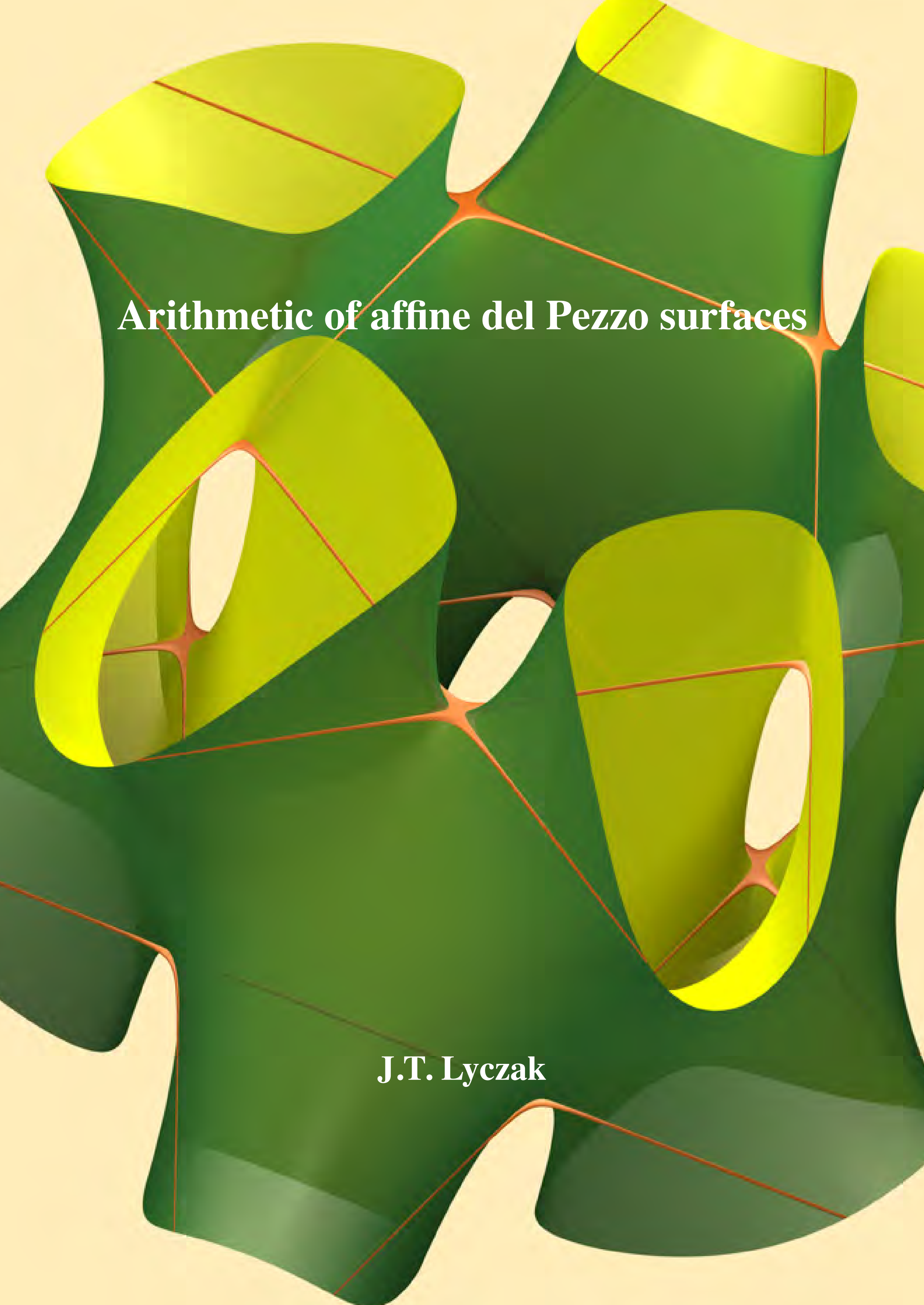
Author: Lyczak, J.T.
Title: Arithmetic of affine del Pezzo surfaces
Issue Date: 2019-10-01

Arithmetic of affine del Pezzo surfaces



J.T. Lyczak

Arithmetic of affine del Pezzo surfaces

Proefschrift

ter verkrijging van
de graad van Doctor aan de Universiteit Leiden,
op gezag van Rector Magnificus prof. mr. C. J. J. M. Stolker,
volgens besluit van het College voor Promoties
te verdedigen op dinsdag 1 oktober 2019
klokke 13.45 uur

door

Julian Tadeusz Lyczak

geboren te Alphen aan den Rijn
in 1988

Promotor:	prof. dr. P. Stevenhagen
Co-promotor:	dr. M. J. Bright

Samenstelling van de promotiecommissie:

Voorzitter:	prof. dr. A. W. van der Vaart	Universiteit Leiden
Secretaris:	prof. dr. B. de Smit	Universiteit Leiden
	prof. dr. S. J. Edixhoven	Universiteit Leiden
	dr. R. D. Newton	University of Reading
	dr. O. Wittenberg	CNRS & Université Paris-Sud

Contents

Introduction **v**

Notation and conventions **ix**

1 Brauer groups **1**
- 1.1 Adelic points, weak and strong approximation 2
- 1.2 Central simple algebras over fields 6
- 1.3 Cyclic algebras 11
- 1.4 Azumaya algebras over schemes 12
- 1.5 Residue and invariant maps 16
- 1.6 The Brauer–Manin obstruction 18
- 1.7 The purity theorem and the ramification locus 20

2 Del Pezzo surfaces **25**
- 2.1 Preliminaries on geometry 26
- 2.2 Generalized and ordinary del Pezzo surfaces 28
- 2.3 Divisor classes of negative self-intersection 30
- 2.4 Curves of negative self-intersection 33
- 2.5 Effective anticanonical divisors 40
- 2.6 Singular del Pezzo surfaces 43
- 2.7 Peculiar del Pezzo surfaces 45
- 2.8 Arithmetic of del Pezzo surfaces 57

3 The Brauer group of ample log K3 surfaces **61**
- 3.1 Ample log K3 surfaces 62
- 3.2 The algebraic Brauer group 63
- 3.3 Uniform bound for the order of the Brauer group 66

4 Order 5 obstructions to the integral Hasse principle **71**
- 4.1 The interesting Galois action 72
- 4.2 Constructions of degree 5 del Pezzo surfaces 77
- 4.3 Models of interesting del Pezzo surfaces 82
  - 4.3.1 Explicit equations for $\mathcal{X}_\alpha$ 83
  - 4.3.2 Geometric construction of $\mathcal{X}_\alpha$ 84

4.3.3 Comparison of the two constructions 87
4.4 The splitting field of the interesting action 90
4.4.1 Number fields of degree 5 90
4.4.2 Number rings $\mathbb{Z}[\alpha]$ of degree 5 92
4.5 The fibres $\mathcal{X}_\ell$ 94
4.5.1 The construction of $\mathcal{X}_\ell$ 94
4.5.2 The -1-curves on $\mathcal{X}_\ell$ 95
4.5.3 The singularities of $\mathcal{X}_\ell$ 97
4.5.4 Fibres with an A_4-singularity 99
4.6 Setup for the following sections 102
4.7 Arithmetic of $\mathcal{U}$ 105
4.8 Computation of the invariant maps 107
4.8.1 The invariant map for unramified primes 108
4.8.2 The invariant map for tamely ramified primes 110
4.8.3 The invariant map for the wildly ramified prime 5 114
4.9 Actual examples of obstructions 117
4.9.1 Obstructions for $\mathcal{U}$ of large conductor 117
4.9.2 Obstructions for $\mathcal{U}$ of conductor 11 118
4.9.3 Obstructions for $\mathcal{U}$ of conductor 25 121

Bibliography **125**

Summary **131**

Samenvatting **135**

Acknowledgements **141**

Curriculum vitae **143**

Introduction

Background

The study of Diophantine equations is concerned with polynomial equations and their solutions in number fields and number rings. This field of research is thousands of years old. For example, triplets of integers x, y and z satisfying the equation $x^2 + y^2 = z^2$ have been studied in several ancient civilizations. A first question can be whether a solution to such an equation exists at all. The existence of solutions is usually proved by providing explicit values satisfying the equation. Next, one can wonder about how many solutions there are and if this number is finite or infinite. If on the other hand no solutions exist, one would want a reason to explain their absence. It is this question with which we concern ourselves in this thesis.

Let X be a scheme over a number field k for which we want to determine whether $X(k)$ is empty or not. A usually first step is to consider a completion k_v of k and note that if $X(k_v)$ is empty then so is $X(k)$. For some schemes this is all one needs to consider; the Hasse–Minkowski theorem states that for a hypersurface $X \subseteq \mathbb{P}^n_k$ defined by a homogeneous quadric equation $F \in k[x_0, \ldots, x_n]$ the set $X(k)$ is non-empty precisely if $X(k_v)$ is non-empty for all completions k_v of v. For other schemes over k this need not be the case and several counterexamples to this principle were published. Then in 1970 Manin [40] introduced a technique which unified all these results. The technique he put forward is now known as the *Brauer–Manin obstruction*.

Up to then all known examples of schemes X over the number field k for which $X(k)$ was empty were either explained by the absence of local points or by the Brauer–Manin obstruction. We now however know that the Brauer–Manin obstruction is not sufficient to explain to the absence of integral points on all schemes. When we restrict to certain subclasses of schemes, such as the quadric hypersurfaces, there do exist positive results. For other types of schemes the study of integral points is still ongoing. This thesis aims to add to the case of *surfaces*, i.e. schemes of dimension two. Although there are still a lot of unanswered questions rational points on surfaces are relatively well understood. We have for example the conjecture by Colliot-Thélène and Sansuc [14, page 174] which states that the Brauer–Manin obstruction is the only one to the Hasse principle on rational surfaces. So despite the many open problems there is an understanding of what the final results should be or at least in what type of language they should be phrased.

Now let us turn our attention to integral points. There is no standard framework yet for working with integral points on surfaces. Even on log rationally connected surfaces, which are not unlike the rationally connected surface mentioned above, Manin's technique does not have to exclude the existence of integral points, as was proved in [16]. In [32] a refined conjecture was put forward, but for the slightly more complex class of log K3 surfaces no conjectures have been formulated yet. In [35] new concepts were introduced to explain the absence of integral solutions. But again, these techniques do not explain all known counter-examples. In the hope to find other techniques and terminology which allow us to formulate reasonable conjectures we still need more examples of the study of integral points on surfaces.

This thesis contributes in that sense to this field of research because one of the main result is the existence of Brauer–Manin obstructions to the integral Hasse principle. What makes these examples stand out is that these are of order 5, in contrast to the previously known examples which are all of order 2 and 3.

Another important result is the uniform bound on the Brauer group of ample log K3 surfaces. This result will help to understand the relevance of the Brauer–Manin obstruction to integral points on such surfaces.

Overview

In the first chapter we recall general notions in the theory of the arithmetic of schemes. We will for example define the *ring of adeles* $\mathbb{A}_k$ associated to a number field k. Now consider a scheme X over k. We will describe the elements of the set of adelic points $X(\mathbb{A}_k)$ as tuples of points $p_v \in X(k_v)$ indexed by all completions k_v of k. This induces an inclusion $X(k) \subseteq X(\mathbb{A}_k)$ which could be used to prove that $X(k)$ is empty. The celebrated Hasse–Minkowski theorem can now be rephrased by saying that this technique is all one needs for quadric hypersurface; consider a projective scheme $X \subseteq \mathbb{P}^d_k$ over a number field defined by a homogeneous quadratic equation. The set $X(k)$ is empty precisely if the set $X(\mathbb{A}_k)$ is empty. One says that quadratic forms satisfy the *Hasse principle*.

For other schemes it is however possible that the set of adelic points is non-empty, but that there are no k-points on X. In this case, one could consider the Brauer group of X. This invariant of schemes can be useful in the following way. Any element $\mathcal{A} \in \operatorname{Br} X$ defines an intermediate subset

$$X(k) \subseteq X(\mathbb{A}_k)^{\mathcal{A}} \subseteq X(\mathbb{A}_k).$$

If this explains the absence of rational points one says that there is a *Brauer–Manin obstruction* to the Hasse principle.

It was recognized in [17] that these techniques can be adapted to explain the absence of $\mathcal{O}_k$-points for a scheme $\mathcal{X}$ defined over $\mathcal{O}_k$. In this case one defines the *ring of integral adeles* $\mathbb{A}_{k,\infty}$ of k. For classes of schemes over $\mathcal{O}_k$ we have the *integral Hasse principle* which says that $\mathcal{X}(\mathcal{O}_k)$ is empty precisely if $\mathcal{X}(\mathbb{A}_{k,\infty})$ is empty. Again, if $\mathcal{X}(\mathcal{O}_k)$ is non-empty then so is $\mathcal{X}(\mathbb{A}_{k,\infty})$ since we have an inclusion $\mathcal{X}(\mathcal{O}_k) \subseteq \mathcal{X}(\mathbb{A}_{k,\infty})$. Also, for any element $\mathcal{A} \in \operatorname{Br} X$ of the Brauer group of the generic fibre $X = \mathcal{X}_k$ we have an intermediate subset

$$\mathcal{X}(\mathcal{O}_k) \subseteq \mathcal{X}(\mathbb{A}_{k,\infty})^{\mathcal{A}} \subseteq X(\mathbb{A}_{k,\infty}).$$

We will use this chain of inclusions to define the *Brauer–Manin obstruction* to the integral Hasse principle.

There are many sources which offer a more complete treatise on the arithmetic of schemes over number rings and number fields. See for example [46] and the introduction in [32].

In the second chapter we review del Pezzo surfaces. We start by treating the surfaces studied by del Pezzo himself in [22] which we will call *ordinary del Pezzo surfaces*. The most common extension in the literature of these surfaces are the *generalized del Pezzo surfaces*. These surfaces are also smooth and are indistinguishable from the ordinary del Pezzo surfaces if one only looks at the geometric Picard group. These two types of surfaces share many geometric properties, one of which is the fact that the anticanonical map turns out to be a morphism into projective space of relatively small dimension for both types of surfaces. The difference is that this morphism is an isomorphism onto its image for precisely the ordinary del Pezzo surfaces; for the generalized del Pezzo surfaces it will merely be a birational morphism onto its image. This brings us to the last common type of del Pezzo surface to be studied in literature. A *singular del Pezzo surface* is the image of a generalized del Pezzo surface under this morphism to projective space.

This chapter also contains the new concept of *peculiar del Pezzo surfaces*. This novel type of surface fits in between the generalized and singular del Pezzo surfaces in the following manner. A generalized del Pezzo surface X over an algebraically closed field k admits a birational morphism $\pi\colon X \to \mathbb{P}^2$ to the projective plane. Let $Y \subseteq \mathbb{P}^d$ be the image of X under the anticanonical morphism to $\mathbb{P}^d$, i.e. the singular del Pezzo surface associated to X. The composition of π with the birational inverse of the anticanonical morphism $X \to Y$ produces a birational map $Y \dashrightarrow \mathbb{P}^2$. Let us consider a splitting of the morphism $X \to X' \to Y$ into two birational morphisms. For each such splitting we have a birational map $X' \dashrightarrow \mathbb{P}^2$. A *peculiar del Pezzo surface* is the minimal X' such that $X' \dashrightarrow \mathbb{P}^2$ is a morphism, i.e. π factors through X'.

We will prove that the generalized del Pezzo surface X, the peculiar del Pezzo surface X', and the singular del Pezzo surface Y all determine each other. The main advantage is that a peculiar del Pezzo surface has a rather direct geometric construction; any peculiar del Pezzo surface is the blowup of the projective plane in a, possibly non-reduced, zero-dimensional scheme. In this manner, many geometric properties of X, X' and Y could be described in terms of this zero-dimensional scheme on $\mathbb{P}^2$. This allows one to solve problems concerning del Pezzo surfaces using the language of points and curves on the projective plane.

In the third chapter we change our attention to log K3 surfaces over a number field k and specially those of ample type. A particular straightforward way to construct such a surface U is by considering a projectively embedded ordinary del Pezzo surface $X \subseteq \mathbb{P}^d$ and taking the complement of a hyperplane, i.e. $U = \mathbb{A}^d \cap X$. We present the results from the author's work joint with Martin Bright [10]. The main theorem in this chapter is that the Brauer group of such surfaces over k are uniformly bounded in the sense that $\#\left(\operatorname{Br} U / \operatorname{Br} k\right) < C$. We will show that the constant C only depends on the degree of the number field k. This result is particularly encouraging with a view towards Várilly-Alvarado's conjecture [56, Conjectures 4.5, 4.6] which states that such a bound should exist for proper K3 surfaces, another class of log K3 surfaces which is disjoint from the ample log K3 surfaces considered in this chapter.

The fourth chapter presents another novel result of the author. It exhibits Brauer–Manin obstructions of order 5 to the integral Hasse principle. These examples are particularly interesting since all other known examples of the Brauer–Manin obstruction, be it to weak approximation or to any form of the Hasse principle, are all of either order 2 or order 3.

These examples arise from an element $\mathcal{A}$ of order 5 in the Brauer group $\operatorname{Br} U$ of the ample log K3 surface U over $\mathbb{Q}$. Note however that the set of integral points on the scheme U over $\mathbb{Q}$ is not well-defined. It depends on the choice of model $\mathcal{U}$ of U, by which we mean a scheme $\mathcal{U}$ over $\mathbb{Z}$ whose fibre $\mathcal{U}_{\mathbb{Q}}$ is isomorphic to U.

A model is often chosen by writing down explicit equations with integral coefficients. For example, homogeneous equations with integral coefficients define a projective scheme over $\mathbb{Z}$. The situation is made even more complicated by the fact that we are dealing with affine schemes. Again, one could simply construct an affine scheme by supplying equations which now should be inhomogeneous. This was done for example in [16], [32], [35], [38], [15]. In each of these papers the application of the Brauer–Manin obstruction to the existence of integral points on certain ample log K3 surfaces is studied. For the examples of Brauer–Manin obstructions to the integral Hasse principle in this thesis another approach is used; we will study the set of integral points on a scheme $\mathcal{U}$ over $\mathbb{Z}$ which is constructed using geometrical tools. This geometric construction will allow for a relatively effortless study of the fibres $\mathcal{U}_\ell$ of $\mathcal{U}$ over a prime $\ell \in \mathbb{Z}$. We will use this to describe the set of integral adelic points $\mathcal{U}(\mathbb{A}_{\mathbb{Q},\infty})$ and compute the subset $\mathcal{U}(\mathbb{A}_{\mathbb{Q},\infty})^{\mathcal{A}}$ corresponding to the class $\mathcal{A} \in \operatorname{Br} U$ of order 5. This subset contains the set $\mathcal{U}(\mathbb{Z})$ of integral points on $\mathcal{U}$. We will give several explicit examples for which $\mathcal{U}(\mathbb{A}_{\mathbb{Q},\infty})^{\mathcal{A}}$ and hence $\mathcal{U}(\mathbb{Z})$ is empty.

Notation and conventions

We list some basic notation and conventions which will reoccur throughout this thesis. Some concepts will also be introduced in the text, but we include them here as well for the reader's convenience.

A *ring* is always assumed to be unitary. All rings will be commutative unless stated otherwise.

A *division ring* is a, not necessarily commutative, ring where each non-zero element has a multiplicative inverse.

Let R be a, not necessarily commutative, topological ring. A *(left) R-module* is an abelian topological group together with a continuous left R-action. A *(left) R-algebra* is a, not necessarily commutative, topological ring together with a continuous left R-action.

Let G be a topological group. A *(left) G-module* is a topological group with a continuous left G-action. The notions of abelian G-modules and $\mathbb{Z}[G]$-modules are equivalent.

Whenever the topology on a ring, module or group is not explicitly stated, we assume the topology to be discrete.

For a field k we will write $\bar{k}$ for a fixed algebraic closure and k^{sep} for the separable closure of k in $\bar{k}$. The absolute Galois group of a field k is endowed with the profinite topology and this topological group is denoted by $G_k = \text{Gal}(k^{\text{sep}}/k)$.

The absolute Galois group of a finite field $\mathbb{F}_q$ is canonically isomorphic to $\widehat{\mathbb{Z}}$ and it is topologically generated by the *Frobenius* $\text{Frob}_q \colon \bar{\mathbb{F}}_q \to \bar{\mathbb{F}}_q$, $x \mapsto x^q$.

A *variety* over a field k is a separated scheme of finite type over $\operatorname{Spec} k$.

A *curve* over a field k is a variety over k of pure dimension 1, it need not be irreducible, reduced or smooth.

A *surface* over a field k is a geometrically integral variety of dimension 2 over k.

A *curve on a surface* over a field k is a closed subscheme of the surface which is a curve over k.

For a scheme X over a field k we will write X_K for the base change $X \times_k K$ for any field extension K of k. The notations X^{sep} and $\bar{X}$ will be synonymous for $X_{k^{\text{sep}}}$ and $X_{\bar{k}}$, respectively.

For a Cartier divisor D on a scheme X we will denote the associated line bundle by $\mathcal{L}(D)$. It comes with a designated rational section denoted by 1_D. The rational section 1_D extends to a global section precisely if D is effective.

The complete linear system of a line bundle $\mathcal{L}$ on a scheme X is denoted by $|\mathcal{L}|_X$. We might suppress the scheme X in the subscript if it is clear from the context. For a Cartier divisor D we write $|D|$ for $|\mathcal{L}(D)|$.

An integral variety X has a unique generic point η. The local ring $\mathcal{O}_{X,\eta}$ is a field which is called the *function field* of X and is denoted by $\kappa(X)$.

The *codimension* of a point x on a scheme X is the dimension of the local ring $\mathcal{O}_{X,x}$. The set of all points of codimension d is denoted by $X^{(d)}$.

The canonical bundle on a normal scheme X over a field k will be denoted by ω_X. The associated divisor class is K_X, and by abuse of notation we also use this notation to denote a divisor representing this class.

The category of abelian groups and group homomorphisms is denoted by **Ab**. The category of schemes and morphisms is denoted by **Sch**.

Chapter 1

Brauer groups and the Brauer–Manin obstruction

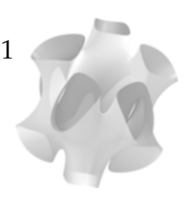
In this chapter we will review important concepts in the theory of rational and integral points on schemes over number fields. We will start by describing several types of adelic rings associated to number fields. These rings are useful for studying all localizations k_v of a number field k simultaneously. In a similar manner one can study the k_v-points on a scheme X for all completions k_v of the number field k at the same time consider the so-called *adelic points* on a scheme X. This set will help us to study the set $X(k)$. In particular, we find a natural inclusion of the set of k-points on X in the adelic points.

In the cases where the set of adelic points is too big to yield information about the set of k-points one can turn to the *Brauer–Manin obstruction*. This technique uses the *Brauer group* of a scheme to identify a subset of the adelic points which contains $X(k)$. We will study the Brauer groups over schemes by first defining the Brauer group of a field in terms of central simple algebras. The definition of these algebras generalizes to sheaves of algebras on schemes. We will show that the Brauer group of a scheme thus constructed coincides with the second étale cohomology group of the scheme with values in $\mathbb{G}_{\mathrm{m}}$, assuming some conditions which will not be too restrictive for our purposes.

We proceed by defining the so-called *residue maps* and stating some of their relevant properties. These group homomorphism are an important tool in the computation and understanding of Brauer groups of local rings and also Brauer groups of schemes. These homomorphisms allow us to define the Brauer–Manin obstruction discussed above.

In the last section of this chapter we review some technical results on Brauer groups and residue maps which we will need throughout the thesis.

Very few results in this chapter are true for all schemes. The main goal will be to study schemes defined by polynomial equations with coefficients in a field. Such schemes will be called varieties. There are however a variety of definitions



in use for this type of scheme. To avoid ambiguity we will fix the definition here for the remainder of the thesis.

DEFINITION 1.0.1. Let k be a field. A *variety over* k is a scheme over k which is separated and of finite type over k.

1.1 Adelic points, weak and strong approximation

Let k be a number field and denote the set of all non-trivial places of k by Ω_k. This set splits into the non-archimedean places Ω_k^{fin} and the archimedean places Ω_k^∞. For a finite subset $S \subseteq \Omega_k$ we write $\mathcal{O}_{k,S}$ or simply $\mathcal{O}_S$ for the elements of k which are v-integral for all finite $v \notin S$. In the case that S contains no finite places we recover the ring of integers $\mathcal{O}_k$ of k.

For a non-archimedean place v of k we write k_v for the completion of k at v and $\mathcal{O}_v$ for the v-integral elements in k_v. We will endow both k_v and $\mathcal{O}_v$ with the v-adic topology, which gives them the structure of topological rings.

We will study the rational points of a scheme X over k by considering the sets of points $X(k_v)$ for all $v \in \Omega_k$. We would like to study these sets of points simultaneously, but this approach loses too much of the global structure of $X(k)$. This is reminiscent of the fact that the product of all k_v does not reflect the important property that an element of k is v-integral for all but finitely many places v. This can be remedied by considering the elements in $\prod_v k_v$ which are v-integral for almost all v. The following definition constructs this ring using the restricted product [12, Section 13], which also incorporates the topologies on the respective completions of k and $\mathcal{O}_k$.

DEFINITION 1.1.1. Let k be a number field and let $T \subseteq \Omega_k$ be a finite set of places of k. The *ring of T-adeles of k* is the ring defined as the restricted product

$$\mathbb{A}_k^T := \prod_{v \in \Omega_k \setminus T}{}' (k_v, \mathcal{O}_v)$$

with respect to the integral elements $\mathcal{O}_v \subseteq k_v$ for all finite places v. We will endow $\mathbb{A}_k^T$ with the restricted product topology, which makes it into a topological ring.

If T is empty we suppress it in the notation and the terminology and we find the *ring of adeles* $\mathbb{A}_k$ of k.

We will often identify $\mathbb{A}_k$ with the subset of elements in $\prod_{v \in \Omega_k} k_v$ which are integral at all but finitely many places. Note that this product and its subset $\mathbb{A}_k$ are topological spaces, and that the inclusion is continuous, but the topology on the ring of adeles will be finer than the subspace topology.

For some applications we will be interested in adeles of k which are integral outside of a given set of places.

DEFINITION 1.1.2. Let S be a finite set of places of a number field k such that the archimedean places Ω_k^∞ are contained in S. We define the *integral adelic points* *of k away from S* as

$$\mathbb{A}_{k,S} := \prod_{v \in S} k_v \times \prod_{v \in \Omega_k \setminus S} \mathcal{O}_v.$$

For a finite set $T \subseteq \Omega_k$ we can repeat the above construction while ignoring any primes in T. This produces the set $\mathbb{A}_{k,S}^T$ of *integral T-adelic points of k away from S*.

The set $\mathbb{A}_{k,S}$ is a subset of both $\mathbb{A}_k$ and $\prod_v k_v$. If we endow $\mathbb{A}_{k,S}$ with the product topology one can show that it is even an open subset of both $\mathbb{A}_k$ and the product $\prod_v k_v$. This shows that the topology on $\mathbb{A}_{k,S}$ coincides with subspace topology coming from either inclusion $\mathbb{A}_{k,S} \subseteq \mathbb{A}_k$ and $\mathbb{A}_{k,S} \subseteq \prod_v k_v$. Note the contrast with the ring of adeles; the topology on $\mathbb{A}_k$ differs from the subspace topology coming from the inclusion $\mathbb{A}_k \subseteq \prod_v k_v$.

The ring of adeles was defined to study rational points of a scheme X over k since the collection of point sets $X(k_v)$ does potentially not contain enough of the global structure of $X(k)$. This can be done by looking at the set $X(\mathbb{A}_k)$ of adelic points on X, which remembers more of the global structure of $X(k)$ than the collection of the $X(k_v)$. One possible setback is that *a priori* the topology on $X(k_v)$ cannot be recovered from $X(\mathbb{A}_k)$. Let us recall how the topology on $X(k_v)$ is defined. The idea is that for an affine scheme X of finite type over a topological ring R one can identify $X(R)$ with a subset of the R-points on an affine space $\mathbb{A}_R^n$, which is in bijection to R^n. Now we proceed by endowing R^n with the product topology and $X(R)$ with the subspace topology.

For a general finite type scheme over a topological ring R we cover X by affine opens X_i. In this case we will require that R is local so as to ensure that $X(R) = \bigcup X_i(R)$. Then we have topologies on the sets $X_i(R)$ and these topologies generate a topology on $X(R)$ and this will be the topology we will consider. One can show that this topology does not depend on the choice of the affine covering and that a morphism of schemes $X \to Y$ over R induces a continuous map $X(R) \to Y(R)$. For more information and the statement that these topologies are in some sense natural, one can consult [18].

To summarize, if R is a topological ring one can define a natural topology on the set $X(R)$ if X is affine over R, or R is a local ring and X is a separated scheme of finite type over R. Note that these constructions do not allow us to topologize the set of adelic points $X(\mathbb{A}_k^T)$, unless X happens to be affine over $\mathbb{A}_k^T$. The following proposition shows however that the set of adelic points on a variety admits a natural bijection to a set which comes with a topology.

PROPOSITION 1.1.3. *Let T be a finite set of places of a number field k and consider the generic fibre $X = \mathcal{X}_k$ of a separated scheme $\mathcal{X}$ of finite type over $\mathcal{O}_{k,T}$. If $v \notin T$ then $\mathcal{X}(\mathcal{O}_v) \to X(k_v)$ is injective and the natural map*

$$X(\mathbb{A}_k^T) \to {\prod_{v \in \Omega_k \setminus T}}' X(k_v)$$

is well-defined and bijective. The product in the codomain is the restricted product of the indicated factors with respect to $\mathcal{X}(\mathcal{O}_v)$ for $v \in \Omega_k^{\mathrm{fin}} \setminus T$.

Proof. See Section 2.3.1 and Exercise 3.4 in [46]. □

Recall that a variety X over k spreads out over an open dense subset U of Spec $\mathcal{O}_k$, see for example [46, Theorem 3.2.1], i.e. there is a separated scheme $\mathcal{X}$ of finite type over $\mathcal{O}_{k,T}$ for some finite set T such that $\mathcal{X}_k$ is isomorphic to X. This proves that any k-variety X is the generic fibre of an $\mathcal{O}_{k,T}$-scheme $\mathcal{X}$ and we can use Proposition 1.1.3 to describe the set of adelic points on X.

Note that the set $X(\mathbb{A}_k^T)$ only depends on the generic fibre X of $\mathcal{X}$, while the restricted product in Proposition 1.1.3 depends *a priori* on the choice of model $\mathcal{X}$ of X. However since for any two models one can identify the two sets of $\mathcal{O}_v$-points for all but finitely many v it follows from the definition of the restricted product that

$$\prod_{v\in\Omega_k\setminus T}{}' X(k_v)$$

is independent of the choice of model. Similarly one proves that the restricted product topology on this product is also independent of the choice of model $\mathcal{X}/\mathcal{O}_{k,T}$. Using the bijection in Proposition 1.1.3 we have produced a topology on the set of adelic points $X(\mathbb{A}_k)$.

DEFINITION 1.1.4. The *adelic topology* on the set $X(\mathbb{A}_k^T)$ of T-adelic points on a variety X over a number field k is the unique topology on this set such that the bijection in Proposition 1.1.3 is a homeomorphism.

There is an important corollary in the case that X is proper, because then the natural map $\mathcal{X}(\mathcal{O}_v) \to X(k_v)$ is an isomorphism for all v and the restricted product becomes the standard product.

COROLLARY 1.1.5. *Let X be a proper variety over a number field k. The natural map*

$$X(\mathbb{A}_k^T) \to \prod_{v\in\Omega_k\setminus T} X(k_v)$$

is a homeomorphism.

For two finite sets of places $T' \subseteq T$ the projection map $\mathbb{A}_k^{T'} \to \mathbb{A}_k^T$ is continuous. In particular we have a map

$$\mathbb{A}_k \to \mathbb{A}_k^T$$

for each such T. Composing with the diagonal map $k \to \mathbb{A}_k$ we get maps $X(k) \to X(\mathbb{A}_k^T)$. Usually, the set of adelic points $X(\mathbb{A}_k^T)$ is easier to work with than the set of rational points $X(k)$ in which we are primarily interested. This motivates our interest in the image of the map $X(k) \to X(\mathbb{A}_k^T)$.

DEFINITION 1.1.6. Let X be a variety over a number field k and let T be a finite set of places of k. If $X(k)$ is dense in $X(\mathbb{A}_k^T)$ we say that X satisfies *strong approximation away from T* and X satisfies *weak approximation away from T* if $X(k)$ is dense in $\prod_{v\in\Omega_k\setminus T} X(k_v)$.

The names of these two concepts seem counterintuitive; being dense in the subset $X(\mathbb{A}_k^T)$ seems weaker than being dense in the larger set $\prod_{v\in\Omega_k\setminus T} X(k_v)$. The terminology stems from the facts that the topology of $X(\mathbb{A}_k^T)$ is not the subspace topology as a subset of

$$\prod_{v\in\Omega_k\setminus T} X(k_v).$$

Using the definition of the restricted product topology on $X(\mathbb{A}_k^T)$ one can show that if X satisfies strong approximation, it will also satisfy weak approximation.

Now let X be a projective variety over a number field k. On such a scheme the notions of weak and strong approximation coincide by Corollary 1.1.5. We have seen that we study the set of adelic points on X, using a model of X over $\mathcal{O}_k$. If X comes with a projective embedding $X \subseteq \mathbb{P}_k^N$ we can define a model as follows; we have that X is a closed subset of $\mathbb{P}_k^N$ which is in turn the generic fibre of $\mathbb{P}_{\mathcal{O}_k}^N$. We can now construct a model $\mathcal{X}/\mathcal{O}_k$ by taking the closure of X in the projective space $\mathbb{P}_{\mathcal{O}_k}^N$. As for any model of a proper scheme one can again identify the $\mathcal{O}_k$-points of $\mathcal{X}$ with the k-points on X.

For a scheme which is not projective, the set of integral points will depend on the model. Hence, the set of integral points on a model can differ from the set of rational points. For this reason we will consider the following setup.

PROPOSITION 1.1.7. *Let S be a finite set of places of k which contains the infinite places Ω_k^∞ of k. Now let $\mathcal{X}$ be a separated scheme of finite type over $\mathcal{O}_{k,S}$. The natural map*

$$\mathcal{X}(\mathbb{A}_{k,S}) \to \prod_{v\in S} X(k_v) \times \prod_{v\notin S} \mathcal{X}(\mathcal{O}_v)$$

is a bijection. Both sets $\mathcal{X}(\mathbb{A}_{k,S})$ and $X(\mathbb{A}_k)$ are naturally subsets of $\prod_{v\in\Omega_k} X(k_v)$ and under this identification we have the inclusion $\mathcal{X}(\mathbb{A}_{k,S}) \subseteq X(\mathbb{A}_k)$. So we have the following chain of inclusions

$$\mathcal{X}(\mathcal{O}_{k,S}) \subseteq \mathcal{X}(\mathbb{A}_{k,S}) \subseteq X(\mathbb{A}_k),$$

where we embed the $\mathcal{O}_{k,S}$-points on $\mathcal{X}$ diagonally into $\mathcal{X}(\mathbb{A}_{k,S})$.

Proof. See Theorem 3.6 in [18]. □

Now let k be a number field and let S be a finite set of places containing the archimedean places Ω_k^∞ of k. If a scheme X does not have a point over any completion k_v of k we see that $\mathcal{X}(\mathbb{A}_{k,S})$ is empty for any model $\mathcal{X}/\mathcal{O}_k$. In this case we also conclude that there are no $\mathcal{O}_{k,S}$-points on $\mathcal{X}$.

By Proposition 1.1.7 we can check whether $\mathcal{X}(\mathbb{A}_{k,S})$ is non-empty by checking the existence of solutions over local fields and rings. However, if $\mathcal{X}(\mathcal{O}_{k,S})$ is empty it can still be that $\mathcal{X}(\mathbb{A}_{k,S})$ is non-empty. This motivates the following definition.

DEFINITION 1.1.8. A separated and finite type $\mathcal{O}_{k,S}$-scheme $\mathcal{X}$ satisfies the *S-integral Hasse principle* if the existence of $\mathbb{A}_{k,S}$-points on $\mathcal{X}$ implies the existence of $\mathcal{O}_{k,S}$-points.

Similarly, a k-variety X is said to satisfy the *Hasse principle* if the existence of adelic points on X implies the existence of k-points on X.

A scheme satisfies the Hasse principle if it either admits a global solution or if it is not locally soluble at at least one place. Usually we will consider the Hasse principle for family of schemes, which means that for each scheme in the family one of the two mutually exclusive statements holds, but not necessarily the same statement is true for all schemes in the family under consideration.

We will later see how the Brauer group of a scheme can in some cases be used to prove that a variety X does not satisfy weak or strong approximation, or that $X(k)$ or $\mathcal{X}(\mathcal{O}_k)$ is empty.

1.2 Central simple algebras over fields

In this section we will define the Brauer group of a field. The main notion will be that of a central simple algebra. The content of this section and much more can be found in [28].

DEFINITION 1.2.1. A *central simple algebra* over a field k is a finite-dimensional k-algebra A which is simple as a ring and central as a k-algebra, i.e. A has precisely two two-sided ideals, and the centre of A is the image of the natural inclusion $k \hookrightarrow A$.

Here are some examples of central simple algebras.

Example. The following algebras are central and simple over the indicated base field:

» any field over itself.

» the quaternions $\mathbb{H}$ over $\mathbb{R}$, i.e. the four-dimensional vector space over $\mathbb{R}$ with basis $1, i, j, ij$, such that multiplication is given by $i^2 = j^2 = -1$ and $ij = -ji$.

Now let A be a central simple algebra over a field k. The following algebras are also central and simple:

» the opposite algebra A^{opp} over k.

» the matrix ring $\mathrm{Mat}_n(A)$ over k for each positive n, in particular $\mathrm{Mat}_n(k)$.

» the base change $A \otimes_k K$ over K for any finite field extension K/k.

The statement follows directly from the definition and a direct computation for the first three cases. For the fourth case one can use the fact that there is a correspondence between the ideals of A and $\mathrm{Mat}_n(A)$, and that the centre $Z(\mathrm{Mat}_n(A))$ consists of the diagonal matrices aI_n where $a \in Z(A)$ and I_n is the identity matrix in $\mathrm{Mat}_n(A)$. The proof that $A \otimes_k K$ is central and simple over K involves some manipulations of the tensor product. See for example [28, Lemma 2.2.2].

The following theorem is an important result if one wants to classify central simple algebras.

THEOREM 1.2.2 (Wedderburn's theorem). *Let A be a finite-dimensional simple k-algebra. There exists a unique integer $r > 0$ and a division k-algebra D such that*

$$\mathrm{Mat}_r(D) \cong A.$$

Proof. See [28, Theorem 2.1.3]. □

This theorem allows us to classify all central simple algebras over two important classes of fields.

COROLLARY 1.2.3. *A central simple algebra A over a finite field k is isomorphic to a matrix algebra* $\mathrm{Mat}_r(k)$ *for some positive integer r.*

Proof. Let D be the division ring such that $\mathrm{Mat}_r(D) \cong A$, hence $Z(D) = k$. Since k is a finite field and D is a finite-dimensional k-algebra we see that D has finitely many elements. A famous theorem by Wedderburn says that every finite division ring is a field showing in particular that D is commutative and we find $D = Z(D) = k$. □

COROLLARY 1.2.4. *Let k be an algebraically closed field. Any central simple algebra over k is a matrix algebra* $\mathrm{Mat}_r(k)$.

Proof. Let A be a central simple algebra over the field k and let D be the division ring such that $A \cong \mathrm{Mat}_r(D)$. We will prove that the natural morphism $k \hookrightarrow D$ is surjective. Pick an element $d \in D$. Since A is of finite dimension over k we see that there is relation over k between finitely many of the $1, d, d^2, \ldots$. This proves that $k(d)$ is a field extension of k of finite degree. Since k is algebraically closed we conclude $d \in k$. □

Corollary 1.2.4 implies the following result about the dimension of a central simple algebra over a general field.

COROLLARY 1.2.5. *The dimension of a central simple algebra over a field is always a square.*

Proof. Let A be a central and simple algebra over a field k. Since $\dim_k A = \dim_{\bar{k}} \left(A \otimes_k \bar{k}\right)$ we can assume that k is algebraically closed. By Corollary 1.2.4 we see that there is a positive integer r and an isomorphism $\mathrm{Mat}_r(k) \cong A$. The statement now follows from the fact $\dim_k \mathrm{Mat}_r(k) = r^2$. □

We have already seen that the finite-dimensional matrix algebras over a field are always central and simple. Hence central simple algebras over finite fields and algebraically closed fields are as simple as they possibly can be. Although the matrix algebras are in some sense the trivial central simple algebras, they play a pivotal role in the study of all central simple algebras. We have for example seen that after base changing to an appropriate field extension every central simple algebra becomes isomorphic to a matrix algebra. One can study a central algebra by considering these fields. This motivates the following definition.

DEFINITION 1.2.6. Let A be a central simple algebra over a field k. A *splitting field* for A is a field extension K/k, such that there exist a positive integer n and an isomorphism

$$A \otimes_k K \cong \mathrm{Mat}_n(K).$$

We have seen that any algebraic closure of k is a splitting field for any central simple algebra over k. The following theorem shows that each central simple algebra has a separable splitting field of finite degree and that any finite-dimensional algebra with this property is both central and simple.

THEOREM 1.2.7. *Let A be a finite-dimensional algebra over a field k. Then A is central and simple precisely if there exists a separable field extension K/k of degree n such that*

$$A \otimes_k K \cong \mathrm{Mat}_n(K).$$

Proof. This is a combination of Theorem 2.2.1 and Theorem 2.2.7 in [28]. □

The following result follows immediately from this theorem.

COROLLARY 1.2.8. *Let A and B be two central simple algebras over a field k. The tensor product $A \otimes_k B$ is a central simple algebra over k.*

Proof. By Theorem 1.2.7 we see that both A and B have a separable splitting field of finite degree over k. This implies that the compositum K of these two fields is also separable and splits both central simple algebras. We then see that

$$(A \otimes_k B) \otimes_k K \cong (A \otimes_k K) \otimes_K (B \otimes_k K) \cong \mathrm{Mat}_m(K) \otimes \mathrm{Mat}_n(K) \cong \mathrm{Mat}_{mn}(K)$$

and we conclude from Theorem 1.2.7 that $A \otimes_k B$ is a central simple algebra over the field k. □

A splitting field of a central simple algebra can be taken to be a splitting field of the associated division ring D from Theorem 1.2.2. The following result tells us where to look for a finite separable splitting field for division rings.

PROPOSITION 1.2.9. *Let A be central simple algebra over k. The algebra A is split by any subfield K of A, i.e. a subalgebra of A which is also a field, satisfying*

$$[K\colon k]^2 = \dim_k A.$$

A central division k-algebra D contains a separable element x of degree $\sqrt{\dim_k D}$, and hence $k(x)$ is a splitting field for all central simple algebras $\mathrm{Mat}_r(D)$ over k.

Proof. The two statements are precisely Proposition 2.2.9 and Proposition 2.2.10 in [28]. □

Using Corollary 1.2.8 we can prove the following result, which is very useful for generalizing central simple algebras to rings and even schemes.

THEOREM 1.2.10. *Let A be a finite-dimensional algebra over a field k. The morphism*

$$\varphi\colon A \otimes_k A^{\mathrm{opp}} \to \mathrm{End}_k(A), \quad \sum_i a_i \otimes b_i \mapsto \left(x \mapsto \sum_i a_i x b_i \right)$$

of vector spaces over k is an isomorphism precisely when A is a central simple algebra over k.

Proof. Suppose that A is a central simple algebra. Then by Corollary 1.2.8 we see that so is $A \otimes_k A^{\mathrm{opp}}$. Because φ is not identically zero, we conclude that the kernel of φ is trivial. By comparing the dimensions we see that φ must be bijective and hence an isomorphism of k-algebras.

Now suppose that φ is an isomorphism of algebras over k. If A is not central, then pick an $a \in Z(A) \backslash k$ and note that $a \otimes 1 - 1 \otimes a$ is non-zero and lies in the kernel of φ. This contradicts the existence of such an a and we conclude that A is central. Now let I be a two-sided ideal of A. Considering I as a subspace of A we find a subspace J of A such that $I \oplus J \cong A$. The morphism

$$\varphi\colon (I \oplus J) \otimes (I \oplus J) \to \mathrm{End}_k(A)$$

restricts to an injective linear map

$$(I \otimes I) \oplus (I \otimes J) \oplus (J \otimes I) \to \mathrm{Hom}_k(A, I).$$

Now let n, i and j be the respective dimensions of A, I and J over k, then we find that $i^2 + ij + ji \leq ni$. So either $i = 0$ or $i + j + j \leq n$ and we see that either $i = 0$ or $j = 0$. These cases correspond to I being either the zero ideal or the whole of A. □

Now we can define an important invariant for fields.

DEFINITION 1.2.11. Let k be a field and A and B two central simple algebras over k. We say that A and B are *similar* if there are integers m and n, such the central simple algebras $\mathrm{Mat}_m(A)$ and $\mathrm{Mat}_n(B)$ over k are isomorphic.

The *Brauer group* $\mathrm{Br}\, k$ *of* k is the set of similarity classes of central simple algebras over k. Let $[A]$ and $[B]$ be two classes of the Brauer group represented by central simple algebras. Multiplication on $\mathrm{Br}\, k$ is defined by

$$[A] \cdot [B] = [A \otimes_k B]$$

and inverses are given by $[A]^{-1} = [A^{\mathrm{opp}}]$.

Let K/k be a finite Galois extension. The *Brauer group of k relative to K* is the subset of $\mathrm{Br}\, k$ consisting of the classes which are split by K and is denoted by $\mathrm{Br}(K/k)$.

Multiplication in $\mathrm{Br}\, k$ is clearly associative and commutative. It follows from $[A] \cdot [k] = [A \otimes_k k] = [A]$ that the class of k is the unit element. Furthermore, Theorem 1.2.10 shows that $[A] \cdot [A]^{-1} = [A] \cdot [A^{\mathrm{opp}}] = [A \otimes A^{\mathrm{opp}}] = [\mathrm{End}_k(A)]$. Now let n^2 be the dimension of A over k then the algebra of k-linear endomorphism of A is isomorphic to $\mathrm{Mat}_{n^2}(k)$ after choosing a k-linear basis for A. So we find $[A] \cdot [A]^{-1} = [\mathrm{Mat}_{n^2}(k)] = [k]$. This shows that $[A]^{-1}$ is the multiplicative inverse of $[A]$.

Note that similar central simple algebras are split by the same fields. Now let K/k be a finite Galois extension. The Brauer group $\mathrm{Br}(K/k)$ of k relative to K is a subgroup of $\mathrm{Br}\, k$, and it follows from Theorem 1.2.7 that the Brauer group of k is the union over all finite Galois extensions of these subgroups.

Sometimes the following cohomological interpretation of the Brauer group is in some cases more useful than the definition using central simple algebras.

THEOREM 1.2.12. *Let k be a field and let K be a finite Galois extension. There are natural maps*

$$\mathrm{Br}(K/k) \to \mathrm{H}^2(\mathrm{Gal}(K/k), K^\times) \quad \textit{and} \quad \mathrm{Br}\, k \to \mathrm{H}^2(k, k^{\mathrm{sep},\times})$$

which are isomorphisms of groups.

Proof. We will sketch the construction of these isomorphisms. The details can be found in Sections 2.4 and 4.4 in [28]. One can find this precise statement there as Theorem 4.4.7.

We start by recalling that central simple algebras over k can be classified as the k-algebras which are forms of $\mathrm{Mat}_n(k)$ by Theorem 1.2.7. Using the fact that the automorphism group of $\mathrm{Mat}_n(K)$ is $\mathrm{PGL}_n(K)$ one can show that the set of isomorphism classes of central simple algebras over k split by K is isomorphic as a pointed set to $\mathrm{H}^1(\mathrm{Gal}(K/k), \mathrm{PGL}_n(K))$. Using the short exact sequence

$$0 \to K^\times \to \mathrm{GL}_n(K) \to \mathrm{PGL}_n(K) \to 0$$

we obtain a map $\mathrm{H}^1(\mathrm{Gal}(K/k), \mathrm{PGL}_n(K)) \to \mathrm{H}^2(\mathrm{Gal}(K/k), K^\times)$. One proceeds by showing that these maps are compatible and combine to give an isomorphism of groups

$$\mathrm{Br}(K/k) \to \mathrm{H}^2(\mathrm{Gal}(K/k), K^\times).$$

For the second statement one uses the fact that every central simple algebra over k is split by a finite Galois extension K. This proves that the isomorphism

$$\mathrm{Br}\, k \to \mathrm{H}^2(k, k^{\mathrm{sep},\times})$$

is the limit of the first isomorphism over all finite Galois extensions K of k. $\square$

A consequence of this theorem is that we can bound the order of an element of the Brauer group in terms of its splitting field.

COROLLARY 1.2.13. *Let K/k be a finite Galois extension of degree n and suppose that A is a central simple algebra over k which is split by K. In the Brauer group* $\operatorname{Br} k$ *we have $n[A] = 0$.*

Proof. Since $\operatorname{Gal}(K/k)$ is of order n, we see that $n\mathrm{H}^2(\operatorname{Gal}(K/k), K^\times) = 0$ by a classical result in group cohomology, see [43, Proposition II.1.31] for example. So $[A]$ has order dividing n in $\operatorname{Br}(K/k) \subseteq \operatorname{Br} k$. □

We can rephrase Corollaries 1.2.3 and 1.2.4 directly in terms of Brauer groups.

PROPOSITION 1.2.14. *We have* $\operatorname{Br} k = 0$ *when k is algebraically closed or a finite field.*

Let us now give a first example of a field whose Brauer group is non-trivial.

PROPOSITION 1.2.15. *It holds that* $\operatorname{Br} \mathbb{R} = \mathbb{Z}/2\mathbb{Z}$ *and it is generated by the class of the only non-trivial central division algebra over $\mathbb{R}$, namely the quaternions* $\mathbb{H}$.

Proof. Let us write $G = \operatorname{Gal}(\mathbb{C}/\mathbb{R})$. Since every central simple algebra over $\mathbb{R}$ is split by $\mathbb{C}$ we see that $\operatorname{Br} \mathbb{R} = \operatorname{Br}(\mathbb{C}/\mathbb{R}) = \mathrm{H}^2(G, \mathbb{C}^\times)$ by Theorem 1.2.12. Group cohomology of cyclic groups is well-understood, see for example [58, Theorem 6.2.2], and we find

$$\mathrm{H}^2(G, \mathbb{C}^\times) \cong (\mathbb{C}^\times)^G / |\mathbb{C}^\times| = \mathbb{R}^\times / \mathbb{R}_{>0}$$

which is naturally isomorphic to $\mathbb{Z}/2\mathbb{Z}$. One can check that $\mathbb{R}$ and $\mathbb{H}$ are division rings whose centres equal $\mathbb{R}$, so their classes are different elements of $\operatorname{Br} \mathbb{R}$, which proves the statement. □

1.3 Cyclic algebras

We will now consider an important class of algebras.

DEFINITION 1.3.1. Let K/k be a cyclic extension of degree n. Fix a generator $\sigma \in \operatorname{Gal}(K/k)$ and an element $a \in k^\times$. We define a k-algebra $(a, K/k, \sigma)$ as follows: consider a K-vector space with basis $1, x, \ldots, x^{n-1}$ endowed with a multiplication determined by the properties $x^n = a$ and $cx^i \cdot dx^j = c\sigma^i(d)x^{i+j}$ for $c, d \in K$. We call this algebra a *cyclic algebra* over k.

When no confusion is possible we might suppress either the field extension or the generator in the notation.

PROPOSITION 1.3.2. *A cyclic algebra $A = (a, K/k, \sigma)$ over a field is a central simple algebra, which is split by the field K.*

The map $k^\times \to$ $\operatorname{Br} k$ *which maps an element $a \in k^\times$ to the class of $(a, K/k, \sigma)$ is a homomorphism of groups. The kernel of this morphism equals* $\operatorname{Nm}_{K/k}(K^\times)$. *In particular, we see that $(a, K/k, \sigma)$ splits over k precisely when a is a norm from K, i.e. $a \in \operatorname{Nm}_{K/k}(K^\times)$.*

Proof. One can check that A is central and simple using Theorem 1.2.10. We also see that K is a subfield of A and by Proposition 1.2.9 we find that K splits A.

The last statement follows from Equation 1.5.21 and Proposition 1.5.23 in [46]. □

The only non-split central simple algebra we have seen so far is an example of a cyclic algebra.

Example. Let $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ denote complex conjugation. Then the map

$$(-1, \mathbb{C}/\mathbb{R}, \sigma) \mapsto \mathbb{H}$$

defined by mapping i to i, and x to j is an isomorphism of algebras over $\mathbb{R}$.

This example illustrates the following general fact, which gives us a condition for when a central simple algebra can be written using only cyclic algebras.

THEOREM 1.3.3 (Merkurjev–Suslin). *Let k be a field which contains a primitive n-th root of unity. An element of* $\mathrm{Br}\, k$ *whose order divides n is the product of classes of cyclic algebras over k.*

Proof. See [28, Theorem 2.5.7]. □

We will be more interested in the following fields, which need not satisfy the condition of the previous statement, but do satisfy an even stronger property.

PROPOSITION 1.3.4. *Let k be a local or a global field. Every central simple algebra over k is a cyclic algebra.*

Proof. This is the content of Theorem 1.5.34 (iii) and Theorem 1.5.36 (iii) in [46]. □

We will see a complete description of Brauer groups of local and global fields in Proposition 1.5.3 and Proposition 1.5.4.

1.4 Azumaya algebras over schemes

We will now generalize the notion of a central simple algebra over a field to the notion of an Azumaya algebra over a scheme. One can think of such an algebra as a family of central simple algebras. The following proposition shows that under this interpretation Theorem 1.2.7 and Theorem 1.2.10 are satisfied locally.

PROPOSITION 1.4.1. *Let $\mathcal{A}$ be a coherent $\mathcal{O}_X$-algebra on a scheme X, such that all stalks $\mathcal{A}_x$ are non-trivial. The following statements are equivalent.*

(i) The sheaf $\mathcal{A}$ is a locally free $\mathcal{O}_X$-module and for each point $x \in X$ the fibre $\mathcal{A}(x) := \mathcal{A} \otimes_{\mathcal{O}_X} \kappa(x)$ is a central simple algebra over $\kappa(x)$.

(ii) *There is an étale covering $U_i \to X$ such that for each i there is an $r_i > 0$ and an isomorphism*
$$\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_{U_i} \cong M_{r_i}(\mathcal{O}_{U_i}).$$

(iii) *The sheaf $\mathcal{A}$ is a locally free $\mathcal{O}_X$-module and the morphism*
$$\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A}^{\mathrm{opp}} \to \mathscr{E}nd_{\mathcal{O}_X}(\mathcal{A})$$
defined on sections by sending $a \otimes b$ to the endomorphism $x \mapsto axb$ is an isomorphism.

Proof. See Definition 6.6.12 in [46]. $\square$

A sheaf satisfying one, and hence all, of the statements in Proposition 1.4.1 is the generalization of central simple algebra with which we will work.

DEFINITION 1.4.2. An *Azumaya algebra* over a scheme X is a coherent $\mathcal{O}_X$-algebra with non-zero stalks satisfying the equivalent statements in Proposition 1.4.1.

Two Azumaya algebras $\mathcal{A}$ and $\mathcal{A}'$ on X are called *similar* if there are locally free coherent $\mathcal{O}_X$-modules $\mathcal{E}$ and $\mathcal{E}'$ with non-zero stalks such that there is an isomorphism
$$\mathcal{A} \otimes_{\mathcal{O}_X} \mathscr{E}nd(\mathcal{E}) \cong \mathcal{A}' \otimes_{\mathcal{O}_X} \mathscr{E}nd(\mathcal{E}')$$
of $\mathcal{O}_X$-algebras.

The *Azumaya Brauer group* $\mathrm{Br}_{\mathrm{Az}}\, X$ of a scheme X is the set of similarity classes of Azumaya algebras on X. The product of two classes $[\mathcal{A}]$ and $[\mathcal{B}]$ is defined as $[\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{B}]$. The class $[\mathcal{O}_X]$ is a two-sided identity element for this multiplication and an inverse of a class $[\mathcal{A}]$ is given by $[\mathcal{A}]^{-1} := [\mathcal{A}^{\mathrm{opp}}]$. This makes $\mathrm{Br}_{\mathrm{Az}}$ into a functor $\mathbf{Sch} \to \mathbf{Ab}^{\mathrm{opp}}$.

Because we require $\mathcal{A}$ to be locally free, the rank of $\mathcal{A}_x$ as an $\mathcal{O}_{X,x}$-module is locally constant. So on connected components of X we see that the rank is the square of a positive integer, just as the dimension of central simple algebras over fields. There are more similarities between Azumaya algebras on schemes and central simple algebras for fields. It might for example be convenient to have a cohomological interpretation of the Azumaya Brauer group of a scheme as we did for fields in Theorem 1.2.12.

DEFINITION 1.4.3. Let X be a scheme. The *cohomological Brauer group* $\mathrm{Br}\, X$ is defined as $\mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_\mathrm{m})$, which defines a functor $\mathbf{Sch} \to \mathbf{Ab}^{\mathrm{opp}}$.

Unlike for fields, the cohomological Brauer group is not always isomorphic to the Azumaya Brauer group. There is however a natural transformation
$$\mathrm{Br}_{\mathrm{Az}} \to \mathrm{Br}$$
constructed similarly as in the case for fields: the isomorphism classes of Azumaya algebras over a scheme X of rank r^2 are classified by the cohomology group $\mathrm{H}^1(X, \mathrm{PGL}_r)$. The short exact sequence
$$0 \to \mathbb{G}_\mathrm{m} \to \mathrm{GL}_r \to \mathrm{PGL}_r \to 0$$

of étale sheaves on X gives rise to a homomorphism

$$\mathrm{H}^1(X, \mathrm{PGL}_r) \to \mathrm{H}^2(X, \mathbb{G}_\mathrm{m}) = \operatorname{Br} X.$$

One can check that the image in $\operatorname{Br} X$ of an Azumaya algebra $\mathcal{A}$ in $\mathrm{H}^1(X, \mathrm{PGL}_r)$ only depends on its class $[\mathcal{A}]$ in $\operatorname{Br}_{\mathrm{Az}} X$. This gives us a map $\operatorname{Br}_{\mathrm{Az}} X \to \operatorname{Br} X$ for any scheme X.

PROPOSITION 1.4.4. *Consider the family of homomorphisms* $\operatorname{Br}_{\mathrm{Az}} X \to \operatorname{Br} X$ *constructed above.*

(a) *This defines a natural transformation* $\operatorname{Br}_{\mathrm{Az}} \to \operatorname{Br}$ *of functors* $\mathbf{Sch} \to \mathbf{Ab}^{\mathrm{opp}}$.

(b) *The homomorphism* $\operatorname{Br}_{\mathrm{Az}} X \to \operatorname{Br} X$ *is injective for every scheme* X.

(c) *Let* k *be a field. The three notions of the Brauer group of* k*, namely* $\operatorname{Br} k$, $\operatorname{Br}(\operatorname{Spec} k)$ *and* $\operatorname{Br}_{\mathrm{Az}}(\operatorname{Spec} k)$*, are all naturally isomorphic.*

(d) *Assume that* X *has an ample line bundle. The natural map* $\operatorname{Br}_{\mathrm{Az}} X \to (\operatorname{Br} X)_{\mathrm{tors}}$ *is an isomorphism.*

We see that $\operatorname{Br} k$ and $\operatorname{Br}(\operatorname{Spec} k)$ are naturally isomorphic and we will keep with the convention that an affine scheme will be denoted by its coordinate ring if no confusion is likely, in particular we will write $\operatorname{Br} R$ for the Brauer group of the affine scheme $\operatorname{Spec} R$ for any commutative ring R.

Proof. The fact that each map $\operatorname{Br}_{\mathrm{Az}} X \to \operatorname{Br} X$ is an injective group homomorphism can be found in [46, Theorem 6.6.17(i)]. A first isomorphism

$$\operatorname{Br} k \to \operatorname{Br}(\operatorname{Spec} k)$$

for the third statement follows from Theorem 1.2.12. It is quite straightforward to exhibit a natural correspondence between Azumaya algebras on $\operatorname{Spec} k$ and central simple algebras over k which preserves similarity of algebras. This proves the existence of the isomorphism $\operatorname{Br} k \to \operatorname{Br}_{\mathrm{Az}}(\operatorname{Spec} k)$. The last statement is Theorem 6.6.17(iii) in [46]. $\square$

Here are some more properties of the two notions of Brauer groups.

PROPOSITION 1.4.5. *Let* X *be a scheme.*

(a) *Suppose that a class* $[\mathcal{A}] \in \operatorname{Br}_{\mathrm{Az}} X$ *is represented by an Azumaya algebra of rank* r^2*. Then* $r[\mathcal{A}]$ *is zero in* $\operatorname{Br}_{\mathrm{Az}} X$.

(b) *The Azumaya Brauer group of a scheme with finitely many connected components is torsion.*

For the next two statement we require that X *is a regular integral noetherian scheme.*

(c) *The natural map* $\operatorname{Br} X \to \operatorname{Br} \kappa(X)$ *is injective.*

(d) *The cohomological Brauer group* $\operatorname{Br} X$ *is a torsion group.*

Proof. The first two statements are precisely Theorem 6.6.17(ii) in [46]. For the last two statement we refer to Theorem 6.6.7 in [46]. □

So, in many cases we can identify all the types of Brauer groups we have seen so far and in those cases we will simply write $\operatorname{Br} X$ for this group. In particular we have the following important corollary to the previous proposition.

COROLLARY 1.4.6. *Let X be a regular quasi-projective variety over a field then we have*

$$\operatorname{Br} X \cong (\operatorname{Br} X)_{\mathrm{tors}} \cong \operatorname{Br}_{\mathrm{Az}} X.$$

Now that we have defined Brauer groups for general schemes, we have in particular done so for rings. Let us consider the following example.

PROPOSITION 1.4.7. *Let R be a complete local ring with maximal ideal m. The natural morphism*

$$\operatorname{Br} R \to \operatorname{Br}(R/m)$$

is an isomorphism.

Proof. See Proposition 6.9.1 in [46]. □

Using the fact that the Brauer group of a finite field is trivial, see Proposition 1.2.14, we find the following result.

COROLLARY 1.4.8. *Let $\mathcal{O}$ be the ring of integers in a non-archimedean local field. Then* $\operatorname{Br} \mathcal{O} = 0$ *and in particular* $\operatorname{Br} \mathbb{Z}_p = 0$.

Let us now look at some examples of Brauer groups of, not necessarily affine, schemes.

PROPOSITION 1.4.9. *Suppose that C is a smooth integral curve over an algebraically closed field then* $\operatorname{Br} C = 0$.

Proof. By Tsen's theorem [46, Theorem 1.5.33] we see that the Brauer group of a field of transcendence degree 1 over an algebraically closed field is trivial. By the inclusion $\operatorname{Br} C \to \operatorname{Br} \kappa(C)$ from Proposition 1.4.5 we see that $\operatorname{Br} C$ must be trivial too. □

Most Brauer groups we have seen thus far are trivial. Starting from a field k with a non-trivial Brauer group one can construct examples of schemes with a non-trivial Brauer group.

PROPOSITION 1.4.10. *Suppose that the k-scheme $\pi\colon X \to \operatorname{Spec} k$ admits a k-point. The natural morphism $\pi^*\colon \operatorname{Br} k \to \operatorname{Br} X$ is injective.*

Proof. A k-point p on X is a section of the structure morphism π. This shows that the composition

$$\operatorname{Br} k \xrightarrow{\pi^*} \operatorname{Br} X \xrightarrow{p^*} \operatorname{Br} k$$

is the identity on $\operatorname{Br} k$ and π^* must be injective. □

The following lemma gives us another situation when the induced map between Brauer groups coming from a morphism between two schemes is injective.

LEMMA 1.4.11. *Let* $X \to Y$ *be a birational morphism between two regular integral noetherian schemes. The induced morphism* $\operatorname{Br} Y \to \operatorname{Br} X$ *is injective.*

Proof. By Proposition 1.4.5 the morphisms $\operatorname{Br} X \to \operatorname{Br} \kappa(X)$ and $\operatorname{Br} Y \to \operatorname{Br} \kappa(Y)$ are injective and we have an isomorphism $\operatorname{Br} \kappa(Y) \to \operatorname{Br} \kappa(X)$ induced by the birational morphism $X \to Y$. By functoriality we have the following commutative square.

$$\begin{tikzcd}
\operatorname{Br} Y \arrow[r] \arrow[d] & \operatorname{Br} \kappa(Y) \arrow[d, "\cong"] \\
\operatorname{Br} X \arrow[r] & \operatorname{Br} \kappa(X)
\end{tikzcd}$$

The injectivity of $\operatorname{Br} Y \to \operatorname{Br} X$ is immediate. □

This lemma is one of the main ingredients for the following proposition.

PROPOSITION 1.4.12. *Let* k *be a field of characteristic* 0 *and* n *a positive integer. The natural morphisms* $\operatorname{Br} k \to \operatorname{Br} \mathbb{A}^n_k$ *and* $\operatorname{Br} k \to \operatorname{Br} \mathbb{P}^n_k$ *are isomorphisms.*

Proof. It follows from Proposition 1.4.9 and [4, Theorem 7.5] that the morphism $\operatorname{Br} L \to \operatorname{Br} \mathbb{A}^1_L$ is an isomorphism for a perfect field L. The first statement follows from induction by considering $\mathbb{A}^n_k$ as the affine line over $\mathbb{A}^{n-1}_k$. It is this step where the condition on the characteristic comes in. Details can be found in [13, Proposition 1.3].

For the last statement, one notes that the embedding $\mathbb{A}^n_k \to \mathbb{P}^n_k$ satisfies the conditions of Lemma 1.4.11. □

Even for a birational map we have the following result for proper schemes.

PROPOSITION 1.4.13. *Let* X *and* X' *be birational regular integral noetherian schemes which are projective over a field of characteristic* 0*. The Brauer groups* $\operatorname{Br} X$ *and* $\operatorname{Br} X'$ *are isomorphic.*

Proof. This follows from Corollaire 7.5 in [30]. □

1.5 Residue and invariant maps

Now we will study residue maps. These are an important tool for working with Brauer groups of local and global fields. They will also turn out to be relevant when studying the Brauer group of a scheme $X \backslash C$ which is the complement of divisor $C \subseteq X$. The more general notion is the following.

PROPOSITION 1.5.1. *Let R be a discrete valuation ring with fraction field K and perfect residue field $\mathbb{F}$. There exists an exact sequence*

$$0 \to \operatorname{Br} R \to \operatorname{Br} K \xrightarrow{\partial_R} \mathrm{H}^1(\mathbb{F}, \mathbb{Q}/\mathbb{Z}) \to 0.$$

The map ∂_R is called the residue map of R.

Proof. This proposition is exactly Proposition 6.8.1 of [46]. The construction of the residue map ∂_R can be found there too. □

If $\mathbb{F}$ is not perfect we lose the surjectivity of ∂_R, but the rest remains true after excluding the p-primary parts of all groups under consideration. See for a more complete treatment Section 6.8.1 in [46].

DEFINITION 1.5.2. Let k be a non-archimedean local field with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}$. The absolute Galois group of $\mathbb{F}$ is $\widehat{\mathbb{Z}}$ with $\mathrm{Frob}_{\mathbb{F}}$ as a topological generator. We can identify the cohomology group $\mathrm{H}^1(\mathbb{F}, \mathbb{Q}/\mathbb{Z})$ with the group $\mathrm{Hom}_{\mathrm{cnt}}(\widehat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z})$ of 1-cocycles, since the action of $\widehat{\mathbb{Z}}$ on $\mathbb{Q}/\mathbb{Z}$ is trivial. The *invariant map* inv_k of k is the composition

$$\operatorname{Br} k \xrightarrow{\partial_{\mathcal{O}}} \mathrm{H}^1(\mathbb{F}, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}_{\mathrm{cnt}}(\widehat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\cong} \mathbb{Q}/\mathbb{Z},$$

where the isomorphism $\mathrm{Hom}_{\mathrm{cnt}}(\widehat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$ is defined by evaluating at $\mathrm{Frob}_{\mathbb{F}}$. When k is either $\mathbb{R}$ or $\mathbb{C}$ we define inv_k to be the unique injective map $\operatorname{Br} k \to \mathbb{Q}/\mathbb{Z}$.

Using these invariant maps we can describe the Brauer groups of both local and global fields.

PROPOSITION 1.5.3. *Let k be a local field. The invariant maps $\mathrm{inv}_k\colon \operatorname{Br} k \hookrightarrow \mathbb{Q}/\mathbb{Z}$ satisfy*

$$\operatorname{Im} \mathrm{inv}_k = \begin{cases} 0 & \text{if } k = \mathbb{C}; \\ \frac{1}{2}\mathbb{Z}/\mathbb{Z} & \text{if } k = \mathbb{R}, \text{ and} \\ \mathbb{Q}/\mathbb{Z} & \text{if } k \text{ is non-archimedean.} \end{cases}$$

If k'/k is a finite extension of local fields, then the diagram in (1.1) *commutes.*

$$\begin{tikzcd} \operatorname{Br} k \arrow[r, "\mathrm{inv}_k"] \arrow[d] & \mathbb{Q}/\mathbb{Z} \arrow[d, "\cdot [k' : k]"] \\ \operatorname{Br} k' \arrow[r, "\mathrm{inv}_{k'}"] & \mathbb{Q}/\mathbb{Z} \end{tikzcd} \tag{1.1}$$

Proof. See [46, Theorem 1.5.34]. □

PROPOSITION 1.5.4. *Let k be a global field. An element $A \in \operatorname{Br} k$ is split by k_v for all but finitely many places v.*

Let inv_v *be the invariant map associated to the completion* k_v *of* k *using the place* v. *We have the following exact sequence*

$$0 \to \operatorname{Br} k \to \bigoplus_v \operatorname{Br} k_v \xrightarrow{\sum \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \to 0.$$

Proof. See [46, Theorem 1.5.36]. □

For more information one can consult Section 1.5.9 in [46]. Here one can find a construction of a cyclic algebra over a fixed local field with a given invariant. Also a construction is given for a cyclic algebra over a global field with a given set of local invariants. This last construction works precisely if the local invariants sum to 0 in $\mathbb{Q}/\mathbb{Z}$, which is the best possible result given the short exact sequence in Proposition 1.5.4.

1.6 The Brauer–Manin obstruction

In this section we fix a number field k and k-variety X.

PROPOSITION 1.6.1. *Let* v *be a place of* k *and* $\mathcal{A}$ *an Azumaya algebra on* X.

(a) The evaluation map $\mathrm{ev}_{\mathcal{A}}\colon X(k_v) \to \operatorname{Br} k_v$ *is locally constant in the* v*-adic topology on* $X(k_v)$.

(b) For an $(x_v) \in X(\mathbb{A}_k)$ *we have that* $\mathrm{ev}_{\mathcal{A}}(x_v)$ *is trivial for almost all* v.

(c) For an adele (x_v) *in the image of* $X(k) \to X(\mathbb{A}_k)$ *we have that*

$$\sum_v \mathrm{inv}_v \, \mathcal{A}(x_v) = 0 \in \mathbb{Q}/\mathbb{Z}.$$

Proof. These are Proposition 8.2.9, Proposition 8.2.1 and Proposition 8.2.2 in [46]. □

The last statement in Proposition 1.6.1 gives us a property of the elements in the image of $X(k)$ in $X(\mathbb{A}_k)$. Although this property can be shared with other adelic points than those coming from $X(k)$, it motivates the following definition.

DEFINITION 1.6.2. Consider a scheme X over a number field k. For an element $\mathcal{A} \in \operatorname{Br} X$ and $x = (x_v) \in X(\mathbb{A}_k)$ define

$$(\mathcal{A}, x) = \sum_{v \in \Omega_k} \mathrm{inv}_v \, \mathcal{A}(x_v).$$

This map $\operatorname{Br} X \times X(\mathbb{A}_k) \to \mathbb{Q}/\mathbb{Z}$ is called the *Brauer–Manin pairing of* X.

For a subset $B \subseteq \operatorname{Br} X$ the *Brauer–Manin set associated to* B is defined as

$$X(\mathbb{A}_k)^B := \{x \in X(\mathbb{A}_k) \mid (\mathcal{A}, x) = 0 \text{ for all } \mathcal{A} \in B\}.$$

The Brauer–Manin sets give an inclusion-reversing map from subsets of $\operatorname{Br} X$ to subsets of $X(\mathbb{A}_k)$ and satisfy the following properties.

COROLLARY 1.6.3. *Let $\mathcal{A}$ be an element of* $\operatorname{Br} X$ *and let $B \subseteq \operatorname{Br} X$ be a subset.*

(a) The map $\sum \operatorname{inv}_v \mathcal{A}\colon X(\mathbb{A}_k) \to \mathbb{Q}/\mathbb{Z}$ is locally constant and the Brauer–Manin set $X(\mathbb{A}_k)^{\mathcal{A}}$ associated to $\mathcal{A}$ is open and closed in $X(\mathbb{A}_k)$.

(b) The subset $X(\mathbb{A}_k)^B$ of $X(\mathbb{A}_k)$ is closed.

(c) We have the following inclusions

$$X(k) \subseteq \overline{X(k)} \subseteq X(\mathbb{A}_k)^B \subseteq X(\mathbb{A}_k).$$

Where we have written $\overline{X(k)}$ for the topological closure of $X(k)$ in $X(\mathbb{A}_k)$.

Some subsets of the Brauer group of a scheme are of particular interest.

DEFINITION 1.6.4. The classes of $\operatorname{Br} X$ in the image of $\operatorname{Br} k \to \operatorname{Br} X$ are called *constant classes*, which form the subgroup $\operatorname{Br}_0 X \subseteq \operatorname{Br} X$. The *algebraic Brauer group* $\operatorname{Br}_1 X$ *of* X is the subgroup consisting of the classes in the kernel of the natural homomorphism $\operatorname{Br} X \to \operatorname{Br} \bar{X}$. The elements of $\operatorname{Br}_1 X$ are called *algebraic classes* and an element of $\operatorname{Br} X \setminus \operatorname{Br}_1 X$ is called a *transcendental class*.

For the Brauer–Manin set associated to either $\operatorname{Br}_1 X$ or $\operatorname{Br} X$ we omit the X from the notation and write $X(\mathbb{A}_k)^{\operatorname{Br}_1}$ and $X(\mathbb{A}_k)^{\operatorname{Br}}$ respectively. The following proposition shows why we do not introduce this notation for $\operatorname{Br}_0 X$. It also shows that in some cases the algebraic part of the Brauer group can be explicitly calculated.

PROPOSITION 1.6.5. *(a) We have* $\operatorname{Br}_0 X \subseteq \operatorname{Br}_1 X \subseteq \operatorname{Br} X$.

(b) The constant Brauer classes of X lie in the right kernel of the Brauer–Manin pairing, i.e. $X(\mathbb{A}_k)^{\operatorname{Br}_0 X} = X(\mathbb{A}_k)$.

(c) Let X be a variety over a number field k such that all global invertible functions over an algebraic closure are constant, i.e. $\mathbb{G}_{\mathrm{m}}(X^{\mathrm{sep}}) = k^{\mathrm{sep},\times}$. There is an isomorphism

$$\operatorname{Br}_1 X / \operatorname{Br}_0 X \cong \mathrm{H}^1(G_k, \operatorname{Pic} X^{\mathrm{sep}}).$$

Proof. For the first statement we only need to prove that constant classes are algebraic. This follows from the fact that the Brauer group of an algebraically closed field is trivial, see 1.2.14. The second statement follows from the fact that the sum of local invariants of $A \in \operatorname{Br} k$ is always zero, Proposition 1.5.4. The third statement follows from the Hochschild–Serre spectral sequence for details one is referred to [46, Corollary 6.7.8]. □

These Brauer–Manin sets allow us in some cases to prove that a scheme X does not satisfy the Hasse principle or weak approximation.

PROPOSITION 1.6.6. *Let X be a scheme over a number field k and let B be a subset of* $\operatorname{Br} X$.

- *If $X(\mathbb{A}_k)^B = \emptyset$ then $X(k) = \emptyset$. If $X(\mathbb{A}_k) \neq \emptyset$ but $X(\mathbb{A}_k)^B = \emptyset$ we say that there is a Brauer–Manin obstruction to the Hasse principle on X.*
- *If $X(\mathbb{A}_k)^B \subsetneq X(\mathbb{A}_k)$ then X does not satisfy strong approximation. We say that there is a Brauer–Manin obstruction to strong approximation on X.*

We call either obstruction an algebraic obstruction if it occurs for a $B \subseteq \operatorname{Br}_1 X$.

The Brauer group can also be used to explain the absence of integral points.

DEFINITION 1.6.7. Consider a finite set S of places of a number field k and suppose that S contains the infinite places Ω_k^∞. Let $\mathcal{X}$ be a separated scheme of finite type over $\mathcal{O}_{k,S}$ and let $B \subseteq \operatorname{Br} X$ be a subgroup of the Brauer group of the generic fibre $X = \mathcal{X}_k$. The *integral Brauer–Manin set associated to B* is denoted by $\mathcal{X}(\mathbb{A}_{k,S})^B$ and defined as

$$X(\mathbb{A}_k)^B \cap \mathcal{X}(\mathbb{A}_{k,S}),$$

where the intersection is taken in $X(\mathbb{A}_k)$.

PROPOSITION 1.6.8.

- *We have the following chain of inclusions*

$$\mathcal{X}(\mathcal{O}_{k,S}) \subseteq \mathcal{X}(\mathbb{A}_{k,S})^B \subseteq \mathcal{X}(\mathbb{A}_{k,S}) \subseteq X(\mathbb{A}_k).$$

- *If $\mathcal{X}(\mathbb{A}_{k,S})^B = \emptyset$ then $\mathcal{X}(\mathcal{O}_{k,S}) = \emptyset$. If also $\mathcal{X}(\mathbb{A}_{k,S}) \neq \emptyset$ we say that there is a Brauer–Manin obstruction to the S-integral Hasse principle on $\mathcal{X}$.*

Note that even if there is no Brauer–Manin obstruction to the integral Hasse principle, the absence of integral points can in some cases be explained by proving that $X(k)$ is the empty set.

1.7 The purity theorem and the ramification locus

We have seen that the residue map can be used to compute the Brauer groups of certain rings. We will now apply residue maps in the setting of schemes; in Proposition 1.4.5 we have seen that for a regular integral noetherian scheme X the natural map $\operatorname{Br} X \to \operatorname{Br} \kappa(X)$ is injective. Not every element of $\operatorname{Br} \kappa(X)$ will necessarily define an Azumaya algebra on X, but it will always do so on a dense open subset.

PROPOSITION 1.7.1. *Let $\mathcal{A} \in \operatorname{Br} \kappa(X)$ be a central simple algebra over the function field of a regular integral noetherian scheme X over a field of characteristic* 0. *There exists an open and dense subset U, such that $\mathcal{A}$ lies in the image* $\operatorname{Br} U \to \operatorname{Br} \kappa(X)$.

Proof. Consider the inverse filtered system of schemes given by open dense subsets $U_i \subseteq X$. We know that $\lim_{\leftarrow} U_i \cong \operatorname{Spec} \kappa(X)$. Corollaire VII.5.9 in [2] tells us that for such a system the natural morphism $\lim_{\rightarrow} \operatorname{Br} U_i \to \operatorname{Br} \kappa(x)$ is an isomorphism. This shows that an element $\mathcal{A} \in \operatorname{Br} \kappa(X)$ comes from an element of $\operatorname{Br} U$ for some open dense $U \subseteq X$. □

A next question might be whether one can enlarge such an open subset U. The following two results answer this question in general.

THEOREM 1.7.2 (Purity theorem). *Let X be a noetherian regular integral scheme over a field of characteristic 0 and consider an element $\mathcal{A} \in \operatorname{Br} \kappa(X)$. There exists an open subscheme $U \subseteq X$ whose complement is either empty or of pure codimension 1, such that $\mathcal{A}$ lies in the image of $\operatorname{Br} U \to \operatorname{Br} \kappa(X)$.*

Proof. Let $U' \subseteq X$ be an open and dense subscheme such that $\mathcal{A}$ lies in the image of $\operatorname{Br} U' \to \operatorname{Br} \kappa(X)$. Consider the complement $Z' = X \backslash U'$ and let Z be the union of the irreducible components of Z' of codimension 1. Then by Corollaire 6.2 in [30] we see that the map $\operatorname{Br}(X\backslash Z) \to \operatorname{Br}(X \backslash Z') = \operatorname{Br} U'$ is an isomorphism. This proves that one can take $U = X \backslash Z$. □

For a cyclic algebra $\mathcal{A} = (g, \kappa(X_K)/\kappa(X), \sigma)$ over the field $\kappa(X)$ Theorem 1.7.2 can be made explicit; the Azumaya class of $\mathcal{A}$ lies in the Brauer group of the dense open subset where g is defined and invertible. The following lemma allows us to check when this class actually lives in the subgroup $\operatorname{Br} X$.

LEMMA 1.7.3. *Consider a smooth and geometrically integral variety X over a field k satisfying $\mathbb{G}_m(\bar{X}) = \bar{k}^\times$. Fix a finite cyclic extension K/k, a generator $\sigma \in \operatorname{Gal}(K/k)$, and an element $g \in \kappa(X)^\times$.*

The cyclic algebra $\mathcal{A} = (g, \kappa(X_K)/\kappa(X), \sigma)$ lies in the image of $\operatorname{Br} X \to \operatorname{Br} \kappa(X)$ precisely if $\operatorname{div} g = N_{K/k}(D)$ for some divisor D on X_K. If k, and hence K, is a number field, and X is everywhere locally soluble then $\mathcal{A}$ is constant exactly when D can be taken to be principal.

Proof. This lemma is similar to Proposition 4.17 from [5]. The difference is that the projectivity assumption is replaced by the weaker condition $\mathbb{G}_m(\bar{X}) = \bar{k}^\times$. One can check that under this assumption the proof presented in [5] is still valid. □

This result exhibits a general principle. To explicitly write down classes of $\operatorname{Br} X$ one usually uses the inclusion $\operatorname{Br} X \hookrightarrow \operatorname{Br} \kappa(X)$ and starts with a central simple algebra over the field $\kappa(X)$. Finding an open subscheme U of X for which $\mathcal{A}$ comes from $\operatorname{Br} U$ is usually straightforward. We would want to identify the irreducible components of the complement Z of U in X on which we can extend the Azumaya algebra. We have seen in the proof of Theorem 1.7.2 that we can extend $\mathcal{A}$ on any irreducible components of Z of codimension at least 2. For the irreducible components of Z of codimension 1 we can use residue maps.

THEOREM 1.7.4. *Suppose that X is a regular integral noetherian scheme over a field of characteristic 0. For each point x of X of codimension 1 we have a residue map*

$$\partial_x \colon \operatorname{Br} \kappa(X) \to \mathrm{H}^1(\kappa(x), \mathbb{Q}/\mathbb{Z}).$$

Fix an element $\mathcal{A} \in \operatorname{Br} \kappa(X)$. The residue $\partial_x(\mathcal{A})$ is trivial for all but finitely many codimension 1 *points $x \in X^{(1)}$ and the following sequence is exact*

$$0 \to \operatorname{Br} X \to \operatorname{Br} \kappa(X) \xrightarrow{\oplus \partial_x} \bigoplus_{x \in X^{(1)}} \mathrm{H}^1(\kappa(x), \mathbb{Q}/\mathbb{Z}).$$

Proof. See Theorem 6.8.3 in [46]. □

Again, one can show a more general statement for schemes not necessarily defined over a field, although one might need to exclude the p-primary part of all groups if there exist codimension 1 points x for which $\kappa(x)$ is imperfect.

In the proof of Proposition 1.7.1 we have seen the important result that the Brauer group does not change when we cut out a subscheme of codimension at least 2. When we cut out a subscheme of codimension 1, the Brauer group can change and the residue maps allow us to quantify this change.

PROPOSITION 1.7.5. *Let D be a geometrically irreducible regular divisor on a regular integral noetherian scheme X over a field of characteristic* 0. *Write U for the complement of D in X. The image of the residue map $\partial_D \colon \operatorname{Br} U \to \mathrm{H}^1(\kappa(D), \mathbb{Q}/\mathbb{Z})$ lies in the subgroup $\mathrm{H}^1_{\text{ét}}(D, \mathbb{Q}/\mathbb{Z})$ and the sequence in* (1.2) *is exact.*

$$0 \to \operatorname{Br} X \to \operatorname{Br} U \xrightarrow{\partial_D} \mathrm{H}^1_{\text{ét}}(D, \mathbb{Q}/\mathbb{Z}) \tag{1.2}$$

Proof. See Corollaire 6.2 in [30]. □

This sequence is only functorial in some cases.

PROPOSITION 1.7.6. *Let $f \colon X' \to X$ be a morphism between two regular integral noetherian schemes over a field of characteristic* 0. *Let D be a regular integral divisor on X, such that $D' = f^{-1}(D)$ is also regular and integral. Define U and U' as the complement of D in X and D' in X' respectively. The diagram in* (1.3) *is commutative.*

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Br} X \arrow[r] \arrow[d] & \operatorname{Br} U \arrow[r] \arrow[d] & \mathrm{H}^1_{\text{ét}}(D, \mathbb{Q}/\mathbb{Z}) \arrow[d] \\ 0 \arrow[r] & \operatorname{Br} X' \arrow[r] & \operatorname{Br} U' \arrow[r] & \mathrm{H}^1_{\text{ét}}(D', \mathbb{Q}/\mathbb{Z}) \end{tikzcd} \tag{1.3}$$

Proof. The rows are exact by Proposition 1.7.5.

Now consider for a regular closed subscheme D of codimension 1 on a regular scheme X the first terms of the Gysin sequence, see for example Corollary 5.2 in [8],

$$0 \to \mathrm{H}^2_{\text{ét}}(X, \mu_n) \to \mathrm{H}^2_{\text{ét}}(X \backslash D, \mu_n) \to \mathrm{H}^1_{\text{ét}}(D, \mathbb{Z}/n\mathbb{Z}).$$

In the discussion following Lemma 5.4 in [8] we see that the Gysin sequence is functorial for a morphism $f \colon X' \to X$ for which $D' = f^{-1}(D)$ is also regular and of codimension 1.

One can derive the exact sequence in Proposition 1.7.5 from the n-torsion from the Gysin sequence, i.e.

$$0 \to \operatorname{Br} X[n] \to \operatorname{Br} U[n] \xrightarrow{\partial_D} \mathrm{H}^1_{\text{ét}}(D, \mathbb{Q}/\mathbb{Z})[n]$$

is exact. Note however that the map $\mathrm{H}^2_{\text{ét}}(X \backslash D, \mu_n) \to \mathrm{H}^1_{\text{ét}}(D, \mathbb{Z}/n\mathbb{Z})$ can differ from the residue map $\operatorname{Br} U[n] \to \mathrm{H}^1_{\text{ét}}(D, \mathbb{Q}/\mathbb{Z})[n]$ by a sign. Furthermore, the functoriality of the Gysin sequence carries over to these sequences and we have the commutative diagram as shown in (1.4).

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Br} X[n] \arrow[r] \arrow[d] & \operatorname{Br} U[n] \arrow[r] \arrow[d] & \mathrm{H}^1_{\text{ét}}(D, \mathbb{Q}/\mathbb{Z})[n] \arrow[d] \\ 0 \arrow[r] & \operatorname{Br} X'[n] \arrow[r] & \operatorname{Br} U'[n] \arrow[r] & \mathrm{H}^1_{\text{ét}}(D', \mathbb{Q}/\mathbb{Z})[n] \end{tikzcd} \tag{1.4}$$

To prove the proposition we recall that the Brauer group of a regular integral noetherian scheme is torsion, see Proposition 1.4.5. □

Consider an element $\mathcal{A} \in \operatorname{Br} \kappa(X)$ and let Z be the union of all irreducible curves D on X for which $\partial_D(\mathcal{A})$ is non-zero. The results in this section show that $\mathcal{A}$ lies in the image of $\operatorname{Br} U \to \operatorname{Br} \kappa(X)$ for $U = X \backslash Z$. The closed subscheme Z is called the *ramification locus* of $\mathcal{A}$.

Chapter 2

Del Pezzo surfaces

In this chapter we will study the many different flavours of del Pezzo surfaces. First we will define and classify generalized del Pezzo surfaces. These surfaces are appropriately named as they generalize the important subclass of ordinary del Pezzo surfaces which were studied by del Pezzo in 1887 [22]. Next we proceed by studying the Picard group of these smooth surfaces and certain effective divisor classes with negative self-intersection. We will describe how to contract a collection of curves in such classes and we will give conditions for the constructed surface to be normal or even smooth.

Using these results we define singular del Pezzo surfaces. These normal projective surfaces are obtained from a generalized del Pezzo surface by contracting all curves with self-intersection equal to -2 and are the generalization of projectively embedded ordinary del Pezzo surfaces using the anticanonical bundle.

The next section describes a novel type of algebraic surface, namely the class of peculiar del Pezzo surfaces. These surfaces are again normal and obtained from contracting a subset of the curves with self-intersection -2 on a generalized del Pezzo surface. This shows that peculiar del Pezzo surfaces fit in between generalized and singular del Pezzo surfaces. This new type of surface was defined by the author to describe certain aspects of the geometry of both generalized and singular del Pezzo surfaces in a simpler manner.

We will show that the minimal desingularization of both a singular and a peculiar del Pezzo surface is a generalized del Pezzo surface. Using this result we can prove that there is a correspondence between generalized, peculiar and singular del Pezzo surfaces. For the classical case of ordinary del Pezzo surfaces the three notions coincide: an ordinary del Pezzo surface X is a generalized, peculiar and singular del Pezzo surface. Note the inescapable confusion: a singular del Pezzo need not be singular.

Now consider a projective family of ordinary del Pezzo surfaces over an open U of some base space B. If one can extend the family to B the fibres over $B \backslash U$ need not be ordinary del Pezzo surfaces as well. Depending on one's interpretation of ordinary del Pezzo surfaces these fibres can be generalized, peculiar or singular del Pezzo surfaces. The properties of ordinary del Pezzo surfaces with which we will work will be directly applicable to at least one of these three more general notions.

In the last section we give a short summary of the arithmetic of ordinary del Pezzo surfaces over a number field.

2.1 Preliminaries on geometry

Before we consider del Pezzo surfaces let us first define the following geometric objects and concepts. Recall that we defined a variety over a field k to be a scheme which is separated and of finite type over k.

DEFINITION 2.1.1. Let k be a field. A *curve over k* is a variety over k of pure dimension 1. A *surface over k* is a geometrically integral variety of dimension 2 over k. By a *curve C on a surface S over k* we mean a closed subscheme $C \subseteq S$ which is a curve over k.

Note that our definition of curve is less restrictive than our definition of surface. This stems from the fact that we will work with reasonably well-behaved surfaces. Practically all surfaces we will encounter are normal, and in this chapter all surfaces will be projective. A one-dimensional subscheme of such a surface can definitely be less elegant from a geometric point of view. For this reason we will want to allow curves to be reducible, non-reduced or singular.

Let us turn to our conventions on divisors. We will use the word divisor to refer to *Cartier divisors*. Since surfaces are integral by definition the *Picard group* $\operatorname{Pic} S$ of a surface S is isomorphic to both the group of isomorphism classes of line bundles and the group of linear equivalence classes of divisors. For a divisor $D \in \operatorname{Pic} X$ the associated line bundle is denoted by $\mathcal{L}(D)$. This line bundle comes with a specified rational section 1_D and the divisor D is effective precisely if 1_D lies in $H^0(S, \mathcal{L}(D))$.

Since most of our surfaces will be normal they are regular in codimension one and this makes it more convenient to also consider Weil divisors. Recall that a *prime divisor* on a surface S is just an integral curve on S, and a *Weil divisor* on S is an element of the free abelian group generated by all prime divisors on S. In this case Cartier divisors are precisely the locally principal Weil divisors. On a surface which is also smooth the two notions of divisors coincide completely.

Let us state the definition of the intersection product between two divisors.

DEFINITION 2.1.2. Let S be a surface which is proper over a field k. For an integral curve C on S and a divisor D on S their *intersection product $C \cdot D$* is defined as the degree of the restriction of the line bundle $\mathcal{L}(D)$ to C.

We recall the definition of the degree of a line bundle $\mathcal{L}$ on a, possibly singular, integral curve C which is projective over a field k. First associate a Weil divisor to $\mathcal{L}$ following the procedure in Section 2.2 of [27]. This divisor is well-defined up to rational equivalence and the proper map $C \to \operatorname{Spec} k$ allows us to push this Weil divisor class forward to a Weil divisor class on $\operatorname{Spec} k$. The group of Weil divisors on $\operatorname{Spec} k$ modulo rational equivalence is canonically isomorphic to $\mathbb{Z}$. The integral number associated to $\mathcal{L}$ in this manner is what we define as the *degree* of $\mathcal{L}$ on C.

A complete treatment of intersection cycles and Weil divisor class groups can be found in [27]. We will use the following properties: the intersection product is linear in both arguments, it is preserved under arbitrary field extensions, and on a projective surface S the intersection pairing descends to a bilinear pairing on $\operatorname{Pic} S$.

Although it is not directly apparent the intersection product is important for both notions in the following definition.

DEFINITION 2.1.3. Let D be a divisor on a projective surface over a field k. We say that D is *nef* if for all integral curves C on X we have $C \cdot D \geq 0$. The divisor D is called *big* if the rational map $X \dashrightarrow \mathbb{P}_k^N$ associated to the complete linear system of a sufficiently large multiple of D defines a birational map from X to its scheme-theoretic image.

It is indeed clear that it can be checked numerically whether a divisor is nef. Corollary 2.2.8 in [36] shows that the same is true for big. We will however use the following proposition which gives a sufficient and necessary numerical condition for a divisor to be big and nef. It is similar to the Nakai–Moishezon criterion [33, Theorem V.1.10] for checking if a divisor on a surface is ample.

PROPOSITION 2.1.4. *Let D be a divisor on a projective surface X over a field k. The line bundle $\mathcal{L}(D)$ is big and nef precisely if $D^2 > 0$ and for all integral curves C on X we have $C \cdot D \geq 0$.*

Proof. Let us first prove the statement in the case that k is algebraically closed. We apply Theorem 2.2.16 of [36] to see that D is big and nef if and only if $D^2 > 0$ and $C \cdot D \geq 0$ for all integral curves C on X. Note that although the standing convention in [36] is that schemes are defined over $\mathbb{C}$ one can check that the statement is true over any algebraically closed field.

It is clear from the definition that being big is preserved under arbitrary field extensions. The same holds for being nef. We will prove this for the field extension $\bar{k}/k$.

Assume that the pullback $\bar{D}$ of D to $\bar{X} = X \times_k \bar{k}$ is nef. Let C be an integral curve on X and write $\bar{C}$ for its pullback to $\bar{X}$. Note that $\bar{C}$ defines an effective Weil divisor W on $\bar{X}$ and hence we find

$$C \cdot D = W \cdot \bar{D} \geq 0.$$

Now consider the case that D is nef on X and let C' be an integral curve on $\bar{X}$. The scheme-theoretic image C of C' under $\bar{X} \to X$ pulls back to a Weil divisor W on $\bar{X}$ which is supported on the finitely many conjugates C'_i of C' under the absolute Galois group G_k. Notes that this group acts transitively on the set of C'_i since C' is irreducible and trivially on W. Hence $\bar{D} \cdot C'_i$ and the multiplicity m_i

of C_i' in W is independent of i. Since the intersection pairing is preserved under base change we find that there exists a positive constant m such that

$$m(C_i' \cdot \bar{D}) = W \cdot \bar{D} = C \cdot D \geq 0$$

for all i. This proves that $\bar{D}$ is nef on X.

The result now follows from the fact that the intersection pairing is also preserved under field extensions. $\square$

This property shows that an ample divisor on a projective surface is both big and nef. The converse is not true; it need not even be the case that a big and nef line bundle is semiample. This means that no rational map $S \dashrightarrow \mathbb{P}_k^N$ associated to a multiple of a big and nef divisor extends to a morphism on the whole of S. For an example the reader is referred to [36, Section 2.3.A]. Practically all big and nef divisors we will encounter will however be semiample. Once we know that generalized del Pezzo surfaces are rational smooth projective surfaces with a big anticanonical divisor this can be explained by Lemma 2.6 in [54].

Now consider a semiample, big and nef divisor D on a surface X. This defines a birational morphism $X \to X' \subseteq \mathbb{P}_k^N$. One can show that the curves C on X which are contracted by this birational morphism are precisely those for which $C \cdot D = 0$.

We will need one more geometric concept. Recall that any smooth scheme X admits a canonical line bundle ω_X. We will write the associated divisor class as K_X. Now consider a normal scheme Y. The singular locus Σ is of codimension at least 2. This proves that there is an isomorphism between Weil divisors on Y and $Y \backslash \Sigma$. This allows us to define the canonical Weil divisor on the normal scheme Y. Take the closure of a canonical divisor $K_{Y \backslash \Sigma}$ in Y as a Weil divisor. We will denote this Weil divisor on Y by K_Y.

If the Weil divisor K_Y on a normal scheme is actually a Cartier divisor, then we denote the associated line bundle $\mathcal{L}(K_Y)$ by ω_Y.

2.2 Generalized and ordinary del Pezzo surfaces

We now have the terminology to define generalized and ordinary del Pezzo surfaces.

DEFINITION 2.2.1. A *generalized del Pezzo surface* is a smooth projective surface X over a field k for which the anticanonical divisor $-K_X$ is big and nef. The surface X is an *ordinary del Pezzo surface* if the anticanonical divisor is moreover ample.

The *degree* d of a generalized del Pezzo surface is defined as the canonical self-intersection number $d = K_X \cdot K_X$.

The following theorem classifies generalized del Pezzo surfaces over algebraically closed fields.

THEOREM 2.2.2. *Let X be a generalized del Pezzo surface over an algebraically closed field k. The surface X is rational and isomorphic to either $\mathbb{P}^2_k$, $\mathbb{P}^1_k \times \mathbb{P}^1_k$, the Hirzebruch surface $\mathbb{F}_2$, or there exists an integer $1 \leq r \leq 8$ such that X can be written as*

$$X = X_r \to X_{r-1} \to \ldots \to X_1 \to X_0 = \mathbb{P}^2_k$$

where each map is the blowup in a reduced closed point which does not lie on a curve of self-intersection -2. The degree of these four types of generalized del Pezzo surfaces are respectively 9, 8, 8 and $9 - r$. In particular, the degree of a generalized del Pezzo surface is a positive integer $d \leq 9$.

Proof. The main ingredients for this proof come from [23], but in the classification the Hirzebruch surface $\mathbb{F}_2$ is erroneously left out. A corrected and completed classification can be found in [19, Proposition 0.4]. □

In this last case we say that X is the blowup of $\mathbb{P}^2_k$ in r points in *almost general position*. If we strengthen the condition that none of the centres of the blowups lies on a curve of self-intersection -2, to curves of self-intersection -1 we say that X is the blowup of the projective plane in r points in *general position*. Using this terminology we can identify the ordinary del Pezzo surfaces in the previous theorem.

THEOREM 2.2.3. *Let X be an ordinary del Pezzo surface over an algebraically closed field k. The surface X is isomorphic to either $\mathbb{P}^2_k$, $\mathbb{P}^1_k \times \mathbb{P}^1_k$ or a blowup of $\mathbb{P}^2_k$ in r points in general position for some $1 \leq r \leq 8$.*

Proof. See [41, Theorem 24.3]. □

The following proposition shows that a surface remains a del Pezzo surface after base extension. This means in particular that Theorem 2.2.2 and Theorem 2.2.3 can be used to classify del Pezzo surfaces over any field.

PROPOSITION 2.2.4. *Let X be a surface over a field k, and let K/k be a field extension. The surface X is a generalized del Pezzo surface precisely if $X_K = X \times_k K$ is a generalized del Pezzo surface. The same statement holds for ordinary del Pezzo surfaces.*

Proof. We have seen in the proof of Proposition 2.1.4 that both bigness and nefness are preserved under field extensions. A similar proof shows that the same holds for ampleness. □

We have seen that blowing up closed points is a principal operation one uses to produce generalized del Pezzo surfaces. The following proposition shows that we can always invert this process.

PROPOSITION 2.2.5. *Let X be a generalized del Pezzo surface X of degree d over a field k, and let L be a geometrically integral rational curve on X of self-intersection -1. There exists a generalized del Pezzo surface X' of degree $d + 1$ together with a morphism $X \to X'$ such that the following properties are satisfied:*

» *L maps to a point $p \in X'(k)$; and*

» *the morphism $X \to \mathrm{Bl}_p X'$ obtained from the universal property of the blowup is an isomorphism.*

If X is an ordinary del Pezzo surface, then so is X'.

Proof. By Castelnuovo's Theorem [33, Theorem V.5.7] there is a smooth surface X' over k, a morphism $X \to X'$ and a point $p \in X'$ such that $X \to \mathrm{Bl}_p X'$ is an isomorphism and L is the exceptional curve of this blowup $\pi\colon X \to X'$. We only need to check that X' is indeed a generalized (respectively ordinary) del Pezzo surface. For generalized del Pezzo surfaces this can be done using Proposition 2.1.4.

The line bundle $L - K_X$ is the pullback of the line bundle $-K_{X'}$ by Proposition V.3.3 in [33]. Let C be an integral curve on X'. The intersection number $-K_{X'} \cdot C$ equals $\pi^*(-K_{X'}) \cdot \pi^*C = (L - K_X) \cdot \pi^*C$. Since L is the exceptional curve of π we have $L \cdot \pi^*C = 0$, and since $-K_X$ is big and nef and π^*C is an effective divisor we find $-K_X \cdot \pi^*C \geq 0$. We also see that

$$K_{X'}^2 = (\pi^*K_{X'})^2 = (L - K_X)^2 = L^2 - 2L \cdot K_X + K_X^2 = -1 + 2 + d = d + 1 > 0.$$

Proposition 2.1.4 now implies that $-K_{X'}$ is big and nef and hence X' is a generalized del Pezzo surface.

For ordinary del Pezzo surfaces we use the Nakai–Moishezon criterion [33, Theorem V.1.10]. The proof is similar to the proof above, but the inequality should be replaced by a strict inequality. □

These curves of negative self-intersection will turn out to be important. Before looking into them we will first consider their divisor classes in the next section.

2.3 Divisor classes of negative self-intersection

Since the Picard group of the blowup of a surface in a point is well understood, we can describe the Picard group of a generalized del Pezzo surface over an algebraically closed field.

PROPOSITION 2.3.1. *Let X be a generalized del Pezzo surface of degree d over an algebraically closed field k. If X is not isomorphic to $\mathbb{P}^1_k \times \mathbb{P}^1_k$ or $\mathbb{F}_2$, then there exist $10 - d$ divisor classes $L_0, L_1, \ldots, L_r \in \mathrm{Pic}\, X$ such that*

» $L_0^2 = 1$;

» $L_i^2 = -1$ *for $i > 0$;*

» $L_i \cdot L_j = 0$ *for $i \neq j$; and*

» $\mathrm{Pic}\, X$ *is freely generated by the L_i.*

The class of the anticanonical bundle $-K_X$ *equals* $3L_0 - \sum_{i=1}^{r} L_i$.

For any generalized del Pezzo surface X *of degree* d *we have* $\operatorname{Pic} X \cong \mathbb{Z}^{10-d}$.

Note that this proves that the lattice isomorphism class of the geometric Picard group $\operatorname{Pic} \bar{X}$ of a generalized del Pezzo surface X over any field only depends on the degree d.

Proof. See [23, II, Section 4]. □

The classes L_i for $i > 0$ have negative self-intersection, but these need not be all of them. Generalized del Pezzo surfaces do not have prime divisors of self-intersection smaller than -2 by nefness of $-K_X$ and the adjunction formula. With this in mind we will consider classes of self-intersection -1 and -2. To control the behaviour of curves representing these classes under the anticanonical embedding we add a condition on the intersection number of these classes with the canonical class.

DEFINITION 2.3.2. Suppose that X is a generalized del Pezzo surface over a field k and let s be either -1 or -2. An *s-class* is a divisor class D on $\bar{X}$ such that $D^2 = s$ and $D \cdot K_X = -2 - s$.

The following proposition shows that we could equally consider the base change to the separable closure.

PROPOSITION 2.3.3. *Let X be a smooth rational surface over a field k. The natural map* $\operatorname{Pic} X^{\mathrm{sep}} \to \operatorname{Pic} \bar{X}$ *is an isomorphism.*

Proof. Lemma 3.1 of [9] contains the same statement for K3 surfaces. The proof only uses that $\mathrm{H}^1(X, \mathcal{O}_X) = 0$. This is however also true for smooth rational surfaces. □

The following lemma shows that there are only finitely many s-classes on any generalized del Pezzo surface.

LEMMA 2.3.4. *Let X be a generalized del Pezzo surface over an algebraically closed field k. Then X has only finitely many s-classes.*

There are no s-classes on $\mathbb{P}^2_k$ and $\mathbb{P}^1_k \times \mathbb{P}^1_k$. The only s-classes on the Hirzebruch surface $\mathbb{F}_2$ are the class of the base curve and its negative. The number of s-classes on the projective plane blown up in r points in almost general position can be found in Table A.

Now suppose that X is written as the blowup $\pi \colon X \to \mathbb{P}^2_k$ of the projective plane in $r \leq 7$ points in almost general position. The pushforward of an s-class on X to $\mathbb{P}^2_k$ is either trivial, $\mathcal{O}(1)$ or $\mathcal{O}(2)$.

The last statement is not true for s-classes on generalized del Pezzo surfaces of degree 1 or 2; the pushforward of an s-class to the projective plane can also be the line bundle $\mathcal{O}(3)$.

r	8	7	6	5	4	3	2	1
$s = -1$	240	56	27	16	10	6	3	1
$s = -2$	240	126	72	40	20	8	2	0

Table A: The number of s-classes on the projective plane blown up in r points in almost general position.

Proof. The numbers of s-classes in the table follow from the description in Proposition 2.3.1 of the Picard group of the projective plane blown up in r points in almost general position. The details can be found in [23, II, Section 5]. Enumerating all s-classes on a generalized del Pezzo surface of degree $d \geq 3$ shows that for an s-class $R \in \operatorname{Pic} \bar{X}$ we have $0 \leq R \cdot L_0 \leq 2$, where L_0 is the first element of a basis of $\operatorname{Pic} \bar{X}$ as given in Proposition 2.3.1.

For the remaining cases the geometric Picard group is either $\mathbb{Z}$, $\mathbb{Z}^2$ with the Euclidean intersection pairing, or $\mathbb{Z} \cdot C + \mathbb{Z} \cdot F$ with the pairing given by $C^2 = -2$, $C \cdot F = 1$ and $F^2 = 0$, see for example Proposition 2.3 and Proposition 2.9 in [33]. In these cases the computation of the number of s-classes is straightforward. □

Now let X be a generalized del Pezzo surface over a general field k. The absolute Galois group G_k of k acts naturally on $\bar{X}$ and this endows the geometric Picard group $\operatorname{Pic} \bar{X}$ with the structure of a G_k-module. An element $\sigma \in G_k$ acts on $\operatorname{Pic} \bar{X}$ in a specific way; it will preserve K_X and the intersection pairing. The following theorem shows that this cannot happen in many ways.

PROPOSITION 2.3.5. *Let X be a generalized del Pezzo surface of degree d over an algebraically closed field k and let A_X be the subgroup of* $\operatorname{Aut}(\operatorname{Pic} X)$ *consisting of the elements which preserve the canonical class K_X and the intersection pairing. The group A_X is finite.*

Now suppose that X is the blowup of $\mathbb{P}^2_k$ in $r = 9 - d$ points in almost general position. The group A_X permutes the -1-classes in $\operatorname{Pic} X$ *and the induced map from A_X to the group consisting of the intersection pairing preserving permutations of the -1-classes is an isomorphism.*

Proof. The finiteness of A_X is part a) of Théorème 2 in [23, II]. This result does not address the surfaces $\mathbb{F}_2$ and $\mathbb{P}^1_k \times \mathbb{P}^1_k$, but for those cases the statement is easily verified.

For the second statement recall that s-classes are defined in terms of their self-intersection and the intersection with the anticanonical class. It follows that A_X preserves s-classes by definition. Now consider the group homomorphism from A_X to the group consisting of the intersection pairing preserving permutations of the -1-classes. The surjectivity of this homomorphism follows from part b) of Proposition 5 in [23, II]. The injectivity is trivial if $d \geq 8$ and follows for $d \leq 7$ from the fact that the -1-classes generate $\operatorname{Pic} X$. □

The second statement is also true for $\mathbb{F}_2$, but not for $\mathbb{P}^1_k \times \mathbb{P}^1_k$.

Note that for generalized del Pezzo surfaces which are the blowup of the projective plane in r points in almost general position the group A_X only depends on the degree d, or equivalently r. For $r \leq 6$ this group has been identified in [41, Theorem 25.4] as the Weyl group W_r of a certain root system. For these del Pezzo surface we will write W_r instead of A_X, although one should be careful since W_r does not come with a fixed action on $\operatorname{Pic} X$. For more information about the groups W_r one can refer to §26 in [41].

Let X be a generalized del Pezzo surface over a general field k which is geometrically the blowup of the projective plane in $r \geq 3$ points in almost general position. After fixing the action of W_r on $\operatorname{Pic} \bar{X}$ the action of G_k on $\operatorname{Pic} \bar{X}$ induces a group homomorphism $G_k \to W_r$. The image of this homomorphism is the smallest subgroup W of W_r such that the action of G_k factors through the induced action of W on $\operatorname{Pic} \bar{X}$. This subgroup of W_r contains much information about the action of Galois on the geometric Picard group. An important corollary to Proposition 2.3.5 is that in this sense there are only finitely many possible actions of Galois on the geometric Picard group.

2.4 Curves of negative self-intersection

In the previous section we have studied the s-classes in the Picard group of a generalized del Pezzo surface. In this section we will study the integral curves in such divisor classes.

DEFINITION 2.4.1. Let X be a generalized del Pezzo surface over a field k. An integral curve C on $\bar{X}$ whose class in $\operatorname{Pic} \bar{X}$ is an s-class will be called a *geometric s-curve*. A curve C on X is called an *s-curve* if its base change $\bar{C} \subseteq \bar{X}$ to an algebraic closure $\bar{k}$ is a geometric s-curve.

Similar to Proposition 2.3.3 we could equivalently have used the separable closure of k instead of an algebraic closure. This fact becomes important once we start looking at the action of G_k on the geometric s-curves.

PROPOSITION 2.4.2. *Let X be a smooth rational surface over a field k. Any geometric s-curve is defined over k^{sep}.*

Proof. We again refer to [9]. The proof of Corollary 3.2 also proves this statement. □

By the adjunction formula we see that every s-curve is rational and the following results directly from Lemma 2.3.4.

LEMMA 2.4.3. *Let X be a generalized del Pezzo surface over a field k. Each s-class contains at most one geometric s-curve and in particular we see that there are finitely many geometric s-curves on X.*

Now suppose that X is a generalized del Pezzo surface which is written as the blowup $\pi\colon X \to \mathbb{P}^2_k$ of the projective plane in $r \leq 7$ points in almost general position. Any

s-curve on X is contracted by π or it is the strict transform along π of a line or an integral conic on $\mathbb{P}^2_k$.

For generalized del Pezzo surfaces of degree 1 and 2 an s-curve can also be the strict transform of a cubic plane curve.

Proof. In general, a class of negative self-intersection has at most one irreducible representative. This proves the first statement together with Lemma 2.3.4. The second statement follows from the same lemma. □

Lemma 2.3.4 also shows that the study of geometric s-curves is trivial on a generalized del Pezzo surface which is geometrically isomorphic to $\mathbb{P}^2_k$, $\mathbb{P}^1_k \times \mathbb{P}^1_k$ or $\mathbb{F}_2$. So in this section X will be a surface given as the composition of blowups. In this case we can easily identify at least some of the geometric s-curves on X.

PROPOSITION 2.4.4. *Let X be a generalized del Pezzo surface over an algebraically closed field k, which is written as the blowup $\pi\colon X \to \mathbb{P}^2_k$ of the projective plane in r points in almost general position. Let X_p be the fibre of π above a point $p \in X(k)$. Then X_p is either a single point or there exists a positive integer $m \le r$ such that X_p is the union of -2-curves $E_1, E_2, \dots, E_{m-1}$ and a -1-curve E_m which satisfy*

$$E_i \cdot E_j = \begin{cases} 1 & \text{if } |i-j| = 1; \\ 0 & \text{otherwise.} \end{cases}$$

More precisely, if X_p is a positive-dimensional fibre of $\pi\colon X \to \mathbb{P}^2_k$, then $\pi^{-1}p$ is a locally principal subscheme of X. The associated Cartier divisor of this subscheme equals

$$E_1 + E_2 + \ldots + E_{m-1} + E_m.$$

Proof. This follows by induction. It is obviously true for the generalized del Pezzo surface $\mathbb{P}^2_k$. Now suppose that the statement is true for a generalized del Pezzo surface X_{r-1} obtained from blowing up the projective plane in $r-1$ points in almost general postion. We know that X_r is the blowup of X_{r-1} in a closed point p_{r-1}. Let p_0 be the image of p_{r-1} in $\mathbb{P}^2_k$. The fibre of $X_r \to \mathbb{P}^2_k$ over any point $p' \neq p_0$ is isomorphic to the fibre over p' of $X_{r-1} \to \mathbb{P}^2_k$.

Now consider the fibre of X_r over p_0. The fibre F_{r-1} of $X_{r-1} \to \mathbb{P}^2_k$ over p_0 is a chain of a non-negative number of -2-curves and one -1-curve E. By Theorem 2.2.2 we see that p_{r-1} cannot lie on one of the -2-curves, so it lies on the -1-curve E. Blowing up this point, the strict transform of E becomes a -2-curve and the exceptional curve of the blowup $X_r \to X_{r-1}$ becomes the new -1-curve in the fibre $X_r \to \mathbb{P}^2_k$ over p_0.

The last statement follows again by induction. □

In the notation of this proposition, we will be more interested in the divisor

$$(\pi^{-1}p)^{\text{pec}} = E_1 + 2E_2 + \ldots + (m-1)E_{m-1} + mE_m$$

above the point p than in the divisor $\pi^{-1}p$. The reason for these unlikely multiplicities will become apparent in the following sections.

DEFINITION 2.4.5. Let X be a generalized del Pezzo surface over a field k, given as the blowup $\pi\colon X \to \mathbb{P}^2_k$ of the projective plane in r points in almost general position. The sum of the positive-dimensional fibres of π is denoted by E_π and is called the *exceptional divisor of* π. The sum of $(\pi^{-1}p)^{\mathrm{pec}}$ over all points p with a positive-dimensional fibre is called the *peculiar divisor of* π and denoted by E_π^{pec}.

Note that the exceptional and the peculiar divisor of π have the same support. The number of prime divisors in this support equals $9-d$, where d is the degree of X. By comparing with the numbers in Table A on page 32 we see that there can be more s-curves than those in the support of E_π. However, any s-curve which is not in E_π will be the strict transform of a plane curve C. This curve C is either a line or a smooth conic by Lemma 2.4.3. Assume for a moment that k is an algebraically closed field. We can decompose the morphism into blowups $X = X_r \to X_{r-1} \to \dots \to X_0 = \mathbb{P}^2_k$ with centres $x_i \in X_i(k)$ and consider the strict transform $C_i \subseteq X_i$ of $C \subseteq \mathbb{P}^2_k$ at every level. By induction we see that the self-intersection of $\tilde{C} = C_r$ can be computed as follows

$$\tilde{C}^2 = C^2 - \#\{i \mid x_i \in C_i(k)\}$$

since $C_{i+1}^2 = C_i^2 - 1$ if $x_i \in C_i(k)$ and $C_{i+1}^2 = C_i^2$ otherwise. Here we have used that C and hence each C_i is a smooth curve.

So if X is given as a composition of blowups of the projective plane, then we can usually identify the remaining s-curves. For ordinary del Pezzo surfaces, this is even more straightforward as the following proposition shows.

PROPOSITION 2.4.6. *On ordinary del Pezzo surfaces every -1-class contains a geometric -1-curve and there are no geometric -2-curves.*

Proof. It is enough to assume that k is algebraically closed. For an ordinary del Pezzo surface $\pi\colon X \to \mathbb{P}^2_k$, written as the blowup of r points in general position, each -1-class is either one of the r irreducible components of E_π or the strict transform of a plane curve of prescribed degree and multiplicities at the blowup centres. These curves are all different and one can count that the number of these curves equals the number of -1-classes. For details see [41, Theorem 26.2]. □

With a little more work one can adapt the proof to show that all -1-classes on a generalized del Pezzo surface are effective over an algebraic closure. However, not every -1-class needs to be represented by a prime divisor, i.e. an integral curve. For example, the class of a connected component of the divisor E_π in Definition 2.4.5 represents a -1-class. However, such a component need not be a prime divisor.

Now let us consider the same problem for -2-classes. We will restrict to generalized del Pezzo surfaces of degree $d \leq 7$ to ensure that there are -2-classes. We already saw that there are no -2-curves on ordinary del Pezzo surfaces. For generalized del Pezzo surfaces we see in general that not all -2-classes will be effective; the negative $-R$ of a -2-class R is also a -2-class, and R and $-R$

cannot both be effective. The following lemma gives an upper bound for the number of -2-curves.

LEMMA 2.4.7. *Let X be a generalized del Pezzo surface of degree d over a field k. The number of -2-curves on X is at most $9-d$.*

Proof. We will prove the result in the case that k is algebraically closed. The general results follow directly from there.

The result is trivial if $d = 9$ and for $d = 8$ there is at most one -2-curve in all possible cases as one can easily check. If $d \leq 7$ then X is the blowup of the projective plane in $r = 9-d$ points in almost general position and the geometric Picard group $\operatorname{Pic} X$ only depends on the degree d. The intersection product on $\operatorname{Pic} X$ defines a negative definite pairing on the orthogonal complement $K_X^{\perp}$ in $\mathbb{R} \otimes \operatorname{Pic} X$ by Proposition 25.2 in [41]. Now let $R_1, \ldots R_t$ be t distinct integral curves with self-intersection -2 on X. We will prove that they are linearly independent in $K_X^{\perp} \subseteq \mathbb{R} \otimes \operatorname{Pic} X$.

Suppose that $\sum_{i \in I} \alpha_i R_i = \sum_{j \in J} \alpha_j R_j$ in $K_X^{\perp}$ where all α_i and α_j are positive and the index sets $I, J \subseteq \{1, 2, \ldots, t\}$ are disjoint. This implies that $R_i \cdot R_j \geq 0$ for all $i \in I$ and $j \in J$ and we find

$$\left(\sum_{i \in I} \alpha_i R_i\right)^2 = \left(\sum_{i \in I} \alpha_i R_i\right)\left(\sum_{j \in J} \alpha_j R_j\right) \geq 0.$$

We see that $\sum_{i \in I} \alpha_i R_i = 0$ and hence $\alpha_i = 0$ for all i. Similarly we find that $\alpha_j = 0$ for all $j \in J$.

We conclude that $R_1, \ldots R_t$ are linearly independent in $K_X^{\perp}$ which is of dimension $9-d$. $\square$

Now that we have discussed the number of s-classes on generalized del Pezzo surfaces we will look at the possible geometric configurations. Let X be a generalized del Pezzo surface of degree $d \leq 7$ over an algebraically closed field k. Consider the graph whose vertices are the -1-curves on X. Between two distinct vertices corresponding to the -1-curves L and L' we have precisely $L \cdot L'$ edges. Since L and L' are integral and different we see that this is a non-negative integer. This graph is called the *intersection graph of -1-curves on X*.

We could also have looked at the *intersection graph of the -1-classes on X*. It follows from Lemma 2.3.1 that the Picard groups of two generalized del Pezzo surfaces of the same degree are isomorphic as lattices. This proves that the intersection graph of the -1-classes on a generalized del Pezzo surface depends only on the degree of X. We conclude that the intersection graph of all -1-curves is a subgraph of the intersection graph of all -1-classes. Proposition 2.4.6 implies that these graphs even coincide on an ordinary del Pezzo surface.

We could have also defined these objects starting from all s-curves or even all s-classes on X. Note that for the s-classes this does not produce a graph; the intersection number between two different -2-classes can be negative. Because of the obvious analogy with the situation of -1-classes we will stick with the terminology of graphs.

It follows as above that this intersection graph of the s-classes again only depends on the degree d of X if $d \leq 7$. This need no longer be true if we consider the intersection graph of all s-curves on X; note that this is an actual graph. We do have the following result if we restrict to the -2-curves.

LEMMA 2.4.8. *Let $\mathcal{R}$ be a set of -2-curves on a generalized del Pezzo surface over a field k. Any connected component of the intersection graph on the elements of $\mathcal{R}$ is a subgraph of the graph $\mathcal{G}$ shown in Figure I.*

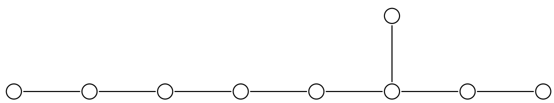


Figure I: The graph $\mathcal{G}$.

Note that the complete graph $\mathcal{G}$ is not possible by Lemma 2.4.7.

Proof. Again we can assume that k is algebraically closed. Since k is algebraically closed we can find a point on X which does not lie on any s-curve. We blow up X in this point and by induction we see that there is a generalized del Pezzo surface $W \to X$ of degree 1 with the same intersection graph of -2-curves as X. This shows that we can assume that X is a generalized del Pezzo surface of degree 1.

The anticanonical map of X has a single base point $p \in X(k)$ [23, Proposition III.2] and hence all effective anticanonical divisors on X pass through p. The anticanonical map on X defines a rational map $X \dashrightarrow \mathbb{P}^1_k$ which is defined away from p. If we blow up p on X we find a smooth surface $X' = \mathrm{Bl}_p X \dashrightarrow \mathbb{P}^1_k$. Since blowing up separates the effective anticanonical divisors at p, the rational map on X extends to a morphism $X' \to \mathbb{P}^1_k$. This makes X' into an elliptic surface with a section given by the exceptional divisor of the blowup $X' \to X$.

Now let R be a -2-curve on X and let R' be the pullback of R back to X'. Since $R \cdot K_X = 0$ we see that p does not lie on R. This also proves that R' lies in a fibre of $X' \to \mathbb{P}^1_k$. This fibre is the strict transform of an effective anticanonical divisor D on X. By Corollaire IV.2 in [23] this effective anticanonical divisor contains the connected component of R in the intersection graph of -2-curves on X. Let us write S for the sum of all -2-curves in this connected component of R. The same corollaire also shows that there is a unique -1-curve L on X such that $D = S + L$. Since p does not lie on any -2-curve it lies on L and this shows that the fibre of $X' \to \mathbb{P}^1_k$ containing R' is a sum of -2-curves, namely the sum of the strict transform of S and the strict transform of L.

The possible singular fibres of elliptic surfaces are classified and can for example be found in [48, Table 15.1]. To recover the connected components of -2-curves one still has to remove one component: the strict transform of L. We find the possible graphs A_n and D_n for any positive integer n, and E_6, E_7 and E_8.

One now uses Lemma 2.4.7 to conclude the proof. □

Now consider a del Pezzo surface over a general field k. We have looked at the action of the absolute Galois group G_k on the s-classes. This action induces an action on the geometric s-curves on X. Understanding this action is important for the following two results.

PROPOSITION 2.4.9. *Let X be a generalized del Pezzo surface of degree d over a field k. Suppose that $\mathcal{L}$ is a Galois-invariant set of -1-curves on $\bar{X}$ such that the curves in $\mathcal{L}$ are pairwise skew, i.e. $L_1 \cdot L_2 = 0$ for all $L_1 \neq L_2$ in $\mathcal{L}$. There exists a unique generalized del Pezzo surface X' defined over k such that $\bar{X}'$ is obtained from $\bar{X}$ by contracting the curves in $\mathcal{L}$. The degree of X' is $d + \#\mathcal{L}$.*

If X is an ordinary del Pezzo surface, then so is X'.

By contracting curves which are not defined over the base field k, we mean that $\bar{X}'$ is obtained from $\bar{X}$ by contracting the elements of $\mathcal{L}$. The proof is by contracting the -1-curves on $\bar{X}$ and then descending this surface back along $\operatorname{Spec} \bar{k} \to \operatorname{Spec} k$. We actually need not pass to an algebraic closure $\bar{k}$ of k; any field K over which all the lines in $\mathcal{L}$ are defined will do. This ensures that we can assume that K/k is a finite Galois extension and this makes the morphism $\operatorname{Spec} K \to \operatorname{Spec} k$ into an fpqc cover.

We will use the following equivalent formulation of an fpqc descent datum along a Galois extension of fields.

PROPOSITION 2.4.10. *Let K/k be a finite Galois extension and let G be the Galois group* $\operatorname{Gal}(K/k)$. *Let $\sigma^* \colon \operatorname{Spec} K \to \operatorname{Spec} K$ be the isomorphism induced by the element $\sigma \in G$.*

Let $\tilde{X}$ be a scheme over K. An fpqc descent datum on $\tilde{X}$ relative to K/k is equivalent to a set of isomorphisms of schemes

$$\tilde{\sigma} \colon \tilde{X} \to \tilde{X},$$

indexed by $\sigma \in G$ such that

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\sigma}} & \tilde{X} \\ \downarrow & & \downarrow \\ \operatorname{Spec} K & \xrightarrow{\sigma^*} & \operatorname{Spec} K \end{array}$$

commutes and $\tilde{\sigma}\tilde{\tau} = \widetilde{\tau\sigma}$ for all σ and τ in G.

Proof. See Proposition 4.4.2 in [46]. □

Let X be a scheme over the field k. In this interpretation the morphisms $\tilde{\sigma}$ corresponding to the effective descent datum on the base change X_K are given by the base change of $\sigma^* \colon \operatorname{Spec} K \to \operatorname{Spec} K$ along $X_K \to \operatorname{Spec} K$.

Proof of Proposition 2.4.9. Let K/k be a finite Galois extension over which all the lines in $\mathcal{L}$ are defined. On the surface X_K we can blow down each curve in $\mathcal{L}$ to obtain a smooth surface $\tilde{X}$ over K [33, Theorem V.5.7] with $x_i \in \tilde{X}(K)$ the image of the contracted curves. Let $\gamma\colon X_K \to \tilde{X}$ be the morphism which contracts all curves in $\mathcal{L}$.

We will make the effective descent datum of X_K into a descent datum on $\tilde{X}$. The morphism $\tilde{\sigma}\colon X_K \to X_K$ associated to an element $\sigma \in \mathrm{Gal}(K/k)$ fits into the commutative diagram in (2.1).

$$\begin{tikzcd}
X_K \arrow[rr, "\tilde{\sigma}"] \arrow[dr, "\gamma"] \arrow[dd] & & X_K \arrow[dr, "\gamma"] \arrow[dd] & \\
& \tilde{X} \arrow[dl] & & \tilde{X} \arrow[dl] \\
\operatorname{Spec} K \arrow[rr, "\sigma^*"] & & \operatorname{Spec} K &
\end{tikzcd} \tag{2.1}$$

Since γ is a birational morphism and $\tilde{\sigma}$ an isomorphism we find a birational map $\tilde{\sigma}\colon \tilde{X} \dashrightarrow \tilde{X}$ which makes the diagram in (2.1) commute. This map $\tilde{\sigma}$ is clearly defined on the complement of the points x_i and the composition $\gamma \circ \tilde{\sigma}$ contracts each curve in $\mathcal{L}$ to a closed point. We conclude from [50, Tag 0C5J] that $\tilde{\sigma}$ descends to an actual morphism $\tilde{\sigma}\colon \tilde{X} \to \tilde{X}$. This morphism makes the diagram in (2.2) commute.

$$\begin{tikzcd}
\tilde{X} \arrow[r, "\tilde{\sigma}"] \arrow[d] & \tilde{X} \arrow[d] \\
\operatorname{Spec} K \arrow[r, "\sigma^*"] & \operatorname{Spec} K
\end{tikzcd} \tag{2.2}$$

The morphisms $\tilde{\sigma}\tilde{\tau}$ and $\widetilde{\tau\sigma}$ restrict to the same automorphisms on $\tilde{X}\backslash\{x_i\}$. This proves that the morphisms themselves are the same, since they agree on an open dense subset.

This proves both conditions of Proposition 2.4.10 for the set of isomorphisms $\tilde{\sigma}\colon \tilde{X} \to \tilde{X}$. It follows that there is a surface X' over k such that X'_K is isomorphic to $\tilde{X}$ over K. The surface X' is a generalized del Pezzo surface by Proposition 2.2.4, because X'_K is a generalized del Pezzo surface. The same proposition also proves that X' is an ordinary del Pezzo surface if X is an ordinary del Pezzo surface.

The morphism $\gamma\colon X_K \to \tilde{X}$ commutes with the Galois descent morphisms $\sigma^*\colon \tilde{X} \to \tilde{X}$ and hence descends to a morphism $X' \to X$ [46, Theorem 4.3.5(i)]. It follows directly that on the complement of the -1-curves in $\mathcal{L}$ this morphism is an isomorphism onto its image, because this is the case for its base change morphism $X_K \to \tilde{X}$. □

We also have the following result about contracting -2-curves, but unlike the case of -1-curves the constructed surface can be singular. Because the possible configurations of contracted curves is limited by Proposition 2.4.9 the surface will however still be normal.

PROPOSITION 2.4.11. *Let $\mathcal{R}$ be a Galois-invariant set of -2-curves on a generalized del Pezzo surface X over a field k. There is a normal surface Y together with a birational proper morphism $\gamma\colon X \to Y$ such that*

» *the integral curves on $\bar{X}$ which are contracted by γ are precisely the elements $R \in \mathcal{R}$;*

» *the Weil divisor K_Y on Y is a Cartier divisor; and*

» *the pullback of the associated line bundle ω_Y along γ is the canonical line bundle ω_X on X.*

Proof. The proof is similar to the proof of Proposition 2.4.9. Let K be a finite Galois extension over which all the -2-curves in $\mathcal{R}$ are defined. We will first contract the -2-curves on X_K.

Let us consider the intersection matrix $(R_i \cdot R_j)$ of the -2-curves of $\mathcal{R}$. We will prove that this matrix is negative definite. This statement is true for -2-curves with intersection graph A_8, D_8 and E_8. It is easily checked that any connected subgraph of the graph $\mathcal{G}$ in Figure I is a subgraph of one of these three graphs. Using Lemma 2.4.8 and the fact that any principal minor of a negative definite matrix is again negative definite we see that this property is satisfied for any collection of geometric -2-curves on a generalized del Pezzo surface.

Since the matrix $(R_i \cdot R_j)$ is negative definite we can apply Theorem 2.7 in [1]. This produces a normal surface $\tilde{X}$ together with a proper birational morphism $\tilde{\gamma}\colon X_K \to \tilde{X}$ which contracts precisely the -2-curves in $\mathcal{R}$. As before we can descend this to a morphism $X \to Y$ over k, which is an isomorphism away from the contracted -2-curves. Using Corollaire 9.10 in [31] we see that Y is normal, because $\tilde{X}$ is.

The statements about the canonical divisor and canonical line bundle on Y also follow from Theorem 2.7 in [1]. □

2.5 Effective anticanonical divisors

In this section we will give several characterizations of the effective anticanonical divisors on a generalized del Pezzo surface X. We will need the following result which we will state as a lemma.

LEMMA 2.5.1. *Let $\pi\colon S' \to S$ be a proper birational morphism of projective surfaces and assume that S is normal. The natural map $\mathcal{O}_S \to \pi_*\mathcal{O}_{S'}$ is an isomorphism.*

More generally, the natural morphism $\mathcal{F} \to \pi_*\pi^*\mathcal{F}$ coming from the adjunction between π_* and π^* is an isomorphism for any quasi-coherent sheaf $\mathcal{F}$ on S.

Proof. The isomorphism $\mathcal{O}_S \to \pi_* \mathcal{O}_{S'}$ follows from [50, Tag 0AY8]. This proves the first claim. We now have the following chain of isomorphisms

$$\begin{aligned}\pi_*\pi^*\mathcal{F} &\cong \pi_*\left(\pi^*\mathcal{F} \otimes \mathcal{O}_{S'}\right)\\ &\cong \mathcal{F} \otimes \pi_*\mathcal{O}_{S'}\\ &\cong \mathcal{F} \otimes \mathcal{O}_S\\ &\cong \mathcal{F},\end{aligned}$$

where we have used the projection formula [33, Exercise III.8.3] in the second isomorphism. One can check locally that this isomorphism is given by the natural morphism $\mathcal{F} \to \pi_*\pi^*\mathcal{F}$ coming from the adjunction between π_* and π^*. □

We can now prove the following proposition and state the main definition of this section. This also explains the coefficients in the definition of E_π^{pec} in Definition 2.4.5.

PROPOSITION 2.5.2. *Let X be a generalized del Pezzo surface of degree d over a field k together with a birational morphism $\pi\colon X \to \mathbb{P}^2_k$.*

There is an isomorphism of line bundles $\pi^*\omega_{\mathbb{P}^2_k} = \omega_X \otimes \mathcal{L}(-E_\pi^{\mathrm{pec}})$ over X and an isomorphism of k-vector spaces

$$\mathrm{H}^0(\mathbb{P}^2_k, \omega^\vee) \to \mathrm{H}^0(X, \omega_X^\vee \otimes \mathcal{L}(E_\pi^{\mathrm{pec}})).$$

This last map is given on divisors by mapping a divisor C to its total transform π^*C. Here we have identified the complete linear system $|D|_X$ of a divisor D on X with the global sections $\mathrm{H}^0(X, \mathcal{L}(D))$ up to scaling by elements in $k^\times$.

The morphism $\mathrm{H}^0(X, \omega_X^\vee) \to \mathrm{H}^0(X, \omega_X^\vee \otimes \mathcal{L}(E_\pi^{\mathrm{pec}}))$ defined by taking the tensor product with the designated global section $1_{E_\pi^{\mathrm{pec}}} \in \mathrm{H}^0(X, \mathcal{L}(E_\pi^{\mathrm{pec}}))$ is injective. This injection is given on divisors by adding the effective divisor E_π^{pec} on X.

Proof. The statement is purely geometric so we can assume that π decomposes as the blowup $X = X_r \to X_{r-1} \to \ldots \to X_1 \to \mathbb{P}^2_k$ of the projective plane in $r = 9 - d$ points in almost general position, where the centre of each blowup is a closed k-point.

Define $\pi_i\colon X_i \to \mathbb{P}^2_k$ to be the composition of the first i blowups. So we get that π_0 is the identity on the projective plane and that $\pi_r = \pi$. One can now prove by induction that $\pi_i^*\omega_{\mathbb{P}^2_k} = \omega_{X_i} \otimes \mathcal{L}(-E_{\pi_i}^{\mathrm{pec}})$. The base case is trivial and for the induction step one uses Proposition V.3.3 in [33].

We apply Lemma 2.5.1 to the quasi-coherent sheaf $\omega_{\mathbb{P}^2_k}^\vee$. We find an isomorphism $\omega_{\mathbb{P}^2_k}^\vee \to \pi_*\pi^*\omega_{\mathbb{P}^2_k}^\vee$. Since the global sections of the pushforward $\pi_*\mathcal{F}$ are the same as the global sections of the original sheaf $\mathcal{F}$ we conclude that there is an isomorphism on global sections

$$\mathrm{H}^0(\mathbb{P}^2_k, \omega_{\mathbb{P}^2_k}^\vee) \to \mathrm{H}^0(X, \pi^*\omega_{\mathbb{P}^2_k}^\vee)$$

induced by π. If we consider the associated divisors of a global section in the domain and codomain of this isomorphism we see that a Cartier divisor associated to a global section in $\mathrm{H}^0(\mathbb{P}^2_k, \omega^\vee)$ gets sent to the Cartier divisor on X locally defined by the same functions after identifying $\kappa(X)$ and $\kappa(\mathbb{P}^2_k)$ along π. This maps a divisor C on $\mathbb{P}^2_k$ to the divisor on X which by definition is the total transform of C along π.

Now consider the inclusion $\mathcal{O}_X \hookrightarrow \mathcal{L}(E^{\mathrm{pec}}_\pi)$ which maps the unit of $\mathcal{O}_X$ to $1_{E^{\mathrm{pec}}_\pi}$. A locally free sheaf is flat so we find the inclusion of line bundles $\omega^\vee_X \hookrightarrow \omega^\vee_X \otimes \mathcal{L}(E^{\mathrm{pec}}_\pi)$. This induces an inclusion on global sections. The last statement about divisors follows from considering the inclusion $\mathrm{H}^0(X, \omega^\vee_X) \to \mathrm{H}^0(X, \omega^\vee_X \otimes \mathcal{L}(E^{\mathrm{pec}}_\pi))$ locally. $\square$

DEFINITION 2.5.3. Let $V_X \subseteq \mathrm{H}^0(\mathbb{P}^2_k, \omega^\vee)$ be the image of the composition of the inclusion $\mathrm{H}^0(X, \omega^\vee) \hookrightarrow \mathrm{H}^0(X, \omega^\vee \otimes \mathcal{L}(E^{\mathrm{pec}}_\pi))$ with the isomorphism

$$\mathrm{H}^0(X, \omega^\vee \otimes \mathcal{L}(E^{\mathrm{pec}}_\pi)) \xrightarrow{\cong} \mathrm{H}^0(\mathbb{P}^2_k, \omega^\vee).$$

Since $\omega_{\mathbb{P}^2_k}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^2_k}(-3)$ we can identify global sections of the anticanonical bundle with homogeneous cubic polynomials in three variables. We will consider the linear subsystem of cubic plane curves associated to these polynomials. The next proposition describes the relation between this linear subsystem $|V_X|_{\mathbb{P}^2_k}$ of cubics on $\mathbb{P}^2_k$ and the effective anticanonical divisors on X. Note that by Definition 2.5.3 both linear systems are of dimension $9 - d$.

PROPOSITION 2.5.4. *Let $\pi\colon X \to \mathbb{P}^2_k$ be a birational morphism from a generalized del Pezzo surface to the projective plane. Let $C \subseteq \mathbb{P}^2_k$ be a cubic plane curve and let $\tilde{C}$ be the strict transform of C along π. The following three statements are equivalent.*

(i) *The divisor C lies in the linear subsystem $|V_X|_{\mathbb{P}^2_k}$.*

(ii) *The anticanonical divisor $\pi^*C - E^{\mathrm{pec}}_\pi$ on X is effective.*

(iii) *There exists an effective divisor D on X supported on the peculiar divisor E^{pec}_π of π such that the class of $\tilde{C} + D$ in the Picard group of X is the anticanonical class $-K_X$.*

Note that $\pi^*C - E^{\mathrm{pec}}_\pi$ is an anticanonical divisor for any cubic plane curve C. Statement (ii) is purely about the effectiveness of this divisor. We will also use that $\pi^*C - \tilde{C}$ is an effective divisor on X whose prime divisors lie in the support of E^{pec}_π.

Proof. The equivalence between (i) and (ii) follows directly from Definition 2.5.3.

Assume statement (ii) holds for a cubic plane curve C. The divisor $D = \pi^*C - E^{\mathrm{pec}}_\pi - \tilde{C}$ then satisfies the conditions of (iii). Now assume (iii). We have two anticanonical divisors $\pi^*C - E^{\mathrm{pec}}_\pi$ and $\tilde{C} + D$. That implies that the difference $(\pi^*C - \tilde{C}) - E_\pi^{\mathrm{pec}} - D$ is a principal divisor. Since $\pi^*C - \tilde{C}$ and D are supported on E_π^{pec} we see that the same holds for $(\pi^*C - \tilde{C}) - E_\pi^{\mathrm{pec}} - D$. Now consider a function $f \in \kappa(X)$ such that $\mathrm{div}_X f = (\pi^*C - \tilde{C}) - E_\pi^{\mathrm{pec}} - D$. We see that f has a trivial divisor on $X \backslash E_\pi^{\mathrm{pec}}$. Using the birational morphism $\pi\colon X \to \mathbb{P}_k^2$ we find a function $f \in \kappa(\mathbb{P}_k^2)$ which has a trivial divisor on the complement of a finite set of points. Since $\mathbb{P}_k^2$ is normal, we see that $\mathrm{div}_{\mathbb{P}_k^2} f = 0$ and hence f must be constant both in $\kappa(\mathbb{P}_k^2)$ and in $\kappa(X)$. This shows that $(\pi^*C - \tilde{C}) - E_\pi^{\mathrm{pec}} - D = 0$ and hence $\pi^*C - E_\pi^{\mathrm{pec}} = \tilde{C} + D$ is an effective anticanonical divisor. □

Note that the proof also shows that the complementary divisor D in (iii) is unique.

We will see in Proposition 2.7.16 another way to identify the cubic plane curves in the linear system of V_X.

2.6 Singular del Pezzo surfaces

A generalized del Pezzo surface comes by definition with a morphism to a projective space, namely the morphism associated to the complete linear system of a sufficiently large multiple of the anticanonical divisor. For del Pezzo surfaces of high degree it is even enough to consider $-K_X$ itself.

THEOREM 2.6.1. *Let X be a generalized del Pezzo surface of degree d over a field k. If $d \geq 3$ then the complete linear system $|-K_X|$ does not have base points and $-K_X$ is very ample outside of the -2-curves. The associated morphism contracts all -2-curves on X and embeds the obtained surface as a degree d surface in $\mathbb{P}_k^d$.*

Proof. See Proposition V.1 of [23]. □

COROLLARY 2.6.2. *The anticanonical map embeds an ordinary del Pezzo surface of degree $d \geq 3$ as a smooth surface in $\mathbb{P}_k^d$ of degree d.*

For ordinary del Pezzo surfaces of degrees 1 and 2 it is known that although the class $-K_X$ is ample it will not be very ample. The smallest multiples of the anticanonical line bundle which are very ample are $-3K_X$ and $-2K_X$ respectively. Theorem 2.6.1 is also true for generalized del Pezzo surfaces of degree 1 and 2 if one considers these multiples of the anticanonical line bundles.

We will now consider the image of a generalized del Pezzo surface under this morphism to a projective space over k. Such a surface will be normal by Proposition 2.4.11.

DEFINITION 2.6.3. The projective normal surface obtained from contracting all the -2-curves on a generalized del Pezzo surface X is called a *singular del Pezzo surface*.

Note the unfortunate terminology: an ordinary del Pezzo surface X is also a singular del Pezzo surface, although X is actually non-singular. In general the morphism $X \to Y$ is not an isomorphism, but it will be a birational morphism. We have seen in Proposition 2.4.11 how to construct Y from X. One can also recover the generalized del Pezzo surface X from the singular del Pezzo surface Y, but we will need the following notion.

DEFINITION 2.6.4. Let Y be a scheme. A scheme X together with a proper birational map $\gamma\colon X \to Y$ is called a *desingularization* of Y if X is regular.

A desingularization $\gamma\colon X \to Y$ of Y is said to be a *minimal desingularization* if for any desingularization X' the morphism $X' \to Y$ factors through $X \to Y$.

Finding desingularizations can be quite hard in general and even minimal desingularizations need not exist [3, Section 3]. For surfaces they are well enough understood and we have the following proposition.

PROPOSITION 2.6.5. *Let Y be a surface over a field k. Suppose that Y has a desingularization $\gamma\colon X \to Y$. Then Y has a unique minimal desingularization.*

A desingularization $X \to Y$ is minimal if and only if all integral curves E on X which map to a point on Y satisfy $E^2 \leq -2\chi(E)$.

Because of this result we will usually talk about the minimal desingularization of a surface instead of a minimal desingularization.

Proof. See Corollary 27.3 in [37]. □

This gives us the result we were looking for.

COROLLARY 2.6.6. *Let X be a generalized del Pezzo surface over a field k and let Y be the associated singular del Pezzo surface over k. The morphism $\gamma\colon X \to Y$ which contracts all -2-curves on X is the minimal desingularization of Y.*

Proof. The surface X is smooth over k and we see by [33, Corollary II.4.8] that the morphism γ is proper. This shows that X is a desingularization of Y by definition. We will show that X is the minimal desingularization of Y.

The integral curves on X mapping to a point on Y are precisely the -2-curves on X. A -2-curve $E \subseteq X$ satisfies $\chi(E) = 1$ because E is a smooth curve of genus 0. This means that we have $E^2 \leq -2\chi(E)$ and we conclude from Proposition 2.6.5 that X is the minimal desingularization of Y. □

A singular del Pezzo surface will only have isolated singularities since it is normal. Such a singularity p on a singular del Pezzo surface Y can be studied by looking at the fibre of p of the minimal desingularization $\gamma\colon X \to Y$; the type of singularity is encoded by the graph of intersection of the components of the fibre $\gamma^{-1}p$.

COROLLARY 2.6.7. *A singularity on a singular del Pezzo surface over an algebraic closed field is an isolated singularity of type A_n or D_n for $n \leq 8$, or E_6, E_7 or E_8.*

Proof. This corollary is proved by listing all connected subgraphs of the graph in Lemma 2.4.8. □

2.7 Peculiar del Pezzo surfaces

The first type of del Pezzo surfaces to be studied were the surfaces which with our definitions would be called ordinary del Pezzo surfaces of degree $d \geq 3$. Del Pezzo in [22] used Corollary 2.6.2 as his definition; he studied smooth surfaces in $\mathbb{P}_k^d$ of degree d. In this terminology singular del Pezzo surfaces appear to be a natural generalization of ordinary del Pezzo surfaces. On the other hand generalized del Pezzo surface seem to be an obvious extension if one considers ordinary del Pezzo surfaces as the projective plane blown up in r points in general position.

In this section we will study a new type of del Pezzo surfaces: peculiar del Pezzo surfaces. We will see that they fit in between the classes of singular and generalized del Pezzo surfaces. They generalize the notion of ordinary del Pezzo surfaces in the following way.

Let X be an ordinary del Pezzo surface over a field k and suppose that it is explicitly written as the blowup $X = X_r \to X_{r-1} \to \ldots \to X_1 \to X_0 = \mathbb{P}_k^2$ of the projective plane in r points in general position. As the centre $p_i \in X_i(k)$ of the blowup $X_{i+1} \to X_i$ does not lie on a curve with negative self-intersection on X_i, we find that for $0 \leq j < i$ the point p_i does not map to p_j under $X_i \to X_j$. Let Z be the union of the images of all p_i in $X_0 = \mathbb{P}_k^2$. By the commutativity of blowing up two closed subschemes [24, Lemma IV-41] we find that X and $\mathrm{Bl}_Z \mathbb{P}_k^2$ are isomorphic.

This approach has the following consequence: we do not need the intermediate surfaces X_i with $0 < i < r$ to study X; the geometry of X is completely determined by information on $\mathbb{P}_k^2$. For example, the -1-curves on an ordinary del Pezzo surface X of degree $d \geq 3$ are either a component of the exceptional divisor of $\beta\colon X = \mathrm{Bl}_Z \mathbb{P}_k^2 \to \mathbb{P}_k^2$ or the strict transform along β of a line on $\mathbb{P}_k^2$ which meets Z in two points or a conic which meets Z in five points. Recall that for ordinary del Pezzo surfaces of degree 1 or 2 we would also have to consider cubic plane curves. In any case, the intersection graph of all s-curves on an ordinary del Pezzo surface is determined by the configuration of Z on the projective plane.

Let us mimic the construction of Z for a generalized del Pezzo surface X written as the blowup $X = X_r \to X_{r-1} \to \ldots \to X_1 \to X_0 = \mathbb{P}_k^2$ of the projective plane in r points in almost general position: let q_α be the images of the blowup centres $p_i \in X_i$ under the composition $X_i \to \mathbb{P}_k^2$. We will let n_α be the number of i such that p_i maps to q_α. We would want Z to be a zero-dimensional scheme supported in the q_α such that the geometrically irreducible component at each q_α is of length n_α. This presents two problems.

The first problem is that $\mathrm{Bl}_Z \mathbb{P}_k^2$ will not be isomorphic to X in general. Consider the first example in Section VI.2.3 from [24]. They consider the blowup of the projective plane in the non-reduced point defined by (x^2, y). This ideal contain the information of the origin p_0 and the direction v defined at p_0 by the line $y = 0$. If we blow up the point p_0 we get a surface $X_1 \to \mathbb{P}^2_k$ with an exceptional divisor E_1. Now consider v as a k-point on E_1. If we blowup X_1 in v, then we get a scheme $X_2 \to X_1$ with the exceptional divisor E_2. Let us denote the strict transform of E_1 along $X_2 \to X_1$ also by E_1. If we compose the two blowup morphisms we get a birational morphism $\pi\colon X = X_2 \to X_1 \to X_0 = \mathbb{P}^2_k$ with peculiar divisor $E^{\mathrm{pec}}_\pi = E_1 + 2E_2$. The morphism π restricts to an isomorphism

$$X \backslash E^{\mathrm{pec}}_\pi \to \mathbb{P}^2_k \backslash p_0.$$

Proposition IV.40 in [24] states that $\mathrm{Bl}_Z \mathbb{P}^2_k$ is obtained from X by contracting the -2-curve E_1. So in this case $\mathrm{Bl}_Z \mathbb{P}^2_k$ is not the generalized del Pezzo surface we started with, but rather the associated singular del Pezzo surface. For some zero-dimensional schemes Z the blowup $\mathrm{Bl}_Z \mathbb{P}^2_Z$ will be neither a generalized or a singular del Pezzo surface. This is where the peculiar del Pezzo surfaces come in.

The second problem is that the support and local lengths do not determine the zero-dimensional subscheme uniquely. It would if we could embed Z into a smooth curve. So let us recall that the associated zero-dimensional scheme Z associated to an ordinary del Pezzo surface naturally lies on a certain important cubic plane curve.

It follows from Proposition 2.3.1 that the pushforward $\beta_* D$ of an effective anticanonical divisor D along $\beta\colon \mathrm{Bl}_Z \mathbb{P}^2_k \to \mathbb{P}^2_k$ is a cubic curve. One can prove that this cubic curve passes through Z. This even defines a bijection between the effective anticanonical divisors on the ordinary del Pezzo surface $\mathrm{Bl}_Z \mathbb{P}^2_k$ and the cubic curves passing through Z. This proves that Z could equivalently be defined as the intersection of all these cubic curves.

Now consider a generalized del Pezzo surface X and construct the points q_α with the multiplicities n_α. With this data we could determine Z if the pushforward of an anticanonical divisor on X were a cubic curve which passes through each q_α and is furthermore smooth at these points. This is precisely the following result.

LEMMA 2.7.1. *Let $X = X_r \to X_{r-1} \to \ldots \to X_1 \to X_0 = \mathbb{P}^2_k$ be a generalized del Pezzo surface over a field k written as the blowup of the projective plane in r points in almost general position. Let $p_i \in X_i(k)$ be the centre of the blowup $X_i \to X_{i-1}$. There exists an irreducible cubic curve $C \subseteq \mathbb{P}^2_k$ such that the strict transform of C_i along $X_i \to X_0 = \mathbb{P}^2_k$ passes through p_i and is also smooth at p_i.*

For generalized del Pezzo surfaces over fields of characteristic zero one can even find an irreducible curve C which is everywhere smooth using Bertini's theorem. We include the more general result so that our results are true over any field.

Proof. This is part (b') of Théorème III.1 in [23] in the version stated in the introduction of part IV. $\square$

So let us first look into zero-dimensional schemes on curves and define what it means for such a subscheme to be in almost general position.

DEFINITION 2.7.2. A zero-dimensional subscheme Z on a surface S is called *curvilinear* if it can be embedded in a curve $C \subseteq S$ which is smooth in the support of Z.

Let k be a field and consider a curvilinear subscheme $Z \subseteq \mathbb{P}^2_k$ of degree $r \leq 6$. We say that Z lies *in almost general position* if the scheme-theoretic intersection of Z with any integral curve L of degree 1 satisfies $\deg(Z \cap L) \leq 3$.

If Z is reduced and we have

$$\deg(Z \cap D) \leq \begin{cases} 2 & \text{if } \deg D = 1; \\ 5 & \text{if } \deg D = 2 \end{cases}$$

for all effective divisors D of degree at most 2, we say that Z lies *in general position*.

In the definition of almost general position one would want to add the condition that for all curves C of degree 2 we have $\deg(Z \cap C) \leq 6$. Since we have restricted to $r \leq 6$ this condition is trivially satisfied. If one considers zero-dimensional subschemes of degrees 7 and 8 in almost general position, one would require that $\deg(Z \cap C) \leq 6$. A more complication condition would also be needed on cubic plane curves.

A zero-dimensional scheme supported on a plane curve of low degree will always lie in almost general position.

LEMMA 2.7.3. *Let k be a field and Z a zero-dimensional supported in the smooth locus of a cubic plane curve $C \subseteq \mathbb{P}^2_k$. Then Z lies in almost general position.*

To construct curvilinear subschemes more generally one can use the fact that a geometrically irreducible zero-dimensional subscheme on a given smooth curve C is uniquely defined by its support and its degree. This warrants the following definition.

DEFINITION 2.7.4. Let m be a positive integer and C a curve which lies on a surface S over a field k. Fix a k-point x on S which is smooth as a point of both C and S over k. We will write $\mathcal{I}_{C,x,m}$ for the ideal sheaf defining the unique zero-dimensional subscheme of C of degree m which is supported at x.

When $X \to \mathbb{P}^2_k$ is the blowup of the projective plane in r points in general position we will consider the degree r zero-dimensional subscheme Z of $\mathbb{P}^2_k$ such that $\mathrm{Bl}_Z\, \mathbb{P}^2_k$ is isomorphic to X over $\mathbb{P}^2_k$. This proves that for ordinary del Pezzo surfaces we can freely shift between the set of points and the subscheme Z.

The main objective of this section is to relate the blowup $X' = \mathrm{Bl}_Z\, \mathbb{P}^2_k$ for a zero-dimensional scheme Z in almost general position to generalized and singular del Pezzo surfaces. Let us start by giving these surfaces X' a name.

DEFINITION 2.7.5. Let k be a field. A *peculiar del Pezzo surface* over k is the blowup of $\mathbb{P}^2_k$ in a subscheme $Z \subseteq \mathbb{P}^2_k$ in almost general position. The degree of a peculiar del Pezzo surface is $9 - \deg Z$.

We have seen that ordinary del Pezzo surfaces are also peculiar del Pezzo surfaces. Note however that peculiar del Pezzo surface can be singular surfaces, but on the other hand they are not necessarily singular del Pezzo surfaces. We will now describe the relation between generalized and peculiar del Pezzo surfaces.

PROPOSITION 2.7.6. *Let X' be a peculiar del Pezzo surface of degree $9 - r$ over a field k. It has the minimal desingularization X and the surface X is a generalized del Pezzo surface given as the blowup of $\mathbb{P}^2_k$ in r points in almost general position. Furthermore, the del Pezzo surfaces X' and X have the same degree.*

For the proof we will need several lemmas. First we will describe the blowup of a surface in a curvilinear subscheme. Then we will consider how the geometry of a surface changes under a blowup in a possibly non-reduced curvilinear scheme.

LEMMA 2.7.7. *Let m be a positive integer and C a curve which lies on a surface S over a field k. Fix a k-point x on S which is smooth as a point of both C and S over k. Let Z be the zero-dimensional subscheme of S defined by the ideal sheaf $\mathcal{I}_{C,x,m}$.*

The blowup $B = \mathrm{Bl}_Z S$ of S in the subscheme Z can be computed as follows: define $S_0 = S$, $x_0 = x$, $C_0 = C$ and recursively the blowup $\pi_{i+1} \colon S_{i+1} \to S_i$ in x_i with exceptional curve E_{i+1}, the strict transform C_{i+1} of C_i along π_{i+1}, and x_{i+1} the unique intersection between E_{i+1} and C_{i+1}. Then B is obtained from S_m by contracting the strict transforms of E_i for all $1 \leq i \leq m-1$.

This construction also shows that the positive-dimensional fibres of $S_m \to S$ are of the same form as described in Proposition 2.4.4. It now follows from Proposition 2.6.5 that S_m is the minimal desingularization of B.

Proof. Consider the completed local ring $\widehat{\mathcal{O}}_{S,x}$ at x and let $\widehat{x}$, $\widehat{C}$ and $\widehat{Z}$ be the pullbacks of x, C and Z along the morphism $\operatorname{Spec} \widehat{\mathcal{O}}_{S,x} \to S$. As $\widehat{Z}$ is the subscheme of $\widehat{C}$ of length m which is supported at $\widehat{x}$ and we recover a situation similar to the one in the lemma; we will compute the blowup of the two-dimensional scheme $\operatorname{Spec} \widehat{\mathcal{O}}_{S,x}$ in the zero-dimensional scheme $\widehat{Z}$, which is contained in $\widehat{C}$ and supported in the point $\widehat{x}$. We will first prove the result in this case. To that end let $\widehat{\pi}_i$, $\widehat{S}_i$, $\widehat{x}_i$ and $\widehat{C}_i$ be the objects mentioned in the lemma when applied to computing the blowup $\widehat{B} = \mathrm{Bl}_{\widehat{Z}} \widehat{S}$.

By smoothness we can choose an isomorphism $\widehat{\mathcal{O}}_{S,x} \to k[[u, v]]$ such that $\widehat{C}$ is given by the vanishing of v. Then Z is given by the ideal (u^m, v). Using these equations one can prove by explicit calculations that $\widehat{S}_m \to \widehat{B}$ is the contraction of all but the last of the exceptional curves $\widehat{E}_i$. A particularly nice way to prove this is using toric geometry: the blowup in the ideal (u^m, v) gives us the completion of a singular toric variety covered by two affine charts. To find a desingularization of $\widehat{B}$ one can use the procedure in [21, Section 10.1], which coincides with the process described in the lemma.

Blowing up commutes with flat base change, so we have the following tower of cartesian squares combining the situation above with the general case.

$$
\begin{tikzcd}
\widehat{S}_m \arrow[rr] \arrow[d, "\widehat{\pi}_m"'] \arrow[dr] & & S_m \arrow[d, "\pi_m"'] & \\
\vdots \arrow[d, "\widehat{\pi}_2"'] & \widehat{B} \arrow[rr] \arrow[ddl, "\widehat{\beta}"'] & \vdots \arrow[d, "\pi_2"'] & B \arrow[ddl, "\beta"'] \\
\widehat{S}_1 \arrow[rr] \arrow[d, "\widehat{\pi}_1"'] & & S_1 \arrow[d, "\pi_1"'] & \\
\widehat{S}_0 = \operatorname{Spec} \widehat{\mathcal{O}}_{S,x} \arrow[rr] & & S_0 = S &
\end{tikzcd}
$$

Let U be the complement of x in S and identify it with the corresponding open $U \times_S S_m \subseteq S_m$. Now consider the map $U \coprod \widehat{S}_0 \to S_0$, which is an fpqc cover, so the same holds for the base change $U \coprod \widehat{S}_m \to S_m$. The fibred product of $U \coprod \widehat{S}_m$ over S_m with itself is the disjoint union of the S_m-schemes $U \times_{S_m} U$, $\widehat{S}_m \times_{S_m} \widehat{S}_m$ and $U \times_{S_m} \widehat{S}_m$. The projection maps of the first product are isomorphisms as U is an open of S_m. Because taking the fibred product of $T \to S$ with the completion of S in x we get the completion of T in the pullback of x we see that the projections $\widehat{S}_0 \times_{S_0} \widehat{S}_0 \to \widehat{S}_0$ are isomorphisms too. Pulling back this isomorphism to S_m we find that $\widehat{S}_m \times_{S_m} \widehat{S}_m$ and $\widehat{S}_m$ are also naturally isomorphic. Lastly, $U \times_{S_m} \widehat{S}_m$ is the complement of the closed point in $\widehat{S}_m$. So we have maps $U, \widehat{S}_m \to B$ over S_m which agree on the product over S_m described above. As representable functors on the category of S-schemes are sheaves in the fpqc topology [25, Theorem 2.55] the morphism $U \coprod \widehat{S}_m \to B$ descends to the morphism $S_m \to B$ of S-schemes we were looking for. $\square$

We have the composition $S_m \to B \to S_0$ and similarly to the proof of Proposition 2.4.4 we see that the positive-dimensional fibre of $S_m \to S$ is a union of -2-curves and one -1-curve. We conclude that the desingularization $S_m \to B$ contracts a chain of -2-curves and Proposition 2.6.5 proves that S_m is the minimal desingularization of B. Similarly to the proof of Proposition 2.4.11 one can prove that if S_m is a projective normal surface, then so is B.

We will also need to know how intersection numbers of curves on S behave under pullback to S_m. The following lemma describes local intersection numbers for the blowup of S in a closed point. For a reference on the notions of intersection numbers and multiplicities of curves one can consult Sections 3.2 and 3.3 in [26].

LEMMA 2.7.8. *Consider a surface S over a field k with a smooth k-point x. Let m be a positive integer, $C \subseteq S$ a curve on S which is smooth at x, and $Z \subseteq C$ the zero-*

dimensional subscheme defined by $\mathcal{I}_{C,x,m}$. Consider an integral curve D on S such that any common component of C and D does not pass through x. It holds that

$$\deg(D \cap Z) = \min(i_x(C,D), \deg Z).$$

Let E be the exceptional divisor of the blowup $\pi\colon S' \to S$ of S at x. Define $\tilde{C}$ and $\tilde{D}$ as the strict transforms of C and D along π, and let x' be the unique intersection point of $\tilde{C}$ with E. If s is the multiplicity of D at x then we have

$$i_x(C,D) = i_{x'}(\tilde{C},\tilde{D}) + s.$$

Proof. We will work with the completed local rings $\widehat{\mathcal{O}}_{S,x}$ and $\widehat{\mathcal{O}}_{S',x'}$. We can pick coordinates u and v for $\widehat{\mathcal{O}}_x$, such that C is given by $v = 0$, and hence Z is given by $u^m = 0 = v$. Then we can use coordinates u and V on $\widehat{\mathcal{O}}_{x'}$, such that the map $\widehat{\mathcal{O}}_x \to \widehat{\mathcal{O}}_{x'}$ induced by π maps u to u and v to Vu. Now let D be defined at x by a polynomial $g(u,v)$.

By definition we have

$$\deg(D \cap Z) = \dim_k \widehat{\mathcal{O}}_x/(u^m, v, g) = \dim_k \widehat{\mathcal{O}}_x/(u^m, v, g(u,0)),$$

$$\deg Z = \dim_k \widehat{\mathcal{O}}_x/(u^m, v) = m$$

and

$$i_x(C,D) = \dim_k \widehat{\mathcal{O}}_x/(v,g) = \dim_k \widehat{\mathcal{O}}_x/(v, g(u,0))$$

which proves the first statement.

For the second statement we interpret

$$i_x(C,D) = \dim_k \widehat{\mathcal{O}}_x/(v, g(u,0))$$

as the smallest t such that u^t is a monomial of g.

Since C is smooth at x and E is defined by the vanishing of u we see that $\tilde{C}$ is defined by $V = 0$. Similarly, $\tilde{D}$ is given by the vanishing of the polynomial $\tilde{g} = \frac{g(u,Vu)}{u^s}$. So we have

$$i_{x'}(\tilde{C},\tilde{D}) = \dim_k \widehat{\mathcal{O}}_{x'}/(V,\tilde{g}).$$

This is the smallest integer t' such that $u^{t'}$ is a monomial of $\tilde{g}$. We have a correspondence between monomials of g and $\tilde{g}$ by associating $u^\alpha v^\beta$ to $u^{\alpha+\beta-s}V^\beta$. This implies that $t = t' + s$. $\square$

We can now compute the self-intersection of the strict transform $\tilde{D} \subseteq S_m$ of an integral curve $D \subseteq S$, which we will need to prove Proposition 2.7.6.

LEMMA 2.7.9. *Let $Z \subseteq S$ be a curvilinear subscheme of degree r on a smooth surface S over a field k. Let $X' \to S$ be the blowup of S in Z and let $X \to X'$ be its minimal desingularization. For an integral curve D on S we define $\tilde{D}$ to be the strict transform of D along $X \to X' \to S$. If D is smooth in the support of Z we have*

$$\tilde{D}^2 = D^2 - \deg(D \cap Z).$$

Proof. Let $C \subseteq S$ be a curve on S which contains Z and is smooth in the support of Z. It is enough to prove the result in the case that Z is supported in a single point $x \in C$. We use the notation of Lemma 2.7.7: the morphism $X \to X' \to S$ equals the composition $X = S_m \to S_{m-1} \to \ldots \to S_1 \to S_0 = S$ of m blowups in k-points $x_i \in S_i(k)$. Let $E_i \subseteq S_i$ be the exceptional curve of $S_i \to S_{i-1}$.

We also define $C_i, D_i \subseteq S_i$ as the strict transforms of C and D along $S_i \to S_0$. We see that $x_i \in C_i$. We will determine the i for which x_i lies on D_i.

We will prove by induction that

$$i_{x_i}(C_i, D_i) = \max(i_{x_0}(C_0, D_0) - i, 0).$$

Indeed, D_i is either smooth in x_i or it does not pass through x_i which correspond to the conditions $i_{x_i}(C_i, D_i) > 0$ and $i_{x_i}(C_i, D_i) = 0$. The result now follows from Lemma 2.7.8; $i_{x_i}(C_i, D_i)$ decreases by 1 as i increases by 1 until it is zero.

Define $m' = \deg(Z \cap D)$. Note that we have proved that D_i passes through x_i for $i = 0, 1, \ldots, m' - 1$, but not for $i > m'$. Since D is smooth at each of the x_i for $i < m'$, we find $D_i^2 = D_0^2 - i$ for $i < m'$. From the fact that D_i does not pass through x_i for $i \geq m'$ it follows that $D_i^2 = D_0^2 - m'$ for those i. Since $Z \cap D \subseteq Z$ we see that $m' = \deg(Z \cap D) \leq \deg Z = m$ and from the first result in Lemma 2.7.8 we conclude

$$\tilde{D}^2 = D_n^2 = D_0^2 - m' = D^2 - \deg(D \cap Z). \qquad \square$$

While proving this last lemma we have also proved the following statement.

COROLLARY 2.7.10. *Let k be a field, m a positive integer, and C a curve on a surface S over k. Let $x \in C(k)$ be a point which is smooth as a point of C and of S. Define Z to be the curvilinear scheme of S corresponding to the ideal $\mathcal{I}_{C,x,m}$. Consider $X' = \mathrm{Bl}_Z S$ and let $X \to X'$ be its minimal desingularization. Now write $E_1 + E_2 + \ldots + E_m$ for the unique positive-dimensional fibre over $\pi \colon X \to S$ as in Proposition 2.4.4. Let $D \subseteq S$ be a curve which passes through x and is smooth at x. The strict transform $\tilde{D} \subseteq X$ of D along π passes through exactly one E_i namely the one with $i = \deg(D \cap Z)$.*

We will now prove that the minimal desingularization of a peculiar del Pezzo surface is a generalized del Pezzo surface.

Proof of Proposition 2.7.6. Fix a zero-dimensional subscheme $Z \subseteq \mathbb{P}^2_k$ in almost general position such that X' is isomorphic to $\mathrm{Bl}_Z \mathbb{P}^2_k$ and let $\pi' \colon X' \to \mathbb{P}^2_k$ be the composition of this isomorphism with the blowup morphism. Now let K be a finite Galois extension of k such that the geometric components of Z are defined over K. We can apply Lemma 2.7.7 to each component of Z and find a smooth surface over K, which descends to a smooth surface X over k with a map $\gamma' \colon X \to X'$. Proposition 2.6.5 shows that γ' is the minimal desingularization of the peculiar del Pezzo surface X' since it is so locally.

Now fix an algebraic closure $\bar{k}$ of k and let $\bar{X}$ be the base change of X to $\bar{k}$. We will prove that there are no integral curves on $\bar{X}$ with self-intersection smaller than -2. Suppose that there is an integral curve $\tilde{D}$ on $\bar{X}$ with self-intersection less than -2. Fix a decomposition $\bar{X} = X_r \to X_{r-1} \to \dots X_1 \to X_0 = \mathbb{P}^2_{\bar{k}}$ into blowups in $\bar{k}$-points $x_i \in X_i(\bar{k})$. Let n be the smallest positive integer such that there is an integral curve $D' \subseteq X_n$ passing through x_n, such that $D'^2 = -2$. Let $D = \pi_* D'$ be the pushforward of D' to the projective plane. Now $\tilde{D}$ is the strict transform of D along $X \to \mathbb{P}^2_{\bar{k}}$. This implies that $\tilde{D}^2 < D'^2 = -2$. By minimality of n we have that X_n is a generalized del Pezzo surface over $\bar{k}$ of degree $d \geq 3$ and hence D is either a line or a conic on $\mathbb{P}^2_{\bar{k}}$ by Lemma 2.4.3. Because D is integral it is smooth and we can apply Lemma 2.7.9.

If D is a line we see that

$$\deg(D \cap Z) = D^2 - \tilde{D}^2 > 1 - (-2) = 3$$

which contradicts Z lying in almost general position. If D is an integral conic we have

$$\deg(D \cap Z) = D^2 - \tilde{D}^2 > 4 - (-2) = 6,$$

which contradicts $\deg Z \leq 6$.

This implies that $\bar{X}$ and hence X is a generalized del Pezzo surface and its degree is also $9 - r = d$. $\square$

Note that for every peculiar del Pezzo surface X' there is a birational morphism from X' to the projective plane, and by composition we find a birational morphism $X \to \mathbb{P}^2_k$ for its associated generalized del Pezzo surface. The next proposition shows that associating a generalized del Pezzo surface to a peculiar del Pezzo surface in this manner defines a bijection.

PROPOSITION 2.7.11. *Let $\pi\colon X \to \mathbb{P}^2_k$ be a birational morphism from a generalized del Pezzo surface X of degree $d \geq 3$ to the projective plane. The natural map*

$$\pi^*\colon \mathrm{H}^0(\mathbb{P}^2_k, \mathcal{O}(1)) \to \mathrm{H}^0(X, \pi^*\mathcal{O}(1))$$

is an isomorphism of k-vector spaces, which identifies the complete linear systems of $\mathcal{O}(1)$ and $\mathcal{L} = \pi^*\mathcal{O}(1)$. The morphism π is associated to the complete linear system of $\mathcal{L}$. The line bundle $\mathcal{L} \otimes \omega_X^\vee$ is big and nef and the image X' of the complete linear system of a sufficiently large multiple of $\mathcal{L} \otimes \omega_X^\vee$ is a peculiar del Pezzo surface.

One can show that the peculiar del Pezzo surface is the image of the complete linear system associated to the line bundle $\mathcal{L} \otimes \omega_X^\vee$ itself. For del Pezzo surfaces of degree 1 and 2 with a birational morphism $\pi\colon X \to \mathbb{P}^2_k$ the contraction of all -2-curves in the support of the exceptional divisor E_π of π is given by the powers $(\mathcal{L} \otimes \omega_X^\vee)^3$ and $(\mathcal{L} \otimes \omega_X^\vee)^2$ respectively.

Proof of Proposition 2.7.11. The isomorphism

$$\pi^*\colon \mathrm{H}^0(\mathbb{P}^2_k, \mathcal{O}(1)) \to \mathrm{H}^0(X, \pi^*\mathcal{O}(1))$$

follows from the isomorphism $\mathcal{O}(1) \to \pi_*\pi^*\mathcal{O}(1)$ in Lemma 2.5.1. This also proves that π is associated to the complete linear system of $\mathcal{L}$.

In particular we see that the complete linear system of $\mathcal{L}$ is non-empty. Let Λ be an effective divisor in the complete linear system of $\mathcal{L}$ and let $-K_X$ be an effective anticanonical divisor on X. It follows from Definition 2.1.3 that $\Lambda - K_X$ is nef, since Λ and $-K_X$ are both nef. By Proposition 2.1.4 we only need to compute

$$(\Lambda - K_X)^2 = \Lambda^2 - 2K_X \cdot \Lambda + K_X^2 = 1 + 2 \cdot 3 + d = d + 7 > 0$$

to conclude that $\Lambda - K_X$ is also big. Here we have used that Λ is the pullback of some line $L \subseteq \mathbb{P}^2_k$ and by the projection formula we find

$$K_X \cdot \Lambda = K_X \cdot \pi^* L = \pi_* K_X \cdot L = K_{\mathbb{P}^2_k} \cdot L = -3.$$

Let $\gamma' \colon X \to X'$ be the birational proper morphism associated to a sufficiently large multiple of $\Lambda - K_X$. The integral curves $C \subseteq X$ which are contracted are those for which $C(\Lambda - K_X) = 0$. We see by Proposition 2.1.4 that $C \cdot \Lambda \geq 0$ and $C \cdot -K_X \geq 0$ and it follows that the curves contracted by γ' are the curves which are contracted by both π and the anticanonical map. These are precisely the -2-curves in the support of E_π.

Now let X' be the image of the anticanonical map on X. We will need to prove that X' is a peculiar del Pezzo surface. We will first show that X' is the blowup of the projective plane in a curvilinear subscheme Z. Note that the statement is purely geometric so we may assume that k is algebraically closed.

By Lemma 2.7.1 there exists a cubic curve $C \subseteq \mathbb{P}^2_k$ such that its strict transform $C_i \subseteq X_i$ is smooth at p_i for all i. Note that in particular C must be reduced as it is irreducible, locally principal and smooth in at least one point.

Let us write the birational morphism $X \to \mathbb{P}^2_k$ as the composition $X = X_r \to X_{r-1} \to \ldots \to X_0 = \mathbb{P}^2_k$ of blowups in closed points $p_i \in X_i(k)$. Consider the multiset I consisting of the images of the points p_i in $\mathbb{P}^2_k$. This means that I is a set of smooth points q_α on C with some multiplicities n_α. Now let Z_α be the zero-dimensional subscheme of C defined by $\mathcal{I}_{C,q_\alpha,n_\alpha}$ and set $Z = \cup Z_\alpha$. It follows from Lemma 2.7.7 that X' is the blowup of $\mathbb{P}^2_k$ in Z.

We will prove that Z lies in almost general position. For a line L on the projective plane we define $\tilde{L}$ to be the strict transform. Since L is smooth we may apply Lemma 2.7.9 and we find

$$\deg(Z \cap L) = L^2 - \tilde{L}^2 \leq 1 - (-2) = 3$$

so Z lies in almost general position. □

The zero-dimensional scheme Z constructed in the proof will be called *the zero-dimensional scheme $Z \subseteq \mathbb{P}^2_k$ associated to the morphism $X \to \mathbb{P}^2_k$*. The scheme Z does not only depend on X, but also on the morphism to the projective plane. When it is clear from the context how X is given as the blowup of the projective plane in r points in almost general position we will say Z is the associated subscheme of X.

In many cases we will also need the curve C which is smooth in the support of Z. Usually the existence is enough, but occasionally we will assume that C is a cubic curve, which is allowed as one can see in the proof above.

Let us put together the main results on peculiar del Pezzo surfaces we have proved thus far.

COROLLARY 2.7.12. *Fix an integer $1 \leq d \leq 9$ and let k be a field. There is a bijection from the set of isomorphism classes of peculiar del Pezzo surfaces to the set of isomorphism classes of generalized del Pezzo surfaces of degree d with a birational morphism to $\mathbb{P}^2_k$ defined by mapping a peculiar del Pezzo surface to its minimal desingularization. The inverse is given by sending a generalized del Pezzo surface with a birational morphism $\pi\colon X \to \mathbb{P}^2_k$ to the peculiar del Pezzo surface obtained by contracting all -2-curves on X which are contracted by π.*

Proof. This follows from Proposition 2.7.6 and Proposition 2.7.11. □

We also have the following result.

COROLLARY 2.7.13. *A peculiar del Pezzo surface $\mathrm{Bl}_Z \mathbb{P}^2_k$ over a field k is a nonsingular surface precisely if Z is reduced.*

Proof. Let Z be a curvilinear subscheme of the projective plane. We can compute the blowup $\mathrm{Bl}_Z \mathbb{P}^2_k$ using Lemma 2.7.7 for each geometrically irreducible component of Z. It follows that it is smooth precisely if all geometrically irreducible components of Z are of degree 1. This is equivalent to Z being reduced. □

As for most statements about generalized del Pezzo surfaces, we are interested in the application to ordinary del Pezzo surfaces.

COROLLARY 2.7.14. *A peculiar del Pezzo surface $\mathrm{Bl}_Z \mathbb{P}^2_k$ over a field k is an ordinary del Pezzo surface precisely if Z lies in general position.*

Proof. It is clear that Z lies in (almost) general position precisely if $\bar{Z}$ lies in (almost) general position. This together with Proposition 2.2.4 implies that we can assume that k is algebraically closed.

Write $X' = \mathrm{Bl}_Z \mathbb{P}^2_k$ and let $X \to X'$ be the minimal desingularization of X'.

Suppose that X' is an ordinary del Pezzo surface. This implies that there are no -2-curves on X. This shows that Z is reduced, otherwise there are -2-curves in the fibres contracted by $X \to X'$ by Lemma 2.7.7. We saw in Lemma 2.3.4 that a -2-curve on X is the strict transform of a line or a conic along $X \to \mathbb{P}^2_k$. Lemma 2.7.9 now shows that there are no curves of self-intersection less than -1 precisely if Z lies in general position.

If follows similarly as discussed in the introduction that if Z lies in general position, then $X' = X$ is an ordinary del Pezzo surface. □

Let $X \to \mathbb{P}^2_k$ be a birational morphism from a generalized del Pezzo surface to the projective plane. Let X' and Y be the corresponding peculiar and singular del Pezzo surfaces. Then there exist birational morphisms $X \xrightarrow{\gamma'} X' \xrightarrow{\gamma''} Y$ such that the composition γ contracts all -2-curves on X. We have also seen that γ' contracts the -2-curves in E_π^{pec} and hence γ'' contracts precisely the remaining -2-curves.

We have also seen in Corollary 2.7.14 that if Z lies in general position then $\mathrm{Bl}_Z \mathbb{P}^2_k$ is an ordinary del Pezzo surface. This implies that γ, γ' and γ'' are isomorphisms. Corollary 2.7.13 shows that γ' is an isomorphism precisely if Z is reduced. Note that Z being a reduced is one of the conditions in the definition for Z to lie in general position. One could wonder what the relation is between the second condition and γ''. It is what one would expect: γ'' is an isomorphism precisely if $\deg(Z \cap L) \leq 2$ for lines on $\mathbb{P}^2_k$ and $\deg(Z \cap C) \leq 5$ for conics. We will not need this precise result, but the following corollary will be useful.

PROPOSITION 2.7.15. *Let $Z \subseteq \mathbb{P}^2_k$ be a zero-dimensional subscheme of degree* $\deg Z = r \leq 5$ *in almost general position. If Z lies on a conic, then the peculiar del Pezzo surface $X' = \mathrm{Bl}_Z \mathbb{P}^2_k$ is a singular del Pezzo surface.*

Proof. Let $X \to X'$ be the minimal desingularization of X'. This implies that X is a generalized del Pezzo surface. The morphism $X \to X'$ contracts precisely the -2-curves in the fibres of $\pi\colon X \to \mathbb{P}^2_k$. We will show that there are no other -2-curves. Suppose that $\tilde{D} \subseteq X$ is a -2-curve. The pushforward $D = \pi_* \tilde{D}$ is a line or a smooth conic on the projective plane. In either case we have $\deg(D \cap Z) \leq \deg(D \cap C) = 2 \deg D$. We now use Lemma 2.7.9 to find

$$-2 = \tilde{D}^2 = D^2 - \deg(D \cap Z) \geq \deg D(\deg D - 2) \geq (\deg D - 1)^2 - 1$$

which is a contradiction for any degree of D. □

An advantage of introducing peculiar del Pezzo surfaces is the following generalization of a well-known fact of ordinary del Pezzo surfaces to generalized ones. It allows us to identify the linear system V_X of cubics in Proposition 2.5.4 in terms of Z.

PROPOSITION 2.7.16. *Let X be a generalized del Pezzo surface over a field k given as a blowup $\pi\colon X \to \mathbb{P}^2_k$ of the projective plane in r points in almost general position. Let $Z \subseteq \mathbb{P}^2_k$ be the associated zero-dimensional subscheme such that $X' = \mathrm{Bl}_Z \mathbb{P}^2_k$ is the associated peculiar del Pezzo surface of $X \to \mathbb{P}^2_k$.*

The linear system of the cubic curves passing through Z is the linear system $|V_X|_{\mathbb{P}^2_k}$ described in Proposition 2.5.4.

We will use the following result.

LEMMA 2.7.17. *Let Z be a curvilinear scheme of degree r supported in a smooth k-point x on a surface S. Let $\pi\colon X \to X' \to S$ be the composition of the minimal desingularization X of $X' = \mathrm{Bl}_Z S$ and the blowup morphism $X' \to S$. Let E_i for $1 \leq i \leq r$ be the components of the exceptional divisor E_π of π as described in Proposition 2.4.4. In particular, $E_i^2 = -2$ for $1 \leq i < r$ and $E_r^2 = -1$.*

Let C be an effective Cartier divisor on S and let $\pi^* C$ be its pullback to X. The following statements are equivalent.

(i) *The zero-dimensional scheme Z lies on C.*

(ii) *The multiplicity of π^*C on E_r is at least r.*

(iii) *The multiplicity of π^*C on E_i is at least i for all $1 \leq i \leq r$.*

Proof. We work with the completed local ring $\widehat{\mathcal{O}}_x \cong k[[u,v]]$ where the isomorphism is chosen such that Z is defined by the ideal (u^r, v). Define $\widehat{S} = \operatorname{Spec} \widehat{\mathcal{O}}_x$ and $\widehat{C} = C \times_S \widehat{S}$. Since localization is flat we can compute the multiplicities of π^*C along E_i by pulling the everything back to $\widehat{X} = X \times_S \widehat{S}$. The scheme $\widehat{X}$ contains affine opens $U_i = \operatorname{Spec} k[[u, V_i]]$ such that the map $U_i \to \widehat{X} \to \widehat{S}$ induced by π is defined by $u \mapsto u$ and $V_i \mapsto u^i v$. The open U_i contains the generic point of E_i and E_i is defined by the vanishing of u on this open.

Now let $\widehat{C}$ be defined by the vanishing of $f \in k[[u,v]]$. The lemma is equivalent to the following immediate statement. Let $0 \leq i \leq r$ be an integer. We have that f lies in the ideal (u^i, v) precisely if the multiplicity of u in $f(u, u^i v)$ is at least i. □

Proof of Proposition 2.7.16. We will prove that the linear subsystems of cubics passing through Z and the linear system $|V_X|_{\mathbb{P}^2_k}$ described in Proposition 2.5.4 are the same. We will do so by showing that a cubic curve $C \subseteq \mathbb{P}^2_k$ passes through Z precisely if $E_\pi^{\mathrm{pec}} \leq \pi^*C$. This follows from applying Lemma 2.7.17 to each point in the support of Z. □

We conclude the following generalization of Corollary V.4.4 in [33].

COROLLARY 2.7.18. *Let $Z \subseteq \mathbb{P}^2_k$ be a zero-dimensional subscheme in almost general position of degree* $\deg Z \leq 6$. *The linear subsystem of cubics through Z is of dimension* $9 - \deg Z$.

Consider the situation where the curvilinear subscheme $Z \subseteq S$ of degree r is supported in a single point x. Let X' be the blowup $\operatorname{Bl}_Z S$, $X \to X'$ the minimal desingularization and $\pi\colon X \to X' \to S$ the composition of these two morphisms. Recall from Proposition 2.4.4 that the fibre of π above x is the union of -2-curves and a single -1-curve E_r. In the above proof we have identified the effective Cartier divisors $C \subseteq S$ which satisfy $\pi^*C \geq E_\pi^{\mathrm{pec}}$ as the locally principal curves passing through Z. This means that $\pi^*C - E_\pi^{\mathrm{pec}}$ is an effective divisor. We would like to be able to describe when this effective divisor is supported on the -1-curve E_r on E_π^{pec}.

LEMMA 2.7.19. *An effective Cartier divisor $C \subseteq S$ satisfies $\pi^*C \geq E_\pi^{\mathrm{pec}} + E_r$ if and only if $\mathcal{I}_C \subseteq \mathcal{I}_x \mathcal{I}_{C,x,r}$.*

Proof. We will use the notation in the proof of Lemma 2.7.17. We will need to show that the multiplicity of π^*C along E_r is at least $r+1$ precisely if $\mathcal{I}_C$ is a ideal subsheaf of $\mathcal{I}_x \mathcal{I}_{C,x,r}$. The multiplicity of π^*C along E_r is the multiplicity of u in $f(u, u^r V_r)$. This is at least $r+1$ if and only if $f \in (u^{r+1}, uv, v) = (u,v)(u^r, v)$. The ideals (u,v) and (u^r, v) define the respective ideals $\mathcal{I}_x$ and $\mathcal{I}_{C,x,r}$ on S. □

In this section we have defined peculiar del Pezzo surfaces associated to zero-dimensional subschemes in almost general position. We have also seen how to construct the associated generalized del Pezzo surface from just this subscheme. The last lemma of this section tells us how to determine the associated singular del Pezzo surface.

LEMMA 2.7.20. *Let $Z \subseteq \mathbb{P}^2_k$ be a zero-dimensional subscheme in almost general position of degree $r \leq 6$.*

The closure of the image Y of the birational map $\mathbb{P}^2_k \dashrightarrow Y \subseteq \mathbb{P}^d_k$ defined by the linear subsystem of cubics passing through Z is the singular del Pezzo surface Y associated to Z.

Proof. Let $\pi\colon X \to \mathbb{P}^2_k$ be the generalized del Pezzo surface associated to Z. Since the degree of X is $9 - r \geq 3$ the image of the anticanonical morphism $\gamma\colon X \to \mathbb{P}^d_k$ is the associated singular del Pezzo surface Y.

Note that π restricts to an isomorphism $\pi^{-1}\colon \mathbb{P}^2_k \backslash Z \xrightarrow{\cong} X \backslash E_\pi$. The pullback of π^{-1} identifies the divisors in V_X restricted to $\mathbb{P}^2_k \backslash Z$ with the divisors in $\mathrm{H}^0(X, \omega^\vee)$ restricted to $X \backslash E_\pi$. This proves that the birational map $\gamma \circ \pi^{-1}$ is defined by the cubics in V_X, i.e. the cubics which pass through Z. To conclude the proof we note that the closure of the scheme-theoretic image of this map is the same as the scheme-theoretic image of the map γ. $\square$

2.8 Arithmetic of del Pezzo surfaces

In Chapters 3 and 4 we will study arithmetic properties of surfaces which are closely related to del Pezzo surfaces. We will use some results on the arithmetic of ordinary del Pezzo surfaces over number fields which we will review in this section. The aim is not to give a complete overview of the topic. The focus lies on the results we will need later on. A more complete overview of the arithmetic of del Pezzo surfaces can be found in [55] which is the main source for this section. For similar results on singular del Pezzo surfaces one is referred to [7] and [19].

THEOREM 2.8.1. *Let X be an ordinary del Pezzo surface of degree $d \geq 5$ over a field k. If $X(k)$ is non-empty then X is birational over k to $\mathbb{P}^2_k$.*

Now let k be a number field. Then X satisfies both weak approximation and the Hasse principle.

Proof. The proof is different for each degree. A very clear and detailed exposition can be found in [55, Lecture 2]. $\square$

The proofs for the cases $d = 5$ and $d = 7$ also yield the following important result.

PROPOSITION 2.8.2. *Let X be an ordinary del Pezzo surface of degree 5 or 7 over a field k. Then X has a k-rational point and hence X is birational over k to $\mathbb{P}^2_k$.*

Proof. See Lecture 2 in [55] for the proof of the existence of the rational points. The last statement now follows from Theorem 2.8.1. □

Note that this proposition also shows that ordinary del Pezzo surfaces of degree 5 and 7 satisfy the Hasse principle trivially. This is not the case for ordinary del Pezzo surfaces of degrees 6, 8 and 9.

Example. Let k be a number field and let $C \subseteq \mathbb{P}^2_k$ be a conic such that $C(k) = \emptyset$. Since $\bar{C}$ is isomorphic to $\mathbb{P}^1_{\bar{k}}$ we see that $X = C \times C$ is an ordinary del Pezzo surface of degree 8 over k, such that $X(k) = \emptyset$.

Also, any ordinary del Pezzo surface X degree 9 is a Brauer–Severi variety, i.e. $\bar{X} \cong \mathbb{P}^2_{\bar{k}}$. By a result of Châtelet we see that X is isomorphic to $\mathbb{P}^2_k$ precisely when $X(k) \neq \emptyset$, see for example [46, Proposition 4.5.10]. The existence of a surface X over k with these properties follows from the correspondence between isomorphism classes of Severi–Brauer varieties of dimension 2 and central simple algebras over k of dimension 3^2, see [46, Section 4.5.1].

For ordinary del Pezzo surfaces of low degree it is known that weak approximation or the Hasse principle need not hold, except for the following case.

PROPOSITION 2.8.3. *Let X be a generalized del Pezzo surface of degree 1 over a number field k. Then $X(k) \neq \emptyset$ and X satisfies the Hasse principle trivially.*

Proof. We see from [23, Proposition III.2] that the anticanonical map on a generalized del Pezzo surface of degree 1 has a unique base point. This base point is hence defined over k which proves that $X(k) \neq \emptyset$. □

For the degrees $2 \leq d \leq 4$ surfaces are known for which the Hasse principle does not hold, and we also have counterexamples for weak approximation on ordinary del Pezzo surfaces of degree $1 \leq d \leq 4$. These counterexamples are clearly presented in Table 3 of [55].

One might wonder if the failure of the Hasse principle or weak approximation can be explained by a Brauer–Manin obstruction. Let us to that end compute the Brauer groups of del Pezzo surfaces over a number field. We first have the following general result, which shows that the transcendental part of the Brauer group of a generalized del Pezzo surface over a field of characteristic 0 is trivial.

PROPOSITION 2.8.4. *Let X be a rational projective smooth variety over an algebraically closed field k of characteristic* 0. *The Brauer group* $\operatorname{Br} X$ *is trivial.*

Proof. Since X is rational it is by definition birational to $\mathbb{P}^n_k$ where n is the dimension of X. This shows that X is irreducible and we can apply Proposition 1.4.13. This shows that $\operatorname{Br} \mathbb{P}^2_k$ and $\operatorname{Br} X$ are isomorphic. By Proposition 1.4.12 and Proposition 1.2.14 we see that $\operatorname{Br} X$ must be trivial. □

COROLLARY 2.8.5. *Let X be a generalized del Pezzo surface over a field k of characteristic zero. We have* $\operatorname{Br} X = \operatorname{Br}_1 X$.

Degree	Possibilities for $\operatorname{Br} X / \operatorname{Br} k$
any $\mathbf{d}$	1
$\mathbf{d} \leq \mathbf{4}$	$2 \quad 2^2$
$\mathbf{d} \leq \mathbf{3}$	$3 \quad 3^2$
$\mathbf{d} \leq \mathbf{2}$	$2^3 \quad 2^4 \quad 2^5 \quad 2^6 \quad 2 \cdot 4 \quad 2^2 \cdot 4 \quad 4 \quad 4^2$
$\mathbf{d} = \mathbf{1}$	$2^7 \quad 2^8 \quad 3^3 \quad 3^4 \quad 2 \cdot 4^2 \quad 2^2 \cdot 4^2 \quad 2^3 \cdot 4$
	$2^4 \cdot 4 \quad 5 \quad 5^2 \quad 2 \cdot 6 \quad 3 \quad 6 \quad 6^2$

Table B: Possible group structures of $\operatorname{Br} X / \operatorname{Br} k$. For example, $2^2 \cdot 4$ means $(\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/4\mathbb{Z}$.

Proof. By Proposition 2.8.4 we see that $\operatorname{Br} \bar{X}$ is trivial. This implies that

$$\operatorname{Br}_1 X = \ker(\operatorname{Br} X \to \operatorname{Br} \bar{X}) = \operatorname{Br} X. \qquad \square$$

We now use Proposition 1.6.5 to compute the algebraic Brauer group modulo constants of a del Pezzo surface.

PROPOSITION 2.8.6. *Let X be a generalized del Pezzo surface over a number field k. The Brauer group modulo constants* $\operatorname{Br} X / \operatorname{Br} k$ *is one of the groups in Table B.*

Proof. This result was proved by Manin for $d \geq 5$ [41], Swinnerton-Dyer for $d = 4$ and 3 [52], and $d = 2$ and 1 by Corn in [20, Theorem 1.4.1] for ordinary del Pezzo surfaces. The result also holds since the geometric Picard group of a generalized del Pezzo surface of degree $d \geq 7$ depends only on d.

During a calculation done for Proposition 3.2.1 the possible Brauer groups modulo constants were recomputed by a MAGMA program available at [39]. For completeness we will explain the code.

Since G_k acts on $\operatorname{Pic} \bar{X}$ it factors through the Weyl group W_{9-d}. Let $W \subseteq W_{9-d}$ be the minimal subgroup of W_{9-d} through which this action factors. By the inflation–restriction sequence we find that the inflation morphism

$$\mathrm{H}^1(W, \operatorname{Pic} \bar{X}) \to \mathrm{H}^1(G_k, \operatorname{Pic} \bar{X})$$

is an isomorphism, since the kernel of $G_k \to W$ acts trivially on the geometric Picard group. One now enumerates the subgroups W of W_{9-d} and compute the first cohomology group of the action of W on $\operatorname{Pic} \bar{X}$.

The computation is simplified by a general result in group cohomology: the cohomology group $\mathrm{H}^1(W, \operatorname{Pic} \bar{X})$ only depends on the conjugacy class of W in W_{9-d} by [58, Example 6.7.7].

The possible groups we find are precisely the groups in Table B. $\square$

One sees that it is only possible to have a Brauer–Manin obstruction to the Hasse principle or weak approximation on a del Pezzo surface if $d \leq 4$. Of course, for $d \geq 5$ we would not expect this since we know from Theorem 2.8.1 that ordinary del Pezzo surfaces of large degree satisfy both the Hasse principle and weak approximation.

All the known examples of failure of either of these local–global principles are explained by a Brauer–Manin obstruction and this led Colliot-Thélène and Sansuc [14, page 174] to ask the following question.

QUESTION 2.8.7. Let X be an ordinary del Pezzo surface of degree $d \leq 4$ over a number field k. If $X(k) = \emptyset$ does this imply $X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$? Also, if the subset $X(k) \subseteq X(\mathbb{A}_k)$ is not dense, does this imply $X(\mathbb{A}_k)^{\mathrm{Br}} \subsetneq X(\mathbb{A}_k)$?

In other words, is the Brauer–Manin obstruction the only obstruction to weak approximation and the Hasse principle on ordinary del Pezzo surfaces?

Chapter 3

The Brauer group of ample log K3 surfaces

This chapter is based on the author's article [10] written jointly with Martin Bright.

In the previous chapter we have looked at the arithmetic of del Pezzo surfaces. Let us briefly discuss the more complicated arithmetic of K3 surfaces. These surfaces will not play an important role in this chapter or even this thesis, but we will use them to put this chapter in context.

Let X be a K3 surface over a number field k. Here $\operatorname{Br} \bar{X}$ is infinite, but it was proved by Skorobogatov and Zarhin [49] that the quotient $\operatorname{Br} X / \operatorname{Br}_1 X$ is finite. The question then arises of trying to bound this finite group; there has been quite a body of work on this in recent years. Ieronymou, Skorobogatov and Zarhin proved in [34] that, when X is a diagonal quartic surface over the field $\mathbb{Q}$ of rational numbers, the order of $\operatorname{Br} X / \operatorname{Br} \mathbb{Q}$ divides $2^{25} \times 3^2 \times 5^2$. When X is the Kummer surface associated to $E \times E$, with $E/\mathbb{Q}$ an elliptic curve with full complex multiplication, Newton [44] described the odd-order part of $\operatorname{Br} X / \operatorname{Br} \mathbb{Q}$. When X is the Kummer surface associated to a curve of genus 2 over a number field k, Cantoral Farfán, Tang, Tanimoto and Visse [11] described an algorithm for computing a bound for $\operatorname{Br} X / \operatorname{Br} k$. More generally, Várilly-Alvarado [56, Conjectures 4.5, 4.6] has conjectured that there should be a uniform bound on $\operatorname{Br} X / \operatorname{Br} k$ for any K3 surface X, depending only on the geometric Picard lattice of the surface. Recent progress towards this conjecture has been made by Várilly-Alvarado and Viray for certain Kummer surfaces associated to non-CM elliptic curves [57, Theorem 1.8] and by Orr and Skorobogatov for K3 surfaces of CM type [45, Corollary C.1].

So far we have been discussing proper varieties. However, non-proper varieties are also of arithmetic interest. A particular case is that of log K3 surfaces; the arithmetic of integral points on log K3 surfaces shows several features analogous to those of rational points on proper K3 surfaces. See [32] for an introduction to the arithmetic of log K3 surfaces. One example of a log K3 surface is the complement of an anticanonical divisor on an ordinary del Pezzo surface, and it is that case with which we concern ourselves in this chapter.

Some calculations of the Brauer groups of such varieties have already appeared in the literature. In [16], Colliot-Thélène and Wittenberg computed explicitly the Brauer group of the complement of a plane section in certain cubic surfaces. In [35], Jahnel and Schindler carried out extensive calculations in the case of an ordinary del Pezzo surface of degree 4. In this chapter, we compute the possible algebraic Brauer groups of these surfaces, and use uniform boundedness of torsion of elliptic curves to bound the possible transcendental Brauer groups, resulting in Theorem 3.3.4, which is the main result of this chapter.

3.1 Ample log K3 surfaces

We defined a surface in Definition 2.1.1 as a geometrically integral variety over a field k of dimension two. In this chapter all surfaces will be smooth over k. By [33, Proposition 6.11] we see that this implies that we can identify Weil and Cartier divisors and we will use the notions interchangeably.

The goal of this chapter is to describe the Brauer groups of certain surfaces over a number field. With this in mind we may restrict to surfaces over a field of characteristic 0 in the general definitions and statements leading up to the main results. Note for example that this is a standing convention in [32].

DEFINITION 3.1.1. Let C be a divisor on a smooth surface X over a field k. Let C_i be the geometrically irreducible components of $\bar{C}$. We say that C has *simple normal crossings* if the following three conditions are satisfied:

» each component C_i is smooth;

» every geometric point x of C lies on at most two components C_i; and

» any two distinct components C_i and C_j meet transversally.

Most of the divisors we will be interested in will only have one geometrically irreducible smooth component. Such divisors obviously have simple normal crossings. We have included this notion to give the general definition of log K3 surfaces.

DEFINITION 3.1.2. Let U be a smooth surface over a field k. A *log K3 structure* on U is a triple (X, C, i) consisting of a proper smooth surface X over k, an effective anticanonical divisor C on X with simple normal crossings and an open embedding $i\colon U \to X$, such that i induces an isomorphism between U and $X \backslash C$. A log K3 structure is called *ample* if C is ample.
A log K3 surface is a simply connected, smooth surface U over k together with a choice of log K3 structure (X, C, i) on U. An *ample log K3 surface* is a surface together with an ample log K3 structure.

We will often use the notation U for $X\backslash C$ and assume the log K3 structure is understood.

PROPOSITION 3.1.3. *Let U be a log K3 surface over a field k with log K3 structure (X, C, i) such that C is not the trivial divisor on X. Then $\bar{X}$ is a rational surface over $\bar{k}$.*

Proof. See Proposition 2.0.18 in [32]. The standing convention in this paper is that k should be of characteristic 0. The proof works for log K3 surface over general fields. □

If C is the trivial divisor then $U \xrightarrow{\cong} X$ is a proper smooth surface. These types of surfaces are precisely the *K3 surfaces* touched upon in the introduction of this chapter. We will however be interested in log K3 structures for which the divisor is not trivial. We will usually even assume that C is an ample divisor. Proposition 3.1.3 shows that in this case the compactification X is an ordinary del Pezzo surface.

DEFINITION 3.1.4. Let $1 \leq d \leq 9$ be an integer. A *log K3 surface of $dP(d)$ type* or an *ample log K3 surface of degree d* is a log K3 surface U with log K3 structure (X, C, i) such that X is an ordinary del Pezzo surface of degree d. If the locally principal subscheme of X associated to the effective divisor C is geometrically irreducible, we say that U is a *log K3 surface of geometrically irreducible type*.

We will be concerned with ample log K3 surfaces, and in particular those of geometrically irreducible type.

Note that if (X, C, i) is a log K3 structure such that C is geometrically irreducible, then $\bar{C}$ has a unique irreducible component which must be smooth by the definition of normal crossing divisor. Equivalently we could assume that C is geometrically integral or geometrically connected.

3.2 The algebraic Brauer group

In the next sections we will compute Brauer groups of certain log K3 surfaces or bound the order of these groups. If an ample log K3 surface is of geometrically irreducible type we can compute the algebraic Brauer group modulo constants using Proposition 1.6.5.

PROPOSITION 3.2.1. *Let k be a number field and $U = X\backslash C$ an ample log K3 surface of geometrically irreducible type of degree d at most 7 over k, i.e. X is an ordinary del Pezzo surface of degree d and $U \subseteq X$ is the complement of a geometrically irreducible curve $C \in |-K_X|$. Then $\mathrm{Br}_1 U / \mathrm{Br}\, k$ depends only on $\mathrm{Pic}\, \bar{X}$ as a Galois module and its order is at most 256. For $d = 1$, the natural map $\mathrm{Br}_1 X \to \mathrm{Br}_1 U$ is an isomorphism. For $2 \leq d \leq 7$, the possible combinations of $\mathrm{Br}\, X / \mathrm{Br}\, k = \mathrm{Br}_1 X / \mathrm{Br}\, k$ and $\mathrm{Br}_1 U / \mathrm{Br}\, k$ are as shown in Table C.*

We will see that the proof does not use the fact that the geometrically irreducible C is smooth. The result is also true for any geometrically irreducible

Degree	$\mathrm{Br}_1 X / \mathrm{Br}\, k$	Possibilities for $\mathrm{Br}_1 U / \mathrm{Br}\, k$
$\mathbf{d = 7}$	1	1
$\mathbf{d = 6}$	1	1 2 3 6
$\mathbf{d = 5}$	1	1 5
$\mathbf{d = 4}$	1	1 2 2^2 2^3 2^4 4 $2 \cdot 4$
	2	2 2^2 2^3 2^4 $2 \cdot 4$
	2^2	2^3 2^4 $2^2 \cdot 4$
$\mathbf{d = 3}$	1	1 3 3^2
	2	2 6
	2^2	2^2 $2 \cdot 6$
	3	3 3^2
	3^2	3^3
$\mathbf{d = 2}$	1	1 2
	2	2 2^2 $2 \cdot 4$ 4
	2^2	2^2 2^3 $2 \cdot 4$ $2^2 \cdot 4$ 4^2
	2^3	2^3 2^4 $2^2 \cdot 4$
	2^4	2^4 2^5 $2^3 \cdot 4$
	2^5	2^6
	2^6	2^7
	3	3 6
	3^2	3^2 $3 \cdot 6$
	$2 \cdot 4$	$2 \cdot 4$ $2^2 \cdot 4$ 4^2
	$2^2 \cdot 4$	$2^2 \cdot 4$ $2^3 \cdot 4$
	4	$2 \cdot 4$ 4
	4^2	$2 \cdot 4^2$

Table C: Possible group structures of $\mathrm{Br}_1 U / \mathrm{Br}\, k$. The notation is the same as in Table B on page 59. For example, $2 \cdot 4^2$ means $\mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/4\mathbb{Z})^2$.

effective anticanonical divisor C on a del Pezzo surface X over a number field k.

Note that our computations agree with those of Jahnel and Schindler [35, Remark 4.7i)] on ordinary del Pezzo surfaces of degree 4.

To prove Proposition 3.2.1 we will use the following lemma.

LEMMA 3.2.2. *Let $U = X\backslash C$ be a log K3 surface of geometrically irreducible type over a number field k of degree $d \leq 7$. Let W be the minimal subgroup of the Weyl group W_{9-d} such that the action of G_k on* $\operatorname{Pic}\bar{X}$ *factors through the induced action of W on* $\operatorname{Pic}\bar{X}$. *The group* $\operatorname{Br}_1 U/\operatorname{Br} k$ *only depends on the conjugacy class of W as a subgroup of W_{9-d}.*

Proof. As C is geometrically irreducible, a section of $\mathbb{G}_\mathrm{m}$ on $\bar{U}$ corresponds to a rational function on $\bar{X}$ whose divisor is a multiple of C. The intersection of this principal divisor with C must be zero and we see that $\mathrm{H}^0(\bar{U}, \mathbb{G}_\mathrm{m}) = \bar{k}^\times$. This means that we can apply Proposition 1.6.5 to compute the algebraic Brauer group modulo constants.

By definition of W we have a surjective group homomorphism $G_k \twoheadrightarrow W$, such that G_k acts on $\operatorname{Pic}\bar{X}$ through $W \subseteq W_{9-d}$. We also find that the action of the kernel of $G_k \to W$ on $\operatorname{Pic}\bar{X}$ is trivial. Now we will determine the induced action of these groups on $\operatorname{Pic}\bar{U}$. By [33, Proposition II.6.5] we have an exact sequence of Galois modules

$$0 \to \mathbb{Z} \to \operatorname{Pic}\bar{X} \to \operatorname{Pic}\bar{U} \to 0$$

where the first map sends 1 to the anticanonical class in $\operatorname{Pic}\bar{X}$. This shows that the Galois module $\operatorname{Pic}\bar{U}$ only depends on the Galois module $\operatorname{Pic}\bar{X}$, since the class of C is the anticanonical class.

Let G' be the kernel of the group homomorphism $G_k \to W$. The inflation-restriction sequence for the actions of G_k and its normal subgroup G' on $\operatorname{Pic}\bar{U}$ gives the exact sequence

$$0 \to \mathrm{H}^1(G_k/G', (\operatorname{Pic}\bar{U})^{G'}) \to \mathrm{H}^1(G_k, \operatorname{Pic}\bar{U}) \to \mathrm{H}^1(G', \operatorname{Pic}\bar{U})^{G_k/G'}.$$

The kernel G' of $G_k \to W$ acts trivially on $\operatorname{Pic}\bar{X}$ and hence also on its quotient $\operatorname{Pic}\bar{U}$. Since G' is a torsion group and $\operatorname{Pic}\bar{U}$ is a free $\mathbb{Z}$-module we see that $\mathrm{H}^1(G', \operatorname{Pic}\bar{U})$ is trivial. From the minimality of W we derive that $G_k \to W$ is surjective and hence G_k/G' is isomorphic to W. We conclude that

$$\mathrm{H}^1(W, \operatorname{Pic}\bar{U}) \to \mathrm{H}^1(G_k, \operatorname{Pic}\bar{U})$$

is an isomorphism.

Similarly to the proof of Proposition 2.8.6 using [58, Example 6.7.7] we see that the cohomology group $\mathrm{H}^1(W, \operatorname{Pic}\bar{U})$ only depends on the conjugacy class of W in W_{9-d} and by Proposition 1.6.5 this determines $\operatorname{Br}_1 U/\operatorname{Br} k$. □

Proof of Proposition 3.2.1. By the discussion following Proposition 2.3.5 there are essentially finitely many Galois actions on $\operatorname{Pic}\bar{X}$ as each action factors through a unique minimal subgroup W of the finite Weyl group W_{9-d}. Enumerating all possible subgroups W of W_{9-r}, or even just the conjugacy classes of subgroups, and computing the induced actions of W on $\operatorname{Pic} \bar{U}$ allows us to calculate the possible cohomology groups. For MAGMA code to accomplish this calculation, see [39]. In the case $d = 1$, the following lemma, its corollary and the Table B on page 59 spares us from what would be a lengthy calculation. □

LEMMA 3.2.3. *The natural map* $\mathrm{H}^1(k, \operatorname{Pic} \bar{X}) \to \mathrm{H}^1(k, \operatorname{Pic} \bar{U})$ *is injective and the cokernel has exponent dividing* δ*, where* δ *is the minimal non-zero value of* $|D \cdot K_X|$ *for* D *a divisor on* X*.*

Proof. Let D be a divisor with $D \cdot K_X = \delta$. As above, we have the exact sequence of Galois modules

$$0 \to \mathbb{Z} \xrightarrow{j} \operatorname{Pic} \bar{X} \to \operatorname{Pic} \bar{U} \to 0.$$

The map $E \mapsto E \cdot D$ gives a map $s\colon \operatorname{Pic} \bar{X} \to \mathbb{Z}$ with the property that $s \circ j$ is multiplication by $-\delta$. Consider the following part of the long exact sequence associated to this short exact sequence:

$$0 \to \mathrm{H}^1(k, \operatorname{Pic} \bar{X}) \to \mathrm{H}^1(k, \operatorname{Pic} \bar{U}) \to \mathrm{H}^2(k, \mathbb{Z}) \underset{s}{\overset{j}{\rightleftarrows}} \mathrm{H}^2(k, \operatorname{Pic} \bar{X})$$

We see that the cokernel of $\mathrm{H}^1(k, \operatorname{Pic} \bar{X}) \to \mathrm{H}^1(k, \operatorname{Pic} \bar{U})$ is isomorphic to the kernel of j, which is contained in the kernel of $s \circ j$; but this map is multiplication by $-\delta$. □

COROLLARY 3.2.4. *If* X *is an ordinary del Pezzo surface of degree* 1 *or* X *contains a* -1*-curve defined over* k*, then the map* $\operatorname{Br} X / \operatorname{Br} k = \operatorname{Br}_1 X / \operatorname{Br} k \to \operatorname{Br}_1 U / \operatorname{Br} k$ *is an isomorphism.*

We draw special attention to the possible algebraic Brauer groups modulo constants for log K3 surfaces of dP_5 type. The following proposition gives a criterion for computing $\operatorname{Br}_1 U / \operatorname{Br} k$.

PROPOSITION 3.2.5. *Let* X *be a del Pezzo surface of degree* 5 *and let* W *be the minimal subgroup of* W_4 *through which the action of* G_k *on* $\operatorname{Pic} \bar{X}$ *factors. The group* $\operatorname{Br}_1 U / \operatorname{Br} k$ *is cyclic of order* 5 *precisely for* W *in one conjugacy class of subgroups of* W_4*. In all other cases* $\operatorname{Br}_1 U / \operatorname{Br} k$ *is trivial.*

One can check that W_4 has 19 conjugacy classes of subgroups and only in one of those cases we find a non-trivial algebraic Brauer group modulo constants. We will study this specific action more in Chapter 4 and use it to produce examples of Brauer–Manin obstructions of order 5 to the integral Hasse principle.

3.3 Uniform bound for the order of the Brauer group

Next we will study the whole Brauer group of ample log K3 surfaces. The Brauer group modulo constants was computed for some instances of these surfaces by

Colliot-Thélène and Wittenberg [16], Jahnel and Schindler [35] and Harpaz [32]. For ample log K3 surfaces U over a number field k we will use the techniques used by Colliot-Thélène and Wittenberg [16] to give a bound for the order of $\operatorname{Br} U / \operatorname{Br} k$ in terms of the degree of k.

We will use the following important result.

PROPOSITION 3.3.1 (Merel). *Let m be a positive number. There exists an effective computable number $N(m)$ such that for an elliptic curve E over a number field of degree m the order of a torsion point in $E(k)$ is bounded by $N(m)$.*

Proof. See [42]. $\square$

This bound will be the basis for practically all bounds in this section. Now let us first look at the case where at least one of the -1-curves on $\bar{X}$ is defined over k. Note that in this situation by Corollary 3.2.4 the image of $\operatorname{Br} X$ in $\operatorname{Br} U$ coincides with the algebraic Brauer group $\operatorname{Br}_1 U$.

LEMMA 3.3.2. *Let $U = X \backslash C$ be an ample log K3 surface of geometrically irreducible type such that a -1-curve $L \subseteq X$ is defined over k. Let m denote the degree $[k : \mathbb{Q}]$. Then the restriction map $\operatorname{Br} X \to \operatorname{Br} U$ is injective, and the order of its cokernel is bounded by $N(m)^2$.*

Proof. Since U of geometrically irreducible type we see that C is geometrically integral. Because C is a strict normal crossings divisor it is smooth, and we have the exact sequence

$$0 \to \operatorname{Br} X \to \operatorname{Br} U \xrightarrow{\partial_C} \mathrm{H}^1(C, \mathbb{Q}/\mathbb{Z})$$

from Proposition 1.7.5. So it is enough to bound the image of the residue map ∂_C.

We have $C \cdot L = 1$ and this implies that $C \cap L$ is geometrically integral zero-dimensional subscheme P on X. This means that we can apply Proposition 1.7.6 and we find the commutative diagram shown in (3.1).

$$\begin{tikzcd}
\operatorname{Br} X \arrow[r] \arrow[d] & \operatorname{Br} U \arrow[r, "\partial_C"] \arrow[d] & \mathrm{H}^1(C, \mathbb{Q}/\mathbb{Z}) \arrow[d, "\alpha"] \\
\operatorname{Br} L \arrow[r] & \operatorname{Br}(L \setminus P) \arrow[r, "\partial_C"] & \mathrm{H}^1(P, \mathbb{Q}/\mathbb{Z})
\end{tikzcd} \tag{3.1}$$

We have $L \cong \mathbb{P}^1_k$ and $L \setminus P \cong \mathbb{A}^1_k$, both of which have Brauer group isomorphic to $\operatorname{Br} k$ by Proposition 1.4.12; exactness of the bottom row shows that ∂_P is the zero map. This implies that the image of ∂_C is contained in the kernel of the homomorphism α. The first cohomology group $\mathrm{H}^1(C, \mathbb{Q}/\mathbb{Z})$ classifies cyclic Galois covers of C, and $\ker \alpha$ corresponds to those cyclic Galois covers $D \to C$ for which the fibre above P is a trivial torsor for the structure group $\mathbb{Z}/n\mathbb{Z}$. So we consider such covers whose kernel is a disjoint union of k-points.

We first bound the degree of such a cover. Let $\pi \colon D \to C$ be a cyclic cover of degree n, and suppose that the fibre $F = \pi^{-1}(P)$ is trivial, so that F consists of

n distinct k-points. The Riemann–Hurwitz formula shows that D has genus 1. Pick a point Q in the fibre F. If we regard D and C as elliptic curves with base points Q and P respectively, then π is an isogeny of elliptic curves, and $F(k) = \ker \pi$ is a cyclic subgroup of order n in $D(k)$. In particular, since D is an elliptic curve over k with a point of order n, we have $n \leq N(m)$.

We now fix n to be the maximal order of an element of $\ker \alpha$. The exponent of a finite abelian group is equal to the maximal order of its elements, so every element of $\ker \alpha$ has order dividing n.

Looking at the long exact sequence in cohomology associated to the short exact sequence of sheaves

$$0 \to \mathbb{Z}/n\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \xrightarrow{\times n} \mathbb{Q}/\mathbb{Z} \to 0$$

shows that the natural map $\mathrm{H}^1(C, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{H}^1(C, \mathbb{Q}/\mathbb{Z})$ is injective. This identifies $\mathrm{H}^1(C, \mathbb{Z}/n\mathbb{Z})$ with the n-torsion in $\mathrm{H}^1(C, \mathbb{Q}/\mathbb{Z})$, which contains $\ker \alpha$. The Hochschild–Serre spectral sequence gives a short exact sequence

$$0 \to \mathrm{H}^1(k, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\beta} \mathrm{H}^1(C, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{H}^1(\bar{C}, \mathbb{Z}/n\mathbb{Z})$$

in which the map α induces a left inverse to β. Thus $\ker \alpha$ is identified with a subgroup of $\mathrm{H}^1(\bar{C}, \mathbb{Z}/n\mathbb{Z})$, which is isomorphic to the n-torsion in $\operatorname{Pic} \bar{C}$ and so has order n^2. Combining this with the above bound on n gives the claimed bound. □

COROLLARY 3.3.3. *Under the conditions of Lemma 3.3.2, the order of* $\operatorname{Br} U / \operatorname{Br} k$ *is bounded by* $2^6 N(m)^2$.

Proof. If we blow down the -1-curve on X we find a del Pezzo surface X' of degree $d+1$, such that $\operatorname{Br} X \cong \operatorname{Br} X'$. So we see in Table B on page 59 that $\#(\operatorname{Br} X / \operatorname{Br} k) = \#(\operatorname{Br} X' / \operatorname{Br} k) \leq 64$ since $\deg X' \geq 2$. Corollary 3.2.4 gives an isomorphism $\operatorname{Br} X / \operatorname{Br} k \cong \operatorname{Br}_1 U / \operatorname{Br} k$. Combining this with Lemma 3.3.2 gives the bound for $\operatorname{Br} U / \operatorname{Br} k$. □

Now we can prove a bound for a general ample log K3 surface of geometrically irreducible type over a number field.

THEOREM 3.3.4. *Let k be a number field of degree m. For an ample log K3 surface of geometrically irreducible type $U = X \backslash C$ of degree d at most 7 the order of* $\operatorname{Br} U / \operatorname{Br} k$ *is bounded by*

$$2^{14} N(240m)^2.$$

Proof. Let K be a finite extension of k such that at least one -1-curve L on $\bar{X}$ is defined over K. The orbit-stabilizer theorem shows that we can always take the degree $[K : k]$ no larger than the number of -1-curves on $\bar{X}$. Since the maximal number of -1-curves on a del Pezzo surface is 240, we find $[K : \mathbb{Q}] \leq 240m$.

By Corollary 3.3.3 we have a bound on $\operatorname{Br} U_K / \operatorname{Br} K$. On the other hand, the kernel of the morphism $\operatorname{Br} U / \operatorname{Br} k \to \operatorname{Br} U_K / \operatorname{Br} K$ is contained in $\operatorname{Br}_1 U / \operatorname{Br} k$ and hence bounded by 256 by Proposition 3.2.1.

Combining these two bounds, we find that

$$\#\left(\operatorname{Br} U / \operatorname{Br} k\right) < 2^{14} N(240m)^2. \qquad \square$$

Remark. There are, of course, many ways in which the constants appearing in this bound could be improved, especially if we were to separate the various different degrees. For example, the group $\operatorname{Br} U_K / \operatorname{Br} K$ and the kernel of the homomorphism $\operatorname{Br}_1 U / \operatorname{Br} k \to \operatorname{Br}_1 U_K / \operatorname{Br} K$ are far from independent. Our interest here has been in showing the existence of a uniform bound, rather than in making that bound as small as possible.

Chapter 4

Order 5 obstructions to the integral Hasse principle

In this chapter we will discuss new examples of the Brauer–Manin obstruction to the integral Hasse principle. The most remarkable property of these obstructions is that they come from a single element of order 5. This is exceptional since all other known examples of the Brauer–Manin obstructions use only elements of order 2 and 3.

The results in this chapter all use an element of order 5 which can occur in the Brauer group of ample log K3 surfaces of dP_5 type described in Proposition 3.2.5. So let us write $U = X \backslash C$ where X is an ordinary del Pezzo surface of degree 5 over $\mathbb{Q}$ and C is an effective anticanonical divisor. In Proposition 3.2.5 we have seen that U has a non-trivial algebraic Brauer group for a specific action of the absolute Galois group G_k on the geometric Picard group $\operatorname{Pic} \bar{X}$. We will first study this action and the induced action on the -1-classes. We will see for example that $\operatorname{Pic} \bar{X}$ will be isomorphic to $\operatorname{Pic} X_K$ for a specific number field K of degree 5.

The next step in producing our examples is recovering an ample log K3 surface U over $\mathbb{Q}$ from this action or even from only its splitting field K. We use the construction described in [29] to first construct a del Pezzo surface X of degree 5 over $\mathbb{Q}$. We start off with a conic Γ_0 on the projective plane $\mathbb{P}^2_{\mathbb{Q}}$ and five distinct geometric points P_i on the conic with a specific action of Galois. We blow up these five points and recover an ordinary del Pezzo surface $\beta\colon B \to \mathbb{P}^2_{\mathbb{Q}}$ of degree 4. The strict transform Γ of Γ_0 along the blowup morphism β is a -1-curve on B. We contract this curve along the morphism $B \to X$ to obtain an ordinary del Pezzo surface of degree 5 over $\mathbb{Q}$. Now for any smooth anticanonical curve $C \subseteq X$ we see that $U = X \backslash C$ is an ample log K3 surface with an algebraic Brauer group modulo constants of order 5. These are the surfaces for which we will consider the set of integral points.

However, U is a surface over $\mathbb{Q}$ and it does not make sense to consider $\mathbb{Z}$-

points; the set of integral points on a scheme over a number field depends on the choice of model over the ring of integers. We can obtain a model $\mathcal{U}/\mathbb{Z}$ for U in the following natural way: the anticanonical map embeds the ordinary del Pezzo surface X into $\mathbb{P}^5_{\mathbb{Q}}$ which restricts to an embedding $U \to \mathbb{A}^5_{\mathbb{Q}}$. Consider a set of equations defining the subscheme $U \subseteq \mathbb{A}^5_{\mathbb{Q}}$. We rescale these equations such that they have integral coefficients. These equations now define a subscheme $\mathcal{U} \subseteq \mathbb{A}^5_{\mathbb{Z}}$ which is affine over $\mathbb{Z}$.

In order to study the geometry of the model $\mathcal{U}$ it is useful to have another description. In the above construction of $\mathcal{X}$ we first blew up points and contracted a curve defined over $\mathbb{Q}$ and then passed to a scheme over the integers. We will now reverse the steps of this process: first we pass over to the integers and then we blow up the projective plane over the integers in a subscheme which is flat of relative dimension zero over $\mathbb{Z}$. We obtain a scheme $\mathcal{B}$ over $\mathbb{Z}$ whose fibres are peculiar del Pezzo surfaces over either a finite field or the rational numbers. We proceed by contracting a subscheme $\Gamma \subseteq \mathcal{B}$ over $\mathbb{Z}$ to obtain a scheme $\mathcal{X}$ over $\mathbb{Z}$. This does not mean that Γ is contracted to a single point, but rather that the image of Γ in $\mathcal{X}$ is a relative point of $\mathcal{X} \to \mathbb{Z}$. This implies that a fibre of Γ over a prime $\ell \in \mathbb{Z}$ is a curve of negative self-intersection on a peculiar del Pezzo surface $\mathcal{B}_\ell$, and the morphism $\mathcal{B}_\ell \to \mathcal{X}_\ell$ is the contraction of this curve to obtain another del Pezzo surface. The scheme $\mathcal{X}$ will naturally embed in $\mathbb{P}^5_{\mathbb{Z}}$. Now consider the complement $\mathcal{U} \subseteq \mathbb{A}^5_{\mathbb{Z}}$ of a hyperplane section $\mathcal{H} \subseteq \mathbb{P}^5_{\mathbb{Z}}$ in $\mathcal{X}$.

We will show that this relative affine surface $\mathcal{U}/\mathbb{Z}$ is naturally isomorphic to the subscheme $\mathcal{U} \subseteq \mathbb{A}^5_{\mathbb{Z}}$ constructed above. The first construction gives us explicit equations while the second construction makes it easier to understand the geometry of the closed fibres of $\mathcal{U}$ over $\mathbb{Z}$.

Now that we have constructed a scheme $\mathcal{U}$ which is affine over $\mathbb{Z}$ we can consider the integral points on $\mathcal{U}$. Using our explicit geometric construction we study the fibres of $\mathcal{U}$ over $\mathbb{Z}$; almost all of which are ample log K3 surfaces of dP_5 type. We proceed by computing the invariant map at a prime ℓ using our description of the fibre $\mathcal{U}_\ell$ and the factorization of ℓ in the number field K. By combining all these local results we obtain several examples of order 5 Brauer–Manin obstructions to the integral Hasse principle.

4.1 The interesting Galois action

In this chapter we will construct an affine scheme $\mathcal{U} \subseteq \mathbb{A}^5_{\mathbb{Z}}$ which has a Brauer–Manin obstruction to the integral Hasse principle. This obstruction is given by an element of order 5 in the Brauer group $\mathrm{Br}\, U$ of the generic fibre $U = \mathcal{U}_{\mathbb{Q}}$. In all our examples we will construct $\mathcal{U}$ in such a way such that $U = X \backslash C$ is an ample log K3 surface of dP_5 type. The existence of an element of order 5 in $\mathrm{Br}\, U / \mathrm{Br}\, \mathbb{Q}$ is then implied by Proposition 3.2.5 for a specific action of $G_{\mathbb{Q}}$ on $\mathrm{Pic}\, \bar{X}$.

Let X be any del Pezzo surface of degree 5 over a field k. We recall some results about the action of Galois on the geometric Picard group $\mathrm{Pic}\, \bar{X}$. We defined

W_4 as the group of intersection-preserving automorphisms of $\operatorname{Pic}\bar{X}$ which map the anticanonical class to itself. If C is geometrically irreducible this induces an action of W_4 on $\operatorname{Pic}\bar{U}$. In Proposition 3.2.5 we have seen that there is a special conjugacy class $\mathcal{W}$ of subgroups of W_4. This conjugacy class $\mathcal{W}$ is precisely the set of all subgroups of $W \subseteq W_4$ with the property that the cohomology group $\mathrm{H}^1(W, \operatorname{Pic}\bar{U})$ is non-trivial. We will study the action of such a subgroup W on $\operatorname{Pic}\bar{X}$, $\operatorname{Pic}\bar{U}$ and the -1-curves on X more closely in this section. Let us first define this interesting action.

DEFINITION 4.1.1. Let X be an ordinary del Pezzo surface of degree 5 over a field k. Let K be the minimal Galois extension of k over which all -1-curves on X are defined. We say that X is *interesting* if $[K\colon k] = 5$. A log K3 surface of dP_5 type $U = X \backslash C$ is called *interesting* if X is an interesting del Pezzo surface and C is geometrically irreducible.

The field K is called the *splitting field* of the interesting surfaces X and U.

Consider an interesting log K3 surface $U = X \backslash C$. By definition of a log K3 surface we see that C is smooth. The curve C is also geometrically irreducible since U is interesting. The results in this chapter are also true for the complement of a geometrically irreducible anticanonical curve C on an ordinary del Pezzo surface X of degree 5. To be able to use the language of log K3 surface we do keep the superfluous condition that C is smooth.

The following lemma shows that an interesting action corresponds to a unique conjugacy class of subgroups of W_4.

LEMMA 4.1.2. *Consider an interesting del Pezzo surface X over a field k. The action of G_k on* $\operatorname{Pic}\bar{X}$ *is uniquely determined up to conjugacy.*

On an interesting del Pezzo surface there are two Galois orbits of geometric -1-curves, each of size 5. *The sum of the -1-curves in one such orbit is an anticanonical divisor.*

Proof. Let K be the minimal Galois extension of k such that all -1-curves on X are defined over K. Since X is interesting the extension K/k is of degree 5.

We have seen in Proposition 2.3.1 that there is a basis $(L_0, L_1, L_2, L_3, L_4)$ of $\operatorname{Pic}\bar{X} \cong \mathbb{Z}^5$ such that $L_0^2 = 1$, $L_0 \cdot L_i = 0$ and $L_i^2 = -1$ for $i \neq 0$. The -1-classes are the classes L_i for $i \neq 0$ and $L_{ij} := L_0 - L_i - L_j$ for $1 \leq i < j \leq 4$. The intersection graph of the -1-curves on a generalized del Pezzo surface of degree 5 is the so-called Petersen graph shown in Figure II.

Let us first prove that $\mathrm{Gal}(K/k)$ does not fix any of the ten geometric -1-curves. If it does fix a -1-curve L consider the three -1-curves L intersects. The Galois action then permutes these three -1-curves since the action preserves the intersection pairing. However there are no non-trivial group homomorphisms $\mathbb{Z}/5\mathbb{Z} \to S_3$ so the action of Galois actually fixes these three -1-curves. Since the graph in Figure II is connected we conclude that all geometric -1-curves are defined over k contradicting the minimality of K. So no geometric -1-curve is fixed by G_k and there must be two orbits of size 5. After choosing a possible different basis of $\operatorname{Pic}\bar{X}$ we see that these two orbits are the two regular pentagons

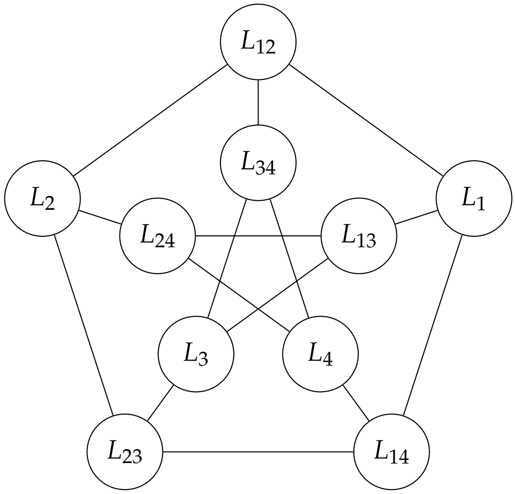


Figure II: The intersection graph of -1-curves on a generalized del Pezzo surface of degree 5.

in Figure II and that there is a $\sigma \in \mathrm{Gal}(K/k)$ which acts on the outer pentagon by rotating counter-clockwise. Since σ preserves the intersection pairing it will also rotate the inner pentagon counter-clockwise. This determines the action of σ on the -1-classes

$$L_1 \mapsto L_{12} \mapsto L_2 \mapsto L_{23} \mapsto L_{14} \mapsto L_1,$$

$$L_3 \mapsto L_4 \mapsto L_{13} \mapsto L_{34} \mapsto L_{24} \mapsto L_3.$$

This proves that $L_0 = L_{12} + L_1 + L_2$ gets mapped to $2L_0 - L_1 - L_2 - L_3$. Since a different choice of such a basis differs by an automorphism in W_4 by Proposition 2.3.5 this determines an action of G_k on $\mathrm{Pic}\,\bar{X}$ up to conjugacy.

For the last statement one needs to check that both the divisor classes of $L_1 + L_{12} + L_2 + L_{23} + L_{14}$ and $L_3 + L_4 + L_{13} + L_{34} + L_{24}$ equal the anticanonical class $3L_0 - L_1 - L_2 - L_3 - L_4$. □

If we consider a log K3 surface of geometrically irreducible dP_5 type over a number field k we can compute its algebraic Brauer group modulo constants using Proposition 1.6.5. The following proposition shows that the action of G_k on $\mathrm{Pic}\,\bar{X}$ is interesting precisely when $\mathrm{Br}_1 U/\mathrm{Br}\,k$ is non-trivial. We conclude that the actions mentioned in Proposition 3.2.5 are precisely the interesting ones.

PROPOSITION 4.1.3. *Let $U = X \backslash C$ be a log K3 surface of geometrically irreducible dP_5 type over a number field k. We have*

$$\mathrm{Br}_1 U/\mathrm{Br}\,k \cong \begin{cases} \mathbb{Z}/5\mathbb{Z} & \textit{if } U \textit{ is interesting;} \\ 1 & \textit{otherwise.} \end{cases}$$

Proof. Let the action of G_k on $\operatorname{Pic} \bar{X}$ factor through the minimal subgroup W of W_4. Lemma 3.2.2 states that $\operatorname{Br}_1 U / \operatorname{Br} k$ only depends on the conjugacy class of $W \subseteq W_4$ and that for precisely one conjugacy class of subgroups of W_4 the algebraic Brauer group modulo constants is non-trivial. We have seen that every interesting action of G_k on $\operatorname{Pic} \bar{X}$ factors through the same conjugacy class of subgroups of W_4 of order 5. This means that it will suffice to prove that $\operatorname{Br}_1 U / \operatorname{Br} k$ is non-trivial for interesting del Pezzo surfaces. So suppose that X is an interesting del Pezzo surface over k. We will fix a basis $(L_0, L_1, L_2, L_3, L_4)$ of $\operatorname{Pic} \bar{X}$ as in the proof of Lemma 4.1.2.

In general, since $C \subseteq X$ is geometrically irreducible we find the following exact sequence of Galois modules

$$0 \to \mathbb{Z} \xrightarrow{j} \operatorname{Pic} \bar{X} \to \operatorname{Pic} \bar{U} \to 0,$$

where j maps n to $-nK_X$. This shows that $\operatorname{Pic} \bar{U} \cong \operatorname{Pic} \bar{X}/\mathbb{Z}C \cong \mathbb{Z}^4$, since the anticanonical divisor class $-K_X = 3L_0 - L_1 - L_2 - L_3$ is primitive. So $\operatorname{Pic} \bar{U}$ is torsion free and from the inflation–restriction sequence we conclude that the inflation homomorphism induces an isomorphism

$$\mathrm{H}^1(\mathrm{Gal}(K/k), \operatorname{Pic} U_K) \xrightarrow{\inf} \mathrm{H}^1(G_k, \operatorname{Pic} \bar{U}).$$

Given the basis of $\operatorname{Pic} \bar{X}$ we saw in the proof of Lemma 4.1.2 the specific action of a generator $\sigma \in \mathrm{Gal}(K/k)$ on $\operatorname{Pic} \bar{X}$. We will compute the action of σ on the quotient $\operatorname{Pic} \bar{U}$ of $\operatorname{Pic} \bar{X}$. The classes $[L_0]$, $[L_1]$, $[L_2]$ and $[L_3]$ in $\operatorname{Pic} U_K$ form a basis and in this basis the class of L_4 becomes $[L_4] = 3[L_0] - [L_1] - [L_2] - [L_3]$. So σ acts on $\operatorname{Pic} \bar{U}$ as

$$\sigma = \begin{pmatrix} 2 & 1 & 1 & 3 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 \end{pmatrix}$$

By results on group cohomology of cyclic groups [58, Theorem 6.2.2] we see that we get

$$\mathrm{H}^1(G, \operatorname{Pic} \bar{U}) \cong \ker(1 + \sigma + \sigma^2 + \sigma^3 + \sigma^4)/\mathrm{Im}(1 - \sigma).$$

Since $1 + \sigma + \sigma^2 + \sigma^3 + \sigma^4 = 0$ and the image of $1 - \sigma$ is generated by $(1, 0, 0, 2)$, $(0, 1, 0, 4)$, $(0, 0, 1, 4)$ and $(0, 0, 0, 5)$ we find

$$\operatorname{Br}_1 U / \operatorname{Br} k \cong \mathbb{Z}/5\mathbb{Z}. \qquad \square$$

Consider an interesting log K3 surface U. On the compactification X of U we have three important effective anticanonical divisors. First of all $C = X \backslash U$, but also the two divisors supported on -1-curves as described in Lemma 4.1.2. We will associate to these effective divisors anticanonical sections using [33, II, Proposition 7.7b] and we will use these elements to construct explicit generators of $\operatorname{Br}_1 U / \operatorname{Br} k$ for an interesting log K3 surface $U = X \backslash C$.

LEMMA 4.1.4. *Let $U = X\backslash C$ be an interesting log K3 surface over a number field k. Let K be the corresponding Galois extension of degree 5 of k. Fix a generator σ of* $\mathrm{Gal}(K/k) \cong \mathbb{Z}/5\mathbb{Z}$. *Let $h \in \mathrm{H}^0(X, \omega_X^\vee)$ be a global section whose divisor of zeroes is C, and let l_1 and l_2 be sections in $\mathrm{H}^0(X, \omega_X^\vee)$ associated to the two anticanonical divisors supported in geometric -1-curves from Lemma 4.1.2.*

The cyclic $\kappa(X)$-algebras

$$\left(\frac{l_1}{h}, \sigma\right) \quad \text{and} \quad \left(\frac{l_2}{h}, \sigma\right)$$

are similar over $\kappa(X)$, their class lies in the subgroup $\mathrm{Br}\, U \subseteq \mathrm{Br}\, \kappa(X)$ *and generates* $\mathrm{Br}_1 U / \mathrm{Br}\, k$.

Proof. As $\mathrm{div}_U(\frac{l_1}{h})$ and $\mathrm{div}_U(\frac{l_2}{h})$ are orbits of -1-curves defined over K it follows from Lemma 1.7.3 that the cyclic algebras lie in the subgroup $\mathrm{Br}\, U$. From Proposition 1.3.2 we see that the algebras $\left(\frac{l_1}{h}, \sigma\right) \otimes \left(\frac{l_2}{h}, \sigma\right)^{\mathrm{opp}}$ and $\left(\frac{l_1}{l_2}, \sigma\right)$ are similar. Now $\mathrm{div}_U(\frac{l_1}{l_2})$ is the norm of a principal divisor on U since this is even the case on X. Indeed, the divisors $L_{14} + L_1 - L_2$ and L_{24} are linearly equivalent on X, and their norms $\mathrm{Nm}_{K/k}(L_{14} + L_1 - L_2)$ and $\mathrm{Nm}_{K/k}(L_{24})$ are the divisors of zeroes of l_1 and l_2. It follows again from Lemma 1.7.3 that $\left(\frac{l_1}{l_2}, \sigma\right)$ is trivial in $\mathrm{Br}\, U$, and hence that the two cyclic algebras lie in the same class.

We have seen in Proposition 1.3.2 that $\mathcal{A}$ is split by the degree 5 extension K. Now Corollary 1.2.13 implies that $\mathcal{A}$ is either trivial or of order 5. Suppose that the class of $\mathcal{A}$ is trivial. Then by Lemma 1.7.3 any -1-curve L on U_K in the support of $\mathrm{div}_U l_1$ is principal, i.e. there is a $g \in \kappa(U_K)$ such that $\mathrm{div}_U g = L$. Consider g as a function on X. Then $\mathrm{div}_X g = L + nC$ for some non-negative integer n, since C is geometrically irreducible. We conclude that

$$0 = K_X \cdot \mathrm{div}_X g = K_X \cdot L + n K_X \cdot C = -1 + 5n,$$

which is a contradiction. □

The anticanonical sections l_1 and l_2 are so important we will repeat their definition.

DEFINITION 4.1.5. Let $X \subseteq \mathbb{P}^5_k$ be an anticanonically embedded interesting del Pezzo surface of degree 5. Let $l_1, l_2 \in \mathcal{O}_{\mathbb{P}^5_k}(1)$ be the linear forms over k in six variables which restrict to the anticanonical sections supported on geometric -1-curves from Lemma 4.1.4.

Note that l_1 and l_2 are only defined up multiplication by an element of $k^\times$. From now on we will denote the class in Lemma 4.1.4 by $\mathcal{A} \in \mathrm{Br}_1(U)$ which is uniquely defined up to an element in $\mathrm{Br}\, k$. Fix for the moment an interesting del Pezzo surface X. We will consider the class $\mathcal{A}_h$ on U_h as h varies over all hyperplane sections. We have seen that $\mathcal{A}_h$ is of order 5 if h cuts out a geometrically irreducible curve. The next lemma shows that this only fails for specific choices of h.

LEMMA 4.1.6. *Let $X \subseteq \mathbb{P}^5_k$ be an interesting del Pezzo surface over a field k. A hyperplane section given by the vanishing of an $h \in \mathrm{H}^0(\mathbb{P}^5_k, \mathcal{O}(1))$ fails to be geometrically irreducible if and only if h is a scalar multiple of either l_1 or l_2.*

Proof. Consider a hyperplane section $C \subseteq X$. Let D be a k-irreducible component of C and consider a geometric -1-curve L. It follows that $L \cdot \bar{D} = \sigma(L) \cdot \bar{D}$ and as the Galois orbit of L is an anticanonical divisor, we find

$$5 \geq -K_X \cdot D = 5L \cdot \bar{D} > 0,$$

since the degree of $D \subseteq \mathbb{P}^5_k$ is positive and at most the degree of C, which equals 5. This proves that $L \cdot \bar{D} = 1$ for all geometric -1-curves L and hence $C - D$ is an effective divisor of degree 0. We conclude that $C = D$ and this proves that any anticanonical section C is irreducible over k.

If C is not geometrically irreducible, then it must have at least two geometrically irreducible components of the same degree d since the Galois group acts on the set of geometrically irreducible components of C. Since C is of degree 5 we find $2d \leq 5$ and hence d is either 1 or 2. But in both cases we see that C contains a geometrically irreducible curve of degree 1, which must be a geometric -1-curve L. Then C also contains all conjugates of L and hence C is the Galois orbit of a geometric -1-curve. This proves that C is defined by the vanishing of either l_1 or l_2. □

4.2 Constructions of degree 5 del Pezzo surfaces

We have seen that the action of Galois on the -1-curves on an ordinary del Pezzo surface X over a field k determines many arithmetic properties of X. In this section we will describe the results in [29] which state that for ordinary del Pezzo surfaces of degree 5 a stronger result is true. Namely, there is a correspondence between isomorphism classes of ordinary del Pezzo surfaces X over k and conjugacy classes of group homomorphisms $G_k \to W_4$.

An isomorphism class of X over k induces an action of G_k on $\operatorname{Pic} \bar{X}$. Now let W be the minimal subgroup of the Weyl group W_4 such that the action of G_k on the geometric Picard group factors through W. This gives us a homomorphism from $G_k \to W \to W_4$. We will follow [29] in constructing an ordinary del Pezzo surface X of degree 5 over a field k starting from a homomorphism $G_k \to W_4$. Let us first show that W_4 is a well understood finite group.

LEMMA 4.2.1. *The group W_4 is isomorphic to S_5 and this isomorphism is unique up to conjugacy.*

Proof. We have seen in Proposition 2.3.5 that W_4 is isomorphic to the automorphism group of the intersection graph of the -1-classes of a generalized del Pezzo surface of degree 5. This graph is the Petersen graph shown in Figure II. The computation of the automorphism group of this graph is more easily done using the following interpretation. Consider the graph whose vertices are the ten subsets $\{a,b\} \in \{1,2,3,4,5\}$ with precisely two elements. The vertices $\{a,b\}$ and $\{c,d\}$ are connected precisely if the two sets are disjoint. This graph is easily seen to be isomorphic to the Petersen graph.

It is clear that the elements of S_5 induce different natural automorphisms on this graph. Also, from an automorphism of the graph we can recover a permutation of $\{1,2,3,4,5\}$; for example, the image of the element 1 is the unique element in the intersections of the images of $\{1,2\}$ and $\{1,3\}$.

The last statement follows from the fact that every automorphism of S_5 is inner. $\square$

Now fix an isomorphism $W_4 \cong S_5$. We have remarked that such an isomorphism is unique up to conjugation. For our applications this will mean that the actual choice of this isomorphism does not matter as long we consistently use the same one.

DEFINITION 4.2.2. Let k be a field. Let $\xi\colon G_k \to S_5$ be a group homomorphism and let $m \in k[T]$ be a monic and square free polynomial of degree 5 with roots α_i for $1 \le i \le 5$. We say that m is a *seed* for ξ if $\sigma \in G_k$ acts on $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ as $\xi(\sigma)$ acts on the set of indices $\{1,2,3,4,5\}$.

The following results shows that seeds always exist.

LEMMA 4.2.3. *In the notation of Definition 4.2.2, there is a seed $m \in k[T]$ for each group homomorphism $\xi\colon G_k \to S_5$.*

Proof. See Lemma 15 in [29]. $\square$

Note that whether m is a seed of ξ depends on the indexing of the roots of m. A different choice for the indices will produce a seed for an S_5-conjugate of ξ. This shows that we should consider m to be a seed of the S_5-conjugacy class of group homomorphisms $G_k \to S_5$.

Also note that we are actually interested in homomorphisms from G_k to W_4. Using an isomorphism between W_4 and S_5, which is unique up to conjugacy, we see can define a *seed* of a conjugacy class of homomorphisms $G_k \to W_4$.

We can use this terminology to produce ordinary del Pezzo surface X of degree 5.

Step 1. Suppose that a homomorphism $\xi\colon G_k \to S_5$ is given. Fix a seed m for ξ.
Step 2. Consider the projective plane $\mathbb{P}^2_k$ over k together with the five distinct points $P_i = (\alpha_i\colon \alpha_i^2\colon 1)$. Let $Q \subseteq k[x,y,z]$ be the subset of quintic homogeneous polynomials which are singular at each P_i.

LEMMA 4.2.4. *The k-linear subspace $Q \subseteq k[x,y,z]$ is of dimension* 6.

Proof. See Theorem 5 in [29]. $\square$

Step 3. Fix a k-basis $q_0, q_1, \ldots, q_5 \in Q$ and define a rational map $\vartheta\colon \mathbb{P}^2_k \dashrightarrow \mathbb{P}^5_k$ by $[x\colon y\colon z] \mapsto [q_i(x,y,z)]_i$. Let $X \subseteq \mathbb{P}^5_k$ be the closure of the image of ϑ.

PROPOSITION 4.2.5. *The scheme $X \subseteq \mathbb{P}^5_k$ over k constructed above is an ordinary del Pezzo surface of degree 5 over k which is anticanonically embedded. The morphism ϑ defines a birational morphism $\mathbb{P}^2_k \to X$ which we will also denote by ϑ.*

We will call X the ordinary del Pezzo surface of degree 5 over k associated to the seed m.

Proof. See Theorem 5 in [29]. □

We now have produced an ordinary del Pezzo surface of degree 5 over k from a homomorphism $G_k \to S_5$. If we on the other hand start off with an ordinary del Pezzo surface X of degree 5 over a field k we have seen that the action of G_k on $\operatorname{Pic} \bar{X}$ defines a homomorphism $G_k \to S_5$. We would like these maps to be inverses to each other. One sees that for this to be true one would have to consider isomorphism classes of ordinary del Pezzo surfaces, because isomorphic surfaces have isomorphic actions of Galois on the set of -1-curves. Also, seeds can be defined for S_5-conjugacy classes of group homomorphisms $G_k \to S_5$. The following proposition shows that up to these equivalences the maps constructed above are actually inverses to each other.

Recall that we have fixed an isomorphism between S_5 and the automorphism group of $\operatorname{Pic} \bar{X}$.

PROPOSITION 4.2.6. *Let k be a perfect field. The map from the set of isomorphism classes of ordinary del Pezzo surfaces of degree 5 over k to the set of conjugacy classes of homomorphisms $G_k \to S_5$ defined by sending an ordinary del Pezzo surface X over k to the action of G_k on the -1-curves is a bijection.*

The inverse is given by mapping a homomorphism $\xi \colon G_k \to S_5$ to the isomorphism class of the ordinary del Pezzo surface of degree 5 over k associated to a seed of ξ.

Proof. See Lemma 14 in [29]. □

We will now describe another construction of a del Pezzo surface X' using a seed m. In Proposition 4.2.9 we will see that X' is isomorphic to the del Pezzo surface X associated m. For now we will write X and X' to distinguish the two constructions. Again we consider the five distinct points $P_i = (\alpha_i \colon \alpha_i^2 \colon 1)$ on $\mathbb{P}^2_k$ associated to the seed m which lie on a unique conic $\Gamma_0 \subseteq \mathbb{P}^2_k$. Let B be the blow up of $\mathbb{P}^2_k$ in the Galois-invariant set $\{P_i\}$ and let Γ be the strict transform of Γ_0 along $B \to \mathbb{P}^2_k$. By Corollary 2.7.14 B is an ordinary del Pezzo surface of degree 4 over k. Also, $\Gamma \subseteq B$ is a -1-curve by Lemma 2.7.9. If we contract this -1-curve using Proposition 2.2.5 we obtain an ordinary del Pezzo surface X' of degree 5.

Let us make this more precise. We will consider a line $\Lambda_0 \subseteq \mathbb{P}^2_k$ which does not meet any of the points P_i and pull it back to the effective divisor $\Lambda \subseteq B$. A careful study of the proof of Castelnuovo's theorem used in Proposition 2.2.5 shows that the morphism $B \to X'$ which contracts $\Gamma \subseteq B$ is the morphism associated to the complete linear system of the line bundle $\Lambda + 2\Gamma$.

PROPOSITION 4.2.7. *The composition $B \xrightarrow{\beta} \mathbb{P}^2_k \overset{\vartheta}{\dashrightarrow} X \subseteq \mathbb{P}^5_k$ extends to a morphism on the whole of B. This morphism $\rho\colon B \to X \subseteq \mathbb{P}^5_k$ corresponds to the complete linear system of the divisor $\Delta = \Lambda + 2\Gamma$. Furthermore, the only curve contracted by this morphism is Γ.*

Proof. This proposition is proved in the proof of Theorem 5 of [29]. □

Note that Γ_0 is the conic defined by the equation $x^2 = yz$. We will also fix an equation for Λ_0. It is clear that the line $z = 0$ does not meet any of the points P_i. We can now prove the following corollary.

COROLLARY 4.2.8. *The morphism β is a birational morphism and induces an isomorphism $\beta^*\colon \kappa(\mathbb{P}^2_k) \to \kappa(B)$. This isomorphism identifies the following two linear subspaces*

$$\frac{1}{z(x^2-yz)^2} Q \subseteq \kappa(\mathbb{P}^2_k) \quad \textit{and} \quad \mathrm{H}^0(B, \mathcal{L}(\Delta)) \subseteq \kappa(B).$$

Proof. Define the divisor $\Delta_0 = \Lambda_0 + 2\Gamma_0$ on $\mathbb{P}^2_k$. We see that

$$\beta^*\Delta_0 = \Delta + 2E_\beta$$

where E_β is the sum of the five exceptional curves of the blowup morphism $\beta\colon B \to \mathbb{P}^2_k$. The spaces of global sections $\mathrm{H}^0(\mathbb{P}^2_k, \mathcal{L}(\Delta_0))$ and $\mathrm{H}^0(B, \mathcal{L}(\beta^*\Delta_0))$ embed naturally in the respective function fields $\kappa(\mathbb{P}^2_k)$ and $\kappa(B)$. The isomorphism β^* on these function fields respects these linear subspaces and we have a natural morphism

$$\beta^*\colon \mathrm{H}^0(\mathbb{P}^2_k, \mathcal{L}(\Delta_0)) \to \mathrm{H}^0(B, \mathcal{L}(\beta^*\Delta_0)).$$

By definition we have $Q \subseteq \mathrm{H}^0(\mathbb{P}^2_k, \mathcal{O}(5))$ and the isomorphism $\mathcal{O}(5) \to \mathcal{L}(\Delta_0)$ is uniquely defined up to multiplication by an invertible element of $\mathcal{O}_{\mathbb{P}^2_k}(\mathbb{P}^2_k) = k$. This implies that Q corresponds to the linear subsystem

$$\frac{1}{z(x^2-yz)^2} Q \subseteq \mathrm{H}^0(\mathbb{P}^2_k, \mathcal{L}(\Delta_0)) \subseteq \kappa(\mathbb{P}^2_k)$$

for any isomorphism $\mathcal{O}(5) \to \mathcal{L}(\Delta_0)$. The morphisms ϑ and ρ are defined by the respective linear systems

$$\frac{1}{z(x^2-yz)^2} Q \subseteq \mathrm{H}^0(\mathbb{P}^2_k, \mathcal{L}(\Delta_0)) \quad \text{and} \quad \mathrm{H}^0(B, \mathcal{L}(\Delta)) \subseteq \mathrm{H}^0(B, \mathcal{L}(\Delta) \otimes \mathcal{L}(2E_\beta)).$$

$$\begin{tikzcd} & B \arrow[dl, "\rho"'] \arrow[dd, "\beta"] \\ \mathbb{P}^5_k & \\ & \mathbb{P}^2_k \arrow[ul, dashed, "\vartheta"] \end{tikzcd} \tag{4.1}$$

From the commutative diagram in (4.1) we see that these linear systems are identified under the isomorphism

$$\beta^*\colon \mathrm{H}^0(\mathbb{P}^2_k, \mathcal{L}(\Delta_0)) \to \mathrm{H}^0(B, \mathcal{L}(\Delta) \otimes \mathcal{L}(2E_\beta)). \qquad \square$$

PROPOSITION 4.2.9. *Consider the surfaces X and X' over k constructed from the same seed m. They are isomorphic over k.*

These surfaces are isomorphic as subschemes of $\mathbb{P}^5_k$ up to an automorphism of $\mathbb{P}^5_k$.

Proof. The schemes X and X' are the respective closures of the images of the maps $\mathbb{P}^2_k \dashrightarrow \mathbb{P}^5_k$ and $B \to \mathbb{P}^5_k$ defined by choosing bases of Q and $\mathrm{H}^0(B, \mathcal{L}(\Delta))$. If we pick bases corresponding to each other using the isomorphism in Corollary 4.2.8 then X and X' are the same subscheme of $\mathbb{P}^5_k$. If we make a different choice of basis of Q then we recover an isomorphic k-scheme X, but the embedding $X \subseteq \mathbb{P}^5_k$ changes by an automorphism of $\mathbb{P}^5_k$. The same holds for X' and the result follows. $\square$

Let us turn our attention back to interesting del Pezzo surfaces over k. We can use the correspondence in Proposition 4.2.6 to classify interesting del Pezzo surfaces over a field k.

PROPOSITION 4.2.10. *Let k be a perfect field and fix an algebraic closure $\bar{k}$. The map which sends an isomorphism class of interesting del Pezzo surfaces over k to its splitting field $K \subseteq \bar{k}$ is a bijection to the set of degree 5 Galois extensions of k contained in $\bar{k}$.*

Proof. Let us first show that we have a correspondence between degree 5 Galois extensions $K \subseteq \bar{k}$ and conjugacy classes of homomorphisms $G_k \to S_5$ whose image is of order 5.

To any homomorphism $\xi\colon G_k \to S_5$ whose image is of order 5, we can consider the kernel which corresponds to a Galois extensions K/k of degree 5.

Now consider a Galois extension K/k of degree 5 and let $G_K \subseteq G_k$ be the corresponding subgroup of index 5. We can construct a group homomorphism $G_k \to G_k/G_K \to S_5$, by sending a generator of G_k/G_K to any element $s \in S_5$ of order 5. Since every element of order 5 in S_5 is of the same cycle type, a different choice of $s \in S_5$ produces a conjugate homomorphism $G_k \to S_5$. So we have assigned a conjugacy class of homomorphisms $G_k \to S_5$ with an image of order 5 to a degree 5 Galois extension K/k.

It is easily checked that both procedures are inverse to each other.

Let us now prove the proposition by constructing an inverse to the described map. So again, let K be a degree 5 Galois extension of k and let $\xi\colon G_k \to S_5$ be a homomorphism in the corresponding conjugacy class of homomorphisms as above. By Proposition 4.2.6 we can produce a del Pezzo surface X by choosing a seed m of ξ, whose action on the -1-curves is uniquely determined by ξ. By construction of ξ we see that the action of G_k on $\operatorname{Pic} \bar{X}$ is non-trivial, but the corresponding action of G_K on $\operatorname{Pic} \bar{X}$ is trivial. By definition we see that X is an interesting del Pezzo surface.

It is clear that these maps define a correspondence between interesting del Pezzo surfaces over k and degree 5 Galois extensions of k. □

DEFINITION 4.2.11. Let K/k be a Galois extension of degree 5. The isomorphism class of interesting del Pezzo surfaces of degree 5 over k which are split by K is denoted by $\mathrm{dP}_5(K)$.

So the isomorphism class of an interesting del Pezzo surface over a field k is uniquely determined by the splitting field K. We have also seen two constructions for del Pezzo surfaces in such an isomorphism class. We will be interested in how the geometric -1-curves can be found on such a surface.

PROPOSITION 4.2.12. *Let m be a seed for a general homomorphism $\xi\colon G_k \to S_k$. Consider the five points $P_i = (\alpha_i\colon \alpha_i^2\colon 1)$ constructed from the roots of m, and let $\vartheta\colon \mathbb{P}^2_k \dashrightarrow X$ be the birational map from Proposition 4.2.5. Define $L_{ij} \subseteq \mathbb{P}^2_{\bar{k}}$ as the line through P_i and P_j for $i \neq j$.*

The strict transform of L_{ij} along ϑ is a -1-curve on $\bar{X}$.

Proof. See Theorem 5 in [29]. □

Note that this allows us to produce all 10 curves on $\bar{X}$ of self-intersection -1. For interesting del Pezzo surfaces this implies the following result.

PROPOSITION 4.2.13. *Let X be an interesting del Pezzo surface of degree 5 over a field k. Consider the divisors*

$$\mathcal{L}_1 = \sum_{i=1}^{5} L_{i,i+1} \quad \textit{and} \quad \mathcal{L}_2 = \sum_{i=1}^{5} L_{i,i+2}$$

on $\mathbb{P}^2_k$ where the indices are considered modulo 5. These are quintic plane curves singular at the P_i, so they correspond to hyperplane sections of $X \subseteq \mathbb{P}^5_k$. These are precisely the anticanonical sections given by l_1 and l_2.

Proof. The group G_k permutes the points P_i cyclically by definition. We see that $\mathcal{L}_1$ and $\mathcal{L}_2$ are defined over k. This means that each divisor pulls back to a divisor supported on a Galois-invariant set of five -1-curves on $\bar{X}$. The only such sets are cut out by l_1 and l_2. □

4.3 Models of interesting del Pezzo surfaces

Let K be a cyclic number field of degree 5 and let $X \subseteq \mathbb{P}^5_{\mathbb{Q}}$ be the anticanonical image of the del Pezzo surface over $\mathbb{Q}$ of degree 5 associated to the field extension $K/\mathbb{Q}$. We will construct a model $\mathcal{X} \subseteq \mathbb{P}^5_{\mathbb{Z}}$ such that the generic fibre $\mathcal{X}_{\mathbb{Q}} \subseteq \mathbb{P}^5_{\mathbb{Q}}$ is isomorphic to X.

Note that although X only depends on the degree 5 extension K the model $\mathcal{X}$ will definitely depend on an explicit generator $\alpha \in K$. We will start by fixing an element $\alpha \in K$ such that $\mathbb{Q}(\alpha) = K$ and we will then give two constructions for $\mathcal{X}$. The first construction will be very useful in determining explicit equations for $\mathcal{X}$ as a subscheme of $\mathbb{P}^5_{\mathbb{Z}}$; the second is of a more geometrical nature and makes it easier to understand the fibres $X = \mathcal{X}_{\mathbb{Q}}$ and $\mathcal{X}_\ell$ for a prime ℓ.

4.3.1 Explicit equations for $\mathcal{X}_\alpha$

As said above we can construct a model $\mathcal{X}/\mathbb{Z}$ of X over $\mathbb{Q}$ for each $\alpha \in K$. To simplify the construction we will assume that α is integral.

Step 1. Start with a cyclic extension $K/\mathbb{Q}$ of degree 5 and fix a generator of $\mathrm{Gal}(K/\mathbb{Q})$. Next choose an element $\alpha \in \mathcal{O}_K$ such that $\mathbb{Q}(\alpha) = K$ and let α_i denote the conjugates of α. We will write m_α and m_{α^2} for the minimal polynomials of α and α^2 over $\mathbb{Z}$. Let $Z \subseteq \mathbb{P}^2_{\mathbb{Q}}$ be the reduced subscheme defined by the homogeneous ideal

$$(x^2 - yz, z^5 m_\alpha(x/z), z^5 m_{\alpha^2}(y/z)) \subseteq \mathbb{Q}[x,y,z].$$

This subscheme is zero-dimensional and supported in the five distinct points $P_i = (\alpha_i \colon \alpha_i^2 \colon 1)$ on the projective plane defined over K.

Step 2. Let $\mathcal{Q} \subseteq \mathbb{Z}[x,y,z]$ be the subset consisting of all quintic polynomials $q \in \mathbb{Z}[x,y,z]$ such that the associated degree 5 curve $C \subseteq \mathbb{P}^2_{\mathbb{Q}}$ is singular at the five points P_i, i.e. C_K has multiplicity at least 2 at each point P_i.

Note that $\mathcal{Q}$ is the intersection of $\mathbb{Z}[x,y,z]$ and Q in $\mathbb{Q}[x,y,z]$.

LEMMA 4.3.1. *The subset $\mathcal{Q} \subseteq \mathbb{Z}[x,y,z]$ is a free $\mathbb{Z}$-module of rank 6 and $\mathbb{Z}[x,y,z]/\mathcal{Q}$ is torsion free.*

Proof. Since $\mathbb{Z}[x,y,z]$ is a free $\mathbb{Z}$-module and $\mathbb{Z}$ is a principal ideal domain, we find that the sub-$\mathbb{Z}$-module $\mathcal{Q}$ is also a free $\mathbb{Z}$-module. We have seen in Lemma 4.2.4 that $\mathcal{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ is a 6-dimensional vector space over $\mathbb{Q}$ so $\mathcal{Q}$ is a free of rank 6 over $\mathbb{Z}$.

Now one has to check that if $nq \in \mathcal{Q}$ for an integer $n \neq 0$ and $q \in \mathbb{Z}[x,y,z]$, then $q \in \mathcal{Q}$. Since $nq \in \mathcal{Q}$ and $n \neq 0$ we see that q is a quintic polynomial. It is clear that q is singular at the points P_i precisely when nq has this property. $\square$

Step 3. Fix a basis $q_0, q_1, \ldots, q_5 \in \mathcal{Q}$ and define the rational map $\vartheta \colon \mathbb{P}^2_{\mathbb{Z}} \dashrightarrow \mathbb{P}^5_{\mathbb{Z}}$ by $[x \colon y \colon z] \mapsto [q_i(x,y,z)]_i$.

DEFINITION 4.3.2. The closure of the image of the rational map $\vartheta \colon \mathbb{P}^2_{\mathbb{Z}} \dashrightarrow \mathbb{P}^5_{\mathbb{Z}}$ defined by $q_0, q_1, \ldots, q_5$ is denoted by $\mathcal{X}_\alpha$ or simply $\mathcal{X}$ if no confusion is possible. Denote the generic fibre of $\mathcal{X}_\alpha$ by X_α or X.

Note that as a $\mathbb{Z}$-scheme $\mathcal{X}$ only depends on α. As a subscheme of $\mathbb{P}^5_{\mathbb{Z}}$ it does depend on the choice of basis of $\mathcal{Q}$. This construction does however show that for a given α the closed embedding $\mathcal{X} \subseteq \mathbb{P}^5_{\mathbb{Z}}$ is unique up to an automorphism of $\mathbb{P}^5_{\mathbb{Z}}$.

The following theorem tells us that this construction does what we want.

THEOREM 4.3.3. *The scheme $\mathcal{X}_\alpha \subseteq \mathbb{P}^5_{\mathbb{Z}}$ is integral. The generic fibre of $\mathcal{X}/\mathbb{Z}$ is isomorphic to* $\mathrm{dP}_5(K)$ *and every fibre of $\mathcal{X} \subseteq \mathbb{P}^5_{\mathbb{Z}}$ over a finite prime ℓ of $\mathbb{Z}$ is a singular del Pezzo surface of degree* 5 *anticanonically embedded. A fibre $\mathcal{X}_\ell$ over a prime ℓ is an ordinary del Pezzo surface precisely when the reduction of Z modulo ℓ is reduced.*

The above construction is convenient for computing explicit equations defining $\mathcal{X}$ as a subscheme of $\mathbb{P}^5_{\mathbb{Z}}$. The MAGMA code which takes $\alpha \in K$ as an input and computes the equations for $\mathcal{X}$ can be found at [39]. The computation uses the following result which follows from the theorem.

COROLLARY 4.3.4. *Consider the generic fibre $X \subseteq \mathbb{P}^5_{\mathbb{Q}}$ of $\mathcal{X} \subseteq \mathbb{P}^5_{\mathbb{Z}}$. The scheme $\mathcal{X}$ is the flat closure of X in $\mathbb{P}^5_{\mathbb{Z}}$.*

Proof. It is clear that $\mathcal{X}$ is a closed subscheme of $\mathbb{P}^5_{\mathbb{Z}}$ which contains X. The result follows from the fact that $\mathcal{X}$ is an integral scheme. The closure of a subscheme of the generic fibre is always flat by [33, Proposition III.9.7]. □

To prove Theorem 4.3.3 we will consider a different construction of $\mathcal{X} \subseteq \mathbb{P}^5_{\mathbb{Z}}$. We will prove that both constructions coincide while studying the fibres $\mathcal{X}_{\alpha,\ell}$ of $\mathcal{X}_\alpha$.

4.3.2 Geometric construction of $\mathcal{X}_\alpha$

The construction of $\mathcal{X}_\alpha$ using the quintic polynomials is reminiscent of the construction we have seen for X_α. Now consider the other construction of X_α; start with five distinct points on the projective plane. Blow up these points and consider the strict transform of the conic through the five points. This conic turns into a -1-curve on the blowup. Now contract this -1-curve.

We will repeat this construction on the projective plane over the integers.

DEFINITION 4.3.5. Define Γ_0 and Λ_0 as the subschemes of $\mathbb{P}^2_{\mathbb{Z}}$ defined by the respective equations $x^2 - yz = 0$ and $z = 0$. Let $\mathcal{Z} \subseteq \mathbb{P}^2_{\mathbb{Z}}$ be the closure of the zero-dimensional scheme $Z \subseteq \mathbb{P}^2_{\mathbb{Q}} \subseteq \mathbb{P}^2_{\mathbb{Z}}$.

Since we assumed that α is integral one can show that $\mathcal{Z}$ is actually defined by the ideal

$$(x^2 - yz, z^5 m_\alpha(x/z), z^5 m_{\alpha^2}(y/z)) \subseteq \mathbb{Z}[x, y, z].$$

If one thinks of $\mathbb{P}^2_{\mathbb{Z}}$ as a family of projective planes over $\mathbb{F}_\ell$ and $\mathbb{Q}$, then Λ_0 and Γ_0 are respectively a line and a conic on each fibre. In this setup $\mathcal{Z}$ is a family over $\mathbb{Z}$ of five relative points on Γ_0. These five points on the fibre $\mathbb{P}^2_{\mathbb{F}_\ell}$ over ℓ are the points P_i on $\mathbb{P}^2_{\mathbb{Q}}$ reduced modulo a prime $\mathfrak{l}$ of K which divides ℓ. It is possible that two points P_i and P_j reduce to the same point modulo $\mathfrak{l}$. This information is captured by $\mathcal{Z}_\ell$ which is a zero-dimensional scheme of degree 5 for all ℓ. The fact that $\Gamma_{0,\ell}$ passes through $\mathcal{Z}_\ell$ helps to identify $\mathcal{Z}_\ell$ in the following way: fix a prime ℓ and mark each point $Q \in \Gamma_{0,\ell}(\mathbb{F}_\mathfrak{l}) \subseteq \mathbb{P}^2_{\mathbb{F}_\ell}(\mathbb{F}_\mathfrak{l})$ with the $n(Q)$ number of points $P_i \in \mathbb{P}^2_{\mathbb{Q}}$ which reduce to Q. Then $\mathcal{Z}_\ell$ is the unique zero-dimensional degree 5 subscheme of $\Gamma_{0,\ell}$ whose degree at each point $Q \in \Gamma_{0,\ell}(\mathbb{F}_l)$ equals precisely $n(Q)$.

PROPOSITION 4.3.6. *The subscheme $\mathcal{Z}$ lies on Γ_0. The subscheme Λ_0 does not meet $\mathcal{Z}$. The fibre $\mathcal{Z}_\ell$ over a finite prime ℓ is a zero-dimensional subscheme of $\mathbb{P}^2_{\mathbb{F}_\ell}$ of degree* 5.

Proof. We know that Z is a subscheme of Γ_0 and this implies that $\mathcal{Z}$ also lies on Γ_0. The second statement uses the defining equations for Λ_0 and $\mathcal{Z}$, and the fact that α is an integral element of K.

We see, for example from Proposition III.9.7 in [33], that $\mathcal{Z} \to \mathbb{Z}$ is a flat morphism. This proves that the Hilbert Polynomial P_ℓ of the fibre $\mathcal{Z}_\ell$ over a prime $\ell \in \mathbb{Z}$ is independent of ℓ [33, Theorem III.9.9]. We know that over the generic point $(0) \in \operatorname{Spec} \mathbb{Z}$ the fibre $Z = \mathcal{Z}_{\mathbb{Q}}$ is zero-dimensional of degree 5 and this proves that the result is true for all fibres. □

We see that on each fibre $\mathbb{P}^2_{\mathbb{F}_\ell}$ we have a zero-dimensional subscheme $\mathcal{Z}_\ell$ which lies on the conic $\Gamma_{0,\ell}$. Because the intersection number between a line L and the conic $\Gamma_{0,\ell}$ equals 2 we conclude from Definition 2.7.2 that $\mathcal{Z}_\ell \subseteq \mathbb{P}^2_{\mathbb{F}_\ell}$ lies in almost general position. We will consider the peculiar del Pezzo surface we get by blowing up $\mathcal{Z}_\ell$.

DEFINITION 4.3.7. Let $\mathcal{B}$ be the blowup $\operatorname{Bl}_{\mathcal{Z}} \mathbb{P}^2_{\mathbb{Z}}$. Write Λ and Γ for the strict transforms of Λ_0 and Γ_0 along the blowup $\beta\colon \mathcal{B} \to \mathbb{P}^2_{\mathbb{Z}}$. Define the divisor $\Delta = \Lambda + 2\Gamma$ on $\mathcal{B}$.

PROPOSITION 4.3.8. *Let ℓ be a rational prime. The fibre $\mathcal{B}_\ell$ is integral and isomorphic to the blowup of $\mathbb{P}^2_{\mathbb{F}_\ell}$ in $\mathcal{Z}_\ell$.*

For the generic fibre we have a similar statement which follows from the fact that blowing up commutes with flat base change. This shows that the blowup $\beta_{\mathbb{Q}}$ on the generic fibre $B \to \mathbb{P}^2_{\mathbb{Z}}$, where $B = \mathcal{B}_{\mathbb{Q}}$ is the generic fibre of $\mathcal{B}$, is the blowup in the zero-dimensional subscheme Z.

Proof of Proposition 4.3.8. Let us first deduce some preliminary results. Let $\mathcal{E} \subseteq \mathcal{B}$ be the exceptional divisor of the blowup $\mathcal{B} \to \mathbb{P}^2_{\mathbb{Z}}$. Since $\mathbb{P}^2_{\mathbb{Z}}$ is a Cohen–Macaulay scheme and $\mathcal{Z} \subseteq \mathbb{P}^2_{\mathbb{Z}}$ is locally defined by a regular embedding we deduce that $\mathcal{E} \to \mathcal{Z}$ is locally a $\mathbb{P}^1$-bundle. Since $\mathcal{Z}$ is faithfully flat over $\operatorname{Spec} \mathbb{Z}$ we see that the same is true for $\mathcal{E}$. These results also allow us to prove that both $\mathcal{Z}$ and $\mathcal{E}$ are Cohen–Macaulay schemes. Following the proof of Theorem II.8.24 in [33] we conclude that $\mathcal{B}$ is Cohen–Macaulay as well.

Define $F = \mathbb{P}^2_{\mathbb{F}_\ell}$ to be the fibre of $\mathbb{P}^2_{\mathbb{Z}}$ over ℓ. The blowup $\operatorname{Bl}_{\mathcal{Z}_\ell} F$ is the scheme-theoretic closure of $F \backslash \mathcal{Z}_\ell$ in $\mathcal{B}_\ell$ by [24, Proposition IV-21]. We will first show that $\mathcal{B}_\ell$ is irreducible. This proves that $\operatorname{Bl}_{\mathcal{Z}_\ell} F$ is isomorphic to the topological closure of $F \backslash \mathcal{Z}_\ell$ in $\mathcal{B}_\ell$ with its reduced scheme structure, since F is reduced. Then we continue by proving that $\mathcal{B}_\ell$ is also reduced and we conclude that the closed embedding $\operatorname{Bl}_{\mathcal{Z}_\ell} F \to \mathcal{B}_\ell$ is actually an isomorphism.

So we will first prove that $\mathcal{B}_\ell$ is integral. By Krulls Hauptidealsatz we see that every irreducible component of $\mathcal{B}_\ell$ is of codimension 1 in $\mathcal{B}$, and the same holds for irreducible components of $\mathcal{E}_\ell$ in $\mathcal{E}$. Since $\mathcal{E}$ is of pure codimension 1 in $\mathcal{B}$, we see that $\mathcal{E}_\ell$ is of pure codimension 2 in $\mathcal{B}$. This proves that the generic point of an irreducible component of $\mathcal{B}_\ell$ does not lie in $\mathcal{E}_\ell$ and since $\mathcal{B}_\ell \backslash \mathcal{E}_\ell \cong F \backslash \mathcal{Z}_\ell$ is irreducible we conclude that $\mathcal{B}_\ell$ is irreducible.

To prove that $\mathcal{B}_\ell$ is also reduced we first note that it is Cohen–Macaulay since it is a Cartier divisor on the Cohen–Macaulay scheme $\mathcal{B}$. In particular $\mathcal{B}_\ell$ satisfies Serre's condition S_1 and we see that being reduced is equivalent to being generically reduced. The result now follows from the fact that $\mathcal{B}_\ell$ is birational to the reduced scheme F. □

This shows that the each $\mathcal{B}_\ell$ is a peculiar del Pezzo surface over $\mathbb{F}_\ell$. Let $\rho_\ell \colon \tilde{\mathcal{B}}_\ell \to \mathcal{B}_\ell$ be the associated generalized del Pezzo surface, i.e. the minimal desingularization of $\mathcal{B}_\ell$. We will show that actually all -2-curves on $\tilde{\mathcal{B}}_\ell$ are contracted by ρ_ℓ. This proves that $\mathcal{B}_\ell$ is even a singular del Pezzo surface over $\mathbb{F}_\ell$. For all but a finite number of primes ℓ the fibre $\mathcal{B}_\ell$ will even be an ordinary del Pezzo surface. In this case the morphism ρ_ℓ will actually be an isomorphism and we can identify $\tilde{\mathcal{B}}_\ell$ and $\mathcal{B}_\ell$. This will in particular be the case for the generic fibre $\mathcal{B}_\mathbb{Q}$.

We will also look into the divisor Γ_ℓ on $\mathcal{B}_\ell$ by computing the self-intersection of the strict transform of $\tilde{\Gamma}_\ell$ on $\tilde{\mathcal{B}}_\ell$.

COROLLARY 4.3.9. *The fibres of $\mathcal{B}/\mathbb{Z}$ are singular del Pezzo surfaces of degree* 4*. The strict transform $\tilde{\Gamma}_\ell$ of Γ_ℓ is a -1-curve on the minimal desingularization $\tilde{\mathcal{B}}_\ell$ of $\mathcal{B}_\ell$ and so is $\Gamma_\mathbb{Q}$ on $\mathcal{B}_\mathbb{Q}$.*

As mentioned above, we will show that actually all but finitely many fibres of $\mathcal{B} \to \operatorname{Spec} \mathbb{Z}$ are ordinary del Pezzo surfaces.

Proof. We will prove the corollary for fibres over finite primes ℓ. The statements over the generic fibre $(0) \in \operatorname{Spec} \mathbb{Z}$ are similar or even simpler.

By Proposition 4.3.8 we see that $\mathcal{B}_\ell$ is isomorphic to $\mathrm{Bl}_{\mathcal{Z}_\ell}(\mathbb{P}^2_{\mathbb{F}_\ell})$. We know that $\mathcal{Z}_\ell$ lies on the conic $\Gamma_{0,\ell} \subseteq \mathbb{P}^2_{\mathbb{F}_\ell}$. It follows from Proposition 2.7.15 that $\mathrm{Bl}_{\mathcal{Z}_\ell}(\mathbb{P}^2_{\mathbb{F}_\ell})$ is a singular del Pezzo surface.

Now consider $\Gamma_{0,\ell} \subseteq \mathbb{P}^2_{\mathbb{F}_\ell}$. We know that $\deg(\Gamma_{0,\ell} \cap \mathcal{Z}_\ell) = \deg \mathcal{Z}_\ell = 5$, so the strict transform $\tilde{\Gamma}_\ell \subseteq \tilde{\mathcal{B}}_\ell$ of the conic $\Gamma_{0,\ell}$ has self-intersection -1 by Lemma 2.7.9. □

We would like to contract Γ_ℓ. Of course we can contract $\tilde{\Gamma}_\ell$ on $\tilde{\mathcal{B}}$, but this only descends to $\mathcal{B}_\ell$ if $\tilde{\Gamma}_\ell$ does not meet any -2-curves on $\tilde{\mathcal{B}}_\ell$.

LEMMA 4.3.10. *The divisor Γ_ℓ does not pass through any singular points on $\mathcal{B}_\ell$.*

Proof. We will prove that on the minimal desingularization $\tilde{\mathcal{B}}_\ell$ the -1-curve $\tilde{\Gamma}_\ell$ does not meet any -2-curves.

Fix a basis $L_0, \ldots, L_5$ as in Proposition 2.3.1 for the geometric Picard group of the generalized del Pezzo surface $\tilde{\mathcal{B}}_\ell$. Note that by Corollary 4.3.9 all -2-curves on $\tilde{\mathcal{B}}_\ell$ are contracted by $\tilde{\beta}\colon \tilde{\mathcal{B}}_\ell \to \mathcal{B}_\ell \to \mathbb{P}^2_{\mathbb{Z}}$. So any -2-curve lies in the support of the exceptional divisor $E_{\tilde{\beta}}$ and we conclude that it must be linear equivalent to a divisor of the form $L_i - L_j$ for distinct and positive i and j. The class of $\tilde{\Gamma}_\ell$ is $2L_0 - L_1 - L_2 - L_3 - L_4 - L_5$ which proves that the intersection number of $\tilde{\Gamma}_\ell$ with any -2-curve is zero. $\square$

We will now consider the image of the map to a projective space associated to the divisor $\Delta = \Lambda + 2\Gamma$ on $\mathcal{B}$.

PROPOSITION 4.3.11. *The $\mathbb{Z}$-module $\mathrm{H}^0(\mathcal{B}, \Delta)$ is free of rank 6. The composition*

$$\delta\colon \mathcal{B} \to \mathbb{P}^2_{\mathbb{Z}} \overset{\vartheta}{\dashrightarrow} \mathcal{X}$$

is a map associated to the complete linear system of the divisor Δ.

The map δ extends to a birational morphism $\mathcal{B} \to \mathcal{X}$. Furthermore, δ is an isomorphism on an open dense subset of each fibre of $\mathcal{B}$ over $\mathbb{Z}$.

We will prove this proposition in the next section, together with Theorem 4.3.3.

4.3.3 Comparison of the two constructions

We have seen two ways to construct $\mathcal{X}$; as the scheme-theoretic closure of the image of $\vartheta\colon \mathbb{P}^2_{\mathbb{Z}} \dashrightarrow \mathbb{P}^5_{\mathbb{Z}}$ and as the scheme-theoretic closure of the image of a map associated to the complete linear system of the divisor Δ on $\mathcal{B}$. In this section we will prove Theorem 4.3.3 and Proposition 4.3.11. We will first prove the following proposition which is the integral version of Corollary 4.2.8.

PROPOSITION 4.3.12. *Consider the blowup morphism $\beta\colon \mathcal{B} \to \mathbb{P}^2_{\mathbb{Z}}$. The pullback morphism β^* is an isomorphism on the function fields $\kappa(\mathbb{P}^2_{\mathbb{Z}}) \to \kappa(\mathcal{B})$. This morphism induces an isomorphism on the sub-$\mathbb{Z}$-modules*

$$\frac{1}{z(x^2 - yz)^2}\mathcal{Q} \subseteq \kappa(\mathbb{P}^2_{\mathbb{Z}}) \quad \textit{and} \quad \mathrm{H}^0(\mathcal{B}, \mathcal{L}(\Delta)) \subseteq \kappa(\mathcal{B}).$$

Proof. The inclusion of the generic fibre $\mathbb{P}^2_{\mathbb{Q}} \to \mathbb{P}^2_{\mathbb{Z}}$ induces an isomorphism $\kappa(\mathbb{P}^2_{\mathbb{Q}}) \to \kappa(\mathbb{P}^2_{\mathbb{Z}})$. We will identify these two fields along this isomorphism. In the same way we will identify $\kappa(B)$ and $\kappa(\mathcal{B})$. Since β is a birational morphism of integral schemes it does indeed induce an isomorphism $\kappa(\mathbb{P}^2_{\mathbb{Z}}) \to \kappa(\mathcal{B})$ and we have seen in Corollary 4.2.8 that under this isomorphism we can identify

$$\frac{1}{z(x^2 - yz)^2}Q \subseteq \kappa(\mathbb{P}^2_{\mathbb{Q}}) \quad \text{and} \quad \mathrm{H}^0(B, \mathcal{L}(\Delta_{\mathbb{Q}})) \subseteq \kappa(B).$$

Since $\beta_\ell\colon \mathcal{B}_\ell \to \mathbb{P}^2_{\mathbb{F}_\ell}$ is a birational morphism of integral varieties over $\mathbb{F}_\ell$ by Proposition 4.3.8 we see that β induces an isomorphism of the local rings $\mathcal{O}_{\mathcal{B},B_\ell}$

and $\mathcal{O}_{\mathbb{P}^2_{\mathbb{Z}},\mathbb{P}^2_{\mathbb{F}_\ell}}$ which proves that $\beta^*\colon \kappa(\mathbb{P}^2_{\mathbb{Z}}) \to \kappa(\mathcal{B})$ preserves the valuations along the prime divisors $\mathcal{B}_\ell$ on $\mathcal{B}$ and $\mathbb{P}^2_{\mathbb{F}_\ell}$ on $\mathbb{P}^2_{\mathbb{Z}}$.

This implies that β^* identifies the submodule of $\frac{1}{z(x^2-yz)^2}Q$ consisting of the elements with a non-negative valuation along $\mathbb{P}^2_{\mathbb{F}_\ell}$ for all primes ℓ with the submodule of $\mathrm{H}^0(B, \mathcal{L}(\Delta_{\mathbb{Q}}))$ with a no poles along $\mathcal{B}_\ell$ for all ℓ. Let us prove these submodules are precisely $\frac{1}{z(x^2-yz)^2}\mathcal{Q}$ and $\mathrm{H}^0(B, \mathcal{L}(\Delta))$.

By definition we see that $\mathrm{H}^0(\mathcal{B}, \mathcal{L}(\Delta)) \subseteq \mathrm{H}^0(B, \mathcal{L}(\Delta_{\mathbb{Q}})) \subseteq \kappa(\mathcal{B})$ after identifying the function fields of $\mathcal{B}$ and its generic fibre B. This identifies the prime divisors of B with the horizontal prime divisors of $\mathcal{B}$ and we see that $\mathrm{H}^0(\mathcal{B}, \mathcal{L}(\Delta))$ consists precisely of the elements in $\mathrm{H}^0(B, \mathcal{L}(\Delta_{\mathbb{Q}}))$ with a non-negative valuation along the vertical prime divisors $\mathcal{B}_\ell$.

We also have $\frac{1}{z(x^2-yz)^2}\mathcal{Q} \subseteq \frac{1}{z(x^2-yz)^2}Q \subseteq \kappa(\mathbb{P}^2_{\mathbb{Z}})$ using the identification of $\kappa(\mathbb{P}^2_{\mathbb{Z}})$ with $\kappa(\mathbb{P}^2_{\mathbb{Q}})$. Let $\frac{q}{z(x^2-yz)^2}$ be a quotient of quintics in $\mathbb{Q}[x, y, z]$, it can be written as $c\frac{q'}{z(x^2-yz)^2}$ with $c \in \mathbb{Q}$ and $q' \in \mathbb{Z}[x, y, z]$ a quintic polynomial with coprime coefficients. The valuation along $\mathbb{P}^2_{\mathbb{F}_\ell}$ is now simply the valuation of c at ℓ. This means that a $\frac{q}{z(x^2-yz)^2}$ has non-negative valuation along all vertical fibres if and only if q actually has integral coefficients. $\square$

This allows us to prove Proposition 4.3.11.

Proof of Proposition 4.3.11. The scheme $\mathcal{X}_\alpha \subseteq \mathbb{P}^5_{\mathbb{Z}}$ is defined as the image of a rational map $\vartheta\colon \mathbb{P}^2_{\mathbb{Z}} \dashrightarrow \mathbb{P}^5_{\mathbb{Z}}$ defined by the chosen basis elements $q_i \in \mathcal{Q}$. By definition we have $\delta = \vartheta \circ \beta$ and using the identification in Proposition 4.3.12 we see that δ is defined using the basis of $\mathrm{H}^0(\mathcal{B}, \mathcal{L}(\Delta))$ corresponding to the chosen basis of $\mathcal{Q}$. By functoriality of maps to projective spaces coming from divisors [24, Theorem III-37], we see that the morphism $\mathcal{B}_\ell \to \mathbb{P}^5_{\mathbb{F}_\ell}$ coming from the divisor Δ_ℓ is simply the fibre of $\delta\colon \mathcal{B} \to \mathbb{P}^5_{\mathbb{Z}}$.

Let $\gamma_\ell\colon \tilde{\mathcal{B}}_\ell \to \mathcal{B}_\ell$ be the minimal desingularization of $\mathcal{B}_\ell$ and define $\tilde{\Gamma}_\ell$, $\tilde{\Lambda}_\ell$ and $\tilde{\Delta}_\ell$ to be the pullbacks of Γ_ℓ, Λ_ℓ and Δ_ℓ along γ_ℓ. By Lemma 2.5.1 we have an isomorphism between the global sections of $\mathcal{L}(\Delta_\ell)$ and $\mathcal{L}(\tilde{\Delta}_\ell)$. This proves that the composition $\delta'_\ell = \delta_\ell \circ \gamma_\ell$ is the map associated to the complete linear system of $\tilde{\Delta}_\ell$. We will first prove that δ'_ℓ is actually a morphism, i.e. the linear system of $\tilde{\Delta}_\ell$ is base point free.

Note that $\Lambda_{0,\ell} + \Gamma_{0,\ell} \subseteq \mathbb{P}^2_{\mathbb{F}_\ell}$ is a cubic plane curve passing through $\mathcal{Z}_\ell$ and this curve is smooth in the support of $\mathcal{Z}_\ell$. Proposition 2.5.4 shows that $\Lambda_\ell + \Gamma_\ell$ is an anticanonical divisor on $\mathcal{B}_\ell$ and $\tilde{\Lambda}_\ell + \tilde{\Gamma}_\ell$ is an anticanonical divisor on $\tilde{\mathcal{B}}_\ell$. Since $\tilde{\mathcal{B}}_\ell$ is a generalized del Pezzo surface of degree 4 we see that the complete linear system of $\tilde{\Lambda}_\ell + \tilde{\Gamma}_\ell$ defines a birational morphism which contracts the -2-curves on $\tilde{\mathcal{B}}_\ell$. In particular, we see that $\mathcal{L}(\tilde{\Lambda}_\ell + \tilde{\Gamma}_\ell)$ is globally generated. This proves that $\mathcal{L}(\tilde{\Lambda}_\ell + \tilde{\Gamma}_\ell + \tilde{\Gamma}_\ell)$ is globally generated away from Γ_ℓ. To see that $\mathcal{L}(\tilde{\Delta}_\ell)$ is globally generated on Γ_ℓ one can proceed as in Step 2 of the proof of

Theorem 5.7 in [33]. For this one needs $\mathrm{H}^1(\tilde{\mathcal{B}}_\ell, \omega^\vee) = 0$. The vanishing of this cohomology group follows for example from the Riemann–Roch theorem for smooth surfaces over a field.

So far we have the commutative diagram shown in (4.2).

$$\begin{tikzcd} & \tilde{\mathcal{B}}_\ell \arrow[dl, "\delta'"'] \arrow[d, "\rho"] \\ \mathcal{X}_\ell & \mathcal{B}_\ell \arrow[l, dashed, "\delta"] \end{tikzcd} \tag{4.2}$$

We will now prove that δ' contracts $\tilde{\Gamma}_\ell$ and all -2-curves on $\tilde{\mathcal{B}}_\ell$. We know that the complete anticanonical linear system on a generalized del Pezzo surface of degree 4 separates points and tangent vectors away from the -2-curves. We have seen that $\tilde{\Lambda}_\ell + \tilde{\Gamma}_\ell$ is an effective anticanonical divisor on $\tilde{\mathcal{B}}_\ell$ and we will write $\tilde{\Delta}_\ell = -K_{\tilde{\mathcal{B}}_\ell} + \tilde{\Gamma}_\ell$. We conclude that the complete linear system of $\tilde{\Delta}_\ell$ separates points and tangent vectors away from the -2-curves and $\tilde{\Gamma}_\ell$. This implies that $\tilde{\Delta}_\ell$ is big and nef and the only possible curves contracted by the associated birational map are the -2-curves and $\tilde{\Gamma}_\ell$. We saw in Lemma 4.3.10 that any -2-curve on $\tilde{\mathcal{B}}_\ell$ does not intersect $\tilde{\Gamma}_\ell$. So for any -2-curve R we have $\tilde{\Delta}_\ell \cdot R = -K_{\tilde{\mathcal{B}}_\ell} \cdot R + \tilde{\Gamma}_\ell \cdot R = 0$. We also have $\tilde{\Delta}_\ell \cdot \tilde{\Gamma}_\ell = -K_{\tilde{\mathcal{B}}_\ell} \cdot \Gamma_\ell + \tilde{\Gamma}_\ell^2 = 1 - 1 = 0$, since $\tilde{\Gamma}_\ell$ is a -1-curve on $\tilde{\mathcal{B}}_\ell$. This proves that δ'_ℓ is the morphism that contracts precisely $\tilde{\Gamma}_\ell$ and the -2-curves on $\tilde{\mathcal{B}}_\ell$. So δ_ℓ is defined off Γ_ℓ.

We also know that ρ_ℓ also contracts all -2-curves. This allows us to deduce that δ_ℓ is defined away from the singularities and we conclude that δ_ℓ is a morphism.

This proof also shows that δ_ℓ is an isomorphism away from Γ_ℓ. □

Now we can prove Theorem 4.3.3.

Proof of Theorem 4.3.3. We see from either construction that the generic fibre of $\mathcal{X}$ is isomorphic to $\mathrm{dP}_5(K)$ by Proposition 4.2.9. We will prove that the fibres $\mathcal{X}_\ell$ are anticanonically embedded del Pezzo surfaces using the second construction of $\mathcal{X}$. We have already seen in Corollary 4.3.9 that the fibres of $\mathcal{B}$ are singular del Pezzo surfaces of degree 4. In the proof of Proposition 4.3.11 we have seen that $\delta_\ell \colon \mathcal{B}_\ell \to \mathcal{X}_\ell$ contracts the curve Γ_ℓ to a point. The composition $\tilde{\mathcal{B}}_\ell \xrightarrow{\rho} \mathcal{B}_\ell \xrightarrow{\delta} \mathcal{X}_\ell$ first contracts the -2-curves and then the curve Γ_ℓ. We can also do so in the opposite order. So let us first contract the -1-curve $\tilde{\Gamma}_\ell$ on $\tilde{\mathcal{B}}_\ell$. This gives us a morphism $\tilde{\delta} \colon \tilde{\mathcal{B}}_\ell \to \tilde{\mathcal{X}}_\ell$ to a generalized del Pezzo surface of degree 5. By Lemma 4.3.10 the -1-curve Γ_ℓ does not meet any -2-curve and so contracting it does not change the set of -2-curves. The pullback of an anticanonical divisor on $\tilde{\mathcal{X}}_\ell$ is the divisor $\tilde{\Delta}_\ell$ on $\tilde{\mathcal{B}}_\ell$ so the anticanonical morphism $\gamma_\ell \colon \tilde{\mathcal{X}}_\ell \to \mathcal{X}_\ell$ makes the diagram in (4.3) commutative.

$$\begin{tikzcd} \tilde{\mathcal{X}}_\ell \arrow[d, "\gamma_\ell"] & \tilde{\mathcal{B}}_\ell \arrow[l, "\tilde{\delta}_\ell"'] \arrow[d, "\rho_\ell"] \\ \mathcal{X}_\ell & \mathcal{B}_\ell \arrow[l, "\delta_\ell"'] \end{tikzcd} \tag{4.3}$$

This proves that $\mathcal{X}_\ell$ is a singular del Pezzo surface of degree 5 over $\mathbb{F}_\ell$.

Note that $\mathcal{X}_\ell$ is the fibre of the scheme $\mathcal{X}$ over $\mathbb{Z}$. On the other hand, the scheme $\tilde{\mathcal{X}}_\ell$ is defined as a scheme over $\mathbb{F}_\ell$ and we have not constructed the scheme $\tilde{\mathcal{X}}$ over $\mathbb{Z}$. Although it exists we will not need it. The same holds for the scheme $\tilde{\mathcal{B}}_\ell$ and for the morphisms ρ_ℓ, γ_ℓ and $\tilde{\delta}_\ell$.

We see that $\mathcal{X}_\ell$ is smooth precisely when $\mathcal{B}_\ell$ is smooth, since the image of the contracted -1-curve Γ_ℓ is a smooth point. We see from Corollary 2.7.14 that $\mathcal{B}_\ell$ is an ordinary del Pezzo surface precisely when $\mathcal{Z}_\ell$ is reduced. So if $\mathcal{Z}_\ell$ is non-reduced we see that $\mathcal{B}_\ell$ is singular. We already saw in Corollary 4.3.9 that $\mathcal{B}_\ell$ is a singular del Pezzo surface, so in this case $\mathcal{B}_\ell$ and hence also $\mathcal{X}_\ell$ are singular del Pezzo surfaces. □

Now that we have our model $\mathcal{X}/\mathbb{Z}$ of $X/\mathbb{Q}$ we would like to extend some objects we have on X to $\mathcal{X}$. Recall the two anticanonical sections l_1 and l_2 on X from Proposition 4.2.13 which cut out the -1-curves. We will redefine l_1 and l_2 as sections of $\mathcal{O}_{\mathbb{P}^5_{\mathbb{Z}}}(1)$.

DEFINITION 4.3.13. The elements $l_1, l_2 \in \mathcal{O}_{\mathbb{P}^5_{\mathbb{Q}}}(1)$ can be rescaled using an element of $\mathbb{Q}^\times$ to obtain elements $l_1, l_2 \in \mathcal{O}_{\mathbb{P}^5_{\mathbb{Z}}}(1)$. We will assume that l_1 and l_2 are rescaled such that they do not vanish on vertical prime divisors of $\mathcal{X}/\mathbb{Z}$.

Before we could think of each l_i as a linear form over $\mathbb{Q}$ in six variables. From now on l_i will be rescaled such that the six coefficients are integral and coprime.

We will now look more closely at the fibre of $\mathcal{X}$ over a prime $\ell \in \mathbb{Z}$. We have seen that this depends on $\mathcal{Z}_\ell$ and this scheme depends heavily on the factorization of ℓ in K. So let us first study the splitting field K more closely.

4.4 The splitting field of the interesting action

The action of Galois on the geometric Picard group of an interesting del Pezzo surface X factors through a quotient $\mathbb{Z}/5\mathbb{Z}$ which must be $\mathrm{Gal}(K/k)$ for a cyclic degree 5 extension of k. We will restrict to the case that the base field k is the field of rational numbers. First we will classify such number fields and then we will look at the arithmetic of monogenic number rings $\mathbb{Z}[\alpha]$ in $\mathcal{O}_K$.

4.4.1 Number fields of degree 5

Let K be a Galois extension of $\mathbb{Q}$ of degree 5. By the Kronecker–Weber theorem we get that the field K is contained in a cyclotomic extension $\mathbb{Q}(\zeta_n)$ for some integer n. So the isomorphism classes of del Pezzo surfaces of degree 5 over $\mathbb{Q}$ split by a degree 5 extension correspond to open subgroups of index 5 in $\widehat{\mathbb{Z}}^\times$. In particular we see that every subgroup of $(\mathbb{Z}/n\mathbb{Z})^\times$ of index 5 gives us such a surface.

DEFINITION 4.4.1. Consider an open subgroup $N \subseteq \widehat{\mathbb{Z}}^\times$ of index 5 and define $K \subseteq \mathbb{Q}^{\mathrm{cyc}}$ to be the associated number field of degree 5. The isomorphism class of the corresponding del Pezzo surface of degree 5 over $\mathbb{Q}$ will be denoted by $\mathrm{dP}_5(N)$.

If n is a positive integer such that $(\mathbb{Z}/n\mathbb{Z})^\times$ has a unique subgroup N' of index 5, then we get such a subgroup N by taking the pre-image under the projection $\widehat{\mathbb{Z}}^\times \to (\mathbb{Z}/n\mathbb{Z})^\times$. In this situation we will write $\mathrm{dP}_5(n)$ for $\mathrm{dP}_5(N)$.

The *conductor* of a $\mathrm{dP}_5(N)$ is the minimal n such that N is the pullback of a subgroup $N' \subseteq (\mathbb{Z}/n\mathbb{Z})^\times$.

Note that the conductor will always exist since we are considering open subgroups of $\widehat{\mathbb{Z}}^\times$.

The construction of these degree 5 fields also gives us information about the splitting of rational primes.

PROPOSITION 4.4.2. *Let K be the number field associated to the open subgroup N of $\widehat{\mathbb{Z}}^\times$ of index 5 and let n be the conductor. The primes $\ell \in \mathbb{Q}$ which ramify in K are precisely those which divide n.*

Proof. Consider a prime $\mathfrak{l}$ above a prime $\ell \in \mathbb{Z}$. We have the commutative diagram shown in (4.4) from [53, Section 6] relating the global and local reciprocity maps.

$$\begin{tikzcd} \mathbb{Q}_\ell^\times \arrow[rr] \arrow[d] & & \mathrm{Gal}(K_{\mathfrak{l}}/\mathbb{Q}_\ell) \arrow[d] \\ \mathbb{I}_{\mathbb{Q}} \arrow[r] & \widehat{\mathbb{Z}}^\times \arrow[r] & \mathrm{Gal}(K/\mathbb{Q}) \end{tikzcd} \tag{4.4}$$

We have used that the bottom map factors through $\mathrm{Gal}(\mathbb{Q}^{\mathrm{cyc}}/\mathbb{Q}) \cong \widehat{\mathbb{Z}}^\times$. The composite map $\mathbb{Q}_\ell^\times \to \widehat{\mathbb{Z}}^\times$ is on $\mathbb{Z}_\ell^\times$ given by the inclusion $\mathbb{Z}_\ell^\times \hookrightarrow \widehat{\mathbb{Z}}^\times$ [53, Example 5.7].

Let ϕ be the diagonal map $\mathbb{Q}_\ell^\times \to \mathrm{Gal}(K/\mathbb{Q})$. We know that $\mathfrak{l}/\ell$ is unramified exactly when the inertia subgroup of $\mathfrak{l}$ is trivial. This group is equal to $\phi(\mathbb{Z}_\ell^\times)$ by [47, Theorem 1, Section 4.1]. So we see that $\mathfrak{l}$ is unramified precisely when $\#\phi(\mathbb{Z}_\ell^\times) = 1$ or equivalently $\mathrm{Im}(\mathbb{Z}_\ell^\times \to \widehat{\mathbb{Z}}^\times)$ lies in N. Now pick the minimal n such that $N \subseteq \widehat{\mathbb{Z}}^\times$ is the pullback of a subgroup N' of $(\mathbb{Z}/n\mathbb{Z})^\times$. Then we see that $\mathfrak{l}$ is unramified when $\mathrm{Im}(\mathbb{Z}_\ell^\times \to (\mathbb{Z}/n\mathbb{Z})^\times)$ lies in N' which by the minimality of n happens precisely for the ℓ which do not divide n. $\square$

COROLLARY 4.4.3. *In the notation of Proposition 4.4.2, the conductor is the product of distinct primes $p \equiv 1 \mod 5$ and possibly a factor 25.*

For the proof of this corollary we will use the following result.

LEMMA 4.4.4. *Let $p \equiv 1 \mod 5$ be a prime. An element in $\mathbb{Z}_p^\times$ is a fifth power precisely when it is modulo p. The fifth powers in $\mathbb{Z}_5^\times$ are precisely the lifts of the fifth powers modulo* 25.

Proof. The first statement follows directly from Hensel's lemma. For the proof of the second statement one need to work modulo 5^3 to be able to apply Hensel's lemma. The fifth powers in $\mathbb{Z}/125\mathbb{Z}$ are easily checked to be all the classes which reduce to $\pm 1, \pm 7 \mod 25$. □

Proof of Corollary 4.4.3. We saw that ℓ is unramified precisely when the image of $\mathbb{Z}_\ell^\times \to \widehat{\mathbb{Z}}^\times$ is a subgroup of $N \subseteq \widehat{\mathbb{Z}}^\times$. This means that ℓ is ramified exactly if the group homomorphism $\mathbb{Z}_\ell^\times \to \widehat{\mathbb{Z}}^\times/N \cong \mathbb{Z}/5\mathbb{Z}$ is not constant, and hence surjective. The kernel of this morphism is an open index 5 subgroup of $\mathbb{Z}_\ell^\times$. This implies that it is the pullback of an index 5 subgroup of $(\mathbb{Z}/\ell^e\mathbb{Z})^\times$ for some $e \geq 1$ and we find $5 \mid \varphi(\ell^e) = \ell^{e-1}(\ell - 1)$. So the only possible ramified primes are the primes $p \equiv 1 \mod 5$ and 5.

Now let N_ℓ be the kernel of $\mathbb{Z}_\ell^\times \to \widehat{\mathbb{Z}}^\times/N$. To prove the statement, we will show if ℓ is either 5 or $1 \mod 5$ the index 5 subgroup $N_\ell \subseteq \mathbb{Z}_\ell^\times$ must be equal to $\left(\mathbb{Z}_\ell^\times\right)^5$. Since N_ℓ is of index 5, we see that $\left(\mathbb{Z}_\ell^\times\right)^5 \subseteq N_\ell$. For $\ell \equiv 1 \mod 5$ we see that $\mathbb{Z}_\ell^\times/(\mathbb{Z}_\ell^\times)^5 \to \mathbb{F}_\ell^\times/(\mathbb{F}_\ell^\times)^5$ is an isomorphism by Lemma 4.4.4. This implies that N_ℓ and $\left(\mathbb{Z}_\ell^\times\right)^5$ are both of index 5 in $\mathbb{Z}_\ell^\times$. So N_ℓ is the subgroup of fifth powers in $\mathbb{Z}_\ell^\times$ and by Lemma 4.4.4 it is the inverse image of the fifth powers modulo ℓ.

For $\ell = 5$ we can compute the index of $\left(\mathbb{Z}_5^\times\right)^5$ in $\mathbb{Z}_5^\times$ in a similar manner to conclude that N_5 can only be the pullback of the subgroup of fifth powers modulo 25 along the reduction morphism $\mathbb{Z}_5^\times \to \mathbb{Z}/25\mathbb{Z}$. □

Now consider the surfaces for which the conductor has only one prime divisor p. If $p \equiv 1 \mod 5$ then the conductor must be equal to p by Corollary 4.4.3. In this case there is a unique such number field K since the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \cong (\mathbb{Z}/p\mathbb{Z})^\times$ is cyclic. So for each prime $p \equiv 1 \mod 10$ there is an isomorphism class $\mathrm{dP}_5(p)$ of interesting del Pezzo surfaces over $\mathbb{Q}$. In the last section we will also consider a surface in the isomorphism class $\mathrm{dP}_5(25)$. This class exists because also $\mathrm{Gal}(\mathbb{Q}(\zeta_{25})/\mathbb{Q}) \cong (\mathbb{Z}/25\mathbb{Z})^\times$ has a unique subgroup of index 5.

4.4.2 Number rings $\mathbb{Z}[\alpha]$ of degree 5

We have seen that an interesting del Pezzo surface X is uniquely determined by its splitting field K. The model $\mathcal{X}$ over $\mathbb{Z}$ however depends on the choice of $\alpha \in \mathcal{O}_K$. Many arithmetic and geometric properties of $\mathcal{X}$ can be described in terms of the number ring $\mathbb{Z}[\alpha]$. Let us look at the primes of $\mathbb{Z}[\alpha]$ and relate this to the possible factorizations of m_α modulo a prime ℓ.

Recall that the *inertia degree* of a prime $\mathfrak{l}$ of a number ring $R \subseteq \mathcal{O}_K$ over a prime $\ell \in \mathbb{Z}$ is defined as the degree of the residue field $\mathbb{F}_{\mathfrak{l}} = R/\mathfrak{l}$ as a field extension of $\mathbb{F}_\ell$.

THEOREM 4.4.5. *Let $m \in \mathbb{Z}[s]$ be an irreducible monic polynomial with integral coefficients. Let $\alpha \in \bar{\mathbb{Q}}$ be a root of m and let ℓ be a prime. Pick monic polynomials $m_i \in \mathbb{Z}[s]$ whose reductions modulo ℓ are irreducible and pairwise distinct such that there exist integral numbers $e_i > 0$ with the property that*

$$m \equiv \prod_i m_i^{e_i} \mod \ell.$$

(a) The prime ideals of $\mathbb{Z}[\alpha]$ which lie above ℓ are $\mathfrak{l}_i = (\ell, m_i(\alpha))$. The inertia degree of the prime ideal $\mathfrak{l}_i$ over ℓ equals the degree of m_i.

(b) Let $r_i \in \mathbb{Z}[s]$ be the remainder of m upon division by m_i. The ideal $\mathfrak{l}_i$ is invertible precisely if $e_i = 1$ or $r_i \not\equiv 0 \mod \ell^2$.

(c) The identity

$$\ell\mathbb{Z}[\alpha] = \prod_i \mathfrak{l}_i^{e_i}$$

holds if and only if each ideal $\mathfrak{l}_i$ is invertible.

Proof. See Theorem 8.2 in [51]. □

Every prime $\tilde{\mathfrak{l}}$ of K lies above a single prime $\mathfrak{l}$ of $\mathbb{Z}[\alpha]$. The following lemma is useful in going from primes of $\mathbb{Z}[\alpha]$ to primes of K.

LEMMA 4.4.6. *Consider the inclusion $\mathbb{Z}[\alpha] \subseteq \mathcal{O}_K$ and let $\mathfrak{l}$ be an invertible prime of $\mathbb{Z}[\alpha]$. There exists a unique prime $\tilde{\mathfrak{l}}$ of $\mathcal{O}_K$ lying above $\mathfrak{l}$.*

Proof. Let $\tilde{\mathfrak{l}}$ be a prime above $\mathfrak{l}$. We get an extension of local rings $\mathbb{Z}[\alpha]_{\mathfrak{l}} \subseteq \mathcal{O}_{K,\tilde{\mathfrak{l}}} \subseteq K$. Since $\mathfrak{l}$ is invertible we see that $\mathbb{Z}[\alpha]_{\mathfrak{l}}$ is a discrete valuation ring with field of fraction K. Since $O_{K,\tilde{\mathfrak{l}}}$ is not a field we conclude that $\mathbb{Z}[\alpha]_{\mathfrak{l}} = \mathcal{O}_{K,\tilde{\mathfrak{l}}}$. This shows that $\tilde{\mathfrak{l}} = \mathcal{O}_K \cap \mathfrak{l}\,\mathbb{Z}[\alpha]_{\mathfrak{l}}$ is uniquely determined by $\mathfrak{l}$. □

The following result will also be useful.

LEMMA 4.4.7. *The inertia degree of a prime $\mathfrak{l}$ of $\mathbb{Z}[\alpha]$ is either* 1 *or* 5.

Proof. Let ℓ be the prime $\mathfrak{l} \cap \mathbb{Z}$ and let $\tilde{\mathfrak{l}}$ be a prime of K lying above $\mathfrak{l}$. Since $K/\mathbb{Q}$ is a Galois extension of degree 5 we know that the inertia degree of $\tilde{\mathfrak{l}}$ divides 5. So we have a the following extensions of fields

$$\mathbb{F}_\ell \subseteq \mathbb{F}_{\mathfrak{l}} \subseteq \mathbb{F}_{\tilde{\mathfrak{l}}}.$$

This proves the statement. □

Note that this also proves that the reduction of m_α modulo a prime ℓ is either irreducible or it splits in linear factors. Note however that these linear factors need not be distinct. The extremal case where $m_\alpha \mod \ell$ is the fifth power of a linear polynomial over $\mathbb{F}_\ell$ will play a special role

4.5 The fibres $\mathcal{X}_\ell$

In Section 4.3 we used the two constructions of the interesting del Pezzo surface X over $\mathbb{Q}$ to produce a model $\mathcal{X}$ of X over $\mathbb{Z}$. In this section we study the fibres of $\mathcal{X}$ over a prime ℓ. We will also come across the fibres $\mathcal{B}_\ell$ of the scheme $\mathcal{B}$ defined in Definition 4.3.7.

4.5.1 The construction of $\mathcal{X}_\ell$

We will start by exhibiting the relation between the fibres $\mathcal{X}_\ell$, $\mathcal{B}_\ell$ and $\mathbb{P}^2_{\mathbb{F}_\ell}$.

PROPOSITION 4.5.1. *Let ℓ be a rational prime and let $\mathfrak{l}$ be a prime of K which divides ℓ. The geometrically irreducible components of $\mathcal{Z}_\ell$ are defined over the residue field $\mathbb{F}_\mathfrak{l}$.*

The fibre $\mathcal{X}_\ell$ can be constructed as follows: we will define generalized del Pezzo surfaces $\mathcal{B}_{\ell,i}$ for $0 \leq i \leq 5$ of degree $9 - i$ over the field $\mathbb{F}_\mathfrak{l}$. Simultaneously we define a curve Γ_i on $\mathcal{B}_{\ell,i}$. We start off with $\mathcal{B}_{\ell,0} = \mathbb{P}^2_{\mathbb{F}_\mathfrak{l}}$ together with $\Gamma_0 = \Gamma_{0,\ell}$. Now consider the reduction Q_i of the point $P_i = (\alpha_i : \alpha_i^2 : 1)$ modulo $\mathfrak{l}$ on $\mathbb{P}^2_{\mathbb{F}_\mathfrak{l}}$. There is a unique $\mathbb{F}_\mathfrak{l}$-point R_i on $\Gamma_i \subseteq \mathcal{B}_{\ell,i}$ whose image under the morphism $\mathcal{B}_{\ell,i} \to \mathbb{P}^2_{\mathbb{F}_\mathfrak{l}}$ is Q_i. Let $\mathcal{B}_{\ell,i+1}$ be the blowup of $\mathcal{B}_{\ell,i}$ in R_i and let Γ_{i+1} be the strict transform of Γ_i along this blowup. Then $\mathcal{B}_{\ell,5}$ is a generalized del Pezzo surface of degree 4 over $\mathbb{F}_\mathfrak{l}$ which descends to the minimal desingularization $\tilde{\mathcal{B}}_\ell$ of $\mathcal{B}_\ell$ over $\mathbb{F}_\ell$.

The strict transform Γ of Γ_0 to $\mathcal{B}_\ell$ can be contracted to obtain $\mathcal{X}_\ell$. This shows that the composition

$$\tilde{\mathcal{B}}_\ell \to \mathcal{B}_\ell \to \mathcal{X}_\ell$$

first contracts the -2-curves and then the -1-curve $\tilde{\Gamma}_\ell$. We can also first contract $\tilde{\Gamma}_\ell$ to obtain a generalized del Pezzo surface $\tilde{\mathcal{X}}_\ell$ of degree 5 and then contract the -2-curves to recover $\mathcal{X}_\ell$.

Proof. Proposition 4.3.8 states that $\mathcal{B}_\ell$ is the blowup of the projective plane in the curvilinear subscheme $\mathcal{Z}_\ell$. We have seen in Lemma 2.7.7 that the composition $\tilde{\mathcal{B}}_\ell \to \mathcal{B}_\ell \to \mathbb{P}^2_{\mathbb{F}_\ell}$ decomposes into blowups in closed points at least when the geometrically irreducible components of Z are defined over the base field. This is the case over the field $\mathbb{F}_\mathfrak{l}$.

The fact that $\delta_\ell \colon \mathcal{B}_\ell \to \mathcal{X}_\ell$ is the contraction of the curve Γ_ℓ was shown in the proof of Proposition 4.3.11. □

This result does not only help us to understand the geometry of $\mathcal{X}_\ell$, but also its arithmetic.

LEMMA 4.5.2. *Consider a finite prime ℓ and let r be the number of distinct roots of m_α in $\mathbb{F}_\ell$. Then we have*

$$\#\mathcal{X}(\mathbb{F}_\ell) = \ell^2 + r\ell + 1.$$

Proof. Factor m_α in $\bar{\mathbb{F}}_\ell$ as $\prod(s - \beta_i)^{e_i}$ and consider the blowup $\beta \colon \mathcal{B}_\ell \to \mathbb{P}^2_{\mathbb{F}_\ell}$. This map is an isomorphism away from the points $(\beta_i : \beta_i^2 : 1)$ and the fibre over each of these points is a projective line defined over $\mathbb{F}_\ell(\beta_i)$. We see that each root of m_α in $\mathbb{F}_\ell$ adds exactly ℓ points and other roots do not change the number of $\mathbb{F}_\ell$-points. Finally we contract the -1-curve Γ_ℓ which is defined over $\mathbb{F}_\ell$ and see that

$$\#\mathcal{X}(\mathbb{F}_\ell) = (\ell^2 + \ell + 1) + r\ell - \ell = \ell^2 + r\ell + 1. \qquad \square$$

4.5.2 The -1-curves on $\mathcal{X}_\ell$

Let us also study the curves of negative self-intersection of $\mathcal{X}_\ell$. We will do this by first considering the s-curves on $\tilde{\mathcal{B}}_\ell$ and on the minimal desingularization $\tilde{\mathcal{X}}_\ell$ of $\mathcal{X}_\ell$.

PROPOSITION 4.5.3. *Let $Q_i \in \mathcal{Z}(\mathbb{F}_{\mathfrak{l}})$ be the reduction of the point $P_i \in Z(K)$ modulo a prime $\mathfrak{l}$ of K lying above ℓ. These 5 points need not be all distinct.*

Consider the minimal desingularization $\tilde{\mathcal{X}}_\ell$ of the fibre of $\mathcal{X}$ over a prime ℓ. A -1-curve on $\tilde{\mathcal{X}}_\ell$ is the strict transform of the line through Q_i and Q_j along the birational morphisms $\tilde{\mathcal{X}}_\ell \leftarrow \tilde{\mathcal{B}}_\ell \rightarrow \mathbb{P}^2_{\mathbb{F}_\ell}$. Let us write $\mathcal{L}_{i,j,\ell}$ for this -1-curve, both on $\tilde{\mathcal{X}}_\ell$ and on $\tilde{\mathcal{B}}_\ell$.

If $Q_i = Q_j$, then $\mathcal{L}_{i,j,\ell}$ is the tangent line at Q_i to $\Gamma_{0,\ell}$.

Proof. Since intersection numbers can only go up when blowing down a -1-curve $\tilde{\mathcal{B}}_\ell \rightarrow \tilde{\mathcal{X}}_\ell$ we will first determine the s-curves on the minimal desingularization $\tilde{\mathcal{B}}_\ell \rightarrow \mathcal{B}_\ell$ of the fibre of $\mathcal{B}$ over ℓ. Consider the composition $\tilde{\beta}\colon \tilde{\mathcal{B}}_\ell \rightarrow \mathcal{B}_\ell \rightarrow \mathbb{P}^2_{\mathbb{F}_\ell}$. By Proposition 2.4.4 we have s-curves in the support of the exceptional divisor $E_{\tilde{\beta}}$. Recall that each component of $E_{\tilde{\beta}}$ has a unique -1-curve. This implies that $\tilde{\Gamma}_\ell$ intersects every component of $E_{\tilde{\beta}}$ in this -1-curve since $\tilde{\Gamma}_\ell$ does not intersect any -2-curves by Lemma 4.3.10. This means that after contracting $\tilde{\Gamma}_\ell$ we lose these -1-curves in the sense that their images on $\tilde{\mathcal{X}}_\ell$ have self-intersection zero. On the other hand, the -2-curves on $\tilde{\mathcal{X}}_\ell$ remain -2-curves when considering their strict transforms on $\tilde{\mathcal{B}}_\ell$.

Any other s-curve on $\tilde{\mathcal{B}}_\ell$ is the strict transform of a line or a conic on $\mathbb{P}^2_{\mathbb{F}_\ell}$. Since we blow up five points we see from Lemma 2.7.9 that apart from $\tilde{\Gamma}_\ell$ there are only -1-curves on $\tilde{\mathcal{B}}_\ell$ and each is the strict transform of a line L with the property that $\deg(L \cap \mathcal{Z}_\ell) = 2$. These are precisely the lines through Q_i and Q_j or the tangent line of $\Gamma_{0,\ell}$ in $Q_i = Q_j$.

For a line L through $Q_i \neq Q_j$ we see that its strict transform $\tilde{L}$ to $\tilde{\mathcal{B}}_\ell$ does not meet $\tilde{\Gamma}_\ell$ since blowing up once separates curves which intersect transversally. For the tangent in $Q_i = Q_j$ we will need to blow up at least two times, which is precisely what happens since P_i and P_j both reduce to $Q_i = Q_j$. This proves that these -1-curves on $\tilde{\mathcal{B}}_\ell$ remain -1-curves on $\tilde{\mathcal{X}}_\ell$ after contracting $\tilde{\Gamma}_\ell$. $\square$

Other important objects are the sections l_1 and l_2. We will now describe the subscheme on $\mathcal{X}_\ell$ and $\mathcal{B}_\ell$ cut out by these forms.

PROPOSITION 4.5.4. *The reduction of l_1 modulo ℓ cuts out an effective divisor of $\mathcal{X}_\ell$ supported in geometric -1-curves.*

Obviously the result also holds for l_2.

Proof. The section l_1 cuts out five geometric -1-curves on the generic fibre X of $\mathcal{X}$. By definition of l_1 over $\mathbb{Z}$ we see that l_1 cuts out the flat closure of these five geometric -1-curves over K in $\mathbb{P}^5_{\mathcal{O}_K}$. Let us consider the closure $\mathcal{L}$ in $\mathbb{P}^5_{\mathcal{O}_K}$ of one of these -1-curves L over $\mathbb{Q}$. The scheme $\mathcal{L}$ is flat over $\mathcal{O}_K$, so the fibre $\mathcal{L}_\mathfrak{l}$ over $\mathfrak{l}$ will be a degree 1 curve on $\mathcal{X}_\mathfrak{l} \subseteq \mathbb{P}^5_{\mathbb{F}_\mathfrak{l}}$ of genus 0. This shows that on the minimal desingularization $(\tilde{\mathcal{X}}_{\mathcal{O}_K})_\mathfrak{l} = \tilde{\mathcal{X}}_\ell \times_{\mathbb{F}_\ell} \mathbb{F}_\mathfrak{l}$ this curve $\mathcal{L}_\mathfrak{l}$ is a -1-curve on $(\tilde{\mathcal{X}}_{\mathcal{O}_K})_\mathfrak{l}$. This proves that l_1 vanishes on five, not necessarily distinct, geometric -1-curves on $\tilde{\mathcal{X}}_\ell$. Since the degree of the divisor of zeroes of l_1 equals five, it vanishes only on these curves. □

Note that this proves that l_1 vanishes on the strict transform along the birational morphisms $\tilde{\mathcal{X}}_\ell \leftarrow \tilde{\mathcal{B}}_\ell \to \mathbb{P}^2_{\mathbb{F}_\ell}$ of the line through Q_i and Q_{i+1}. Here we consider the indices modulo 5. Similar l_2 vanishes on the strict transform of the line through Q_i and Q_{i+2}.

Understanding the sections which are cut out by l_1 and l_2 on the fibre of $\mathcal{X}/\mathbb{Z}$ over a prime ℓ is also important for the following reason.

LEMMA 4.5.5. *Let Q be a singular $\bar{\mathbb{F}}_\ell$-point on the fibre $\mathcal{X}_\ell$. Then Q lies on the divisor of zeroes of both l_1 and l_2.*

We will see in Proposition 4.5.8 that all singular points on $\mathcal{X}_\ell$ are already defined over the base field $\mathbb{F}_\ell$.

Proof. A singular point Q corresponds to a connected set of -2-curves on $\tilde{\mathcal{X}}_\ell$ or equivalently $\tilde{\mathcal{B}}_\ell$. This is a chain of -2-curves which lies above a geometric point $Q_i = (\beta_i \colon \beta_i^2 \colon 1)$ of $\mathcal{Z}_\ell \subseteq \mathbb{P}^2_{\mathbb{F}_\ell}$. Now let e_i be the local degree of $\mathcal{Z}_\ell$ at Q_i. Since there is a singular point above Q_i we have that $e_i > 1$.

If $e_i = 2$, then assume without loss of generality that P_1 and P_2 reduce to the same point $Q_1 = Q_2$ modulo $\mathfrak{l}$, but the points P_3, P_4 and P_5 do not reduce to Q_1 modulo $\mathfrak{l}$. Now l_1 vanishes on the secant $\mathcal{L}_{2,3,\ell}$ and l_2 vanishes on the secant $\mathcal{L}_{1,3,\ell}$. By Corollary 2.7.10 these lines intersect a -2-curve on $\tilde{\mathcal{B}}_\ell$ above Q_1.

If $e_i > 2$, then assume without loss of generality that $Q_1 = Q_2 = Q_3$. This proves that the tangent lines $\mathcal{L}_{1,2,\ell}$ and $\mathcal{L}_{1,3,\ell}$ in Q_i at $\Gamma_{0,\ell}$ are the same line and this line is cut out by both l_1 and l_2. We again use Corollary 2.7.10 to conclude the statement. □

We have now seen that the geometry of the fibre $\mathcal{X}_\ell$ is determined by the factorization of m_α modulo ℓ. We will look into how factors of this factorization determine the singular points on $\mathcal{X}_\ell$.

4.5.3 The singularities of $\mathcal{X}_\ell$

We have seen that $\mathcal{B}_\ell$ and hence $\mathcal{X}_\ell$ is smooth precisely if $\mathcal{Z}_\ell$ is reduced. This result can be directly related to the splitting of m_α in $\bar{\mathbb{F}}_\ell$.

LEMMA 4.5.6. *A fibre $\mathcal{X}_\ell$ is non-singular precisely if m_α is separable over $\bar{\mathbb{F}}_\ell$.*

Proof. We have seen in Theorem 4.3.3 that $\mathcal{X}_\ell$ is an ordinary del Pezzo surface precisely if $\mathcal{Z}_\ell$ is reduced. Now write $m_\alpha = \prod_i (s-\beta_i)^{e_i}$ over $\bar{\mathbb{F}}_\ell$. Recall that $\mathcal{Z}_\ell$ is supported in the points $(\beta_i \colon \beta_i^2 \colon 1)$ and the geometrically irreducible component supported in this point is of degree e_i. This shows that $\mathcal{Z}_\ell$ is reduced precisely when m_α is separable in $\bar{\mathbb{F}}_\ell$. $\square$

Although the factorization over $\bar{\mathbb{F}}_\ell$ determines the geometry of $\mathcal{X}_\ell$ we have also seen that arithmetic properties are determined by the factors of m_α over $\mathbb{F}_\ell$. From Lemma 4.4.7 we deduce that m_α is either irreducible modulo ℓ or it splits into, not necessarily distinct, linear factors over $\mathbb{F}_\ell$. Let us describe $\mathcal{X}_\ell$ in both cases.

LEMMA 4.5.7. *Let ℓ be a prime such that m_α is irreducible modulo ℓ. The fibre $\mathcal{X}_\ell$ is a smooth del Pezzo of degree 5 with two Galois orbits of exceptional curves of size 5.*

Proof. In this case the procedure to determine $\mathcal{X}_\ell$ is similar to how we got X starting with the projective plane over $\mathbb{Q}$; we blow up five conjugate points and then contract the strict transform of the unique conic through these five points. The result follows. $\square$

PROPOSITION 4.5.8. *Suppose that m_α splits into linear factors modulo ℓ. The fibre of $\mathcal{X}$ above ℓ has a singular point of type A_{e_i-1} for each factor with multiplicity $e_i > 1$ in this factorization. In particular we see that there are at most 2 singular points on $\mathcal{X}_\ell$ and these are defined over $\mathbb{F}_\ell$.*

Proof. Write $m_\alpha = \prod_i (x-\beta_i)^{e_i}$ where β_i are the distinct roots of m_α in $\mathbb{F}_\ell$. The morphism $\mathcal{B}_\ell \to \mathbb{P}^2_{\mathbb{F}_\ell}$ restricts to an isomorphism on the complement of the points $(\beta_i, \beta_i^2, 1)$. Using the procedure described in Proposition 4.5.1 we see that the fibre above such a point is obtained by contracting the -2-curves from a chain of $e_i - 1$ curves with self-intersection -2 and one -1-curve. This shows that $\mathcal{B}_\ell$ has a singularity of type A_{e_i-1} above the point $(\beta_i, \beta_i^2, 1)$.

Contracting the -1-curve $\Gamma_\ell \subseteq \mathcal{B}_\ell$ does not produce additional singular points or change the type of singularities on $\mathcal{B}_\ell$, since these points do not lie on Γ_ℓ by Lemma 4.3.10.

As a quintic polynomial can have at most 2 multiple irreducible factors we see this is the maximal number of singularities. $\square$

Let us now relate the possible factorization of m_α modulo ℓ to the factorization of ℓ in $\mathcal{O}_K$. Let us first treat the unramified primes.

LEMMA 4.5.9. *Let $\ell \in \mathbb{Z}$ be a prime which is inert in K. The fibre $\mathcal{X}_\ell$ either*

» *is smooth, in the case that m_α is irreducible modulo ℓ, or*

» *has a single singular point which is of type A_4, when m_α is a fifth power of a linear function modulo ℓ.*

In particular, we see that $\mathcal{X}_\ell$ has at most one singular point if ℓ is inert.

Proof. By assumption K has a unique prime $\tilde{\mathfrak{l}}$ lying above ℓ. This implies that $\mathfrak{l} = \tilde{\mathfrak{l}} \cap \mathbb{Z}[\alpha]$ is the unique prime of $\mathbb{Z}[\alpha]$ lying above ℓ. The inertia degree of this prime is either 5 or 1. This corresponds precisely to the two cases described in the lemma. $\square$

LEMMA 4.5.10. *Let $\ell \in \mathbb{Z}$ be a prime which splits completely in K. This implies that m_α splits into linear factors over $\mathbb{F}_\ell$, and $\mathcal{X}_\ell$ is a surface as described in Proposition 4.5.8.*

Proof. As ℓ splits completely we have an isomorphism of residue fields $\mathbb{F}_\ell \to \mathbb{F}_{\tilde{\mathfrak{l}}}$ for any prime $\tilde{\mathfrak{l}}$ of K lying above ℓ. This implies that in the tower $\mathbb{Z} \subseteq \mathbb{Z}[\alpha] \subseteq \mathcal{O}_K$ each prime of $\mathcal{O}_K$ which divides ℓ comes from a prime $\mathfrak{l}$ in $\mathbb{Z}[\alpha]$ with the same inertia degree. So all primes $\mathfrak{l}/\ell$ have inertia degree 1 and by the Kummer–Dedekind theorem, Theorem 4.4.5, we see that m_α must split into linear factors modulo ℓ. $\square$

We now consider the case that ℓ is ramified in K. We will usually denote such a prime by p.

LEMMA 4.5.11. *Let p be a prime which is ramified in K. The minimal polynomial m_α is the fifth power of a linear polynomial over $\mathbb{F}_p$. The fibre $\mathcal{X}_p$ is a singular del Pezzo surface of degree* 5 *with a single singular point which is of type A_4. This surface contains precisely one line.*

Proof. We will need to prove that the first statement. The last statements then follow from Propositions 4.5.8 and 4.5.3.

Let $\tilde{\mathfrak{p}}$ be the prime of $\mathcal{O}_K$ dividing p. Since it is ramified and $K/\mathbb{Q}$ is a Galois extension we find $e(\tilde{\mathfrak{p}}/p) = 5$. This proves that p totally ramifies in K and that $\tilde{\mathfrak{p}}$ is the unique prime of $\mathcal{O}_K$ dividing p. Now let $\mathfrak{p}$ be the prime $\tilde{\mathfrak{p}} \cap \mathbb{Z}[\alpha]$ of $\mathbb{Z}[\alpha]$. Since the inertia degree of $\tilde{\mathfrak{p}}$ is one, so must the inertia degree of $\mathfrak{p}$. This proves the statement. $\square$

Note that m_α can be a fifth power modulo ℓ in all three cases and this will always yield a singular del Pezzo surface of degree 5 with one singular point of type A_4, and one -1-curve. We will look more closely at this surface.

4.5.4 Fibres with an A_4-singularity

Consider a fibre of $\mathcal{X}$ over $\mathbb{Z}$ with a singularity of type A_4. We can encounter such a fibre over any prime ℓ, but we have seen in Lemma 4.5.11 that the fibre over any ramified prime p will be of this type. Since we will mainly discuss this fibre over ramified primes we will write p for the prime in this section, although theses results are true for any fibre with a singularity of type A_4.

PROPOSITION 4.5.12. *Let $\mathcal{X}_p$ be a fibre of $\mathcal{X}$ over $\mathbb{Z}$ with a singularity of type A_4. The surface $\mathcal{X}_p$ is the unique singular del Pezzo surface of degree 5 with a single -1-curve E_4 and one singular point which is of type A_4. Furthermore, the complement of this -1-curve E_4 in $\mathcal{X}_p$ is isomorphic to $\mathbb{A}^2_{\mathbb{F}_p}$.*

Let Z' be a curvilinear subscheme of degree 4 supported in the inflection point p_0 of a cubic plane curve $C \subseteq \mathbb{P}^2_{\mathbb{F}_p}$. The surface $\mathcal{X}_p$ is the singular del Pezzo surface over $\mathbb{F}_p$ associated to Z'.

The choice for notation of this unique -1-curve will become clear during the proof.

Proof of Proposition 4.5.12. Let us prove the first statement. It follows from Proposition 4.5.8 that $\mathcal{Z}_p$ is a degree 5 subscheme of $\Gamma_{0,p}$ supported in a single point Q. The procedure described in Proposition 4.5.1 shows that we can find $\tilde{\mathcal{B}}_p$ by first blowing up $\mathbb{P}^2_{\mathbb{F}_p}$ five times to get a configuration of three -1-curves and four -2-curves as shown in Figure III.

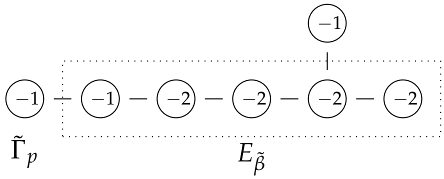


Figure III: Curves of negative self-intersection on $\tilde{\mathcal{B}}_p$.

The -1-curve on the left is $\tilde{\Gamma}_p \subseteq \tilde{\mathcal{B}}_p$. Consider the tangent line $L_{0,p} \subseteq \mathbb{P}^2_{\mathbb{F}_p}$ to $\Gamma_{0,p}$ in Q. The strict transform L of $L_{0,p}$ on $\tilde{\mathcal{B}}_p$ is a -1-curve by Lemma 2.7.9. The position of L in Figure III is determined by Lemma 2.7.10. This accounts for the -1-curve on top. The remaining four -2-curves and the -1-curve are the components of the exceptional divisor $E_{\tilde{\beta}}$. Then we contract $\tilde{\Gamma}_p$, which becomes a smooth point and makes the unique -1-curve on $E_{\tilde{\beta}}$ into a curve of self-intersection 0. Finally we contract all the -2-curves which gives an A_4 singularity lying on the remaining curve of negative self-intersection.

We see that all s-curves on $\tilde{\mathcal{X}}_p$ are defined over $\mathbb{F}_p$. This proves that there is an $\mathbb{F}_p$-morphism $\tilde{\pi}\colon \tilde{\mathcal{X}}_p \to \mathbb{P}^2_{\mathbb{F}_p}$ which is the blowup of the projective plane in 4 points in almost general position. Let $Z' \subseteq \mathbb{P}^2_{\mathbb{F}_p}$ be the associated zero-dimensional subscheme of this composition of blowups. Since we have only one -1-curve on $\tilde{\mathcal{X}}_p$ we conclude from Proposition 2.4.4 that Z' is supported in only one point p_0. This accounts for the -1-curve and three -2-curves, because they occur in the exceptional divisor $E_{\tilde{\pi}}$. Let us rename these divisors as in Proposition 2.4.4; the -2-curves are E_1, E_2 and E_3, and the -1-curve is E_4. Let us call the remaining -2-curve R. The intersection graph on s-classes on $\tilde{\mathcal{X}}_p$ is now shown in Figure IV.

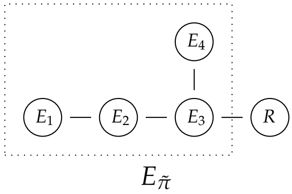


Figure IV: Curves of negative self-intersection on $\tilde{\mathcal{X}}_p$.

Let L be the image of R in $\mathbb{P}^2_{\mathbb{F}_p}$. We will now prove that L is a line. We know that

$$-2 = R^2 = L^2 - \deg(Z' \cap L) \geq L^2 - 3$$

since Z' is supported on a cubic curve C which is smooth at p_0. This proves that $L^2 \leq 1$ and hence $\deg L = 1$ and $\deg(Z' \cap L) = 3$. This proves that L is the tangent line at an inflection point of C. In the diagram in (4.5) of birational morphisms we get isomorphisms after removing the -1-curve on $\mathcal{X}_p$, which by abuse of notation will also be denoted by E_4, in the middle all the E_i and R, and on the right the line L.

$$\mathcal{X}_p \longleftarrow \tilde{\mathcal{X}}_p \longrightarrow \mathbb{P}^2_{\mathbb{F}_p} \tag{4.5}$$

This proves that the complement of the -1-curve E_4 in $\mathcal{X}_p$ is isomorphic to $\mathbb{A}^2_{\mathbb{F}_p}$. $\square$

COROLLARY 4.5.13. *The surface $\tilde{\mathcal{X}}_p$ is the generalized del Pezzo surface associated to Z'. Write $\tilde{\pi}\colon \tilde{\mathcal{X}}_p \to \mathbb{P}^2_{\mathbb{F}_p}$ for the associated blowup of the projective plane in* 4 *points in almost general position.*

The generalized del Pezzo surface $\tilde{\mathcal{X}}_p$ has five s-curves. Three -2-curves and a -1-curve form the support of the exceptional divisor $E_{\tilde{\pi}}$, and a -2-curve R is the strict transform of the tangent line L in p_0 to C along $\tilde{\pi}$.

We will fix coordinates for p_0 and the equation for the line L. This does not determine the equation for C uniquely. It does however uniquely determine the zero-dimensional ideal Z' up to an automorphism of $\mathbb{P}^2_{\mathbb{F}_p}$ which fixes p_0 and L.

LEMMA 4.5.14. *Assume without loss of generality that p_0 is the point $(0\colon 1\colon 0)$ and that L is given by $z = 0$. Let C be a cubic curve which passes through x, is smooth there, has L as the tangent line to C at x, and has an inflection point at x. Let Z' be the curvilinear subscheme associated to $\mathcal{I}_{C,p_0,4}$.*

There is a unique $\lambda \in \mathbb{F}_p^\times$ such that the cubic curve $x^3 - \lambda y^2 z = 0$ passes through Z'.

Proof. Consider the homogeneous cubic $f_{\mathrm{hom}} \subseteq \mathbb{F}_p[x, y, z]$ defining C. We will consider the corresponding affine curve defined by $f = f_{\mathrm{hom}}(x, 1, z)$. Since this polynomial $f(x, z)$ defines a inflection point at $(0, 0)$ it must contain the monomial x^3 and no other monomials which only contain x. Since L is the unique tangent line at $(0, 0)$ the linear term of f is given by a multiple of z. So we can rescale f to $f = x^3 + z\kappa(x, z) - \lambda z$ where the degree of every monomial of κ is at least 1. Now note that xz and z^2 pass through Z' so this means that $f - z\kappa(x, z) = x^3 - \lambda z$ defines a cubic curve which passes through Z'.

Now if we had two such curves given by $x^3 = \lambda y^2 z$ and $x^3 = \lambda' y^2 z$ for different λ and λ', then we would find that the line L defined by $(\lambda - \lambda')z = 0$ passes through Z'. This would mean that $\deg(Z' \cap L) = \deg Z' = 4$, which is impossible since Z' lies in the intersection of L and C. □

We can assume without loss of generality that $\lambda = 1$ by rescaling the coordinate z. The curve defined by $x^3 = y^2 z$ has a inflection point at the smooth point $(0\colon 1\colon 0)$ so we can take this curve to be C.

COROLLARY 4.5.15. *The subscheme Z' is defined by the homogeneous ideal*

$$(x^3 - y^2 z, xz, z^2) \subseteq \mathbb{F}_p[x, y, z].$$

The birational map $\psi\colon \mathbb{P}^2_{\mathbb{F}_p} \dashrightarrow \tilde{\mathcal{X}}_p \to \mathcal{X}_p \subseteq \mathbb{P}^5_{\mathbb{F}_p}$ is defined by a basis of the vector space over $\mathbb{F}_p$ of cubic polynomials in the ideal $(x^3 - y^2 z, xz, z^2)$.

Proof. It is clear that the subscheme defined by $(x^3 - y^2 z, xz, z^2)$ lies on C. Also, any point on this scheme would have to satisfy $z^2 = 0$ and $x^3 = yz^2$. This shows that the subscheme defined by $(x^3 - y^2 z, xz, z^2)$ is only supported in p_0. Let us compute the degree of this subscheme on the local chart $y = 1$. We can rewrite the ideal as $(x^3 - z, xz, z^2) = (x^3 - z, x^4, x^6) = (x^3 - z, x^4)$, which clearly defines a degree 4 subscheme.

The last statement follows from Lemma 2.7.20. □

COROLLARY 4.5.16. *We have the following equality of effective divisors*

$$\mathrm{div}_{\mathcal{X}_p} l_1 = \mathrm{div}_{\mathcal{X}_p} l_2 = 5E_4.$$

Proof. We know that the divisor of zeroes of l_1 is of degree 5 and supported in -1-curves. By Proposition 4.5.12 we know that there is a unique -1-curve E_4 on $\mathcal{X}_\ell$ and this implies the result. □

Since the map $\psi\colon \mathbb{P}^2_{\mathbb{F}_p} \dashrightarrow \mathcal{X}_p \subseteq \mathbb{P}^5_{\mathbb{F}_p}$ is given by a basis of the cubics in the ideal $(x^3 - y^2z, xz, z^2)$ we see that there is a correspondence between the cubics in the ideal $(x^3 - y^2z, xz, z^2) \subseteq \mathbb{F}_p[x,y,z]$ and the sections in $\mathrm{H}^0(\mathcal{X}_p, \omega^\vee)$.

LEMMA 4.5.17. *The hyperplane section on $\mathcal{X}_p$ cut out by l_1 corresponds to the cubic z^3 up to a factor of $\mathbb{F}_p^\times$.*

Proof. Note that l_1 does not vanish on $\mathcal{X}_p \backslash E_4$. This proves that the associated cubic curve on $\mathbb{P}^2_{\mathbb{F}_p}$ does not meet $\mathbb{A}^2_{\mathbb{F}_p} = \mathbb{P}^2_{\mathbb{F}_p} \backslash L$. We see that this cubic curve must be given by a multiple of z^3. □

After fixing a multiple of z^3 which corresponds to l_1 we get an actual correspondence between the elements of $\mathrm{H}^0(\mathcal{X}_p, \omega^\vee)$ and the homogeneous cubic polynomials in $(x^3 - y^2z, xz, z^2)$.

DEFINITION 4.5.18. Let h be a linear form on $\mathcal{X} \subseteq \mathbb{P}^5_{\mathbb{Z}}$ with coprime coefficients and let p be a prime for which $\mathcal{X}_p$ has a singular point of type A_4. The birational map $\pi\colon \mathcal{X}_p \dashrightarrow \tilde{\mathcal{X}}_p \to \mathbb{P}^2_{\mathbb{F}_p}$ induces an isomorphism $\kappa(\mathbb{P}^2_{\mathbb{F}_p}) \cong \kappa(\mathcal{X}_p)$. Let f_{hom} be the homogeneous polynomial over $\mathbb{F}_p$ such that $\frac{h}{l_1}$ corresponds to $\frac{f_{\mathrm{hom}}}{z^3}$ under this isomorphism. We will call f_{hom} and $f = f_{\mathrm{hom}}(x,y,1)$ the *associated homogeneous and inhomogeneous polynomials* of h at p.

We will study hyperplane sections on $\mathcal{X}_p \subseteq \mathbb{P}^5_{\mathbb{F}_p}$ by looking at the corresponding cubic polynomial in three variables.

The following lemma is the first example and identifies the effective anticanonical divisors of $\mathcal{X}_p$ as cubic plane curves.

LEMMA 4.5.19. *The hyperplane sections of $\mathcal{X}_p \subseteq \mathbb{P}^5_{\mathbb{F}_p}$ which contain E_4 are those for which the associated homogeneous polynomial lies in the ideal*

$$(zx^2, z^2).$$

Proof. Let us work on the affine part given by $y = 1$. We see from Lemma 2.7.19 that the cubic forms for which the associated anticanonical divisor is supported on E_4 lie in the ideal

$$(x,z)(x^3 - z, xz, z^2) = (x^4 - xz, z^2, x^2z).$$

The homogeneous part of this ideal of degree 3 is clearly generated by zx^2 and z^2. □

4.6 Setup for the following sections

We have introduced a lot of notation over the last few sections. For convenience of the reader we will group the notation we will fix for the remainder of this chapter.

SETUP 4.6.1. We start by fixing one, hence both, of the following equivalent objects

- N, a subgroup of $\widehat{\mathbb{Z}}$ of index 5;
- K, a degree 5 Galois extension of $\mathbb{Q}$.

The correspondence between these objects is described in Definition 4.4.1.
Then we fix

- σ, a generator of $\mathrm{Gal}(K/\mathbb{Q})$;
- α, an element of $\mathcal{O}_K$ generating K.

This produces the following intermediate objects, which we will refer to

- m_α and m_{α^2}, the minimal polynomials of α and α^2;
- $\mathcal{Z}$, the subscheme of $\mathbb{P}^2_{\mathbb{Z}}$ defined by
$$(x^2 - yz, z^5 m_\alpha(x/z), z^5 m_{\alpha^2}(y/z)) \subseteq \mathbb{Z}[x,y,z];$$
- Γ_0, the subscheme of $\mathbb{P}^2_{\mathbb{Z}}$ defined by $x^2 = yz$;
- $\mathcal{B}$, the blowup of $\mathbb{P}^2_{\mathbb{Z}}$ in $\mathcal{Z}$;
- β, the blowup morphism $\mathcal{B} \to \mathbb{P}^2_{\mathbb{Z}}$;
- Γ, the strict transform of Γ_0 along $\beta \colon \mathcal{B} \to \mathbb{P}^2_{\mathbb{Z}}$.

These objects were used in the process of defining

- $\mathcal{X}_\alpha$ or $\mathcal{X}$, which is a subscheme of $\mathbb{P}^5_{\mathbb{Z}}$;
- ϑ, the birational map $\vartheta \colon \mathbb{P}^2_{\mathbb{Z}} \dashrightarrow \mathcal{X}$;
- δ, the morphism $\delta \colon \mathcal{B} \to \mathcal{X}$ which contracts Γ.

We also fix the following notation for schemes and morphisms defined over either $\mathbb{Q}$ or $\mathbb{F}_\ell$.

- X_α or X, the generic fibre of $\mathcal{X}$;
- Z and B, the generic fibres of $\mathcal{Z}$ and $\mathcal{B}$;
- $\tilde{\mathcal{B}}_\ell$, the minimal desingularization of $\mathcal{B}_\ell$;
- $\tilde{\mathcal{X}}_\ell$, the minimal desingularization of $\mathcal{X}_\ell$;
- $\tilde{\Gamma}_\ell$, the strict transform of $\tilde{\beta}_\ell \colon \tilde{\mathcal{B}} \to \mathbb{P}^2_{\mathbb{F}_\ell}$;
- $\tilde{\delta}_\ell$, the morphism $\tilde{\mathcal{B}}_\ell \to \tilde{\mathcal{X}}_\ell$ which contracts the -1-curve $\tilde{\Gamma}_\ell$;

» ρ_ℓ, the contraction of -2-curves $\tilde{\mathcal{B}}_\ell \to \mathcal{B}_\ell$;

» γ_ℓ, the contraction of -2-curves $\tilde{\mathcal{X}}_\ell \to \mathcal{X}_\ell$.

It is even possible to define schemes $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{X}}$ over $\mathbb{Z}$, such that their fibres over $\mathbb{Z}$ are the minimal desingularizations of the corresponding fibres of $\mathcal{X}$ and $\mathcal{B}$. Also the morphisms ρ_ℓ, γ_ℓ have $\tilde{\delta}_\ell$ relative versions over $\mathbb{Z}$. We will not define these objects and one should keep in mind that part of the diagram in (4.6) is only defined on the fibres over $\mathbb{Z}$.

$$\begin{tikzcd}
& \tilde{\mathcal{X}} \arrow[dl, "\gamma"] & & \tilde{\mathcal{B}} \arrow[ll, "\tilde{\delta}"'] \arrow[dl, "\rho"] \arrow[dd, "\tilde{\beta}"] \\
\mathcal{X} & & \mathcal{B} \arrow[ll, "\delta"'] \arrow[dr, "\beta"] & \\
& & & \mathbb{P}^2_{\mathbb{Z}}. \arrow[lllu, dashed, "\vartheta"]
\end{tikzcd} \tag{4.6}$$

Other notation which will reappear in this chapter:

» n, the conductor of N and K;

» α_i, the conjugates of α in K;

» $P_i = (\alpha_i : \alpha_i^2 : 1)$, the K-points on $\mathbb{P}^2_{\mathbb{Q}}$ in the support of Z;

» $\mathrm{dP}_5(N)$, $\mathrm{dP}_5(K)$ or when it makes sense $\mathrm{dP}_5(n)$, for the isomorphism class of X;

» l_1 and l_2, linear forms over $\mathbb{Z}$ with coprime coefficients considered as elements of $\mathcal{O}_{\mathbb{P}^5_{\mathbb{Z}}}(1)$.

We will want to consider integral points on an affine open of $\mathcal{X}$ by choosing

» h, a linear form over $\mathbb{Z}$ with coprime coefficients considered as an element of $\mathcal{O}_{\mathbb{P}^5_{\mathbb{Z}}}(1)$;

» $\mathcal{C}$, the hyperplane section on $\mathcal{X}$ defined by $h = 0$;

» $\mathcal{U}$, the complement of $\mathcal{C}$ in $\mathcal{U}$;

» C and U, the generic fibres of $\mathcal{C}$ and $\mathcal{U}$.

This allows us to consider

» $\mathcal{A}$, the class of the Azumaya algebra $\left(\frac{l_1}{h}, K/k, \sigma\right)$ in $\operatorname{Br} U$.

We will also write

- » ℓ, for a general prime number;
- » p, for a prime number which is totally ramified in K;
- » $\mathfrak{l}$, a prime of K lying above ℓ;
- » $\mathfrak{p}$, the prime of K lying above p.

Let $\mathcal{X}_p$ be a fibre with a singular point of type A_4. When dealing with such a fibre we use the following notation introduced in Section 4.5.4

- » C, the cubic curve on $\mathbb{P}^2_{\mathbb{F}_p}$ given by $x^3 = y^2 z$;
- » p_0, the $\mathbb{F}_p$-point $(0\colon 1\colon 0)$ on C;
- » L, the line $z = 0$ tangent to C in p_0;
- » Z', the curvilinear subscheme associated to the ideal sheaf $\mathcal{I}_{C,p_0,4}$;
- » π, the birational map $\mathcal{X}_p \dashrightarrow \mathbb{P}^2_{\mathbb{F}_p}$ associated to Z';
- » ψ, the birational inverse of π;
- » E_i and R, the -2-curves E_1, E_2, E_3 and R, and the -1-curve E_4 on $\tilde{\mathcal{X}}_p$;
- » E_4, the image of $E_4 \subseteq \tilde{\mathcal{X}}_p$ along $\delta_p\colon \tilde{\mathcal{X}}_p \to \mathcal{X}_p$;
- » f and f_{hom}, the associated inhomogeneous and homogeneous polynomial of h at p.

4.7 Arithmetic of $\mathcal{U}$

Using the projective model $\mathcal{X}$ we can construct a model for affine surfaces with non-trivial algebraic Brauer group as described in Lemma 4.1.4.

DEFINITION 4.7.1. Let h be a primitive linear form defining a hyperplane in $\mathbb{P}^5_{\mathbb{Z}}$. Denote by $\mathcal{C}$ the curve on $\mathcal{X}$ defined by this hyperplane and write $\mathcal{U}$ for the complement of $\mathcal{C}$ in $\mathcal{X}$. We will denote the generic fibres of these objects by C and U. If we want to stress the dependence on h we might add it as a subscript.

Note that if C is geometrically irreducible then $\mathrm{Br}_1 U / \mathrm{Br}\, k$ is a cyclic group of order 5 by Proposition 4.1.3. To determine whether this might give an obstruction to the Hasse principle for integral points we are first interested in points on $\mathcal{U}$ over all completions of $\mathbb{Z}$.

LEMMA 4.7.2. *We have that $U(\mathbb{Q})$ is not empty and the same holds for $\mathcal{U}(\mathbb{Z}_\ell)$ if ℓ is a prime which is at least 7. The same is true for all odd ℓ for which the fibre $\mathcal{X}_\ell$ is smooth.*

Proof. Rational points on an ordinary del Pezzo surface of degree 5 are Zariski dense by Proposition 2.8.2. We conclude that the set of rational points on X cannot be contained in C. This proves the first statement.

Now let $\ell \geq 7$ be a prime. Since h is primitive we find that $\mathcal{C}_\ell$ is a degree 5 curve of arithmetic genus 1 in $\mathbb{P}^5_{\mathbb{F}_\ell}$. From the degree we see that $\mathcal{C}_\ell$ consists of at most 5 geometrically irreducible components. From the genus we deduce that at most one of these components is of arithmetic genus 1. The number of $\mathbb{F}_\ell$-points on an irreducible genus 0 curve is bounded by $\ell + 1$ and for a genus 1 curve it is at most $\ell + 1 + 2\sqrt{\ell}$. It now follows using the result in Lemma 4.5.2 that

$$\#\mathcal{U}(\mathbb{F}_\ell) = \#\mathcal{X}(\mathbb{F}_\ell) - \#\mathcal{C}(\mathbb{F}_\ell) \geq (\ell^2 + 1) - (5\ell + 5 + 2\sqrt{\ell}). \tag{4.7}$$

It is easily checked that the right hand side is at least 4 for $\ell \geq 7$. As there are at most two singular points on each fibre by Proposition 4.5.8 we find a smooth $\mathbb{F}_\ell$-point on $\mathcal{U}_\ell$ which lifts using Hensel's lemma to a $\mathbb{Z}_\ell$-point.

Now assume that $\mathcal{X}_\ell$ is smooth. That implies that m_α modulo ℓ is separable and either all roots are defined over $\mathbb{F}_\ell$ or over $\mathbb{F}_{\ell^5}$. In the first case the approach in (4.7), using the exact count $\#\mathcal{X}(\mathbb{F}_\ell) = \ell^2 + 5\ell + 1$, yields $\#\mathcal{U}(\mathbb{F}_\ell) \geq 1$. In the second case the fibre $\mathcal{X}_\ell$ is an interesting del Pezzo surface of degree 5 over $\mathbb{F}_\ell$. It follows from Lemma 4.1.6 that the geometrically irreducible components of a hyperplane section are all of degree 1 or 5. In the first case we find that $\mathcal{C}_\ell$ is the orbit of a geometric -1-curve which is defined over $\mathbb{F}_{\ell^5}$ and there are obviously no $\mathbb{F}_\ell$-points on $\mathcal{C}$. In the latter case we find that the hyperplane section is a geometrically irreducible curve of arithmetic genus 1. So we find

$$\#\mathcal{U}(\mathbb{F}_\ell) = \#\mathcal{X}(\mathbb{F}_\ell) - \#\mathcal{C}(\mathbb{F}_\ell) \geq (\ell^2 + 1) - (\ell + 1 + 2\sqrt{\ell})$$

which is positive for $\ell \geq 3$. □

For primes over which the fibre $\mathcal{X}_\ell$ is singular there might be better bounds if one fixes the multiplicities of the factors of m_α modulo ℓ. We will only need the following result.

LEMMA 4.7.3. *Let p be an odd prime such that $\mathcal{X}_p$ has a singularity of type A_4. The set $\mathcal{U}(\mathbb{Z}_p)$ is non-empty.*

In particular using Corollary 4.4.3 we see that if p is ramified in K, then $\mathcal{U}_p$ admits a $\mathbb{Z}_p$-point

Proof. Recall that $\mathcal{X}_p \subseteq \mathbb{P}^5_{\mathbb{F}_p}$ is a singular del Pezzo surface and $\mathcal{U}_p$ is the intersection of $\mathcal{X}_p$ with a fixed affine chart $\mathbb{A}^5_{\mathbb{F}_p}$ defined by $h \neq 0$. Furthermore, E_4 is a line in $\mathbb{P}^5_{\mathbb{F}_p}$ which lies on $\mathcal{X}_p$.

From the isomorphism between $\mathcal{X}_p \backslash E_4$ and $\mathbb{A}^2_{\mathbb{F}_p}$ we see that any $\mathbb{F}_p$-point of $\mathcal{U}_p \backslash E_4$ is smooth. So we can lift such a point to a $\mathbb{Z}_p$-point. So let us prove that $(\mathcal{U}_p \backslash E_4)(\mathbb{F}_p) \cong (\mathbb{A}^2_{\mathbb{F}_p} \backslash V(f))(\mathbb{F}_p)$ is non-empty, where f is the associated inhomogeneous polynomial associated to h. So it is enough to prove that this associated polynomial f of h is non-zero at some point of the affine plane. We know that f is non-zero and of degree at most 3. A geometrically irreducible affine plane cubic, conic or line with a rational point at infinity contains at most $p + 2\sqrt{p}$, p or p points respectively. So we see that f vanishes in at most $2p$ points which happens when f cuts out two parallel lines. Since $\#\mathbb{A}^2(\mathbb{F}_p) = p^2$ there is an $\mathbb{F}_p$-point on the affine plane which is not a zero of f. □

On the other hand, the bound for the smooth fibres cannot be sharpened in general; we will see examples where $\mathcal{X}_2$ is smooth but $\mathcal{U}(\mathbb{Z}_2)$ is empty.

4.8 Computation of the invariant maps

The following section is about computing the invariant maps for the element $\mathcal{A}$ of the Brauer group of U. At the infinite prime of $\mathbb{Q}$ we see that the map

$$\operatorname{inv}_\infty \mathcal{A} \colon U(\mathbb{R}) \to \frac{1}{2}\mathbb{Z}/\mathbb{Z}$$

is zero because $\mathcal{A}$ is of odd order. So from now on we will only consider the finite primes. At points in $\mathcal{U}(\mathbb{Z}_\ell)$ for a finite prime ℓ we have the following result.

LEMMA 4.8.1. *Fix a prime ℓ and let P be a point in $\mathcal{U}(\mathbb{Z}_\ell)$ such that $\frac{l_1}{h}(P)$ is defined and invertible modulo ℓ. Then*

$$\operatorname{inv}_\ell \mathcal{A}(P)$$

is $0 \in \mathbb{Q}/\mathbb{Z}$ precisely if the image of $\frac{l_1}{h}(P) \in \mathbb{Z}_\ell^\times$ under the homomorphism $\mathbb{Z}_\ell^\times \to \widehat{\mathbb{Z}}^\times$ lies in N.

Proof. Let P be such a point. Then $\mathcal{A}(P) \in \operatorname{Br}(\mathbb{Q}_\ell)$ is simply the class of the cyclic algebra $\left(\frac{l_1}{h}(P), \sigma\right)$. Proposition 1.3.2 tells us that the cyclic algebra $\left(\frac{l_1}{h}(P), \sigma\right)$ over $\mathbb{Q}_\ell$ is trivial in the Brauer group precisely when $\frac{l_1}{h}(P)$ is a norm in the extension $K_{\mathfrak{l}}/\mathbb{Q}_\ell$ for a prime $\mathfrak{l}$ lying above l. The group $N_{K_{\mathfrak{l}}/\mathbb{Q}_\ell}(K_{\mathfrak{l}}^\times)$ is the kernel of the top map in (4.4) in the proof of Proposition 4.4.2. This group is exactly the kernel of $\mathbb{Q}_\ell^\times \to \operatorname{Gal}(K/\mathbb{Q}) \cong \widehat{\mathbb{Z}}^\times/N$ since the map $\operatorname{Gal}(K_{\mathfrak{l}}/\mathbb{Q}_\ell) \to \operatorname{Gal}(K/\mathbb{Q})$ is an inclusion. □

Since the image of $\mathbb{Z}_\ell^\times \to \widehat{\mathbb{Z}}^\times$ is a subset of N for all but finitely primes we have the following result.

COROLLARY 4.8.2. *Consider a model $\mathcal{U}/\mathbb{Z}$ as before of conductor n. Let ℓ be a prime which does not ramify in K and $P \in \mathcal{U}(\mathbb{Z}_\ell)$ be a point where l_1 or l_2 is invertible modulo ℓ. Then we have* $\operatorname{inv}_\ell \mathcal{A}(P) = 0$.

Proof. We saw in the proof of Proposition 4.4.2 that ℓ is unramified in K precisely when the image of $\mathbb{Z}_\ell^\times \to \widehat{\mathbb{Z}}^\times$ lies in N. This proves the claim. □

For primes $p \equiv 1 \mod 5$ which divide the conductor of such a subgroup N, i.e. the tamely ramified primes of K, we have the following result.

COROLLARY 4.8.3. *Let p a prime which is tamely ramified in K. This implies that $p \equiv 1 \mod 5$. Write W for the set of $\mathbb{Z}_p$-points P on $\mathcal{U}$ where $\frac{l_1}{h}(P)$ is invertible in $\mathbb{Z}_p$ and let $q\colon W \to (\mathcal{U}_p \backslash E_4)(\mathbb{F}_p)$ be the reduction map. This map is surjective and the invariant map $\operatorname{inv}_p \mathcal{A}$ on W can be computed as*

$$W \overset{q}{\twoheadrightarrow} (\mathcal{U}_p \backslash E_4)(\mathbb{F}_p) \xrightarrow{\frac{l_1}{h}} \mathbb{F}_p^\times/(\mathbb{F}_p^\times)^5 \hookrightarrow \mathbb{Q}/\mathbb{Z}$$

where the second map sends a point P to the class of $\frac{l_1}{h}(P)$ modulo fifth powers and the last map is a group isomorphism onto the unique subgroup of $\mathbb{Q}/\mathbb{Z}$ of order 5.

In particular we see that for a point $P \in \mathcal{U}(\mathbb{Z}_p)$ which does not reduce modulo p to a point on E_4 that $\operatorname{inv}_p \mathcal{A}(P) = 0$ precisely if $\frac{l_1}{h}(P)$ is a fifth power modulo p.

Proof. Consider an $\mathbb{F}_p$-point $\bar{P}$ in $\mathcal{U}_p \backslash E_4$. As $\frac{l_1}{h}(\bar{P})$ is invertible the point $\bar{P}$ cannot lie in the zero locus of l_1. By Lemma 4.5.5 all singular points lie on the intersection of the zero loci of l_1 and l_2. So by assumption $\bar{P}$ must be a smooth point. This implies that $\bar{P}$ lifts using Hensel's lemma to a point P in W. This shows that q is surjective.

The last statement follows from Lemma 4.4.4. □

These last two results are a restatement of the fact that in unramified extensions of local fields every norm is a unit, and agrees with the fact that in a totally but tamely ramified extension every principal unit is a norm.

For the wildly ramified prime 5 the situation is a little different.

COROLLARY 4.8.4. *Suppose that 5 is ramified in K and let $P \in \mathcal{U}(\mathbb{Z}_5)$ be a point which does not lie in the zero locus of both l_1 and l_2 modulo 5. Then P lies in the kernel of $\operatorname{inv}_5 \mathcal{A}$ precisely if for one i, or equivalently both, we have $\frac{l_i}{h}(P) \in \{\pm 1, \pm 7\}$ modulo 5^2.*

Proof. This proof is similar to the proof of the previous corollary and again relies on Lemma 4.4.4. □

We see that in a point P over $\mathbb{Z}_\ell$ not reducing to the zero locus of both l_1 and l_2 modulo ℓ the value of $\operatorname{inv}_\ell \mathcal{A}(P)$ follows from these three corollaries. We will now show how to compute the invariant map on the remaining points or why we should not bother to do so.

4.8.1 The invariant map for unramified primes

The invariant map is often best understood at unramified primes. This is also the case here.

LEMMA 4.8.5. *Suppose a prime ℓ splits completely in K. The map*

$$\operatorname{inv}_\ell \mathcal{A} \colon \mathcal{U}(\mathbb{Z}_\ell) \to \mathbb{Q}/\mathbb{Z}$$

is identically equal to zero.

Proof. Let P be a $\mathbb{Z}_\ell$-point of $\mathcal{U}$. The evaluation map $\operatorname{Br} U \to \operatorname{Br} \mathbb{Q}_\ell, \mathcal{A} \mapsto \mathcal{A}(P)$ factors through $\operatorname{Br} U_{\mathbb{Q}_\ell}$. For a prime $\tilde{\mathfrak{l}}$ above a completely split prime ℓ we find that $K_{\tilde{\mathfrak{l}}} = \mathbb{Q}_\ell$. Since $\mathcal{A}_{\mathbb{Q}_\ell} \in \operatorname{Br} U_{\mathbb{Q}_\ell}$ is split by $K_{\tilde{\mathfrak{l}}}$ we see that it is trivial and so is the image $\mathcal{A}(P)$ in $\operatorname{Br} \mathbb{Q}_\ell$. □

For inert primes we have similar, but slightly weaker, result.

LEMMA 4.8.6. *Suppose that ℓ is an inert prime. The map* inv_ℓ *is identically equal to zero if $\mathcal{U}_\ell$ is smooth over $\mathbb{F}_\ell$. If $\mathcal{U}_\ell$ is singular, then* inv_ℓ *is zero on the points in $\mathcal{U}(\mathbb{Z}_\ell)$ which do not reduce to the singular point of $\mathcal{U}_\ell$.*

Note that by Lemma 4.5.9 that $\mathcal{X}_\ell$ has at most one singular point. So the same is true for $\mathcal{U}_\ell$.

Proof. Note that inv_ℓ being constant on the indicated point sets follows from [6, Theorem 1]. We will however need to prove a stronger result.

There is a correspondence between the factors of m_α over $\mathbb{Q}_\ell$ and primes of K dividing ℓ. Since ℓ is inert in K there is a unique prime $\mathfrak{l}$ in K lying above ℓ. This shows that m_α is irreducible over $\mathbb{Q}_\ell$ and hence $U_{\mathbb{Q}_\ell}$ is an interesting del Pezzo surface. In particular, there are no $\mathbb{Q}_\ell$-points on U in the zero locus of both l_1 and l_2.

Consider a point $P \in \mathcal{U}(\mathbb{Z}_\ell)$. By the above discussion we see that $\frac{l_i}{h}(P) \neq 0 \in \mathbb{Z}_\ell$ for at least one i. Assume without loss of generality that $i = 1$. We will use the representation $\left(\frac{l_1}{h}, \sigma\right)$ of $\mathcal{A}$ to prove that $\operatorname{inv}_\ell \mathcal{A}(P)$ is equal to zero if P does not reduce to a singular point on $\mathcal{U}_\ell$. We will have to show that $\frac{l_1}{h}(P)$ is the of an element in $\mathcal{O}_{K_{\mathfrak{l}}}$. We have seen in Corollary 4.8.2 that only the valuation of $\frac{l_1}{h}(P)$ matters. We will show that this valuation is always a multiple of 5.

If $\frac{l_1}{h}(P)$ is a unit in $\mathbb{Z}_\ell$ then this is clear. So suppose ℓ divides $\frac{l_1}{h}(P)$ and let $\bar{P} \in \mathcal{U}(\mathbb{F}_\ell)$ be the reduction of P modulo ℓ. This means that $\bar{P}$ lies on the zero locus of l_1 on $\mathcal{U}_\ell \subseteq \mathcal{X}_\ell$. By Lemma 4.5.9 there are two possibilities for this zero locus.

If m_α is irreducible in $\mathbb{F}_\ell$ then $\mathcal{X}_\ell$ is an interesting del Pezzo surface. So the zero locus of l_1 consists of five conjugate lines. In particular there are no $\mathbb{F}_\ell$-points in the zero divisor of l_1.

If m_α reduces to the fifth power of a linear function modulo ℓ then $\mathcal{X}_\ell$ contains one line E_4 and one singular point which lies on this line. Locally around $\bar{P}$ the closed subscheme E_4 is Cartier since $\bar{P}$ is assumed to be smooth. This proves that in the local ring at $\bar{P}$ the function $\frac{l_1}{h}$ is the fifth power of the function defining E_4, since $\operatorname{div}_{\mathcal{X}_\ell} l_1 = 5E_4$. This proves that the valuation of $\frac{l_1}{h}(P)$ is a multiple of 5.

We conclude that $\operatorname{inv}_\ell \mathcal{A}(P) = 0$ for all indicated points $P \in \mathcal{U}(\mathbb{Z}_\ell)$. □

4.8.2 The invariant map for tamely ramified primes

Let us consider the case of a prime p which is ramified in K. By construction of the field K, the only primes which can be ramified are the primes 5 and the ones which are $1 \mod 10$. The prime 5 is wildly ramified, but we will first consider the tamely ramified primes. By Corollary 4.8.3 we see that the invariant map on $\mathbb{Z}_p$-points is largely determined by the $\mathbb{F}_p$-points of $\mathcal{U}_p$ (see also [6, Theorem 1]).

Unlike for the unramified primes we will see that $\mathrm{inv}_p \mathcal{A}$ is only constant in an obvious situation and in all other cases it takes many values. To prove these results we will first look at the following lemmas on affine curves over $\mathbb{F}_p$ which we will translate back to $\mathcal{X}_p$ using the isomorphism in Proposition 4.5.12.

LEMMA 4.8.7. *Fix a prime $p \geq 5$ and let $f \in \mathbb{F}_p[x, y]$ be a non-constant irreducible polynomial of degree $d \leq 3$. Also assume that*

» *if $d = 3$ the projective closure of the corresponding affine curve is geometrically integral and intersects the line at infinity in a single point with multiplicity 3;*

» *if $d = 2$ the corresponding curve either has two distinct rational points at infinity, or it has a single point at infinity and is geometrically integral.*

Then the map $f \colon \mathbb{F}_p^2 \to \mathbb{F}_p$ is surjective.

Proof. Let us write $f_{\mathrm{hom}} \in \mathbb{F}_p[x, y, z]$ for the homogenization of f. We can now consider the following cases depending on the degree of f.

Case 1: f is a linear polynomial. In this case f is a non-constant linear map, and it is clear that it takes all values in $\mathbb{F}_p$ on $\mathbb{F}_p^2$.

Case 2: f is a quadratic polynomial. There are two cases to consider. First assume f_{hom} has two distinct rational solutions at $z = 0$. Without loss of generality these points are $(1\colon 0\colon 0)$ and $(0\colon 1\colon 0)$ and then f must be of the form $\kappa xy + \lambda x + \mu y + \nu$, where $\kappa \neq 0$. Now fix y_0 such that $\kappa y_0 + \lambda$ is non-zero. Then we see that $f(x, y_0) = x(\kappa y_0 + \lambda) + \mu y_0 + \nu$ assumes all values in $\mathbb{F}_p$.

Now consider the case where f_{hom} is geometrically integral and has a single solution with $z = 0$. Assume without loss of generality that this point is $(0\colon 1\colon 0)$. This implies that f can be written as $f = \kappa x^2 + \lambda x + \mu y + \nu$. Since the curve defined by f_{hom} is geometrically integral we find $\mu \neq 0$ which directly implies that f is surjective.

Case 3: f is a cubic polynomial. We know that f has a single point at infinity of multiplicity 3. Assume that this point is $(0\colon 1\colon 0)$. We will show that f is surjective by proving that the projective cubic plane curve C_λ defined by the cubic form $f_{\mathrm{hom}} - \lambda z^3$ has an $\mathbb{F}_p$-point satisfying $z \neq 0$ for all $\lambda \in \mathbb{F}_p$.

We will first show that each of these p plane curves C_λ does not split into three lines over $\bar{\mathbb{F}}_p$. Note that a cubic plane curve defined by a cubic homogeneous polynomial F passes through $(0\colon 1\colon 0)$ with multiplicity three and splits into three lines precisely when F is independent of y. Since C_0 is geometrically integral by assumption we see that f_{hom} does depend on y. So the same holds for all $f_{\text{hom}} - \lambda z^3$ and none of the curves C_λ splits into three lines over $\bar{\mathbb{F}}_p$.

So each plane curve C_λ is either geometrically integral or factors into a linear and quadratic factor. We will prove in each case that the curve C_λ has at least two $\mathbb{F}_p$-points, and since it has single point $(0\colon 1\colon 0)$ at infinity it will have an $\mathbb{F}_p$-points satisfying $z \neq 0$.

A geometrically integral cubic curve has at least $p + 1 - 2\sqrt{p}$ points and as $p \geq 5$ there must be a point away from infinity. In the second case we have a projective curve consisting of line and a conic defined over $\mathbb{F}_p$, both with the given point $(0\colon 1\colon 0)$ at infinity. Either component will have rational points on the affine part defined by $z \neq 0$.

We see that for all $\lambda \in \mathbb{F}_p$ the curve C_λ has a point satisfying $z \neq 0$ and we conclude that f is surjective on the affine plane. □

If f defines a non-reduced conic we have the following weaker statement.

LEMMA 4.8.8. *Let $f \in \mathbb{F}_p[x]$ be the square of a non-constant linear function over $\mathbb{F}_p$ with root ρ. The map $f\colon \mathbb{F}_p \backslash \{\rho\} \to \mathbb{F}_p^\times / (\mathbb{F}_p^\times)^5$ is surjective.*

LEMMA 4.8.9. *Fix a prime p and let f_{hom} be a homogeneous cubic polynomial in the ideal $(x^3 - y^2 z, xz, z^2)$ over $\mathbb{F}_p$. The polynomial $f = f_{hom}(x, y, 1)$ either*

- » *is constant;*
- » *vanishes on two distinct parallel lines on $\mathbb{A}^2_{\mathbb{F}_p}$;*
- » *satisfies the conditions in Lemma 4.8.7, or*
- » *is independent of y and satisfies the conditions in Lemma 4.8.8.*

In the second case we allow the lines to be conjugated over $\mathbb{F}_p$.

Proof. Let us write $f_{\text{hom}} = \kappa x^3 + z(\mu(x,y)x + \nu(x,y,z)z - \kappa y^2)$ where μ and ν are homogeneous linear polynomials in the indicated variables. In the case that $\kappa = 0$ the polynomial f is of degree at most 2 and all points with $z = 0$ are rational points. All these polynomials are covered among the four cases.

If $\kappa \neq 0$ we have a cubic polynomial $f = \kappa x^3 + (\mu(x,y)x + \nu(x,y,1) - \kappa y^2)$, which has an inflection point at $(0\colon 1\colon 0)$ and the result of Lemma 4.8.7 holds for such f. □

We have seen a surjectivity statement for the last two of these four cases. In the first case we will not find such a result and in the second we have the following proposition, which applies since f is going to be the product of two distinct lines precisely when it is independent of y.

LEMMA 4.8.10. *Let p be a prime and $f \in \mathbb{F}_p[x]$ a quadratic polynomial with distinct roots ρ_1 and ρ_2 in $\bar{\mathbb{F}}_p$. The homomorphism $f\colon \mathbb{F}_p \backslash \{\rho_1, \rho_2\} \to \mathbb{F}_p^\times / (\mathbb{F}_p^\times)^5$ is surjective when $p \geq 31$.*

Proof. Write $f = ax^2 + bx + c$ with $a \neq 0$ and fix a $\lambda \in \mathbb{F}_p^\times$. We will prove that if $p \geq 31$ there exist $u, v \in \mathbb{F}_p$ with v invertible, such that $f(u) = \lambda v^5$. Consider the affine hyperelliptic curve $f(x) = \lambda y^5$, which is smooth because f has distinct roots. The smooth birational model for this curve is of genus 2 and has at least $p + 1 - 4\sqrt{p}$ points, one of which does not lie on the affine part given by $f(x) = \lambda y^5$ and at most two points satisfy $y = 0$. So the number of $u, v \in \mathbb{F}_p$ with v invertible satisfying $f(u) = \lambda v^5$ is at least $p + 1 - 4\sqrt{p} - 3$. This is positive if $p \geq 31$, which proves the statement. □

Remark. One can check that if $p = 11$ the map $f \colon \mathbb{F}_p \backslash \{\rho_1, \rho_2\} \to \mathbb{F}_p^\times / (\mathbb{F}_p^\times)^5$ is never surjective; the image of $f \colon \mathbb{F}_{11} \backslash \{\rho_1, \rho_2\} \to \mathbb{F}_{11}^\times / (\mathbb{F}_{11}^\times)^5$ for any separable quadratic polynomial f is of size 4.

We have seen that the cubics in $(x^3 - y^2 z, xz, z^2)$ correspond to the effective divisors of degree 3 on $\mathbb{P}^2_{\mathbb{F}_p}$ which pull back to effective anticanonical divisors on $\mathcal{X}_p$. Considering the different bounds in Lemma 4.8.7, Lemma 4.8.8 and Lemma 4.8.10 we get a trichotomy of effective anticanonical divisors on $\mathcal{X}$.

DEFINITION 4.8.11. Let p be a prime which is ramified in K, h an irreducible linear form on $\mathcal{X} \subseteq \mathbb{P}^5_{\mathbb{Z}}$ and f the associated inhomogeneous polynomial of h at p. We say that h at the prime p is of

» *class I* if f is constant;

» *class II* if f vanishes on two distinct parallel lines in $\mathbb{A}^2_{\mathbb{F}_p}$;

» *class III* in all other cases.

We have seen that these three classes partition the five-dimensional projective space of linear forms on $\mathcal{X}_p$ over $\mathbb{F}_p$. As the invariant map for tamely ramified primes is determined by the reduction of a point modulo p we get the following result.

THEOREM 4.8.12. *Let h be a linear form on the surface $\mathcal{X} \subseteq \mathbb{P}^5_{\mathbb{Z}}$ constructed using a degree 5 number field K. Let p be a prime which is tamely ramified in K.*

The invariant map $\mathrm{inv}_p \mathcal{A} \colon \mathcal{U}_h(\mathbb{Z}_p) \to \frac{1}{5}\mathbb{Z}/\mathbb{Z}$

» *is constant if h is of class I;*

» *misses exactly one value in $\frac{1}{5}\mathbb{Z}/\mathbb{Z}$ if h is of class II and $p = 11$;*

» *is surjective if h is of class II and $p \geq 31$;*

» *is surjective if h is of class III.*

Since p is congruent to 1 modulo 10 this covers all possible cases.

Proof. We have a birational map $\pi \colon \mathcal{X}_p \dashrightarrow \mathbb{P}^2_{\mathbb{F}_p}$ such that the isomorphism on function fields identifies $\frac{l_1}{h}$ on $\mathcal{X}_p$ with $\frac{z^3}{f_{\mathrm{hom}}}$ on $\mathbb{P}^2_{\mathbb{F}_p}$. On the projective plane we have the line L given by $z = 0$ and on $\mathcal{X}_p$ the -1-curve E_4 given by $l_1 = 0$, such that the birational map π restricts to an isomorphism $\mathcal{X}_p \backslash E_4 \cong \mathbb{P}^2_{\mathbb{F}_p} \backslash L \cong \mathbb{A}^2_{\mathbb{F}_p}$ as we saw in Proposition 4.5.12.

This proves the following chain of isomorphisms

$$\mathcal{U}_p \backslash E_4 \cong \mathcal{X}_p \backslash (V(h) \cup E_4) \cong \mathbb{P}^2_{\mathbb{F}_p} \backslash (V(f_{\mathrm{hom}}) \cup L) \cong \mathbb{A}^2_{\mathbb{F}_p} \backslash V(f).$$

Define the set W of $\mathbb{Z}_p$-points P on $\mathcal{U}$ where $\frac{l_1}{h}(P)$ is invertible in $\mathbb{Z}_p$. The reduction map $q\colon W \to (\mathcal{U}_p \backslash E_4)(\mathbb{F}_p)$ is surjective by Corollary 4.8.3.

We dehomogenize the function fields on the affine schemes $\mathcal{U}_p$ and $\mathbb{A}^2_{\mathbb{F}_p}$ using $l_1 = 1$ and $z = 1$ to find the commutative diagram in (4.8) which can be used to compute the invariant map for points in W.

$$\begin{tikzcd}
W \arrow[r, "q"] & (\mathcal{U}_p \backslash E_4)(\mathbb{F}_p) \arrow[r, "\frac{l_1}{h} = \frac{1}{h}"] \arrow[d, "\cong"] & \mathbb{F}_p^\times / (\mathbb{F}_p^\times)^5 \arrow[r] \arrow[d, equal] & \mathbb{Q}/\mathbb{Z} \arrow[d, equal] \\
 & \left(\mathbb{A}^2_{\mathbb{F}_p} \backslash V(f)\right)(\mathbb{F}_p) \arrow[r, "\frac{1}{f}"] & \mathbb{F}_p^\times / (\mathbb{F}_p^\times)^5 \arrow[r] & \mathbb{Q}/\mathbb{Z}
\end{tikzcd} \tag{4.8}$$

Now if h is of class I or II we get from Lemma 4.5.19 that h vanishes along E_4 and so $W = \mathcal{U}_p(\mathbb{Z}_p)$. By the surjectivity of $q\colon \mathcal{U}_p(\mathbb{Z}_p) \to (\mathcal{U}_p \backslash E_4)(\mathbb{F}_p)$ and the injectivity of $\mathbb{F}_p^\times / (\mathbb{F}_p^\times)^5 \to \mathbb{Q}/\mathbb{Z}$ we see that $\mathrm{inv}_p\, \mathcal{A}\colon \mathcal{U}_p(\mathbb{Z}_p) \to \mathbb{Q}/\mathbb{Z}$ has the same image as the map $\frac{1}{f}\colon \left(\mathbb{A}^2_{\mathbb{F}_p} \backslash V(f)\right)(\mathbb{F}_p) \to \mathbb{F}_p^\times / (\mathbb{F}_p^\times)^5$. If h is of class I at p, then f is constant by definition. The statements for h of class II follow from Lemma 4.8.10 and the remark following this lemma.

Now assume that h is of class III. The surjectivity of q, Lemma 4.8.7 and Lemma 4.8.8 combine to show that $\mathrm{inv}_p\, \mathcal{A}$ is surjective on the set W of $\mathbb{Z}_p$-points of $\mathcal{U}$ not reducing to E_4 so it is definitely surjective on $\mathcal{U}(\mathbb{Z}_p)$. □

In the cases where $\mathrm{inv}_p\, \mathcal{A}\colon \mathcal{U}(\mathbb{Z}_p) \to \frac{1}{5}\mathbb{Z}/\mathbb{Z}$ is surjective we cannot get a Brauer–Manin obstruction to the integral Hasse principle. In particular we have the following result coming from Lemma 4.5.19.

COROLLARY 4.8.13. *Let p be a prime which is tamely ramified in K and let h define a hyperplane section on $\mathcal{X} \subseteq \mathbb{P}^5_{\mathbb{Z}}$ such that the -1-curve $E_4 \subseteq \mathcal{X}_p$ does not lie in the zero locus of the reduction of h modulo p. The invariant map $\mathrm{inv}_p\, \mathcal{A}\colon \mathcal{U}_h(\mathbb{Z}) \to \frac{1}{5}\mathbb{Z}/\mathbb{Z}$ is surjective.*

This shows that a Brauer–Manin obstruction to the integral Hasse principle will not exist if h does not cut out the unique -1-curve over each tamely ramified prime. If it does then by the isomorphism in Proposition 4.5.12 we see that $\mathcal{U}_p$ is isomorphic to an open subscheme of $\mathbb{A}^2_{\mathbb{F}_p}$ which generally simplifies the situation.

4.8.3 The invariant map for the wildly ramified prime 5

We have looked at the invariant maps at unramified and tamely ramified primes. The last case is that of wildly ramified primes. The only prime which can be wildly ramified in our number fields K of degree 5 is the prime 5 which happens precisely if 5 divides the conductor n.

We have already seen a useful result in Corollary 4.8.4. We will also make frequent use of the following easy lemma.

LEMMA 4.8.14. *Two different lifts to $(\mathbb{Z}/25\mathbb{Z})^\times$ of an element $(\mathbb{Z}/5\mathbb{Z})^\times$ lie in different cosets of the subgroup $\{\pm 1, \pm 7\} \subseteq (\mathbb{Z}/25\mathbb{Z})^\times$.*

Now let us show that in many cases the invariant map is surjective.

LEMMA 4.8.15. *Suppose that 5 is ramified in K. Then*

$$\mathrm{inv}_5\, \mathcal{A} \colon \mathcal{U}_h(\mathbb{Z}_5) \to \frac{1}{5}\mathbb{Z}/\mathbb{Z}$$

is surjective if h is not of class I.

Proof. Pick coordinates u_i on $\mathbb{P}^5_{\mathbb{F}_5}$ and x, y and z on $\mathbb{P}^2_{\mathbb{F}_5}$ such that the birational map

$$\psi \colon \mathbb{P}^2_{\mathbb{F}_5} \dashrightarrow \mathcal{X}_5$$

is given by $(x, y, z) \mapsto (z^3, xz^2, yz^2, x^2z, xyz, y^2z - x^3)$ and we find an isomorphism after restricting to the affine opens $\mathbb{A}^2_{\mathbb{F}_5}$ and $\mathcal{X}_5 \backslash E_4$ respectively given by $z = 1$ and $u_0 = 1$. The size of the image of $\mathrm{inv}_5\, \mathcal{A} \colon \mathcal{U}_h(\mathbb{Z}_5) \to \frac{1}{5}\mathbb{Z}/\mathbb{Z}$ only depends on $\mathcal{A}$ up to constants. Since we also know that $l_1 \mod 5$ is a multiple of u_0, we can assume without loss of generality that $u_0 \equiv l_1 \mod 5$.

Pick an $\mathbb{F}_5$-point $\bar{P}$ on $\mathcal{V} := \mathcal{U}_5 \backslash E_4$. Recall that the map ψ restricts to an isomorphism on $\mathcal{V}$ to an open subscheme of $\mathbb{A}^5_{\mathbb{F}_5}$. Then $\bar{P}$ is an $\mathbb{F}_5$-point of $\mathcal{X}$ which does not lie in the zero locus of h and l_1. This guarantees that $\bar{P}$ is a smooth point and $\lambda := \frac{l_1}{h}(\bar{P})$ is defined and invertible modulo 5. We will show that at the $\mathbb{Z}_5$-points P reducing to $\bar{P}$ the function $\frac{l_1}{h}$ assumes either one or five values modulo 25.

Let us work with $\frac{h}{l_1}$ which on $\mathcal{V}$ becomes

$$h_{\mathrm{aff}} = a_0 + a_1 u_1 + a_2 u_2 + a_3 u_3 + a_4 u_4 + a_4 u_5.$$

Now let $\vec{x} = (x_1, x_2, x_3, x_4, x_5)$ be a 5-tuple of integers reducing to $\bar{P} \in \mathcal{V}$. We will first show that the lifts of $\bar{P}$ to points in $\mathcal{X}(\mathbb{Z}/25\mathbb{Z})$ are $\vec{x} + 5\vec{w}$ where $\vec{w}$ is any vector in a translation of the tangent space of $\mathcal{V}$ at $\bar{P}$.

Suppose that $\mathcal{X}$ is given by polynomials g_j in the variables u_i. The tangent space at $\bar{P}$ is by definition

$$T_{\bar{P}}\mathcal{V} = \left\{ \vec{v} \in \mathbb{F}_5^5 : \sum_{i=1}^{5} \frac{dg_j}{du_i}(\bar{P}) v_i \equiv 0 \mod 5 \right\}$$

and if $\vec{x} + 5\vec{w} \in \mathcal{X}(\mathbb{Z}/25\mathbb{Z})$ then for all j

$$0 \equiv g_j(\vec{x} + 5\vec{w}) \equiv g_j(\vec{x}) + 5\sum_{i=1}^{5} \frac{dg_j}{du_i}(\vec{x})w_i \mod 25$$

which proves the claim.

To compute h_{aff} at these lifts, let us write $\vec{a} = (a_1, a_2, a_3, a_4, a_5) \in \mathbb{Z}^5$. Then we find

$$h_{\mathrm{aff}}(\vec{x} + 5\vec{w}) \equiv h_{\mathrm{aff}}(\vec{x}) + 5\vec{a} \cdot \vec{w} \mod 25.$$

So we find all possible lifts modulo 25 of $h_{\mathrm{aff}}(\vec{x}) \mod 5$ when $\vec{a} \cdot \vec{w}$ is not constant modulo 5 or equivalently that there exists a $\vec{v} \in T_P\mathcal{V}$ such that $5 \nmid \vec{a} \cdot \vec{v}$. This clearly does not happen when $\vec{a} \equiv 0 \mod 5$, so let us assume that one of the components of $\vec{a}$ is not divisible by 5.

We will prove that under this assumption we can find a point $\bar{P} \in \mathcal{V}(\mathbb{F}_p)$ and a tangent vector $\vec{v} \in T_{\bar{P}}\mathcal{V}$ such that $5 \nmid \vec{a} \cdot \vec{v}$. Using ψ we can translate this problem to $\mathbb{A}^2_{\mathbb{F}_5} \backslash \{f = 0\}$. Using the equations of ψ we see that the tangent space at $\psi(x, y)$ is generated by the vectors $\vec{v}_1 = (1, 0, 2x, y, -3x^2)$ and $\vec{v}_2 = (0, 1, 0, x, 2y)$. We are looking for a point (x, y) such that $f(x, y) = h_{\mathrm{aff}} \circ \psi(x, y) \neq 0$ and $\vec{a} \cdot \vec{v}_1$ or $\vec{a} \cdot \vec{v}_2$ is non-zero.

Note that the degrees of $\vec{a} \cdot \vec{v}_1$ and $\vec{a} \cdot \vec{v}_2$ as polynomials in x and y are at most 2 and 1 and that at least one is non-zero since we assumed that $\vec{a} \neq 0 \mod 5$. If the linear equation $\vec{a} \cdot \vec{v}_2$ is non-zero it vanishes in at most five points of $\mathbb{A}^2(\mathbb{F}_5)$. If it is the zero polynomial then $\vec{a} \cdot \vec{v}_1$ is a non-zero linear polynomial. We see that $\vec{a} \cdot \vec{v}_1$ and $\vec{a} \cdot \vec{v}_2$ vanish simultaneously in at most 5 points. We have seen in the proof of Lemma 4.7.3 that f vanishes in at most $2p = 10$ points on $\mathbb{A}^2_{\mathbb{F}_p}$. So if $\vec{a} \not\equiv 0 \mod 5$ then there exists a point $\bar{P} \in \mathcal{V}(\mathbb{F}_5)$ and a vector $\vec{v}$ in its tangent space such that $\vec{a} \cdot \vec{v} \not\equiv 0 \mod 5$.

In particular if $\vec{a} \not\equiv 0 \mod 5$ we see that there exists a point $\bar{P} \in \mathcal{V}(\mathbb{F}_5)$ such that $\vec{a} \cdot \vec{v} \mod 5$ is surjective on the tangent space of $\bar{P}$, proving that h_{aff} assumes five values modulo 25 on $\mathbb{Z}_5$-points P reducing to $\bar{P}$. We see from Lemma 4.8.14 that h_{aff} assumes all values in $(\mathbb{Z}/25\mathbb{Z})^\times$ modulo fifth powers and so do $\frac{h}{l_1}$ on $\mathcal{U}_5$ and $\frac{l_1}{h}$ on $\mathcal{U}$. This proves the surjectivity of $\mathrm{inv}_5\, \mathcal{A}$ in this case.

If on the other hand $\vec{a} \equiv 0 \mod 5$ we find that h_{aff} assumes only one value modulo 25 on $\mathbb{Z}_p$-points reducing to a fixed point $\bar{P} \in \mathcal{V}(\mathbb{F}_5)$. We also see that 5 cannot divide the coefficient of u_0 in h since it already divides all the other coefficients and so h is of class I. □

In particular we have proved the following result similar to Lemma 4.8.13.

COROLLARY 4.8.16. *Suppose that* 5 *divides the conductor n and consider a hyperplane section given by* $h = 0$ *on* $\mathcal{X} \subseteq \mathbb{P}^5_{\mathbb{Z}}$ *such that the* -1*-curve* $E_4 \subseteq \mathcal{X}_5$ *does not lie on the zero locus of the reduction of h modulo* 5*. The invariant map* $\mathrm{inv}_5\, \mathcal{A}\colon \mathcal{U}_h(\mathbb{Z}) \to \frac{1}{5}\mathbb{Z}/\mathbb{Z}$ *is surjective.*

We can even classify all of the remaining hyperplane sections for which the evaluation map is not surjective.

THEOREM 4.8.17. *Suppose that* 5 *is ramified in K. Then* $\mathrm{inv}_5 \mathcal{A} \colon \mathcal{U}(\mathbb{Z}_5) \to \frac{1}{5}\mathbb{Z}/\mathbb{Z}$ *is not surjective precisely when there exist integers* λ, c_1 *and* c_3 *satisfying* $5 \nmid \lambda$, *and* $5 \mid c_1, c_3$ *or* $5 \nmid c_1$ *such that*

$$h \equiv \lambda l_1 + 5(c_1 u_1 + c_3 u_3) \mod 25$$

for specific hyperplane sections $u_1, u_3 \in \mathcal{O}_{\mathbb{P}^5_{\mathbb{Z}}}(1)$. *The invariant map is constant when* $5 \mid c_1, c_3$ *and otherwise the size of its image is* 3.

Proof. Let us again fix coordinates u_i for $1 \leq i \leq 5$ on $\mathcal{X}_5 \subseteq \mathbb{P}^5_{\mathbb{F}_5}$ such that their reduction modulo 5 corresponds to $(z^3, xz^2, yz^2, x^2z, xyz, y^2z - x^3)$ under the isomorphism of function fields described in Proposition 4.5.12.

Since l_1 and h are both of class I at 5 we see that there exist an integer λ such that $5 \nmid \lambda$ and $h \equiv \lambda l_1 \mod 5$. This implies the existence of a polynomial k over $\mathbb{Z}$ such that $h - \lambda l_1 = 5k$.

It follows for any point $P \in \mathcal{U}(\mathbb{Z}_5)$ that $\frac{h}{l_1}(P) \equiv \lambda \mod 5$ and that $\frac{h}{l_1}(P) \mod 25$ only depends on the image of P in $\mathcal{U}(\mathbb{F}_5)$.

Let us compute $\frac{h}{l_1}$ at P modulo 25:

$$\begin{aligned} \frac{h}{l_1}(P) &= \frac{\lambda l_1 + 5k}{l_1}(P) \\ &= \lambda + 5\frac{k}{l_1}(P). \end{aligned}$$

We know that $z^3 \frac{k}{l_1}$ corresponds to a homogeneous cubic polynomial f_{hom} in the ideal $(x^3 - y^2z, xz, z^2)$ of $\mathbb{F}_5[x, y, z]$. Now write

$$k = c_0u_0 + c_1u_1 + c_2u_2 + c_3u_3 + c_4u_4 + c_5u_5$$

such that the cubic polynomial becomes

$$f_{\mathrm{hom}} = c_0z^3 + c_1xz^2 + c_2yz^2 + c_3x^2z + c_4xyz + c_5(y^2z - x^3).$$

Now consider $f = f_{\mathrm{hom}}(x, y, 1)$. We have seen in Lemma 4.8.7 that f is surjective to $\mathbb{F}_5^\times$ if it describes a line, a conic with two distinct rational points at infinity, a geometrically integral conic with a single point at infinity, or a cubic curve. The remaining cases are the constant functions and the quadratics which are independent of y. This shows that $k \equiv c_0u_0 + c_1u_1 + c_3u_3 \mod 5$. Let us show that we can assume that $c_0 = 0$. First note that l_1 reduces to a multiple of u_0 modulo 5. Let us write $\lambda' l_1 \equiv u_0 \mod 5$ where $\lambda' \in \mathbb{F}_5^\times$. This gives

$$(\lambda + 5c_0\lambda')l_1 \equiv \lambda l_1 + 5c_0\lambda' l_1 \equiv \lambda l_1 + 5c_0u_0 \mod 25.$$

So we can indeed assume that $c_0 = 0$.

The hyperplane section of $\mathbb{P}^5_{\mathbb{F}_5}$ defined by $k \equiv c_1u_1 + c_3u_3 \mod 5$ corresponds to the polynomial $c_1x + c_3x^2$ on $\mathbb{A}^2_{\mathbb{F}_5}$ which is quadratic if $c_3 \neq 0$ and constant if $c_1 = c_3 = 0$. By symmetry we see that a quadratic in one variable over $\mathbb{F}_5$ assumes exactly 3 values. And obviously if $h \equiv \lambda l_1 \mod 25$ then $c_1u_1 + c_3u_3$ is constant modulo 5.

So we see that $\frac{h}{l_1}(P)$ is constant modulo 5, and assumes either one, three or five values modulo 25. By Lemma 4.8.14 we see that $\frac{h}{l_1}(P)$ and hence $\frac{l_1}{h}(P)$ also assumes one, three or five values in $(\mathbb{Z}/25\mathbb{Z})^\times$ modulo fifth powers and so the same holds for $\mathrm{inv}_5\,\mathcal{A}$ by Lemma 4.4.4.

This proves that precisely in the specified cases the invariant map is not surjective. $\square$

4.9 Actual examples of obstructions

We will now combine the results on the invariant maps to produce some explicit examples of Brauer–Manin obstructions of order 5 and discuss some cases in which no algebraic obstruction can exist.

4.9.1 Obstructions for $\mathcal{U}$ of large conductor

Let us first show that in most cases there does not exist an algebraic obstruction to integral points if the conductor has a large prime divisor.

PROPOSITION 4.9.1. *Let $\mathcal{U}_h/\mathbb{Z}$ be a model as before of an interesting log K3 surface $U = X\backslash C$ of conductor n. If h is not of class I at a prime divisor $p > 11$ of n, then the invariant map $\mathrm{inv}_p\,\mathcal{A}\colon \mathcal{U}_p(\mathbb{Z}_p) \to \frac{1}{5}\mathbb{Z}/\mathbb{Z}$ is surjective. This implies that in these cases there does not exist an algebraic Brauer–Manin obstruction to the Hasse principle for integral points on $\mathcal{U}_h$.*

Proof. It follows from Theorem 4.8.12 that $\mathrm{inv}_p\,\mathcal{A}\colon \mathcal{U}_h(\mathbb{Z}_p) \to \frac{1}{5}\mathbb{Z}/\mathbb{Z}$ is surjective. This also proves that $\sum_\ell \mathrm{inv}_\ell\,\mathcal{A}\colon \mathcal{U}(\mathbb{A}_\mathbb{Z}) \to \frac{1}{5}\mathbb{Z}/\mathbb{Z}$ is surjective so in particular $\mathcal{X}(\mathbb{A}_{\mathbb{Q},\infty})^{\mathcal{A}}$ is not empty. $\square$

If h is of class I at a prime divisor $p \equiv 1 \mod 5$ of the conductor n, then it is possible to get an obstruction.

THEOREM 4.9.2. *Fix a prime $p \equiv 1 \mod 5$. There exists a scheme $\mathcal{U}_h/\mathbb{Z}$ such that h is of class I at p which divides the conductor such that the surface $\mathcal{U}_h$ is locally soluble, but there is a Brauer–Manin obstruction to the Hasse principle for integral points.*

Proof. We will describe how to produce such examples.

Start off with a model $\mathcal{X}$ for $\mathrm{dP}_5(p)$ for p by picking an $\alpha \in \mathbb{Z}[\zeta_p]$ such that $K = \mathbb{Q}(\alpha)$ is a number field of degree 5. Such an α is not unique and a different choice of α will yield a different model for $\mathrm{dP}_5(p)$ over $\mathbb{Z}$. We will use the relatively large degree of freedom in the choice of h.

For $\ell \in \{2,3,5\}$ pick a smooth point $P_\ell \in \mathcal{X}(\mathbb{Z}_\ell)$ and a linear form $h_\ell \in \mathbb{F}_\ell[u_0, u_1, u_2, u_3, u_4, u_5]$ such that $h_\ell(P_\ell) \not\equiv 0 \mod \ell$. For inert primes ℓ which divide the discriminant of m_α we have a unique singular point S_ℓ on $\mathcal{X}_\ell$. We also impose the conditions $h_\ell(S_\ell) \equiv 0 \mod \ell$, i.e. S_ℓ does not lie on $\mathcal{U}_\ell$. Note that these conditions can be satisfied simultaneously in the case that ℓ is a small inert prime for which $\mathcal{X}_\ell$ has an A_4-singularity.

Now fix an invertible quintic non-residue q modulo p and let h_p be a linear form over $\mathbb{F}_p$ which satisfies $h_p \equiv q l_1 \mod p$. Let h be a primitive linear form over $\mathbb{Z}$ which is a lift of these h_ℓ.

By the condition modulo p we see that h is different from $\pm l_1$ and $\pm l_2$ so by Lemma 4.1.6 the hyperplane section defined by h is geometrically irreducible. Furthermore, $\mathcal{U}_h$ is locally soluble as it is has $\mathbb{Z}_\ell$-points for $\ell \leq 5$ by choice of h and it has local points for the remaining primes by Lemma 4.7.2.

Let us prove the theorem by showing that $\sum \mathrm{inv}_\ell \mathcal{A}$ is constant and non-zero. For primes $\ell \neq p$ we see from Lemmas 4.8.5 and 4.8.6 that $\mathrm{inv}_\ell \mathcal{A}$ is identically zero. For the ramified prime p we use Lemma 4.8.3, and as $\frac{l_1}{h} \equiv \frac{1}{q} \mod p$ is constant and not a fifth power modulo p we see that $\mathrm{inv}_p \mathcal{A}$ is constant and non-zero. $\square$

For $p = 11$, there are examples of such obstructions of a different nature.

4.9.2 Obstructions for $\mathcal{U}$ of conductor 11

We will use the model of $\mathrm{dP}_5(11)$ constructed using the element

$$\alpha = \zeta_{11} + \zeta_{11}^{-1} - 2 \in \mathbb{Q}(\zeta_{11})$$

with minimal polynomial $m_\alpha = s^5 - 11s^4 + 44s^3 - 77s^2 + 55s - 11$. This model $\mathcal{X}$ over $\mathbb{Z}$ is given by the five equations

$$u_0^2 - 492u_0u_3 - 52838u_0u_5 - u_1u_3 - 22u_1u_4 - 121u_1u_5 \\ + 1952u_2u_4 + 412071u_2u_5 - 971038u_3u_5 - 1771110u_5^2,$$

$$u_0u_2 - 22u_0u_3 - 4u_0u_4 - 3267u_0u_5 - u_1u_4 - 11u_1u_5 \\ + 88u_2u_4 + 20504u_2u_5 + 16u_3u_4 - 39017u_3u_5 - 78089u_5^2,$$

$$u_0u_3 + 169u_0u_5 + u_1u_5 - u_2^2 - 4u_2u_4 - 451u_2u_5 - 500u_3u_5 + 220u_5^2,$$

$$u_0u_4 - 11u_0u_5 - u_2u_3 + 121u_2u_5 - 4u_3u_4 - 363u_3u_5 - 594u_5^2,$$

$$u_0u_5 - u_2u_4 + u_3^2 - 11u_2u_5 + 40u_3u_5 + 55u_5^2.$$

As $\mathrm{disc}(m_\alpha) = 11^4$ we see that all fibres $\mathcal{X}_\ell$ are smooth for $\ell \neq 11$, and since $m_\alpha \equiv s^5 \mod 11$ we see that $\mathcal{X}_{11}$ has one singular point.

One can check that each of the two hyperplane sections given by

$$l_1 = u_1 + 187u_0 - 759u_2 + 693u_3 - 979u_4 + 4114u_5,$$

$$l_2 = u_1 + 187u_0 - 770u_2 + 792u_3 - 1199u_4 + 4840u_5$$

consist of five conjugate exceptional curves defined over the degree 5 number field $K = \mathbb{Q}(\zeta_{11})^+ = \mathbb{Q}(\zeta_{11} + \zeta_{11}^{-1})$ and the only ramified prime is 11.

Let us group the results on local solubility and computing the invariant maps on this surface for a general h. Again we consider the affine surface $\mathcal{U}$ over $\mathbb{Z}$ given by the complement of a hyperplane section $\mathcal{C} = \{h = 0\}$. The generic fibres of these two schemes are denoted by X and U.

LEMMA 4.9.3. *The affine surface $\mathcal{U}_h$ is everywhere locally soluble precisely when*

$$h \not\equiv u_2 + u_3 \mod 2.$$

Proof. A quick computer check shows that $\mathcal{X}(\mathbb{F}_2)$ consists of five points which lie on a unique hyperplane given by $u_2 + u_3$. Local solubility at 11 is given in Lemma 4.7.3 and the existence of points over the other completions is precisely Lemma 4.7.2, since all the fibres over $\ell \neq 11$ are smooth. □

LEMMA 4.9.4. *Consider a geometrically irreducible hyperplane section given by a primitive h. Let ℓ be a prime and let $\mathcal{A}$ be a generator for* $\operatorname{Br} U_h / \operatorname{Br} \mathbb{Q}$. *We consider the invariant map*

$$\operatorname{inv}_\ell \mathcal{A} \colon \mathcal{U}_h(\mathbb{Z}_\ell) \to \mathbb{Q}/\mathbb{Z}.$$

(a) If $\ell \neq 11$, then the invariant map is identically zero.

(b) If $\ell = 11$, then h is of

- *class I precisely when h is a multiple of u_1;*
- *class II precisely when h is not of class I, but it is in the $\mathbb{F}_{11}$-span of u_0, u_1 and u_3;*
- *class III in all other cases.*

The value $0 \in \mathbb{Q}/\mathbb{Z}$ does not lie in the image of $\operatorname{inv}_{11} \mathcal{A}$ precisely when h is class I or II and the associated polynomial

$$f = h(x + 4x^2, 1, y, x^2, xy, y^2 - x^3).$$

does not assume the values ± 1 modulo 11 for $x, y \in \mathbb{F}_{11}$.

Note that for h of class I the associated polynomial f is constant and whether or not ± 1 lies in the image of f is immediate. If h is of class II, then f is independent of y and one only needs to evaluate f at 11 points.

Proof. We treat the cases of unramified and the ramified prime separately.

(a) This follows directly from Lemmas 4.8.5 and 4.8.6 since $\mathcal{X}_{11}$ is the only singular fibre of $\mathcal{X}$ over $\mathbb{Z}$.

(b) One can check that the rational map $\psi\colon \mathbb{P}^2_{\mathbb{F}_{11}} \dashrightarrow \mathcal{X}_{11}$ is defined by

$$(x, y, z) \mapsto (xz^2 + 4x^2z, z^3, yz^2, x^2z, xyz, y^2z - x^3).$$

Its inverse π is given by

$$(u_0 \colon u_1 \colon u_2 \colon u_3 \colon u_4 \colon u_5) \mapsto (u_0 + 7u_3 \colon u_2 \colon u_1).$$

This shows that the associated polynomial of h at 11 is given by $f(x, y) = h(x + 4x^2, 1, y, x^2, xy, y^2 - x^3)$. It follows that h is of

» class I precisely when f is constant;

» class II precisely when f is independent of y, but not of x; and

» class III otherwise.

For an h of class III the map $\operatorname{inv}_{11} \mathcal{A}\colon \mathcal{U}(\mathbb{Z}_{11}) \to \frac{1}{5}\mathbb{Z}/\mathbb{Z}$ is surjective, so there is a $\mathbb{Z}_{11}$-point P such that $\operatorname{inv}_{11} \mathcal{A}(P) = 0$.

We saw in Lemma 4.8.3 how to compute the invariant map on the set $W \subseteq \mathcal{U}(\mathbb{Z}_{11})$ of $\mathbb{Z}_{11}$-points P such that $\frac{l_1}{h}(P)$ is defined and invertible in $\mathbb{Z}_{11}$. In the proof of Theorem 4.8.12 we saw that for h of class I or II these are all $\mathbb{Z}_{11}$-points on $\mathcal{U}$, so we see that 0 does not lie in the image of $\operatorname{inv}_{11} \mathcal{A}$ precisely if f does not assume a fifth power modulo 11 on $\mathbb{A}^2_{\mathbb{F}_{11}} \setminus V(f)$, or equivalently f does not assume the values ± 1 on the whole of $\mathbb{A}^2_{\mathbb{F}_{11}}$. □

We can now apply the above results to compute the Brauer–Manin obstruction for a fixed h and find actual algebraic obstructions of order 5 to the integral Hasse principle.

THEOREM 4.9.5. *Let $\mathcal{H}$ be the hyperplane in $\mathbb{P}^5_{\mathbb{Z}}$ given by the vanishing of u_0. The complement $\mathcal{U} = \mathcal{X} \setminus \mathcal{H}$ has points over $\mathbb{Q}$ and every $\mathbb{Z}_\ell$, but there is an algebraic Brauer–Manin obstruction to the existence of integral points.*

Proof. The local solubility follows from Lemma 4.9.3 and the invariant map for $\ell \neq 11$ is identically zero by Lemma 4.9.4.

Let us consider the only remaining prime 11. We will follow the outline in Lemma 4.9.4 to check for a possible obstruction. The associated inhomogeneous polynomial of $h = u_0$ is found to be $f = 4x^2 + x$, which is easily checked to never assume the values ± 1. So the element $0 \in \frac{1}{5}\mathbb{Z}/\mathbb{Z}$ does not lie in the image of $\operatorname{inv}_{11} \mathcal{A}$. This shows that for all $(P_\ell)_\ell \in \mathcal{U}(\mathbb{A}_{\mathbb{Q},\infty})$ we have $\sum_\ell \operatorname{inv}_\ell \mathcal{A}(P_\ell) = \operatorname{inv}_{11} \mathcal{A}(P_{11}) \neq 0$ and hence

$$\mathcal{U}(\mathbb{A}_{\mathbb{Q},\infty})^{\mathcal{A}} = \emptyset$$

and so $\mathcal{U}(\mathbb{Z}) = \emptyset$. □

A careful analysis of the above proof yields the following result.

THEOREM 4.9.6. *Let $\mathcal{U}_h$ be the complement in $\mathcal{X}$ of a geometrically irreducible hyperplane section given by a primitive linear form $h \in \mathbb{Z}[u_0, u_1, \ldots, u_5]$. The class of h modulo 2 determines whether the affine surface $\mathcal{U}_h$ is locally soluble. The existence of an algebraic obstruction to the Hasse principle for integral points depends only on the reduction of h modulo 11. Out of the $11^6 - 1 = 1771560$ possible reductions of h modulo 11 precisely 228 give an obstruction.*

Note that this does not mean that the reduction of h modulo 2 and 11 is the only condition; the proof still uses the assumption that h is primitive. It follows from Lemma 4.1.6 that the condition that the section is geometrically irreducible is immediately satisfied if h does not reduce to $\pm u_1$. For hyperplanes h reducing to either of these two form it is easily shown that $\operatorname{inv}_{11} \mathcal{A}$ is identically equal to 0 on $\mathcal{U}_h(\mathbb{Z}_{11})$.

Proof of Theorem 4.9.6. We already saw the statement about local solubility in Lemma 4.9.3 and in Lemma 4.9.4 we saw that the existence of an obstruction only depends on the associated polynomial f which only depends on the reduction of h modulo 11.

Let us count the non-zero linear forms h over $\mathbb{F}_{11}$ for which such an obstruction exists. In Theorem 4.8.12 we saw that we get no obstruction unless h is of class I or class II. Let $f \in \mathbb{F}_{11}[x, y]$ be the associated polynomial of h. We must consider the cases where f is constant or a separable quadratic polynomial.

If f is constant then we see that $\operatorname{inv}_{11} \mathcal{A}$ is constant and we get an obstruction if f is one of the 8 non-fifth powers modulo 11.

For h of class II we see that $f \colon \mathbb{F}_{11} \backslash \{\rho_1, \rho_2\} \to \mathbb{F}_{11}^\times / (F_{11}^\times)^5, x \mapsto f(x)$ misses exactly one value. If f misses the value $q \in \mathbb{F}_{11}^\times / (\mathbb{F}_{11}^\times)^5$ we see that λf for $\lambda \in \mathbb{F}_{11}^\times$ misses the class of λq.

There are $10 \cdot 11^2$ quadratic polynomials over $\mathbb{F}_{11}$ and $10 \cdot 11$ of these are inseparable. The group $\mathbb{F}_{11}^\times$ acts on the remaining $10^2 \cdot 11$ quadratic polynomials by multiplication. All orbits have size 10 and in such an orbit exactly 2 miss the unit element in $\mathbb{F}_{11}^\times / (\mathbb{F}_{11}^\times)^5$. This proves that for an h of class II at 11 there is an obstruction if h reduces to one of these $2 \cdot 10 \cdot 11 = 220$ separable quadratic polynomials. $\square$

4.9.3 Obstructions for $\mathcal{U}$ of conductor 25

It is also possible to find obstructions of order 5 to the integral Hasse principle when $\mathcal{X}$ is a model of $\mathrm{dP}_5(25)$ so then K has 5 as a ramified prime. For example, define the field $K \subseteq \mathbb{Q}(\zeta_{25})$ as the splitting field of the polynomial

$$m_\alpha = s^5 - 20s^4 + 100s^3 - 125s^2 + 50s - 5.$$

This produces the projective surface $\mathcal{X}$ over the integers given by the five equations

$$\begin{aligned}
&u_1^2 - u_0u_3 - 1600u_1u_3 - 40u_0u_4 - 400u_0u_5 - 524950u_1u_5 \\
&\qquad\qquad + 7251005u_2u_5 - 30039800u_3u_5 - 16150000u_5^2, \\
&u_1u_2 - 40u_1u_3 - u_0u_4 - 20u_0u_5 - 18125u_1u_5 \\
&\qquad\qquad + 200050u_2u_5 - 750995u_3u_5 - 403750u_5^2, \\
&u_2^2 - u_1u_3 - u_0u_5 - 500u_1u_5 + 1875u_2u_5, \\
&u_2u_3 - u_1u_4 + 20u_1u_5 - 400u_2u_5 + 1875u_3u_5 + 995u_5^2, \\
&u_3^2 - u_2u_4 + u_1u_5 - 20u_2u_5 + 100u_3u_5 + 50u_5^2.
\end{aligned}$$

The two hyperplane sections over $\mathbb{Z}$ cutting out the two quintuples of -1-curves are

$$l_1 = u_0 + 575u_1 - 3550u_2 + 8650u_3 - 4285u_4 + 10475u_5,$$
$$l_2 = u_0 + 525u_1 - 2575u_2 + 4325u_3 - 2285u_4 + 11100u_5.$$

As in the previous case, local solubility is immediate at most primes.

LEMMA 4.9.7. *The surface $\mathcal{U}_h$ is everywhere locally soluble precisely when*

$$h \not\equiv u_2 + u_3 + u_5 \mod 2.$$

Proof. Since $\operatorname{disc}(m_\alpha) = 5^8 7^6$, there are only two singular fibres and the local solubility at primes $\ell \geq 3$ follows from Lemmas 4.7.2 and 4.7.3. For the prime 2 we find, just like in the proof of Lemma 4.9.3, that the five points on $\mathcal{X}_2$ lie on a unique hyperplane in $\mathbb{P}^5_{\mathbb{F}_2}$. This hyperplane is given by $u_2 + u_3 + u_5 \equiv 0$. $\square$

THEOREM 4.9.8. *Consider the invariant map*

$$\operatorname{inv}_\ell \mathcal{A} \colon \mathcal{U}_h(\mathbb{Z}_\ell) \to \mathbb{Q}/\mathbb{Z}.$$

(a) *If $\ell \neq 5$, then the invariant map is identically zero.*

(b) *If $\ell = 5$ then either $\operatorname{Im}(\operatorname{inv}_5 \mathcal{A}) = \frac{1}{5}\mathbb{Z}/\mathbb{Z}$ or there are integers λ, c_1 and c_3 satisfying $5 \nmid \lambda$ and either $5 \mid c_1, c_3$ or $5 \nmid c_1$ such that $h \equiv \lambda(u_0 + 15u_4) + 5(c_1u_1 + c_3u_3) \mod 25$. In this case the value $0 \in \mathbb{Q}/\mathbb{Z}$ lies in the image of $\operatorname{inv}_{11} \mathcal{A}$ precisely when $\lambda 1 + 5(c_1x + c_3x^2)$ assumes one of values $\pm 1, \pm 7$ modulo 25 for $x \in \mathbb{Z}$.*

Proof. The first statement follows as before. For the second statement one checks that the birational map $\mathcal{X}_5 \dashrightarrow \mathbb{P}^2_{\mathbb{F}_5}$ which restricts to the isomorphism from Proposition 4.5.12 is given by $(u_0, u_1, u_2, u_3, u_4, u_5) \mapsto (u_1, u_2, u_0)$ and the inverse by $(x, y, z) \mapsto (z^3, xz^2, yz^2, x^2z, xyz, y^2z - x^3)$. We see that under this map the hyperplanes u_1 and u_3 reduce to x^2 and x on the affine plane over $\mathbb{F}_5$. The statement now follows from Lemma 4.8.17 since $l_1 \equiv u_0 + 15u_4 \mod 25$ and $\frac{h}{l_1}(P)$ reduces to one of $\pm 1, \pm 7$ modulo 25 precisely when $\frac{l_1}{h}(P)$ does. $\square$

For completeness we will give an example of a hyperplane for which the associated affine scheme over the integers does not have integral solutions.

THEOREM 4.9.9. *Consider an h which cuts out a geometrically irreducible hyperplane section such that* 0 *does not lie in the image of* $\mathrm{inv}_5\,\mathcal{A}$. *The reduction of h modulo* 25 *is one of* 176 *out of the* $(5^2)^6 - 5^6 = 244125000$ *possible hyperplanes over* $\mathbb{Z}/25\mathbb{Z}$. *For example, the surface $\mathcal{U}_h/\mathbb{Z}$ for $h = 2u_0 + 10u_3 + 5u_4$ admits a Brauer–Manin obstruction of order* 5 *to the existence of integral points.*

Proof. Let $(\mathbb{Z}/25\mathbb{Z})^\times$ act on the hyperplanes modulo 25 for which $\mathrm{inv}_5\,\mathcal{A}$ is not surjective by multiplication. This translates the image of the invariant map by an element of $\frac{1}{5}\mathbb{Z}/\mathbb{Z}$ depending on the class of $(\mathbb{Z}/25\mathbb{Z})^\times$ modulo fifth powers. So if the size of the image of an invariant map corresponding to a hyperplane has one element, then $\frac{4}{5}$ of the scalar multiples of h do not have 0 in the image. For invariant maps whose image is of size 3 precisely $\frac{2}{5}$ of the scalar multiples have this property. This means that the number of hyperplanes modulo 25 for which 0 does not lie in the image of the invariant map is $\frac{4}{5} \cdot 20 + \frac{2}{5} \cdot 20 \cdot 4 \cdot 5 = 176$.

Now consider the hyperplane $h = 2u_0 + 10u_3 + 5u_4$. The affine surface $\mathcal{U}_h$ is locally soluble by Lemma 4.9.7. The result follows from the previous theorem; take $\lambda = 2$, $c_1 = 0$ and $c_3 = 2$ and note that $2(1 + 5x^2)$ only assumes the values $2, 12, 17 \mod 25$. So 0 does not lie in the image of the invariant map at 5 and the invariant maps at the other primes are all constant zero. $\square$

Bibliography

[1] M. Artin. *Some numerical criteria for contractability of curves on algebraic surfaces*. Amer. J. Math., 84:485–496, 1962.

[2] M. Artin, A. Grothendieck, and J.-L. Verdier, editors. *Tome 2* of *Séminaire de géométrie algébrique du Bois-Marie 1963–1964. Théorie des topos et cohomologie étale des schémas (SGA4)* - (Avec la collaboration de N. Bourbaki, P. Deligne, B. Saint-Donat), exposés V–VIII. Volume 270 of *Lecture Notes in Mathematics*. Springer, Berlin–New York, 1972.

[3] M. F. Atiyah. *On analytic surfaces with double points*. Proc. Roy. Soc. London Ser. A, 247:237–244, 1958.

[4] M. Auslander and O. Goldman. *The Brauer group of a commutative ring*. Trans. Amer. Math. Soc., 97:367–409, 1960.

[5] M. J. Bright. *Computations on diagonal quartic surfaces*. Ph.D. Thesis, University of Cambridge, 2002.

[6] M. J. Bright. *Efficient evaluation of the Brauer–Manin obstruction*. Math. Proc. Cambridge Philos. Soc., 142(1):13–23, 2007.

[7] M. J. Bright. *Bad reduction of the Brauer–Manin obstruction*. J. Lond. Math. Soc. (2), 91(3):643–666, 2015.

[8] M. J. Bright, T. D. Browning, and D. Loughran. *Failures of weak approximation in families*. Compos. Math., 152(7):1435–1475, 2016.

[9] M. J. Bright, A. Logan, and R. M. van Luijk. *Finiteness theorems for K3 surfaces over arbitrary fields*. To appear in Eur. J. Math., arXiv:1810.04905v3, 2019.

[10] M. J. Bright and J. T. Lyczak. *A uniform bound on the Brauer groups of certain log K3 surfaces*. Michigan Math. J., 68(2):377–384, 2019.

[11] V. Cantoral-Farfán, Y. Tang, S. Tanimoto, and H. D. Visse. *Effective bounds for Brauer groups of Kummer surfaces over number fields*. J. Lond. Math. Soc. (2), 97(3):353–376, 2018.

[12] J. W. S. Cassels. *Global Fields* in *Algebraic Number Theory* - (proc. instructional conf., Brighton, 1965), pages 42–84. Thompson, Washington, D.C., 1967.

[13] J.-L. Colliot-Thélène. *Formes quadratiques multiplicatives et variétés algébriques: deux compléments*. Bull. Soc. Math. France, 108(2):213–227, 1980.

[14] J.-L. Colliot-Thélène. *Points rationnels sur les fibrations* in *Higher Dimensional Varieties and Rational Points*, pages 171–221. Volume 12 of *Bolyai Society Mathematical Studies*. Springer, Berlin, 2003.

[15] J.-L. Colliot-Thélène, D. Wei, and F. Xu. *Brauer–Manin obstruction for Markoff surfaces*. To appear in Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), arXiv:1808.01584v4, 2019.

[16] J.-L. Colliot-Thélène and O. Wittenberg. *Groupe de Brauer et points entiers de deux familles de surfaces cubiques affines*. Amer. J. Math., 134(5):1303–1327, 2012.

[17] J.-L. Colliot-Thélène and F. Xu. *Brauer–Manin obstruction for integral points of homogeneous spaces and representation by integral quadratic forms*. Compos. Math., 145(2):309–363, 2009.

[18] B. Conrad. *Weil and Grothendieck approaches to adelic points*. Enseign. Math. (2), 58(1-2):61–97, 2012.

[19] D. F. Coray and M. A. Tsfasman. *Arithmetic on singular Del Pezzo surfaces*. Proc. London Math. Soc. (3), 57(1):25–87, 1988.

[20] P. K. Corn. *Del Pezzo surfaces and the Brauer–Manin obstruction*. Ph.D. Thesis, University of California, 2005.

[21] D. A. Cox, J. B. Little, and H. K. Schenck. *Toric Varieties*. Volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.

[22] P. del Pezzo. *Sulle superficie dell'nmo ordine immerse nello spazio din dimensioni*. Rend. Circ. Mat. Palermo (1), 1:241–271, 1887.

[23] M. Demazure. *Surfaces de Del Pezzo: I–V* in *Séminaire sur les Singularités des Surfaces*, pages 21–69. Volume 777 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin–New York, 1980.

[24] D. Eisenbud and J. D. Harris. *The Geometry of Schemes*. Volume 197 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.

[25] A. Vistoli. *Grothendieck topologies, fibered categories and descent theory* in *Fundamental algebraic geometry* - (Grothendieck's FGA explained), pages 1–104. Volume 123 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005.

[26] W. Fulton. *Algebraic curves* - (An Introduction to Algebraic Geometry). Addison-Wesley Publ. Co., Redwood City, CA, 1989.

[27] W. Fulton. *Intersection theory*, second edition. Volume 2 of *Ergeb. Math. Grenzgeb. (3)*. Springer-Verlag, Berlin, 1998.

[28] P. Gille and T. Szamuely. *Central simple algebras and Galois cohomology*. Volume 101 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2006.

[29] J. González-Sánchez, M. Harrison, I. Polo-Blanco, and J. Schicho. *Algorithms for Del Pezzo surfaces of degree 5 (construction, parametrization)*. J. Symbolic Comput., 47(3):342–353, 2012.

[30] A. Grothendieck. *Le groupe de Brauer. III. Exemples et compléments* in *Dix exposés sur la cohomologie des schémas*, pages 88–188. Volume 3 of *Advanced Studies in Pure Mathematics*. North-Holland Publ. Co., Amsterdam, 1968.

[31] A. Grothendieck. *Revêtements étales et groupe fondamental (SGA 1)*. Volume 224 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin–New York, 1971.

[32] Y. Harpaz. *Geometry and arithmetic of certain log K3 surfaces*. Ann. Inst. Fourier (Grenoble), 67(5):2167–2200.

[33] R. Hartshorne. *Algebraic Geometry*. Volume 52 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1977.

[34] E. Ieronymou, A. N. Skorobogatov, and Y. G. Zarhin. *On the Brauer group of diagonal quartic surfaces*. J. Lond. Math. Soc. (2), 83(3):659–672, 2011.

[35] J. Jahnel and D. Schindler. *On integral points on degree four del Pezzo surfaces*. Israel J. Math., 222(1):21–63, 2017.

[36] R. K. Lazarsfeld. *Positivity in algebraic geometry, I.* Classical setting: line bundles and linear series. Volume 48 of *Ergeb. Math. Grenzgeb. (3)*. Springer-Verlag, Berlin, 2004.

[37] J. Lipman. *Rational singularities, with applications to algebraic surfaces and unique factorization*. Inst. Hautes Études Sci. Publ. Math., (36):195–279, 1969.

[38] D. Loughran and V. Mitankin. *Integral Hasse principle and strong approximation for Markoff surfaces*. arXiv:https://arxiv.org/abs/1807.10223v3, 2018.

[39] J. T. Lyczak. *Magma code for computing algebraic Brauer groups*. `http://www.julianlyczak.nl/#code`.

[40] Y. I. Manin. *Le groupe de Brauer-Grothendieck en géométrie diophantienne* in *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1*, pages 401–411. Gauthier-Villars, Paris, 1971.

[41] Y. I. Manin. *Cubic forms: algebra, geometry, arithmetic*, second edition. Volume 4 of *North-Holland Mathematical Library*. North-Holland Publ. Co., Amsterdam, 1986.

[42] L. Merel. *Bornes pour la torsion des courbes elliptiques sur les corps de nombres*. Invent. Math., 124(1-3):437–449, 1996.

[43] J. S. Milne. *Class Field Theory*, version 4.02. Available at www.jmilne.org/math/, 2013.

[44] R. D. Newton. *Transcendental Brauer groups of products of CM elliptic curves*. J. Lond. Math. Soc. (2), 93(2):397–419, 2016.

[45] M. Orr and A. N. Skorobogatov. *Finiteness theorems for K3 surfaces and abelian varieties of CM type*. Compos. Math., 154(8):1571–1592, 2018.

[46] B. Poonen. *Rational Points on Varieties*. Volume 186 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2017.

[47] J.-P. Serre. *Local Class Field Theory* in *Algebraic Number Theory* - (proc. instructional conf., Brighton, 1965), pages 128–161. Thompson, Washington, D.C., 1967.

[48] J. H. Silverman. *The Arithmetic of Elliptic Curves*, second edition. Volume 106 of *Graduate Texts in Mathematics*. Springer, Dordrecht, 2009.

[49] A. N. Skorobogatov and Y. G. Zarhin. *A finiteness theorem for the Brauer group of abelian varieties and K3 surfaces*. J. Algebraic Geom., 17(3):481–502, 2008.

[50] The Stacks Project Authors. *Stacks Project*. `https://stacks.math.columbia.edu`, 2019.

[51] P. Stevenhagen. *The arithmetic of number rings* in *Algorithmic number theory: lattices, number fields, curves and cryptography*, pages 209–266. Volume 44 of *Math. Sci. Res. Inst. Publ.*. Cambridge Univ. Press, Cambridge, 2008.

[52] H. P. F. Swinnerton-Dyer. *The Brauer Group of Cubic Surfaces*. Math. Proc. Camb. Phil. Soc., 113:449–460, 1993.

[53] J. T. Tate. *Global Class Field Theory* in *Algebraic Number Theory* - (proc. instructional conf., Brighton, 1965), pages 162–203. Thompson, Washington, D.C., 1967.

[54] D. Testa, A. Várilly-Alvarado, and M. Velasco. *Big rational surfaces*. Math. Ann., 351(1):95–107, 2011.

[55] A. Várilly-Alvarado. *Arithmetic of del Pezzo surfaces* in *Birational geometry, rational curves, and arithmetic*, pages 293–319. Volume 1 of *Simons symposia*. Springer, Berlin–New York, 2013.

[56] A. Várilly-Alvarado. *Arithmetic of K3 surfaces* in *Geometry over nonclosed fields*, pages 197–248. Volume 5 in *Simons symposia*. Springer, Cham, 2017.

[57] A. Várilly-Alvarado and B. L. Viray. *Abelian n-division fields of elliptic curves and Brauer groups of product Kummer and abelian surfaces*. Forum of Mathematics, Sigma, 5:e26, 2017.

[58] C. A. Weibel. *An introduction to homological algebra*. Volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

Summary

This thesis concerns the mathematical area of arithmetic geometry. I will first give a general introduction to this field before touching upon the more specific topics and results discussed in this thesis.

What is arithmetic geometry?

In *number theory* it is common to study rational solutions to equations such as $x^2 + y^3 = 5$. This means that one is interested in pairs of rational numbers x and y for which this equation is true. Some of these solutions can be found by simply trying some small values for x and y. For example, the solutions $x = 2$ and $y = 1$, and $x = -2$ and $y = 1$ could be found by simple inspection.

In *arithmetic geometry* one approaches such questions using geometric techniques and terminology. For example, one can draw all the real points on the xy-plane for which the equation $x^2 + y^3 = 5$ is satisfied. We recover the curved line as shown in the diagram on the next page and we see that the solution $x = 2$ and $y = 1$ corresponds to the point $(2, 1)$ on this geometric object. This interpretation shows why a solution to an equation is often called a *point*; it is a point on the geometric object defined by the equation. We will use the terms point and solution interchangeably.

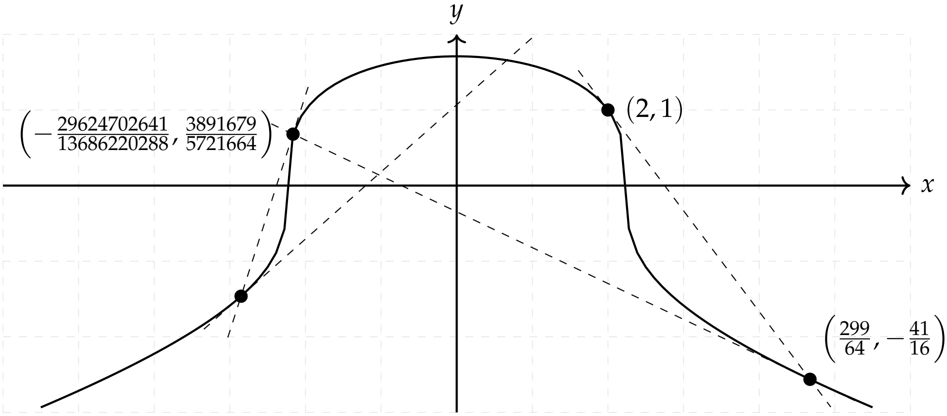
We will now show that the equation $x^2 + y^3 = 5$ has infinitely many solutions over the rational numbers. It has been shown that if (x, y) is a rational solution, then another solution is given by

$$\left(\frac{y^3+10}{2x} - \frac{3y^3}{8x}\frac{y^3+40}{y^3-5}, \frac{y}{4}\frac{y^3+40}{y^3-5}\right).$$

If we start with the point $(x, y) = (2, 1)$ we can iterate this procedure to find the following points

$$(2,1) \mapsto \left(\frac{299}{64}, -\frac{41}{16}\right) \mapsto \left(-\frac{29624702641}{13686220288}, \frac{3891679}{5721664}\right) \mapsto \dots$$

A suspicious reader is invited to check that these pairs are really solutions to the equation $x^2 + y^3 = 5$.



These formulas for determining new solutions are as ugly as they are mysterious, but this procedure has a very nice geometric interpretation: consider the geometric object defined by $x^2 + y^3 = 5$. Then draw the tangent line in the point $(2, 1)$ and intersect it again with the curve. This new intersection point is $\left(\frac{299}{64}, -\frac{41}{16}\right)$; precisely the point obtained from applying the procedure described above. To compute the next point one draws the tangent line in this new point $\left(\frac{299}{64}, -\frac{41}{16}\right)$ and again intersects it with the curve. We can repeat this procedure indefinitely to find infinitely many rational points.

This is a first example of how geometric techniques are used to study equations. We also use geometric properties to classify equations. The equation above describes a 'bent line', which is called a *curve*. This thesis however is about *surfaces* which look more like 'bent, twisted or curved planes' such as on the cover of this thesis.

Surfaces

There are several ways to move from equations describing curves to equations describing surfaces. The first way is to add a third variable, for which we will use z. For example, the equation $x^3 + y^3 + z^3 = 4$ describes a surface which has many solutions such as

$$\left(\frac{1}{21}, \frac{5}{3}, -\frac{6}{7}\right).$$

Another way to write down equations for surfaces is by not only increasing the number of variables, but also the number of equations. Consider the system of equations with the four variables w, x, y en z:

$$\begin{cases} w^2 + x^2 + y^2 = 6, \\ w^2 + xy + yz = 2. \end{cases}$$

We can then also study the quadruple (w, x, y, z) for which *both* equations are satisfied simultaneously. One can check that $(1, 2, 1, -1)$ and $(\frac{1}{5}, -2, -\frac{7}{5}, -\frac{21}{5})$ are solutions to this set of equations.

These two surfaces are examples of so-called *ample log K3 surfaces*. Occasionally, ample log K3 surfaces are referred to, albeit slightly imprecisely, as *affine del Pezzo surfaces* which is the term used in the title of this thesis.

Another example of an ample log K3 surface is given by the system of five equations

$$\begin{cases} 1 + 1952wy + 412071wz = \\ \qquad 121v + 492x + 52838z + vx + 22vy + 971038xz + 1771110z^2, \\ w + 88wy + 20504wz + 16xy = \\ \qquad 11v + 22x + 4y + 3267z + vy + 39017xz + 78089z^2, \\ v + x + 169z + 220z^2 = w^2 + 4wy + 451wz + 500xz, \\ y + 121wz = 11z + wx + 4xy + 363xz + 594z^2, \\ z + x^2 + 40xz + 55z^2 = wy + 11wz \end{cases} \tag{$\star$}$$

in the five variables v, w, x, y and z. The surface it describes has many rational points, such as

$$\left(-20, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, 0\right) \quad \text{and} \quad \left(\frac{23}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, 0\right).$$

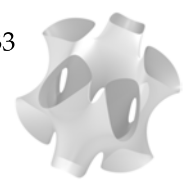
Although the equations are quite hideous this surface is geometrically very pleasing. One of its geometrically interesting features of this surface is even shown on the cover: although the surface is very twisted it is possible to walk in a straight line along this surface. Note that there are precisely ten such straight lines.

The number of lines is an aspect that distinguishes this surface from the previous two examples, since one can show that those surfaces contain respectively 27 and 16 lines. The reader is advised at this point to look up a picture of the 27 lines on the 'Clebsch surface' online.

We will consider ample log K3 surfaces with 10 lines. There are technical reasons we will not go into to call these surfaces *ample log K3 surfaces of degree* 5. In contrast the other examples are log K3 surfaces of degree 3 and 4, respectively. So now we used the geometric property of the number of lines to classify equations.

Integral points

Let us now focus on the ample log K3 surface of degree 5 defined by the system ($\star$). We saw that it has rational points. A next question is whether some of these rational solutions are in fact solutions in integers. In this case will speak of *integral points* and *integral solutions*.



The two rational solutions mentioned above are clearly not integral, but this is in no way a guarantee that all rational solutions are not integral. But one would do well to also consider the possibility that there are no integral solutions at all!

Any technique which proves that there can not be any points is called an *obstruction*. One possible obstruction to the existence of solutions was developed by Manin. He used a technical object called the *Brauer group* of the system of equations. If such an obstruction excludes the existence of integral solutions one says that there is a *Brauer–Manin obstruction to the existence of integral points*. In that case we can conclude that there are no integral solutions. This is what I did in this thesis for the system ($\star$).

THEOREM 1. *There is a Brauer–Manin obstruction to the existence of integral solutions to the system of equations* ($\star$). *This proves that the system does not have an integral solution.*

Chapter 4 gives a general method for deciding whether an ample log K3 surface of degree 5 admits a Brauer–Manin obstruction. Using this method, I found infinitely many systems of equations which do not admit an integral solution.

Bounding the Brauer group

One can now also wonder if it would be possible to have a computer determine whether or not a system of equations has a Brauer–Manin obstruction. Usually this depends on the complexity of the Brauer group. Unfortunately it is known that the Brauer group of surfaces can be arbitrarily complex, in the sense that for every surface there is another surface for which the Brauer group is more complex. The main theorem of Chapter 3 states that this is not the case for ample log K3 surface.

THEOREM 2. *There is a bound on the complexity of the Brauer group of ample log K3 surfaces.*

This bound shows that the Brauer group of an ample log K3 surface can not be arbitrarily complex. This is important information for developing an algorithm that determines whether there is Brauer–Manin obstruction to the existence of integral solutions to complicated systems of equations such as the system ($\star$).

Samenvatting

Dit proefschrift met de titel 'De arithmetiek van affiene del Pezzo oppervlakken' valt binnen de tak *arithmetische meetkunde* van de wiskunde. Deze samenvatting begint met een algemene introductie van dit vakgebied om vervolgens de specifieke onderwerpen en de belangrijkste resultaten uit dit proefschrift kort te behandelen.

Wat is arithmetische meetkunde?

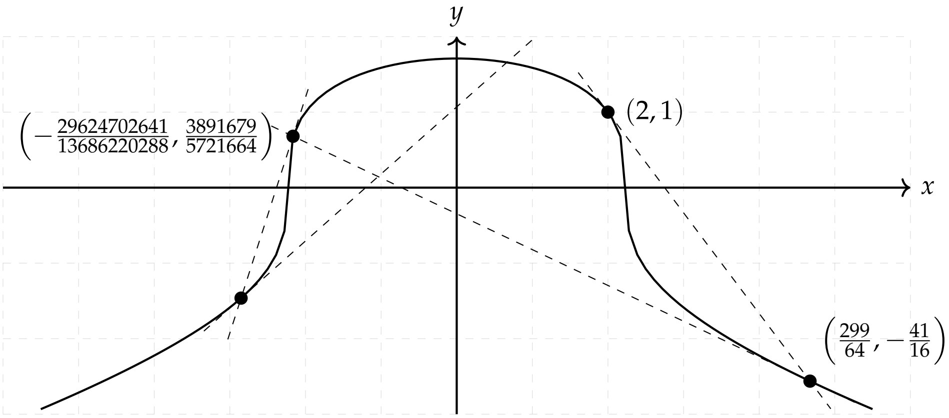
In de *getaltheorie* is het gebruikelijk om vergelijkingen zoals $x^2 + y^3 = 5$ op te lossen in rationale getallen. Dat wil zeggen dat men geïnteresseerd is in paren van breuken x en y waarvoor deze vergelijking klopt. Zo'n oplossing noemt men een *rationale oplossing*. Sommige rationale oplossingen kunnen gevonden worden door kleine getallen te proberen voor x en y. De oplossingen $x = 2$ en $y = 1$, en $x = -2$ en $y = 1$ van de bovenstaande vergelijking zijn daar voorbeelden van.

In de *arithmetische meetkunde* worden zulke vraagstukken benaderd met behulp van meetkundige technieken en terminologie. Ter illustratie; men kan alle reële punten op het xy-vlak tekenen waarvoor de vergelijking $x^2 + y^3 = 5$ waar is. Dit levert een gekromde lijn op zoals in de figuur op de volgende pagina. Het feit dat $x = 2$ en $y = 1$ een oplossing is van de vergelijking, laat zich nu vertalen naar de uitspraak dat het punt $(2, 1)$ op dit meetkundige object ligt.

Deze interpretatie verklaart waarom een oplossing van een vergelijking ook vaak een *punt* wordt genoemd; het is eenvoudigweg een punt op het meetkundige object gedefinieerd door de vergelijking. De termen *oplossingen* en *punt* worden dan ook uitwisselbaar gebruikt.

We zullen nu laten zien dat de vergelijking $x^2 + y^3 = 5$ oneindig veel rationale oplossingen heeft. Men kan laten zien dat als (x, y) een rationale oplossing is, dat de volgende formule ook een andere rationale oplossing geeft:

$$\left(\frac{y^3 + 10}{2x} - \frac{3y^3}{8x}\frac{y^3 + 40}{y^3 - 5}, \frac{y}{4}\frac{y^3 + 40}{y^3 - 5}\right).$$



Als we beginnen met het punt $(x, y) = (2, 1)$ dan kunnen we deze formule herhaaldelijk gebruiken en vinden we de volgende oplossingen

$$(2,1) \mapsto \left(\frac{299}{64}, -\frac{41}{16}\right) \mapsto \left(-\frac{29624702641}{13686220288}, \frac{3891679}{5721664}\right) \mapsto \dots$$

De achterdochtige lezer staat het vrij om te controleren dat deze paren van rationale getallen daadwerkelijke oplossingen zijn van de vergelijking $x^2 + y^3 = 5$.

Deze formule om nieuwe oplossingen te construeren is zowel lelijk als mysterieus, maar deze procedure heeft een uiterst mooie meetkundige interpretatie: neem het meetkundige object gegeven door de vergelijking $x^2 + y^3 = 5$. Teken de raaklijn aan dit object in het punt $(2, 1)$ en doorsnij deze lijn opnieuw met het object. Dit snijpunt is $\left(\frac{299}{64}, -\frac{41}{16}\right)$; precies het punt dat we verkregen met behulp van bovenstaande vergelijkingen. Om het volgende punt te bepalen tekenen we vervolgens de raaklijn in het punt $\left(\frac{299}{64}, -\frac{41}{16}\right)$ en doorsnijden dit wederom met het meetkundige object. Deze procedure kunnen we blijven herhalen om zo oneindig veel rationale punten te vinden.

Dit is een eerste voorbeeld van hoe meetkundige technieken gebruikt kunnen worden bij het bestuderen van vergelijkingen. Daarnaast worden ook meetkundige eigenschappen gebruikt om vergelijkingen te classificeren. De vergelijking hierboven beschrijft een 'gebogen lijn' en zo'n meetkundig object noemen we een *kromme*. Dit proefschrift gaat echter over *oppervlakken* welke er meer uitzien als 'gebogen, gedraaide en getwiste vlakken' zoals de figuur op de omslag.

Oppervlakken

Er zijn meerdere manieren om van vergelijkingen van krommen over te gaan op vergelijkingen van oppervlakken. Een eerste manier is door een derde variabele toe te voegen en die zullen we hier z noemen. De vergelijking $x^3 + y^3 + z^3 = 4$ beschrijft bijvoorbeeld een oppervlak. Bovendien heeft het veel oplossingen zoals onder andere

$$\left(\frac{1}{21}, \frac{5}{3}, -\frac{6}{7}\right).$$

Een tweede manier om oppervlakken te beschrijven is door niet alleen variabelen, maar ook vergelijkingen toe te voegen. Neem het systeem van de volgende twee vergelijkingen met de vier variabelen w, x, y en z.

$$\begin{cases} w^2 + x^2 + y^2 = 6, \\ w^2 + xy + yz = 2. \end{cases}$$

We kunnen de viertallen (w, x, y, z) bestuderen waarvoor *beide* vergelijkingen kloppen. Men kan controleren dat $(1, 2, 1, -1)$ en $(\frac{1}{5}, -2, -\frac{7}{5}, -\frac{21}{5})$ oplossingen zijn van dit systeem van vergelijkingen.

Deze twee oppervlakken zijn voorbeelden van zogenaamde *ampele log K3 oppervlakken*. Soms worden ampele log K3 oppervlakken aangeduid met, al zij het enigzins onnauwkeurig, *affiene del Pezzo oppervlakken*, zoals ook in de titel van dit proefschrift.

Een derde voorbeeld van een ampel log K3 oppervlak wordt gegeven door het systeem van vijf vergelijkingen

$$\begin{cases} 1 + 1952wy + 412071wz = \\ \qquad 121v + 492x + 52838z + vx + 22vy + 971038xz + 1771110z^2, \\ w + 88wy + 20504wz + 16xy = \\ \qquad 11v + 22x + 4y + 3267z + vy + 39017xz + 78089z^2, \\ v + x + 169z + 220z^2 = w^2 + 4wy + 451wz + 500xz, \\ y + 121wz = 11z + wx + 4xy + 363xz + 594z^2, \\ z + x^2 + 40xz + 55z^2 = wy + 11wz \end{cases} \tag{$\star$}$$

in de vijf variabelen v, w, x, y en z. Het oppervlak beschreven door dit systeem heeft veel rationale punten, zoals bijvoorbeeld

$$\left(-20, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, 0\right) \quad \text{en} \quad \left(\frac{23}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, 0\right).$$

Ondanks de afzichtelijke vergelijkingen is de meetkunde van dit oppervlak erg mooi. Zo siert een interessante meetkundige eigenschap van dit oppervlak de omslag van dit proefschrift. Namelijk, ondanks dat dit oppervlak erg kronkelt is het toch mogelijk om langs een rechte lijn over dit oppervlak te lopen. Merk op dat er precies tien van zulke rechte lijnen op dit oppervlak liggen.

Het aantal rechte lijnen onderscheidt dit oppervlak van de eerste twee voorbeelden. Het is namelijk al aangetoond dat er op die oppervlakken respectievelijk 27 en 16 lijnen liggen. De lezer wordt geadviseerd om een plaatje te

bekijken van de 27 lijnen op het Clebsch oppervlak. Een online zoektocht met de Engelse termen 'lines Clebsch surface' levert al snel een overdaad aan schitterende plaatjes op.

Ampele log K3 oppervlakken met 10 lijnen zijn een belangrijk onderwerp van dit proefschrift. Er zijn technische redenen, waar we niet op in zullen gaan, om deze oppervlakken aan te duiden als *ampele log K3 oppervlakken van graad* 5. Dit in tegenstelling tot de twee eerdere voorbeelden van ampele log K3 oppervlakken die van respectievelijk graad 3 en 4 zijn. Ook dit is een goed voorbeeld van hoe een meetkundige eigenschap, namelijk het aantal rechte lijnen op een oppervlak, wordt gebruikt om vergelijkingen te classificeren.

Gehele punten

We zullen ons nu beperken tot het ampele log K3 oppervlak van graad 5 gedefinieerd door het systeem van vergelijkingen ($\star$). We hebben al gezien dat er op dit oppervlak rationale punten liggen. Een vervolgvraag is of sommige van deze rationale oplossingen zelfs oplossingen in gehele getallen zijn. In dat geval praten we over *gehele punten* en *gehele oplossingen*.

De twee rationale oplossingen hierboven zijn overduidelijk niet geheel, maar dit is natuurlijk geen garantie dat geen van de rationale punten geheel is. Het is even goed mogelijk dat er geen enkel geheel punt is.

Een verklaring voor de afwezigheid van punten noemt men een *obstructie*. Een voorbeeld van zo'n obstructie was ontdekt door Manin. Hij gebruikte een zeer technisch object namelijk de *Brauer-groep* van een systeem van vergelijkingen. Als zo'n obstructie het bestaan van gehele punten uitsluit, dan praten we over een *Brauer–Manin-obstructie voor het bestaan van gehele punten*. In dat geval kunnen we dus ook daadwerkelijk bewijzen dat er geen gehele punten zijn. In dit proefschrift heb ik deze techniek toegepast op het systeem ($\star$).

STELLING 1. *Er is een Brauer–Manin-obstructie voor het bestaan van gehele oplossingen voor het systeem van vergelijkingen* ($\star$). *Dit systeem heeft dus geen gehele oplossingen.*

In Hoofdstuk 4 wordt een algemene methode beschreven om te bepalen of een ampel log K3 oppervlak van graad 5 een Brauer–Manin-obstructie heeft. Met behulp hiervan heb ik oneindig veel systemen van vergelijkingen gevonden welke geen gehele oplossingen hebben.

Begrenzen van de Brauer-groep

Je zou je ook kunnen afvragen of het in het algemeen mogelijk is om een computer te laten bepalen of een systeem van vergelijkingen een Brauer–Manin-obstructie heeft. Over het algemeen hangt het antwoord van deze vraag af van de complexiteit van de Brauer-groep. Helaas is het bekend dat de Brauer-groep van een oppervlak willekeurig complex kan zijn. Dit wil zeggen dat voor elk oppervlak er een ander oppervlak is met een nog complexere Brauer-groep. Het belangrijkste resultaat uit Hoofdstuk 3 zegt dat dit gelukkig niet het geval is als men zich beperkt tot de ampele log K3 oppervlakken van willekeurige graad.

STELLING 2. *Er is een grens aan de complexiteit van Brauer-groepen van ampele log K3 oppervlakken.*

Deze begrenzing geeft aan dat de Brauer-groepen van ampele log K3 oppervlakken niet willekeurig complex kunnen zijn. Dit is belangrijke informatie bij het ontwikkelen van een algoritme dat bepaalt of er een Brauer–Manin-obstructie is voor gecompliceerde systemen van vergelijkingen zoals systeem ($\star$).

Acknowledgements

This thesis was written with the help and support of a great many people. I would like to thank all of them; both those who contributed to the content of and the research leading up to this thesis, and those in my personal life who brought me the confidence and joy needed for completing it. It might not be possible to name each and everyone of them, but at the risk of leaving anyone out I will mention a few people who played a special role.

First and foremost I would like to thank my supervisor; Martin, your guidance and supervision has of course been fundamental to the results in my thesis, but more importantly it put me well on my way on becoming an independent researcher. The patience with which you answered my repetitive questions and the understanding you showed when my research was not going well eventually led to novel results. It was also our joined project, which features in this thesis as Chapter 3, which paved the way for me to come up with my own ideas and research projects. Besides our academic interactions I also enjoyed our many conversations on non-mathematical topics, such as music, puzzles and languages. I am looking forward to working more with you in the future on mathematical projects and the cryptic crossword puzzles which you usually bring to conferences.

I also want to thank my reading committee: Bas, Olivier and Rachel. You all made suggestions which considerably improved my thesis. Your careful work is not reflected by my mistakes which are still present, but by all the imprecise or incorrect statements which did not survive your meticulous reading.

Next, I would like to acknowledge the open atmosphere at the Mathematical Institute in Leiden. The many kind and helpful researchers and supportive staff helped me feel welcome even while I was a student from Utrecht University. This was also one of the reasons for me to come to Leiden for my PhD. I appreciate that the doors in the algebra, geometry and number theory group are always open and I think there are only a few of those doors on which I did not call during the last few years. Two people who helped me a lot are Bas and David. Bas, bedankt voor alle tijd die je nam voor mijn meetkundige vragen. Ik kwam altijd weg met meer antwoorden dan waar ik voor kwam. David, you helped me with practically every statement in this thesis involving the word 'blowup'; one of the more central geometric techniques in this thesis. Thank you so much.

There are many colleagues which I am happy to count among my friends. Erik, jij was eigenlijk vanaf dag één bij mijn promotie betrokken. We hebben zoveel gedeeld, van een kantoor tot conferenties en van de organisatie van de barbecue en seminaria tot het spelen van bordspellen, dat ik echt niet alles hier kan opnoemen. Het was echter niet ongebruikelijk dat onze meningen niet overeen kwamen, maar dat maakt een gesprek met jou altijd interessant en gegarandeerd leerzaam. Rosa, academisch zusje, ik heb met jou zoveel lol gehad. Daarbij kan de roadtrip naar Frankrijk natuurlijk niet onvermeld blijven. Zoals je weet, is er bepaalde muziek die mij daar altijd aan zal blijven herinneren. Wouter, ik waardeer onze vele gesprekken tot over de meest kleine details enorm. Jij hebt dan ook over heel veel dingen een goed onderbouwde mening. Jouw argumenten heb ik dan ook meermaals gebruikt om mijn eigen intuïtieve voorkeuren te verantwoorden. Raymond, naast al onze nuttige gesprekken over wiskunde wil ik je bedanken voor alle afleiding met het oplossen van leuke problemen, het trainen van leerlingen voor wiskundewedstrijden en het bedenken van wiskundewedstrijdopgaven. Stefan, ik denk nog altijd met veel plezier terug aan ons verblijf in Berlijn. Het was het begin van veel voetbalgerelateerde activiteiten, gezelligheid en humor.

Then there is the group consisting of Anna, Chloe, Erik, Garnet, Rosa and Wouter. With each of you I had a lot of fun over the last years and shared unforgettable conversations on a variety of topics. Whether it was about balloons, board games, dancing, linguistics, mathematics, music, politics or staplers, fun was guaranteed. But I mostly enjoyed spending time with all of you as a group. Although some of us have moved or will move abroad I do hope that we will stay in touch.

Ik wil ook Sanne enorm bedanken. Jij hebt mij volledig gesteund tijdens de laatste twee jaar van mijn promotie. Je hebt nog nooit een punt gemaakt van mijn slechte planning; bijvoorbeeld als ik weer eens ‘s avonds of in het weekend zat te werken en zelfs als er op vakantie nog een dankwoord geschreven moest worden. Bovendien heb je mij er ook regelmatig aan herinnerd dat kwaliteit niet belangrijker is dan de verhouding tussen kwaliteit en geïnvesteerde tijd. Zonder jou was dit proefschrift mogelijk een heel klein beetje meer volmaakt geweest, maar zonder twijfel nog lang niet klaar.

Tot slot wil ik graag mijn ouders, Paul en Dominique bedanken. Jullie hebben mij gesteund tijdens mijn promotie, maar ook tijdens de lange weg die mij hier gebracht heeft. Dit onderzoek komt voornamelijk voort uit een algemene interesse en nieuwsgierigheid die ik vanuit huis heb meegekregen, alle uitdagende problemen en puzzels die we gedeeld hebben en natuurlijk dat jullie mij leerden hoe daar mee om te gaan.

Curriculum vitae

Julian Lyczak was born in Alphen aan den Rijn in The Netherlands on the 5th of August 1988. After completing his secondary education at the Groene Hart Lyceum in 2006 he chose to combine his studies in mathematics at Utrecht University with physics and astronomy. He completed both bachelor programs in 2011 after which he decided to shift his focus entirely to mathematics. He successfully defended his Master's thesis at the end of 2014. Next he started his doctorate in Leiden under the supervision of dr. Martin Bright. The result of this research is the thesis you are currently reading.

For his Bachelor's, Master's and PhD degree Julian wrote consecutive theses in the fields of algebraic topology, algebraic geometry and arithmetic geometry. He has continued his mathematical journey in number theory by starting a postdoctoral research position at the Institute for Science and Technology Austria near Vienna in the group of prof. dr. Tim Browning.

Julian has been active with mathematical competitions for both high school and university students first as a student and later as trainer and team leader. As a student he won the Dutch National Mathematical Olympiad in 2005 and represented the Netherlands at the International Mathematical Olympiad in 2006 in Ljubljana, Slovenia. While at university he won several prizes at different international mathematical contests. As a team leader he trained and accompanied students to such contests all over the world. A personal highlight occurred in 2016 when he was the Team Leader of the dutch delegation at the International Mathematical Olympiad in Hong Kong.

Besides his mathematical interests Julian also plays the trombone and related instruments. A personal favourite being the bass trumpet.