transformation $t \to t + \pi$, with the scalar degree of freedom being described by π and obeying the following equations of motion:

$$A\pi'' + B\pi' + C\pi + k^2 D\pi + H_0 E = 0, \tag{5.4}$$

where A, B, C, D, E are functions of time and k and their explicit expressions can be found in [80]. H_0 is the present day value of the Hubble parameter and primes are derivatives with respect to conformal time.

The corresponding mathematical conditions are [80]:

- if $B^2 - 4A(C + k^2 D) > 0$ then

$$\frac{-B \pm \sqrt{B^2 - 4A(C + k^2 D)}}{2A} < H_0\,, \tag{5.5}$$

- if $B^2 - 4A(C + k^2 D) < 0$ then

$$\frac{-B}{2A} < H_0\,. \tag{5.6}$$

The no-ghost and no-gradient conditions, as well as the mathematical ones, are implemented in the publicly available version of the Einstein-Boltzman solver *EFTCAMB* [22, 80], respectively under the name of *physical* and *mathematical* (math) conditions. Although it would be reasonable to add the no-tachyon conditions under the umbrella of physical conditions, in this Chapter we stick to the original convention and retain the term physical for the pair of no-ghost and no-gradient. We always refer separately to the no-tachyon one as the *mass* condition.

In this work, we extend the stability module of *EFTCAMB* to include the mass conditions in terms of the mass eigenvalues μ_i and proceed to study the impact of the latter on the parameter space of different scalar-tensor theories within Horndeski gravity. Our first goal is to show that the physical plus mass conditions form a complete set of physically motivated, rigorously derived requirements that do guarantee stability on the whole range of linear cosmological scales. We also compare the mass and math conditions in terms of performance, showing that the latter can be safely disregarded in favor of the former. Finally we study the effects of the different conditions on the parameter space of the different theories, identifying some noteworthy features.

5.3 Models

We consider several classes of scalar-tensor models of gravity:

- f(R) [11]: specifically designer $f(R)$ [12, 81] with a wCDM background. For any value of the equation of state, w_0, the different models reproducing the corresponding expansion history can be labeled by the present value of the Compton wavelength of the scalaron, namely $B_0 = B(z = 0)$, where

$$B(z) = \frac{f_{RR}}{1 + f_R}\frac{H\dot{R}}{\dot{H}}\,. \tag{5.7}$$

  Hence we have a two-dimensional model parameter space, i.e. $\{w_0, B_0\}$. These models can be fully mapped in the EFToDE/MG language, and the only corresponding non-zero EFT functions are: $\Omega = f_R$ and $\Lambda = \frac{m_0^2}{2}(f - Rf_R)$.

- Generalized Brans Dicke (GBD): non-minimally coupled scalar-tensor theories with a canonical kinetic term. f(R) gravity is a sub-class of these theories, with a fixed coupling to matter. When one allows the coupling to vary, a representative class is that of Jordan-Brans-Dicke (JBD) models [139]. These models correspond, in the EFToDE/MG language, to non-zero $\{\Omega, \Lambda, c\}$. In this work we adopt the so-called 'pure EFT' approach, where we simply explore several different choices for $\{\Omega, \Lambda, c\}$ as functions of time, thus creating a large ensemble of GBD models. For more details on this approach we refer the reader to [140, 141].

- Horndeski (Hor): the full class of second order scalar-tensor theories as identified by Horndeski [78]. Within the EFToDE/MG formalism, we can explore them by turning on the full set of EFT functions in action (5.1), i.e. $\{\Omega, \Lambda, c, M_2^4, \bar{M}_1^3, \bar{M}_3^2\}$. We also consider separately the subset of Horndeski for which the speed of sound of tensor is equal to that of light, $c_t^2 = 1$. Most of the modifications happen in the scalar sector of the theory, hence we refer to this class as H_S. This specific class has become of great interest after the detection of the gravitational wave GW170817 and its electromagnetic counter part GRB170817A [142–144], which has set tight constraints on the speed of propagation of tensor modes. We can create large ensembles of these models in the pure EFToDE/MG approach, by turning on the following EFT functions: $\{\Omega, \Lambda, c, M_2^4, \bar{M}_1^3\}$.

5.4 Methodology

We aim at studying in detail the way different sets of stability conditions affect the parameter space of the models under consideration. We always impose the set of physical stability conditions, i.e. no-ghost and no-gradient, as a baseline; on top, we separately switch on the checks for either the math (5.5 or 5.6) or the mass condition. For one class of models, namely $f(R)$, we consider an additional condition, as desrcibed in the following.

f(R)-gravity is among the models for which the stability conditions have been extensively investigated [12, 81, 145–147]. Besides the usual no-ghost and no-gradient conditions, an additional important requirement has been identified in the literature by demanding the high curvature regime to be stable against small perturbations. This translates into requiring a positive mass squared for the scalaron, in the limit of $|Rf_{RR}| \ll 1$ and $f_R \to 0$

$$m^2_{f_R} \approx \frac{1+f_R}{3f_{RR}} \approx \frac{1}{3f_{RR}}\,, \tag{5.8}$$

which leads to the well known $f_{RR} > 0$ constraint, often dubbed as the tachyon condition for $f(R)$. Since such condition has been obtained under specific hypothesis, it does not coincide analytically with the general no-tachyon conditions discussed in Sec. 5.2. In the analysis of $f(R)$, we include this latter condition as one of the case studies, to compare with the mass and math ones.

To study the viable parameter space of all the models under the different sets of stability conditions, we use the numerical framework adopted in [140, 141]. It consists of a Monte Carlo (MC) code which samples the space of the EFT functions, building a statistically significant ensemble of viable models. To compute wether a sampled model is stable (and thus accepted by the sampler) or not, we interface the MC code with the publicly available Einstein-Boltzmann solver *EFTCAMB* [22, 23]. Starting from the background solution, the code undergoes a built-in check for the stability of the model.

In *EFTCAMB*, $f(R)$ is implemented via the so-called 'mapping' mode, i.e. as a specific model after being mapped to the EFToDE/MG language [80]. Currently both the Hu-Sawicki model [82] and designer $f(R)$ models are available. For our study we focus on the latter one, choosing a wCDM background. For the remaining three classes of models we adopt the 'pure EFT' approach, where we explore many different choices for the time-dependence of the corresponding EFT functions. Specifically, following [140, 141, 148], we parametrize the relevant EFT functions using a Padé expansion:

$$f(a) = \frac{\sum_{n=1}^{N} \alpha_n \left(a-a_0\right)^{n-1}}{1+\sum_{m=1}^{M} \beta_m \left(a-a_0\right)^{m}}\,, \tag{5.9}$$

where the truncation orders are given by N and M. The coefficients α_n and β_m are sampled with uniform prior in the range $[-1, 1]$ and we verified that the results are not sensitive to the prior range. Furthermore, the convergence of the results is reached at $N + M = 9$, meaning that each EFT function has 9 free parameters. We consider, with equal weight, expansions around $a_0 = 0$ and $a_0 = 1$ to represent thawing and freezing models, respectively. For further details about the sampling procedure we refer the reader to [140, 141].

In this pure EFToDE/MG approach any choice of Λ and Ω produces a different background expansion history, which can be solved for using the Friedmann equation, as explained in [140]. The remaining background

EFT function, c, can be determined in terms of Λ, Ω and $H(a)$, and does not need to be sampled independently.

Following this procedure, we build large numerical samples of viable models for the GBD, H_S and Hor classes reaching a samples size of $\sim 10^4$ accepted models.

For $f(R)$ gravity we study the impact of stability conditions on the $\{w_0, B_0\}$ parameter space, comparing also to previous results [23]. For the remaining classes of models the dimension of the parameter space is very high (e.g. GBD has 27 additional parameters) and, furthermore, the individual parameters in the Padé expansion do not have any direct physical meaning. For such reasons we look at their predictions for the phenomenology of Large Scale Structure, studying the cuts of different stability criteria on the, physical, space of the phenomenological functions (μ, Σ) defined in the usual way [149]:

$$k^2\Psi = -4\pi G\mu(a,k)a^2\rho\Delta \ , \tag{5.10}$$

$$k^2(\Phi+\Psi) = -8\pi G\,\Sigma(a,k)\,a^2\rho\Delta \ , \tag{5.11}$$

where ρ is the background matter density, $\Delta = \delta + 3aHv/k$ is the comoving density contrast, and Ψ and Φ are the scalar perturbations, respectively, to the time-time and spatial components of the metric in conformal Newtonian gauge. From their definition, $\mu = \Sigma = 1$ in ΛCDM, but in general they are functions of time and scale. The function μ directly affects the clustering and the peculiar motion of galaxies, hence it is well constrained by galaxy clustering and redshift space distortion measurements [150–152]. On the other hand, Σ affects the geodesics of light and is directly measured by Weak Lensing, Cosmic Microwave Background and galaxy number counts experiments [152–154].

The *EFTCAMB* software allows us, in principle, to evolve the full dynamics of linear perturbations and extract the exact form of Σ and μ for each model in our ensembles. This however would be highly time-consuming, hence we opt for the Quasi Static (QS) analytical expressions of these functions, worked out from the modified Einstein equations after reducing them to an algebraic set (in Fourier space) by neglecting time derivatives of the scalar degrees of freedom [15, 155]. In [141] the authors have compared the QS and exact (Σ, μ) for the same ensembles of models as those considered in this Chapter, finding that the agreement is excellent for all scales below the typical Compton wavelength of the sampled model.

In order to visualize the effect of different stability cuts, we show the predictions of a given ensemble of models in the (Σ, μ) plane for a given value of scale and redshift. We set our output scale at $k = 0.01h$ Mpc^{-1}, which has been shown to be safely inside the Compton scale for

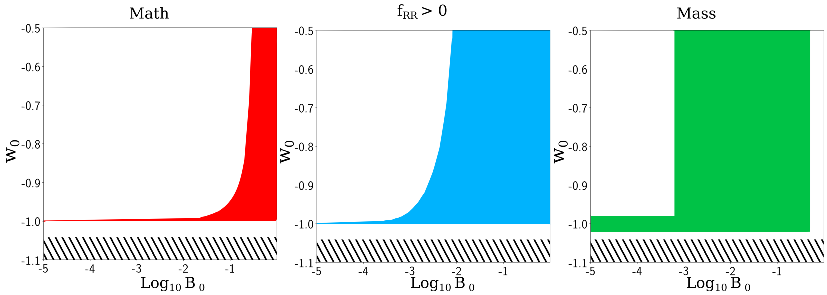

Figure 5.1: The viable (B_0, w_0) parameter space of designer $f(R)$ gravity on a wCDM background. We plot the regions allowed by different stability conditions. The physical conditions are imposed as a baseline in all cases and cut the lower region highlighted by black lines. On top of them, we apply separately the math (left panel), $f_{RR} > 0$ (central panel) and the mass (right panel) conditions. The corresponding viable regions are indicated with solid colors, respectively in red for math, blue for $f_{RR} > 0$ and green for mass.

all models considered in the present analysis [141]. For this scale, we show the results at a given value of the scale factor that we choose to be $a = 0.9$, corresponding to a redshift $z \approx 0.1$. This choice is more realistic than $a = 1$ in terms of measurements from upcoming surveys.

5.5 Results

We now proceed to discuss the outcome of our analysis, focusing on the cuts in the different parameter spaces considered. When studying the GBD and Horndeski classes, through the large ensemble of models generated via the Monte Carlo sampling, we also analyze the acceptance rates of models when the different conditions are turned on.

5.5.1 Designer $f(R)$ on wCDM background

We studied designer $f(R)$ on a wCDM background, considering flat priors on $w_0 \in [-1.1, -0.5]$ and $Log_{10}\, B_0 \in [-5, 0]$, while fixing the cosmological parameters to Planck 2015 ΛCDM values [128]. We also tested that our results do not change when assuming the values from Planck 2018 [156], as they are mostly compatible with the 2015 release.

In Fig. 5.1 we show the impact of the math, $f_{RR} > 0$ and mass conditions on the (B_0, w_0) space. As it was already known, the physical conditions do not constrain B_0 while they clearly constrain the equation of state to $w_0 > -1.04$. Going beyond the baseline, it is evident that the math condition constrains the parameter space most severely, pushing

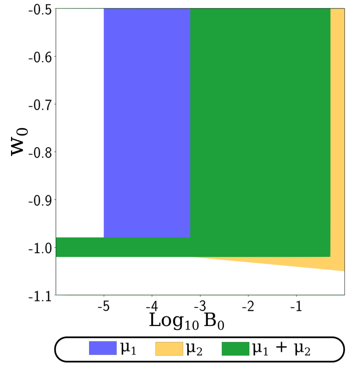

Figure 5.2: Allowed parameter space for designer $f(R)$ gravity with wCDM background, when applying the physical conditions jointly with the mass conditions considered separately, i.e. the μ_1 (blue) or μ_2 (yellow) stability cuts. In green we show the combined viable region.

w_0 towards $w_0 = -1$ for values of $B_0 \lesssim 10^{-2}$. On the other hand, $f_{RR} > 0$ and mass condition have a very similar, and less severe, impact. We have studied the cosmology of a number of models excluded by the math conditions but allowed by the f_{RR} and mass conditions. They all exhibited stable behaviors, hence we infer the math conditions for $f(R)$ are too stringent.

The viable parameter space of the same designer $f(R)$ was studied in [23] under the no-ghost, no-gradient and the $f_{RR} > 0$ requirements. Their findings are in line with our results. Let us now look at Fig. 5.2, where we consider separately the constraints coming from the individual mass eigenvalues. One thing that can be noticed, is that both mass conditions are important, cutting regions of the parameter space that are partially complementary. The combined effect is similar to that of the f_{RR} condition, and this is a direct result of the mixing of the scalar and matter d.o.f., which plays an important role in the determination of the full no-tachyon conditions.

5.5.2 Horndeski

The results for Generalized Brans Dicke models (GBD), Horndeski with $c_t^2 = 1$ (H_S) and full Horndeski (Hor) are presented as marginalized 2D and 1D distributions for the phenomenological functions $\Sigma - 1$ and $\mu - 1$, at $a = 0.9$ and $k = 0.01h\text{Mpc}^{-1}$. In all cases the sampling was done till 10^4 models were accepted by the set of stability conditions under consideration.

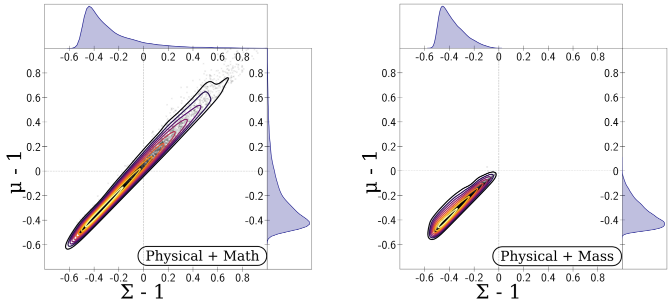

Figure 5.3: Marginalized 2D and 1D distributions for the phenomenological functions $\Sigma - 1$ and $\mu - 1$ for the GBD models, computed at $a = 0.9$ and $k = 0.01h\mathrm{Mpc}^{-1}$. In the two panels we show the results of the different stability cuts: on the *left* physical+math and on the *right* physical+mass. Black shaded points represent the values computed for the single sampled models, while the contour lines cut the 2D distribution at 10%, 20%, . . . , 90% of the total sample. These results are computed within the QSA.

In Fig. 5.3 we show the marginalized 2D and 1D distributions for (Σ, μ) for the ensemble of GBD models, when applying the math and mass stability cuts, on top of the usual physical baseline. It becomes instantly clear that the mass condition has a stronger impact on the plane than the math conditions. While the math conditions generally allow models satisfying the relation $(\mu - 1)(\Sigma - 1) > 0$, the mass conditions break the degeneracy and allow purely models with $(\mu - 1), (\Sigma - 1) < 0$.* In terms of EFT functions this result translates into the mass conditions requiring $\Omega > 0$, since in the QSA $\Sigma \sim \frac{1}{1+\Omega}$. The difference between mass and math conditions is quite relevant when analyzed in the (Σ, μ) plane, yet in terms of acceptance rates, the two conditions do not differ much, as shown in Table 5.1. In retrospect, one notices that in fact the bulk of viable models were in the $(\mu - 1), (\Sigma - 1) < 0$ quadrant already for the math conditions case. Still, it is noteworthy that the mass conditions in GBD models forbid more distinctively the first quadrant.

In Fig. 5.4 the same combinations of stability conditions are studied in the phenomenological plane of H_S and Hor models. In this case, the difference between the mass and math conditions is not so evident in terms of allowed regions in the (Σ, μ) plane. Nevertheless, we can notice that the math conditions allow the ensembles of models to have a tail in the second quadrant ($\mu > 1$ and $\Sigma < 1$), while this tail is drastically cut by the requirement of mass stability. In fact the models lying in the second quadrant are reduced from $\sim 1\%$ in the former case to $\sim 0.1\%$ in

*This boundary when imposed by the math condition is rather sharp, while imposed by the mass condition can be violated by a statistically negligible number of models (0.01% of the total sample).

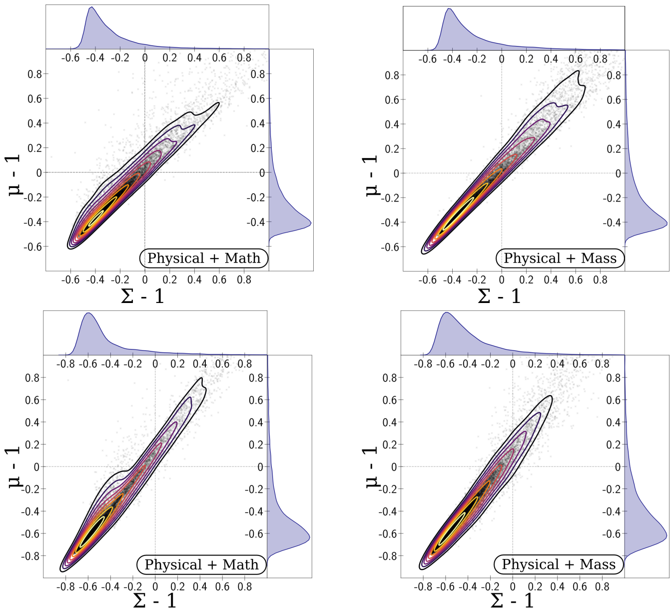

Figure 5.4: Marginalized 2D and 1D distributions for the phenomenological functions $\Sigma - 1$ and $\mu - 1$ for H_S (top panels) and Hor (bottom panels), computed at $a = 0.9$ and $k = 0.01h\text{Mpc}^{-1}$. In the two panels we show the results of the different stability cuts: on the *left* physical+math and on the *right* physical+mass. Black shaded points represent the values computed for the single sampled models, while the contour lines cut the two dimensional distribution at 10%, 20%, ..., 90% of the total sample. These results are computed within the QSA.

the latter.

As we discussed in Sec. 5.4, we adopt a Monte Carlo sampling technique to create ensembles of models that obey different sets of stability conditions. It is quite informative to compare the acceptance rates for different stability conditions and we present their percentage values in Table 5.1. It can be immediately noticed that the baseline of physical conditions has a quite strong impact on Horndeski models, a result previously discussed in [141]. Focusing on the mass and math conditions a general trend emerges, the mass conditions are more stringent than the math conditions. For GBD and H_S, this effect is stronger than for Hor where the impact of the two conditions is fairly similar. Finally, let us notice that, while in the (Σ, μ) plane of H_S and Hor there were no striking differences between mass and math conditions, looking at the acceptance rates we do see some tangible differences. This can be easily understood in terms of the large number of free EFT functions that describe these models. In other words, mass conditions do cut more

	GBD (%)	H_S (%)	Hor (%)
physical	18	8.3	1.2
physical + math	15	1.9	0.3
physical + mass	13	1.1	0.2

Table 5.1: Acceptance rates for the GBD, H_S and Hor classes of models as subjected to the different sets of stability requirements: physical, physical + math and physical + mass.

models than math conditions, yet when we look at the phenomenology, i.e. at the (Σ, μ) plane, there is high degeneracy among the models and we can not associate specific regions to these cuts. This is in stark contrast to GBD where the additional cut contributed by mass has a specific direction in the (Σ, μ) plane due to the fact that mainly one EFT function, Ω, is being constrained.

5.6 Conclusions

In the final work presented in this thesis we test the impact of the conditions for the avoidance of tachyon instabilities in scalar-tensor theories, concluding the line of work. These conditions are crucial to guarantee the stability of theories on the whole range of linear scales, complementing the no-ghost and no-gradient conditions. The latter are the ones most commonly implemented in Einstein-Boltzmann solvers, but being intrinsically high-k conditions, they can not guarantee stability on all scales. So far this shortcoming was addressed with a set of *mathematical* conditions built on the additional scalar field equation, in such a way to filter out models with exponentially growing solutions that would have escaped the no-ghost and no-gradient check.

A complete derivation of the mass condition in Horndeski, and more general modified gravity models, was carried out in Chapter 2 and we have used those results as the basis for our analysis, implementing the corresponding conditions in the stability module of *EFTCAMB*. We then have carried out an extensive study of the impact of the new conditions on the parameter space of Horndeski gravity, in particular comparing the corresponding cuts to those previously contributed by the mathematical conditions, as well as the improvement brought upon the incomplete set of no-ghost and no-gradient.

Overall, we show that, as expected, the mass condition provides the missing constraining power at low-k. Combined with the no-ghost and no-gradient, they form a complete set of conditions that guarantee the stability of any theory over all cosmological scales. Additionally, while the mathematical conditions depended on a number of simplifying assumptions about the nature of the dynamical equation for the extra degree of freedom, the mass conditions are more rigorously defined from a physical point of view. They are obtained from the stability analysis of the full action and do not rely on any such simplifying assumptions.

In our analysis, we considered a set of different scalar-tensor theories, as implemented in *EFTCAMB*, namely $f(R)$ gravity, Generalized Brans Dicke (GBD) models, the full Horndeski theory, as well as the subset of it that does not alter the speed of tensor modes. In all cases, we used the combination of no-ghost and no-gradient as the baseline and, on top of that, compared the performance of mathematical versus mass conditions. As mentioned above, the mass condition proves to be a very reliable substitute of the mathematical condition in all cases. On top of this, there are some features peculiar to specific models that are worth summarizing. One of the most evident results of this work is that, in GBD models, the mass condition removes more efficiently models away from the $\Sigma\,,\,\mu > 1$ region, clearly cutting the tail of models that were allowed by the mathematical conditions. This indicates that GBD models with either μ and/or Σ bigger than one would develop a tachyon instability. While this feature is interesting and clearly stands out in the (Σ, μ) plane of Fig. 5.3 it is important to notice that, from the acceptance rates in Table 5.1, the tail is a small fraction of the whole ensemble of models which tend to live in the $\Sigma < 1\,,\,\mu < 1$ quadrant. Interestingly the latter would be severely constrained if one imposes consistency with local tests of gravity, as shown in [141].

It is also worth stressing that in the case of $f(R)$ gravity, we compared the mass condition not only with the mathematical one, but also with the popular $f_{\rm RR} > 0$ condition, which is based on the stability of the theory in the high-curvature regime. As shown in Fig. 5.1, we found that the ad-hoc mathematical condition is too stringent in the $f(R)$ case, while the $f_{\rm RR} > 0$ and mass ones contribute an almost equivalent, and more generous, cut to the parameter space.

For the full Horndeski class, as well as the sub-class obeying $c_t^2 = 1$ at all times, we do not report significant differences between the impact of mathematical and mass conditions. But we highlight that the mass condition completes the no-ghost and no-gradient into a reliable set of conditions that guarantees stability on all linear scales, while being physically informed.

As we have shown, the combination of no-ghost, no-gradient and no-tachyon forms a theoretically rigorous and practically important set of conditions that guarantees stability on all linear cosmological scales. Finally, let us notice that, while in this work we focused on Horndeski gravity, the mass conditions derived in Chapter 2 and implemented in the stability module, can cover beyond Horndeski models as well.

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Summary

The discovery of the accelerating expansion of our universe, dubbed cosmic acceleration, has been one of the biggest in recent history and as such sent ripples through the cosmological community. Combined with that of Dark Matter, it led to the formulation of a concordance model, dubbed ΛCDM. In order to explain cosmic acceleration, that model postulates the existence of a, so-called, Cosmological Constant in the action of General Relativity. First introduced by Albert Einstein, this constant can be interpreted as the energy of the vacuum. The Dark Matter sector is represented by a presureless fluid composed of slowly moving particles, also called Cold Dark Matter. Up to this day, ΛCDM has been very succesfull in explaining the vast quantity of observations, a remarkable feat in view of the simplicity of the model.

Despite its success, ΛCDM is problematic from a theoretical point of view. In particular, the interpretation of the Cosmological Constant as the energy of the vacuum carries a fundamental issue. The observed value, in terms of the Planck Mass, is of the order of $\Lambda_{obs} \sim (10^{-30} M_{pl})^4$. On the other hand, when calculating the expected value of the Cosmological Constant within the Standard Model, the value turns out to be larger by 60 orders of magnitude. This discrepancy motivated the search for alternatives to the Cosmological Constant, creating a vast landscape of gravitational models which explain the observed acceleration in a multitude of ways.

Most models explaining cosmic acceleration introduce new degrees of freedom in addition to those of General Relativity, ranging from scalar fields all the way to including additional tensor fields. The resulting wealth of models make the challenge of ΛCDM very inefficient, as one needs to compare each model individually to the available observational data. This process can be streamlined by constraining the parameter space of models through a set of theoretical conditions, which eliminate models containing theoretical instabilities.

In this thesis we present the derivation, and the subsequent test, of a complete set of conditions which any model extending General Relativity must satisfy, in order to be considered a viable cosmological candidate. Our work focused on the subcase of models which introduces an additional scalar field in addition to General Relativity, as it is the most diverse and broadly populated set of models. An example of such a scalar-tensor

theory is the Horndeski theory which, for a long time, was considered the most general scalar-tensor theory with second order equations of motion. In order to achieve our goal we employ the unifying and effective framework provided by the Effective Field Theory of Dark Energy and Modified Gravity. This allows us to construct the desired conditions in a model-independent and efficient way.

The set of conditions under consideration guarantees the absence of ghost, gradient and tachyonic instabilities. In the literature the tachyonic instability has been severely neglected and therefore deserves particular attention. In the first three Chapters we proceed to derive this set of conditions in, subsequently, vacuum, in the presence of two dominant matter fields, radiation and Cold Dark Matter, and in the de-Sitter limit.

While the set of conditions derived in the presence of matter fields is the physically interesting one, it is a substantially more involved problem and one usually chooses the vacuum case for determining the conditions. Our work covers this deficiency and allows us to compare the two cases and to quantify the errors when neglecting matter. It is found that the ghost and gradient conditions remain largely unchanged with respect to the vacuum case, partially validating the assumption. In the case of beyond Horndeski, an extension of the standard Horndeski models, the previous statement does not hold as the speed of propagation of the new scalar degree of freedom is modified by the presence of the radiation field. In contrast, the condition guaranteeing the absence of the tachyonic instability changes drastically as every additional field introduces a new, non-trivial, condition, modifying the vacuum results beyond recognition. This is a direct effect of the gravitational interactions between the new scalar degree of freedom and the matter fields. The new tachyonic conditions constitute the main results of this thesis with a high potential impact on future work.

Having derived the new conditions guaranteeing the absence of tachyonic instabilities it is paramount to test and exhibit their constraining power. In Chapter five we proceed to do this with the help of EFTCAMB, a patch of the Boltzmann solver CAMB, which incorporates the EFToDE/MG. Up till this work, in order to guarantee the stability of models at all length scales, a set of ad-hoc mathematical conditions were employed which, while seemingly effective, were based on some limiting assumptions. Thus we wished to substitute them with the tachyonic conditions and compare the results. The comparison yields a similar constraining power between the two types of conditions with the tachyonic ones being slightly more constraining. Subsequently, we proceed to study the impact on the (μ, Σ) parameter space which encodes the deviations from General Relativity in the Poisson and lensing equation respectively. This leads to the observation of some noticeable effects, in particular for non-minimally coupled theories with a canonical kinetic term, known as Generalised Brans Dicke (GBD). As the tachyonic conditions are theoretically well motivated, in contrast to the mathematical ones, it is evident that they deserve to be chosen as the conditions guaranteeing a stable theory at large length scales.

In parallel to the work on the viability conditions we enhance the EFToDE/MG framework in a number of ways. In the first Chapter we present an expansion of the EFToDE/MG formalism to include the Hořava gravity model, a potential candidate for both quantum gravity as well as cosmic acceleration. Subsequently, we provide a complete dictionary mapping individual models into the unifying framework. This includes all known scalar field models, with the exception of the newly developed DHOST models which require an additional operator, not taken into consideration. Finally, we present a new basis for the operators, inspired by the ReParametrized Horndeski or "alpha" basis which aims to make the physical effects of the operators clearer.

The work presented in this thesis represents a research line which has reached its natural end. It is now important to turn to applications of the presented theoretical results. An initial step in this direction is presented in Chapter five where a number of parameter space studies are presented. These studies will be extended as more data become available and it becomes increasingly possible to constrain models for cosmic acceleration. Furthermore, the vast amount of data allows one to directly reconstruct cosmological functions in a model-independent way. These reconstructions can then use the derived theoretical conditions as a guiding principle, avoiding parameter space sections which cannot belong to a viable extension of gravity. It is thus our hope that this body of work will play an important role in the ongoing golden era of cosmology.

Samenvatting

De ontdekking van de versnellende expansie van ons universum is één van de grootste in de geschiedenis van de natuurkunde en heeft verregaande gevolgen binnen de kosmologische sector. Samen met de ontdekking van Koude Donkere Materie heeft dit geleid tot de ontwikkeling van hét kosmologische concordantiemodel, ΛCDM. In dit model wordt de kosmische expansie uitgelegd door middel van een Kosmologische Konstante in de uitdrukking van de actie van de Algemene Relativiteitstheorie. Als eerste geïntroduceerd door Albert Einstein, kan deze konstante geïnterpreteerd worden als de energie van het vacuum. De donkere materie sector wordt gerepresenteerd door middel van een drukloze vloeistof, bestaand uit langzaam bewegende deeltjes, genaamd Koude Donkere Materie (Cold Dark Matter). Tot vandaag blijkt ΛCDM zeer succesvol in het beschrijven van onze kosmologische waarnemingen, een opmerkelijk resultaat gezien de eenvoud van het model.

Echter, ΛCDM is, vanuit een theoretisch oogpunt, problematisch. Om specifiek te zijn, de interpretatie van de Kosmologische Konstante als de energie van het vacuum gaat gepaard met een fundamenteel probleem. De waargenomen waarde van deze konstante, in termen van de Planck Massa, is van de orde $\Lambda_{obs} \sim (10^{-30} M_{pl})^4$. Wanneer men, via het Standaard Model, de verwachtingswaarde van de Kosmologische Konstante berekent blijkt er een verschil te zijn van 60 ordes van grootte. Dit verschil ligt aan de basis van een intensieve zoektocht naar alternatieven. Dit heeft geresulteerd in een weids landschap van zwaartekracht modellen die de waargenomen kosmische versnelling op allerlei manieren proberen uit te leggen.

De meeste modellen introduceren nieuwe vrijheidsgraden, naast die van de Algemene Relativiteitstheorie, variërend van scalaire velden tot extra tensor velden. Zo is een bijna onbeperkte verzameling theoriën van zwaartekracht ontstaan. Het is onmogelijk al deze theorieën efficient met ΛCDM te vergelijken, aangezien elk model individueel met de beschikbare kosmologische data moet worden geconfronteerd. Dit process kan worden gestroomlijnd door de parameter ruimte in te perken door te eisen dat theoretische instabiliteiten moeten worden vermeden.

In dit proefschrift geef ik een afleiding van een komplete verzameling voorwaarden, waar elke uitbreiding van de Algemene Relativiteitstheorie aan moet voldoen om een levensvatbaar kosmologisch model te zijn. En vervolgens toets ik mijn resultaat. Ik heb mij beperkt tot de groep van modellen die een extra scalair veld toevoegen aan de Algemene Relativiteitstheorie, aangezien dit de grootste groep is. Een bekend voorbeeld hiervan is de Horndeski theorie die lang als de meest algemene scalaire-tensor theorie met tweedegraads bewegingsvergelijkingen werd gezien. Om ons doel te bereiken heb ik gebruik gemaakt van het verenigend en effectief kader van de “Effective Field Theory of Dark Energy and Modified Gravity” . Dit maakt het mogelijk om de voorwaarden op een modelonafhankelijke en efficiënte wijze te construeren.

De voorwaarden moeten garanderen dat er geen spook-, gradiënt- of tachyon-instabiliteiten ontstaan. De tachyon-instabiliteit is totaal verwaarloosd in de literatuur en krijgt uitgebreid de aandacht. In de eerste drie Hoofdstukken worden de voorwaarden afgeleid in,achtereenvolgens, vacuüm, in het bijzijn van twee dominante materie velden, straling en Koude Donkere Materie, en de De Sitter limiet.

Alhoewel de voorwaarden onder de invloed van materie vanuit een natuurkundig oogpunt de meest interessante zijn is de afleiding daarvan erg ingewikkeld. Daarom wordt meestal de aanname gemaakt dat de materie verwaarloosd kan worden. Ons werk lijdt niet aan deze tekortkoming en geeft ons de mogelijkheid om de invloed ervan te kwantificeren. De voorwarden om spook- en gradiënt- instabiliteiten te vermijden blijven grotendeels onveranderd. In het geval van het “beyond Horndeski model”, een uitbreiding van het Horndeski model, is wel een afwijking gevonden. De geluidssnelheid van de scalaire vrijheidsgraad verandert onder invloed van een stralings veld. In tegenstelling tot de spook- en gradient-instabiliteiten, is de tachyon instabiliteit wel degelijk sterk afhankelijk van eventuele materie velden. Elk meegenomen materie veld introduceert een extra voorwaarde. Dit is een direct resultaat van de gravitationele interactie tussen de scalaire vrijheidsgraad en de materie velden. Deze, nieuwe, tachyonische voorwaarden zijn het belangrijkste resultaat van dit proefschift, met mogelijk grote gevolgen voor toekomstig werk.

Na de afleiding van de nieuwe voorwaarden, die de absentie van mogelijke tachyon instabiliteiten garanderen, is het belangrijk om ze te toetsen en hun kracht in het inperken van de parameter ruimte aan te tonen. In Hoofdstuk vijf doen we dit met behulp van EFTCAMB, een patch van de bekende Boltzmann solver, CAMB, waarin de EFToDE/MG is toevoegd. Om de stabiliteit van de modellen op alle lengteschalen te garanderen, werden, tot dan, een aantal ad-hoc wiskundige voorwaarden gebruikt die gebaseerd waren op een aantal limiterende aannames. Het praktische doel van mijn onderzoek was om deze wiskundige aannames te vervangen door de nieuwe tachyonische voorwaarden en de resultaten te vergelijken. Daarom hebben wij de invloed van deze voorwaarden op de (μ, Σ) parameter ruimte getest. Deze ruimte encodeert de afwijking van de Algemene Relativiteitstheorie in, respectievelijk, de Poisson en lens vergelijking. Deze toets laat sterke afwijkingen zien in de niet-minimaal gekoppelde velden, beter bekend onder de naam "Generalised Brans Dicke" (GBD). Aangezien de tachyonische voorwaarden een sterke theoretische fundering hebben, in tegenstelling tot de wiskundige voorwaarden, is het duidelijk dat zij de voorkeur genieten als de voorwaarden die stabiliteit op grote lengteschalen garanderen .

Naast het werk aan de stabiliteits voorwaarden heb ik ook gewerkt aan het formalisme van EFToDE/MG. In het eerste Hoofdstuk heb ik een uitbreiding van het EFToDE/MG formalisme gepresenteerd die het Hořava gravitatiemodel toevoegt. Dit model is interessant omdat het een kandidaat is voor zowel kwantum gravitatie als kosmische versnelling. Vervolgens heb ik een compleet "woordenboek" opgebouwd dat duidelijk maakt hoe individuele modellen in the EFToDE/MG formalisme beschreven kunnen worden. Dit woordenboek geldt voor alle bekende scalaire modellen met uitzondering van de nieuwe DHOST-modellen die nieuwe, niet meegenomen, operatoren vereisen. Tot slot hebik een nieuwe basis voor de operatoren, geinsipireerd door de "ReParametrized Horndeski" basis, gepresenteerd die de fysische interpretatie van de operatoren duidelijker maakt.

Dit proefschrift vertegenwoordigt een onderzoeksrichting die zijn natuurlijk einde heeft bereikt. Het is nu belangrijk om met de theoretische resultaten aan het werk te gaan. In Hoofdstuk vijf hebben wij hiermee een begin gemaakt door een aantal parameter ruimtes te bestuderen. Deze studies zullen uitgebreid worden en, met behulp van hoge precisie kosmologische data, een leidende rol spelen in het inperken van het weidse gravitationele landschap. Verder is het mogelijk om kosmologische functies te reconstrueren uit de beschikbare data. Deze reconstructies kunnen gebruik maken van de ontwikkelde voorwaarden zodat ze instabiele delen van de parameter ruimte vermijden. Ik hoop dat het onderzoek dat in dit proefschrift is beschreven een belangrijke rol zal spelen in het gouden tijdperk van de kosmologie.

Περίληψη

Η ανακάλυψη της επιταχυνόμενης διαστολής του σύμπαντος είναι μια από της σημαντικότερες στην ιστορία της φυσικής και ως εκ τούτου άλλαξε της ισσοροπίες στο πεδίο της κοσμολογίας. Σε συνδυασμό με την ανακάλυψη της μαύρης ύλης, η παραπάνω ανακάλυψη οδήγησε στην δημιουργεία του κοσμολογικού μοντέλου ονομάτι ΛCDM . Το συγκεκριμένο μοντέλο υποθέτει την ύπαρξη μιας Κοσμολογικής Σταθεράς στη δράση της Γενικής Σχετικότητας. Η Κοσμολογική Σταθερά εισήχθη για πρώτη φορά απο το Άλμπερτ Άινσταιν και μπορεί να ερμηνευθεί ως την ενέργεια του κενού. Ο τομέας της Μαύρης Ύλης εκπροσωπείτε απο ένα ρευστό χωρίς πίεση, το οποίο απαρτίζεται από αργά κινούμενα σωματίδια, ονομάτι Κρύα Μαύρη Ύλη. Έως σήμερα το μοντέλο ΛCDM έχει υπάρξει πολύ επιτυχημένο στην επεξήγηση των κοσμολογικών παρατηρήσεων, ένα αξιοσημείωτο γεγονός, λόγω της απλότητας του.

Παρά την επιτυχία του, το μοντέλο ΛCDM είναι προβληματικό από θεωρητικής άποψης. Συγκεκριμένα, η ερμηνεία της Κοσμολογικής Σταθεράς ως την ενέργεια του κενού παρουσιάζει ένα σημαντικό θεωρητικό πρόβλημα. Η παρατηρηθείσα τιμή, σε τιμές της μάζας Planck, ανέρχεται σε $\Lambda_{obs} \sim (10^{-30}M_{pl})^4$. Από την θεωρητική πλευρά, ο υπολογισμός της Κοσμολογικής Σταθεράς μέσω του Καθιερωμένου Σωματιδιακού Προτύπου, οδηγεί σε μια τιμή ανώτερη κατά 60 τάξεις μεγέθους. Η παραπάνω διαφορά οδήγησε σε μια εκτεταμένη έρευνα για εναλλακτικές λύσεις, που κατέληξε σε ένα τεράστιο βαρυτικό τοπίο απαρτιζόμενο από μοντέλα τα οποία εξηγούν την κοσμολογική διαστολή με διάφορους τρόπους.

Τα περισσότερα μοντέλα που δημιουργήθηκαν εισάγουν νεους βαθμούς ελευθερίας επάνω στη θεωρία της Γενικής Σχετικότητας, απο βαθμωτά έως τανιστυκά πεδία. Αυτή η εξέλιξη οδήγησε σε ένα μεγαλο και δισεπίλυτο βαρυτικό τοπίο, κάνοντας κάθε προσπάθεια σύγκρισης με το ΛCDM, αναποτελεσματική μιας και κάθε νεο μοντέλο πρέπει να ελεγχθει ατομικά με τα διαθέσιμα κοσμολογικά δεδομένα. Αυτή η διαδικασία μπορεί να διευκολυνθεί με τη βοήθεια ενός συνόλου θεωρητικών προϋποθέσεων. Το συγκεκριμο σύνολο εξαλείφει ασταθή μοντέλα και ως εκ τούτου μειώνει δραστικά το χώρο παραμέτρου των μοντέλων που επεκτείνουν την θεωρία της Γενικής Σχετικότητας.

Στη συγκεκριμένη πτυχιακή παρουσιάσαμε τον υπολογισμό, και τον επακόλουθο έλεγχο, ενος ολοκληρωμένου συνόλου προϋποθέσεων το οποίο πρέπει να ικανοποιεί κάθε μοντέλο που επεκτείνει την θεωρία της Γενι-

κής Σχετικότητας, ώστε να θεωρηθεί ένα βιώσιμο κοσμολογικό μοντέλο. Στο έργο μας εστιάσαμε τη προσοχή στην υποκατηγορία μοντέλων που εισάγουν ένα βαθμωτό πεδίο, μιας και είναι η πιό κατοικημένη και ποίκιλη υποκατηγορία. Ένα παράδειγμα μιας βαθμωτής-τανυστικής θεωρίας είναι το μοντέλο Horndeski το οποίο, μέχρι προσφάτως, θεωρούνταν η πιο γενική θεωρία με εξισώσεις κίνησης δευτέρου βαθμού. Για να πετύχουμε το στόχο μας χρησιμοποιήσαμε το αποτελεσματικό και ενοποιητικό πλαίσιο της Αποτελεσματική Πεδιακής Θεωρίας της Μαύρης Ενέργειας και της Μεταλλαγμένης Βαρύτητας (ΑΠΘτΜΕ/ΜΒ). Αυτή η επιλογή μας επέτρεψε να δημιουργήσουμε τις επιθυμητές προϋποθέσεις χωρίς να επιλέξουμε κάποιο συγκεκριμένο μοντέλο.

Το σύνολο των προϋποθέσεων που λάβαμε υπόψιν εγγυώνται την αποφυγή των ghost, gradient και ταχυονικών ασταθειών. Από τις παραπάνω, η ταχυονική αστάθεια έχει παραμεληθεί στην επιστημονική βιβλιογραφία και χρειάζεται εκτεταμένη μέριμνα. Στα τρια πρώτα Κεφάλαια υπολογίσαμε το σύνολο των προϋποθέσεων στο κενό, ύπο την επίρρεια δύο υλικών πεδίων, ακτινοβολία και κρύα μαύρη ύλη, και στο όριο de-Sitter.

Παρότι το σύνολο των προϋποθέσεων ύπο την επίρρεια υλικών πεδίων είναι η πιο ενδιαφέρουσα, ο βαθμός δυσκολίας του υπολογισμού τους συχνά οδηγεί στην παραμέλησή τους. Το έργο μας καλύπτει το συγκεκριμένο θεωρητικό κενό και μας επιτρέπει να αναλύσουμε εις βάθος την επιρροή των υλικών πεδίων. Στην περίπτωση των ghost και gradient ασταθειών βρήκαμε ότι, σε γενικές γραμμές, οι δύο περιπτώσεις δεν διαφέρουν και η παραμέληση των υλικών πεδίων είναι δεκτή. Η μοναδική εξαίρεση είναι η περίπτωση των beyond Horndeski μοντέλων μιας και η ταχύτητα διαδοσής τους εξαρτάται απο το πεδίο ακτινοβολίας, ενα φαινόμενο που επηρεάζει την gradient αστάθεια. Η ταχυονική αστάθεια, απο την άλλη, εξαρτάται σε σημαντικό βαθμό από τα υλικά πεδία ανεξαρτήτως περίπτωσης. Συγκεκριμένα, κάθε υλικό πεδίο εισάγει μια νέα προϋπόθεση, αλλειώνοντας εις βάθος τα αποτελέσματα που υπολογίσαμε στο κενό. Αυτό είναι το άμεσο αποτέλεσμα της βαρυτικής αλληλοεπίδρασης μεταξύ των υλικών πεδίων και του νέου βαθμωτού πεδίου. Η νέες ταχυονικές προϋποθέσεις είναι το κυρίως καινοτόμο αποτέλεσμα στην συγκεκριμένη πτυχιακή με υψηλή μελλοντική επίδραση στο πεδίο της μαύρης ενέργειας και της μεταλλαγμένης βαρύτητας.

Ο πρωταρχικός στόχος, μετά την ολοκλήρωση του υπολογισμού των προϋποθέσεων που εγγυώνται την έλλειψη ταχυονικών ασταθειών, είναι ο έλεγχος και η προβολή της επιρροής τους στο βαρυτικό τοπίο. Στο Κεφάλαιο 5 πραγματοποιήσαμε το συγκεκριμένο στόχο με την βοήθεια του προγράμματος EFTCAMB , ενα patch του Boltzmann κώδικα CAMB , που ενσωματώνει την ΑΠΘτΜΕ/ΜΒ. Μέχρι προσφάτως, για να εγγυηθεί η σταθερότητα των κοσμολογικών μοντέλων, ενα σύνολο ad-hoc μαθηματικών **προυποθέσεων** είχαν προστεθεί στις ghost και gradient προϋποθέσεις. Οι συγκεκριμένες ad-hoc προϋποθέσεις ήταν αποτελεσματικές αλλα βασίζονταν σε προβληματικές υποθέσεις. Ο τελικός στοχος είναι η αντικαταστασή τους με τις ταχυονικές προϋποθέσεις και την ανάλυση της διαφοράς. Η ανάλυση έδειξε ότι οι δυο επιλογές έχουν παρόμοια δύναμη περιορισμού του βαρυτικού τοπίου με τις ταχυονικές προϋποθέσεις ελαφρώς πιο ισχυρές. Στη συνέχεια αναλύσαμε την επιρροή στο χώρο παραμέτρου (μ,Σ) το οποίο κωδικοποιει την απόκλιση απο τη Γενική Σχετικότητα στις εξισώσεις Poisson και βαρυτικής εστίασης, αντιστοίχως. Αυτό οδήγησε στον ενοτπισμό μιας σημαντικής απόκλισης στα μοντέλα μηελάχιστης σύζευξης με κανονικό κινετικό όρο, ονομάτι Generalised Brans Dicke (GBD). Είναι ξεκάθαρο οτι οι ταχυονικές προϋποθέσεις, μιας και είναι θεωρητικά πιο αποδεκτές, αξίζουν να χρησιμοποιηθουν σε συνδυασμό με τις γηοστ και γραδιεντ.

Παράλληλα με την αποπάνω εργασία πάνω στη θεωρητική σταθερότητα προχωρήσαμε σε μερικές βελτιώσεις του πλαισίου της ΑΠΘτΜΕ/ΜΒ. Στο πρώτο Κεφάλαιο παρουσιάσαμε μια επέκταση της ΑΠΘτΜΕ/ΜΒ ώστε να συμπεριληφθεί το βαρυτικό μοντέλο Hořava. Το συγκεκριμένο μοντέλο είναι υποψήφιο για την κβαντική βαρύτητα και την κοσμική διαστολή. Στη συνέχεια δημιουργήσαμε ενα «λεξικό» το οποίο παρουσιάζει την μορφή συγκεκριμένων βαρυτικών μοντέλων μέσα στο πλαίσιο της ΑΠΘτΜΕ/ΜΒ. Συγκεκριμένα συμπεριλαμβάνουμε ολα τις γνωστές βαθμωτές τανυστικές θεωρίες με εξαίρεση τη θεωρία DHOST που χρειάζεται ένα παραπάνω τελεστή που δεν συμπεριλάβαμε. Στο τέλος παρουσιάσαμε μια νέα βάση για τους τελεστές, βασισμένη στη βαση ReParametrized Horndeski ή άλφα που προσπαθεί να ξεκαθαρίσει τη φυσική σημασία του κάθε τελεστή.

Το έργο το οποίο παρουσιάζει η συγκεκριμένη πτυχιακή καλύπτει μια γραμμή έρευνας που έφτασε στο τέλος της. Στο πέμπτο Κεφάλαιο κάναμε μια επίδειξη των πιθανών εφαρμογων των αποτελεσμάτων. Στο μέλλον οι εφαρμογές θα επεκταθούν , καθώς όλο και περισσότερα κοσμολογικά δεδομένα γίνονται διαθέσιμα και η δυνατότητα περιορισμού του βαρυτικού τοπίου γινεται πιό εφικτή. Επιπλέον, τα νέα κοσμολογικά δεδομένα κάνουν εφικτή την ανακατεσκευή των συναρτήσεων της ΑΠΘτΜΕ/ΜΒ με μεθόδους που δεν χρειάζονται την επιλογή ενός μοντέλου. Οι συγκεκριμένες ανακατεσκευές θα χρησιμοποιήσουν τις θεωρητικές προϋποθέσεις ωστε να αποφύγουν περιόχες των χωρών παραμέτρου που δεν επιτρέπουν βιώσιμα κοσμολογικά μοντέλα. Ως εκ τούτου ελπίζουμε το έργο μας να παίξει ένα σημαντικό ρόλο στην τρέχον χρυσή εποχή της κοσμολογίας.

Curriculum Vitæ

I was born and raised in Rethymnon, Greece, on the 31st of August 1990. Situated on the island of Crete, this town was the location where I attended primary school. During my secondary education, at the high school named «Πειραματικό Λύκειο Ρεθύμνου» , I followed the science track, in preparation for my university degree.

In September 2009, I enrolled in the Physics department at the University of Utrecht where I finished my studies in 2012 with the Bachelor thesis *"Influence of the excess ion polarizability on the Electric Double Layer (EDL)"* under the supervision of Prof. dr. R.H.H.G. van Roij. I continued my studies in the Theoretical Physics Master program at the same university, graduating in October 2014. I did my Msc. reasearch under the supervision of dr. Umut Gürsoy resulting in a thesis with the title *"Magnetic Catalysis in Holographic Quantum Chromodynamics"*. After my graduation I stayed three more months while working on the same project.

In June 2015, I started my PhD research in the group of Prof. dr. Ana Achúcarro and dr. Alessandra Silvestri. The results of that work are described in this thesis. During these four years I focused my attention on one central topic which spans all my work. I collaborated intensively with dr. Antonio de Felice, dr. Noemi Frusciante, Simone Peirone and my direct supervisor, dr. Alessandra Silvestri.

During these years. I attended a number of schools in different countries and got the opportunity to present my work at conferences and during seminars in the Netherlands, Portugal, Spain, United Kingdom and the United States.

Finally, conform to my teaching responsibilities, I was the student assistant of the courses "Quantum Mechanics I" and "Relativistic Electrodynamics".

List of publications

The thesis is based on the following publications:

- N. Frusciante, G. Papadomanolakis and A. Silvestri, "An Extended action for the effective field theory of dark energy: a stability analysis and a complete guide to the mapping at the basis of EFTCAMB," JCAP **1**607, 018 (2016).
- A. De Felice, N. Frusciante and G. Papadomanolakis, "On the stability conditions for theories of modified gravity in the presence of matter fields," JCAP **1**703, 027 (2017).
- A. De Felice, N. Frusciante and G. Papadomanolakis, "de Sitter limit analysis for dark energy and modified gravity models," Phys. Rev. D **96**, 024060 (2017).
- N. Frusciante, G. Papadomanolakis, S. Peirone and A. Silvestri, "The role of the tachyonic instability in Horndeski gravity,"JCAP **1**902, 029 (2019).

Other publications by the author:

- N. Frusciante and G. Papadomanolakis, "Tackling non-linearities with the effective field theory of dark energy and modified gravity," JCAP **1**712, 014 (2017).