Ruin formulas for a delay renewal risk model with general dependence

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Abstract

In this paper, we derive the explicit expressions and an upper bound of the ruin probability for the compound delayed renewal risk model taking into account the dependence within the inter-claim times, the claim sizes and between the claims and their inter-claim times. We use a specific mixture of exponential distributions to define the dependence structure between the inter-occurrence times and the claim sizes.

Keywords: Ruin Probability, mixing, delay renewal process, Archimedean copulas, Laplace transform.

1 Introduction

The last two decades research on ruin probabilities and related quantities has seen an intensification (see Albrecher and Dominik [3], Cai and Dickson [5], Delbaen and Haezendonck [9], Taylor [23], Waters [24], Sundt and Teugels [22], Gerber [11]). Related to our purpose, Ramsay [6] considered the ordinary risk model and assumed that the claim sizes have Pareto distribution with integer parameter. He then derived the Laplace transform of the integral equation of the ruin probability and solved this equation using an exponential integral combined with an inverse Laplace transform. He showed further in his paper that the solution can be expressed as the expected value of a function of a two parameter gamma random variable. Wei and Yang [18] extended the result of Ramsay [6] by considering the case where the inter-arrival times have Erlang (2) or Erlang (n) distributions and the claim amounts have a Pareto distribution. As in Ramsay [6] they showed that the ultimate ruin probability can be expressed as the expected value of a function of a two parameter gamma random variable. Albrecher and Teugels [2] considered an insurance portfolio situation in which there is possible dependence between the waiting time for a claim and its actual size. By employing the underlying random walk structure they obtained explicit exponential estimates for infinite- and finite-time ruin probabilities in the case of light-tailed claim sizes. Willmot [12] introduced the delayed renewal risk process, which is the process where the number of claim process is assumed to have a modified renewal process. In this model the time until the first claim occurs is assumed to have a distribution different to the forthcoming inter-arrival times distributions. Cossette et al. [7] analyzed dependent risk models using copulas. In their study, the dependence structure is specified by an Archimedean copula. Using this approach with specific marginals, they derived the explicit expressions for the probability density function of the aggregate risk and other related quantities and used those results to investigate risk models in regard to the aggregation, capital allocation and ruin problems. Fouad et al. [10] incorporated the dependence structure among the inter-arrival times, the claim sizes, and also assumed a dependence structure between the inter-claim times and the claim sizes and derived a closed expression for the high moments in the case of a mixing exponential model. They also gave some numerical illustrations to analyze the impact of the dependency on the moments of the discounted aggregate claim amounts. Sendova and Zitikis [20] suggested a general method for analyzing aggregate insurance claims that arrive according to a very general point process, they also allowed for the process to govern claim sizes via a general dependence structure that relates claim sizes to claim or inter-claim times and obtained general results with special closed-form formulas.

Aggregate claim process and its related functionals has been the subject of several studies, especially within the renewal context. The reader is referred to Jang [13], Léveillé and Adékambi [14], Léveillé et al. [17], Léveillé and Garrido [16, 15], Willmot [25]. In particular, Sarabia et al. [19] derived analytic expressions for the probability density function (pdf) and the cumulative distribution function (cdf) of aggregated risks, modelled according to a mixture of exponential distributions.

Using mixing ideas, Albrecher et al. [1] established the explicit formulas for the ruin probabilities and gave some examples for the compound Poisson risk model with completely monotone marginal claim sizes distributions that are dependent, modelled this dependence structure via an Archimedean survival copula and also studied the case where the dependence is among the inter-occurrence times.

In this paper we consider a general mixed risk model which is an extension of the risk model described in Albrecher et al. [1]. Our objective is to derive the ruin formulas and to achieve this, we use derivative and Laplace transform techniques. As we assume that the claim amounts follow an exponential distribution with a continuous non negative random parameter Θ and the inter-claim times also follow an exponential distribution with a continuous non negative random parameter Λ, we first condition on $(\Lambda \ \& \ \Theta)$ and take the first derivative of the integral equation for the ruin probability in the case of ordinary renewal model and for the delayed renewal model we use the assumption that after the first occurrence of the claim the process becomes an ordinary process. Then, we use the Laplace transform technique to solve these differential equations. Finally by assuming a specific distribution of Λ and Θ and using a specific copula to model the dependence structure, we derive the ruin probability.

To the best of our knowledge, there is no study in the literature which used a mixed exponential distribution for the claim amounts and the inter- arrival time in the ordinary and the delayed renewal risk model and assumed that there exists dependence structure between the parameter of those distributions. This dependence is relevant to account for the heterogeneity within the portfolio, moreover, since the distributions of the claim amounts and the counting process are generally unknown in advance, it is reasonable to assume that conditional on some factors $(\Lambda \ \& \ \Theta)$, the claim amounts and the inter-occurrence times follow a specific distribution and are independent as these factors $(\Lambda \ \& \ \Theta)$ can be used to describe the segmentation for a given line of business.

This paper is structured as follows: In Section (2), we describe the dependence structure of our risk model. The explicit expressions for the ruin formulas are derived in the case of an Archimedean copula in Section (3). Section (4) deals with the delayed risk model and provides some numerical examples to illustrate the impact of the delayed time on the ruin probability. Section (5) concludes the paper.

2 The model

In this section, we introduce the ordinary and the delayed renewal process with dependence.

The surplus process is represented by $\mathbf{U}(t)$, $t \geq 0$ with $\mathbf{U}(0) = u$ and

$$\mathbf{U}(\mathbf{t}) = u + ct - \mathbf{S}_t, \ \text{ or } \ \mathbf{U}(\mathbf{t}) = u + ct - \mathbf{S}^d_t, \quad t \geq 0 \tag{1}$$

where $\mathrm{S}_t = \sum_{i=1}^{\mathrm{N}(t)} X_i$, $\mathrm{S}_t^d = \sum_{i=1}^{\mathrm{N}_d(t)} X_i$, $\mathrm{S}_t = 0$ if $\mathrm{N}(t) = 0$ and $\mathrm{S}_t^d = 0$ if $\mathrm{N}_d(t) = 0$. The claims counting processes $\mathrm{N}(t), t \geq 0$ and $\mathrm{N}_d(t), t \geq 0$ form, respectively, an ordinary and a delayed renewal process and, for $k \in \mathbb{N} = 1, 2, 3, \cdots$ we define T_k to be the corresponding claim occurrence times and $\mathrm{W}_j = \mathrm{T}_j - \mathrm{T}_{j-1}$, $j = 1, 2, \cdots$ to be the subsequent inter-claim times. The claim severities $\{\mathrm{X}_k\}_{k=1}^{\infty}$ are assumed to form an independent identically distributed (i.i.d.) sequence and their higher moments exist over a subset $\mathbf{U} \subset \mathbb{R}$ including a neighborhood of the origin.

In the usual ordinary renewal risk process, the sequences $\{\mathrm{W}_j\}_{j=1}^{\infty}$ are assumed to be mutually independent. However, in this paper, we assume that $\mathrm{W}_1, \mathrm{W}_2, \cdots$ are dependent and their dependence structure modelled via Archimedean copulas.

Let Λ be a random variable with pdf $f_\Lambda(\lambda)$, assume that its Laplace transform given by
$f_\Lambda^\star(s) = \int_0^\infty e^{-s\lambda} f_\Lambda(\lambda)\mathrm{d}\lambda$ exists over a subset $k \subset \mathbb{R}$ including a neighborhood of the origin.

For a general set up, the results obtained above by using an exponential distribution for the conditional distribution of the time between successive claims, can be extended to other conditionally independent distributions. For example, the conditional distribution of the inter-claims time can be written in the power form:

$$\mathbf{Pr}\big[\mathrm{W}_i \geq x_i \big| \Lambda = \lambda\big] = \big[\overline{\mathrm{H}}(x_i)\big]^\lambda \text{ for some distribution function } \mathrm{H}(x_i) \text{ and}$$

$$\mathbf{Pr}\big[\mathrm{W}_1 \geq x_1, \mathrm{W}_2 \geq x_2, \cdots, \mathrm{W}_n \geq x_n \big| \Lambda = \lambda\big] = \prod_{i=1}^n \big[\overline{\mathrm{H}}(x_i)\big]^\lambda. \tag{2}$$

For all n, i.e. Λ is the common mixing parameter, then

$$\begin{aligned}
\overline{\mathrm{F}}_{\mathrm{W}_1,\cdots,\mathrm{W}_n}(w_1, \cdots, w_n) &= \int_0^\infty \mathbf{Pr}\big[\mathrm{W}_1 \geq x_1, \cdots, \mathrm{W}_n \geq x_n \big| \Lambda = \lambda\big] f_\lambda(\lambda)\mathrm{d}\lambda, \\
&= \int_0^\infty \big(\overline{\mathrm{H}}(x_1)\big)^\lambda \big(\overline{\mathrm{H}}(x_2)\big)^\lambda \cdots \big(\overline{\mathrm{H}}(x_n)\big)^\lambda f_\Lambda(\lambda)\mathrm{d}\lambda, \\
&= f_\Lambda^\star\big(-\log \overline{H}(x_1) - \cdots - \log \overline{H}(x_1)\big), \\
&= f_\Lambda^\star\big(f_\Lambda^{\star-1}(\overline{\mathrm{F}}_{\mathrm{W}_1}) + \cdots + f_\Lambda^{\star-1}(\overline{\mathrm{F}}_{\mathrm{W}_n})\big).
\end{aligned} \tag{3}$$

The form of the dependence structure is again Archimedean with generator $\emptyset(t) = f_\Lambda^{\star-1}(t)$, where $\overline{\mathrm{F}}_{W_j}(x_{j)} = f_\Lambda^\star\big(-\log \overline{H}(x_j)\big)$.

Remark: Specific mixture of exponential distributions

2.1 Ordinary renewal case

Let the random variables (r.v) $\mathrm{W}_1, \mathrm{W}_2, \cdots, \mathrm{W}_n$ be an n dependent, positive and continuous r.v. Assume that, given $\Lambda = \lambda$, the r.v's $\mathrm{W}_1, \mathrm{W}_2, \cdots, \mathrm{W}_n$ are conditionally independent and distributed as $\mathrm{Exp}(\lambda)$ then,

$$\begin{aligned}
\mathbf{Pr}\big[\mathrm{W}_1 \geq x_1, \cdots, \mathrm{W}_n \geq x_n \big| \Lambda = \lambda\big] &= \mathbf{Pr}\big[\mathrm{W}_1 \geq x_1 \big| \Lambda = \lambda\big] \cdots \mathbf{Pr}\big[\mathrm{W}_n \geq x_n \big| \Lambda = \lambda\big], \\
&= e^{-\lambda x_1} \cdots e^{-\lambda x_n}.
\end{aligned} \tag{4}$$

It follows that

$$\overline{\mathrm{F}}_{W_i}(x_i) = \mathbf{Pr}\big(W_i \geq x_i\big) = \int_0^\infty e^{-\lambda x} f_\Lambda(\lambda)\mathrm{d}\lambda = f_\Lambda^*(x) = \int_0^\infty \mathbf{Pr}\big(W_i \geq x_i \,|\Lambda = \lambda\big) f_\Lambda(\lambda)\mathrm{d}\lambda. \tag{5}$$

The joint tail distribution of $\mathrm{W}_1, \mathrm{W}_2, \cdots, \mathrm{W}_n$ can be written as

$$\begin{aligned}
\overline{\mathrm{F}}_{W_1,\cdots,W_n}(x_1,\cdots x_n) &= \int_0^\infty \mathbf{Pr}\big(W_1 \geq x_1,\cdots W_n \geq x_n \mid \Lambda = \lambda\big) f_\Lambda(\lambda)\mathrm{d}\lambda, \\
&= \int_0^\infty e^{-\lambda \sum_{i=1}^n x_i} f_\Lambda(\lambda)\mathrm{d}\lambda, \\
&= f_\Lambda^*\bigg(\sum_{i=1}^n x_i\bigg), \\
&= f_\Lambda^*\big(f_\Lambda^{*-1}\big(\overline{\mathrm{F}}_{W_1}(x_1)\big) + \cdots + f_\Lambda^{*-1}\big(\overline{\mathrm{F}}_{W_n}(x_n)\big)\big). \qquad (6)
\end{aligned}$$

Otherwise from Sklar's theorem, see e.g. Sklar [21],

$$\mathbf{Pr}\big(W_1 \geq x_1,\cdots,W_n \geq x_n\big) = \tilde{C}\big(\overline{\mathrm{F}}_{W_1}(x_1),\cdots,\overline{\mathrm{F}}_{W_n}(x_n)\big). \qquad (7)$$

It follows that $\tilde{C}(u_1,\cdots,u_n)$ is an Archimedean copula with

$$\tilde{C}(u_1,\cdots,u_n) = f_\Lambda^*\big(f_\Lambda^{*-1}(u_1) + \cdots + f_\Lambda^{*-1}(u_n)\big) = \emptyset^{-1}\big(\emptyset(u_1) + \cdots + \emptyset(u_n)\big), \qquad (8)$$

where $\emptyset^{-1}(t) = f_\Lambda^{*-1}(t)$ is the generator of the Archimedean copula $\tilde{C}$.

2.2 The delayed renewal dependence case

Let $(\Lambda_1,\ \Lambda_2)$ be a positive random vector with cumulative distribution $\mathrm{F}_{\Lambda_1,\Lambda_2}$. Assume that its Laplace transform given by $f^*_{\Lambda_1,\Lambda_2}(s,t) = \int_0^\infty\int_0^\infty e^{-\lambda_1 s-\lambda_2 t}\mathrm{dF}_{\Lambda_1,\Lambda_2}(\lambda_1,\lambda_2)$ exists over a subset $\mathrm{K}\times\mathrm{K}\subset\mathbb{R}^2$ including a neighborhood of the origin. Assume further that $\mathrm{W}_1, \mathrm{W}_2, \cdots, \mathrm{W}_n$ are n dependent random variables, conditionally independent given $(\Lambda_1,\ \Lambda_2)$ and distributed as an exponential distribution, $\mathrm{W}_1|\Lambda_1 = \lambda_1$ and $\mathrm{W}_i|\Lambda_2 = \lambda_2 \sim \exp(\lambda_2)$, $i = 2,\cdots,n$, such that the joint conditional survival function is given by:

$$\begin{aligned}
\mathbf{Pr}\big(W_1 \geq x_1,\cdots,W_n \geq x_n|\Lambda_1 = \lambda_1,\Lambda_2 = \lambda_2\big) &= \mathbf{Pr}\big(W_1 \geq x_1|\Lambda_1 = \lambda_1\big)\cdots\mathbf{Pr}\big(W_n \geq x_n|\Lambda = \lambda_2\big), \\
&= e^{-\lambda_1 x_1}e^{-\lambda_2 x_2}\cdots e^{-\lambda_2 x_n}. \qquad (9)
\end{aligned}$$

The marginal distribution of W_1 is given by

$$\overline{\mathrm{F}}_{W_1}(x_1) = \mathbf{Pr}(W_1 \geq x_1) = \int_0^\infty \mathbf{Pr}(W_1 \geq x_1|\Lambda_1 = \lambda_1)f_{\Lambda_1}(\lambda_1)\mathrm{d}\lambda_1 = f^*_{\Lambda_1}(x).$$

It follows that the joint survival function of $(\mathrm{W}_1,\cdots,\mathrm{W}_n)$ can be written as

$$\begin{aligned}
\overline{\mathrm{F}}_{W_1,\cdots,W_n}(x_1,\cdots,x_n) &= \int_0^\infty\int_0^\infty \mathbf{Pr}\big(W_1 \geq x_1,\cdots,W_n \geq x_n|\Lambda_1 = \lambda_1,\Lambda_2 = \lambda_2\big)\mathrm{dF}_{\Lambda_1,\Lambda_2}(\lambda_1,\lambda_2), \\
&= \int_0^\infty\int_0^\infty e^{-\lambda_1 x_1}e^{-\lambda_2\sum_{i=2}^n x_i}\mathrm{dF}_{\Lambda_1,\Lambda_2}(\lambda_1,\lambda_2), \\
&= f^*_{\Lambda_1,\Lambda_2}\bigg(x_1,\sum_{i=2}^n x_i\bigg), \qquad (10)
\end{aligned}$$

and the survival function of the marginal distribution of W_i for $i = 2, \cdots, n$ is given by:

$$\overline{\mathrm{F}}_{W_i}(x_i) = f^*_{\Lambda_2}(x_i).$$

Note that all the marginal distributions are not identically distributed. The first inter-arrival time has distribution $\overline{\mathrm{F}}_{W_1}(x_1) = f^*_{\Lambda_1}(x_1)$, while all subsequent inter-arrival times have a cdf of the form $\overline{\mathrm{F}}_{W_i}(x_i) = f^*_{\Lambda_2}(x_i)$, for $i = 2, \cdots, n$. For special case when $\overline{\mathrm{F}}_{W_1}(x_1) = \overline{\mathrm{F}}_{W_i}(x_i)$, for $i = 2, \cdots, n$, we get the ordinary renewal process.
From (10), we get

$$\overline{\mathrm{F}}_{W_1,\cdots,W_n}(x_1, \cdots, x_n) = f^*_{\Lambda_1,\Lambda_2}\big(f^{*-1}_{\Lambda_1}(\mathrm{F}_{W_1}(x_1)), f^{*-1}_{\Lambda_2}(\mathrm{F}_{W_2}(x_2)), \cdots, f^{*-1}_{\Lambda_2}(\mathrm{F}_{W_n}(x_n))\big)$$

By applying Sklar's theorem, see e.g. Sklar [21], the survival distribution function of $\mathrm{W}_1, \mathrm{W}_2, \cdots, \mathrm{W}_n$ is given by

$$\mathbf{Pr}\big(W_1 \geq x_1, \cdots, W_n \geq x_n\big) = \tilde{C}\big(\overline{\mathrm{F}}_{W_1}(x_1), \cdots, \overline{\mathrm{F}}_{W_n}(x_n)\big).$$

Therefore, we have a factor copula of the form

$$\tilde{C}(u_1, \cdots, u_n) = f^*_{\Lambda_1,\Lambda_2}\big(f^{*-1}_{\Lambda_1}(u_1), f^{*-1}_{\Lambda_2}(u_2) + \cdots + f^{*-1}_{\Lambda_2}(u_n)\big) = \Psi^{-1}\big(\Psi_1(u_1), \emptyset(u_2) + \cdots + \emptyset(u_n)\big), \quad (11)$$

where $\emptyset(t) = f^{*-1}_{\Lambda_2}(t)$, $\Psi_1(t) = f^{*-1}_{\Lambda_1}(t)$ is the generator of the Archimedean copula $\tilde{\mathrm{C}}$.

Let $(\Theta, \Lambda_1, \Lambda_2)$ be a positive random vector with cumulative distribution $\mathrm{F}_{\Theta,\Lambda_1,\Lambda_2}$. Assume that its Laplace transform given by $f^*_{\Theta,\Lambda_1,\Lambda_2}(x, y, z) = \int_0^\infty \int_0^\infty \int_0^\infty e^{-\theta x - \lambda_1 y - \lambda_2 z} \mathrm{dF}_{\Theta,\Lambda_1,\Lambda_2}(\theta, \lambda_1, \lambda_2)$ exists over a subset $\mathrm{K} \times \mathrm{K} \times \mathrm{K} \subset \mathbb{R}^3$ including a neighborhood of the origin. Assume further that $X_1, \cdots, X_n, \mathrm{W}_1, \mathrm{W}_2, \cdots, \mathrm{W}_n$ are dependent random variables, conditionally independent given $(\Theta, \Lambda_1, \Lambda_2)$ and distributed as an exponential distribution, $X_i|\Theta \sim \exp(\theta)$, $i = 1, \cdots, n$, $\mathrm{W}_1|\Lambda_1 = \lambda_1 \sim \exp(\lambda_1)$ and $\mathrm{W}_j|\Lambda_2 = \lambda_2 \sim \exp(\lambda_2)$, $j = 2, \cdots, n$, such that the joint conditional survival function is given by:

$$\mathbf{Pr}\big(X_1 \geq x_1, \cdots, X_n \geq x_n, W_1 \geq t_1, \cdots, W_n \geq t_n | \Theta = \theta, \Lambda_1 = \lambda_1, \Lambda_2 = \lambda_2\big) = e^{-\lambda_1 x_1} e^{-\theta \sum_{i=1}^n x_i} e^{-\lambda_2 \sum_{j=2}^n x_j}. \quad (12)$$

It follows that the joint survival function of $(X_1, \cdots, X_n, \mathrm{W}_1, \cdots, \mathrm{W}_n)$ can be written as

$$\begin{aligned}\overline{\mathrm{F}}_{X_1,\cdots,X_n,W_1,\cdots,W_n}(x_1, \cdots, x_n, t_1, \cdots, t_n) &= \int_0^\infty \int_0^\infty \int_0^\infty e^{-\lambda_1 x_1} e^{-\theta \sum_{i=1}^n x_i} e^{-\lambda_2 \sum_{j=2}^n t_j} \mathrm{dF}_{\Theta,\Lambda_1,\Lambda_2}(\lambda_1, \lambda_2), \\ &= f^*_{\Theta,\Lambda_1,\Lambda_2}\Big(\sum_{i=1}^n x_i, t_1, \sum_{j=2}^n t_i\Big).\end{aligned} \quad (13)$$

From (13), we get

$$\begin{aligned}\overline{\mathrm{F}}_{X_1,\cdots,X_n,W_1,\cdots,W_n}(x_1, \cdots, x_n, t_1, \cdots, t_n) = f^*_{\Theta,\Lambda_1,\Lambda_2}\Big(& f^{*-1}_{\Theta}(\mathrm{F}_{X_1}(x_1)), \cdots, f^{*-1}_{\Theta}(\mathrm{F}_{X_n}(x_n)), \\ & f^{*-1}_{\Lambda_1}(\mathrm{F}_{W_1}(t_1)), f^{*-1}_{\Lambda_2}(\mathrm{F}_{W_2}(t_2)), \cdots, f^{*-1}_{\Lambda_2}(\mathrm{F}_{W_n}(t_n))\Big)\end{aligned}$$

By applying Sklar's theorem, see e.g. Sklar [21], the survival distribution function of $X_1, \cdots, X_n, \mathrm{W}_1, \mathrm{W}_2, \cdots, \mathrm{W}_n$ is

$$\mathbf{Pr}\big(X_1 \geq x_1, \cdots, X_n \geq x_n, W_1 \geq t_1, \cdots, W_n \geq t_n\big) = \tilde{C}\big(\overline{\mathrm{F}}_{X_1}(x_1), \cdots, \overline{\mathrm{F}}_{X_n}(x_n), \overline{\mathrm{F}}_{W_1}(t_1), \cdots, \overline{\mathrm{F}}_{W_n}(t_n)\big).$$

Therefore, we have a factor copula of the form

$$\begin{aligned}\tilde{C}(u_1,\cdots,u_n,v_1,\cdots,v_n) &= f^*_{\Theta,\Lambda_1,\Lambda_2}\Big(\sum_{i=1}^{n} f^{*-1}_{\Theta}(u_i), f^{*-1}_{\Lambda_1}(v_1), \sum_{j=1}^{n} f^{*-1}_{\Lambda_2}(v_j)\Big), \\ &= \Psi^{-1}\big(\Psi_1(u_1)+\cdots+\Psi_1(u_n), \emptyset_1(v_1), \emptyset_2(v_2)\cdots+\emptyset_2(v_n)\big),\end{aligned} \tag{14}$$

where $\emptyset_1(t) = f^{*-1}_{\Lambda_1}(t)$, $\emptyset_2(t) = f^{*-1}_{\Lambda_2}(t)$, $\Psi_1(t) = f^{*-1}_{\Theta}(t)$ is the generator of the Archimedean copula $\tilde{\mathrm{C}}$. Hence the dependence structure among W_1, W_j for $j = 2,\cdots,n$ and X_i is generated by a copula C_{123} given by

$$\mathrm{C}_{123}(x,t,s) = f^*_{\Theta,\Lambda_1,\Lambda_2}\big(f^{*-1}_{\Theta}(x), f^{*-1}_{\Lambda_1}(t), f^{*-1}_{\Lambda_2}(s)\big). \tag{15}$$

Remark 1. *Letting $x_i \to 0$ in (13) for all i yields (10). If $\Lambda_1 \to \Lambda_2$, then the equation (13) is consistent with the result of Fouad et al. [10]. Taking $t_1 = 0$ in (13) yields the joint survival function of $(X_1,\cdots,X_n,\mathrm{W}_2,\cdots,\mathrm{W}_n)$. Taking $t_j = 0$ for $j = 2,\cdots,n$ in (13) gives the joint distribution of $(X_1,\cdots,X_n,\mathrm{W}_1)$.*

3 Ruin probability in the ordinary case

The conditional infinite ruin probability is defined as

$$\psi_{\Lambda,\Theta}(\lambda,\theta,u) = \mathbf{Pr}\Big[\inf_{t\geq 0}\mathbf{U}(t) < 0 \,\Big|\, \mathbf{U}(0) = u,\ \Lambda = \lambda,\ \Theta = \theta\Big],\ \ u \geq 0, \tag{16}$$

and the infinite ruin probability is defined as

$$\psi(u) = \mathbf{Pr}\Big[\inf_{t\geq 0}\mathbf{U}(t) < 0 \,\Big|\, \mathbf{U}(0) = u\Big],\ \ u \geq 0. \tag{17}$$

Consider the model defined by equation (1), Albrecher et al. [1] have proved that Conditional on Λ and Θ the infinite ruin probability is given by

$$\psi_{\Lambda,\Theta}(\lambda,\theta,u) = \min\left\{\frac{\lambda}{\theta c}\exp\big(-(\theta-\frac{\lambda}{c})u\big), 1\right\},$$

where c represents the premium rate and u the initial reserve.
From equation (17) we have:

$$\psi(u) \;=\; \int_{\lambda}\int_{\theta}\psi_{\Lambda,\Theta}(\lambda,\theta,u)f_{\Lambda,\Theta}(\theta,\lambda)d\theta d\lambda.$$

From the non-ruin condition, we have $\dfrac{c}{\lambda} = c\mathbf{E}(\mathrm{W}_1) > \mathbf{E}(X) = \dfrac{1}{\theta}$ then $\dfrac{c}{\lambda} > \dfrac{1}{\theta} \iff \dfrac{\lambda}{c} < \theta$. Hence, if $\theta \leq \dfrac{\lambda}{c}$ the ruin is certain $\big(\psi_{\Lambda,\Theta}(u) = 1\big)$. It follows that

$$\psi(u) = \int_0^{\infty}\int_0^{\frac{\lambda}{c}} f_{\Lambda,\Theta}(\theta,\lambda)d\theta d\lambda + \int_0^{\infty}\int_{\frac{\lambda}{c}}^{\infty}\psi_{\Lambda,\Theta}(\lambda,\theta,u)f_{\Lambda,\Theta}(\theta,\lambda)d\theta d\lambda. \tag{18}$$

3.1 Explicit ruin probability for product copula

The following theorem gives the formula of the ruin probability in the ordinary renewal process where Λ and Θ are independent.

Theorem 1. *Consider the model defined by equation (1) such that conditional on Λ the counting process follows Poisson process. $\left(X_k\right)_{k\,\geq\,0}$ are such that conditional on Θ, they are i.i.d and follow exponential distribution. Assume that the random variable Θ has a mixing Erlang distribution with density $f_\Theta(\theta) = \sum_{i=1}^{p} \frac{a_i\gamma_i^n}{(n-1)!}\theta^{n-1}\exp(-\gamma_i\theta)$. Assume further that Λ has a mixing Erlang distribution with density $f_\Lambda(\lambda) = \sum_{j=1}^{q} \frac{b_j\beta_j^n}{(n-1)!}\lambda^{n-1}\exp(-\beta_j\lambda)$ where $\sum_{i=1}^{p} a_i = 1$, $\sum_{j=1}^{q} b_j = 1$, $p, q \in \mathbb{N}$, $(a_i)_{i=1}^{p} \in \mathbb{N}$ and $(a_i)_{i=1}^{p} \in \mathbb{N}$. If (Θ, Λ) are independent, then the infinite ruin probability is given by:*

$$
\begin{aligned}
\psi(u) = {} & \sum_{i=1}^{p}\sum_{j=1}^{q} a_i b_j \left\{ 1 - \left[\frac{c\beta_j}{c\beta_j+\gamma_i}\right]^{2n-1} \sum_{k=0}^{n-1} \binom{2n-1}{k} \left[\frac{\gamma_i}{c\beta_j}\right]^{k} \right\} \\
& + \sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=0}^{n-1} a_i b_j \frac{n+k}{n-1} \binom{n+k-1}{k} \left[\frac{c\gamma_i\beta_j}{(\gamma_i+u)(c\beta_j+\gamma_i)}\right]^{n} \left[\frac{\gamma_i+u}{c\beta_j+\gamma_i}\right]^{k+1},
\end{aligned}
\tag{19}
$$

where u is the initial reserve and c is the premium rate.

Proof. Using equation (18), the infinite ruin probability is given by

$$\begin{aligned}
\psi(u) &= \int_0^\infty \int_0^{\frac{\lambda}{c}} f_\Theta(\theta) f_\Lambda(\lambda)\mathrm{d}\theta\mathrm{d}\lambda + \int_0^\infty \int_{\frac{\lambda}{c}}^\infty \psi(u,\lambda,\theta) f_\Theta(\theta) f_\Lambda(\lambda)\mathrm{d}\theta\mathrm{d}\lambda, \\
&= \int_0^\infty f_\Lambda(\lambda)\Big(\int_0^{\frac{\lambda}{c}} \sum_{i=1}^p \frac{a_i\gamma_i^n}{(n-1)!}\theta^{n-1}\exp(-\gamma_i\theta)\mathrm{d}\theta\Big)\mathrm{d}\lambda \\
&+ \frac{1}{c}\int_0^\infty \lambda f_\Lambda(\lambda)\exp\big(\frac{\lambda}{c}u\big)\int_{\frac{\lambda}{c}}^\infty \sum_{i=1}^p \frac{a_i\gamma_i^n}{(n-1)!}\theta^{n-2}\exp(-(\gamma_i+u)\theta)\mathrm{d}\theta\mathrm{d}\lambda, \\
&= \int_0^\infty f_\Lambda(\lambda)\Big(1-\sum_{i=1}^p a_i\exp\big(-\frac{\gamma_i}{c}\lambda\big)\sum_{k=0}^{n-1}\frac{(\lambda\gamma_i)^k}{c^k k!}\Big)\mathrm{d}\lambda \\
&+ \frac{1}{c}\int_0^\infty \lambda f_\Lambda(\lambda)\exp\big(\frac{\lambda}{c}u\big)\sum_{i=1}^p \frac{a_i\gamma_i^n}{(n-1)(\gamma_i+u)^{n-1}}\sum_{k=0}^{n-1}\frac{\big((\gamma_i+u)\lambda\big)^k}{c^k k!}\exp\big(-(\gamma_i+u)\frac{\lambda}{c}\big), \\
&= 1-\sum_{i=1}^p\sum_{j=1}^q\sum_{k=0}^{n-1}\int_0^\infty \frac{a_i b_j\beta_j^n\gamma^k}{(n-1)!k!}\frac{\lambda^{n+k-1}}{c^k}\exp\big(-(\frac{c\beta_j+\gamma_i}{c})\lambda\big)\mathrm{d}\lambda \\
&+ \sum_{i=1}^p\sum_{j=1}^q\sum_{k=0}^{n-1}\frac{a_i\gamma_i^n b_j\beta_j^n}{(n-1)!k!}\frac{(\gamma_i+u)^{k+1-n}}{(n-1)c^{k+1}}\int_0^\infty \lambda^{n+k}\exp\big(-(\frac{c\beta_j+\gamma_i}{c})\lambda\big)\mathrm{d}\lambda, \\
&= 1-\sum_{i=1}^p\sum_{j=1}^q\sum_{k=0}^{n-1}\frac{a_i b_j\beta_j^n\gamma^k}{(n-1)!k!c^k}\frac{(n+k-1)!}{(c\beta_j+\gamma_j)^{n+k}}c^{n+k} \\
&+ \sum_{i=1}^p\sum_{j=1}^q\sum_{k=0}^{n-1}\frac{a_i\gamma_i^n b_j\beta_j^n}{(n-1)!k!}\frac{(\gamma_i+u)^{k+1-n}}{(n-1)c^{k+1}}(n+k)!\frac{c^{n+k+1}}{(c\beta_j+\gamma_i)^{n+k+1}}, \\
&= 1-\sum_{i=1}^p\sum_{j=1}^q\sum_{k=0}^{n-1}\frac{a_i b_j(c\beta_j)^n\gamma_i^k}{(c\beta_j+\gamma_i)^{n+k}}\binom{n+k-1}{k} \\
&+ \sum_{i=1}^p\sum_{j=1}^q\sum_{k=0}^{n-1}\frac{n+k}{n-1}\frac{a_i b_j(c\gamma_i\beta_j)^n}{(\gamma_i+u)^{n-k-1}(c\beta_j+\gamma_i)^{n+k+1}}\binom{n+k-1}{k}, \\
&= 1-\sum_{i=1}^p\sum_{j=1}^q a_i b_j\Big[\frac{c\beta_j}{c\beta_j+\gamma_i}\Big]^{2n-1}\sum_{k=0}^{n-1}\binom{2n-1}{k}\Big[\frac{\gamma_i}{c\beta_j}\Big]^k \\
&+ \sum_{i=1}^p\sum_{j=1}^q\sum_{k=0}^{n-1} a_i b_j\frac{n+k}{n-1}\binom{n+k-1}{k}\Big[\frac{c\gamma_i\beta_j}{(\gamma_i+u)(c\beta_j+\gamma_i)}\Big]^n\Big[\frac{\gamma_i+u}{c\beta_j+\gamma_i}\Big]^{k+1}.
\end{aligned}$$

This proves the statement. ∎

3.2 Ruin probability with copula

If we assume that the dependence structure between Λ and Θ is given by an Archimedean copula, then the ruin probability is given by:

$$\psi(u) = \int_0^\infty \int_0^{\frac{\lambda}{c}} f_{\Theta,\Lambda}(\theta,\lambda)\mathrm{d}\theta\mathrm{d}\lambda + \int_0^\infty \int_{\frac{\lambda}{c}}^\infty \frac{\lambda}{\theta c} \exp\big(-\big(\theta - \frac{\lambda}{c}\big)u\big) f_{\Theta,\Lambda}(\theta,\lambda)\mathrm{d}\theta\mathrm{d}\lambda, \tag{20}$$

where $f_{\Theta,\Lambda}(\theta,\lambda)$ is the joint distribution of Λ and Θ and is given by

$$f_{\Theta,\Lambda}(\theta,\lambda) = \frac{\partial^2}{\partial\theta\partial\lambda}\mathrm{C}\big(F_\Theta(\theta), F_\Lambda(\lambda)\big) f_\Theta(\theta) f_\Lambda(\lambda).$$

and C(u,v) is a copula, $(u,v) \in [0,1]^2$.
The integral (20) is difficult to evaluate in most of cases with some copulas. In the next subsection we derive an upper bound of the ruin probability under the dependence assumption.

3.2.1 Upper bound of the ruin probability with dependence

The next theorem gives an upper bound of the ruin formula in the ordinary case where the random variables Θ and Λ are dependent.

Theorem 2. *Consider the model defined by equation (1) such that conditional on Λ the counting process follows Poisson process. $(\mathrm{X}_k)_{k\,\geq\,0}$ are such that conditional on Θ, they are i.i.d and follow exponential distribution. Assume that $\big(\Lambda,\ \Theta\big)$ are dependent and also the Laplace's transform $\big(f^*_{\Lambda,\ \Theta}(s,t)\big)$ of Λ and Θ exists, then the upper bound of the ruin is given by:*

$$\psi(u) \leq f^*_{\Theta,\Lambda}\big(u, -\frac{u}{c}\big), \tag{21}$$

where u is the initial reserve and c is the premium rate.

Proof.

$$\begin{aligned}\psi(u) &= \int_0^\infty \int_0^{\frac{\lambda}{c}} f_{\Theta,\Lambda}(\theta,\lambda)\mathrm{d}\theta\mathrm{d}\lambda + \int_0^\infty \int_{\frac{\lambda}{c}}^\infty \psi(u,\lambda,\theta) f_{\Theta,\Lambda}(\theta,\lambda)\mathrm{d}\theta\mathrm{d}\lambda, \\ &= \int_0^\infty \int_0^{\frac{\lambda}{c}} f_{\Theta,\Lambda}(\theta,\lambda)\mathrm{d}\theta\mathrm{d}\lambda + \int_0^\infty \int_{\frac{\lambda}{c}}^\infty \frac{\lambda}{c\theta} e^{-\left(\theta-\frac{\lambda}{c}\right)u} f_{\Theta,\Lambda}(\theta,\lambda)\mathrm{d}\theta\mathrm{d}\lambda.\end{aligned}$$

In one hand, from the non ruin condition $\frac{c}{\lambda} \geq \frac{1}{\theta} \Longleftrightarrow \frac{\lambda}{c\theta} \leq 1$ which implies,

$$\psi(u) \leq \int_0^\infty \int_0^{\frac{\lambda}{c}} f_{\Theta,\Lambda}(\theta,\lambda)\mathrm{d}\theta\mathrm{d}\lambda + \int_0^\infty \int_{\frac{\lambda}{c}}^\infty e^{-\left(\theta-\frac{\lambda}{c}\right)u} f_{\Theta,\Lambda}(\theta,\lambda)\mathrm{d}\theta\mathrm{d}\lambda.$$

In the other hand $\lambda \geq c\theta \Longleftrightarrow e^{-\left(\theta-\frac{\lambda}{c}\right)u} \geq 1$ which implies,

$$\begin{aligned}\psi(u) &\leq \int_0^\infty \int_0^{\frac{\lambda}{c}} e^{-\left(\theta-\frac{\lambda}{c}\right)u} f_{\Theta,\Lambda}(\theta,\lambda)\mathrm{d}\theta\mathrm{d}\lambda + \int_0^\infty \int_{\frac{\lambda}{c}}^\infty e^{-\left(\theta-\frac{\lambda}{c}\right)u} f_{\Theta,\Lambda}(\theta,\lambda)\mathrm{d}\theta\mathrm{d}\lambda,\\
&\leq \int_0^\infty \int_0^\infty e^{-\left(\theta-\frac{\lambda}{c}\right)u} f_{\Theta,\Lambda}(\theta,\lambda)\mathrm{d}\theta\mathrm{d}\lambda,\\
&\leq f^*_{\Theta,\Lambda}\big(u, -\frac{u}{c}\big).\end{aligned}$$

□

3.2.2 Numerical illustration with Clayton copula.

Let Θ follows exponential distribution with parameter $a = 5 \times 10^{-3}$, Λ follows exponential distribution with parameter $b = 10^{-3}$. Assume that the dependence structure between Θ and Λ is modelled by Clayton copula with parameter α.

Tables [1 - 2] below give some numerical values of the ruin probability and its upper bound for some values of α with respect to the initial surplus u.

Table 1: Ruin probability with Clayton copula parameter $\alpha = 0.75$ and premium rate c = 25%

u	0	10	25	50	75	100	150	200	250
Numerical value	0.7007	0.6939	0.6846	0.6708	0.6587	0.6480	0.6298	0.6149	0.6024
Upper bound	1	0.9895	0.9717	0.9379	0.9002	0.8598	0.7755	0.6910	0.6064

Table 2: Ruin probability with Clayton copula parameter $\alpha = 1$ and premium rate c = 25%.

u	0	10	25	50	75	100	150	200	250
Numerical value	0.6893	0.6828	0.6736	0.6602	0.6484	0.6380	0.6205	0.6062	0.5942
Upper bound	1	0.9901	0.9756	0.9524	0.9302	0.9091	0.8696	0.8333	0.8000

For a fixed initial reserve u, increasing the parameter α increases the ruin probability which means that high losses are expected under the dependence scenarios.

4 Ruin probability in the delayed renewal case

In this section we derive the Laplace transform for the ruin probability with the delayed renewal process. An explicit expression of the ruin under the following assumptions:

- The claim sizes are exponentially distributed $\Big(\big(X_k \big| \Theta = \theta\big)_{k\geq 0} \sim \exp(\theta)\Big)$;
- The inter-arrival time $\mathrm{W}_1 \big| \Lambda = \lambda_1 \sim \exp(\lambda_1)$ and $\big(\mathrm{W}_i \big| \Lambda = \lambda_2\big)_{i\geq 2} \sim \exp(\lambda_2)$ are independent and identically distributed,

is also given.

4.1 Laplace transform for conditional ruin probability in the delayed renewal process

The following proposition expresses the Laplace transform of the ruin probability when the counting process is assumed to be a delayed renewal process.

Proposition 1. *Consider the mixed delayed Poisson process for the claim arrival where the claim severities given Θ are i.i.d with density $f_{X\mid\Theta}(x,\theta)$. Assume that the distribution of the first occurrence time is $\exp(\Lambda_1)$ and the forthcoming claim times are i.i.d. and follow exponential distribution with parameter Λ_2. Furthermore, assume that the Laplace transform for all quantities involved exist. Then the Laplace transform for the delayed ruin probability is given by:*

$$\tilde{\psi}^d_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,p)=\frac{c}{cp-\lambda_1}\Big[\psi^d_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,0)-\frac{\lambda_1}{cp}\Big(1+\frac{cp\big(\psi^o_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,0)-1\big)}{\lambda_2\tilde{f}(p,\theta)+cp-\lambda_2}\tilde{f}(p,\theta)\Big)\Big], \quad (22)$$

where $\psi^o_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,u)$ is the ruin probability for the ordinary renewal process and c represents the premium rate.

Proof.

$$\begin{aligned}\psi^d_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,u)&=\mathbf{P}\big[X_1>u+cT_1\,\big|\Lambda_1=\lambda_1,\ \Lambda_2=\lambda_2,\ \Theta=\theta\,\big]\\&+\int_0^\infty\int_0^{u+ct}\psi^o_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,u+ct-x)f_{X_1\mid\Theta}(x,\theta)f_{T_1\mid\Lambda_1=\lambda_1}(t)dxdt,\\&=\int_0^\infty\int_{u+ct}^\infty f_{X_1\mid\Theta}(x,\theta)f_{T_1\mid\Lambda_1=\lambda_1}(t)dxdt\\&+\int_0^\infty\int_0^{u+ct}\psi_o(\lambda_1,\lambda_2,\theta,u+ct-x)f_{X_1\mid\Theta}(x,\theta))f_{T_1\mid\Lambda_1=\lambda_1}(t)dxdt.\end{aligned}$$

Let $z=u+ct$ then we have

$$\begin{aligned}\psi^d_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,u)&=\frac{\lambda_1}{c}\exp\big(\frac{\lambda_1}{c}u\big)\int_u^\infty\int_z^\infty f_{X_1\mid\Theta}(x,\theta)\exp\big(-\lambda_1\frac{z}{c}\big)dxdz\\&+\frac{\lambda_1}{c}\exp\big(\frac{\lambda_1}{c}u\big)\int_u^\infty\int_0^z\psi^o_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,z-x)f_{X_1\mid\Theta}(x,\theta)\exp\big(-\lambda_1\frac{z}{c}\big)dxdz,\end{aligned}$$

taking the derivative of the above expression yields

$$\begin{aligned}\big(\psi^d_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,u)\big)'&=\frac{\lambda_1}{c}\psi^d_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,u)-\frac{\lambda_1}{c}\int_u^\infty f_{X_1\mid\Theta}(x,\theta)dx\\&-\frac{\lambda_1}{c}\int_0^u\psi^o_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,u-x)f_{X_1\mid\Theta}(x,\theta)dx,\\&=\frac{\lambda_1}{c}\psi^d_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,u)-\frac{\lambda_1}{c}\overline{F}_{X_1\mid\Theta}(u,\theta)\\&-\frac{\lambda_1}{c}\int_0^u\psi^o_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,u-x)f_{X_1\mid\Theta}(x,\theta)dx.\end{aligned}$$

By taking the Laplace transform on both side of the above equation, we have

$$\begin{aligned}p\tilde{\psi}^d_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,p)-\psi^d_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,0)&=\frac{\lambda_1}{c}\tilde{\psi}^d_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,p)-\frac{\lambda_1}{c}\tilde{H}(p,\theta)\\&-\frac{\lambda_1}{c}\tilde{\psi}_o(\lambda_1,\lambda_2,\theta,p)\tilde{f}(p,\theta),\end{aligned}$$

which gives by rearranging

$$\tilde{\psi}^{d}_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,p)=\frac{c}{cp-\lambda_1}\left[\psi^{d}_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,0)-\frac{\lambda_1}{c}\left(\tilde{H}(p,\theta)+\tilde{\psi}^{o}_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,p)\tilde{f}(p,\theta)\right)\right]. \tag{23}$$

Similarly:

$$\begin{aligned}p\tilde{\psi}^{o}_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,p)-\psi^{o}_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,0)&=\frac{\lambda_2}{c}\tilde{\psi}^{o}_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,p)-\frac{\lambda_2}{c}\tilde{H}(p,\theta)\\&-\frac{\lambda_1}{c}\tilde{\psi}^{o}_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,p)\tilde{f}(p,\theta).\end{aligned}$$

By rearranging, we get:

$$\tilde{\psi}^{o}_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,p)=\frac{c\psi^{o}_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,0)-\lambda_2\tilde{H}(p,\theta)}{\lambda_2\tilde{f}(p,\theta)+cp-\lambda_2}\quad\text{where }\tilde{H}(p,\theta)=\frac{1-\tilde{f}(p,\theta)}{p}, \tag{24}$$

and $\tilde{f}(p,\theta)$ is the Laplace transform of $f_{X_1\mid\Theta}(x,\theta)$. Then substituting (24) in (23) yields the result □

4.2 Explicit formula for the conditional ruin probability in the delayed renewal process

The following corollary expresses the Laplace transform of the ruin probability when the claims are mixed exponentially distributed.

Corollary 1. *In the delayed Poisson risk model, where the claims severities are i.i.d. and follow exponential distribution with parameter Θ the Laplace transform of the ruin probability is given by:*

$$\tilde{\psi}^{d}_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,p)=\frac{c}{cp-\lambda_1}\left[\psi^{d}_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,0)-\frac{\lambda_1}{c}\left(\frac{1}{p+\theta}+\frac{\lambda_2}{c}\frac{1}{\left(p+\left(\theta-\frac{\lambda_2}{c}\right)\right)\left(p+\theta\right)}\right)\right], \tag{25}$$

where c represents the premium rate.

Proof. First recall that when the claims are conditionally exponential then $\psi^{o}_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,0)=\dfrac{\lambda_2}{c\theta}$. For the proof, we refer to Albrecher et al. [1]
Therefore using equation (24), we have:

$$\begin{aligned}\tilde{\psi}^{o}_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,p)&=\frac{c\psi^{o}_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,0)-\frac{\lambda_2}{p+\theta}}{\frac{\lambda_2\theta}{p+\theta}+cp-\lambda_2},\\&=\frac{\lambda_2 p}{\theta\left(cp^2+(c\theta-\lambda_2)p\right)},\\&=\frac{\lambda_2}{c\theta}\frac{1}{p+\left(\theta-\frac{\lambda_2}{c}\right)}.\end{aligned}$$

Substituting the above equation into (23) with $\tilde{f}(p,\theta)=\frac{\theta}{p+\theta}$ and $\tilde{H}(p,\theta)=\frac{1}{p+\theta}$ yields:

$$\begin{aligned}\tilde{\psi}^{d}_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,p)&=\frac{c}{cp-\lambda_1}\left[\psi^{d}_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,0)-\frac{\lambda_1}{c}\left(\frac{1}{p+\theta}+\frac{\lambda_2}{c\theta}\frac{1}{p+\left(\theta-\frac{\lambda_2}{c}\right)}\frac{\theta}{p+\theta}\right)\right],\\&=\frac{c}{cp-\lambda_1}\left[\psi^{d}_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,0)-\frac{\lambda_1}{c}\left(\frac{1}{p+\theta}+\frac{\lambda_2}{c}\frac{1}{\left(p+\left(\theta-\frac{\lambda_2}{c}\right)\right)(p+\theta)}\right)\right].\end{aligned}$$

□

Theorem 3. *Under the assumptions of Corollary (1) the ruin probability is given by*

$$\psi^d_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,u) = \min\left\{\frac{\lambda_1}{\lambda_1-\lambda_2+c\theta}\exp\big(-\big(\theta-\frac{\lambda_2}{c}\big)u\big),1\right\}. \tag{26}$$

Proof. The elementary decomposition of equation (25) is given by:

$$\begin{aligned}\tilde{\psi}^d_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,p) &= \frac{\psi^d_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,0)}{p-\frac{\lambda_1}{c}} \\ &\quad -\frac{\lambda_1}{c}\left[\frac{\alpha_1}{p-\frac{\lambda_1}{c}}+\frac{\alpha_2}{p+\theta}+\frac{\lambda_2}{c}\left(\frac{\alpha_3}{p-\frac{\lambda_1}{c}}+\frac{\alpha_4}{p+\big(\theta-\frac{\lambda_2}{c}\big)}+\frac{\alpha_5}{p+\theta}\right)\right],\end{aligned}$$

where

$$\begin{aligned}&\alpha_1=\frac{c}{\lambda_1+c\theta},\qquad \alpha_2=-\frac{c}{\lambda_1+c\theta},\qquad \alpha_3=\frac{c^2}{(\lambda_1+c\theta)(\lambda_1-\lambda_2+c\theta)}\\ &\alpha_4=\frac{c^2}{\lambda_2(\lambda_2-\lambda_1-c\theta)},\qquad \alpha_5=\frac{c^2}{\lambda_2(\lambda_1+c\theta)}\end{aligned}$$

Taking the inverse Laplace transform of each term, yields

$$\begin{aligned}\psi^d_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,u) &= \big[\psi^d_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,0)-\frac{\lambda_1}{c}\big(\alpha_1+\frac{\lambda_2}{c}\alpha_3\big)\big]\exp\big(\frac{\lambda_1}{c}u\big)\\ &\quad -\frac{\lambda_1\lambda_2}{c^2}\alpha_4\exp\big(-\big(\theta-\frac{\lambda_2}{c}\big)u\big)-\frac{\lambda_1}{c}\big(\alpha_2+\frac{\lambda_2}{c}\alpha_5\big)\big]\exp(-\theta u),\\ &=\left[\psi^d_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,0)-\frac{\lambda_1}{\lambda_1+c\theta}\left(1+\frac{\lambda_2}{\lambda_1-\lambda_2+c\theta}\right)\right]\exp\big(\frac{\lambda_1}{c}u\big)\\ &\quad -\frac{\lambda_1}{\lambda_2-\lambda_1-c\theta}\times\exp\big(-\big(\theta-\frac{\lambda_2}{c}\big)u\big)\\ &\quad -\frac{\lambda_1}{c}\left(-\frac{c}{\lambda_1+c\theta}+\frac{\lambda_2}{c}\frac{c^2}{\lambda_2(\lambda_1+c\theta)}\right)\exp(-\theta u),\\ &=\big[\psi^d_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,0)-\frac{\lambda_1}{\lambda_1-\lambda_2+c\theta}\big]\exp\big(\frac{\lambda_1}{c}u\big)\\ &\quad +\frac{\lambda_1}{\lambda_1-\lambda_2+c\theta}\exp\big(-\big(\theta-\frac{\lambda_2}{c}\big)u\big).\end{aligned}$$

since $\lim_{u\to\infty}\psi^d_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,u)=0$ and due to the fact that $\left[\psi^d_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,0)-\frac{\lambda_1}{\lambda_1-\lambda_2+c\theta}\right]$ is independent of u, the necessary and sufficient condition is that

$$\psi^d_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,0)-\frac{\lambda_1}{\lambda_1-\lambda_2+c\theta}=0,$$

therefore:

$$\psi^d_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,0)=\frac{\lambda_1}{\lambda_1-\lambda_2+c\theta}\quad\text{hence}\quad \psi^d_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,u)=\frac{\lambda_1}{\lambda_1-\lambda_2+c\theta}\exp\big(-\big(\theta-\frac{\lambda_2}{c}\big)u\big),$$

provided that $c\theta \geq \lambda_2$ □

Remark 2. *If $\lambda_1 = \lambda_2 = \lambda$ then*

$$\psi^d_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,u) = \psi^o_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,u) = \frac{\lambda}{c\theta}\exp\left[-\left(\theta-\frac{\lambda}{c}\right)u\right] \text{ with } c\theta \ \geq \ \lambda.$$

The ruin probability in the delayed model is consistent with the ordinary model.

4.3 Unconditional ruin probability in the delayed renewal risk model

In the following section, we derive an analytical expression of the unconditional ruin probability for a given distribution of Λ_1, Λ_2 and θ in the delayed renewal risk model.

Theorem 4. *Assume that Λ_1, Λ_2 and θ follow a general mixing exponential distribution with densities given by*

$$f_\Theta(x) = \sum_{i=1}^{n} a_i\mu_i\exp(-\mu_i x), \ \ f_{\Lambda_1}(y) = \sum_{j=1}^{m} b_j\gamma_j\exp(-\gamma_j y), \ \ f_{\Lambda_2}(z) = \sum_{k=1}^{q} r_k\alpha_k\exp(-\alpha_k z),$$

where $\sum_{i=1}^n a_i = 1$, $\sum_{j=1}^m b_j = 1$, $\sum_{k=1}^q r_k = 1$ and $(n,m,q) \in \mathbb{N}^3$, $(a_i)_{i=1}^n \in \mathbb{R}_+$,$(b_j)_{j=1}^m \in \mathbb{R}_+$ and $(r_i)_{k=1}^q \in \mathbb{R}_+$. Then, the ruin probability formula in the delayed renewal process is given by

$$\begin{aligned}\psi^d(u) = &\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{q}\frac{a_i\mu_i b_j r_k}{u_i + c\gamma_j + c\alpha_k} \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{q}\sum_{l=1}^{n}\frac{a_i\mu_i b_{j\gamma_j} r_k\alpha_k a_l}{c\alpha_k+\mu_i}\left[a_2^{i,j,l}\ln\left(\frac{c\gamma_j+\mu_l}{u+\mu_i}\right) + \frac{a_1^{i,j,l}}{\gamma_j+\frac{\mu_l}{c}}\right],\end{aligned} \tag{27}$$

where $a_2^{i,j,l} = \dfrac{1}{\gamma_j + \frac{\mu_l-\mu_i-u}{c}}$ and $a_1^{i,j,l} = \dfrac{1}{\frac{\mu_i+u-\mu_l}{c}-\gamma_j}$,$u$ is the initial reserve and c is the premium rate.

Proof. From the expression of the conditional ruin probability of the delayed renewal risk process defined by (26), the unconditional ruin probability can be expressed by:

$$\begin{aligned}\psi^d(u) &= \int_0^\infty\int_0^{\frac{\lambda_2}{c}}\int_0^{\frac{\lambda_2}{c}} f_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta)d\theta d\lambda_1 d\lambda_2 \\ &+ \int_0^\infty\int_{\frac{\lambda_2}{c}}^\infty\int_{\frac{\lambda_2}{c}}^\infty \psi^d_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,u) f_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta)d\theta d\lambda_1 d\lambda_2.\end{aligned}$$

From equation (26), we have:

$$\begin{aligned}\psi^d(u) &= \int_0^\infty\int_0^{\frac{\lambda_2}{c}}\int_0^{\frac{\lambda_2}{c}} f_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta)d\theta d\lambda_1 d\lambda_2 \\ &+ \int_0^\infty\int_{\frac{\lambda_2}{c}}^\infty\int_{\frac{\lambda_2}{c}}^\infty \frac{\lambda_1}{\lambda_1-\lambda_2+c\theta}\exp\big(-(\theta-\frac{\lambda_2}{c})u\big) f_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta)d\theta d\lambda_1 d\lambda_2.\end{aligned} \tag{28}$$

Let $\mathrm{S} = c\Theta - \Lambda_2$, $\mathrm{U} = \Lambda_1 e^{-\frac{u}{c}\mathrm{S}}\mathbb{1}_{\{\Lambda_1\leq c\Theta\}}\mathbb{1}_{\{S\geq 0\}}$, and $\mathrm{V} = \Lambda_1 + \mathrm{S}$.
Therefore

$$\psi^d_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta,u) = \frac{\mathrm{U}}{\mathrm{V}}, \text{ where } \mathrm{U} = \Lambda_1\exp\big(-\frac{u}{c}\mathrm{W}\big)\mathbb{1}_{\{\Lambda_1\leq c\Theta\}}\mathbb{1}_{\{S\geq 0\}} \text{ and } \mathrm{V} = \Lambda_1 + \mathrm{S}.$$

Distribution of S = $c\Theta - \Lambda_2$.
Consider the case where S ≥ 0. It follows that

$$\begin{aligned}
\mathbf{P}[S \leq s] &= \mathbf{P}[c\Theta - \Lambda_2 \leq s], \\
&= \mathbf{P}\big[\Theta \leq \frac{1}{c}(\Lambda_2 + s\big], \\
&= \int_0^\infty \Big(\int_0^{\frac{1}{c}z+s} \sum_{i=1}^n a_i\mu_i \exp(-\mu_i x)\mathrm{x}\Big) \sum_{k=1}^q r_k\alpha_k \exp(-\alpha_k z), \\
&= \int_0^\infty \sum_{k=1}^n r_k\alpha_k \exp(-\alpha_k z) \sum_{i=1}^n a_i \Big[1 - \exp\big(-\frac{\mu_i}{c}(z+s)\big)\Big]\mathrm{d}z, \\
&= \int_0^\infty \sum_{i=1}^n a_i \Big[\sum_{k=1}^q r_k\alpha_k \exp(-\alpha_k z) - \sum_{k=1}^q r_k\alpha_k \exp(-\alpha_k z)\exp\big(-z\big(\frac{\mu_i}{c} + \alpha_k\big)\big)\Big]\mathrm{d}z, \\
&= 1 - \sum_{i=1}^n a_i \exp\big(-\frac{\mu_i}{c}s\big) \int_0^\infty \sum_{k=1}^q r_k\alpha_k \exp\big(-z\big(\frac{\mu_i}{c} + \alpha_k\big)\big)\mathrm{d}z, \\
&= 1 - c\sum_{i=1}^n \sum_{k=1}^q \frac{a_i r_k \alpha_k}{c\alpha_k + \mu_i} \exp\big(-\frac{\mu_i}{c}s\big).
\end{aligned}$$

Similarly for S < 0 we have

$$\mathbf{P}[S \leq s] \ = \ \sum_{i=1}^n \sum_{k=1}^q \frac{a_i r_k \mu_i}{c\alpha_k + \mu_i} \exp(\alpha_k s).$$

Therefore

$$f_S(s) = \sum_{i=1}^n \sum_{k=1}^q \frac{a_i\mu_i r_k \alpha_k}{c\alpha_k + \mu_i} \begin{cases} e^{-\frac{\mu_i}{c}s} & \text{si } \ s \geq 0 \\ e^{\alpha_k s} & \text{si } \ s < 0 \end{cases} \tag{29}$$

then,

$$\begin{aligned}
\psi^d(u) &= \int_0^\infty \int_{cz}^\infty \int_{cz}^\infty f_{\Lambda_2}(x) f_{\Lambda_1}(y) f_\Theta(z)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&+ \int_0^\infty \int_0^{cz} \int_0^{cz} \psi^d_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1, \lambda_2, \theta) f_{\Lambda_2}(\lambda_2) f_{\Lambda_1}(\lambda_1) f_\Theta(\theta)\mathrm{d}\lambda_2\mathrm{d}\lambda_1\mathrm{d}\theta, \\
&= \int_0^\infty \int_{cz}^\infty \int_{cz}^\infty f_{\Lambda_2}(x) f_{\Lambda_1}(y) f_\Theta(z)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&+ \int_0^\infty \int_{\{\lambda_1 \leq c\theta\}} \int_{s \geq 0} \frac{\lambda_1 \exp\big(\frac{-u}{c}s\big)}{\lambda_1 + s} f_S(s) f_\Theta(\theta) f_{\Lambda_1}(\lambda_1)\mathrm{d}s\mathrm{d}\theta\mathrm{d}\lambda_1, \\
&= \int_0^\infty \int_{cz}^\infty \int_{cz}^\infty f_{\Lambda_2}(x) f_{\Lambda_1}(y) f_\Theta(z)\mathrm{d}x\mathrm{d}y\mathrm{d}z + \mathbf{E}\Big[\frac{\Lambda_1 \exp\big(\frac{-u}{c}S\big)}{\Lambda_1 + S} \mathbb{1}_{\{\Lambda_1 \leq c\Theta\}} \mathbb{1}_{\{S \geq 0\}}\Big], \\
&= \int_0^\infty \int_{cz}^\infty \int_{cz}^\infty f_{\Lambda_2}(x) f_{\Lambda_1}(y) f_\Theta(z)\mathrm{d}x\mathrm{d}y\mathrm{d}z + \mathbf{E}\Big[\frac{\mathrm{U}}{\mathrm{V}}\Big].
\end{aligned}$$

According to Cressie et al. [8],

$$\mathbf{E}\left[\frac{\mathrm{U}}{\mathrm{V}}\right] = \int_0^\infty \lim_{t_2\to 0^-} \frac{\partial}{\partial t_2}\mathbf{M}_{\mathrm{V},\mathrm{U}}(-t_1,t_2)\mathrm{d}t_1.$$

$$\begin{aligned}
\mathbf{M}_{\mathrm{V},\mathrm{U}}(-t_1,t_2) &= \mathbf{E}\left[e^{-t_1\mathrm{V}+t_2\mathrm{U}}\right],\\
&= \mathbf{E}\left[e^{-t_1(\Lambda_1+\mathrm{S})+t_2(\Lambda_1 e^{-\frac{u}{c}\mathrm{S}})}\mathbb{1}_{\{\Lambda_1\le c\Theta\}}\mathbb{1}_{\{S\ge 0\}}\right],\\
&= \mathbf{E}\left[e^{\left(-t_1+t_2e^{-\frac{u}{c}\mathrm{S}}\right)\Lambda_1-t_1\mathrm{S}}\mathbb{1}_{\{\Lambda_1\le c\Theta\}}\mathbb{1}_{\{S\ge 0\}}\right],\\
&= \int_0^\infty \mathbf{E}\left[e^{\left(-t_1+t_2e^{-\frac{u}{c}\mathrm{S}}\right)\Lambda_1-t_1\mathrm{S}}\mathbb{1}_{\{\Lambda_1\le c\Theta\}}\middle|\mathrm{S}=s\right]f_{\mathrm{S}}(s)\mathrm{d}s,\\
&= \int_0^\infty \mathbf{E}\left[e^{\left(-t_1+t_2e^{-\frac{u}{c}s}\right)\Lambda_1-t_1 s}\mathbb{1}_{\{\Lambda_1\le c\Theta\}}\right]f_{\mathrm{S}}(s)\mathrm{d}s,\\
&= \int_0^\infty e^{-t_1 s}\mathbf{M}_{\Lambda_1\mathbb{1}_{\{\Lambda_1\le c\Theta\}}}\left(-t_1+t_2e^{-\frac{u}{c}s}\right)f_{\mathrm{S}}(s)\mathrm{d}s,\\
\lim_{t_2\to 0^-}\frac{\partial}{\partial t_2}\mathbf{M}_{\mathrm{V},\mathrm{U}}(-t_1,t_2) &= \mathbf{E}\left[e^{-\left(\frac{u}{c}+t_1\right)\mathrm{S}}\Lambda_1 e^{-t_1\Lambda_1}\mathbb{1}_{\{\Lambda_1\le c\Theta\}}\mathbb{1}_{\{S\ge 0\}}\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbf{E}\left[\frac{\mathrm{U}}{\mathrm{V}}\right] &= \int_0^\infty\int_0^\infty\int_0^\infty\int_{\frac{v}{c}}^\infty v e^{-\left(\frac{u}{c}+t_1\right)s-t_1 v} f_\Theta(\theta)f_{\Lambda_1}(v)f_{\mathrm{S}}(s)\mathrm{d}\theta\mathrm{d}v\mathrm{d}s\mathrm{d}t_1,\\
&= \int_0^\infty\int_0^\infty f_S(s)e^{-\left(\frac{u}{c}+t_1\right)s}\int_0^\infty v e^{-t_1 v}\sum_{i=1}^n \exp\left(-\frac{\mu_i}{c}v\right)\sum_{j=1}^m b_j\gamma_j\exp(-\gamma_j v)\mathrm{d}v\mathrm{d}s\mathrm{d}t_1,\\
&= \int_0^\infty\int_0^\infty\sum_{i=1}^n\sum_{k=1}^q\frac{a_i\mu_i r_k\alpha_k}{c\alpha_k+\mu_i}e^{-\left(t_1+\frac{\mu_i+u}{c}\right)s}\int_0^\infty\sum_{i=1}^n\sum_{j=1}^m a_i b_j\gamma_j v e^{-\left(t_1+\gamma_j+\frac{\mu_i}{c}\right)v}\mathrm{d}v\mathrm{d}s\mathrm{d}t_1,\\
&= \int_0^\infty\sum_{i=1}^n\sum_{k=1}^q\frac{a_i\mu_i r_k\alpha_k}{(c\alpha_k+\mu_i)}\frac{1}{\left(t_1+\frac{\mu_i+u}{c}\right)}\times\sum_{i=1}^n\sum_{j=1}^m a_i b_j\gamma_j\frac{1}{\left(t_1+\gamma_j+\frac{\mu_i}{c}\right)^2}\mathrm{d}t_1,\\
&= \sum_{i=1}^n\sum_{j=1}^m\sum_{k=1}^q\sum_{l=1}^n\frac{a_i a_l b_j r_k\mu_i\gamma_j\alpha_k}{c\alpha_k+\mu_i}\int_0^\infty\frac{\mathrm{d}t_1}{\left(t_1+\frac{\mu_i+u}{c}\right)\left(t_1+\gamma_j+\frac{\mu_l}{c}\right)^2},\\
&= \sum_{i=1}^n\sum_{j=1}^m\sum_{k=1}^q\sum_{l=1}^n\frac{a_i a_l b_j r_k\mu_i\gamma_j\alpha_k}{c\alpha_k+\mu_i}\int_0^\infty\frac{a_2^{i,j,l}\mathrm{d}t_1}{t_1+\frac{\mu_i+u}{c}}+\frac{a_1^{i,j,l}\mathrm{d}t_1}{\left(t_1+\gamma_j+\frac{\mu_l}{c}\right)^2}+\frac{a_2^{i,j,l}\mathrm{d}t_1}{t_1+\gamma_j+\frac{\mu_l}{c}},
\end{aligned}$$

where $a_2^{i,j,l} = \dfrac{1}{\left(\gamma_j+\frac{\mu_l-\mu_i}{c}\right)^2}$, $a_1^{i,j,l} = \dfrac{1}{\frac{u+\mu_i-\mu_l}{c}-\gamma_j}$ and $a_2^{i,j,l} = -a_0^{i,j,l}$. Therefore,

$$\mathbf{E}\left[\frac{\mathrm{U}}{\mathrm{V}}\right] = \sum_{i=1}^n\sum_{j=1}^m\sum_{k=1}^q\sum_{l=1}^n\frac{a_i\mu_i b_{j\gamma_j}r_k\alpha_k a_l}{c\alpha_k+\mu_i}\left[a_2^{i,j,l}\ln\left(\frac{c\gamma_j+\mu_l}{u+\mu_i}\right)+\frac{a_1^{i,j,l}}{\gamma_j+\frac{\mu_l}{c}}\right]. \tag{30}$$

Moreover,

$$\int_0^\infty \int_{cz}^\infty \int_{cz}^\infty f_{\Lambda_1}(x) f_{\Lambda_2}(y) f_\Theta(z) \mathrm{d}x\mathrm{d}y\mathrm{d}z = \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^q a_i b_j r_k \mu_i \int_0^\infty e^{-(\mu_i + c\gamma_j + c\alpha_k)z} \mathrm{d}z,$$
$$= \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^q \frac{a_i \mu_i b_j r_k}{\mu_i + c\gamma_j + c\alpha_k}. \tag{31}$$

Putting equation (31) & (30) together yields the result. □

Corollary 2. *For the special case where* $\Theta \sim \exp(\mu)$*,* $\Lambda_1 \sim \exp(\gamma)$*,* $\Lambda_2 \sim \exp(\alpha)$ *and they are mutually independent we have,*

$$\psi^d(u) = \frac{\mu}{c\gamma + c\alpha + \mu} + \frac{\gamma\mu\alpha}{c\alpha + \mu}\left[a_2 \ln\left(\frac{\gamma c + \mu}{u + \mu}\right) + \frac{a_1}{\gamma + \frac{\mu}{c}}\right], \tag{32}$$

where $a_2 = \dfrac{1}{\left(\gamma - \frac{u}{c}\right)^2}$*,* $a_1 = \dfrac{1}{\frac{u}{c} - \gamma}$*.*

4.4 Upper bound for the ruin probability in the delayed renewal risk model in the case of dependence

Let assume that the trivariate random variables Λ_1, Λ_2 and Θ are dependent. Moreover assume that this dependence structure is modelled by a copula, then the ruin probability formula is given by:

$$\begin{aligned}
\psi^d(u) &= \int_0^\infty \int_0^{\frac{\lambda_2}{c}} \int_0^{\frac{\lambda_2}{c}} f_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1, \lambda_2, \theta) d\theta d\lambda_1 d\lambda_2 \\
&+ \int_0^\infty \int_{\frac{\lambda_2}{c}}^\infty \int_{\frac{\lambda_2}{c}}^\infty \psi^d_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1, \lambda_2, \theta, u) f_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1, \lambda_2, \theta) d\theta d\lambda_1 d\lambda_2, \\
&= \int_0^\infty \int_0^{\frac{\lambda_2}{c}} \int_0^{\frac{\lambda_2}{c}} f_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1, \lambda_2, \theta) d\theta d\lambda_1 d\lambda_2 \\
&+ \int_0^\infty \int_{\frac{\lambda_2}{c}}^\infty \int_{\frac{\lambda_2}{c}}^\infty \frac{\lambda_1}{\lambda_1 - \lambda_2 + c\theta} \exp\big(-\big(\theta - \frac{\lambda_2}{c}\big)u\big) f_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1, \lambda_2, \theta) d\theta d\lambda_1 d\lambda_2.
\end{aligned}$$

where

$$f_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1, \lambda_2, \theta) = c_{123}\big(F_{\Lambda_1}(\lambda_1), F_{\Lambda_2}(\lambda_2), F_\Theta(\theta)\big) f_{\Lambda_1}(\lambda_1) f_{\Lambda_2}(\lambda_2) f_\Theta(\theta),$$

with

$$c_{123}(x, y, z) = \frac{\partial^3 \mathrm{C}_{123}(x, y, z)}{\partial x \partial y \partial z}.$$

Since it is hard to determine the analytical expression of this integral, in the next theorem we derive an upper bound of the ruin probability in the delayed renewal model with dependence.

4.4.1 Upper bound of the ruin probability with dependence

The following theorem gives an upper bound of the ruin probability where the random variables Θ, Λ_1 and Λ_2 are dependent.

Theorem 5. *Let assume that the random variables Λ_1, Λ_2 and θ are dependent, assume further that the Laplace transform $f^{\star}_{\Lambda_2,\Theta}(x,s)$ of (Λ_2,Θ) exists, then an upper bound of the ruin probability in the delayed renewal process with dependence is given by*

$$\psi^d(u) \leq f^{\star}_{\Theta,\Lambda_2}\left(u, -\frac{u}{c}\right), \tag{33}$$

where c is the premium rate and u the initial reserve.

Proof.

$$\begin{aligned}
\psi^d(u) &= \int_0^\infty \int_0^{\frac{\lambda_2}{c}} \int_0^{\frac{\lambda_2}{c}} f_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta)d\theta d\lambda_1 d\lambda_2 \\
&+ \int_0^\infty \int_{\frac{\lambda_2}{c}}^\infty \int_{\frac{\lambda_2}{c}}^\infty \frac{\lambda_1}{\lambda_1-\lambda_2+c\theta} \exp\big(-\big(\theta-\frac{\lambda_2}{c}\big)u\big) f_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta)d\theta d\lambda_1 d\lambda_2, \\
&\leq \int_0^\infty \int_0^\infty \int_0^{\frac{\lambda_2}{c}} f_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta)d\theta d\lambda_1 d\lambda_2 \\
&+ \int_0^\infty \int_0^\infty \int_{\frac{\lambda_2}{c}}^\infty \frac{\lambda_1}{\lambda_1-\lambda_2+c\theta} \exp\big(-\big(\theta-\frac{\lambda_2}{c}\big)u\big) f_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta)d\theta d\lambda_1 d\lambda_2.
\end{aligned}$$

Since the non-ruin condition implies

$$\begin{aligned}
\theta \ \geq \ \frac{\lambda_2}{c} &\Longleftrightarrow c\theta-\lambda_2 \ \geq \ 0, \\
&\Longleftrightarrow c\theta-\lambda_2+\lambda_1 \ \geq \ \lambda_1, \\
&\Longleftrightarrow \frac{c\theta-\lambda_2+\lambda_1}{\lambda_1} \ \geq \ 1, \\
&\Longleftrightarrow \frac{\lambda_1}{c\theta-\lambda_2+\lambda_1} e^{-\left(\theta-\frac{\lambda}{c}\right)u} \ \leq \ e^{-\left(\theta-\frac{\lambda}{c}\right)u},
\end{aligned} \tag{34}$$

moreover $e^{-\left(\theta-\frac{\lambda}{c}\right)u} \ \geq \ 1$ when $\theta \ \leq \ \frac{\lambda_2}{c}$, combining this relation with (34) gives:

$$\begin{aligned}
\psi^d(u) &\leq \int_0^\infty \int_0^\infty \int_0^{\frac{\lambda_2}{c}} e^{-\left(\theta-\frac{\lambda}{c}\right)u} f_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta)d\theta d\lambda_1 d\lambda_2 \\
&+ \int_0^\infty \int_0^\infty \int_{\frac{\lambda_2}{c}}^\infty \exp\big(-\big(\theta-\frac{\lambda_2}{c}\big)u\big) f_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta)d\theta d\lambda_1 d\lambda_2, \\
&\leq \int_0^\infty \int_0^\infty \int_0^\infty \exp\big(-\big(\theta-\frac{\lambda_2}{c}\big)u\big) f_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta)d\theta d\lambda_1 d\lambda_2, \\
&\leq \int_0^\infty \int_0^\infty \exp\big(-\big(\theta-\frac{\lambda_2}{c}\big)u\big) f_{\Lambda_1,\Lambda_2,\Theta}(\lambda_1,\lambda_2,\theta)d\theta d\lambda_1 d\lambda_2, \\
&\leq f_{\Theta,\Lambda_2}\left(u, -\frac{u}{c}\right).
\end{aligned}$$

Which proves the statement. □

4.4.2 Numerical illustration in the independence case

Let Θ follows exponential distribution with parameter $\mu = 5 \times 10^2$, Λ_2 follows exponential distribution with parameter $\alpha = 10^3$ and Λ_1 follows exponential distribution with parameter γ. Assume further that those random variables are mutually independent.
Tables [3 - 4] below give some numerical values of the ruin probability and its upper bound where $\gamma = 20$ and $\gamma = 200$ with respect to the initial surplus u using matlab for numerical integration.

Table 3: Ruin probability with premium rate $c = 25\%$ in the delayed risk process where $\gamma = 200$.

u	0	10	25	50	75	100	150	200	250
Numerical value	0.6643	0.6639	0.6633	0.6624	0.6616	0.6609	0.6596	0.6586	0.6578
Upper bound	0.6705	0.6699	0.6689	0.6676	0.6664	0.6653	0.6635	0.6620	0.6607

Table 4: Ruin probability where premium rate $c = 25\%$ in the delayed risk process where $\gamma = 20$.

u	0	10	25	50	75	100	150	200	250
Numerical value	0.6240	0.6217	0.6184	0.6133	0.6087	0.6046	0.5973	0.5912	0.5861
Upper bound	0.6717	0.6678	0.6623	0.6539	0.6464	0.6396	0.6279	0.6182	0.6100

For a fixed initial reserve u, increasing the parameter of the distribution of Λ_1, increases the ruin probability.

5 Conclusion

We have found the explicit formulas for the ruin probability with a general mixed exponential risk model in the ordinary and the delayed renewal risk process when the claim's distributions and the inter-claim time's distributions are independent. As the explicit expression of the ruin probability is difficult to obtain in the dependence case, we propose an upper bound for the ruin probability.

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