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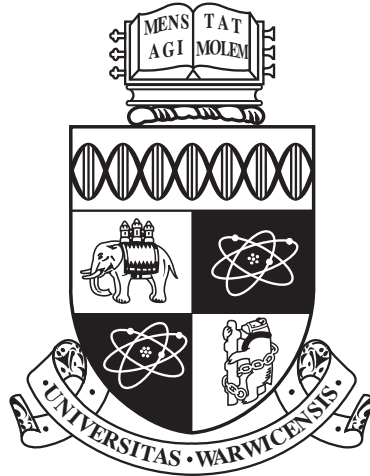
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**Applications of Pseudoholomorphic Subvarieties to
J-Anti-invariant Forms and Spectral Geometry**

by

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Declaration

Chapter 2 contains classical results that are required in subsequent chapters. Chapter 3 is taken from the paper [5] which is joint work with Weiyi Zhang. Chapter 4 will be submitted as a paper in its own right in due course and is the authors own work undertaken during his doctoral studies.

I hereby declare that this submission is my own work and that, to the best of my knowledge and belief, it contains no material previously published or written by another person nor material which has been accepted for the award of any other degree or diploma of the university or other institute of higher learning, except where due acknowledgement has been made in the text.

Abstract

The results contained in this thesis can be split into two categories, namely those involving the analysis of J -anti-invariant forms and those in the realm of spectral geometry.

We primarily study the relation of J -anti-invariant 2-forms with pseudoholomorphic curves in the first half of the thesis. We show that the zero set of a closed J -anti-invariant 2-form on an almost complex 4-manifold supports a J -holomorphic subvariety in the canonical class. This confirms a conjecture of Draghici-Li-Zhang. A higher dimensional analogue is also established. Furthermore, the local model built for the bundle of J -anti-invariant forms can be used to prove that, on an almost complex 4-manifold, the dimension of the cohomology group associated to closed J -anti-invariant 2-forms is a birational invariant in the sense that it is invariant under degree one pseudoholomorphic maps.

In the latter half we study the eigenvalues of the Laplacian of an almost Kähler metric. In particular we find that the bounds established by Kokarev [33] in the case of a Kähler metric with respect to an integrable almost complex structure also hold in the almost Kähler setting. That is, we show that if a compact almost Kähler manifold admits a pseudoholomorphic map into a projective space then the k -th eigenvalue of the Laplacian, with respect to a given Kähler metric, can be bounded above by a constant depending only on dimension, the map into projective space and the Kähler class. We provide examples of strictly almost Kähler manifolds which admit a non-trivial pseudoholomorphic map into a projective space. Similarly to Kokarev [33] we establish a version of the estimate for pseudoholomorphic subvariety. Finally we prove that the estimate holds for almost Kähler manifolds admitting a pseudoholomorphic map into projective space in a class of non-smooth maps. In particular we obtain that the estimate holds for Kähler manifolds which admit a rational map into projective space.

Chapter 1

Introduction

The groundbreaking work of Gromov [24] in the 1980's set the ball rolling on the study of pseudoholomorphic curves in symplectic manifolds. A pseudoholomorphic curve is a smooth map from a Riemann surface into an almost complex manifold and since any symplectic manifold admits an almost complex structure they are natural objects to study. Indeed, their study has allowed the field of symplectic topology to reach new heights, of particular note is the deep theory of symplectic 4-manifolds. Their influence can also be seen further afield, for example in algebraic geometry and string theory to name a few.

A cornerstone in the theory of pseudoholomorphic curves is the positivity of intersections phenomena. This property is, of course, one shared with holomorphic curves in complex manifolds and more generally with all complex subvarieties. To study complex submanifolds it is often convenient to view them as zero loci of analytic functions. This approach is colloquially known as the “*mapping out viewpoint*” and has proven extremely powerful in the complex setting where it is essentially the intersection theory of complex subvarieties. In contrast, Gromov's groundbreaking work on pseudoholomorphic curves has made the “*mapping into*” point of view the most common approach in the study of pseudoholomorphic curves and higher dimensional almost complex submanifolds. Nevertheless Taubes uses the “*mapping out*” approach in his seminal work on the equivalence of SW and Gr to give a criteria for a set to support a pseudoholomorphic curve. Building on these techniques Zhang [60] has initiated a programme to develop the “*mapping out*” point of view for arbitrary almost complex structures. It is the purpose of this thesis to continue this work, exploring its applications to understanding the canonical bundle of an almost complex manifold, the birational invariants of an almost complex manifold and the spectrum of the Laplacian of an almost Kähler manifold.

1.1 Outline and main results

The main results of this thesis, whilst all lying under the broad guise of almost complex geometry, can be split into two distinct areas. Firstly there are results concerning the zero locus of sections of the canonical bundle. This work will make up Chapter 3. Secondly we present estimates on the eigenvalues of the Laplace-Beltrami operator on a closed, almost Kähler manifold, this is the content of Chapter 4.

Chapter 2 is a catch-all background chapter, in that it may have no clear through-line, but it shall provide all of the necessary background to make sense of the subsequent chapters. A brief review of almost complex structures, pseudoholomorphic curve theory and spectral theory will be included. In particular, whilst looking at spectral theory, we recall some fundamental results pertaining to the eigenvalues of the Laplacian on a Riemann surface. These results motivate the estimate of Bourguignon, Li and Yau [7] which is the starting point of Chapter 4.

We now provide an overview of the main results of this thesis. First we turn our attention to a well known folklore theorem (see [29, 35]) which dates back to the 1980's and says that for a generic Riemannian metric on a 4-manifold with positive self-dual second Betti number, the zero set of a self-dual harmonic 2-form is a finite number of embedded circles. It is the starting point of Taubes' attempts, *e.g.* [51], to generalise the identification of Seiberg-Witten invariants and Gromov invariants for symplectic 4-manifolds to general compact oriented 4-manifolds.

Recently Zhang [60] proposed the subsequent philosophy,

1.1.1. *A statement for smooth maps between smooth manifolds in terms of R.Thom's transversality should also have its counterpart in pseudoholomorphic setting without requiring the transversality or genericity, but using the notion of pseudoholomorphic subvarieties.*

Following this philosophy the above genericity statement for the zero set of a self-dual harmonic 2-form in the smooth category should find its counterpart in the almost complex setting without the genericity assumption. It is stated as Question 1.6 in [60] which first appeared in [16]. Let us now make the statement precise.

Let (M^{2n}, J) be an almost complex manifold. The almost complex structure acts on the bundle of real 2-forms Λ^2 as the following involution, $\alpha(\cdot, \cdot) \rightarrow \alpha(J\cdot, J\cdot)$. This involution induces the splitting

$$\Lambda^2 = \Lambda_J^+ \oplus \Lambda_J^-, \quad (1.1)$$

corresponding to the eigenspaces of eigenvalues ± 1 respectively. The sections of these bundles are called J -invariant and J -anti-invariant 2-forms respectively. The bundle Λ_J^- inherits an almost complex structure, still denoted by J , from $J\alpha(X, Y) = -\alpha(JX, Y)$.

On the other hand, for any Riemannian metric g on a 4-manifold, we have the

well-known self-dual, anti-self-dual splitting of the bundle of 2-forms,

$$\Lambda^2 = \Lambda_g^+ \oplus \Lambda_g^-. \quad (1.2)$$

When g is compatible with J , *i.e.* $g(Ju, Jv) = g(u, v)$, we have $\Lambda_J^- \subset \Lambda_g^+$. In particular, it follows that a closed J -anti-invariant 2-form is a g -self-dual harmonic form. Hence, a closed J -anti-invariant 2-form is the natural almost complex refinement of a self-dual harmonic form on an almost complex 4-manifold. Following philosophy (1.1.1) our expectation is that the almost complex counterpart of the aforementioned folklore theorem should be that the zero set of a J -anti-invariant 2-form is a J -holomorphic curve.

Since the complex line bundle Λ_J^- can be viewed as a natural generalisation of the canonical bundle of a complex manifold it is instructive to take a brief digression and consider what is known in the complex setting. On a complex surface, if α is a closed J -anti-invariant 2-form, then $J\alpha$ is also closed and $\alpha + iJ\alpha$ is a holomorphic $(2, 0)$ form. Hence the zero set $\alpha^{-1}(0)$ is a canonical divisor of (M, J) , *e.g.* by the Poincaré-Lelong theorem. This meets our expectations in the case of an integrable almost complex structure.

We are able to confirm our above speculation for any closed, almost complex 4-manifold.

Theorem 1.1.1. *Suppose (M, J) is a closed, connected, almost complex 4-manifold and α is a non-trivial, closed, J -anti-invariant 2-form. Then the zero set, Z , of α supports a J -holomorphic 1-subvariety, Θ_α , in the canonical class K_J .*

Theorem 3.1.1 could be extended to the sections of bundle $\Lambda_{\mathbb{R}}^{n,0}$ of real parts of $(n, 0)$ forms, which has a natural complex line bundle structure induced by the almost complex structure on M . The space of its sections is denoted $\Omega_{\mathbb{R}}^{n,0}$. We have Theorem 3.4.1, which says that the zero set of a non-trivial closed form in $\Omega_{\mathbb{R}}^{n,0}$ supports a pseudoholomorphic subvariety of real codimension 2 up to Question 3.9 of [60].

We also study the relation of J -anti-invariant forms with the birational geometry of almost complex 4-manifolds, in particular we look for birational invariants. To this end recall that we can define cohomology groups, *e.g.* [39],

$$H_J^\pm(M) = \{\mathfrak{a} \in H^2(M; \mathbb{R}) \mid \exists \alpha \in \mathcal{Z}_J^\pm \text{ such that } [\alpha] = \mathfrak{a}\}$$

generalising the real Hodge cohomology groups, where \mathcal{Z}_J^\pm are the spaces of closed 2-forms in Ω_J^\pm . It is proven in [14] that $H_J^+(M) \oplus H_J^-(M) = H^2(M; \mathbb{R})$ when $\dim_{\mathbb{R}} M = 4$. The dimensions of the vector spaces $H_J^\pm(M)$ are denoted as $h_J^\pm(M)$.

In [60] it is shown that the natural candidate for generalising birational morphisms to the almost complex category are degree one pseudoholomorphic maps. We can use the local model built for Theorem 1.1.1 to study the extension properties of closed J -anti-invariant forms. This is the content of Proposition 3.5.1, which should be compared

with the Hartogs extension for pseudoholomorphic bundles over almost complex 4-manifolds established in [11].

With this Hartogs type extension for closed J -anti-invariant 2-forms in hand, we are able to show the dimension of J -anti-invariant cohomology is a birational invariant.

Theorem 1.1.2. *Let $\psi : (M_1, J_1) \rightarrow (M_2, J_2)$ be a degree 1 pseudoholomorphic map between closed, connected, almost complex 4-manifolds. Then $h_{J_1}^-(M_1) = h_{J_2}^-(M_2)$.*

Together with the almost complex birational invariants defined in [11], including plurigenera, Kodaira dimension, and irregularity, we have a rich source of invariants to study the birational geometry of almost complex manifolds.

The second part of this thesis is dedicated to obtaining upper bounds for the eigenvalues of the Laplacian on almost Kähler manifolds.

The jumping off point for us will be a bound of Bourguignon, Li and Yau [7]. They provided an upper bound for the first non-zero eigenvalue for a given Kähler metric on a projective manifold M which depended only on dimension, volume and a holomorphic immersion $\phi : M^n \rightarrow \mathbb{P}^m$. Notice in particular that this bound depends only on the Kähler class $[\omega]$.

Theorem 1.1.3 (Main Theorem of [7]). *Let M^n be an n -dimensional complex manifold admitting a holomorphic immersion $\phi : M \rightarrow \mathbb{P}^N$. Suppose that Φ is full in the sense that $\phi(M)$ is not contained in any hyperplane of \mathbb{P}^N . Then, for any Kähler metric ω on M , the first non-zero eigenvalue $\lambda_1(M, \omega)$ satisfies*

$$\lambda_1(M, \omega) \leq 4n \frac{N+1}{N} d([\phi], [\omega]),$$

where

$$d([\phi], [\omega]) := \frac{\int_M \phi^* \omega_{FS} \wedge \omega^{n-1}}{\int_M \omega^n}.$$

Recently Kokarev [33] has extended their result by giving bounds on the k -th eigenvalue, which depend linearly on k , for a more general class of Kähler manifolds.

Theorem 1.1.4 (Theorem 1.2 of [33]). *Let (M^n, J) be a closed n -dimensional Kähler manifold and $\phi : M^n \rightarrow \mathbb{P}^m$ a non-trivial holomorphic map. Then, for any Kähler metric g on M^n , the eigenvalues of the Laplace-Beltrami operator Δ_g satisfy,*

$$\lambda_k(M^n, g) \leq C(n, m) d([\phi], [\omega_g]) k, \quad \text{for any } k \geq 1, \quad (1.3)$$

where $C(n, m) > 0$ is a constant depending only on n and m and $d([\phi], [\omega_g])$ is defined by,

$$d([\phi], [\omega_g]) := \frac{\int_M \phi^* \omega_{FS} \wedge \omega_g^{n-1}}{\int_M \omega_g^n}. \quad (1.4)$$

We are able to establish that this result in fact holds if the the almost complex structure is not integrable, that is, it holds for almost Kähler manifolds.

Theorem 1.1.5. *Let (M^n, J) be a closed n -dimensional almost Kähler manifold and $\phi : M^n \rightarrow \mathbb{P}^m$ a non-trivial pseudoholomorphic map, where \mathbb{P}^m is taken with its standard complex structure. Then, for any almost Kähler metric g on M^n , the eigenvalues of the Laplace-Beltrami operator Δ_g satisfy,*

$$\lambda_k(M^n, g) \leq C(n, m)d([\phi], [\omega_g])k, \quad \text{for any } k \geq 1, \quad (1.5)$$

where $C(n, m) > 0$ is a constant depending only on n and m and $d([\phi], [\omega_g])$ is defined by,

$$d([\phi], [\omega_g]) := \frac{\int_M \phi^* \omega_{FS} \wedge \omega_g^{n-1}}{\int_M \omega_g^n}. \quad (1.6)$$

Notice that alternatively we can write

$$d([\phi], [\omega_g]) := \frac{(\phi^*[\omega_{FS}] \smile [\omega_g]^{n-1}, [M])}{([\omega_g]^n, [M])}, \quad (1.7)$$

where (\cdot, \cdot) denotes the pairing of de-Rham cohomology and singular homology. It is clear that $d([\phi], [\omega_g])$ depends only on the de-Rham class $[\omega_g] \in H^2(M; \mathbb{R})$ and the induced map on 2-cohomology $\phi^* : H^2(\mathbb{P}^m; \mathbb{Q}) \rightarrow H^2(M; \mathbb{Q})$.

Corollary 1.1.1. *Let $E \rightarrow M$ be a complex vector bundle over a compact almost complex manifold (M, J) . Suppose further that the total space is endowed with an almost complex structure \mathcal{J} and the bundle is globally generated by pseudoholomorphic sections with respect to \mathcal{J} . Then for any almost Kähler metric g on M the eigenvalues of the Laplace-Beltrami operator satisfy,*

$$\lambda_k(M, g) \leq C \frac{(c_1(E) \smile [\omega_g]^{n-1}, [\Sigma^n])}{([\omega_g]^n, [\Sigma^n])} k, \quad \text{for any } k \geq 1, \quad (1.8)$$

where $C > 0$ is a constant depending only on $\dim(M)$, $\text{rank}(E)$ and $\dim(H_{\mathcal{J}}^0(E))$.

As in [33] we can also obtain a version of Theorem 1.1.5 for pseudoholomorphic subvarieties of almost Kähler manifolds. Let $(M^{n+\ell}, J)$ be a closed almost Kähler manifold and Σ^n an irreducible pseudoholomorphic subvariety whose regular part Σ_*^n has complex dimension n . Here we say that $\Sigma^n \subset M^{n+\ell}$ is an irreducible pseudoholomorphic subvariety if it is the image of a somewhere immersed pseudoholomorphic map $\Phi : X \rightarrow M$ where X is a smooth, closed, connected almost complex manifold. Given an almost Kähler metric g on M its restriction to the regular part of Σ yields an incomplete almost Kähler metric on Σ_* . We are interested in the eigenvalues of the Laplacian corresponding to g_{Σ} .

Theorem 1.1.6. *Let $(M^{n+\ell}, J)$ be a closed almost Kähler manifold and $\phi : M^{n+\ell} \rightarrow \mathbb{P}^m$ a non-trivial pseudoholomorphic map. Furthermore let $\Sigma^n \subset M^{n+\ell}$ be an irreducible pseudoholomorphic subvariety such that the restriction of ϕ to Σ is non-trivial. Then, for any almost Kähler metric g on M , the eigenvalues of the Laplacian associated to*

g_Σ satisfy,

$$\lambda_k(\Sigma, g_\Sigma) \leq C(n, m) \frac{\int_\Sigma \phi^* \omega_{FS} \wedge \omega_g^{n-1}}{\int_\Sigma \omega_g^n} k, \quad \text{for any } k \geq 1, \quad (1.9)$$

where $C(n, m) > 0$ is a constant depending only on n and m and ω_g is the Kähler form of g on M .

Finally we are able to show that the pseudoholomorphic map $\phi : M \rightarrow \mathbb{P}^m$ in Theorem 1.1.5 need not be smooth. The precise statement of the weakened regularity conditions can be found in §4.9.1 wherein we state and prove Theorem 4.9.1 which is the low regularity counterpart of Theorem 1.1.5. This is new even in the holomorphic setting. It turns out that rational maps satisfy the regularity conditions in question and hence we obtain the following interesting consequence of Theorem 4.9.1.

Theorem 1.1.7. *Let M^n be a closed Kähler manifold and $L \rightarrow M$ a holomorphic line bundle with base locus $V \subset M$. If V is a subvariety of codimension at least 2 then, for any Kähler metric ω on M , the eigenvalues of the Laplace-Beltrami operator satisfy,*

$$\lambda_k(M, \omega) \leq C \frac{(c_1(L) \smile [\omega]^{n-1}, [M])}{\text{Vol}(M, [\omega])} k, \quad \text{for any } k \geq 1, \quad (1.10)$$

where $C > 0$ is a constant depending only n and m .

Chapter 2

Background

2.1 Almost Complex Structures

An almost complex manifold is a pair, (M, J) , where M is a smooth manifold and $J \in \text{End}(TM)$ is an automorphism of the tangent bundle such that $J^2 = -I$. We call the automorphism J an *almost complex structure*.

A nondegenerate 2-form $\omega \in \Omega^2(M)$ is said to be *tamed* by an almost complex structure J if

$$\omega(X, JX) > 0, \quad \forall X \in \Gamma(TM) \setminus \{0\},$$

and *compatible* with J if

$$\omega(JX, JY) = \omega(X, Y) \quad \forall X, Y \in \Gamma(TM).$$

With these definitions, for a nondegenerate 2-form ω and an almost complex structure J , the bilinear form defined by

$$\langle X, Y \rangle = \omega(X, JY), \quad \forall X, Y \in \Gamma(TM),$$

defines a Riemannian metric on M if and only if ω is tamed by and compatible with J .

When an almost complex manifold (M, J) is equipped with a Riemannian metric g satisfying

$$g(JX, JY) = g(X, Y), \quad \forall X, Y \in \Gamma(TM),$$

then we call the triple (M, J, g) an *almost Hermitian manifold*. Notice that the metric defined by a tamed and compatible nondegenerate 2-form is thus almost Hermitian (with respect to the given almost complex structure). On the other hand given a Hermitian manifold (M, J, g) there is a naturally associated nondegenerate 2-form defined by

$$\omega(X, Y) = g(JX, Y), \quad \forall X, Y \in \Gamma(TM),$$

which is often called the *Hermitian form* or *fundamental form*.

One might initially hope that every oriented, even dimensional manifold admits an almost complex structure. But this is not the case, even if one requires closedness in addition. Indeed Borel and Serre [6] prove that the only spheres which admit an almost complex structure are \mathbb{S}^2 and \mathbb{S}^6 . The next proposition tells us, amongst other things, that manifolds admitting almost complex structures are in one-to-one correspondence with those admitting nondegenerate 2-forms.

Proposition 2.1.1 ([40]). *Let M be a smooth manifold of dimension $2n$, then*

- (i) *for each almost complex structure J there exists a nondegenerate 2-form compatible with J , moreover the space of such forms is contractible;*
- (ii) *for each nondegenerate 2-form ω there exists an almost complex structure J with which ω is compatible, moreover the space of such almost complex structures is contractible.*

Corollary 2.1.1. *Every orientable 2-dimensional manifold admits an almost complex structure.*

Example 2.1.1. The most fundamental example of an almost complex structure is the standard almost complex structure J_0 on \mathbb{R}^{2n} which arises from the identification with \mathbb{C}^n via $z_j = x_j + iy_j$. That is, J_0 acts via multiplication by i on each fibre of $T\mathbb{C}^n$, this can be represented in matrix form as

$$J_0 = \bigoplus^n \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Notice that here we have taken the convention that $J_0 \frac{\partial}{\partial x_j} = -\frac{\partial}{\partial y_j}$, which is the convention we will use throughout this thesis.

The nondegenerate 2-form given by,

$$\omega_0 = \sum_{j=1}^n dx_j \wedge dy_j,$$

is tamed by and compatible with J_0 . Thus $(\mathbb{C}^n, J_0, \omega_0)$ is an almost Hermitian manifold, in fact it is a Kähler manifold the definition of which shall be given in the next section.

Example 2.1.2. We now give an example, first investigated by Kodaira in the 1950's [32], of an almost complex manifold which we shall return to multiple times throughout this thesis. The Kodaira-Thurston surface is given by $X = S^1 \times (\text{Nil}^3/\Gamma)$ where Nil^3 is the Heisenberg group,

$$\text{Nil}^3 = \left\{ A \in \text{GL}(3, \mathbb{R}) \mid A = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbb{R} \right\},$$

and Γ is the subgroup of Nil^3 with integral entries, acting by left multiplication. Letting t denote a coordinate on S^1 an invariant frame of TX is given by

$$\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}, \quad \frac{\partial}{\partial z},$$

with its coframe being

$$dt, \quad dx, \quad dy, \quad dz - xdy.$$

We can define an almost complex structure J on T^*X on X by $J(dx) = dt$, $J(dy) = dz - xdy$, by taking the duals we have defined an almost complex structure on X . Furthermore there is a tamed and compatible 2-form given by

$$\omega = dx \wedge dt + dy \wedge (dz - xdy),$$

thus making (X, J, ω) an almost Hermitian manifold.

To round out this brief section we will discuss the conditions under which an almost complex structure is a complex structure. For this first recall that a complex manifold is a manifold with a holomorphic atlas, *i.e.* transition maps are holomorphic. A complex manifold has a naturally associated almost complex structure which is given locally by Example 2.1.1 and patched together by the holomorphic transition data.

An almost complex structure is said to be *integrable* if it is associated to a holomorphic atlas, in this case we often refer to it simply as a complex structure. One might hope that in fact every almost complex structure is integrable but, alas, this is not the case. The J in Example 2.1.2 is non-integrable whereas J_0 in Example 2.1.1 is integrable.

One can characterise integrability in a number of ways. For now we focus on the so-called Nijenhuis tensor:

$$N_J(X, Y) := \frac{1}{4} ([JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]), \quad X, Y \in TM. \quad (2.1)$$

A straightforward calculation shows that if J is integrable then $N_J \equiv 0$. It turns out that the converse is also true and this is the content of the famed Newlander-Nirenberg theorem.

Theorem 2.1.1. *Let M be a smooth manifold and J a smooth almost complex structure on M . Then there exists a holomorphic atlas associated to J if and only if $N_J \equiv 0$.*

Example 2.1.3. The almost complex structure J_0 on \mathbb{C}^n in Example 2.1.1 was defined using the global holomorphic coordinates z_j and is hence integrable. On the other hand it is easy to calculate that $N_{J_0} \equiv 0$.

Example 2.1.4. Consider now the Kodaira-Thurston surface X with the almost complex structure J given in Example 2.1.2, we claim that this is not integrable.

Indeed, we have that

$$J\left(\frac{\partial}{\partial x}\right) = -\frac{\partial}{\partial t} \quad \text{and} \quad J\left(\frac{\partial}{\partial z}\right) = -\frac{\partial}{\partial y} - x\frac{\partial}{\partial z},$$

hence we can calculate,

$$N_J\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right) = -J\left(\left[\frac{\partial}{\partial x}, -\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}\right]\right) = -\frac{\partial}{\partial y} - x\frac{\partial}{\partial z} \neq 0.$$

Thus this almost complex structure J is not integrable.

On the other hand we can also take the original viewpoint of Kodaira [32] using the fact that $\mathbb{C}^2 = \text{Nil}^3 \times \mathbb{R}$ is a nilpotent group to write $X = \mathbb{C}^2/\tilde{\Gamma}$, where $\tilde{\Gamma}$ is a discrete subgroup which acts holomorphically and preserves the standard symplectic form on \mathbb{C}^2 . With this in mind X is an elliptic surface, in particular X admits an integrable almost complex structure.

Fundamental to the study of complex geometry are the type decompositions of complex differential forms. The decompositions continue to hold for almost complex manifolds but in the non-integrable setting these decompositions do not play nicely when taking derivatives.

Given an almost complex structure J , extending the almost complex structure to the complexified tangent bundle $T^{\mathbb{C}}M = TM \otimes \mathbb{C}$ induces a splitting

$$T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M,$$

into the eigenspaces of J with eigenvalues i and $-i$ respectively. The complexified cotangent bundle thus admits an analogous type decomposition which in turn induces a decomposition of the bundle of complex k -forms

$$\Omega_{\mathbb{C}}^k = \bigoplus_{p+q=k} \Omega^{p,q}.$$

Remark 2.1.1. The sections of $T^{1,0}X$ are precisely the vector fields of the form $Z - iZ$ where Z is a real vector field.

With these splittings we can give another characterisation of integrability of J . Indeed there is a natural $(2,0)$ -form with values in $T^{1,0}M$ given by:

$$\tau : \Gamma(T^{1,0}M) \times \Gamma(T^{1,0}M) \longrightarrow \Gamma(T^{0,1}M), \quad \tau(\xi_1, \xi_2) = [\xi_1, \xi_2]^{0,1}. \quad (2.2)$$

To verify that this is indeed a $(2,0)$ -form it is enough to recall the following identity:

$$[\xi_1, f\xi_2] = f[\xi_1, \xi_2] + \xi_1(f)\xi_2, \quad \forall f \in C^\infty(M).$$

Note that unless otherwise specified scalar functions are to be considered as mapping

into \mathbb{C} . With this identity it is clear that $\tau(\xi_1, f\xi_2) = f\tau(\xi_1, \xi_2)$. We call τ the *torsion of J* .

Suppose that we have a local holomorphic coordinate system z^i , then, since $[\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}] = 0$, we must have $\tau \equiv 0$. It turns out that τ vanishing is equivalent to the vanishing of the Nijenhuis tensor.

Proposition 2.1.2. $\Gamma(T^{1,0}M)$ is closed under the Lie bracket, i.e. $\tau \equiv 0$, if and only if $N_J \equiv 0$.

Proof. The $(1,0)$ vector fields on (M, J) are precisely those of the form $Y - iJY$ where $Y \in \Gamma(TM)$. Thus it suffices to check when the following quantity is of this form:

$$[X - iJX, Y - iJY] = [X, Y] - [JX, JY] - i([JX, Y] + [X, JY]).$$

So we have closure if and only if

$$J([X, Y] - [JX, JY]) = [JX, Y] + [X, JY].$$

This is in turn equivalent to $N_J(X, Y) = 0$. □

A fundamental difference between complex and almost complex geometry is the interaction of the exterior derivative with the natural type decompositions coming from J . Let $d : \Omega_{\mathbb{C}}^k \rightarrow \Omega_{\mathbb{C}}^{k+1}$ be the complex linear extension of the exterior derivative on real forms and define operators

$$\partial := \pi^{p+1,q} \circ d : \Omega^{p,q} \longrightarrow \Omega^{p+1,q}, \quad \bar{\partial} := \pi^{p,q+1} \circ d : \Omega^{p,q} \longrightarrow \Omega^{p,q+1},$$

where $\pi^{p,q} : \Omega_{\mathbb{C}}^k \rightarrow \Omega^{p,q}$ is the projection map.

The torsion, τ , of J yields operators:

$$\tau' : \Omega^{p,q} \longrightarrow \Omega^{p+2,q-1}, \quad \text{and} \quad \tau'' : \Omega^{p,q} \longrightarrow \Omega^{p-1,q+2}.$$

Indeed, if ξ_1, \dots, ξ_m a local frame of $T^{1,0}X$ over $U \subset X$ then τ is given by

$$\tau = \sum_i \tau_i \otimes \bar{\xi}_i, \quad \tau_i \in \Omega^{2,0}(U).$$

Now if $u \in \Omega^{p,q}$ then:

$$\tau' u = \sum_i \tau_i \wedge (\bar{\xi}_i \lrcorner u), \quad \tau'' u = \sum_i \bar{\tau}_i \wedge (\xi_i \lrcorner u).$$

It is straightforward to verify that τ', τ'' are both derivations, more precisely that:

$$\tau'(u \wedge v) = (\tau' u) \wedge v + (-1)^{\deg u} u \wedge (\tau' v),$$

and similarly for τ'' . It turns out that $\tau' + \tau''$ is precisely the quantity that describes how far the exterior derivative is from respecting type decompositions.

Lemma 2.1.1. $d = \partial + \bar{\partial} + \tau' + \tau''$.

Proof. Since all of the operators involved are derivations it suffices to check the formula for 0-forms and 1-forms. But by convention the contraction of a function by a vector field is zero, so for 0-forms $\tau'u = \tau''u = 0$. Since du can have only $(1, 0)$ and $(0, 1)$ parts we have the formula for 0-forms.

Now let $u \in \Omega^1$ and $\xi, \eta \in \Gamma(T^{\mathbb{C}}M)$, then

$$du(\xi, \eta) = \xi(u(\eta)) - \eta(u(\xi)) - u([\xi, \eta]).$$

So if u is of type $(0, 1)$ and ξ, η of type $(1, 0)$, it is easy to see that

$$(du)^{2,0}(\xi, \eta) = -u([\xi, \eta]^{0,1}) = -u(\tau(\xi, \eta)) = (-\tau'u)(\xi, \eta).$$

By definition we have $(du)^{1,1} = \partial u$, $(du)^{0,2} = \bar{\partial}u$ and $\tau''u = 0$. Thus the desired formula holds. By conjugation we have the formula for $(1, 0)$ -forms also. \square

Remark 2.1.2. It will be useful in later chapters to note that when acting on complex valued functions we get the splitting $d = \partial + \bar{\partial}$ even in the non-integrable case.

Since integrability of J is equivalent to the vanishing of τ the Newlander-Nirenberg theorem hence gives the following characterisation of integrability.

Theorem 2.1.2. *An almost complex structure J is integrable if and only if $d = \partial + \bar{\partial}$.*

2.1.1 Almost Kähler Manifolds

In this brief section we shall review some of the remarkable properties that arise when we assume that the fundamental form of a (almost) Hermitian manifold is closed, that is, when the manifold is also symplectic with compatible almost complex structure.

An almost Hermitian manifold is called *almost Kähler* if the fundamental form is closed, *i.e.* $d\omega = 0$, and *Kähler* if the almost complex structure is also integrable. Remark that any symplectic manifold is almost Kähler by Proposition 2.1.1.

Example 2.1.5. The form $\omega_0 = \sum_{j=1}^n dx_j \wedge dy_j$ on (\mathbb{C}^n, J_0) is clearly closed and hence $(\mathbb{C}^n, J_0, \omega_0)$ is a Kähler manifold.

Let us now give an example of a strictly almost Kähler manifold, *i.e.* a symplectic manifold which does not admit an integrable almost Kähler structure. To do this we discuss a basic topological restriction on Kähler manifolds, namely the Hodge diamond.

Theorem 2.1.3 (Hodge Decomposition and Symmetries). *A Kähler manifold admits the following decomposition of its cohomology with complex coefficients,*

$$H^k(M; \mathbb{C}) \cong \bigoplus_{i+j=k} H^{i,j}(M),$$

where $H^{i,j}(M)$ are the Dolbeault cohomology groups. Moreover the cohomology groups $H^{i,j}(M)$ satisfy the following symmetries,

$$H^{i,j}(M) \cong \overline{H^{j,i}(M)}, \quad \text{and} \quad H^{i,j}(M) \cong H^{n-i,n-j}(M),$$

where $n = \dim(M)$. In particular, for $h^{i,j}(M) = \dim H^{i,j}(M)$, we have

$$h^{i,j}(M) = h^{j,i}(M) \quad \text{and} \quad h^{i,j}(M) = h^{n-i,n-j}(M).$$

Remark 2.1.3. These symmetries of $h^{i,j}$ are commonly conveyed by arranging the numbers in a diamond, known as the Hodge diamond.

Notice that, from the symmetry $H^{i,j}(M) \cong \overline{H^{j,i}(M)}$, it follows that $\dim(H^{1,0}(M)) = \dim(H^{0,1}(M))$ and thus from the Hodge theorem we see that the first Betti number of a Kähler manifold is necessarily even,

$$b_1 = 2 \dim H^{1,0}(M).$$

Example 2.1.6. Thurston was the first to observe that the Kodaira-Thurston surface admits a symplectic form but no Kähler structure [53]. The form $\omega = dx \wedge dt + dy \wedge (dz - xdy)$ on the Kodaira-Thurston surface (X, J) is closed thus making (X, J, ω) a strictly almost Kähler manifold.

In fact X cannot admit a Kähler structure despite the fact that it admits both almost Kähler structures and integrable almost complex structures. Indeed, one can explicitly compute the first fundamental group and its commutator subgroup which leads one to conclude that $b_1 = 3$. In particular b_1 is odd and hence X cannot admit a Kähler structure.

Now consider the setting of an almost Hermitian manifold (M, J, g) of dimension $2n$. We say that an affine connection ∇ is an *almost Hermitian connection* if

$$\nabla J = \nabla g = 0.$$

Such connections always exist on an almost Hermitian manifold.

At this point it is interesting to remark that an almost Hermitian manifold is Kähler if and only if the Levi-Civita connection of associated the Riemannian metric is almost Hermitian.

Proposition 2.1.3 (Lemma 4.15 of [41]). *Let (M, J, g) be an almost Hermitian man-*

ifold, ω the fundamental form and ∇^{LC} the Levi-Civita connection of g . Then the following are equivalent,

- (1) $\nabla^{LC}J = 0$,
- (2) $d\omega = 0$ and J is integrable.

It is interesting to put this characterisation in the context of other natural connections available on almost Hermitian manifolds.

Let e_i be a local unitary frame, θ^i its dual coframe and θ_i^j the connection 1-forms associated to a given almost Hermitian connection ∇ . The torsion $\Theta = (\Theta^1, \dots, \Theta^n)$ of a connection ∇ , which is a matrix of 2-forms, can be defined by Cartan's first structure equation

$$d\theta^i = -\theta_j^i \wedge \theta^j + \Theta^i.$$

The so-called *Chern connection* (also sometimes referred to as the *second canonical connection*) is the unique almost Hermitian connection for which the $(1, 1)$ part of the torsion vanishes, that is, each $(\Theta^i)^{(1,1)} = 0$. It is well-known that such a connection always exists, for example consult [20].

Lemma 2.1.2. *Let (M, J, g) be an almost Hermitian manifold, then there exists a unique almost Hermitian connection ∇ such that $\Theta^{(1,1)} \equiv 0$.*

On the other hand there is an equivalent description of the torsion which has become more common in Riemannian geometry textbooks. Namely the torsion T of an affine connection ∇ is defined by,

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \quad X, Y \in \Gamma(TM).$$

In terms of a local frame E_i of TM we have that

$$T(X, Y) = \Theta^i(X, Y)E_i, \quad \forall X, Y \in \Gamma(TM).$$

Thus returning to the local unitary frame e_i we have that,

$$T = 2(\Theta^i e_i + \overline{\Theta^j e_j})$$

holds on $T^{\mathbb{C}}M$.

Let us define functions T_{jk}^i and $N_{\bar{j}\bar{k}}^i$ as the coefficients of the torsion of the Chern connection with respect to the local unitary frame θ^i , that is,

$$(\Theta^i)^{(2,0)} = T_{jk}^i \theta^j \wedge \theta^k, \quad (\Theta^i)^{(0,2)} = N_{\bar{j}\bar{k}}^i \bar{\theta}^j \wedge \bar{\theta}^k.$$

In [54] it is remarked that the $(0, 2)$ part of the torsion is independent of the Hermitian metric and can in fact be regarded as a Nijenhuis tensor for J which maps $T^{0,1}M \times T^{0,1}M \rightarrow T^{1,0}M$ (*c.f.* with the torsion of J defined by (2.2)). Furthermore

it is remarked that these functions give a useful characterisation of the almost Kähler condition $d\omega = 0$ and the quasi-Kähler condition $(d\omega)^{(1,2)} = 0$.

Lemma 2.1.3 (Lemma 2.4 of [54]). *An almost Hermitian manifold (M, J, g) is almost Kähler if and only if*

$$T_{jk}^i = 0, \quad \text{and} \quad N_{\bar{j}k}^i + N_{\bar{k}i}^j + N_{\bar{i}j}^k = 0,$$

and quasi-Kähler if and only if

$$T_{jk}^i = 0.$$

Given an almost Hermitian manifold (M, J, g) endowed with its canonical connection ∇^C one thus has two, quite natural, choices of how to define the Laplacian acting on functions. On one hand we can use the Laplace-Beltrami operator which arises purely from the metric g (see §2.2 for a definition). On the other we can use the canonical connection and define the Laplacian as the trace of the Hessian, which in terms of a local unitary frame e_i is given by

$$\Delta^C f = \sum_i (\nabla^C df)(e_i, \bar{e}_i) + (\nabla^C df)(\bar{e}_i, e_i),$$

for some $f \in C^\infty(M)$.

In general these notions of Laplacian do not agree, but if the almost Hermitian manifold is quasi-Kähler then they do. Indeed, the Laplacian of the Levi-Civita connection acting on a function is given as the trace of the map $F : TM \rightarrow TM$ defined by

$$F(X) = \nabla_X^C(\text{grad } f) + T_{\nabla^C}(\text{grad } f, X),$$

where ∇^{LC} is the Levi-Civita connection, ∇^C is the canonical connection and T_{∇^C} is the torsion of ∇^C . For details see [31]. Now if g is quasi-Kähler then T_{∇^C} is simply the Nijenhuis tensor by Lemma 2.1.3 and hence maps $T^{0,1}M \times T^{0,1}M \rightarrow T^{1,0}M$. Thus it is straightforward to deduce that the trace of the torsion term must be zero in this case and hence we have the following lemma.

Lemma 2.1.4 (Lemma 2.6 of [54]). *Let (M, J, g) be an almost Hermitian manifold and g be quasi-Kähler then the canonical Laplacian is equal to the usual Laplacian of the Levi-Civita connection of g .*

2.1.2 Cauchy-Riemann Type Equations and Pseudoholomorphic Curves

Let us now briefly discuss Cauchy-Riemann type equations and some of their applications on almost complex manifolds with a view to developing some basic pseudoholomorphic curve theory.

In the following we shall assume that (M, J) is a smooth almost complex manifold with J a smooth almost complex structure tamed by some ω and (Σ, j) a compact

Riemann surface. A smooth map $u : \Sigma \rightarrow M$ is said to be a (parameterised) J -holomorphic curve if

$$du \circ j = J \circ du. \quad (2.3)$$

We shall also refer to such curves as pseudoholomorphic curves. Notice that the requirement that u satisfy (2.3) is equivalent to it solving the Cauchy Riemann equation

$$\bar{\partial}_J u := \frac{1}{2} (du + J \circ du \circ j) = 0. \quad (2.4)$$

Remark 2.1.4. Throughout the later chapters of this thesis we shall refer to pseudoholomorphic disks in some given almost complex manifold. Henceforth, by pseudoholomorphic disk we shall be referring to a disk $D_\rho \subset \mathbb{C}$ equipped with the standard almost complex structure, j_0 , and a smooth J -holomorphic map $u : (D_\rho, j_0) \rightarrow (M, J)$.

Take a holomorphic coordinate atlas U_α of Σ and write u_α for the restriction of u to U_α . Then, if $z = s + it$ is a holomorphic coordinate on U_α , the above Cauchy-Riemann equation reduces to the following first order non-linear PDE,

$$\partial_s u_\alpha + J(u_\alpha) \partial_t u_\alpha = 0.$$

Remark 2.1.5. If $u = f + ig$ maps into \mathbb{C}^n equipped with its standard almost complex structure then the above equation is in fact the usual system of Cauchy-Riemann equations

$$\partial_s f = \partial_t g \quad \partial_s g = -\partial_t f.$$

Local Properties

Throughout this section we will consider a smooth almost complex manifold (M, J) , *i.e.* M is a smooth manifold equipped with a smooth almost complex structure J , and smooth J -holomorphic curves. Although all of the statements below hold with varying degrees of lower regularity.

We first establish a unique continuation property following the exposition in [40] since this gives us an excuse to discuss the Carleman Similarity Principle.

Theorem 2.1.4 (Unique Continuation). *Let (M, J) be a smooth almost complex manifold and $\Omega \subset \mathbb{C}$ an open neighbourhood of the origin. If $u, v : \Omega \rightarrow M$ are two J -holomorphic curves which agree to infinite order at the origin, that is*

$$\int_{|z| \leq r} |u - v| = O(r^k), \quad \forall k \in \mathbb{N},$$

then $u \equiv v$ on Ω .

We should briefly remark that a J -holomorphic curve satisfies (2.4) and hence is a

solution to the following second order quasi-linear equation

$$\Delta u = (\partial_t J(u))\partial_s u - (\partial_s J(u))\partial_t u.$$

Thus the unique continuation property can be seen to follow from Aronszajn's unique continuation theorem.

Theorem 2.1.5 (Theorem 2.3.4 of [40]). *Let $\Omega \subset \mathbb{C}$ be a connected open set. Suppose that $w \in W_{loc}^{2,2}(\Omega; \mathbb{R}^m)$ satisfies,*

$$|\Delta w(z)| \leq C(|w| + |\partial_s w| + |\partial_t w|)(z), \quad (2.5)$$

for almost every $z = s + it \in \Omega$ and that w vanishes to infinite order at some $z_0 \in \Omega$. Then $w \equiv 0$.

It will be useful for the work undertaken in later chapters to present a proof which does not rely on Aronszajn's unique continuation theorem. To that end consider now the Carleman Similarity Principle which says, in essence, that one can locally transform a J -holomorphic curve to a holomorphic curve. We shall see in the following that this in fact leads J -holomorphic curves to share many local properties with holomorphic curves.

Theorem 2.1.6 (c.f. Theorem 2.3.5 of [40]). *Let $p > 2$ and $B_\varepsilon \subset \mathbb{C}$ for some $\varepsilon > 0$. Suppose that $C \in L^\infty(B_\varepsilon, \text{End}_{\mathbb{R}}(\mathbb{C}))$ and $v \in W^{1,p}(B_\varepsilon, \mathbb{C})$ is a solution to*

$$\bar{\partial}v(z) + C(z)v(z) = 0. \quad (2.6)$$

Then, for a sufficiently small $\delta > 0$, there exist maps $\Phi \in C^0(B_\delta, \text{End}_{\mathbb{R}}(\mathbb{C}))$ and $\sigma \in C^\infty(B_\delta, \mathbb{C})$ such that $\Phi(z)$ is invertible and on B_δ ,

$$v(z) = \Phi(z)\sigma(z), \quad \bar{\partial}\sigma = 0, \quad \Phi^{-1}(z)J(z)\Phi(z) = J_0.$$

Now Theorem 2.1.4 follows by a simple argument, the full details of which can be found in [40]. Indeed, consider the set up of Theorem 2.1.4 and remark that it suffices to prove the theorem for $u, v : \Omega \rightarrow \mathbb{C}^n$. Define a function $w : \Omega \rightarrow \mathbb{C}^n$ by $w(z) = u(z) - v(z)$ and remark that this satisfies a Cauchy-Riemann type equation of the form (2.6). Hence Theorem 2.1.6 yields a holomorphic function σ on some small ball, B_δ , centred at the origin such that $\sigma = \Phi^{-1}w$ for some invertible function $\Phi \in C^0(B_\delta, \text{End}_{\mathbb{R}}(\mathbb{C}))$. Since w vanishes to infinite order at $z = 0$ and $\Phi(0)$ is invertible it follows that σ vanishes to infinite order. But σ is holomorphic and hence $\sigma \equiv 0$ on B_δ . So $w \equiv 0$ on B_δ , from which it is straightforward to conclude Theorem 2.1.4.

For holomorphic curves it is well known that critical points are isolated. Thus, given a J -holomorphic curve $u : \Omega \rightarrow \mathbb{C}^n$, Theorem 2.1.6 immediately implies that $u^{-1}(x)$ is a finite set for every $x \in M$. Furthermore, one can show that $v := \partial_s u : \Omega \rightarrow \mathbb{C}^n$

also satisfies an equation of the form (2.6) and hence Theorem 2.1.6 can be applied to deduce that its zeroes are isolated. These properties are encapsulated in the following proposition.

Proposition 2.1.4 (Lemma 2.4.1 of [40]). *Let Σ be a compact Riemann surface without boundary, (M, J) a smooth almost complex manifold and $u : \Sigma \rightarrow M$ a smooth, nonconstant pseudoholomorphic curve. Then the set*

$$X := u^{-1}(\{u(z) | z \in \Sigma, du(z) = 0\}),$$

of preimages of critical values is finite. Moreover, $u^{-1}(x)$ is a finite set for every $x \in M$.

In fact by a similar argument (with a sprinkling of point set topology) one can obtain the following characterisation of the intersection of pseudoholomorphic curves.

Proposition 2.1.5 (Proposition 2.4.4 of [40]). *Let (M, J) be a smooth almost complex 4-manifold and for $i = 1, 2$ let $u_i : \Sigma_i \rightarrow J$ be J -holomorphic curves where Σ_i are closed Riemann surfaces. Furthermore assume that $u_1 : \Sigma_1 \rightarrow M$ is nonconstant and $u_1(\Sigma_1) \neq u_2(\Sigma_2)$. Then the set $u_1^{-1}(u_2(\Sigma_2))$ is at most countable and accumulates only at the critical points of u_1 .*

This proposition suggests that the intersection index of pseudoholomorphic curves may be worth exploring. To this end let us take a brief digression to recall the notion of a local intersection index for smooth oriented submanifolds.

Suppose that M is a smooth oriented manifold of dimension n and $u_i : X_i \hookrightarrow M$ are smooth, oriented submanifolds of dimension n_i for $i = 1, 2$. We say that X_1 intersects X_2 transversally at a point $x \in X_1 \cap X_2$ if

$$T_x X_1 \oplus T_x X_2 = T_x M.$$

We say that X_1 intersects X_2 transversally, abbreviated $X_1 \pitchfork X_2$, if they intersect transversally for all $x \in X_1 \cap X_2$.

Now suppose that there is no excess dimension, *i.e.* $n = n_1 + n_2$. In this case notice that if $X_1 \pitchfork X_2$ then the intersection submanifold $X_1 \cap X_2$ is 0-dimensional, *i.e.* consists only of isolated points. We define the local intersection index, $\delta(u_1, u_2; x)$ as follows. If $X_1 \pitchfork X_2$ then we set $\delta(u_1, u_2; x) = \pm 1$, with the sign positive if and only if the natural orientations on either side of the following splitting match

$$T_x M = \text{Im } du_1(x) \oplus \text{Im } du_2(x).$$

More precisely, if e_1, \dots, e_{n_1} is an oriented basis of $T_x X_1$ and $e_{n_1+1}, \dots, e_{n_1+n_2}$ is an oriented basis of $T_x X_2$ then the intersection index at x is $+1$ if e_1, \dots, e_n is an oriented basis of $T_x M$ and -1 otherwise. In the case that the intersection is not transverse we need to appeal to the Transversality Theorem.

Indeed, if X_1 and X_2 do not intersect transversally then by the Transversality Theorem one can (smoothly) deform X_1 by an arbitrarily small amount to, say, $u_1^\varepsilon : X_1^\varepsilon \hookrightarrow M$ such that $X_1^\varepsilon \pitchfork X_2$. Then we define $\delta(u_1, u_2; x) = \delta(u_1^\varepsilon, u_2; x)$. Of course one must check that this is well defined, that is, that $\delta(u_1, u_2; x)$ is a homotopy invariant, see for example [25, 43].

With another application of the Carleman Similarity Principle we can obtain the following proposition (this also appears as Exercise 2.6.1 in [40]) which is the easiest case of Gromov's phenomena of positivity of intersections. For the proof we follow Wendl [58].

Proposition 2.1.6 (Theorem 2.88 [58]). *Let (M, J) be an almost complex manifold and Q a compact codimension 2 J -holomorphic submanifold. Suppose that $u : D \rightarrow M$ is a pseudoholomorphic disk such that $u(0) \in Q$.*

- (1) *Then either $u^{-1}(Q)$ consists of isolated points, or, $u(D) \subset Q$.*
- (2) *Suppose that the intersection points are isolated and, after possibly shrinking D , that $u(0)$ is the unique such point. Define an intersection number $u \cdot Q$ to be the number of points of intersection (counted with multiplicities) with a generic smooth perturbation of u which fixes the boundary ∂D . Then $u \cdot Q \geq 1$ with equality if and only if u is transverse to Q at zero. That is,*

$$\delta(Q, u_1(D); u(0)) \geq 1,$$

with equality if and only if u intersects Q transversally at zero.

Sketch Proof. First remark that the result is local and so it is enough to assume that $M = \mathbb{C}^n$ equipped with some almost complex structure J . Moreover we can choose complex coordinates such that $Q = \mathbb{C}^{n-1} \times \{0\}$, $u(0) = 0 \in \mathbb{C}^n$ and $J = \tilde{J} \oplus i$ along Q , where \tilde{J} is some almost complex structure on \mathbb{C}^{n-1} and i is the standard complex structure on \mathbb{C} .

In this setting the map u has the form $u(\zeta) = (\tilde{u}(\zeta), f(\zeta))$ for some smooth functions $\tilde{u} : D \rightarrow \mathbb{C}^{n-1}$ and $f : D \rightarrow \mathbb{C}$. Thus the intersection $u(D) \cap Q$ is described by the zeroes of f . It turns out that f satisfies a Cauchy-Riemann type equation and hence the hypothesis of the Carleman Similarity Principle. To prove this we employ a neat interpolation trick used by [40] and [58].

For $\tau \in [0, 1]$ define a smooth homotopy between $u_0(\zeta) = (\tilde{u}(\zeta), 0)$ and $u_1 = u$ by $u_\tau(\zeta) = (\tilde{u}(\zeta), \tau f(\zeta))$. Writing $s + it$ for coordinates on D our map u satisfies $\partial_s u + J(u)\partial_t u = 0$. Hence,

$$\begin{aligned} \partial_s u + J(u_0)\partial_t u &= \partial_s u + J(u)\partial_t u + [J(u_0) - J(u)]\partial_t u \\ &= - \left(\int_0^1 \frac{d}{d\tau} J(u, \tau f) d\tau \right) \partial_t u \\ &=: -Af. \end{aligned}$$

Since $J(u_0) = J(\tilde{u}, 0) = i$ the projection $\mathbb{C}^{n-1} \times \mathbb{C} \rightarrow \mathbb{C}$ factors through the above equation and so we can define a smooth function $C : D \rightarrow \text{End}_{\mathbb{R}}(\mathbb{C})$ to be the projection of A onto the second factor and obtain the following Cauchy-Riemann type equation for f

$$\partial_s f + i\partial_t f + Cf = 0.$$

Applying the Carleman Similarity Principle we obtain that, either the zeroes of f are isolated, or, f vanishes identically on a neighbourhood. This proves the first part of the lemma. For the second part suppose that $u(D)$ and Q intersect uniquely at $0 \in \mathbb{C}^{n-1} \times \mathbb{C}$. By the Carleman Similarity Principle this zero of f is isolated and of positive order. Thus we can perturb f by an arbitrarily small amount to a smooth function with only simple zeroes and whose signed count is positive. Since this count is precisely that of the transverse intersections of the resulting perturbation of u with Q we obtain the desired inequality. If we in fact have $\delta(Q, u_1(D); u(0)) = 1$ then the corresponding zero of f is already simple and hence the intersection of u and Q must already be transverse. \square

Writing $\delta(u_1, u_2)$ for the number of all intersection points we can state the more general case of positivity of intersections in 4-manifolds as follows. Unfortunately the proof of this is beyond the scope of this elementary background chapter, rather nice accounts are given in [40, 58].

Theorem 2.1.7 (Positivity of Intersections). *Let (M, J) be a smooth almost complex 4-manifold and $A_1, A_2 \in H_2(M; \mathbb{Z})$ homology classes represented by simple J -holomorphic curves $u_1 : \Sigma_1 \rightarrow M$ and $u_2 : \Sigma_2 \rightarrow M$ respectively. Suppose that $u_1(U_1) \neq u_2(U_2)$ for any nonempty open subsets $U_1 \subset \Sigma_1, U_2 \subset \Sigma_2$, then*

$$\delta(u_1, u_2) \leq A_1 \cdot A_2,$$

with equality if and only if all intersections are transverse.

2.1.3 Local Coordinates

Local holomorphic coordinates provide an extremely powerful tool in complex geometry. An example of this, which is particularly relevant to the work carried out in subsequent chapters, is addressing whether a set has the structure of a complex subspace. More precisely we are referring principally to the work of King [30] in which it is established that the intersection of complex cycles yields another complex cycle. Due in part to the lack of local holomorphic coordinates in the non-integrable case such a result is not yet available in this setting.

In [50] Taubes suggested an alternative to holomorphic coordinates for symplectic 4-manifolds which he used to prove, amongst other things, a useful criterion for determining whether or not a set is an almost complex subvariety. In this setting one

can find a coordinate neighbourhood of a given point which is foliated by embedded J -holomorphic disks in any given direction. As one would expect this can be generalised to higher dimensions and one can always find a coordinate neighbourhood which can be foliated by J -holomorphic disks in any given direction. The following statement is Lemma 3.10 in [60] and the proof there is based on a combination of the proofs in [50] and [55].

Let (M, J) be an almost complex manifold of complex dimension n . For any point $x \in M$ we can find a neighbourhood U of x and a non-degenerate 2-form Ω on U such that J is compatible with Ω in U . This pair (Ω, J) induces an almost Hermitian metric on U .

Remark 2.1.6. We say that an almost complex manifold (M, J) is *locally symplectic* if for any point $x \in M$ there exists an open neighbourhood U of x on which there is a symplectic form compatible with J , *i.e.* a closed, non-degenerate 2-form Ω on U such that J is compatible with Ω . Not all almost complex manifolds are locally symplectic, for example [8] implies that \mathbb{S}^6 equipped with the standard almost complex structure does not have the locally symplectic property. On the other hand it was shown in [46] (although a mistake was noticed and corrected in [36]) that every almost complex 4-manifold is locally symplectic.

Now we can identify a geodesic ball centred at x with a ball in \mathbb{R}^{2n} centred at the origin. We identify $\mathbb{R}^{2n} \cong \mathbb{C}^n$ so that

$$\begin{aligned} \Omega_x = \omega_0 &= dx^1 \wedge dx^2 + \cdots + dx^{2n-1} \wedge dx^{2n} \\ &= \frac{i}{2} (dz^0 \wedge d\bar{z}^0 + \cdots + dz^{n-1} \wedge d\bar{z}^{n-1}). \end{aligned}$$

Here we write complex coordinates $(z^0, \dots, z^{n-1}) = (x^1, x^2, \dots, x^{2n-1}, x^{2n})$. Further we may assume that J is an almost complex structure on \mathbb{C}^n which agrees with the standard complex structure J_0 at the origin.

Lemma 2.1.5. *Let J be an almost complex structure on \mathbb{C}^n which agrees with the standard complex structure J_0 at the origin. Further, let g be a Hermitian metric compatible with J . Then there exists a constant $\rho_0 > 0$ with the following property. Let $0 < \rho < \rho_0$ and $D \subset \mathbb{C}$ the disk of radius ρ . There exists a diffeomorphism $Q : D \times D^{n-1} \rightarrow \mathbb{C}^n$, and constants L, L_m depending only on g and J , such that*

- For all $w \in D^{n-1}$, $Q(D_w)$ is a J -holomorphic curve containing $(0, w)$;
- For all $w \in D^{n-1}$, $|(Q, w) - Q(\zeta, w)| \leq L \cdot \rho \cdot |\zeta|$;
- For all $w \in D^{n-1}$, the derivatives of order m of Q are bounded by $L_m \cdot \rho$;
- For each $\kappa \in \mathbb{C}P^{n-1}$ we can choose Q such that the disk $Q(D_0)$ is tangent at the origin to the line determined by κ .

We use D_w to denote the disk of radius ρ at the point $w \in D^{n-1}$ in the space $D \times D^{n-1}$.

Proof. Since J agrees with J_0 at the origin our strategy is to look for J -holomorphic disks which are perturbations of J_0 -holomorphic disks.

The space of complex directions at the origin in \mathbb{C}^n is parameterised by $\mathbb{C}P^{n-1}$ so for a given direction $\kappa = [1 : \kappa_1 : \dots : \kappa_{n-1}] \in \mathbb{C}P^{n-1}$ consider the J_0 -holomorphic disk $(\zeta, w_1 + \kappa_1 \zeta, \dots, w_{n-1} + \kappa_{n-1} \zeta)$ through the point $w = (w_1, \dots, w_{n-1}) \in D^{n-1}$, where $\zeta \in D$. We search for J -holomorphic perturbations of the form

$$q_{w,\kappa}(\zeta) = (\zeta, w_1 + \kappa_1 \zeta + \tau_1(w, \kappa, \zeta), \dots, w_{n-1} + \kappa_{n-1} \zeta + \tau_{n-1}(w, \kappa, \zeta)),$$

for some smooth functions $\tau_i : D \rightarrow \mathbb{R}^2$. The system of J -holomorphic equations for $q_{w,\kappa}$ are of the form

$$\frac{\partial \tau_i}{\partial \bar{\zeta}} = Q_i(w, \kappa, \tau_1(w, \kappa, \zeta), \dots, \tau_{n-1}(w, \kappa, \zeta)), \quad i = 1, \dots, n-1,$$

and satisfy, after possibly shrinking the disk D , the following estimates for some constants $C_k > 0$,

$$\|Q_i\|_{C^k} \leq C_k \|J - J_0\|_{C^k}, \quad i = 1, \dots, n-1. \quad (2.7)$$

A schematic expression of Q_i can be found in [50] where it is remarked that Q_i can be seen to come from pull-backs of tensors on \mathbb{C}^2 which are constructed from the coefficients of $J - J_0$.

It will be convenient to consider the equation over a slightly larger disk than D hence we introduce a smooth cutoff function $\chi_\rho : \mathbb{C} \rightarrow [0, 1]$ which is identically 1 on D and vanishes for $|z| > \frac{3}{2}\rho$. We now look for solutions to the following system of equations

$$\frac{\partial \tau_i}{\partial \bar{\zeta}} = \chi_\rho Q_i(w, \kappa, \tau_1(w, \kappa, \zeta), \dots, \tau_{n-1}(w, \kappa, \zeta)), \quad i = 1, \dots, n-1,$$

which have the form

$$\tau_i(\zeta) = \frac{1}{\pi} \int \frac{\chi_\rho Q_i(w, \kappa, \tau_1(w, \kappa, \eta), \dots, \tau_{n-1}(w, \kappa, \eta))}{\zeta - \eta} d^2 \eta, \quad i = 1, \dots, n-1.$$

The search will be over the class of $(n-1)$ -tuples of $C^{2, \frac{1}{2}}$ which restrict to the circle of radius 4ρ in the span $\{e^{-i\theta}, e^{-2i\theta}, \dots\}$. This class of functions can be viewed as a Banach space when equipped with the norm

$$\|\tau\| = \sum_{i=1}^{n-1} \left(|\tau_i| + \rho |d\tau_i| + \rho^2 |\nabla d\tau_i| + \rho^{\frac{5}{2}} \sup_{t,s \in \mathbb{C}} \left(\frac{|\nabla d(\tau_i)_t - \nabla d(\tau_i)_s|}{|t-s|^{\frac{1}{2}}} \right) \right).$$

By making $\rho > 0$ small we can make the right hand side of (2.7) arbitrarily small. Thus we can apply contraction mapping theorem on the Banach space described above

to obtain a unique smooth solution $\tau = (\tau_1, \dots, \tau_{n-1})$ which also varies smoothly in w , κ and satisfies the bounds (*c.f.* Lemma 5.5 of [50])

$$\left| \frac{\partial \tau}{\partial w_i} \right| < C\rho, \quad \left| \frac{\partial \tau}{\partial \kappa_i} \right| < C\rho^2,$$

$$\|\tau\|_{C^0} < C(\rho^2 + \rho(|w| + |\kappa|)), \quad \|\tau\|_{C^1} < C(\rho + (|w| + |\kappa|)).$$

With existence of a J -holomorphic perturbation under our belts the lemma will follow by applying the implicit function theorem with κ held constant. Indeed, without loss of generality, assume that $\kappa = [1 : 0 : \dots : 0]$. Then there exists an $\varepsilon > 0$ such that for each $|w| < \varepsilon$ there exists a unique smooth solution τ_w . That is, the perturbed disks $q_{w,\kappa}$ are J -holomorphic. As the pair (ζ, w) vary the map $\sigma(\zeta, w) = q_{w,\kappa}(\zeta)$ defines a map from a neighbourhood of the origin in \mathbb{C}^n to \mathbb{C}^n . Now by taking $\rho > 0$ small we can ensure that

$$\left| \frac{\partial \tau_w}{\partial w} \right| < 1, \quad \text{at } (\zeta, w) = (0, 0).$$

That is, for $\rho > 0$ small the derivative of σ is invertible at the $(0, 0) \in \mathbb{C}^n$ and hence σ is a diffeomorphism on some neighbourhood of the origin. \square

Throughout we refer to a coordinate system arising from this lemma as a J -fibre diffeomorphism. Note that these coordinates are far from unique, it is particularly important to remark that for any given complex direction there is a foliation whose central disk is tangent to this direction at the origin of this coordinate system.

The coordinates given by Lemma 2.1.5 can in fact be improved further to give a convenient coordinate expression for the almost complex structure. First we remark that we can choose coordinates for which the almost complex structure agrees with the standard almost complex structure J_0 along a given embedded pseudoholomorphic disk.

Lemma 2.1.6 (Lemma 2.4.2 of [40]). *Let (M, J) be a smooth almost complex manifold and $u : D \rightarrow M$ an embedded pseudoholomorphic disk. Then there exists a smooth coordinate chart $\psi : U \rightarrow \mathbb{C}^n$ on an neighbourhood of $u(0)$ such that for $z \in \Omega \cap u^{-1}(U)$*

$$\psi \circ u(z) = (z, 0, \dots, 0), \quad d\psi(u(z))J(u(z)) = J_0 d\psi(u(z)).$$

Proof. Writting $z = s + it$ for complex coordinates on D , choose a complex frame bundle

$$\frac{\partial u}{\partial s} =: Z_1, Z_2, \dots, Z_n$$

of the pull-back bundle u^*TM and consider the exponential map $\phi : \Omega \times \mathbb{C}^{n-1} \rightarrow M$

$$\phi(z, w_1, \dots, w_{n-1}) = \exp_{u(z)} \left(\sum_{j=2}^n x_j Z_j(z) + \sum_{j=2}^n y_j J(u(z)) Z_j(z) \right),$$

where $w_j = x_j + iy_j$. There exist neighbourhoods $U \subset M$ and $V \subset \mathbb{C}^n$ such that $\phi : V \rightarrow U$ is a diffeomorphism. Now, since $\phi(z, 0, \dots, 0) = u(z)$ and along the disk $D \times \{(0, \dots, 0)\} \subset \mathbb{C}^n$, it holds that

$$\frac{\partial \phi}{\partial x_j} + J(\phi) \frac{\partial \phi}{\partial y_j} = 0, \quad j = 1, \dots, n,$$

we have that the inverse $\phi^{-1} : U \rightarrow V$ gives the desired coordinate chart. \square

Let us return now to the foliation, say Q , of a neighbourhood in M by embedded J -holomorphic disks given by Lemma 2.1.5. Henceforth it will be constructive to view Q as a map $Q : D \times D \times D^{n-2} \rightarrow \mathbb{C}^n$ and write (ξ, ζ, w) for the associated coordinates, where $\xi, \zeta \in D$ and $w = (w^1, \dots, w^{n-2}) \in D^{n-2}$.

Since the disks of constant (ζ, w) are J -holomorphic the almost complex structure J must decompose, with respect to the splitting $T(D \times D \times D^{n-2}) = TD \oplus TD \oplus TD^{n-2} = \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}^{2n-4}$, as follows:

$$J = \begin{pmatrix} a & b_1 & c_1 \\ 0 & a' & c_2 \\ 0 & b_2 & c_3 \end{pmatrix}.$$

Here $a, a', b_1 \in \mathbb{R}^{2 \times 2}$, $b_2 \in \mathbb{R}^{(2n-4) \times 2}$, $c_1, c_2 \in \mathbb{R}^{2 \times (2n-4)}$ and $c_3 \in \mathbb{R}^{(2n-4) \times (2n-4)}$ are matrix valued functions on D^n such that the condition $J^2 = -I$ is satisfied.

We can further choose coordinates (ξ_1, ζ_1, w_1) such that the disk $\{\xi_1 = 0, w_1 = 0\}$ is J -holomorphic. To see this first remark that from the proof of Lemma 2.1.5 we can find smooth functions $\tau_0, \dots, \tau_{n-2} : D \rightarrow \mathbb{R}^2$ such that $\tau_i(0) = 0$ and the embedding $\zeta \mapsto (\tau_0(\zeta), \zeta, \tau_1(\zeta), \dots, \tau_{n-2}(\zeta))$ is J -holomorphic. By making the change of coordinates

$$(\xi_1, \zeta_1, w_1) := (\xi - \tau_0(\zeta), \zeta, w^1 - \tau_1(\zeta), \dots, w^{n-2} - \tau_{n-2}(\zeta)),$$

we thus have a foliation such that disks of constant (ζ, w) are J -holomorphic as is the disk $\{\xi_1 = 0, w_1 = 0\}$. Finally we can make a further change of coordinates (similarly to Lemma 2.1.6) to (ξ_2, ζ_2, w_2) so that

$$a \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } a'|_{u(D)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Applying this process to the complex directions determined by the $n-2$ components of w_1 , that is, choosing J -holomorphic disk foliations along the directions of w_1 at $x = Q(0, 0, 0)$, we are able to standardize the coordinate along the central disk $\{\xi_2 = 0, w_2 = 0\}$ such that $J|_{\{\xi_2=0, w_2=0\}}$ is a $2n \times 2n$ block matrix with n of the 2×2 matrices

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We use the coordinates defined in this section and slight refinements in Chapter 3.

2.2 Spectral Theory

Since the second half of this thesis concentrates on spectral properties of almost Kähler manifolds we include here some basic definitions and results which will expedite the discussions in Chapter 4.

2.2.1 The Rayleigh Quotient and the Min-Max Theorem

In this section we largely follow the treatment of [9] and [10] wherein a more thorough account can be found.

Let (M, g) be a compact, oriented Riemannian manifold without boundary. First recall that the divergence operator on (M, g) is defined to be the map

$$\operatorname{div} : \Gamma(TM) \rightarrow C^\infty(M), \quad \text{satisfying } (\operatorname{div}(X))dV_g = d(\iota_X(dV_g)),$$

where $\iota_X : \Omega^n(M) \rightarrow \Omega^{n-1}(M)$ denotes contraction by X and dV_g the volume form. The Laplace-Beltrami operator acting on smooth functions is given by,

$$\Delta : C^\infty(M) \rightarrow C^\infty(M), \quad \Delta f = \operatorname{div}(\nabla f),$$

where ∇f denotes the gradient of f . In local coordinates x^i the Laplace-Beltrami operator associated to g has the form,

$$\Delta = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left(\sqrt{|g|} g^{ij} \frac{\partial}{\partial x^j} \right),$$

with $|g| = \det g$ and g^{ij} are the components of the inverse to the matrix with entries $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$.

Remark 2.2.1. We choose to work with oriented Riemannian manifolds above as all manifolds appearing in later chapters will be. Nonetheless the Laplacian can be defined on any Riemannian manifold. Indeed, choosing a local volume form we can define the divergence locally. On the other hand since changing the sign of dV_g does not alter $\operatorname{div}(X)$ the definition extends to the whole manifold and hence one can define the Laplacian as above.

Through a suitable choice of differential form, Stoke's Theorem can be used to obtain the Divergence theorem on a Riemannian manifold.

Theorem 2.2.1 (Divergence Theorem without boundary). *Let (M, g) be a compact Riemannian manifold without boundary, then*

$$\int_M \operatorname{div} X dV_g = 0, \quad \forall X \in \Gamma(TM).$$

For functions $u, v \in C^\infty(M)$, by simply taking the vector field $X = v\nabla u$ in the divergence theorem and using the following identity,

$$\operatorname{div}(v\nabla u) = v\Delta u + \langle \nabla v, \nabla u \rangle,$$

we deduce that Green's formulas also hold.

Corollary 2.2.1 (Green's Identities). *Let (M, g) be a compact Riemannian manifold without boundary and $u, v \in C^\infty(M)$ then*

$$\int_M v\Delta u \, dV_g = - \int_M \langle \nabla v, \nabla u \rangle \, dV_g,$$

and

$$\int_M v\Delta u \, dV_g = \int_M v\Delta u \, dV_g.$$

Notice that for $u, v \in C^\infty(M)$ the first Green's Identity yields that

$$\langle u, -\Delta u \rangle_{L^2(M)} = \int_M |\nabla u|^2 \, dV_g = \|\nabla u\|_{L^2(M)}^2 \geq 0, \quad (2.8)$$

and the second that

$$\langle v, -\Delta u \rangle_{L^2(M)} = \langle u, -\Delta v \rangle_{L^2(M)}. \quad (2.9)$$

We will return to the meaning of these identities shortly.

We let $L^2(M)$ denote the space of measurable functions $u : M \rightarrow \mathbb{R}$ which satisfy,

$$\|u\|_{L^2(M)}^2 = \int_M |u|^2 \, dV_g < \infty.$$

As usual this norm is induced by the following inner product which makes $L^2(M)$ a Hilbert space,

$$\langle u, v \rangle_{L^2(M)} = \int_M uv \, dV_g, \quad u, v \in L^2(M).$$

We denote by $H^k(M)$ the closure of $C^\infty(M)$ with respect to the Sobolev norms

$$\|u\|_{H^k}^2 = \|u\|_{L^2(M)}^2 + \sum_{\ell \leq k} \left\| \nabla^\ell u \right\|_{L^2(M)}^2,$$

where

$$\left\| \nabla^\ell u \right\|_{L^2(M)}^2 = \int_M |\nabla^\ell u|^2 \, dV_g.$$

If the reader is familiar with the more general Sobolev spaces $W^{k,p}(M)$ then one can remark that $H^k(M)$ is an abbreviation for the space of approximable $W^{k,2}$ functions.

Remark 2.2.2. Beware that the Sobolev spaces $H^k(M)$ should not be confused with the cohomology groups of M ; the intended meaning of $H^k(M)$ should be clear from context.

Remark 2.2.3. One must pay attention to the definitions of Sobolev spaces between Riemannian manifolds. Indeed, for compact Riemannian manifolds M, N of dimensions m and n respectively we can give two inequivalent definitions of $W^{k,p}(M, N)$. Firstly we can define it similarly to $H^k(M)$ above, that is, to be the closure of $C^\infty(M, N)$ with respect to the $W^{k,p}$ -norm. On the other hand we can take an isometric embedding $N \hookrightarrow \mathbb{R}^k$ and define a Sobolev space for maps by,

$$W^{1,2}(M, N) := \{u \in W^{1,2}(M, \mathbb{R}^k) \mid u(x) \in N \text{ a.e. } x \in M\}.$$

For $p > m$ these spaces agree, which essentially follows from the Sobolev embedding theorem. Furthermore in the borderline case, $p = m$, Schoen and Uhlenbeck [47, 48] proved that these spaces still agree. But for $p < m$ this is not the case. Consider, for example, the radial projection from the unit ball in \mathbb{R}^3 to its boundary \mathbb{S}^2 . For $2 \leq p < 3$ this is in $W^{1,p}(\mathbb{B}^3, \mathbb{S}^2)$ but does not lie in the closure of $C^\infty(\mathbb{B}^3, \mathbb{S}^2)$ with respect to the $W^{1,p}$ -norm. An enlightening discussion of these spaces is given in [4].

In Chapter 4, to avoid confusion, we shall refer to functions in the closure of $C^\infty(M, N)$ with respect to the $W^{k,p}$ -norm as approximable $W^{k,p}$ functions.

Let us briefly recall some definitions from functional analysis. First that the spectrum of a linear operator $T : \mathcal{D}(T) \subset H \rightarrow H$ defined on a dense subset $\mathcal{D}(T)$ of a Hilbert space H is the set of $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not invertible, where I is the identity operator. Moreover an element λ in the spectrum of T is an eigenvalue if $T - \lambda I = 0_H$, where 0_H denotes the zero operator.

The resolvent of an operator T is $R_\lambda = (T - \lambda I)^{-1}$ for $\lambda \in \mathbb{C}$ not in the spectrum of T . If there exists a λ such that $(T - \lambda I)^{-1}$ is a compact, bounded, linear operator and is defined on a dense subset of the range of T then we say that T has a compact resolvent. Recall that an operator is compact if it maps the unit ball to a precompact set. Using the Spectral Theorem for Compact Operators [44] one can show that if an operator T has a compact resolvent then the spectrum of T is discrete and that any non-zero elements of the spectrum are eigenvalues.

Theorem 2.2.2 ([18, 21]). *Let (M, g) be a compact Riemannian manifold without boundary and Δ the associated Laplace-Beltrami operator defined above, then there exists a unique self-adjoint extension of the Laplacian to a positive, linear operator*

$$-\Delta : H^2(M) \rightarrow L^2(M)$$

satisfying,

$$\langle -\Delta u, v \rangle_{L^2(M)} = \langle \nabla u, \nabla v \rangle_{L^2(M)}, \quad \forall u \in H^2(M), v \in H^1(M).$$

Moreover $-\Delta$ has a compact resolvent and in particular it follows that the spectrum is discrete and has the following properties:

- (i) all eigenvalues are real and have finite multiplicity;
- (ii) ordering eigenvalues as follows, $0 < \lambda_1 \leq \lambda_2 \leq \dots$ we have that $\lambda_k \rightarrow \infty$ as $k \rightarrow \infty$;
- (iii) there exists an orthonormal basis $\{u_k\} \subset L^2(M)$ where each u_k is an eigenfunction corresponding to eigenvalue λ_k .

Some remarks about the proof of this theorem are in order. Firstly, since $C^\infty(M)$ is dense in $L^2(M)$ the identity (2.9) says precisely that $-\Delta$ is formally self adjoint, one can use this fact to further prove that the extension is self-adjoint. Similarly positivity can be seen to arise from the identity (2.8). As mentioned above Theorem 2.2.2 the discreteness of the spectrum follows from the compactness of the resolvent operator. To deduce the compactness of the resolvent we use the energy estimate (see for example [18])

$$\|(-\Delta)^{-1}g\|_{H^1(M)} \leq C \|g\|_{L^2(M)}, \quad \text{for } g \in L^2(M)$$

and since $H^1(M)$ embeds compactly into $L^2(M)$, by the Rellich-Kondrachov Compactness Theorem, we deduce that the resolvent is indeed compact. Notice here that the compactness of M is vital for Rellich-Kondrachov to apply, if the manifold is non-compact then parts of the spectrum may be continuous.

Let u_i, u_j be eigenfunctions associated to distinct eigenvalues λ_i and λ_j respectively. By the second Green's Identity we have,

$$0 = \int_M (u_i \Delta u_j - u_j \Delta u_i) d\text{Vol}_g = (\lambda_i - \lambda_j) \int_M u_i u_j d\text{Vol}_g.$$

Thus we see that distinct eigenspaces are orthogonal with respect to the L^2 inner product. Therefore, we may choose an L^2 -orthonormal sequence $u_0 = (\text{Vol}(M))^{-\frac{1}{2}}, u_1, u_2, \dots$ of eigenfunctions corresponding to the eigenvalues $0, \lambda_1, \lambda_2, \dots$. Then $\{u_i\}$ is a complete orthonormal sequence in $L^2(M)$ and, in particular, we have the so-called Parseval identities.

$$u = \sum_{i=0}^{\infty} \langle u, u_i \rangle_{L^2} u_i, \quad \|u\|_{L^2}^2 = \sum_{i=0}^{\infty} \langle u, u_i \rangle_{L^2}^2, \quad \forall u \in L^2(M).$$

From these considerations it is straightforward to prove Rayleigh's Theorem, for example see [10]. We include a useful variational characterisation in the statement which can also be found in [10].

Theorem 2.2.3 (Rayleigh's Theorem). *Let (M, g) be a compact Riemannian manifold without boundary and Δ the extension of the Laplace-Beltrami operator given by Theorem 2.2.2. Write $0 < \lambda_1 \leq \lambda_2 \leq \dots$ for the eigenvalues and u_1, u_2, \dots for the corresponding L^2 -normalised eigenfunctions. For $k \in \mathbb{N}$ and $E_k := \{u_0, u_1, \dots, u_{k-1}\}^\perp \subset L^2(M)$ it holds that,*

$$\lambda_k = \inf \{ \mathcal{R}_g(u) \mid u \in H^1(M) \cap E_k \}, \quad (2.10)$$

where $\mathcal{R}_g(u)$ is the Rayleigh quotient defined by

$$\mathcal{R}_g(u) = \frac{\|\nabla u\|_{L^2(M)}^2}{\|u\|_{L^2(M)}^2} = \frac{\int_M |\nabla u|^2 d\text{Vol}_g}{\int_M u^2 d\text{Vol}_g}. \quad (2.11)$$

Moreover the infimum is achieved if and only if the function in question is an eigenfunction of λ_k .

Moreover, if for $k \in \mathbb{N}$ we let \mathcal{V}_k be the collection of all $k + 1$ -dimensional subspaces of $C^\infty(M)$, then

$$\lambda_k(g) = \inf_{V \in \mathcal{V}_k} \sup_{u \in V} \mathcal{R}_g(u). \quad (2.12)$$

A useful consequence of this characterisation is that to prove a bound on λ_k it suffices to produce $k + 1$ linearly independent test functions whose Rayleigh quotient satisfies the same bound. Since we will only be considering compact manifolds one can take a constant as one of these test functions reducing the problem to finding k linearly independent test functions. This is the approach used by Kokarev in [33] and the one we take in Chapter 4.

2.2.2 Estimates for Kähler Manifolds

In general, for a given Riemannian manifold (M, g) , one cannot expect to compute eigenvalues explicitly except in very special cases. For example, the eigenvalues of the Laplacian on the sphere with respect to its standard metric can be computed [10] as can those of the Laplacian on \mathbb{P}^m . Despite this one can give estimates on the eigenvalues in terms of geometric quantities associated to (M, g) . For the purpose of this thesis we look only at bounds for Kähler manifolds and this story inevitably starts with Riemann surfaces.

The spectrum of the Laplacian of a Riemann surface is an important invariant and there is a rich history of results. The geometric estimates we are interested in arguably started with the work of Szegő estimating the first eigenvalue of the Laplacian for simply connected domains in \mathbb{R}^2 .

Theorem 2.2.4 ([49]). *Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain with finite area A . The first non-constant Neumann eigenvalue of the Laplacian on Ω satisfies the following bound*

$$\lambda_1(\Omega) \leq C \cdot A^{-1},$$

where $C > 0$ is a computable universal constant.

This work was later generalised by Hersch [28] to give a bound on λ_1 for an arbitrary metric on the 2-sphere which depends only on the area.

Theorem 2.2.5 ([28]). *Let g be a smooth Riemannian metric on \mathbb{S}^2 . Then the first*

eigenvalue of the Laplacian associated to g satisfies the following bound

$$\lambda(\mathbb{S}^2, g) \leq 8\pi \cdot A^{-1},$$

where A is the area of \mathbb{S}^2 with respect to g . Moreover, equality is achieved if and only if g is the round metric on \mathbb{S}^2 .

Berger verified that this estimate also holds for flat metrics on the torus and suggested that it holds for any metric on the torus. This was proven and generalised to any oriented Riemann surface by Yang and Yau [59]. The statement of this result that we give below is chosen to emphasise its similarity to results later in this section and thesis.

Theorem 2.2.6 (Yang-Yau [59], [17]). *Let (M, g) be an orientable Riemann surface of genus γ and area A . Then, there exists a conformal map $\phi : M \rightarrow \mathbb{P}^1$ of degree at most $\left\lceil \frac{\gamma+3}{2} \right\rceil$ and it holds that*

$$\lambda_1(M, g) \leq 8\pi \cdot \deg(\phi) \cdot A^{-1} \leq 8\pi \left\lceil \frac{\gamma+3}{2} \right\rceil A^{-1}.$$

It was pointed out by Berger [3] that such a bound, *i.e.* an upper bound in terms of volume, fails for higher dimensional spheres. On the other hand the non-orientable surface case was considered by Li and Yau [38] where the importance of the conformal class was pointed out, in particular an upper bound for λ_1 in terms of the so-called conformal area is given. These methods lead to what is, as far as the author is aware, the earliest geometric bound on the first eigenvalue of a compact Kähler Manifold (excluding results which apply only to Riemann Surfaces or compact Riemannian manifolds in general).

Theorem 2.2.7 (Li-Yau [38]). *Let M be a compact Kähler Manifold with Kähler form Ω and which admits a meromorphic map into \mathbb{P}^1 . Then*

$$\lambda_1(M) \leq 2V_\Omega(M) \text{Vol}(M, \Omega)^{-1},$$

where $V_\Omega(M) = \inf_f \left\{ \int_M \Omega^{m-1} \wedge f^* \omega_{FS} \mid f : M \rightarrow \mathbb{P}^1 \text{ is meromorphic} \right\}$ and ω_{FS} denotes the Fubini-Study metric defined below.

The Fubini-Study metric on \mathbb{P}^N is the natural metric induced on the quotient $\mathbb{P}^N = \mathbb{S}^{N+1}/\mathbb{S}^1$, in local holomorphic coordinates z_i it has the components

$$(\omega_{FS})_{i\bar{j}} = \frac{1}{\left(1 + \sum_{i=1}^N |z_i|^2\right)^2} \left(\left(1 + \sum_{i=1}^N |z_i|^2\right) \delta_{i\bar{j}} - \bar{z}_i z_j \right). \quad (2.13)$$

Thus Bourguignon, Li and Yau were lead to consider the compact complex manifolds arising from holomorphic maps into projective space, that is algebraic submanifolds.

In a sense one can now see the Kähler class as playing the role of the conformal class for these estimates.

Theorem 2.2.8 (Bourguignon-Li-Yau [7]). *Let M^n be an n -dimensional complex manifold admitting a holomorphic immersion $\Phi : M \rightarrow \mathbb{P}^N$. Suppose that Φ is full in the sense that $\Phi(M)$ is not contained in any hyperplane of \mathbb{P}^N . Then, for any Kähler metric ω on M , the first non-zero eigenvalue $\lambda_1(M, \omega)$ satisfies*

$$\lambda_1(M, \omega) \leq 4n \frac{N+1}{N} d([\Phi], [\omega]),$$

where

$$d([\Phi], [\omega]) := \frac{\int_M \phi^* \omega_{FS} \wedge \omega^{n-1}}{\int_M \omega^n}.$$

A number of years later Arezzo-Ghigi-Loi [1] generalised this result to compact Kähler manifolds admitting globally generated holomorphic line bundles with a stability condition. The main theorem in [1] is the following.

Theorem 2.2.9 (Arezzo-Ghigi-Loi [1]). *Let $E \rightarrow M$ be a holomorphic vector bundle of rank r over a compact Kähler manifold M of complex dimension n . Assume further that E is globally generated and the Gieseker point T_E is stable. Then, for any Kähler metric ω on M the first non-zero eigenvalue $\lambda_1(M, \omega)$ satisfies*

$$\lambda_1(M, \omega) \leq \frac{4\pi h^0(E)}{r(h^0(E) - r)} \cdot \frac{(c_1(E) \smile [\omega]^{n-1}, [M])}{(n-1)! \text{Vol}(M, [\omega])}.$$

Remark 2.2.4. Notice that if $\omega \in 2\pi c_1(L)$ for some line bundle $L \rightarrow M$, then

$$\lambda_1(M, \omega) \leq \frac{2nh^0(E) \deg(E)}{r(h^0(E) - r)c_1(L)^n},$$

where $\deg(E) = c_1(E) \cdot c_1(L)^{n-1}$.

Remark 2.2.5. Rather than understand Gieseker stability fully in the complex algebraic sense it is enough for us to remark that, roughly speaking, a holomorphic, globally generated vector bundle is Gieseker stable if and only if the associated map into the Grassmannian can be moved into a “balanced” condition. A basis of $H^0(E)$ is said to be ω -balanced if and only if (up to multiplication by a constant) the basis is orthonormal with respect to the L^2 inner product induced by the pull-back of the standard metric on the universal subbundle of the Grassmannian and the volume form $\frac{\omega^n}{n!}$. Now a theorem of Wang [57] implies that if E is Gieseker stable then $H^0(E)$ admits an ω -balanced basis.

It is well known from the Kodaira embedding theorem that closed Kähler manifolds which admit a globally generated holomorphic vector bundles can be embedded holomorphically into a complex Grassmannian and hence, via the Plücker embedding,

into some projective space. Arezzo, Ghigi and Loi were able to show that globally generated vector bundles on a Kähler manifold are stable thus arriving at the following generalisation of the Bourguignon-Li-Yau estimate.

Corollary 2.2.2 (Arezzo-Ghigi-Loi [1]). *Let $E \rightarrow M$ be a globally generated holomorphic vector bundle over a compact Kähler manifold M of dimension n . Suppose that $N = h^0(E) = \dim H^0(E)$ and let $\phi_t : M \rightarrow \mathbb{P}^{N-1}$ be the holomorphic embedding arising from Kodaira's embedding theorem. Then, for any Kähler metric g on M , the first eigenvalue of the associated Laplace-Beltrami operator satisfies,*

$$\lambda_1(M, g) \leq \frac{4nN}{N-1} \frac{(\phi_t^*[\omega_{FS}] \smile [\omega_g]^{n-1}, [M])}{([\omega_g]^n, [M])}.$$

Finally we are brought to the most recent results building upon the estimate of Bourguignon, Li and Yau which are due to Kokarev [33]. He uses the min-max characterisation of the eigenvalues of the Laplacian to give an upper bound on the k -th eigenvalue which is linear in k . To obtain this more general result one needs to use cut-off functions to construct linearly independent test functions and hence the explicit constant ends up being sacrificed to obtain this estimate for higher eigenvalues.

Theorem 2.2.10 (Kokarev [33]). *Let (M^n, J) be a closed n -dimensional Kähler manifold and $\phi : M^n \rightarrow \mathbb{P}^m$ a non-trivial holomorphic map. Then, for any Kähler metric g on M^n , the eigenvalues of the Laplace-Beltrami operator Δ_g satisfy,*

$$\lambda_k(M^n, g) \leq C(n, m) d([\phi], [\omega_g]) k, \quad \text{for any } k \geq 1, \quad (2.14)$$

where $C(n, m) > 0$ is a constant depending only on n and m and $d([\phi], [\omega_g])$ is defined by,

$$d([\phi], [\omega_g]) := \frac{\int_M \phi^* \omega_{FS} \wedge \omega_g^{n-1}}{\int_M \omega_g^n}. \quad (2.15)$$

In the case of $n = 1$, i.e. M is a Riemann surface, and $m = 1$ this result in fact recovers the bound of Korevaar [34]. This states that for any Hermitian metric g on a Riemann surface M the Laplace eigenvalues satisfy

$$\lambda_k(M, g) \text{Vol}(M, g) \leq C \deg(\phi) k,$$

where $\phi : M \rightarrow \mathbb{P}^1$ is an arbitrary non-trivial holomorphic map and C is a universal constant. Indeed, for a non-trivial holomorphic map $\phi : M \rightarrow \mathbb{P}^1$ it holds that

$$d([\phi], [\omega_g]) = \deg(\phi) \frac{\text{Vol}(\mathbb{P}^1)}{\text{Vol}(M, g)},$$

from which the estimate of Korevaar follows. Thus Theorem 2.2.10 can be viewed as a generalisation of Korevaar's estimate to higher dimensional Kähler manifolds.

When $\phi : M^n \rightarrow \mathbb{P}^1$, Theorem 2.2.10 can also be seen to be a generalisation of Li and Yau's estimate recalled in Theorem 2.2.7 above. In this case one can again express $d([\phi], [\omega_g])$ in terms of the ratio $\deg(\phi)/\text{Vol}_g(M)$ where $\deg(\phi)$ is taken to be the volume of the generic \mathbb{P}^1 fibre.

Finally we close out this section by recalling that for an n -dimensional Riemannian manifold, (M, g) , the Weyl asymptotic law states that

$$\lambda_k(M, g) \text{Vol}(M, g)^{\frac{1}{n}} \sim C(n)k^{\frac{1}{n}}, \quad \text{as } k \rightarrow +\infty,$$

where $C(n)$ is a constant depending only on n . For $n = 1$ we see that this asymptotic is compatible with (2.14) in the sense that the power of k matches. Of course this is no longer the case for $n > 1$. In fact, as pointed out in [33], estimate (2.14) cannot hold if k is replaced by $k^{\frac{1}{n}}$. Indeed, if this were the case, then in the limit $k \rightarrow +\infty$ one finds that an estimate of the form $\text{Vol}(M, g)^{\frac{1}{n}-1} \leq C(n) \cdot d$ holds with d the numerator of (2.15). Taking any compact Kähler manifold Σ consider the fibration $\phi : \Sigma \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ which forgets the first factor. Working with the product metric $g_\Sigma \oplus g_{\mathbb{P}^1}$ the degree, d , of ϕ is independent of g_Σ from which we see that the $\text{Vol}(M, g)$ estimate above cannot hold. Despite this one can still ask whether an estimate compatible with the Weyl law exists over a given Kähler class.

Chapter 3

J -holomorphic curves from J -anti-invariant forms

Since the 1980s there has been a well known folklore theorem (see [29, 35]) which says that for a generic Riemannian metric on a 4-manifold with positive self-dual second Betti number, the zero set of a self-dual harmonic 2-form is a finite number of embedded circles. It is the starting point of Taubes' attempts, *e.g.* [51], to generalise the identification of Seiberg-Witten invariants and Gromov invariants for symplectic 4-manifolds to general compact oriented 4-manifolds.

Following the philosophy of [60], which is stated as (1.1.1) in §1.1, the above genericity statement for the zero set of a self-dual harmonic 2-form in the smooth category should find its counterpart in the almost complex setting without assuming genericity. It is stated as Question 1.6 in [60] which first appeared in [16]. In this chapter we make this speculation precise and in the process build a local model which allows us to give a higher dimensional version as well. We are also able to apply this local model to study birational invariants of almost complex 4-manifolds.

This chapter is based on [5] which is joint work with Weiyi Zhang.

3.1 Introductory Remarks and an Overview

Let (M^{2n}, J) be an almost complex manifold. The almost complex structure acts on the bundle of real 2-forms Λ^2 as the following involution, $\alpha(\cdot, \cdot) \rightarrow \alpha(J\cdot, J\cdot)$. This involution induces the splitting,

$$\Lambda^2 = \Lambda_J^+ \oplus \Lambda_J^-, \tag{3.1}$$

corresponding to the eigenspaces of eigenvalues ± 1 respectively. The sections of these bundles are called J -invariant and J -anti-invariant 2-forms respectively and the spaces of these sections are denoted by Ω_J^\pm . The bundle Λ_J^- inherits an almost complex structure, still denoted by J , from $J\alpha(X, Y) = -\alpha(JX, Y)$.

On the other hand, for any Riemannian metric g on a 4-manifold, we have the well-known self-dual, anti-self-dual splitting of the bundle of 2-forms,

$$\Lambda^2 = \Lambda_g^+ \oplus \Lambda_g^-. \quad (3.2)$$

When g is compatible with J , *i.e.* $g(Ju, Jv) = g(u, v)$, we have $\Lambda_J^- \subset \Lambda_g^+$. In particular, it follows that a closed J -anti-invariant 2-form is a g -self-dual harmonic form. Hence, a closed J -anti-invariant 2-form is the natural almost complex refinement of a self-dual harmonic form on an almost complex 4-manifold. Following philosophy (1.1.1) our expectation is that the almost complex counterpart of the aforementioned folklore theorem should be that the zero set of a J -anti-invariant 2-form is a J -holomorphic curve.

Since the complex line bundle Λ_J^- can be viewed as a natural generalisation of the canonical bundle of a complex manifold it is instructive to take a brief digression and consider what is known in the complex setting. First recall that the canonical bundle of a complex manifold of complex dimension n is the n -th exterior power $\bigwedge^n \Omega$ of the holomorphic cotangent bundle Ω , notice that $\bigwedge^n \Omega$ is a line bundle. Under the divisor to line bundle correspondence the canonical bundle can be associated to a Weil divisor (up to linear equivalence), say K , the divisor class of K is known as the canonical class and any divisor in this class is known as a canonical divisor. On a complex surface, if α is a closed J -anti-invariant 2-form, then $J\alpha$ is also closed and $\alpha + iJ\alpha$ is a holomorphic $(2,0)$ form. Hence the zero set $\alpha^{-1}(0)$ is a canonical divisor of (M, J) , *e.g.* by the Poincaré-Lelong theorem. This meets our expectations in the case when the almost complex structure is integrable.

In this chapter, we are able to confirm our above speculation for any compact almost complex 4-manifold.

Theorem 3.1.1. *Suppose (M, J) is a closed, connected, almost complex 4-manifold and α is a non-trivial, closed, J -anti-invariant 2-form. Then the zero set, Z , of α supports a J -holomorphic 1-subvariety, Θ_α , in the canonical class K_J .*

We will call the J -holomorphic 1-subvariety Θ_α stated in theorem the *zero divisor* of α .

Here, a closed set $C \subset M$ with finite, nonzero 2-dimensional Hausdorff measure is said to be an irreducible J -holomorphic 1-subvariety [52] if it has no isolated points and if the complement of a finite set of points in C , called the singular points, is a connected smooth submanifold with J -invariant tangent space. A J -holomorphic 1-subvariety is a finite set of pairs $\{(C_i, m_i), 1 \leq i \leq m < \infty\}$, where each C_i is an irreducible J -holomorphic 1-subvariety and each m_i is a positive integer.

The general scheme to prove Theorem 3.1.1 is similar to what is used in [60] where it is proven that the intersection of a compact 4-dimensional pseudoholomorphic subvariety and a compact almost complex submanifold of codimension 2 in a (not necessarily

compact) almost complex manifold is a pseudoholomorphic 1-subvariety. This basic strategy traces back to [30] at least, where it works in complex analytic setting. In the pseudoholomorphic situation, this strategy was worked out by Taubes [50].

More concretely, the plan is to first show that Z has finite 2-dimensional Hausdorff measure, this is done in section 2. The idea is to foliate neighbourhoods of points in Z by J -holomorphic disks. Applying a dimension reduction argument with the help of a unique continuation result, Proposition 3.2.2, we are able to reduce our study to the intersection of Z with J -holomorphic disks. We establish the positivity of such intersections in Lemma 3.2.1 by exhibiting a holomorphic trivialisation of $\Lambda_{\bar{J}}$ over a given J -holomorphic disk. This lemma is the counterpart of Gromov's positivity of intersections of J -holomorphic curves with complex submanifolds of real codimension two, *c.f.* Proposition 2.1.6 and [24].

If, in addition, we can find a "positive cohomology assignment" for Z in the sense of Taubes, which plays the role of intersection number of the set Z with each local disk, we are able to show that Z is a J -holomorphic 1-subvariety by Proposition 6.1 of [50] (stated as Proposition 3.3.1).

Our strategy to associate a positive cohomology assignment to Z is to view J -anti-invariant 2-forms as sections of the bundle $\Lambda_{\bar{J}}$. Now a J -anti-invariant form α defines a 4-dimensional submanifold Γ_{α} in the total space of $\Lambda_{\bar{J}}$ whose intersection with M , as submanifolds of $\Lambda_{\bar{J}}$, describe the zero set of the form. Given a disk in M , whose boundary does not intersect Γ_{α} , we can compose with a section and perturb to obtain a disk $\sigma' : D \rightarrow \Lambda_{\bar{J}}$ which intersects M transversely. Then the oriented intersection number of σ' defines a positive cohomology assignment. A finer study of positive cohomology assignment also gives rise the desired information for the homology class of the zero divisor.

Theorem 3.1.1 could be extended to the sections of bundle $\Lambda_{\mathbb{R}}^{n,0}$ of real parts of $(n, 0)$ forms, which has a natural complex line bundle structure induced by the almost complex structure on M . The space of its sections is denoted $\Omega_{\mathbb{R}}^{n,0}$. We have Theorem 3.4.1, which says that the zero set of a non-trivial closed form in $\Omega_{\mathbb{R}}^{n,0}$ supports a pseudoholomorphic subvariety of real codimension 2 up to Question 3.9 of [60]. The key to establish this result is again a version of Lemma 3.2.1 for the bundle $\Lambda_{\mathbb{R}}^{n,0}$. This is our Lemma 3.4.1.

In Section 3.5 we study the relation of J -anti-invariant forms with birational geometry of almost complex manifolds. Recall we have the cohomology groups [39]

$$H_J^{\pm}(M) = \{\mathfrak{a} \in H^2(M; \mathbb{R}) \mid \exists \alpha \in \mathcal{Z}_J^{\pm} \text{ such that } [\alpha] = \mathfrak{a}\}$$

generalising the real Hodge cohomology groups, where \mathcal{Z}_J^{\pm} are the spaces of closed 2-forms in Ω_J^{\pm} . It is proven in [14] that $H_J^+(M) \oplus H_J^-(M) = H^2(M; \mathbb{R})$ when $\dim_{\mathbb{R}} M = 4$. The dimensions of the vector spaces $H_J^{\pm}(M)$ are denoted as $h_J^{\pm}(M)$.

In [60] it is shown that the natural candidate for generalising birational morphisms

to the almost complex category are degree one pseudoholomorphic maps. Using the local model given by Lemma 3.2.1 together with the foliation-by-disks technique as used to establish Theorem 3.1.1, one can study the extension properties of closed J -holomorphic disks. This gives us Proposition 3.5.1, which should be compared with Hartogs extension for pseudoholomorphic bundles over almost complex 4-manifolds established in [11].

With this Hartogs type extension for closed J -anti-invariant 2-forms in hand, we are able to show the dimension of J -anti-invariant cohomology is a birational invariant.

Theorem 3.1.2. *Let $\psi : (M_1, J_1) \rightarrow (M_2, J_2)$ be a degree 1 pseudoholomorphic map between closed, connected almost complex 4-manifolds. Then $h_{J_1}^-(M_1) = h_{J_2}^-(M_2)$.*

Together with the almost complex birational invariants defined in [11], including plurigenera, Kodaira dimension, and irregularity, we have a rich source of invariants to study the birational geometry of almost complex manifolds.

3.2 Finite 2-dimensional Hausdorff measure

In this section, we assume M is a 4-dimensional closed manifold. The peculiarity of dimension 4 is that the Hodge operator $*_g$ of a Riemannian metric g on M also acts as an involution on Λ^2 . Thus we have the self-dual, anti-self-dual splitting of the bundle of 2-forms

$$\Lambda^2 = \Lambda_g^+ \oplus \Lambda_g^-.$$

On the other hand given an almost complex structure J on M , we also get a splitting of the bundle of 2-forms into J -invariant and J -anti-invariant forms

$$\Lambda^2 = \Lambda_J^+ \oplus \Lambda_J^-.$$

Moreover, we can always choose a compatible g in the sense that g is J -invariant, *i.e.* $g(Ju, Jv) = g(u, v)$. The pair (g, J) induces a J -invariant (in general non-closed) 2-form ω by

$$\omega(u, v) = g(Ju, v).$$

The triple (J, g, ω) defines an almost Hermitian structure. It is straightforward to deduce the decompositions

$$\Lambda_g^+ = \mathbb{R}(\omega) \oplus \Lambda_J^-, \tag{3.3}$$

$$\Lambda_g^- = \mathbb{R}(\omega) \oplus \Lambda_J^+. \tag{3.4}$$

In particular, $\Lambda_J^- \subset \Lambda_g^+$ and it follows that every closed J -anti-invariant form is a harmonic g -self-dual form, *e.g.* Lemma 2.6 of [14].

Also recall that Λ_J^- inherits an almost complex structure, still denoted by J , from $J\beta(X, Y) = -\beta(JX, Y)$. In particular Λ_J^- is a complex line bundle over M .

In this section, we will show that the 2-dimensional Hausdorff measure of the zero locus, Z , of any closed J -anti-invariant 2-form is finite.

To this end let us briefly recall some basic definitions concerning the Hausdorff measure and dimension on compact Riemannian manifolds. For (M, g) a compact Riemannian manifold let d_g be the associated distance function and for any subset $A \subset M$ we denote by $\text{diam}(A)$ the diameter of A ,

$$\text{diam}(A) := \sup\{d_g(p, q) \mid p, q \in A\}, \quad \text{diam}(\emptyset) := 0.$$

For any $A \subset M$ and $\delta > 0$ we define,

$$\mathcal{H}_\delta^k(A) := \inf \left\{ \sum_{i=1}^{\infty} (\text{diam}(U_i))^k \mid A \subset \bigcup_{i=1}^{\infty} U_i, \text{diam}(U_i) < \delta \ \forall i \right\}, \quad k \in [0, \infty),$$

with the infimum being taken over all countable open covers, U_i , of A satisfying $\text{diam}(U_i) < \delta$. The $\mathcal{H}_\delta^k(A)$ are monotone and decreasing in δ and thus the limit as $\delta \rightarrow 0$ exists (although it may be infinite). We can thus define an outer measure by,

$$\mathcal{H}^k(A) := \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^k(A),$$

we call this the k -dimensional Hausdorff measure. Notice that if $\mathcal{H}^k(A) < \infty$ then $\mathcal{H}^\ell(A) = 0$ for all $\ell > k$ and that if $\mathcal{H}^k(A) > 0$ then $\mathcal{H}^\ell(A) = \infty$ for all $\ell < k$. From this we can also define the Hausdorff dimension of a subset $A \subset M$ as,

$$\dim_{\mathcal{H}}(A) := \inf\{k \geq 0 \mid \mathcal{H}^k(A) = 0\},$$

or $\dim_{\mathcal{H}}(A) = \infty$ if $\mathcal{H}^k(A) = 0$ for all $k \geq 0$.

Proposition 3.2.1. *Let (M, J) be a closed, connected, almost complex 4-manifold and suppose that α is a non-trivial, closed, J -anti-invariant 2-form. Then the zero set Z of α is compact, with Hausdorff dimension 2 and finite 2-dimensional Hausdorff measure.*

Remark 3.2.1. Since every closed J -anti-invariant form is a harmonic g -self-dual form for a compatible g , it follows from [2] that the zero locus $Z = \alpha^{-1}(0)$ is a countably 2-rectifiable set. Recall that a subset of an n -dimensional Riemannian manifold M is called countably k -rectifiable if it can be written as a countable union of sets of the form $\phi(X)$, where $X \subset \mathbb{R}^k$ is bounded and $\phi : X \rightarrow M$ is a Lipschitz map. However, it is not clear whether such a set would have finite 2-dimensional Hausdorff measure by [2]. Furthermore it is important for this and future work to have a proof which uses only *pseudoholomorphic properties*.

Considering α as a smooth section of the bundle Λ_J^- the compactness of Z follows immediately from the continuity of α since a closed subset of a compact space is compact. Hence we can cover Z by finitely many balls. We need to show that $C\varepsilon^{-2}$ many

ε -balls will be enough to cover Z . We show this in each ball. These balls may be taken small enough such that they are foliated by J -holomorphic disks as we recalled in the background chapter. Let us cement notation by explicitly recalling the coordinates set up in §2.1.3.

Fix $x \in M$, we can find a neighbourhood U of x and a non-degenerate 2-form Ω on U such that J is compatible with Ω in U . This pair (Ω, J) induce an almost Hermitian metric on U . Now we can identify a geodesic ball centred at x with a ball in \mathbb{R}^4 centred at the origin. Identifying $\mathbb{R}^4 = \mathbb{C}^2$ such that

$$\Omega_x = \omega_0 = dx^1 \wedge dx^2 + dx^3 \wedge dx^4 = \frac{i}{2} (dw^0 \wedge d\bar{w}^0 + dw^1 \wedge d\bar{w}^1).$$

Here we write complex coordinates $(w^0, w^1) = (x^1, x^2, x^3, x^4)$. We may assume that J is an almost complex structure on \mathbb{C}^2 which agrees with the standard complex structure J_0 at the origin.

Let $D_w := \{(\xi, w) \mid |\xi| < \rho\}$, where $w \in D$. Now Lemma 2.1.5 yields a diffeomorphism $Q : D \times D \rightarrow \mathbb{C}^2$, where $D \subset \mathbb{C}^2$ is the disk of radius ρ , such that

- $\forall w \in D$, $Q(D_w)$ is a J -holomorphic submanifold containing $(0, w)$,
- $\forall w \in D$, there exists z depending only on Ω and J such that

$$|(\xi, w) - Q(\xi, w)| \leq z \cdot \rho \cdot |\xi|,$$

- $\forall w \in D$, the derivatives of order m of Q are bounded by $z_m \cdot \rho$, where z_m depends only on Ω and J .

Such diffeomorphisms shall be called J -fibre-diffeomorphisms. It is important to remark that we can change the direction of these disks by rotating the original Gaussian coordinate chart chosen. More precisely given $\kappa \in \mathbb{C}P^1$ we can choose Q such that $Q(D_0)$ is tangent at the origin to the line determined by κ .

Let $u : D \rightarrow M$ be an embedded J -holomorphic disk with $x = u(0)$. We can further choose the coordinate system such that the almost complex structure J behaves particularly well along the image $u(D)$. This is essentially a reformulation of the construction on page 903 of [50] and will be used in Lemma 3.2.1.

Let (ξ, w) be the coordinates associated with the above Q . Since the disks of constant w are J -holomorphic the almost complex structure J must decompose, with respect to the splitting $T(D \times D) = TD \oplus TD = \mathbb{R}^2 \oplus \mathbb{R}^2$, as follows:

$$J = \begin{pmatrix} a & b \\ 0 & a' \end{pmatrix}.$$

Here a, a', b are 2×2 matrix valued functions on $D \times D$ such that the condition $J^2 = -I$ is satisfied.

We can further choose coordinates (ξ_1, w_1) such that $u(D)$ is just $\xi_1 = 0$, at least locally near x . Indeed as remarked previously we can choose the direction the foliation such that $Q(D_0)$ intersects $u(D)$ transversally at $u(0)$. The transversality condition facilitates the application of implicit function theorem to find, after shrinking D if necessary, a smooth map $\tau : D \rightarrow \mathbb{R}^2$ such that $\tau(0) = 0$ and $u(w) = Q(\tau(w), w)$. We let $(\xi_1, w_1) = (\xi - \tau(w), w)$. Thus, in the (ξ_1, w_1) coordinates, the matrix b obeys $b(0, \cdot) = 0$. We can make a further change to coordinates $(\xi_2, w_2) := (g_1(\xi_1, w_1) \cdot \xi_1, g_2(w_1))$, for suitable smooth matrix value functions g_1 and g_2 such that, in addition to the general requirement $J^2 = -I$, we have

$$a \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } a'(0, \cdot) \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

To summarise, the discussion above allows us to take coordinates in a neighbourhood of $u(0)$ such that $J = J_0$ along $u(D)$. Later, we will denote such coordinates, (w_2, ξ_2) , by (x^1, x^2, x^3, x^4) so that $u(D)$ is described by $x^3 = x^4 = 0$ near $u(0)$.

We continue assuming $u : D \rightarrow M$ is an embedded J -holomorphic disk and U is a neighbourhood of $u(0)$ with the coordinates described above. On $u(D) \cap U$ define

$$\phi_0|_{u(D) \cap U} := dx^1 \wedge dx^3 - dx^2 \wedge dx^4. \quad (3.5)$$

This is J -anti-invariant. We notice that

$$-J\phi_0|_{u(D) \cap U} = dx^1 \wedge dx^4 + dx^2 \wedge dx^3, \quad (3.6)$$

where the J refers to the almost complex structure on Λ_J^- .

We can extend ϕ_0 to a section of Λ_J^- on U . Indeed, by shrinking U if necessary, we may assume that Λ_J^- is trivialised over U . So we can take a local basis of Λ_J^- , say $\psi, J\psi$. On $u(D)$ there are functions h_1, h_2 such that

$$\phi_0|_{u(D) \cap U} = h_1\psi|_{u(D) \cap U} + h_2J\psi|_{u(D) \cap U}.$$

Now to extend ϕ_0 we choose any non-zero smooth extensions of h_1 and h_2 to U .

The foliations described above reduce the study of Z to its intersection with embedded J -holomorphic disks. To study such intersections we need to produce an appropriate local trivialisation of Λ_J^- .

Using the almost complex structure on Λ_J^- we can locally choose an orthogonal basis, say, $\phi, J\phi$. We write the J -anti-invariant form α locally in terms of this basis, $\alpha = f\phi + gJ\phi$, where f and g are smooth functions.

Lemma 3.2.1 below establishes a trivialisation for Λ_J^- in which α is holomorphic over an embedded J -holomorphic disk in terms of the chosen basis. This allows us to establish that if a given embedded J -holomorphic disk intersects the zero set non-trivially then the intersection is a finite number of isolated points. Furthermore these

intersections are positive.

Lemma 3.2.1. *Let (M, J) be an almost complex 4-manifold and $u : D \rightarrow M$ a smooth, embedded J -holomorphic disk. Then for any closed, J -anti-invariant 2-form α , there exists a neighbourhood $U \subset M$ of $u(0)$ and a nowhere vanishing $\phi \in \Omega_J^-(U)$ such that for α expressed in terms of the basis $\{\phi, J\phi\}$,*

$$\alpha = f\phi + gJ\phi, \quad (3.7)$$

on U , the function $(f \circ u) + i(g \circ u)$ is holomorphic on $u^{-1}(u(D) \cap U)$.

We will first write α with respect to the local basis ϕ_0 and show that the coefficients satisfy a Cauchy-Riemann type equation. From this point an application of the Carleman Similarity Principle allows us to find a local basis whose coefficients are holomorphic. We only state a weak version of the Carleman Similarity Principle which is enough for our application.

Theorem 3.2.1. *Let $p > 2$ and $B_\varepsilon \subset \mathbb{C}$ for some $\varepsilon > 0$. Suppose that $C_1, C_2 \in L^\infty(B_\varepsilon, \mathbb{C})$ and $v \in W^{1,p}(B_\varepsilon, \mathbb{C})$ is a solution to*

$$\bar{\partial}v(z) + C_1(z)v(z) + C_2(z)\bar{v}(z) = 0. \quad (3.8)$$

Then, for a sufficiently small $\delta > 0$, there exist functions $\Phi \in C^0(B_\delta, \mathbb{C})$ and $\sigma \in C^\infty(B_\delta, \mathbb{C})$ such that $\Phi(z)$ is nowhere zero and on B_δ ,

$$v(z) = \Phi(z)\sigma(z), \quad \bar{\partial}\sigma = 0.$$

Remark 3.2.1. If $C_2 = 0$ then the transformation Φ can be found to depend only on C_1 . But in the general case, Φ will depend on v . This is essentially the hidden reason that our argument would not lead to a divisor-to-section correspondence for J -anti-invariant forms and their divisors even for tamed J .

Proof of Lemma 3.2.1. Take ϕ_0 to be the extension of (3.5) described above and write $\alpha = f_0\phi_0 + g_0J\phi_0$. Since α is closed, we must have

$$0 = d\alpha = df_0 \wedge \phi_0 + f_0 d\phi_0 + dg_0 \wedge J\phi_0 + g_0 d(J\phi_0). \quad (3.9)$$

First remark that the subsequent equalities follow from the definition of ϕ_0 ,

$$\begin{aligned} u^*(\partial_3 \lrcorner \phi_0) &= -ds = u^*(\partial_4 \lrcorner (-J\phi_0)), \\ u^*(\partial_4 \lrcorner \phi_0) &= dt = u^*(\partial_3 \lrcorner (J\phi_0)), \end{aligned}$$

where $z = s + it$ are holomorphic coordinates on (D, J_0) centred at the origin such that $J_0 ds = dt$.

By contracting (3.9) with ∂_3 and pulling back along u we obtain the first of the following expressions of 2-forms on $u^{-1}(U)$. The second is obtained by contracting with ∂_4 instead. Using tilde's to denote quantities which have been pulled back to D we obtain

$$\begin{aligned} d\tilde{f}_0 \wedge ds + \tilde{f}_0 \tilde{\beta} - d\tilde{g}_0 \wedge dt - \tilde{g}_0 \tilde{\gamma} &= -u^* \left[\frac{\partial f_0}{\partial x^3} \phi_0 - \frac{\partial g_0}{\partial x^3} J\phi_0 \right] = 0, \\ -d\tilde{f}_0 \wedge dt + \tilde{f}_0 \tilde{\beta}' - d\tilde{g}_0 \wedge ds - \tilde{g}_0 \tilde{\gamma}' &= -u^* \left[\frac{\partial f_0}{\partial x^4} \phi_0 - \frac{\partial g_0}{\partial x^4} J\phi_0 \right] = 0, \end{aligned}$$

where $\beta := \partial_3 \lrcorner d\phi_0$, $\gamma := \partial_3 \lrcorner dJ\phi_0$, $\beta' := \partial_4 \lrcorner d\phi_0$ and $\gamma' := \partial_4 \lrcorner dJ\phi_0$. The second equality on each line follows from $u^*\phi_0 = u^*J\phi_0 = 0$.

For 1-forms η, λ on D we have the identity $\eta \wedge J_0 \lambda = -J_0 \eta \wedge \lambda$. Thus we can rewrite the equations above as,

$$\begin{aligned} (d\tilde{f}_0 + J_0 d\tilde{g}_0) \wedge ds &= -\tilde{f}_0 \tilde{\beta} + \tilde{g}_0 \tilde{\gamma}, \\ (d\tilde{f}_0 + J_0 d\tilde{g}_0) \wedge dt &= \tilde{f}_0 \tilde{\beta}' - \tilde{g}_0 \tilde{\gamma}'. \end{aligned}$$

Or equivalently in terms of components with respect to the coordinates $z = s + it$ on D ,

$$\begin{aligned} \frac{\partial \tilde{f}_0}{\partial t} + \frac{\partial \tilde{g}_0}{\partial s} &= -\tilde{f}_0 \tilde{\beta}_{12} + \tilde{g}_0 \tilde{\gamma}_{12} \\ \frac{\partial \tilde{f}_0}{\partial s} - \frac{\partial \tilde{g}_0}{\partial t} &= \tilde{f}_0 \tilde{\beta}'_{12} - \tilde{g}_0 \tilde{\gamma}'_{12}. \end{aligned}$$

This is a Cauchy-Riemann type equation for $\tilde{f}_0 + i\tilde{g}_0$.

By Theorem 3.2.1 there exists a $\delta > 0$, a nowhere zero function $\Phi : B_\delta \rightarrow \mathbb{C}$ and a holomorphic function $F : B_\delta \rightarrow \mathbb{C}$ such that

$$F_0 = \Phi F, \tag{3.10}$$

where $F_0 = \tilde{f}_0 + i\tilde{g}_0$. Henceforth we write $F = \tilde{f} + i\tilde{g}$ and $\Phi = \Phi_1 + i\Phi_2$.

Define

$$\phi|_{u(B_\delta)} := (\Phi_1 \circ u^{-1}) \cdot \phi_0 + (\Phi_2 \circ u^{-1}) \cdot J\phi_0$$

and thus

$$J\phi|_{u(B_\delta)} = -(\Phi_2 \circ u^{-1}) \cdot \phi_0 + (\Phi_1 \circ u^{-1}) \cdot J\phi_0.$$

These are nowhere vanishing J -anti-invariant forms on $u(B_\delta)$. Extending them to a neighbourhood of $u(0)$ in M we can thus write

$$\alpha = f\phi + gJ\phi,$$

for some smooth functions $f, g : M \rightarrow \mathbb{R}$. By restricting to $u(B_\delta)$ and applying (3.10),

we have $f \circ u + i g \circ u = F$. The conclusion follows since F is holomorphic. \square

Remark 3.2.2. Above we applied Theorem 3.2.1 to a Cauchy-Riemann equation whose zeroth order term is not a multiple of $\tilde{f}_0 + i\tilde{g}_0$. Thus the basis $\{\phi, J\phi\}$ found in the lemma will depend on α by Remark 3.2.1.

The next lemma establishes a unique continuation result for $Z = \alpha^{-1}(0)$. The result is well known for self-dual harmonic forms [2], alternately it can be regarded as a corollary to Lemma 3.2.1 (*c.f.* proof of Lemma 3.4.2).

Lemma 3.2.2. *Suppose that α is a closed, J -anti-invariant 2-form, then if $\alpha \equiv 0$ on some open set in M , it must vanish identically on the whole of M .*

Proof. For any Riemannian metric g compatible with J , we have $\Lambda_{\bar{J}} \subset \Lambda_g^+$. In particular, any closed J -anti-invariant 2-form is a self-dual harmonic form. Hence any nontrivial, closed, J -anti-invariant 2-form cannot vanish on an open subset of M . In fact, from [2] it is known that such zero set has Hausdorff dimension ≤ 2 . \square

Remark 3.2.3. It is useful to have a proof of the above fact which relies only on pseudoholomorphic properties, for this see the proof of Lemma 3.4.2.

We now have all of the necessary ingredients to locally estimate the Hausdorff measure of the zero set Z in Proposition 3.2.1. In particular, Lemma 3.2.1 serves the role of Lemma 2.2 of [60], *i.e.* Gromov's positivity of intersections of a J -holomorphic disk and a codimension two almost complex submanifold, in the following proof.

Proof of Proposition 3.2.1. This proof follows closely the structure of the proof of Proposition 2.4 in [60].

First we should remark that since M is compact the finiteness of the Hausdorff measure will be independent of the metric we use. Now for any $x \in Z$ we can find a J -fibre-diffeomorphism Q^x of a neighbourhood of x in M . By compactness we can choose finitely many of these diffeomorphisms, say Q^{x_i} , covering Z and such that the disks are all of the same radius. We show that each $Z \cap Q^{x_i}(D \times D)$ has finite 2-dimensional Hausdorff measure.

Pick $x \in Z$ and write Q for Q^x . For each $w \in D$ we know that $Q(D_w)$ intersects Z in finitely many points if it is not totally contained in Z , this is by Lemma 3.2.1. We claim that there are only finitely many $w \in \bar{D}$ such that $Q(D_w) \subset Z$.

Suppose that this is not the case. Then we may assume without loss of generality that 0 is an accumulation point of w . We now foliate a neighbourhood of x by J -holomorphic disks transverse to $Q(D_0)$, whereby producing an open neighbourhood M which is contained in Z . Since this contradicts Lemma 3.2.2 we will then have the claim.

As before take Gaussian coordinates centred at x but now so that $(0, w')$ is identified with $Q(D_0)$. We choose a J -fibre-diffeomorphism $Q' : D' \times D' \rightarrow \mathbb{C}^2$, where D' denotes the disk in \mathbb{C} of radius $\rho' < \rho$, such that

- $\forall w' \in D'$, $Q'(D'_{w'})$ is a J -holomorphic submanifold containing $(0, w')$,
- $\forall w' \in D'$, there exists z depending only on Ω and J such that

$$|(\xi', w') - Q'(\xi', w')| \leq z \cdot \rho' \cdot |\xi'|,$$

- $\forall w' \in D'$, the derivatives of order m of Q' are bounded by $z_m \cdot \rho'$, where z_m depends only on Ω and J .

So all of the disks $Q'(D'_{w'})$ are transverse to $Q(D_0)$. As being transverse is an open condition we have that $Q'(D'_{w'})$ are transverse to $Q(D_w)$ for all $|w| < \varepsilon$. Thus the intersection points of $Q'(D'_{w'})$ and Z are not isolated and so, by Lemma 3.2.1, $Q'(D'_{w'}) \subset Z$. So $Q'(D' \times D') \subset Z$ and since $Q'(D' \times D')$ covers an open neighbourhood of x we have the desired contradiction.

Now we claim that Q may be chosen so that none of the J -holomorphic disks are contained in Z . In fact we show that there are only finitely many complex directions of $T_x M$ such that there are J -holomorphic disks tangent to it and contained in Z . With this the claim follows by rotating the Gaussian coordinate system we chose initially.

Suppose that there are infinitely many such directions. Since the directions in $T_x M$ are parametrised by $\mathbb{C}P^1$ there is at least one accumulative direction v . Choose the Gaussian coordinate system so that $Q(D_0)$ is transverse to v , and hence $Q(D_w)$ are transverse to v for small $|w| < \varepsilon$. This is a contradiction with Lemma 3.2.1 and Lemma 3.2.2 since the intersection numbers of $Q(D_w) \cap Z$ are infinite for $|w| < \varepsilon$.

Hence if we fix x then we can choose a complex direction such that there is no J -holomorphic curve in Z tangent to it. By the perturbative nature of J -fibre diffeomorphisms we can choose Gaussian coordinates and a J -fibre diffeomorphism so that no $Q(D_w)$ is contained in Z for w sufficiently close to 0.

Finally we are able to estimate the Hausdorff measure of the compact set $Z \cap Q(\bar{D} \times \bar{D})$. First remark that, by shrinking D if necessary, we may assume without loss of generality that the distortion of Q on the domain $2D \times 2D$ is bounded by some constant $C > 0$. Also note that, by our choice of Q , for each $w \in \bar{D}$ the set $Z \cap Q(\bar{D}_w)$ is a finite set of points.

Define,

$$g : \bar{D} \rightarrow \mathbb{N} \cup \{0\}, \quad w \mapsto \#(Z \cap Q(\bar{D}_w)).$$

Clearly this is an upper semi-continuous function and hence achieves a maximal value, say N , at some point $w \in \bar{D}$. Since each intersection point contributes positively by Lemma 3.2.1, we know $Z \cap Q(\bar{D}_w)$ contains at most N points for all $w \in \bar{D}$. By the Vitali covering lemma we can take a finite cover of the compact set $Z \cap Q(\bar{D} \times \bar{D})$ by balls of radius ε such that L of these balls are disjoint and the union of L concentric balls with radius dilated by a factor of 3 cover. By our distortion assumption each ε ball intersects $Q(2\bar{D}_w)$ in an open set of area bounded above by $\pi C^2 \varepsilon^2$. The coarea

formula then yields,

$$N\pi C^2 \varepsilon^2 \cdot \pi C^2 (2\rho)^2 > \frac{1}{2} L\pi^2 \varepsilon^4.$$

Hence there is a constant $C' > 0$ such that there can be no more than $C'\varepsilon^{-2}$ balls of radius 3ε covering $Z \cap Q(\bar{D} \times \bar{D})$. This finishes the proof. \square

3.3 Positive cohomology assignment

In this section, we will finish the proof of Theorem 3.1.1.

To establish that the zero set of a closed J -anti-invariant 2-form supports a J -holomorphic curve we use a criteria due to Taubes [50], this is Proposition 3.3.1 below. The strategy underpinning the proof Taubes gives dates back to the work of King [30] at least. The right classical analogy is the following question:

3.3.1. *Let $C \subset \mathbb{C}^2$ be a codimension 2 submanifold with positive local intersection index with all complex lines. Then, is C complex analytic?*

The answer is affirmative and follows by representing C near a point as a graph over its tangent space. If the tangent space is not complex then one can find a complex line which has negative intersection index with C at the point. We wish to apply this style of argument to sets which are not, a priori, oriented submanifolds and hence we cannot directly use the local intersection index.

To this end let us recall the notion of positive cohomology assignment, introduced in [50]. We assume (X, J) is an almost complex manifold, and $C \subset X$ is a set. Let $D \subset \mathbb{C}$ be the standard unit disk. A map $\sigma : D \rightarrow X$ is called *admissible* if C intersects the closure of $\sigma(D)$ inside $\sigma(D)$. Next we define the notion of a positive cohomology assignment to C , which is extracted from section 6.1(a) of [50].

Definition 3.3.1. *A positive cohomology assignment to the set C is an assignment of an integer, $I(\sigma)$, to each admissible map $\sigma : D \rightarrow X$ meeting the following criteria:*

1. *If $\sigma : D \rightarrow X \setminus C$, then $I(\sigma) = 0$.*
2. *If $\sigma_0, \sigma_1 : D \rightarrow X$ are admissible and homotopic via an admissible homotopy (a homotopy $h : [0, 1] \times D \rightarrow X$ where C intersects the closure of $\text{Image}(h)$ inside $\text{Image}(h)$), then $I(\sigma_0) = I(\sigma_1)$.*
3. *Let $\sigma : D \rightarrow X$ be admissible and let $\theta : D \rightarrow D$ be a proper, degree k map. Then $I(\sigma \circ \theta) = k \cdot I(\sigma)$.*
4. *Suppose that $\sigma : D \rightarrow X$ is admissible and that $\sigma^{-1}(C)$ is contained in a disjoint union $\cup_i D_i \subset D$ where each $D_i = \theta_i(D)$ with $\theta_i : D \rightarrow D$ being an orientation preserving embedding. Then $I(\sigma) = \sum_i I(\sigma \circ \theta_i)$.*
5. *If $\sigma : D \rightarrow X$ is admissible and a J -holomorphic embedding with $\sigma^{-1}(C) \neq \emptyset$, then $I(\sigma) > 0$.*

It is constructive to compare this definition with the local intersection index for oriented submanifolds given in §2.1.2. Indeed, in the situation where the set in question, C , is the zero set of a closed J -anti-invariant 2-form our expectation is that it supports a J -holomorphic curve. If this is the case then an open dense subset of C , say \tilde{C} , is a real, oriented, 2-dimensional submanifold of X . Now admissible disks are those which intersect \tilde{C} transversally and the local intersection index defines a positive cohomology assignment.

The following is Proposition 6.1 of [50], which will be used to prove Theorem 3.1.1.

Proposition 3.3.1. *Let (X, J) be a 4-dimensional almost complex manifold and let $C \subset X$ be a closed set with finite 2-dimensional Hausdorff measure and a positive cohomology assignment. Then C supports a compact J -holomorphic 1-subvariety.*

Recall from [46] that a real $2p$ -current C in M is an almost complex integral cycle if it satisfies:

- (i) *Rectifiability:* There exists an at most countable union of disjoint oriented C^1 $2p$ -submanifolds, say $C = \bigcup_i N_i$, and an integer multiplicity $\theta \in L_{loc}^1(C)$ such that for any compactly supported $2p$ -form ψ on M one has,

$$C(\psi) = \sum_i \int_{N_i} \theta \psi.$$

- (ii) *Closedness:* $\partial C = 0$.
- (iii) *Almost Complex:* For \mathcal{H}^{2p} -a.e. point $x \in C$, the approximate tangent plane T_x to the rectifiable set C is invariant under the almost complex structure J .

The proof of Proposition 3.3.1 is divided into two parts. Firstly, Taubes proves that an open dense subset of the set C is a Lipschitz submanifold of X . From this it follows, in particular, that C is an almost complex integral 2-cycle. The second step is to prove that any almost complex integral 2-cycle is in fact a J -holomorphic subvariety. This in fact follows from Almgrens big regularity paper but Taubes [50] provides a proof without recourse to this result. In fact this second step was generalised to higher dimensions by Tian-Rivi re [46], namely it is proven that any almost complex integral 2-cycle in a $2m$ -dimensional almost complex manifold satisfying the locally symplectic property may be viewed as a J -holomorphic subvariety.

Now we shall assign an appropriate positive cohomology assignment to the set $Z = \alpha^{-1}(0)$ for admissible maps. To do this it is convenient to understand J -anti-invariant 2-forms as a smooth sections of the complex line bundle Λ_J^- over M . We shall denote such a section associated to α by $\Gamma_\alpha : M \rightarrow \Lambda_J^-$.

Let $\sigma : D \rightarrow M$ be an admissible map and α a J -anti-invariant 2-form. We assign an integer $I_\alpha(\sigma)$ as follows. Since σ is admissible with respect to the zero set $Z = \alpha^{-1}(0)$, the closure of the image of the composition $\Gamma_\alpha \circ \sigma(D)$ intersects the compact manifold

M , viewed as a submanifold of the total space of the bundle Λ_J^- , inside $\Gamma_\alpha \circ \sigma(D)$. In other words, $\Gamma_\alpha \circ \sigma : D \rightarrow \Lambda_J^-$ is admissible with respect to $M \subset \Lambda_J^-$. There exists an arbitrarily small perturbation of $\Gamma_\alpha \circ \sigma$ which produces a map σ' , homotopic to $\Gamma_\alpha \circ \sigma$ through admissible maps, such that σ' is transverse to M . The set T of intersection points of $\sigma'(D)$ with M is a finite set of signed points. We define $I_\alpha(\sigma)$ to be the sum of these signs.

We now check I_α is a positive cohomology assignment when α is a closed J -anti-invariant 2-form. In particular, the independence of the perturbations we have chosen follows from the assertion (2) of Definition 3.3.1.

Proposition 3.3.2. *Suppose α is a non-trivial closed J -anti-invariant 2-form. The assignment $I_\alpha(\sigma)$ to an admissible map $\sigma : D \rightarrow M$ defines a positive cohomology assignment to $Z = \alpha^{-1}(0)$.*

Proof. We will check the assertions (1)-(5) of Definition 3.3.1 in the following.

If $\sigma(D) \cap \alpha^{-1}(0) = \emptyset$, then $\Gamma_\alpha \circ \sigma(D) \cap M = \emptyset$, which implies $I_\alpha(\sigma) = 0$. This is assertion (1).

Showing assertion (2) is equivalent to showing the following. Let $\sigma'_t : D \rightarrow \Lambda_J^-$, $t \in [0, 1]$, be admissible maps with respect to M . Let σ'_0 and σ'_1 intersect M transversely. Then the intersection numbers (*i.e.* the corresponding sums of the signed intersection points T) $\sigma'_0 \cdot M = \sigma'_1 \cdot M$.

To show this, we look at the admissible homotopy $\sigma' : D \times I \rightarrow \Lambda_J^-$, where $\sigma'(x, t) = \sigma'_t(x)$. Its boundary map $\partial\sigma' : S^2 \rightarrow \Lambda_J^-$ is homotopic to zero. Hence $\partial\sigma' \cdot M = 0$. Since σ'_t are admissible, M intersects $\partial\sigma'$ only at $\sigma'_0(D)$ and $\sigma'_1(D)$. Moreover, $\partial\sigma'$ induces the reverse orientation at $\sigma'_1(D)$. Hence, $\sigma'_0 \cdot M - \sigma'_1 \cdot M = \partial\sigma' \cdot M = 0$. This implies Definition 3.3.1(2), *i.e.* $I_\alpha(\sigma_0) = I_\alpha(\sigma_1)$ if σ_0 and σ_1 are connected via an admissible homotopy.

To show assertion (3), we first choose an admissible map $\sigma' : D \rightarrow \Lambda_J^-$ (with respect to M) transverse to M which is perturbed from $\Gamma_\alpha \circ \sigma$. We can also find a small perturbation θ' of the degree k map $\theta : D \rightarrow D$ such that there is no critical value of θ' mapping to M by σ' . Hence the sum of the signs of the intersection points of $\sigma' \circ \theta' : D \rightarrow \Lambda_J^-$ is k times that of $\sigma' : D \rightarrow \Lambda_J^-$. Since the number I_α is independent of the choice of perturbations by assertion (2), we thus have $I_\alpha(\sigma \circ \theta) = k \cdot I_\alpha(\sigma)$.

For assertion (4), we choose a perturbation $\sigma' : D \rightarrow \Lambda_J^-$ of $\Gamma_\alpha \circ \sigma$ such that $\sigma'|_{D - \cup_i D_i} = \Gamma_\alpha \circ \sigma|_{D - \cup_i D_i}$. Hence $I_\alpha(\sigma) = \sum_i I_\alpha(\sigma \circ \theta_i)$.

For the last assertion, let $\sigma : D \rightarrow M$ be an admissible embedded J -holomorphic disk. For each intersection point $p \in \sigma^{-1}(\sigma(D) \cap Z)$, we can choose a small neighbourhood $D_p \subset D$ such that, for a certain trivialisation of the complex line bundle Λ_J^- over an open neighbourhood $U_p \subset M$ containing $\sigma(D_p)$, the composition $\Gamma_\alpha \circ \sigma$ is a holomorphic function over D_p by Lemma 3.2.1. Hence, if we perturb this holomorphic function to a nearby one, we will get a holomorphic function with single zeros. Thus, without loss, we can assume p is such a single zero. At $\Gamma_\alpha \circ \sigma(p)$, the tangent space has

the following splitting regarding the orientation

$$T_{\Gamma_{\alpha \circ \sigma(p)}} \Lambda_J^- = \Lambda_J^-|_{\sigma(p)} \oplus T_{\sigma(p)}(U_p) = \Lambda_J^-|_{\sigma(p)} \oplus \sigma_*(T_p D_p) \oplus T_{\sigma(p)}(U_p)/\sigma_*(T_p D_p).$$

Here, the fibre of the bundle Λ_J^- is oriented by a local basis $\{\phi, J\phi\}$ as in Section 3.2. Since D_p is a J -holomorphic disk in U_p , the vector space $T_{\sigma(p)}(U_p)/\sigma_*(T_p D_p)$ is a natural complex plane. Hence, the sign associated to the intersection point $\sigma(p)$ is $+1$. This confirms assertion (5). \square

The assignment I_α satisfies the assertions Definition 3.3.1 (1)-(4) for any J -anti-invariant 2-forms. The assumption that α is a closed J -anti-invariant 2-form is only used to show assertion (5).

Before we complete the proof of Theorem 3.1.1, we recall that given a J -holomorphic subvariety $\Theta = \{(C_i, m_i)\}$, there is a natural positive cohomology assignment for its support $|\Theta| = \cup C_i$. Let $C_i = \phi_i(\Sigma_i)$ where each Σ_i is a compact connected complex curve and $\phi_i : \Sigma_i \rightarrow M$ is a J -holomorphic map embedding off a finite set. When $\sigma : D \rightarrow M$ is admissible, there is an arbitrarily small perturbation, σ' , of σ which is homotopic to σ through admissible maps and is transverse to ϕ_i . Each fibre product $T_i := \{(x, y) \in D \times \Sigma_i | \sigma'(x) = \phi_i(y)\}$ is a finite set of signed points of $D \times \Sigma$. We associate weight m_i to each signed point in T_i . The weighted sum of these signs in $\cup T_i$ is a positive cohomology assignment IS_Θ .

Conversely, once a positive cohomology assignment I is given as in Proposition 3.3.1 and $C = \cup C_i$. Then we can associate the positive weight m_i to C_i as $I(\sigma)$ where σ is a J -holomorphic disk intersecting transversally to C_i at a smooth point. The cohomology assignment IS_Θ for the subvariety $\Theta = \{(C_i, m_i)\}$ obtained in this way is equal to the original I .

We will now prove Theorem 3.1.1.

Proof of Theorem 3.1.1. The zero set $Z = \alpha^{-1}(0)$ is a closed set with finite 2-dimensional Hausdorff measure. By Proposition 3.3.2, Z could be endowed with a positive cohomology assignment, $I_\alpha(\sigma)$, for each admissible map $\sigma : D \rightarrow M$. Hence, by Proposition 3.3.1, the zero set $Z = \alpha^{-1}(0)$ supports a J -holomorphic 1-subvariety. Let Θ_α be the J -holomorphic 1-subvariety determined in the manner described above by the cohomology assignment I_α .

The assignment $I_\alpha(\sigma)$ for an admissible map $\sigma : D \rightarrow M$ could be understood in the following equivalent way. We look at the disk $\sigma(D) \subset M \subset \Lambda_J^-$ and the section $\Gamma_\alpha(M)$ inside the total space of the bundle Λ_J^- . Then we perturb the section Γ_α to another one $\Gamma_{\alpha'}$ where α' is a J -anti-invariant 2-form, such that $\Gamma_{\alpha'}$ is transverse to $\sigma(D)$. Moreover, we require $\Gamma_{\alpha'}$ is homotopic to Γ_α through sections α_t such that $\alpha_t^{-1}(0) \cap \partial\sigma = \emptyset, \forall t \in [0, 1]$. The set T' of intersection points of $\sigma(D)$ and $\Gamma_{\alpha'}(M)$ is a finite set of signed points. Suppose σ is of degree k onto its image. Then our $I_\alpha(\sigma)$ is k times the sum of these signs in T' .

When we choose α' such that $\Gamma_{\alpha'}(M) \pitchfork M$ inside the total space of Λ_J^- , we know $\Gamma_{\alpha'}(M) \cap M$ is a smooth submanifold of M representing the Euler class of the bundle Λ_J^- . By Proposition 4.3 of [60], it is the canonical class K_J of the almost complex manifold (M, J) . The sign of each point in T' is equal to the one calculated from the intersection of $\sigma(D)$ with $\Gamma_{\alpha'}(M) \cap M$ inside M if we orient the fibre of the bundle Λ_J^- by a local basis $\{\phi, J\phi\}$ as in Section 3.2.

Any homology class $\xi \in H_2(M, \mathbb{Z})$ is representable by an embedded submanifold, the above claim just implies $\xi \cdot [\Theta_\alpha] = \iota_*(\xi) \cdot [M]$ as integers. Here $\iota_*(\xi)$ denotes the induced class in the second Borel-Moore homology of the total space of Λ_J^- and the latter product is understood as the intersection pairing in Borel-Moore homology. The homology class $[\Theta_\alpha]$ is determined by the intersection pairing with all the classes in $H_2(M, \mathbb{Z})$. As explained in the last paragraph, $\xi \cdot [\Theta_\alpha] = \xi \cdot [\Theta_{\alpha'}] = \xi \cdot K_J, \forall \xi \in H_2(M, \mathbb{Z})$. Hence Θ_α is a J -holomorphic 1-subvariety in the canonical class K_J . \square

The J -holomorphic 1-subvariety Θ_α determined by the positive cohomology assignment I_α corresponding to the closed J -anti-invariant form α is called the *zero divisor* of α .

Finally, we remark that the zero locus $Z = \alpha^{-1}(0)$ is exactly where α is degenerate. In particular, it implies α is almost Kähler on $M \setminus Z$ if α is a closed J -anti-invariant 2-form. It is direct to see from the local expression that the zero locus is exactly the points where α is degenerate. Indeed, for any point $p \in (M, J)$, the tangent space is identified with a 4-dimensional real vector space along with a complex structure J_p . Let x_1, x_2, y_1, y_2 be coordinates centered at p such that $J_p dx_1 = -dy_1$ and $J_p dx_2 = -dy_2$. Now $(\Lambda_J^-)_p$ is spanned by two non-degenerate 2-forms

$$\beta = dx_1 \wedge dx_2 - dy_1 \wedge dy_2, \quad J_p \beta = dx_1 \wedge dy_2 + dy_1 \wedge dx_2.$$

If $\alpha_p = a\beta + bJ_p\beta$ is degenerate, then there exists an $X \in T_p M$ such that $\beta(aX + bJ_p X, \cdot) = 0$. Since β is non-degenerate, we must have $a = b = 0$.

Since the first Chern class $c_1(M \setminus Z, J) = 0$, we know $M \setminus Z$ is an open symplectic Calabi-Yau 4-manifold when α is a closed J -anti-invariant 2-form. If the almost complex structure J is compatible with (or tamed by) a symplectic form on M , we would like to know whether $M \setminus Z$ is a complex symplectic manifold.

3.4 Higher dimensions

Our argument can be applied to sections of the canonical bundle in higher dimensions. Let (M, J) be a closed connected almost complex $2n$ -manifold. As in the four dimensional case there is a natural generalisation of the canonical bundle.

Indeed, first we remark that for an almost complex 4-manifold the canonical bundle Λ_J^- can be viewed as either the bundle of J -anti-invariant 2-forms or as the bundle of the real parts of $(2, 0)$ -forms, *i.e.* $\Lambda_J^- = (\Lambda^{2,0} \oplus \Lambda^{0,2}) \cap \Lambda^2$. Thus one is lead to consider

the line bundle of real parts of $(n, 0)$ forms on an almost complex $2n$ -manifold to be the canonical bundle. We will denote this bundle by $\Lambda_{\mathbb{R}}^{n,0}$. The space of its sections is denoted by $\Omega_{\mathbb{R}}^{n,0}$.

The almost complex structure J on M induces a complex line bundle structure on $\Lambda_{\mathbb{R}}^{n,0}$, we still denote the almost complex structure on $\Lambda_{\mathbb{R}}^{n,0}$ by J . Indeed, J on $\Lambda_{\mathbb{R}}^{n,0}$ can be described concretely by its action on a section β as follows,

$$J\beta(X_1, X_2, \dots, X_n) := -\beta(JX_1, X_2, \dots, X_n).$$

Using the argument given over the previous two sections we are able to prove the following.

Theorem 3.4.1. *Let (M, J) be a closed, connected almost complex $2n$ -manifold and α a non-trivial, closed form in $\Omega_{\mathbb{R}}^{n,0}$. Then the zero set $Z := \alpha^{-1}(0)$ is a set of finite $(2n - 2)$ -dimensional Hausdorff measure admitting a positive cohomology assignment.*

This naturally asks for a generalisation of Proposition 3.3.1 which we phrase as the following question (Question 3.9 in [60]).

Question 3.4.1. Let (M, J) be a closed, connected almost complex $2n$ -manifold and $C \subset M$ a closed set with finite $(2n - 2)$ -dimensional Hausdorff measure and admitting a positive cohomology assignment. Does C support a compact J -holomorphic subvariety of complex dimension $n - 1$?

If the answer to this question is affirmative then Theorem 3.4.1 would imply that the zero set of a closed form α in $\Omega_{\mathbb{R}}^{n,0}$ supports a J -holomorphic $(n - 1)$ -subvariety in the canonical class. Recall a J -holomorphic k -subvariety is a finite set of pairs $\{(V_i, m_i), 1 \leq i \leq m\}$, where each V_i is an irreducible J -holomorphic k -subvariety and each m_i is a positive integer. Here an irreducible J -holomorphic k -subvariety is the image of a somewhere immersed pseudoholomorphic map $\phi : X \rightarrow M$ from a compact connected smooth almost complex $2k$ -manifold X .

The key to the proof of Theorem 3.4.1 is to establish foliations by J -holomorphic disks in higher dimensions, this is the content of §2.1.3. Indeed, given any point $x \in M$ we can find a local Gaussian coordinate chart and hence Lemma 2.1.5 gives a foliation by J -holomorphic disks in a neighbourhood of x .

Fix $x \in M$, we can find a neighbourhood U of x and a non-degenerate 2-form Ω on U such that J is compatible with Ω in U . This pair (Ω, J) induce an almost Hermitian metric on U . Now we can identify a geodesic ball centred at x with a ball in \mathbb{R}^{2n} centred at the origin. Identifying $\mathbb{R}^{2n} = \mathbb{C}^n$ such that

$$\begin{aligned} \Omega_x = \omega_0 &= dx^1 \wedge dx^2 + \dots + dx^{2n-1} \wedge dx^{2n} \\ &= \frac{i}{2} (dz^0 \wedge d\bar{z}^0 + \dots + dz^{n-1} \wedge d\bar{z}^{n-1}). \end{aligned}$$

Here we write complex coordinates $(z^0, \dots, z^{n-1}) = (x^1, x^2, \dots, x^{2n-1}, x^{2n})$. We may as well assume that J is an almost complex structure on \mathbb{C}^n which agrees with the standard complex structure J_0 at the origin.

Lemma 2.1.5 gives a J -fibre diffeomorphism Q and let (ξ, ζ, w) be the associated coordinates, where $\xi, \zeta \in D$ and $w = (w^1, \dots, w^{n-2}) \in D^{n-2}$. Since the disks of constant (ζ, w) are J -holomorphic the almost complex structure J must decompose, with respect to the splitting $T(D \times D \times D^{n-2}) = TD \oplus TD \oplus TD^{n-2} = \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}^{2n-4}$, as follows:

$$J = \begin{pmatrix} a & b_1 & c_1 \\ 0 & a' & c_2 \\ 0 & b_2 & c_3 \end{pmatrix}.$$

Here $a, a', b_1 \in \mathbb{R}^{2 \times 2}$, $b_2 \in \mathbb{R}^{(2n-4) \times 2}$, $c_1, c_2 \in \mathbb{R}^{2 \times (2n-4)}$ and $c_3 \in \mathbb{R}^{(2n-4) \times (2n-4)}$ are matrix valued functions on D^n such that the condition $J^2 = -I$ is satisfied.

We can further choose coordinates (ξ_1, ζ_1, w_1) such that $u(D)$ is the disk $\{\xi_1 = 0, w_1 = 0\}$, at least locally near $x = u(0)$. To see this first remark that by the final part of Lemma 2.1.5 the J -fibre diffeomorphism may be chosen so that $Q(D_0)$ intersects $u(D)$ transversally at $u(0)$. The transversality condition facilitates the application of implicit function theorem to find, after shrinking D if necessary, smooth functions $\tau_0, \dots, \tau_{n-2} : D \rightarrow \mathbb{R}^2$ such that $\tau_i(0) = 0$ and $u(\zeta) = (\tau_0(\zeta), \zeta, \tau_1(\zeta), \dots, \tau_{n-2}(\zeta))$. By making the change of coordinates

$$(\xi_1, \zeta_1, w_1) := (\xi - \tau_0(\zeta), \zeta, w^1 - \tau_1(\zeta), \dots, w^{n-2} - \tau_{n-2}(\zeta)),$$

we ensure that $u(D)$ is described by $\{\xi_1 = 0, w_1 = 0\}$ in a neighbourhood of x . Thus in the (ξ_1, ζ_1, w_1) coordinates we must have $b_1 = 0$ and $b_2 = 0$ along the disk $u(D)$. Finally we can make a further change of coordinates to (ξ_2, ζ_2, w_2) so that

$$a \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } a'|_{u(D)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Applying this process to the complex directions determined by the $n-2$ components of w_1 , that is, choosing J -holomorphic disk foliations along the directions of w_1 at $x = u(0)$ and choosing $u(D)$ to be in the center as above, we are able to standardise the coordinate at $u(D)$ such that $J|_{u(D)}$ is a $2n \times 2n$ block matrix with n 2×2 matrices

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Henceforth we let $(z^1, \dots, z^n) = (x^1, x^2, \dots, x^{2n-1}, x^{2n})$ denote the coordinates (ζ_2, ξ_2, w_2) so that $u(D)$ is defined by $z^2 = \dots = z^n = 0$.

We continue assuming $u : D \rightarrow M$ is an embedded J -holomorphic disk and U is a

neighbourhood of $u(0)$ with the coordinates described above. On $u(D) \cap U$ define

$$\phi_0|_{u(D) \cap U} := \Re [dz^1 \wedge \cdots \wedge dz^n] = \Re [(dx^1 + idx^2) \wedge \cdots \wedge (dx^{2n-1} + idx^{2n})].$$

We can extend ϕ_0 to a form in $\Lambda_{\mathbb{R}}^{n,0}(U)$. Indeed, by shrinking U if necessary, we may assume that $\Lambda_{\mathbb{R}}^{n,0}$ is trivialised over U . So we can take an orthogonal local basis of $\Lambda_{\mathbb{R}}^{n,0}$, say $\psi, J\psi$. On $u(D)$ there are smooth functions h_1, h_2 such that

$$\phi_0|_{u(D) \cap U} = h_1\psi|_{u(D) \cap U} + h_2J\psi|_{u(D) \cap U}.$$

Now to extend ϕ_0 we choose any non-zero smooth extensions of h_1 and h_2 to U .

A straightforward calculation shows that

$$\begin{aligned} J\phi_0|_{u(D) \cap U} &= \Re [idz^1 \wedge \cdots \wedge dz^n] \\ &= \Re [(-dx^2 + idx^1) \wedge \cdots \wedge (dx^{2n-1} + idx^{2n})]. \end{aligned}$$

We can establish positivity of intersections of the zero set with embedded J -holomorphic disks. With the coordinates described above one may derive some generalised Cauchy-Riemann equations for the coefficients of α as in Lemma 3.2.1. Applying Carleman Similarity Principle we obtain the following lemma.

Lemma 3.4.1. *Let (M, J) be an almost complex $2n$ -manifold and $u : D \rightarrow M$ a smooth, embedded J -holomorphic disk. Then for any closed form α in $\Omega_{\mathbb{R}}^{n,0}$ there exists a neighbourhood $U \subset M$ of $u(0)$ and a nowhere vanishing form ϕ in $\Omega_{\mathbb{R}}^{n,0}(U)$ such that for α expressed in terms of the basis $\{\phi, J\phi\}$*

$$\alpha = f\phi + gJ\phi, \tag{3.11}$$

on U , the function $(f \circ u) + i(g \circ u)$ is holomorphic on $u^{-1}(u(D) \cap U)$.

Proof. Let $\alpha = f_0\phi_0 + g_0J\phi_0$ in terms of the basis $\{\phi_0, J\phi_0\}$. Then closedness implies,

$$0 = d\alpha = df_0 \wedge \phi_0 + f_0d\phi_0 + dg_0 \wedge J\phi_0 + g_0d(J\phi_0). \tag{3.12}$$

Following the similarity principle argument used in Lemma 3.2.1 it is enough to verify that $f_0 + ig_0$ satisfies a Cauchy-Riemann type equation.

First remark that

$$\begin{aligned} \Re [\partial_{z_2} \lrcorner \cdots \lrcorner \partial_{z^n} \lrcorner (dz^1 \wedge \cdots \wedge dz^n)] &= -\Re [i\partial_{z_2} \lrcorner \cdots \lrcorner \partial_{z^n} \lrcorner (idz^1 \wedge \cdots \wedge dz^n)] \\ &= (-1)^{\frac{n(n-1)}{2}} dx^1, \\ \Re [\partial_{z_2} \lrcorner \cdots \lrcorner \partial_{z^n} \lrcorner (idz^1 \wedge \cdots \wedge dz^n)] &= \Re [i\partial_{z_2} \lrcorner \cdots \lrcorner \partial_{z^n} \lrcorner (dz^1 \wedge \cdots \wedge dz^n)] \\ &= -(-1)^{\frac{n(n-1)}{2}} dx^2. \end{aligned}$$

This allows us to choose a series of contractions that when applied to (3.12) yields the following pair of equations on D .

$$\begin{aligned} d\tilde{f}_0 \wedge ds + \tilde{f}_0 \tilde{\beta} - d\tilde{g}_0 \wedge dt + \tilde{g}_0 \tilde{\gamma} &= u^* \Psi_1 = 0, \\ -d\tilde{f}_0 \wedge ds + \tilde{f}_0 \tilde{\beta}' - d\tilde{g}_0 \wedge dt + \tilde{g}_0 \tilde{\gamma}' &= u^* \Psi_2 = 0, \end{aligned}$$

where tilde's are used to denote a quantity having been pulled back along u , the forms β, β' are contractions of $d\phi_0$, the forms γ, γ' are contractions of $d(J\phi_0)$ and Ψ_i are error terms which contain no $dx^1 \wedge dx^2$ terms and hence pull back to 0.

Arguing identically as in the proof of Lemma 3.2.1 shows the above pair of equations is a Cauchy-Riemann type system for $\tilde{f}_0 + i\tilde{g}_0$ and that the Similarity Principle gives the desired conclusion. \square

This lemma allows us to deduce a unique continuation result for closed sections of the canonical bundle.

Lemma 3.4.2. *Suppose that α is a closed form in $\Omega_{\mathbb{R}}^{n,0}$, then if $\alpha \equiv 0$ on some open set in M , it must vanish identically on the whole of M .*

Proof. Suppose that α vanishes on an open subset $U \subset M$. We may further assume that U is the largest open subset where α vanishes. By continuity α vanishes on its closure \bar{U} . If $\bar{U} \neq M$, choose a point $x \in \partial U := \bar{U} \setminus U$. Take a neighbourhood \mathcal{N}_x of x such that there is a J -fibre-diffeomorphism $Q : D \times D^{n-1} \rightarrow \mathcal{N}_x$. We can take ρ small enough such that each disk $Q(D_w)$ intersects U . In particular, for each $w \in D^{n-1}$, $Q(D_w) \cap U$ is an open subset in $Q(D_w)$. However, by Lemma 3.4.1, we know α vanishes either at isolated points or totally on $Q(D_w)$. This implies $\alpha|_{Q(D_w)} = 0$ for all $w \in D^{n-1}$ and thus $\alpha|_{\mathcal{N}_x} = 0$. Hence $U \cup \mathcal{N}_x \supsetneq U$, which contradicts the choice of U . Thus α vanishes on whole M . \square

Now Theorem 3.4.1 follows the same argument as Theorem 3.1.1. To show that the $(2n - 2)$ -dimensional Hausdorff measure of Z is finite we follow the argument of Proposition 3.2.1 replacing the appropriate lemma's with higher dimensional versions (*c.f.* Proposition 4.5.2).

Proof of Theorem 3.4.1. Viewing α as a smooth section of the canonical bundle, continuity implies that the zero set is compact.

By compactness we can cover Z by finitely many neighbourhoods which admit J -fibre-diffeomorphisms as in Lemma 2.1.5. So it is enough to show that the intersection of Z with each of these neighbourhoods is of finite $(2n - 2)$ -dimensional Hausdorff measure.

Following the arguments of Proposition 3.2.1 we can choose the J -fibre-diffeomorphisms as follows. Given $x \in Z$ there is a J -fibre-diffeomorphism $Q : D \times D^{n-1} \rightarrow M$ such that $Q(0,0) = x$ and no J -holomorphic disk $Q(D_w)$ is contained in Z . With such a

choice Lemma 3.4.1 implies that, for each $w \in D^{n-1}$, the intersection $Q(D_w) \cap Z$ is a finite set of points.

Further, by shrinking D if necessary, we may assume without loss of generality that the distortion of Q on the domain $2D \times (2D)^{n-1}$ is bounded by some constant $C > 0$.

Define,

$$g: \bar{D} \rightarrow \mathbb{N} \cup \{0\}, \quad \xi \mapsto \#(Z \cap Q(\bar{D}_w)).$$

Clearly this is an upper semi-continuous function and hence achieves a maximal value, say N , at some point $\xi \in \bar{D}$. Thus by Lemma 3.4.1, we know $Z \cap Q(\bar{D}_w)$ contains at most N points for all $\xi \in \bar{D}$. By the Vitali covering lemma we can take a finite cover of the compact set $Z \cap Q(\bar{D} \times \bar{D}^{n-1})$ by balls of radius ε such that L of these balls are disjoint and the union of L concentric balls with radius dilated by a factor of 3 cover. By the distortion assumption each ε ball intersects $Q(2\bar{D}_w)$ in an open set of area bounded above by $\pi C^2 \varepsilon^2$. The coarea formula then yields,

$$N \pi C^2 \varepsilon^2 \cdot \pi C^{2n-2} (2\rho)^{2n-2} > L \omega_{2n} \varepsilon^{2n},$$

where ω_{2n} is the volume of the unit $2n$ -ball. Hence there is a constant $C' > 0$ such that $C' \varepsilon^{-(2n-2)}$ balls of radius 3ε are enough to cover $Z \cap Q(\bar{D} \times \bar{D}^{n-1})$. This finishes the proof that $\mathcal{H}^{2n-2}(Z) < \infty$.

Finally identical to the argument of Proposition 3.3.2, we can verify that the assignment I_α of Section 3.3 defines a positive cohomology assignment for Z in the sense of Definition 3.3.1. \square

Since the first Chern class of the complex line bundle $\Lambda_{\mathbb{R}}^{n,0}$ is K_J (*e.g.* by the same argument as Proposition 4.3 in [60]), if Question 3.4.1 is answered affirmatively, the Poincaré dual of the homology class of the pseudoholomorphic $(n-1)$ -subvariety supported on Z is K_J .

3.5 A birational invariant of almost complex 4-manifolds

A famous question of Donaldson regarding compact almost complex 4-manifolds asks whether an almost complex structure tamed by a symplectic form necessarily admits a compatible symplectic form. Recall that a symplectic form ω is said to be tamed by an almost complex structure J if $\omega(J\cdot, \cdot)$ is positive definite and that it is compatible with J if $\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot)$. The study of this question lead Li and Zhang [39] to define the cohomology groups $H_J^\pm(X) \subset H^2(X, \mathbb{R})$. These generalise the real Hodge cohomology groups, and can be represented by J -invariant and J -anti-invariant 2-forms respectively. We denote by Ω^2 the space of 2-forms on M (C^∞ -sections of the bundle Λ^2), Ω_J^+ the space of J -invariant 2-forms, *etc.* Let also \mathcal{Z}^2 denote the space of closed 2-forms on M

and let $\mathcal{Z}_J^\pm = \mathcal{Z}^2 \cap \Omega_J^\pm$. Then we define the cohomology groups,

$$H_J^\pm(M) = \{\mathfrak{a} \in H^2(M; \mathbb{R}) \mid \exists \alpha \in \mathcal{Z}_J^\pm \text{ such that } [\alpha] = \mathfrak{a}\}.$$

It is proven in [14] that $H_J^+(M) \oplus H_J^-(M) = H^2(M; \mathbb{R})$ when $\dim_{\mathbb{R}} M = 4$. The dimensions of the vector spaces $H_J^\pm(M)$ are denoted as $h_J^\pm(M)$.

These groups are analogous to the Dolbeault cohomology and relate naturally to them when J is integrable [14], in particular it holds that

$$H_J^-(X) = (H_{\bar{\partial}}^{2,0}(X) \oplus H_{\bar{\partial}}^{0,2}(X)) \cap H^2(X; \mathbb{R}). \quad (3.13)$$

Through a series of papers [14, 15, 16] Draghici, Li and Zhang give partial answers to Donaldson's question using these groups, furthermore it is found that as well as the groups themselves the dimensions h_J^\pm are of great significance. In this section we prove that h_J^- is a birational invariant of compact almost complex 4-manifolds.

The results of [60] suggest that the right notion of birational morphism between almost complex four manifolds are degree 1 pseudoholomorphic maps. Indeed for such maps Zariski's main theorem holds and one can obtain a detailed description of the singular set, this is summarised by the following theorem.

Theorem 3.5.1 (Theorem 1.5 [60]). *Let $u : (X, J) \rightarrow (M, J_M)$ be a degree one pseudoholomorphic map between connected almost complex 4-manifolds such that J is almost Kähler. Then there exists a subset $M_1 \subset M$ consisting of finitely many points such that,*

- (1) *the restriction $u|_{X \setminus u^{-1}(M_1)}$ is a diffeomorphism;*
- (2) *at each point of M_1 the preimage is an exceptional curve of the first kind;*
- (3) *$X \cong M \# k\mathbb{C}P^2$ diffeomorphically, where k is the number of irreducible components of the J -holomorphic subvariety $u^{-1}(M_1)$.*

For our purposes it suffices to say that a pseudoholomorphic curve is an exceptional curve of the first kind if its configuration is equivalent to the empty set through topological blowdowns, see Definition 5.11 of [60] and references therein for details. Also we should remark that Zhang believes that the almost Kähler assumption on (X, J) to be removable.

Thus we say that two closed almost complex four manifolds M_1 and M_2 are *birational* if there exist closed almost complex manifolds $X_1, \dots, X_{n+1}, Y_1, \dots, Y_n$ such that $M_1 = X_1$, $M_2 = X_{n+1}$ and there are degree one pseudoholomorphic maps $\phi_i : Y_i \rightarrow X_i$ and $\psi_i : Y_i \rightarrow X_{i+1}$ for all $i = 1, \dots, n$.

In this section we prove that h_J^- is a birational invariant of almost complex four manifolds.

Theorem 3.5.2. *Let $\psi : (M_1, J_1) \rightarrow (M_2, J_2)$ be a degree 1 pseudoholomorphic map between closed, connected, almost complex 4-manifolds. Then $h_{J_1}^-(M_1) = h_{J_2}^-(M_2)$.*

The basic strategy is similar to the proof of Theorem 5.3 in [11]. However, we need a version of Hartog's extension theorem for closed J -anti-invariant forms. This relies on the trivialisation of Λ_J^- over embedded J -holomorphic disks provided by Lemma 3.2.1.

Again it is convenient to view J -anti-invariant 2-forms as a smooth sections of the complex line bundle Λ_J^- over M . We shall denote such a section associated to a J -anti-invariant 2-form α by $\Gamma_\alpha : M \rightarrow \Lambda_J^-$. By Lemma 3.2.1 there is a trivialisation of Λ_J^- over a given embedded J -holomorphic disk $u : D \rightarrow M$ such that $\Gamma_\alpha \circ u$ may be viewed as a holomorphic function $\Gamma_\alpha \circ u : D \rightarrow \mathbb{C}$ when α is closed. Notice that once a trivialisation has been chosen we abuse notation and ignore the holomorphic projection of $\Lambda_J^- \cong D \times \mathbb{C}$ onto its second factor. We identify the basis $\{\phi, J\phi\}$ in Lemma 3.2.1 with 1 and i in \mathbb{C} under the trivialisation.

Before proceeding it is convenient to make some remarks about Lemma 3.2.1. First consider $U \subset M$ an open, connected subset, α a closed J -anti-invariant 2-form defined on $U \setminus \{p\}$ for some $p \in U$ and $u : D \rightarrow M$ an embedded J -holomorphic disk with $u(0) = p$. It follows from the arguments of Lemma 3.2.1 that, after possibly shrinking $u(D)$, there is a holomorphic structure on Λ_J^- over $u(D) \setminus \{p\}$ such that $\Gamma_\alpha \circ u : D \setminus \{0\} \rightarrow \Lambda_J^-$ is holomorphic.

Indeed, by Lemma 3.2.1, we can cover $D \setminus \{0\}$ by subdisks D_i such that $\Lambda_J^-|_{u(D_i)}$ is trivialised with $\Gamma_\alpha \circ u : D_i \rightarrow \Lambda_J^-$ a holomorphic section. Furthermore, we assume the zero locus $\alpha^{-1}(0) \cap u(\partial D_i) = \emptyset$. We look at the transition function $\beta + i\gamma$ of the line bundle $\Lambda_J^-|_{u(D_i)}$ for $D_1 \cap D_2$ say. The form α could be represented in terms of two basis'

$$\alpha = f_1\phi_1 + g_1J\phi_1 = f_2\phi_2 + g_2J\phi_2.$$

By computation, $(f_1 + ig_1) = (f_2 + ig_2)(\beta + i\gamma)$. In other words, writing $h_i = (\Gamma_\alpha \circ u)|_{D_i}$ we can write transition functions as $\tau_{ij} = \frac{h_i}{h_j}$ on $D_{ij} = D_i \cap D_j$. Since the h_i are holomorphic, and the transition functions are nowhere zero, we know τ_{ij} are holomorphic.

This transition data thus defines a holomorphic line bundle structure on Λ_J^- over $u(D) \setminus \{p\}$ such that $\Gamma_\alpha \circ u : D \setminus \{0\} \rightarrow \Lambda_J^-$ is holomorphic. Furthermore $D \setminus \{0\}$ is Stein and hence, by Oka's principle, the bundle is isomorphic to $D \setminus \{0\} \times \mathbb{C}$. This allows one to view $\Gamma_\alpha \circ u : D \setminus \{0\} \rightarrow \mathbb{C}$ as a holomorphic complex valued function. In summary we have found a trivialisation of Λ_J^- over $u(D) \setminus \{p\}$ such that $\Gamma_\alpha \circ u : D \setminus \{0\} \rightarrow \mathbb{C}$ is a holomorphic function.

Secondly, for $\varepsilon \in (0, 1)$, let $u_\varepsilon : D \rightarrow M$ be a smooth family of embedded J -holomorphic disks. For each $\varepsilon \in (0, 1)$ the arguments of §2 provide coordinates x_ε^i such that $J = J_0$ along $u_\varepsilon(D)$. Moreover the x_ε^i vary smoothly in ε . Defining $\phi_{0,\varepsilon}$ by (3.5) and following the arguments of Lemma 3.2.1 we obtain a family of functions

$v_{0,\varepsilon} = f_{0,\varepsilon} + ig_{0,\varepsilon}$ satisfying a Cauchy-Riemann type equation $\bar{\partial}v_{0,\varepsilon} + C_1^\varepsilon v_{0,\varepsilon} + C_2^\varepsilon \bar{v}_{0,\varepsilon} = 0$, where $v_{0,\varepsilon}$ and $C_1^\varepsilon, C_2^\varepsilon$ vary smoothly in ε . Hence the resulting family of holomorphic functions $f_\varepsilon + ig_\varepsilon$ and forms ϕ_ε vary smoothly in ε . That is, the trivialisations over each $u_\varepsilon(D)$ vary smoothly.

Proposition 3.5.1. *Let (M, J) be an almost complex 4-manifold, $U \subset M$ open and $p \in U$. Suppose that α is a closed J -anti-invariant 2-form defined on $U \setminus \{p\}$. Then α extends smoothly to U .*

Proof. First, by shrinking U if necessary, we may assume that there is a J -fibre diffeomorphism $Q : D \times D \rightarrow U$ centred at p such that $Q(\{0\} \times D)$ and each $Q(D_w)$ is an embedded J -holomorphic disk.

We trivialisise $\Lambda_{\bar{J}}$ with respect to α , first along $Q(\{0\} \times D) \setminus \{p\}$ then along each $Q(D_w)$ and $Q(D_0) \setminus \{p\}$. By the remarks preceding the proposition Γ_α may be considered a smooth map $\Gamma_\alpha : (D \times D) \setminus \{(0, 0)\} \rightarrow \mathbb{C}$ such that

- (i) $\Gamma_\alpha(\cdot, w) : D \rightarrow \mathbb{C}$ is holomorphic for each $w \neq 0$,
- (ii) $\Gamma_\alpha(\cdot, 0) : D \setminus \{0\} \rightarrow \mathbb{C}$ is holomorphic,
- (iii) $\Gamma_\alpha(0, \cdot) : D \setminus \{0\} \rightarrow \mathbb{C}$ is holomorphic.

For each $j \in \mathbb{Z}$ define,

$$a_j(w) := \int_{|\xi|=\rho} \frac{\Gamma_\alpha(\xi, w)}{\xi^{j+1}} d\xi.$$

Clearly this is a smooth function $a_j : D \rightarrow \mathbb{C}$ for all $j \in \mathbb{Z}$. Moreover, by (i), we have $a_0(w) = \Gamma_\alpha(0, w)$, $w \neq 0$, and hence $a_0 : D \setminus \{0\} \rightarrow \mathbb{C}$ is holomorphic.

For each $w \neq 0$ the Cauchy Integral formula gives the following Laurent series

$$\Gamma_\alpha(\xi, w) = \sum_{j=-\infty}^{\infty} a_j(w)\xi^j = \sum_{j=0}^{\infty} a_j(w)\xi^j,$$

where the second equality follows from (i). In particular $a_j(w) = 0$ for all $j < 0$ and $w \neq 0$. By smoothness of α on $U \setminus \{p\}$ and the trivialisations along the disks, it follows that $a_j(0) = 0$ for all $j < 0$. Applying Cauchy Integral formula again yields,

$$\Gamma_\alpha(\xi, 0) = \sum_{j=-\infty}^{\infty} a_j(0)\xi^j = \sum_{j=0}^{\infty} a_j(0)\xi^j,$$

proving that $\Gamma_\alpha(\xi, 0)$ is holomorphic on D with $\Gamma_\alpha(0, 0) = a_0(0)$.

Let us now verify that $\Gamma_\alpha(0, w)$ can be extended to a holomorphic function on D with value $a_j(0)$ at the origin. To this end notice that, by smoothness,

$$\frac{\partial}{\partial \bar{w}} \Gamma_\alpha(0, w) = \int_{|\xi|=\rho} \frac{\frac{\partial}{\partial \bar{w}} \Gamma_\alpha(\xi, w)}{\xi} d\xi = 0, \quad \forall w \in D.$$

So $\Gamma_\alpha(0, w)$ extends as a holomorphic function to D and $\Gamma_\alpha(0, 0) = a_0(0)$.

As remarked in Section 3.2 the J -fibre diffeomorphism may be chosen such that $Q(\{0\} \times D)$ is a given J -holomorphic disk and $Q(D_0)$ is tangent at p to a given complex direction $\kappa \in \mathbb{C}P^1$ transverse to $Q(\{0\} \times D)$. Varying κ we produce a family of embedded J -holomorphic disks whose complex tangent directions cover a neighbourhood of κ . Moreover, each of these disks is the D_0 fibre of a J -fibre diffeomorphism. We can choose finitely many such families whose union covers a neighborhood of p , and their tangent directions cover $\mathbb{C}P^1$. Since $Q(\{0\} \times D)$ is fixed the argument above provides a holomorphic extension in each complex direction κ with the same extended value at p . For the disks not transverse to the given J -holomorphic disk, we choose any other disk in the family to complete the proof. \square

With this Hartogs type extension in hand, we are able to prove Theorem 3.5.2.

Proof of Theorem 3.5.2. Since ψ is pseudoholomorphic the pullback of 2-forms along ψ induces a map

$$\psi^* : \mathcal{Z}_{J_2}^-(M_2) \rightarrow \mathcal{Z}_{J_1}^-(M_1).$$

We claim that this induced map is an isomorphism. If this is the case then this induces an isomorphism between $H_{J_1}^-(M_1)$ and $H_{J_2}^-(M_2)$ since \mathcal{Z}_J^- is isomorphic to H_J^- (see *e.g.* [14]).

By Proposition 5.9 of [60] there exists a finite set $Y \subset M_2$ such that $u|_{M_1 \setminus \psi^{-1}(Y)}$ is a diffeomorphism and $\psi^{-1}(y)$ is a pseudoholomorphic subvariety for all $y \in Y$. Thus, given $\alpha \in \mathcal{Z}_{J_2}^-(M_2)$, it follows that if $\psi^*(\alpha) = 0$ then $\alpha|_{M_2 \setminus Y} = 0$ and hence smoothness implies that $\alpha \equiv 0$.

It is left to show that $\psi^* : \mathcal{Z}_{J_2}^-(M_2) \rightarrow \mathcal{Z}_{J_1}^-(M_1)$ is surjective. Since $\psi|_{M_1 \setminus \psi^{-1}(Y)}$ is a diffeomorphism we can pull back a given $\tilde{\alpha} \in \mathcal{Z}_{J_1}^-(M_1)$ to give a J_2 -anti-invariant form $\alpha := (\psi^{-1})^*(\tilde{\alpha}) \in \mathcal{Z}_{J_2}^-(M_2 \setminus Y)$. As Y is a finite set Proposition 3.5.1 gives an extension to a form $\alpha \in \mathcal{Z}_{J_2}^-(M_2)$ which concludes the proof. \square

3.6 Further discussions

In this section we provide a definition of multiplicity of zeros for a continuous function $u : D^2 \rightarrow \mathbb{R}^2$ which generalises the multiplicity of zeros of a holomorphic function.

3.6.1 Multiplicity of zeros for a continuous function $u : D^2 \rightarrow \mathbb{R}^2$

An amusing application is to define the multiplicity of isolated zeros of a continuous function $u : D^2 \rightarrow \mathbb{R}^2$ from the open unit disk D^2 , as a generalisation of the multiplicity of zeros of a holomorphic function. This subsection could also be viewed as some explicit calculations of the intersection number used throughout the chapter.

Consider a trivial bundle \mathcal{O} over D^2 of real rank two. A continuous function $u : D^2 \rightarrow \mathbb{R}^2$, $u(x, y) = (f(x, y), g(x, y))$, is called admissible if $\overline{u^{-1}(0)} \cap \partial D^2 = \emptyset$. By

taking complex coordinate $z = x + iy$ on D and using the standard identification of $\mathbb{R}^2 = \mathbb{C}$ we can write $u(z) = f(z, \bar{z}) + ig(z, \bar{z})$, where f and g are real valued functions. It is clear that this definition of admissibility also works for an admissible function $u : B^n \rightarrow \mathbb{R}^n$.

Example 3.6.1. The function $u(z) = x$ is not admissible. All non-trivial holomorphic functions are admissible. The function $u(z) = |z|^2$ is admissible.

For an admissible function $u : D^2 \rightarrow \mathbb{R}^2$, we define the sum of multiplicities of zeros inside D^2 by perturbation. We perturb u to a smooth function $\tilde{u} : D^2 \rightarrow \mathbb{R}^2$ such that the Jacobian of each zero of \tilde{u} is non-degenerate. It is equivalent to viewing the function u as a map to the total space of the trivial bundle \mathcal{O} , and requiring that the perturbed \tilde{u} has transverse intersection with the zero section. Then the multiplicity $I(u)$ is the sum of the signs of the Jacobian of each zero of \tilde{u} . The multiplicity $I(u)$ is independent of the choice of the perturbation \tilde{u} .

Example 3.6.2. When u is a holomorphic function, $I(u)$ is just the sum of the multiplicities of all the zeros of u inside the unit disk. Each zero contributes positively to the sum.

One may choose a holomorphic perturbation u' such that u' has more zeros than u over \mathbb{R}^2 and each zero will contribute positively to the index. A generic holomorphic perturbation would have $I(u)$ many zeros inside the unit disk.

On the other hand, if u is an anti-holomorphic function, then each zero contributes negatively.

The following provides an explicit example of the multiplicity being independent of the perturbation as long as the Jacobian is non-degenerate at any zero point.

Example 3.6.3. Let $u(z) = |z|^2$. Then $I(u) = 0$. There are many ways of admissible perturbations. For example, if $\tilde{u}(z) = |z|^2 + \varepsilon z$, then it has two zeros $z = 0$ and $z = -\varepsilon$. The Jacobian matrix has determinants $|\varepsilon|^2$ and $-|\varepsilon|^2$ at 0 and $-\varepsilon$ respectively. This implies $I(u) = 0$.

We can also calculate it using other perturbations. A natural one is $\tilde{u}(z) = u(z) + c$. When $c > 0$, there will be no zeros in D , which again implies $I(u) = 0$ immediately. When $c < 0$, it is not a good perturbation to calculate the multiplicity since the Jacobian is degenerate at the zero set.

In fact, our multiplicity is uniquely defined in a natural sense.

Proposition 3.6.1. *The multiplicity $I(u)$ is the unique functional satisfying the following five properties:*

- $I(u) = 0$ if $u(a) \neq 0, \forall a \in D$;
- If $u_0, u_1 : D \rightarrow \mathbb{R}^2$ are admissible and homotopic via an admissible family u_t , then $I(u_0) = I(u_1)$;

- If $\theta : D \rightarrow D$ is a proper degree k map, then $I(u \circ \theta) = k \cdot I(u)$;
- If all the zeros are included in disjoint union $\cup_i D_i \subset D$ where each $D_i = \theta_i(D)$ with embedding $\theta_i : D \rightarrow D$, then $I(u) = \sum I(u \circ \theta_i)$;
- If u is holomorphic, $I(u)$ is the usual multiplicity of zeros for holomorphic functions.

Proof. By Proposition 3.3 in [60] (or Proposition 3.3.2 in this paper), $I(u)$ satisfies the five properties. To show the uniqueness, we first perturb u to \tilde{u} such that all the zeros are non-degenerate. We write the Taylor expansion in terms of z, \bar{z} at each zero of \tilde{u} . By virtue of the fourth item we can, on a small disk around each zero, use a local linear homotopy from \tilde{u} to the linear term of its Taylor expansion. By choosing the disk to be small, no more zeros would be brought in through this homotopy. The linear term at each zero (without loss, we assume the zero is the original point) can be written as

$$\left(\frac{a+d}{2}\right)z + \left(\frac{a-d}{2}\right)\bar{z} + \left(\frac{c-b}{2}\right)iz + \left(\frac{c+b}{2}\right>i\bar{z},$$

where the Jacobian matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is non-degenerate. If the determinant is positive, a linear homotopy

$$\left(\frac{a+d}{2}\right)z + t\left(\frac{a-d}{2}\right)\bar{z} + \left(\frac{c-b}{2}\right)iz + t\left(\frac{c+b}{2}\right>i\bar{z}$$

would lead to a holomorphic function with $I = 1$. Notice, when $t \in [0, 1]$, the Jacobians are all non-degenerate. Similarly, when the determinant is negative, it is homotopic to an anti-holomorphic function. By the third item, an anti-holomorphic has the multiplicity opposite to its holomorphic conjugation. Hence, our multiplicity $I(u)$ is uniquely determined by the classical multiplicity of a holomorphic function and the other four properties. \square

Chapter 4

Eigenvalues of the Laplacian on almost Kähler manifolds

4.1 Introduction

Bourguignon, Li and Yau [7] proved an upper bound for the first non-zero eigenvalue for a given Kähler metric on a projective manifold M which depended only on dimension, volume and a holomorphic immersion $\phi : M^n \rightarrow \mathbb{P}^m$. Recently Kokarev [33] has improved their result giving bounds, for a more general class of Kähler manifolds, on the k -th eigenvalue which depend linearly on k . The class in question are those Kähler manifolds that admit a non-trivial holomorphic map into a projective space \mathbb{P}^m .

The purpose of this chapter is to show that the same bound in fact holds in the almost Kähler setting. More precisely we prove the following theorem.

Theorem 4.1.1. *Let (M^n, J) be a closed $2n$ -dimensional almost Kähler manifold and $\phi : M^n \rightarrow \mathbb{P}^m$ a non-trivial pseudoholomorphic map, where \mathbb{P}^m is taken with its standard complex structure. Then for any almost Kähler metric g on M^n , the eigenvalues of the Laplace-Beltrami operator Δ_g satisfy,*

$$\lambda_k(M^n, g) \leq C(n, m)d([\phi], [\omega_g])k, \quad \text{for any } k \geq 1, \quad (4.1)$$

where $C(n, m) > 0$ is a constant depending only on n and m and $d([\phi], [\omega_g])$ is defined by,

$$d([\phi], [\omega_g]) := \frac{\int_M \phi^* \omega_{FS} \wedge \omega_g^{n-1}}{\int_M \omega_g^n}. \quad (4.2)$$

Alternatively we can write

$$d([\phi], [\omega_g]) := \frac{(\phi^*[\omega_{FS}] \smile [\omega_g]^{n-1}, [M])}{([\omega_g]^n, [M])}, \quad (4.3)$$

where (\cdot, \cdot) denotes the pairing of de-Rham cohomology and singular homology. From this expression it is clear that $d([\phi], [\omega_g])$ depends only on the de-Rham class $[\omega_g] \in$

$H^2(M; \mathbb{R})$ and the induced map on 2-cohomology $\phi^* : H^2(\mathbb{P}^m; \mathbb{Q}) \rightarrow H^2(M; \mathbb{Q})$.

Let (M, J) be an almost complex manifold and $E \rightarrow M$ a complex vector bundle of complex rank r over M . Suppose further that the total space E is endowed with an almost complex structure \mathcal{J} such that the projection map is pseudoholomorphic and that E is globally generated by pseudoholomorphic sections. Here we say that a section $s : M \rightarrow E$ is pseudoholomorphic with respect to \mathcal{J} if $ds \circ J = \mathcal{J} \circ ds$. We can define the Kodaira map $\kappa_E : M \rightarrow \text{Gr}(r, \mathbb{C}^N)$ in the usual way, where $\text{Gr}(r, \mathbb{C}^N)$ denotes the grassmannian and $N = \dim(H_{\mathcal{J}}^0(E))$ is the dimension of the vector space of global pseudoholomorphic sections of E with respect to \mathcal{J} . We provide details of the construction in §3. Composing with the Plücker embedding yields a pseudoholomorphic map $\phi : M \rightarrow \mathbb{P}^m$ and hence Theorem 4.1.1 may be applied to obtain the following corollary.

Corollary 4.1.1. *Let $E \rightarrow M$ be a complex vector bundle over a compact almost complex manifold (M, J) . Suppose further that the total space is endowed with an almost complex structure \mathcal{J} and the bundle is globally generated by pseudoholomorphic sections with respect to \mathcal{J} . Then, for any almost Kähler metric g on M , the eigenvalues of the Laplace-Beltrami operator satisfy,*

$$\lambda_k(M, g) \leq C \frac{(c_1(E) \smile [\omega_g]^{n-1}, [M])}{([\omega_g]^n, [M])} k, \quad \text{for any } k \geq 1, \quad (4.4)$$

where $C > 0$ is a constant depending only on $\dim(M)$, $\text{rank}(E)$ and $\dim(H_{\mathcal{J}}^0(E))$.

Notice that for the map ϕ defined as the composition of the Kodaira map and the Plücker embedding we have $\phi^*[\omega_{\text{FS}}] = a \cdot c_1(E)$ for some constant $a > 0$, where $[\omega_{\text{FS}}]$ denotes the de-Rham class. Indeed writing $U \rightarrow \text{Gr}(r, \mathbb{C}^N)$ for the tautological bundle we have that $c_1(E) = \kappa_E^* c_1(\det U)$. On the other hand one can explicitly calculate that the pull back of the Fubini-Study metric under the Plücker embedding is, up to scaling, the curvature of the Hermitian metric on $\det U$ induced by the constant metric on the fibres of the trivial bundle $\text{Gr}(r, \mathbb{C}^N) \times \mathbb{C}^N$. So, by Chern-Weil theory, one finds that $\phi^*[\omega_{\text{FS}}] = ac_1(E)$ for some constant $a > 0$.

Section 4 is dedicated to providing examples of strictly almost Kähler manifolds to which Theorem 4.1.1 applies. More precisely, we provide examples of strictly almost Kähler manifolds which admit a globally generated pseudoholomorphic vector bundle. These are built from the examples given by Chen-Zhang [11].

Let us briefly discuss the outline of the proof of Theorem 4.1.1. To establish the desired bound (4.1) it suffices to produce $k + 1$ linearly independent test functions for which the bound (4.1) holds. These test functions are constructed on \mathbb{P}^m and hence are identical to those used in [33]. Intuitively they are components of the moment map of the action of the group of isometries of \mathbb{P}^m restricted to carefully chosen annuli. The main technical difficulty arises in trying to control the measure of these annuli with respect to the push-forward of the volume measure Vol_g . Control can be established

using the work of Grigoryan, Netrusov and Yau [23] if the measure $\mu := \phi_*(\text{Vol}_g)$ is non-atomic. In the situation of an integrable complex structure the level sets of a non-trivial holomorphic map are subvarieties of positive codimension which implies that μ must be non-atomic. On the other hand if the almost complex structure is not integrable determining the structure of level sets is an open question. Whilst we cannot establish that level sets are pseudoholomorphic subvarieties we can prove an estimate on their Hausdorff dimension from which it follows that μ is non-atomic. From this point it is routine to verify that the arguments of [33] continue to hold and hence finish the proof of Theorem 4.1.1.

We can also obtain a version of Theorem 4.25 for pseudoholomorphic subvarieties of almost Kähler manifolds. Let $(M^{n+\ell}, J)$ be a closed almost Kähler manifold and Σ^n an irreducible pseudoholomorphic subvariety whose regular part Σ_*^n has complex dimension n . Here we say that $\Sigma^n \subset M^{n+\ell}$ is an irreducible pseudoholomorphic subvariety if it is the image of a somewhere immersed pseudoholomorphic map $\Phi : X \rightarrow M$ where X is a smooth, closed, connected almost complex manifold. Given an almost Kähler metric g on M its restriction to the regular part of Σ yields an incomplete almost Kähler metric, g_Σ , on Σ_* . We are interested in the eigenvalues of the Laplacian corresponding to g_Σ . It is shown that for an appropriate function space $\Delta = \Delta_{g_\Sigma}$ is essentially self-adjoint and has discrete spectrum.

Theorem 4.1.2. *Let $(M^{n+\ell}, J)$ be a closed almost Kähler manifold and $\phi : M^{n+\ell} \rightarrow \mathbb{P}^m$ a non-trivial pseudoholomorphic map. Furthermore let $\Sigma^n \subset M^{n+\ell}$ be an irreducible pseudoholomorphic subvariety such that the restriction of ϕ to Σ is non-trivial. Then, for any almost Kähler metric g on M , the eigenvalues of the Laplacian associated to g_Σ satisfy,*

$$\lambda_k(\Sigma, g_\Sigma) \leq C(n, m) \frac{\int_\Sigma \phi^* \omega_{FS} \wedge \omega_g^{n-1}}{\int_\Sigma \omega_g^n} k, \quad \text{for any } k \geq 1, \quad (4.5)$$

where $C(n, m) > 0$ is a constant depending only on n and m and ω_g is the Kähler form of g on M .

The chapter is structured as follows; in sections 2 and 3 we establish the necessary preliminaries from Riemannian and almost complex geometry followed by some examples of strictly almost Kähler manifolds to which Theorem 4.1.1 applies in §4. Next §5 is dedicated to establishing an estimate on the level sets of pseudoholomorphic maps and hence that the push-forward measure $\phi_*(\text{Vol}_g)$ is non-atomic. Sections 6 and 7 define the test functions and prove Theorem 4.1.1. Then in §8 we give a brief discussion of pseudoholomorphic subvarieties of a almost Kähler manifolds and prove Theorem 4.1.2. Finally we give some possible further directions of study in §9, in particular we prove that the regularity of the map ϕ in Theorem 4.1.1 can be reduced and as a consequence the theorem can be applied when ϕ is only a rational map.

4.2 Preliminaries

4.2.1 Eigenvalues of the Laplacian on an almost Kähler Manifold

Let (M, g) be a compact Riemannian manifold without boundary. In local coordinates x^i the Laplace-Beltrami operator associated to g has the form,

$$\Delta = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left(\sqrt{|g|} g^{ij} \frac{\partial}{\partial x^j} \right),$$

with $|g| = \det g$. The eigenvalues of the Laplacian are real numbers $0 < \lambda_1(g) \leq \dots \leq \lambda_k(g) \leq \dots$ such that

$$-\Delta u = \lambda_k(g)u$$

has a non-trivial solution.

Recall that we can characterise the k -th eigenvalue by

$$\lambda_k(g) = \inf_V \sup_{u \in V} \mathcal{R}_g(u), \quad (4.6)$$

with the infimum taken over all $(k+1)$ -dimensional subspaces of $\text{Lip}(M)$ and $\mathcal{R}_g(u)$ is the Rayleigh quotient defined by,

$$\mathcal{R}_g(u) = \frac{\int_M |\nabla u|^2 d \text{Vol}_g}{\int_M u^2 d \text{Vol}_g}. \quad (4.7)$$

Thus to prove a bound on λ_k it suffices to produce $k+1$ linearly independent test whose Rayleigh quotient satisfies the same bound. By compactness one can take a constant as one of these test functions reducing the problem to finding k linearly independent test functions.

In general, on a Hermitian manifold, the Laplacian of the Levi-Civita connection and the Laplacian of the Chern connection agree only up to a first order term. But as we saw in §2 if we have an almost Kähler manifold (in fact a quasi Kähler manifold is sufficient) then they agree. In light of this, given an almost Kähler manifold, we shall henceforth only refer to the Laplace-Beltrami operator.

4.2.2 Almost Kähler Geometry and Complex Projective Space

Throughout this chapter \mathbb{P}^m will denote the complex projective space of dimension m endowed with the Fubini-Study metric, see (2.13), scaled to have diameter $\frac{\pi}{2}$.

Since the standard action of $SU(m+1)$ on \mathbb{P}^m preserves ω_{FS} it has an associated moment map $\tau : \mathbb{P}^m \rightarrow \mathfrak{su}_{m+1}^*$. By the Killing scalar product $\langle X, Y \rangle = \text{tr} X^* Y = -\text{tr} XY$ we can identify \mathfrak{su}_{m+1}^* and \mathfrak{su}_{m+1} so that in local homogeneous coordinates $[Z] = [z_0 : \dots : z_m]$ we can express the components of τ by

$$\tau_{k\ell}([Z]) = i \frac{z_k \bar{z}_\ell}{|z|^2}. \quad (4.8)$$

The moment map also satisfies the identity,

$$\omega_{\text{FS}} = -\frac{i}{2} \sum_{k\ell} d\tau_{k\ell} \wedge d\tau_{\ell k},$$

see [1] for details.

Lemma 4.2.1. *Let (M^n, J, g) be an almost Kähler manifold and $\phi : M \rightarrow \mathbb{P}^m$ a pseudoholomorphic map. Then the gradient of the matrix valued map $\tau \circ \phi : M \rightarrow \mathbb{C}^{(m+1)^2}$ satisfies,*

$$|\nabla(\tau \circ \phi)|^2 \omega_g^n = n \phi^*(\omega_{\text{FS}}) \wedge \omega_g^{n-1}, \quad (4.9)$$

where $\tau : \mathbb{P}^m \rightarrow \mathfrak{su}_{m+1}$ is the moment map associated to $SU(m+1)$ acting on \mathbb{P}^m

Proof. We claim that for a smooth function $\varphi : M \rightarrow \mathbb{C}$ the following formula holds,

$$|\nabla\varphi|^2 \omega_g^n = in(\partial\varphi \wedge \bar{\partial}\bar{\varphi} + \partial\bar{\varphi} \wedge \bar{\partial}\varphi) \wedge \omega_g^{n-1}. \quad (4.10)$$

Write $T^{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$ for the natural splitting of the complexified tangent space induced by J and let e_1, \dots, e_n be a local unitary frame for $T^{1,0}M$ with respect to g . Writing $\theta^1, \dots, \theta^n$ for the dual coframe we have

$$\omega_g = i\theta^k \wedge \bar{\theta}^k.$$

Furthermore, for a smooth function $\varphi : M \rightarrow \mathbb{C}$ we can write the decomposition of $d\varphi$ into $(1,0)$ and $(0,1)$ parts as

$$d\varphi = \partial\varphi + \bar{\partial}\varphi = \varphi_k \theta^k + \varphi_{\bar{k}} \bar{\theta}^k.$$

In this notation we have,

$$|\nabla\varphi|^2 \omega_g^n = (|\partial\varphi|^2 + |\bar{\partial}\varphi|^2) \omega_g^n = i^n n! \sum_k (\varphi_k \bar{\varphi}_k + \bar{\varphi}_{\bar{k}} \varphi_{\bar{k}}) \theta^1 \wedge \bar{\theta}^1 \wedge \dots \wedge \theta^n \wedge \bar{\theta}^n.$$

On the other hand,

$$\begin{aligned} in(\partial\varphi \wedge \bar{\partial}\bar{\varphi} + \partial\bar{\varphi} \wedge \bar{\partial}\varphi) \wedge \omega_g^{n-1} &= in \left(\sum_{k,\ell} (\varphi_k \bar{\varphi}_\ell + \bar{\varphi}_{\bar{k}} \varphi_{\bar{\ell}}) \theta^k \wedge \bar{\theta}^\ell \right) \wedge \omega_g^{n-1} \\ &= i^n n! \sum_k (\varphi_k \bar{\varphi}_k + \bar{\varphi}_{\bar{k}} \varphi_{\bar{k}}) \theta^1 \wedge \bar{\theta}^1 \wedge \dots \wedge \theta^n \wedge \bar{\theta}^n, \end{aligned}$$

which gives the desired claim.

To proceed let $\varphi_{ij} = \tau_{ij} \circ \phi : M \rightarrow \mathbb{C}$ be the components of the composition

$\tau \circ \phi : M \rightarrow \mathbb{C}^{(2m+2)^2}$. Now,

$$\begin{aligned} \partial\varphi_{ij} &= d\varphi_{ij} - i d\varphi_{ij} \circ J \\ &= d\tau_{ij} \circ d\phi - i d\tau_{ij} \circ d\phi \circ J \\ &= (d\tau_{ij} - i d\tau_{ij} \circ J_{\mathbb{P}^m}) \circ d\phi = \phi^*(d\tau_{ij} - i d\tau_{ij} \circ J_{\mathbb{P}^{2m}}) = \phi^*\partial\tau_{ij}, \end{aligned}$$

where in the second equality we used chain rule and in the third that ϕ is pseudoholomorphic. Similarly we compute that,

$$\bar{\partial}\varphi_{ij} = \phi^*\bar{\partial}\tau_{ij}, \quad \partial\bar{\varphi}_{ij} = \phi^*\partial\bar{\tau}_{ij}, \quad \bar{\partial}\bar{\tau}_{ij} = \phi^*\bar{\partial}\bar{\tau}_{ij}.$$

By (4.10) we have,

$$\begin{aligned} |\nabla\varphi_{ij}|^2\omega_g^n &= in(\partial\varphi_{ij} \wedge \bar{\partial}\bar{\varphi}_{ij} + \partial\bar{\varphi}_{ij} \wedge \bar{\partial}\varphi_{ij}) \wedge \omega_g^{n-1} \\ &= in\phi^*(\partial\tau_{ij} \wedge \bar{\partial}\bar{\tau}_{ij} + \partial\bar{\tau}_{ij} \wedge \bar{\partial}\tau_{ij}) \wedge \omega_g^{n-1}. \end{aligned}$$

Finally we remark that, as τ is the moment map, we have the expression

$$\omega_{\text{FS}} = -\frac{i}{2} \sum_{i,j} d\tau_{ij} \wedge d\tau_{ji} = i \sum_{i,j} (\partial\tau_{ij} \wedge \bar{\partial}\bar{\tau}_{ij} + \partial\bar{\tau}_{ij} \wedge \bar{\partial}\tau_{ij}).$$

Thus,

$$|\nabla\varphi|^2\omega_g^n = \sum_{i,j} |\nabla\varphi_{ij}|^2\omega_g^n = n\phi^*(\omega_{\text{FS}}) \wedge \omega_g^{n-1}.$$

□

Remark 4.2.1. The calculation of ω_{FS} in terms of τ_{ij} only requires the almost complex structure on \mathbb{P}^m to be compatible (but not necessarily tamed) with ω_{FS} . So the lemma would hold for any pseudoholomorphic map $\phi : (M, J) \rightarrow (\mathbb{P}^m, \tilde{J})$ where \tilde{J} is compatible with ω_{FS} .

4.3 Pseudoholomorphic Vector Bundles

Let (M, J) be an almost complex manifold and $E \rightarrow M$ a complex vector bundle of complex rank r over M . Suppose further that the total space E is endowed with an almost complex structure \mathcal{J} such that,

- (1) *the projection map is pseudoholomorphic;*
- (2) *the almost complex structure induced on each fibre is multiplication by i ;*
- (3) *fibrewise multiplication and addition are pseudoholomorphic.*

Such a structure \mathcal{J} is called a bundle almost complex structure and was first introduced by Bartolomeis-Tian [13].

On the other hand one can endow a complex vector bundle with a Cauchy-Riemann type operator which we call a pseudoholomorphic structure.

Definition 4.3.1. *Let (M, J) be an almost complex manifold and $E \rightarrow M$ a complex vector bundle of complex rank r over M . A pseudoholomorphic structure on E is a differential operator $\bar{\partial}_E : \Gamma(M, E) \rightarrow \Gamma(M, (T^*M)^{0,1} \otimes E)$ satisfying the following Leibniz rule,*

$$\bar{\partial}_E(fs) = f \bar{\partial}_E s + \bar{\partial}_J f \otimes s,$$

where $f \in C^\infty(M)$ and $s \in \Gamma(M, E)$.

In [13] it is shown that bundle almost complex structures are in one-to-one correspondence with pseudoholomorphic structures.

Proposition 4.3.1 (de Bartolomeis-Tian). *There is a bijection between bundle almost complex structures and the pseudoholomorphic structures on E .*

We shall write $\bar{\partial}_{\mathcal{J}}$ for the pseudoholomorphic structure on E induced by a bundle almost complex structure \mathcal{J} . Henceforth we shall call a complex vector bundle E equipped with a bundle almost complex structure \mathcal{J} a pseudoholomorphic vector bundle and write $\bar{\partial}_E := \bar{\partial}_{\mathcal{J}}$.

A smooth section $s \in \Gamma(M, E)$ of a pseudoholomorphic bundle E is said to be pseudoholomorphic if $\bar{\partial}_E s = 0$. Note that by following the proof of the aforementioned correspondence established by Bartolomeis-Tian this definition is equivalent to $s : M \rightarrow E$ being a (J, \mathcal{J}) -holomorphic map. We write $H_{\mathcal{J}}^0(E)$ for the space of global pseudoholomorphic sections of E with respect to \mathcal{J} .

Recall that, if (E, h) is a Hermitian bundle over an almost complex manifold (M, J) , then a connection $\nabla : \Gamma(M, E) \rightarrow \Gamma(M, T^*M \otimes E)$ is said to be Hermitian if it is compatible with h , *i.e.* if

$$d(h(s_1, s_2)) = h(\nabla s_1, s_2) + h(s_1, \nabla s_2), \quad \forall s_1, s_2 \in \Gamma(M, E). \quad (4.11)$$

The following is the analogue of Lemma 2.1.2 for general pseudoholomorphic bundles.

Proposition 4.3.2. *Let (E, h) be a Hermitian bundle equipped with a pseudoholomorphic structure $\bar{\partial}_E$, then there exists a unique Hermitian connection ∇ such that $\nabla^{(0,1)} = \bar{\partial}_E$.*

Consider the dual bundle $E^* \rightarrow M$ of a pseudoholomorphic bundle $E \rightarrow M$. One can define a pseudoholomorphic structure on E^* by,

$$(\bar{\partial}_{E^*} \sigma)(s) = \bar{\partial}(\sigma(s)) - \sigma(\bar{\partial}_E s), \quad (4.12)$$

for any $\sigma \in \Gamma(M, E^*)$ and $s \in \Gamma(M, E)$. This in turn induces a bundle almost complex structure on E^* giving a natural sense in which to consider the dual bundle a

pseudoholomorphic bundle. Henceforth whenever we are dealing with the dual of a pseudoholomorphic bundle we shall, unless otherwise stated, assume that the dual bundle is equipped with this bundle almost complex structure.

It is also natural to look at the conjugate bundle \bar{E} . In the following suppose that (E, h) is a Hermitian bundle, $\bar{\partial}_E$ a pseudoholomorphic structure and ∇ the unique Hermitian connection such that $\nabla^{(0,1)} = \bar{\partial}_E$. The conjugate connection $\overline{\nabla^{(0,1)}} : \bar{E} \rightarrow (T^*M)^{0,1} \otimes \bar{E}$ defines a pseudoholomorphic structure on \bar{E} . On the other hand we can identify E^* and \bar{E} using the Hermitian metric h and thus we have an induced pairing of \bar{E} and E . Differentiating this pairing using (4.11) and taking the $(0, 1)$ part we have

$$\bar{\partial}(\sigma(s)) = \sigma(\nabla^{(0,1)}s) + (\overline{\nabla^{(0,1)}\sigma})(s).$$

Comparing with (4.12) we see that $\bar{\partial}_{E^*} = \overline{\nabla^{(0,1)}}$, *i.e.* $\bar{\partial}_{E^*}$ defines a pseudoholomorphic structure on \bar{E} .

We are now in the position to follow the usual route and define a L^2 formal adjoint $\bar{\partial}_E^*$. For this we equip (M, J) with a J -compatible Riemannian metric and write ω_g for the associated Hermitian form. We claim that the operator $\bar{\partial}_E^* := - * \nabla^{(0,1)} *$ defines the formal dual, where $*$ is the following extension of the Hodge star to E valued differential forms

$$* : \Lambda^{p,q} \otimes E \rightarrow \Lambda^{n-p,n-q} \otimes E, \quad *(\alpha \otimes s) := (*\alpha) \otimes s.$$

To see this we calculate, using Stokes theorem, that

$$\begin{aligned} \int_M h(\bar{\partial}_E(\alpha \otimes s), \beta \otimes \sigma) \omega_g^n &= \int_M \bar{\partial}_E(\alpha \otimes s) \wedge \overline{*(\beta \otimes \sigma)} \\ &= \int_M (-1)^{p+q} (\alpha \otimes s) \wedge \bar{\partial}_{\bar{E}}(\overline{*(\beta \otimes \sigma)}) \\ &= \int_M (\alpha \otimes s) \wedge \left[** \nabla^{(0,1)} *(\beta \otimes \sigma) \right], \end{aligned}$$

where $\alpha \otimes s \in \Lambda^{p,q-1} \otimes E$ and $\beta \otimes \sigma \in \Lambda^{p,q} \otimes E$. Note that the factor $(-1)^{p+q}$ appears since $\overline{*(\beta \otimes \sigma)} \in \Lambda^{n-p,n-q} \otimes \bar{E}$ and hence $\bar{\partial}_{\bar{E}}$ appearing in the second line is the natural extension of $\bar{\partial}_{\bar{E}}$ to $(n-p, n-q)$ -forms with values in E . Furthermore the final equality follows from the identity $** = (-1)^{p+q}$ on $\Lambda^{p,q}$.

We define a Laplacian operator

$$\Delta_{\bar{\partial}_E} = \bar{\partial}_E \bar{\partial}_E^* + \bar{\partial}_E^* \bar{\partial}_E.$$

Since local holomorphic coordinates are not available on the base manifold (M, J) one has to work a little harder to prove that $\Delta_{\bar{\partial}_E}$ is an elliptic operator in the non-integrable setting, but nonetheless this is indeed the case [11]. In particular the operator has finite

dimensional kernel, that is, the space

$$\mathcal{H}_{\bar{\partial}_E}^{p,q}(M, E) := \{s \in \Gamma(M, \Lambda^{p,q} \otimes E) \mid \Delta_{\bar{\partial}_E} s = 0\}$$

is finite dimensional for all $0 \leq p, q \leq n$. Since $\bar{\partial}_E^* = 0$ when acting on sections of E it is straightforward to deduce that

$$H_{\mathcal{J}}^0(E) = \mathcal{H}_{\bar{\partial}_E}^{0,0}(M, E)$$

since $\Delta_{\bar{\partial}_E} s = 0$ if and only if $\bar{\partial}_E s = 0$ and $\bar{\partial}_E^* s = 0$. We thus have the following lemma.

Lemma 4.3.1. *Let (M, J) be a closed almost complex four manifold and $E \rightarrow M$ a pseudoholomorphic vector bundle, then $H_{\mathcal{J}}^0(E)$ is a finite dimensional vector space.*

4.3.1 Globally Generated Bundles

Suppose that E is a rank r pseudoholomorphic vector bundle which is globally generated by pseudoholomorphic sections $s_1, \dots, s_N \in H_{\mathcal{J}}^0(E)$. For each point $p \in M$ let V_p be the subspace of $H_{\mathcal{J}}^0(E)$ spanned by sections vanishing at p . As in the complex setting we can define the Kodaira map κ_E by $p \mapsto \text{Ann}(V_p)$, where $\text{Ann}(V_p)$ is the annihilator subspace of V_p , that is the space of linear functionals vanishing on V_p . We can write this map as follows,

$$\kappa_E : M \rightarrow \text{Gr}(r, H_{\mathcal{J}}^0(E)^*), \quad p \mapsto \{s \in H_{\mathcal{J}}^0(E) \mid s(p) = 0\}.$$

Identifying $\text{Gr}(r, H_{\mathcal{J}}^0(E)^*)$ with $\text{Gr}(r, N) = \text{Gr}(r, \mathbb{C}^N)$ via the basis s_1, \dots, s_N we claim that the resulting map $\kappa_E : M \rightarrow \text{Gr}(r, N)$ is pseudoholomorphic.

Lemma 4.3.2. *Let (M, J) be a closed almost complex four manifold and $E \rightarrow M$ a globally generated pseudoholomorphic vector bundle. If $N = \dim H_{\mathcal{J}}^0(E)$ and J_{std} is the standard complex structure on $\text{Gr}(r, \mathbb{C}^N)$, then the Kodaira map $\kappa_E : M \rightarrow \text{Gr}(r, N)$ defined above is (J, J_{std}) -holomorphic.*

Let us first verify the claim in the relatively straightforward case of a line bundle, *i.e.* $r = 1$. Here the map $\kappa_E : M \rightarrow \mathbb{P}^N$ is given by $p \mapsto [s_1(p) : \dots : s_N(p)]$ and it suffices to check that the transition maps $\frac{s_i}{s_j}$ are pseudoholomorphic sections of the trivial bundle over $M \setminus \{s_j^{-1}(0)\}$. It follows from the identity $\frac{s_j}{s_j} = 1$, using (4.12), that $\frac{1}{s_j}$ is a pseudoholomorphic section of E^* over $M \setminus \{s_j^{-1}(0)\}$. Hence it is straightforward to observe, again using (4.12), that $\frac{s_i}{s_j}$ are pseudoholomorphic sections of the trivial bundle over $M \setminus \{s_j^{-1}(0)\}$.

The general case follows similarly but requires a brief digression to recall a holomorphic coordinate system on $\text{Gr}(r, N)$. Given a point $x \in \text{Gr}(r, N)$, if a_1, \dots, a_r are vectors spanning the subspace U_x associated to the point x , then denote by $A(x)$ the $N \times r$ matrix whose columns are a_1, \dots, a_r . We call $A(x)$ a homogeneous coordinate

of $x \in \text{Gr}(r, N)$. Notice that $A(x)$ is of maximal rank and that if $\tilde{A}(x)$ is another homogeneous coordinate of x , *i.e.* another basis of U_x is chosen, then there exists some $g \in \text{GL}(r, \mathbb{C})$ such that $\tilde{A}(x) = A(x)g$. So homogeneous coordinates are only defined up to right multiplication by some element of $\text{GL}(r, \mathbb{C})$.

Now let e_1, \dots, e_N denote the standard basis of \mathbb{C}^N and assume that we take homogeneous coordinates with respect to this basis, that is for $x \in \text{Gr}(r, N)$ we take a homogeneous coordinate $A(x)$ whose columns (with respect to the basis e_i) form a basis of U_x . Given a multi-index $I = (i_1, \dots, i_r)$ of length r such that $1 \leq i_1 < \dots < i_r \leq n$ we take a coordinate chart U_I on $\text{Gr}(r, N)$ to be the set of matrices A such that the $r \times r$ submatrix A_I is invertible. Here A_I denotes the submatrix formed by the i_1 -th to i_r -th rows of A .

Let $x \in U_I$ and $A(x)$ be a homogeneous coordinate for x , then the matrix $A(x)$ can be decomposed as

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix},$$

where $A_1 \in \text{GL}(r, \mathbb{C})$ and A_2 is some $(N - r) \times r$ matrix. Notice that for simplicity of notation we have assumed that $I = (1, 2, \dots, r)$ here. Now holomorphic coordinates on U_I are given by the matrix $A_1^{-1}A_2$.

Returning to the set-up of Lemma 4.3.2, recall that the global generating set s_1, \dots, s_N of E identifies $\text{Gr}(r, H_{\mathcal{J}}^0(E)^*)$ with $\text{Gr}(r, N) = \text{Gr}(r, \mathbb{C}^N)$ and hence we have that κ_E can be viewed as a mapping

$$\kappa_E : M \rightarrow \text{Gr}(r, N).$$

Indeed, first notice that we can cover M by open sets U_I such that s_{i_1}, \dots, s_{i_r} form a local frame for E over U_I .

Remark 4.3.1. In general it is not possible to find local pseudoholomorphic frames but here our bundle is assumed to be globally generated.

Consider now $p \in U_I$ and $s \in H_{\mathcal{J}}^0(E)$. It suffices to consider the case $I = (1, 2, \dots, r)$. Notice that if we write $a_{i\alpha}$ for the complex valued smooth functions such that

$$s_i = \sum_{\alpha=1}^r a_{i\alpha} s_{\alpha}, \tag{4.13}$$

then

$$s = \sum_{\alpha=1}^r \left(\sum_{i=1}^r a_{i\alpha} s^i, s \right) s_{\alpha},$$

where $s^i \in H_{\mathcal{J}}^0(E)^*$ are dual to s_i and (\cdot, \cdot) denotes the natural pairing of $H_{\mathcal{J}}^0(E)$ and

$H_{\mathcal{J}}^0(E)^*$. From the definition of $\kappa_E : M \rightarrow \text{Gr}(r, H_{\mathcal{J}}^0(E)^*)$ we see from the above that

$$\kappa_E(p) = \left[\text{span} \left\{ \sum_{i=1}^N a_{i\alpha} s^i \mid \alpha \in \{1, \dots, r\} \right\} \right].$$

It is straightforward to verify that in fact the right hand side is independent of the local frame initially chosen. That is, the matrix $A = (a_{i\alpha})_{i\alpha}$ is a local expression of $\kappa_E|_{U_I}$ in homogeneous coordinates.

To see that κ_E is pseudoholomorphic it suffices to check that the transition maps, when changing pseudoholomorphic frames, are themselves pseudoholomorphic sections of the bundle $\text{GL}(r, \mathbb{C}) \rightarrow M$. Indeed suppose that $U_I \cap U_J \neq \emptyset$ and write s_α, s_β for the local frames with $\alpha \in I$ and $\beta \in J$. In this case the transition data on $U_I \cap U_J$ is described by $p \mapsto (a_{\alpha\beta}(p))_{\alpha\beta} \in \text{GL}(r, \mathbb{C})$. By applying $\bar{\partial}_{\mathcal{J}}$ to (4.13) we deduce that $a_{\alpha\beta}$ are pseudoholomorphic yielding the desired conclusion.

4.4 Examples

In [11] examples are given of compact, strictly almost complex manifolds which have globally generated pseudoholomorphic line bundles, *i.e.* the pseudoholomorphic sections are base point free. This section is devoted to the discussion of these examples and showing that they in fact generalise to provide examples of globally generated pseudoholomorphic vector bundles over compact, strictly almost Kähler manifolds.

Let us briefly review the almost complex structures and the pseudoholomorphic sections (or lack thereof) of the corresponding canonical bundles found on the Kodaira-Thurston surface [11].

Recall from Chapter 2 the Kodaira-Thurston surface is given by $X = S^1 \times (\text{Nil}^3/\Gamma)$ where Nil^3 is the Heisenberg group,

$$\text{Nil}^3 = \left\{ A \in \text{GL}(3, \mathbb{R}) \mid A = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbb{R} \right\},$$

and Γ is the subgroup of Nil^3 with integral entries, acting by left multiplication. Letting t denote a coordinate on S^1 an invariant frame of TX is given by

$$\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}, \quad \frac{\partial}{\partial z},$$

with its coframe being

$$dt, \quad dx, \quad dy, \quad dz - xdy.$$

For any $a \in \mathbb{R} \setminus \{0\}$ we can define an almost complex structure J_a on X with respect

to the above frame of TX by,

$$J_a = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{a} \\ 0 & 0 & -a & 0 \end{pmatrix}.$$

It is straightforward to compute that the Newlander-Nirenberg tensor is nonzero for all $a \neq 0$ and hence that J_a is not integrable.

Furthermore the symplectic form

$$\omega_a = dx \wedge dt + \frac{1}{a} dy \wedge (dz - xdy)$$

makes (X, J_a, ω_a) an almost Kähler manifold.

Proposition 4.4.1 (c.f. Proposition 6.1 of [11]). *For any $a \in 4\pi\mathbb{Z}$ there exists a non-integrable almost complex structure J_a on the Kodaira-Thurston Surface $X = S^1 \times (\text{Nil}^3/\Gamma)$ such that*

$$\dim(H^0(X, K_{J_a})) = 1.$$

In particular, there exists a pseudoholomorphic map $\phi_a : X \rightarrow \mathbb{P}^1$.

We now take a brief digression to discuss products of pseudoholomorphic vector bundles over products of almost complex manifolds. This is necessary to produce examples in dimensions greater than 4.

Let $\pi_i : (E_i, \mathcal{J}_i) \rightarrow (M_i, J_i)$ be pseudoholomorphic vector bundles of rank r_i over closed almost complex manifolds (M_i, J_i) for $i = 1, 2$. We can equip the product manifold $M_1 \times M_2$ with the product almost complex structure $J_1 \times J_2$ which is defined to be $J_1(p) \oplus J_2(q)$ on $T_{(p,q)}(M_1 \times M_2) \cong T_p M_1 \oplus T_q M_2$. Writing $p_i : M_1 \times M_2 \rightarrow M_i$ for the projection maps we consider the product bundle $p_1^* E_1 \otimes p_2^* E_2$ which we will denote by $E_1 \otimes E_2$ for simplicity. Now consider the product almost complex structure $\mathcal{J}_1 \otimes \mathcal{J}_2$ induced on the product bundle $E_1 \otimes E_2 \rightarrow M_1 \times M_2$. It is straightforward to verify that this is indeed a bundle almost complex structure with respect to the almost complex structure $J_1 \times J_2$ on the base. We now prove a Künneth formula holds for sections of $E_1 \otimes E_2$.

Proposition 4.4.2. *Let $(E_i, \mathcal{J}_i) \rightarrow (M_i, J_i)$ be pseudoholomorphic vector bundles of rank r_i over closed almost complex manifolds (M_i, J_i) for $i = 1, 2$. Letting \mathcal{J} denote the bundle almost complex structure induced on the tensor product bundle $E_1 \otimes E_2 \rightarrow M_1 \times M_2$, where $M_1 \times M_2$ is equipped with the product almost complex structure $J_1 \times J_2$, it holds that*

$$H_{\mathcal{J}}^0(E_1 \otimes E_2) = H_{\mathcal{J}_1}^0(E_1) \otimes H_{\mathcal{J}_2}^0(E_2).$$

Proof. To proceed we equip E_i with a Hermitian metric h_i and $E_1 \otimes E_2$ with the product metric $h_1 \otimes h_2$. Now consider the space of sections $L^2(M_1 \times M_2, E_1 \otimes E_2)$, it is

straightforward to see the the set of L^2 sections of the form $s_1 \otimes s_2$ for $s_i \in L^2(M_i, E_i)$ is dense.

Let $\Delta_{\bar{\partial}_{\mathcal{J}_i}}$ be the Laplacian operator associated to $\bar{\partial}_{\mathcal{J}_i}$ acting on L^2 sections of E_i . The Laplacian operator induced on the product bundle is

$$\Delta_{\bar{\partial}_{\mathcal{J}}} = \Delta_{\bar{\partial}_{\mathcal{J}_1}} \otimes I_{E_2} + I_{E_1} \otimes \Delta_{\bar{\partial}_{\mathcal{J}_2}}.$$

This is a semi-positive, self-adjoint operator.

Let $\{\varphi_i\}$ and $\{\psi_i\}$ be the sets of eigensections of $\Delta_{\bar{\partial}_{\mathcal{J}_1}}$ and $\Delta_{\bar{\partial}_{\mathcal{J}_2}}$ respectively, with corresponding eigenvalues $\{\lambda_i\}$ and $\{\mu_i\}$. Remark that $\{\varphi_i\}$ and $\{\psi_i\}$ form a Hilbert basis of $L^2(M, E_1)$ and $L^2(M, E_2)$ respectively. By positivity of the Laplacian operator we have $\lambda_i, \mu_i \geq 0$ and hence from the above formula for the Laplacian on the product bundle we have,

$$\Delta_{\bar{\partial}_{\mathcal{J}}}(\varphi_i \otimes \psi_j) = (\lambda_i + \mu_j)\varphi_i \otimes \psi_j = 0 \iff \lambda_i = \mu_j = 0.$$

To conclude we simply remark that by denseness of sections of the form $s_1 \otimes s_2$ the set $\{\varphi_i \otimes \psi_j\}$ is a Hilbert basis of $L^2(M_1 \times M_2, E_1 \otimes E_2)$ and hence that $\ker(\Delta_{\bar{\partial}_{\mathcal{J}}}) = \text{Span}(\varphi_i \otimes \psi_j)$. That is to say, we have

$$H_{\mathcal{J}}^0(E_1 \otimes E_2) = H_{\mathcal{J}_1}^0(E_1) \otimes H_{\mathcal{J}_2}^0(E_2),$$

by the Hodge theory in the paragraph preceding Lemma 4.3.1. □

Proposition 4.4.3. *For any positive integers $n, k \geq 2$ there are examples of compact $2n$ -dimensional strictly almost Kähler manifolds admitting globally generated pseudoholomorphic vector bundles of rank k .*

Proof. Consider a closed Riemann surface S with a rank $k - 1$ holomorphic vector bundle. By taking products of the Kodaira-Thurston surface X (equipped with J_a and ω_a for $a \in 4\pi\mathbb{Z}$) with S and applying Proposition 4.4.2 we obtain compact, strictly almost Kähler manifolds admitting globally generated pseudoholomorphic vector bundles of rank k . □

Corollary 4.4.1. *For any positive integer $n \geq 2$ there are examples of compact $2n$ -dimensional strictly almost Kähler manifolds admitting a non-trivial pseudoholomorphic map into some projective space \mathbb{P}^N of dimension N .*

For an explicit example one only needs to give an explicit holomorphic vector bundle on a Riemann surface. Given any closed Riemann surface S there exists a holomorphic embedding $S \hookrightarrow \mathbb{P}^N$ for some $N \geq 3$ (in fact one can take $N = 3$ by projecting). The normal bundle associated to this embedding is a non-trivial, globally generated holomorphic vector bundle of rank $N - 1$ on S .

4.5 Regularity of the Level Sets

The aim of this section is to prove the following proposition which facilitates the construction of the desired test functions.

Proposition 4.5.1. *Let (M^n, J, g) be an almost Kähler manifold and $\phi : M \rightarrow \mathbb{P}^m$ a non-trivial pseudoholomorphic map. Then the push forward of volume measure $\mu := \phi_*(\text{Vol}_g)$ is non-atomic.*

Recall that if (X, d, ν) is a metric measure space then ν is non-atomic if and only if $\nu(\{x\}) = 0$ for any point $x \in X$.

A key property of pseudoholomorphic maps is that they have a unique continuation property. This is well known to experts but we include a proof here for completeness.

Lemma 4.5.1. *For $i = 1, 2$ let (M_i, J_i) be almost complex manifolds of dimension $2n$ and $2m$ respectively and $\phi : M_1 \rightarrow M_2$ be a non-trivial pseudoholomorphic map. If there exists an open set $U \subset M_1$ such that $\phi(U) = y$ for some $y \in M_2$, then $\phi \equiv y$.*

Since the unique continuation property is well known for pseudoholomorphic curves we provide a proof of the lemma using this fact.

Proof. Suppose that ϕ is constant on some open set U , without loss of generality we may assume that $U \neq M$ is a maximal such open set. By continuity ϕ is also constant on \bar{U} .

Notice that given a smooth, embedded J_1 -holomorphic disk $u : D \rightarrow M_1$ we have that $\phi \circ u : D \rightarrow M_2$ is a smooth, not necessarily embedded, J_2 -holomorphic disk. Unique continuation for pseudoholomorphic curves is well known.

To conclude suppose that $x \in \partial U$ and consider a foliation of J_1 -holomorphic disks transverse to ∂U . Details about such foliations may be found in Lemma 2.1.5. By shrinking the disk radius parameter of the foliation we may assume that U intersects each disk in the fibration in some relatively open set.

Now on the restriction to each disk in the foliation ϕ is a pseudoholomorphic disk in (M_2, J_2) . By assumption ϕ is constant on an open set in each fibre and hence, by unique continuation, ϕ is constant on each fibre. Thus ϕ is constant on an open neighbourhood of $x \in \partial U$ contradicting the maximality of U . \square

In the remainder of this section we prove an estimate on the Hausdorff dimension of level sets of pseudoholomorphic maps from which Proposition 4.5.1 follows but which is also of independent interest.

Proposition 4.5.2. *For $i = 1, 2$ let (M_i, J_i) be almost complex manifolds of dimension $2n$ and $2m$ respectively with M_1 compact. If $\phi : M_1 \rightarrow M_2$ is a non-trivial C^1 pseudoholomorphic map, then for any $y \in M_2$ we have that $\mathcal{H}^{2n-2}(\phi^{-1}(y)) < \infty$.*

Since local holomorphic coordinates are not available in the almost complex case we use the local model proposed by Taubes, that is, we foliate neighbourhoods by embedded J -holomorphic disks. It is well known that the preimage of any point on a non-trivial J -holomorphic disk consists of a finite number of points. This leads to the following simple description of the intersection of $\phi^{-1}(y)$ and J -holomorphic disks.

Lemma 4.5.2. *For $i = 1, 2$ let (M_i, J_i) be almost complex manifolds of dimension $2n$ and $2m$ respectively and $\phi : M_1 \rightarrow M_2$ be a pseudoholomorphic map. Further let $u : D \rightarrow M_1$ be a smooth embedded J_1 -holomorphic disk. Then for any $y \in M_2$ either $u(D) \subset \phi^{-1}(y)$, or, the set $u(D) \cap \phi^{-1}(y)$ either consists of a finite number of isolated points.*

Proof. Remark that $\phi \circ u : D \rightarrow M_2$ is a J_2 -holomorphic disk and so it suffices to show that for a J -holomorphic disk the preimage of any point is a finite set. But this is a well known fact about pseudoholomorphic curves, see Proposition 2.1.4. \square

With these two lemma's and the foliations of holomorphic disks of §2.1.3 we are ready to estimate the Hausdorff measure.

Proof of Proposition 4.5.2. As with the proof of Proposition 3.2.1 this proof follows closely the structure of the proof of Proposition 2.4 in [60].

Fix $y \in M_2$ and write $Z = \phi^{-1}(y)$. We show that $\mathcal{H}^{2n-2}(Z) < \infty$. First note that since M_1 is compact the Hausdorff measure will be independent of the metric we use. Now for any $x \in Z$ we can find a J -fibre-diffeomorphism Q^x of a neighbourhood of x in M_1 . By compactness we can choose finitely many of these diffeomorphisms, say Q^{x_i} , covering Z and such that the disks are all of the same radius. We show that each $Z \cap Q^{x_i}(D \times B)$ has finite $(2n - 2)$ -dimensional Hausdorff measure.

Pick $x \in Z$ and write Q for Q^x . For each w we know that $Q(D_w)$ intersects Z in finitely many points if it is not totally contained in Z by Lemma 4.5.2. We claim that there are only finitely many $w \in \bar{D}$ such that $Q(D_w) \subset Z$.

Suppose that this is not the case. Then we may assume, without loss of generality, that 0 is an accumulation point of w . We now foliate a neighbourhood of x by J -holomorphic disks transverse to $Q(D_0)$, whereby producing an open neighbourhood M which is contained in Z . Since this contradicts Lemma 4.5.1 we will then have the claim.

As before take Gaussian coordinates centred at x but now so that $(0, w')$ is identified with $Q(D_0)$. We choose a J -fibre-diffeomorphism $Q' : D' \times B \rightarrow \mathbb{C}^n$, where D' denotes the disk in \mathbb{C} of radius $\rho' < \rho$, such that,

- $\forall w' \in D', Q'(D'_{w'})$ is a J -holomorphic submanifold containing $(0, w')$;
- $\forall w' \in D'$, there exists z depending only on Ω and J such that

$$|(\xi', w') - Q'(\xi', w')| \leq z \cdot \rho' \cdot |\xi'|;$$

- $\forall w' \in D'$, the derivatives of order m of Q' are bounded by $z_m \cdot \rho'$, where z_m depends only on Ω and J .

So all the disks $Q'(D'_{w'})$ are transverse to $Q(D_0)$. As being transverse is an open condition we have that $Q'(D'_{w'})$ are transverse to $Q(D_w)$ for all $|w| < \varepsilon$. Thus the intersection points of $Q'(D'_{w'})$ and Z are not isolated and so, by Lemma 4.5.2, $Q'(D'_{w'}) \subset Z$. So $Q'(D' \times B) \subset Z$ and since $Q'(D' \times B)$ covers an open neighbourhood of x we have the desired contradiction.

Now we claim that Q may be chosen so that none of the J -holomorphic disks are contained in Z . In fact we show that there are only finitely many complex directions of $T_x M$ such that there are J -holomorphic disks tangent to it and contained in Z . With this the claim follows by rotating the Gaussian coordinate system we choose initially.

Suppose that there are infinitely many such directions. Since the directions in $T_x M$ are parametrised by \mathbb{P}^{n-1} there is at least one accumulative direction v . Choose the Gaussian coordinate system so that $Q(D_0)$ is transverse to v , and hence $Q(D_w)$ are transverse to v for small $|w| < \varepsilon$. This is a contradiction with Lemma 4.5.2 and Lemma 4.5.1 since the intersection numbers of $Q(D_w) \cap Z$ are infinite for $|w| < \varepsilon$.

Hence if we fix x then we can choose a complex direction such that there is no J -holomorphic curve in Z tangent to it. By the perturbative nature of J -fibre diffeomorphisms we can choose Gaussian coordinates and a J -fibre diffeomorphism so that no $Q(D_w)$ is contained in Z for w sufficiently close to 0.

Finally we are able to estimate the Hausdorff measure of the compact set $Z \cap Q(\bar{D} \times \bar{B})$. First remark that, by shrinking D and B if necessary, we may assume without loss of generality that the distortion of Q on the domain $2D \times 2B$ is bounded by some constant $C > 0$. Also note that, by our choice of Q , for each $w \in \bar{D}$ the set $Z \cap Q(\bar{D}_w)$ is a finite set of points.

Define,

$$g : \bar{D} \rightarrow \mathbb{N} \cup \{0\}, \quad w \mapsto \#(Z \cap Q(\bar{D}_w)).$$

Clearly this is an upper semi-continuous function and hence achieves a maximal value, say N , at some point $w \in \bar{D}$. Thus by Lemma 4.5.2, we know $Z \cap Q(\bar{D}_w)$ contains at most N points for all $w \in \bar{D}$. By the Vitali covering lemma we can take a finite cover of the compact set $Z \cap Q(\bar{D} \times \bar{B})$ by balls of radius ε such that L of these balls are disjoint and the union of L concentric balls with radius dilated by a factor of 3 cover. By our distortion assumption each ε ball intersects $Q(2\bar{D}_w)$ in an open set of area bounded above by $\pi C^2 \varepsilon^2$. The coarea formula then yields,

$$N \pi C^2 \varepsilon^2 \cdot \pi C^2 (2\rho)^2 > L \omega_{2n} \varepsilon^{2n},$$

where ω_{2n} is the volume of the unit $2n$ -ball. Hence there is a constant $C' > 0$ such that $C' \varepsilon^{-(2n-2)}$ balls of radius 3ε are enough to cover $Z \cap Q(\bar{D} \times \bar{B})$. This finishes the proof. \square

4.6 Test Functions

The functions defined in this section are identical to those of §3.2 of [33]. We include an outline of their definitions and properties for completeness.

Given $[W] \in \mathbb{P}^m$ and a number $t > 0$ consider the \mathbb{C} -linear operator $\Theta_{t,[W]} : \mathbb{C}^{m+1} \rightarrow \mathbb{C}^{m+1}$ defined by,

$$\Theta_{t,[W]}Z = \begin{cases} Z & \text{if } Z \in [W], \\ tZ & \text{if } \langle Z, W \rangle = 0, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Hermitian inner product on \mathbb{C}^{m+1} . This induces a biholomorphism $\theta_{t,[W]} : \mathbb{P}^m \rightarrow \mathbb{P}^m$ via $\theta_{t,[W]}[Z] := [\Theta_{t,[W]}Z]$. Clearly the point $[W]$ and the points $[Z]$ associated to 1-dimensional orthogonal subspaces to $[W]$ are fixed points of $\theta_{t,[W]}$ for any $t > 0$.

Now recall that $\varphi_{[W]} - \frac{1}{m+1}$ is a first eigenfunction of $(\mathbb{P}^m, \omega_{\text{FS}})$, where $\varphi_{[W]} : \mathbb{P}^m \rightarrow \mathbb{R}$ is defined by

$$\varphi_{[W]}([Z]) := \frac{|\langle Z, W \rangle|^2}{|Z|^2|W|^2}. \quad (4.14)$$

The metric balls in $(\mathbb{P}^m, \omega_{\text{FS}})$ can be written,

$$B_{[W]}(r) = \{[Z] \in \mathbb{P}^m \mid \varphi_{[W]}([Z]) < \cos^2 r\},$$

where $r \in (0, \frac{\pi}{2})$. By stereographic projection it is straightforward to verify that the Fubini-Study metric satisfies,

$$\cos(\text{dist}_{\omega_{\text{FS}}}([Z], [W])) = \frac{|\langle Z, W \rangle|}{|Z||W|}.$$

Hence we can write

$$\varphi_{[W]}([Z]) = \cos^2(\text{dist}_{\omega_{\text{FS}}}([Z], [W])). \quad (4.15)$$

Clearly the function $\varphi_{[W]} - \frac{1}{2}$ is positive on $B_{[W]}(\frac{\pi}{4})$ and vanishes on the boundary.

Given $R \in (0, \frac{\pi}{4})$ there exists a unique value $t > 0$, depending only on R , such that $\theta_{t,[W]}(B_{[W]}(2R)) = B_{[W]}(\frac{\pi}{4})$. With this choice of t we define,

$$\Psi_{R,[W]}([Z]) = \begin{cases} \varphi_{[W]}(\theta_{t,[W]}([Z])) - \frac{1}{2}, & \text{if } [Z] \in B_{[W]}(2R), \\ 0, & \text{if } [Z] \notin B_{[W]}(2R). \end{cases} \quad (4.16)$$

Here we collect some necessary facts concerning Ψ .

Lemma 4.6.1. *For any $[W] \in \mathbb{P}^m$ and any $R \in (0, \frac{\pi}{4})$ the function $\Psi_{R,[W]}$ defined by (4.16) has the following properties,*

- $\Psi_{R,[W]}$ is a non-negative Lipschitz function,
- $\text{Supp } \Psi_{R,[W]} \subset B_{[W]}(2R)$,

- $\Psi_{R,[W]} \leq \frac{1}{2}$ on \mathbb{P}^m ,
- $\Psi_{R,[W]} \geq \frac{3}{10}$ on $B_{[W]}(R)$.

We shall see that it is necessary for our test functions to be supported on annuli and so we introduce a second function, distinguished by a bar, which is supported on the complement a metric ball. Again, given $r \in (0, \frac{\pi}{2})$ there exists a unique value $t > 0$ depending only on r such that $\theta_{t,[W]}(B_{[W]}(\frac{r}{2})) = B_{[W]}(\frac{\pi}{4})$. We then define,

$$\bar{\Psi}_{R,[W]}([Z]) = \begin{cases} 0, & \text{if } [Z] \in B_{[W]}(\frac{r}{2}), \\ \varphi_{[W]}(\theta_{t,[W]}([Z]) + 1)^{-1} - \frac{2}{3}, & \text{if } [Z] \notin B_{[W]}(\frac{r}{2}). \end{cases} \quad (4.17)$$

Lemma 4.6.2. *For any $[W] \in \mathbb{P}^m$ and any $r \in (0, \frac{\pi}{2})$ the function $\bar{\Psi}_{r,[W]}$ defined by (4.17) has the following properties,*

- $\bar{\Psi}_{R,[W]}$ is a non-negative Lipschitz function,
- $\text{Supp } \bar{\Psi}_{R,[W]} \subset \mathbb{P}^m \setminus B_{[W]}(\frac{r}{2})$,
- $\bar{\Psi}_{R,[W]} \leq \frac{1}{3}$ on \mathbb{P}^m ,
- $\bar{\Psi}_{R,[W]} \geq \frac{1}{6}$ on $B_{[W]}(r)$.

Finally consider the nested annuli $A \subset 2A$ defined by

$$A := B_{[W]}(R) \setminus B_{[W]}(r), \quad \text{and} \quad 2A := B_{[W]}(2R) \setminus B_{[W]}(\frac{r}{2}),$$

where $0 \leq r < R < \frac{\pi}{4}$ and $[W] \in \mathbb{P}^m$. Now define a function u_A on \mathbb{P}^m by $u_A := \Psi_{R,[W]} \bar{\Psi}_{r,[W]}$. Finally the following lemma lists the properties of u_A that shall be needed in the next section.

Lemma 4.6.3. *For any $[W] \in \mathbb{P}^m$ and any $0 \leq r < R < \frac{\pi}{4}$ the function $u_A := \Psi_{R,[W]} \bar{\Psi}_{r,[W]}$ has the following properties,*

- u_A is a non-negative Lipschitz function,
- $\text{Supp } u_A \subset 2A$,
- $u_A \leq \frac{1}{6}$ on \mathbb{P}^m ,
- $u_A \geq \frac{1}{20}$ on A .

4.7 Proof of Theorem 4.1.1

By the characterisation (4.6) it is enough to produce $k+1$ Lipschitz functions u_i which satisfy,

$$\mathcal{R}_g(u_i) \leq C(n, m)d([\phi], [\omega_g])k. \quad (4.18)$$

In fact is enough to produce k test functions since we can take the $(k + 1)$ -th function to be constant.

Recall the following definition for (X, d) a separable metric space.

Definition 4.7.1. *For an integer $N > 1$ the metric space (X, d) is said to have the N -covering property if each metric ball $B_p(2r)$ can be covered with N balls of radius r .*

The next proposition facilitates the construction of the desired k test functions on $(\mathbb{P}^m, d_{\omega_{\text{FS}}})$. The statement we use appears as Proposition 3.1 in [33] and is a reformulation of Corollary 3.2 in the work of Grigoryan, Netrusov, and Yau [23]. In more detail it provides a collection of k disjoint sets of controlled measure which will form the supports of the test functions. These sets cannot, in general, be taken to be metric balls but can be taken as annuli for which we use the following notation,

$$A = \{x \in X \mid r \leq d(x, p) < R\}, \quad 2A = \{x \in X \mid \frac{r}{2} \leq d(x, p) < 2R\}.$$

Proposition 4.7.1. *Let (X, d) be a separable metric space with the property that all metric balls $B_p(r)$ are precompact. Suppose further that it satisfies the N -covering property for some $N > 1$. Then for any finite, non-atomic measure μ on (X, d) and any positive integer k there exists a collection of k disjoint annuli $2A_i$ such that*

$$\mu(A_i) \geq \frac{c\mu(X)}{k}, \quad \text{for any } 1 \leq i \leq k, \quad (4.19)$$

where $c = c(N) > 0$ is some constant depending only on N .

Now consider the metric space $(\mathbb{P}^m, d_{\omega_{\text{FS}}})$, by volume comparison one can compute that it satisfies the N -covering property with $N = 9^{2m}$. Further we endow $(\mathbb{P}^m, d_{\omega_{\text{FS}}})$ with the measure $\mu := \phi_* \text{Vol}_g$ which is finite and non-atomic by Proposition 4.5.1. Thus, for a given integer $k \geq 1$, there is a collection of k annuli $A_i \subset \mathbb{P}^m$ such that the annuli $2A_i$ are disjoint and

$$\mu(A_i) \geq \frac{c\mu(\mathbb{P}^m)}{k}, \quad (4.20)$$

with $c > 0$ depending only on m .

Finally we can define our test functions to be $u_i = u_{A_i} \circ \phi$ where u_{A_i} are as in Lemma 4.6.3 and A_i are the annuli of the previous paragraph. Since the annuli are disjoint these k functions are linearly independent as each u_i is supported on $\phi^{-1}(2A_i)$.

For each u_i we have the following estimate,

$$\int_M u_i^2 \frac{\omega_g^n}{n!} \geq \frac{\text{Vol}_g(\phi^{-1}(A_i))}{400} = \frac{\mu(A_i)}{400} \geq \frac{c\mu(\mathbb{P}^m)}{400k} \geq \frac{c}{400n!} \frac{1}{k} \int_M \omega_g^n, \quad (4.21)$$

where the first inequality follows from Lemma 4.6.3, the second inequality from (4.20) and the third from the definition of μ . We are thus left to estimate the gradient.

The gradient estimate is identical to [33] so we provide only a brief sketch. Using the definition of the functions u_{A_i} from the previous section the key is to estimate the

following quantity,

$$\int_M |\nabla(\Psi_i \circ \phi)|^2 \omega_g^n \leq \int_M |\nabla(\tau \circ \theta_{t_i, [W_i]} \circ \phi)|^2 \omega_g^n.$$

Here τ denotes the moment map of the action of the group of isometries on $(\mathbb{P}^m, \omega_{\text{FS}})$. The appearance of τ follows from the fact that, after a rotation and scaling, the functions $\varphi_{[W_i]}$ defined by (4.14) can be viewed as components $\tau_{k\ell}$ of τ (see (4.8)). Now Lemma 4.2.1 implies,

$$\int_M |\nabla(\tau \circ \theta_{t_i, [W_i]} \circ \phi)|^2 \frac{\omega_g^n}{n!} \leq \frac{1}{(n-1)!} \int_M \phi^* \omega_{\text{FS}} \wedge \omega_g^{n-1}.$$

Similarly one estimates,

$$\int_M |\nabla(\bar{\Psi}_i \circ \phi)|^2 \frac{\omega_g^n}{n!} \leq \frac{1}{(n-1)!} \int_M \phi^* \omega_{\text{FS}} \wedge \omega_g^{n-1}.$$

Overall this yields,

$$\int_M |\nabla u_i|^2 \frac{\omega_g^n}{n!} \leq \frac{4}{(n-1)!} \int_M \phi^* \omega_{\text{FS}} \wedge \omega_g^{n-1}. \quad (4.22)$$

Combining with estimate (4.21) we have an estimate of the form (4.18).

4.8 Eigenvalues of Pseudoholomorphic Subvarieties

4.8.1 Analysis on Pseudoholomorphic Subvarieties

For the purposes of this section we use the following definition of an irreducible pseudoholomorphic subvariety.

Definition 4.8.1. *Let $(M^{n+\ell}, J)$ be an almost complex manifold of dimension $n+\ell$ then we say that a closed set $\Sigma^n \subset M^{n+\ell}$ is a n -dimensional irreducible pseudoholomorphic subvariety if it is the image of a somewhere immersed smooth pseudoholomorphic map $\Phi : X^n \rightarrow M^{n+\ell}$, where X^n is a n -dimensional smooth, closed, connected almost complex manifold.*

We call the image of the set of points $p \in X$ where the differential $d_p \Phi$ is not of full rank the singular set of Σ and denote it by Σ_{sing} . The regular locus, Σ_ is the complement $\Sigma \setminus \Sigma_{\text{sing}}$.*

Recall the definition of a real $2n$ -current C in M being an almost complex integral cycle given in §3.3. It follows immediately from Definition 4.8.1 that a pseudoholomorphic subvariety defines an almost complex integral cycle.

Let $(M^{n+\ell}, J, g)$ be an almost Kähler manifold and $\Sigma^n \subset M^{n+\ell}$ a n -dimensional irreducible pseudoholomorphic subvariety. Furthermore, write ω_g for the Kähler form which is a closed, non-degenerate 2-form. For any $x \in M$ and any n -dimensional

complex subspace $W \subset T_x M$ we can identify $W \cong \mathbb{C}^n$ on which $\omega_g|_x$ is a non-degenerate 2-form. Hence the Wirtinger inequality, see [12] for example, implies

$$\frac{1}{n!} |\omega_g^n| \leq 1.$$

On the other hand, recall that, for $1 \leq p \leq n$, a p -form α is called a calibration on an Riemannian manifold if it is closed and at any point x and any p -dimensional oriented subspace W of the tangent space at x it holds that

$$|\alpha|_W \leq 1.$$

Moreover we say that a submanifold of N is a calibrated submanifold with respect to α if we have equality above for all points on the submanifold. Hence we see that the $2n$ -form $\frac{1}{n!} \omega_g^n$ is a calibration on M and furthermore, since

$$\frac{1}{n!} \int_{\Sigma} \omega_g^n = \text{Vol}_g(\Sigma),$$

Σ is a calibrated $\frac{1}{n!} \omega_g^n$ -current. Thus by the fundamental work of Harvey-Lawson [26] Σ is a minimal current in (M, g) .

Lemma 4.8.1. *Let $(M^{n+\ell}, J, g)$ be an almost Kähler manifold and $\Sigma^n \subset M^{n+\ell}$ a n -dimensional irreducible pseudoholomorphic subvariety. Then Σ is a minimal current in M . Moreover there exists a positive integer N and a map $F : \Sigma \rightarrow \mathbb{R}^N$ such that $F(\Sigma)$ is a current of bounded mean curvature.*

Since almost complex integral cycles in almost Kähler manifolds are area minimisers Almgren's big regularity theorem applies and the singular set has finite $(2n - 2)$ -Hausdorff dimension. For the case of pseudoholomorphic subvarieties in the sense of definition 4.8.1 we can have the stronger property that the $(2n - 2)$ -Hausdorff measure is finite. It is in fact a corollary of the following theorem of Zhang [60].

Theorem 4.8.1 (Theorem 3.8 of [60]). *Suppose (M^{2n}, J) is an almost complex $2n$ -dimensional manifold, and Z_2 is a codimension 2, compact, connected, almost complex submanifold. Let (M_1, J_1) be a compact, connected, almost complex manifold of dimension $2k < 2n$ and $u : M_1 \rightarrow M$ a pseudoholomorphic map such that $u(M_1) \not\subseteq Z_2$. Then $u^{-1}(Z_2) \subset M_1$ is a closed set with finite $(2k - 2)$ -dimensional Hausdorff measure and a positive cohomology assignment.*

Indeed we can use the framework of 1-jets of pseudoholomorphic maps [19] to view the singular set as the preimage of an almost complex submanifold under a pseudoholomorphic map. This is the approach taken by Zhang in the proof of Theorem 5.5 of [60] and of Proposition 6.1 of [11].

Lemma 4.8.2. *Let $(M^{n+\ell}, J)$ be an almost complex manifold of dimension $n + \ell$*

and $\Sigma^n \subset M^{n+\ell}$ a n -dimensional irreducible pseudoholomorphic subvariety. Then $\mathcal{H}^{2n-2}(\Sigma_{sing}) < \infty$ and in particular $\mathcal{H}^{2n-1}(\Sigma_{sing}) = 0$.

Proof. Since M and X are compact the finiteness of the Hausdorff measure on each space will be independent of the choice of Riemannian metric. Moreover since the defining map Φ is a smooth map between smooth, compact manifolds it is Lipschitz. Thus it suffices to prove that $\mathcal{H}^{2n-2}(\Phi^{-1}(\Sigma_{sing})) < \infty$.

Let $z_1, \dots, z_{n+\ell}$ be complex coordinates on some open set $U \subset M$ and write $V = \Phi^{-1}(U)$. We will study the intersection of the singular set with all possible projections of U onto n of its coordinates.

Consider first the case of projection onto the first n coordinates z_1, \dots, z_n , writing $U_1 = \text{proj}_{z_1, \dots, z_n}(U)$. As explained in [60] the manifold of 1-jets of pseudoholomorphic maps from V to U_1 , say $J^1(V, U_1)$, may be identified with the total space, E_1 , of the complex vector bundle over $V \times U_1$ whose fibre at each point (x, y) is the complex vector space of \mathbb{C} -linear homomorphisms $T_x V \rightarrow T_y U_1$. As shown in [19] there is a canonical almost complex structure \mathcal{J} such that the lift of each pseudoholomorphic map $\Psi : V \rightarrow U_1$, i.e. $\Psi_{E_1}(x) = (x, \Psi(x), d\Psi|_{T_x M})$, is a pseudoholomorphic map $V \rightarrow J^1(V, U_1)$. Taking the fibrewise determinant of E_1 we get a complex line bundle $L_1 = \det E_1$ over $V \times U_1$ whose total space inherits a natural almost complex structure from $J^1(V, U_1)$. A pseudoholomorphic map $\Psi : V \rightarrow U_1$ induces a pseudoholomorphic map $\Psi_{L_1}(x) = (x, \Psi(x), \det d\Psi|_{T_x M})$. For Φ , a defining map associated to Σ , we write $\Phi_1(x) = \text{proj}_{z_1, \dots, z_n}(\Phi(x))$ which is a pseudoholomorphic map $V \rightarrow U$. Hence there is an induced pseudoholomorphic section of L_1 which we call Φ_{1, L_1} .

Repeating the above procedure for all possible projections onto n coordinates one obtains a finite family of line bundles L_i and pseudoholomorphic maps Φ_{i, L_i} . Recalling that $\Phi^{-1}(\Sigma_{sing})$ is the set of points $x \in X$ such that $d_x \Phi$ is not of full rank it follows that $\Phi^{-1}(\Sigma_{sing}) \cap V = \bigcap_i \Phi_{i, L_i}^{-1}(V \times U_i \times \{0\})$.

If $\Phi_{i, L_i}^{-1}(V \times U_i \times \{0\})$ is not the whole of V then we can apply Theorem 4.8.1 to deduce that it has finite $(2n - 2)$ -dimensional Hausdorff measure. On the other hand if every $\Phi_{i, L_i}^{-1}(V \times U_i \times \{0\})$ were the whole of V then this would contradict Φ being a somewhere immersed pseudoholomorphic map. In particular there exists an i such that $\mathcal{H}^{2n-2}(\Phi_{i, L_i}^{-1}(V \times U_i \times \{0\})) < \infty$ and hence $\mathcal{H}^{2n-2}(\Phi^{-1}(\Sigma_{sing}) \cap V) < \infty$. \square

As an application of this fact we can deduce a Sobolev inequality on Σ using the work of Michael and Simon [42].

Lemma 4.8.3. *Let $(M^{n+\ell}, J, g)$ be a closed almost Kähler manifold and $\Sigma^n \subset M^{n+\ell}$ be an irreducible pseudoholomorphic subvariety. Then the inclusion $W^{1,2}(\Sigma) \subset L^2(\Sigma)$ is compact.*

Proof. First recall from the definition of pseudoholomorphic subvarieties that Σ is compact. The argument is standard once we establish an appropriate Sobolev inequality, for example see §5 of [37]. By Lemma 4.8.1 we can view Σ as a compact current

of bounded mean curvature in some Euclidean space \mathbb{R}^N . Thus the Michael-Simon Sobolev inequality [42] applies and we can conclude compactness in the usual way, see for example the arguments in §5 of [37]. \square

Let $(M^{n+\ell}, J, g)$ be an almost Kähler manifold and $\Sigma^n \subset M^{n+\ell}$ a n -dimensional irreducible pseudoholomorphic subvariety. The metric g restricts to a (possibly incomplete) Kähler metric g_Σ on the regular locus Σ_* . We consider the exterior derivative as an operator on the domain

$$\mathcal{D}(d) := \{u \in C^1(\Sigma_*) \mid u, du \in L^2(\Sigma)\}.$$

Further, if δ denotes the co-differential with respect to g_Σ , then we consider δ as an operator on

$$\mathcal{D}(\delta) := \{\alpha \in C^1(\Sigma_*; T^*\Sigma_*) \mid \alpha, \delta\alpha \in L^2(\Sigma)\}.$$

We denote by $\Delta_\Sigma = -\delta d$ the Laplace-Beltrami operator of g_Σ on the following function space

$$\begin{aligned} \mathcal{D}(\Delta_\Sigma) &= \{u \in C^2(\Sigma_*) \mid u \in \mathcal{D}(d), du \in \mathcal{D}(\delta)\} \\ &= \{u \in C^2(\Sigma_*) \mid u, \nabla u, \Delta_\Sigma u \in L^2(\Sigma)\} \subset L^2(\Sigma). \end{aligned}$$

Since the singular set is of real codimension at least 2 we can follow the arguments of Li and Tian [37] leading us to the following lemma.

Lemma 4.8.4. *Let $(M^{n+\ell}, J, g)$ be a closed almost complex manifold of dimension $n+\ell$ and $\Sigma^n \subset M^{n+\ell}$ a n -dimensional irreducible pseudoholomorphic subvariety equipped with g_Σ . Then the closure of the Laplacian $\Delta = \Delta_{g_\Sigma}$ is self-adjoint.*

We thus have all of the ingredients to prove that Δ_Σ is essentially self-adjoint and has discrete spectrum. We refer the reader to Proposition 4.1 of [33] for the proof.

Proposition 4.8.1. *Let $(M^{n+\ell}, J, g)$ be a closed almost complex manifold of dimension $n+\ell$ and $\Sigma^n \subset M^{n+\ell}$ a n -dimensional irreducible pseudoholomorphic subvariety equipped with g_Σ . The Laplace-Beltrami operator associated to the induced metric g_Σ is essentially self-adjoint and has discrete spectrum.*

Proof of Theorem 4.1.2

From the previous section we can make sense of the eigenvalues of the Laplacian. Moreover, as remarked in [33], a standard argument shows that the variational principle continues to hold for these eigenvalues. Thus the proof reduces to finding k linearly independent functions $u_i \in W^{1,2}(\Sigma)$ such that

$$\mathcal{R}(u_i) \leq C(n, m) \frac{\int_\Sigma \phi^* \omega_{\text{FS}} \wedge \omega_g^{n-1}}{\int_\Sigma \omega_g^n} k.$$

For the test functions we again use $u_i = u_{A_i} \circ \phi$ where u_{A_i} is the function constructed on \mathbb{P}^m in §4. Since $\text{Vol}_g(\Sigma^*)$ is finite and u_{A_i} is Lipschitz it follows that $u_i \in W^{1,2}(\Sigma)$. It is now straightforward to verify that the argument of the proof of Theorem 4.1.1 can be followed to the desired conclusion.

4.9 Further Directions

There are several potential extensions of Theorem 4.1.1 one could explore. We have already remarked that Lemma 4.2.1 holds for ϕ a pseudoholomorphic map with respect to any almost complex structure on \mathbb{P}^m compatible with the Fubini-Study form ω_{FS} . With this fact it is easy to deduce that in fact the Theorem 4.1.1 holds for any almost Kähler manifold admitting such a map. At present no examples of strictly almost Kähler manifolds admitting a pseudoholomorphic map to \mathbb{P}^m with a non-standard almost complex structure compatible with the Fubini-Study form are known.

4.9.1 Locally Approximable Pseudoholomorphic Maps

In another direction we could weaken the regularity assumptions of the map ϕ and consider the class of locally approximable pseudoholomorphic maps.

Let (M^{2n}, J_M, h_M) and (N^{2m}, J_N, h_N) be a compact almost Hermitian manifold and consider an isometric embedding $N \hookrightarrow \mathbb{R}^k$. We can define a Sobolev space for maps $u : M \rightarrow N$ by,

$$W^{1,2}(M, N) := \{u \in W^{1,2}(M, \mathbb{R}^k) \mid u(x) \in N \text{ a.e. } x \in M\},$$

where $W^{1,2}(M, \mathbb{R}^k)$ is defined as in Chapter 2. Similarly we define,

$$W_{loc}^{1,2}(M, N) := \{u \in W_{loc}^{1,2}(M, \mathbb{R}^k) \mid u(x) \in N \text{ a.e. } x \in M\}.$$

Following the lead of Rivière and Tian [46] we say that a map $u \in W_{loc}^{1,2}(M, N)$ is locally approximable if for any ball $\bar{B} \subset M$ there exists a sequence $u_i \in C^\infty(M, N)$ such that

$$u_i \rightarrow u \text{ strongly in } W^{1,2}(B, N).$$

It is proven in [4] that $u \in W_{loc}^{1,2}(M, N)$ is locally approximable if and only if

$$\int_M u^* \omega \wedge d\alpha = 0, \quad \forall \alpha \in \Omega^{2n-3}(B) \text{ and } \forall \omega \in \mathcal{Z}^2(N),$$

for any ball $\bar{B} \subset M$. That is, if and only if

$$d(u^* \omega) = 0, \quad \forall \omega \in \mathcal{Z}^2(N),$$

holds in the sense of currents.

Thus, given a locally approximable $u \in W_{loc}^{1,2}(M, \mathbb{P}^m)$, we see that $u^*\alpha$ defines a closed 2-current on M for any closed 2-form $\alpha \in \Omega^2(\mathbb{P}^m)$. By a theorem of de Rham any closed 2-current is cohomologous to a smooth, closed 2-form and thus u induces a well-defined map $u^* : H^2(\mathbb{P}^m) \rightarrow H^2(M)$.

We say that a map $\phi \in W_{loc}^{1,2}(M, \mathbb{P}^m)$ is pseudoholomorphic if

$$J_{\mathbb{P}^m} \circ d\phi(x) = d\phi(x) \circ J, \quad \text{a.e. } x \in M. \quad (4.23)$$

Lemma 4.9.1. *Let (M^{2n}, J, h) be a compact almost Hermitian manifold and $\phi \in W^{1,2}(M, \mathbb{P}^m)$ a pseudoholomorphic map, then*

$$n \omega^{n-1} \wedge \phi^* \omega_{FS} = |\nabla \phi|^2 \omega^n,$$

holds almost everywhere on M . In particular for any $\phi \in W^{1,2}(M, \mathbb{P}^m)$,

$$\int_M \phi^* \omega_{FS} \wedge \omega^{n-1} < \infty.$$

Proof. First remark that for any closed 2-form Ω the following formula holds,

$$(\text{tr}_g h_\Omega) \omega^n = 2n \omega^{n-1} \wedge \Omega,$$

where $h_\Omega(X, Y) := \frac{1}{2} (\Omega(X, JY) + \Omega(Y, JX))$. This follows by direct calculation, for example see [54].

Let $x \in M$ be such that $d\phi(x) \neq 0$ and (4.23) is satisfied at x . Consider now the above formula at the point x with $\Omega_x = \phi^* \omega_{FS}|_x$. To prove the lemma it suffices to notice that $|\nabla \phi|^2(x) = \frac{1}{2} (\text{tr}_g h_\Omega)(x)$. \square

Remark 4.9.1. If $\phi \in W^{1,p}(M, N)$ and $\alpha \in Z^k(N)$ then $\phi^*\alpha$ is in L^1 for all $k \leq p$.

In general one expects such maps to have singularities, even in the holomorphic case. For example, consider the map $(z_1, z_2) \mapsto [z_1, z_2]$ from \mathbb{C}^2 into \mathbb{P}^1 which is holomorphic on $\mathbb{C}^2 \setminus \{0\}$, cannot be extended over $(0, 0) \in \mathbb{C}^2$ and is locally approximable. In the holomorphic case these singularities are relatively well studied [27, 30] but rely heavily on local holomorphic coordinates. The almost complex case was first considered by Rivi re and Tian [46] wherein it was proven that a locally approximable map from a compact almost complex 4-manifold into a projective variety has at most isolated point singularities. They further conjecture that maps from higher dimensional almost complex manifolds into a projective space should have a singular set with zero $(2n - 4)$ -dimensional Hausdorff measure. This was verified and generalised by Wang [56] who proved that this is the case for any stationary, locally approximable map between compact almost complex manifolds. Henceforth, we shall restrict our study to locally approximable pseudoholomorphic maps $\phi \in W^{1,2}(M, \mathbb{P}^m)$.

Lemma 4.9.2. *Let M be a compact almost complex manifold and $\phi \in W^{1,2}(M, \mathbb{P}^m)$*

a locally approximable pseudoholomorphic map. Then the pushforward of the volume measure, $\mu = \phi_* \text{Vol}_g$, is non-atomic.

Proof. Since ϕ is a $W^{1,2}$ pseudoholomorphic map between almost Kähler manifolds Theorem C of [56] applies and hence ϕ is smooth away from a singular set Σ which satisfies $\mathcal{H}^{2m-2}(\Sigma) = 0$. By the argument given in §4.5 $\text{Vol}_g(\phi^{-1}(x)) = 0$ for all $x \in \phi(M) \setminus \phi(\Sigma)$.

On the other hand suppose that $x \in \phi(\Sigma)$ and consider the level set $\phi^{-1}(x)$ which we write as a union of $A = \phi^{-1}(x) \cap \Sigma \subset \Sigma$ and $B = \phi^{-1}(x) \setminus (\phi^{-1}(x) \cap \Sigma) \subset M \setminus \Sigma$. Clearly $\text{Vol}_g(A) = 0$. Moreover since ϕ is regular on B one can apply the argument of Proposition 3.2.1 to deduce that $\text{Vol}_g(B) = 0$ and hence that $\text{Vol}_g(\phi^{-1}(x)) = 0$. \square

Theorem 4.9.1. *Let (M^n, J) be a closed, n -dimensional, almost Kähler manifold and $\phi \in W_{loc}^{1,2}(M, \mathbb{P}^m)$ a non-trivial, locally approximable pseudoholomorphic map, where \mathbb{P}^m is taken with its standard complex structure. Then, for any almost Kähler metric g on M^n , the eigenvalues of the Laplace-Beltrami operator Δ_g satisfy,*

$$\lambda_k(M^n, g) \leq C(n, m) \frac{\int_M \phi^* \omega_{FS} \wedge \omega_g^{n-1}}{\int_M \omega_g^n} k, \quad \text{for any } k \geq 1, \quad (4.24)$$

where $C(n, m) > 0$ is a constant depending only on n and m .

Proof. As in the previous cases the proof reduces to finding $k+1$ linearly independent functions $u_i \in W^{1,2}(M)$ such that

$$\mathcal{R}(u_i) \leq C(n, m) \frac{\int_M \phi^* \omega_{FS} \wedge \omega_g^{n-1}}{\int_M \omega_g^n} k.$$

Firstly remark that since the pushforward measure $\mu = \phi_* \text{Vol}_g$ is non-atomic we can define functions $u_i = u_{A_i} \circ \phi$ as constructed in the discussion preceding Proposition 4.7.1 in §4.7. Since u_{A_i} is a compactly supported Lipschitz function on \mathbb{P}^m and ϕ is $W^{1,2}$ it follows that the composition $u_i \in W^{1,2}(M)$. Remarking the min-max characterisation of the eigenvalues can be done over $W^{1,2}$ we are left to verify the estimates (4.21) and (4.22) continue to hold. This is straightforward and hence will be omitted. \square

In the case of M being a Riemann surface, since $W^{1,2}$ holomorphic maps are necessarily smooth, this theorem contains no new information on top of that provided by Theorem 4.1.1. For $n \geq 2$ one can construct examples of maps satisfying the hypotheses of Theorem 4.9.1 which are not smooth and hence to which Theorem 4.1.1 do not apply. Let us briefly look at a class of examples of particular geometric interest, namely rational maps. Since this is new even in the case of an integrable complex structure we focus on this simpler case.

Recall, e.g. [22, p. 490], that a rational map from a compact, complex manifold into projective space, say $\phi : M \dashrightarrow \mathbb{P}^m$, is given by a holomorphic map $\phi : M \setminus V \rightarrow \mathbb{P}^m$

away from a subvariety, $V \subset M$, of complex codimension at least 2. Considering an isometric embedding $\mathbb{P}^m \hookrightarrow \mathbb{R}^k$ and using that \mathbb{P}^m is compact and $V \subset M$ is of real codimension 4 a standard cut-off argument (*c.f.* the argument in the next paragraph) can be employed to deduce that a rational map $\phi : M \dashrightarrow \mathbb{P}^m$ lies in $W^{1,2}(M, \mathbb{P}^m)$.

Remark 4.9.2. One could define a rational map in the pseudoholomorphic category as in the holomorphic setting, *i.e.* to be a pseudoholomorphic map away from a subvariety of codimension at least 2. In this case Corollary 4.9.1 could also be stated in the pseudoholomorphic category. However at present it is not clear if this is the right notion for dimension $n \geq 3$.

Moreover these maps are locally approximable, indeed it is enough to verify that $d(\phi^*\omega_{\text{FS}}) = 0$ holds in the sense of currents. Without loss of generality we may assume that the indeterminacy locus, V , is connected, otherwise one can apply the following argument to each connected component. First take a tubular neighbourhood of the indeterminacy locus, say $V_\varepsilon \supset V$, and a smooth cut-off function η_ε such that $\eta_\varepsilon \equiv 0$ on V_ε and $\eta_\varepsilon \equiv 1$ on $M \setminus V_{2\varepsilon}$. Furthermore we may arrange that $|\nabla\eta_\varepsilon| \leq \frac{1}{\varepsilon}$. Consider the smooth 2-form $\omega_\varepsilon = \eta_\varepsilon \phi^*\omega_{\text{FS}} \in \Omega^2(M)$, we calculate that

$$d\omega_\varepsilon = d\eta_\varepsilon \wedge \phi^*\omega_{\text{FS}} + \eta_\varepsilon d(\phi^*\omega_{\text{FS}}).$$

Now remark that

$$\int_M |d\eta_\varepsilon \wedge \phi^*\omega_{\text{FS}}|^2 \omega^n \leq C \frac{\text{Vol}(V_{2\varepsilon} \setminus V_\varepsilon)}{\varepsilon^2} \xrightarrow{\varepsilon \rightarrow 0} 0,$$

where $C = \sup_{M \setminus V} |\phi^*\omega_{\text{FS}}|^2 < +\infty$ and we have used that V is of real codimension at least 4 in the limit. It is thus straightforward to conclude that $d(\phi^*\omega_{\text{FS}}) = 0$ holds in the sense of currents. Thus rational maps $\phi : M \dashrightarrow \mathbb{P}^m$ are examples of locally approximable $W^{1,2}(M, \mathbb{P}^m)$ maps.

Recall, *e.g.* [22], that rational maps from a compact complex manifold into a projective space are in one-to-one correspondence with holomorphic line bundles whose base locus is of codimension at least 2. Suppose that $L \rightarrow M$ is a holomorphic line bundle with base locus, V , of codimension at least 2 associated to a given rational map $\phi : M \dashrightarrow \mathbb{P}^m$. By construction, it holds that $\phi^*H = L$ on $M \setminus V$, where H denotes the hyperplane bundle on \mathbb{P}^m . Since ϕ is holomorphic on $M \setminus V$ we have that $c_1(L|_{M \setminus V}) = c_1(\phi^*H) = \phi^*[\omega_{\text{FS}}]$. On the other hand recall that $\phi^*[\omega_{\text{FS}}]$ defines a class in $H^2(M)$. We claim that $c_1(L) = \phi^*[\omega_{\text{FS}}]$ in $H^2(M)$.

Let $s : M \setminus V \rightarrow L$ be a holomorphic section, by the Poincaré-Lelong formula the zero divisor, Z , of s is cohomologous to $c_1(L|_{M \setminus V})$. Since V is of codimension at least 2 we can extend s over V by Hartog's extension theorem. Writing $\tilde{s} : M \rightarrow L$ for the extension and $\tilde{Z} = \{\tilde{s} = 0\}$ the Poincaré-Lelong formula again implies that $c_1(L) = [\tilde{Z}]$. To prove the claim we need only verify that the currents of integration

satisfy the equality $[Z] = [\tilde{Z}]$. But $V \cap \tilde{Z}$ is of codimension at least 1 in \tilde{Z} from which the equality follows. Thus we obtain the following consequence of Theorem 4.9.1.

Corollary 4.9.1. *Let M^n be a closed Kähler manifold and $L \rightarrow M$ a holomorphic line bundle with base locus $V \subset M$. If V is a subvariety of codimension at least 2 then, for any Kähler metric ω on M , the eigenvalues of the Laplace-Beltrami operator satisfy,*

$$\lambda_k(M, \omega) \leq C \frac{(c_1(L) \smile [\omega]^{n-1}, [M])}{\text{Vol}(M, [\omega])} k, \quad \text{for any } k \geq 1, \quad (4.25)$$

where $C > 0$ is a constant depending only on n and m .

Of course there are many examples of compact complex manifolds which do not admit rational map into any projective space. For example the generic Hopf surfaces, which are quotients of $\mathbb{C}^2 \setminus \{0\}$ by an infinite cyclic group $\Gamma \subset \text{GL}(2, \mathbb{C})$, do not admit any global, non-constant meromorphic functions. Nonetheless Corollary 4.9.1 does provide many new examples of Kähler manifolds on which eigenvalues can be bounded over a given Kähler class.

Example 4.9.1. Let M be a rational surface, that is, a surface which is birationally isomorphic to \mathbb{P}^2 . By the arguments above Theorem 4.9.1 applies and for any Kähler metric g with associated Kähler form ω there exists a universal constant $C > 0$ such that

$$\lambda_k(M, \omega) \leq C \frac{(\phi^*[\omega_{\text{FS}}] \smile [\omega], [M])}{\text{Vol}(M, [\omega])} k,$$

where $\phi : M \dashrightarrow \mathbb{P}^2$ is a birational map. In other words, for any class $\alpha \in \mathcal{K} \subset H^2(M, \mathbb{R})$ in the Kähler cone of M there exists a constant $C_\alpha > 0$, depending only on α and $c_1(L)$, such that,

$$\sup_{\omega \in \alpha} \lambda_k(M, \omega) \text{Vol}(M, \alpha) \leq C_\alpha k,$$

where $L \rightarrow M$ is a holomorphic line bundle associated to ϕ .

4.9.2 Variations of the Almost Complex Structure

It is also natural to consider the question of whether one can obtain uniform eigenvalue bounds when the symplectic form is fixed and the almost complex structure is permitted to vary. This question has been considered by Polterovich in [45] where the following theorem is proven,

Theorem 4.9.2 (Theorem 1.2.A. [45]). *Let (M, Ω) be a closed symplectic manifold and $T^4 = \mathbb{R}^4/\mathbb{Z}^4$ the 4-torus endowed with the standard symplectic form σ . Then*

$$\Lambda_1(T^4 \times M, \sigma \oplus \Omega) := \sup \lambda_1(T^4 \times M, g) = +\infty,$$

where the supremum runs over all compatible metrics, that is, Riemannian metrics of the form $g(X, Y) = \sigma \oplus \Omega(X, JY)$ for some almost complex structure J .

Notice that here that the symplectic form is fixed and the almost complex structure is allowed to vary. This is in contrast to Theorem 4.1.1 where the almost complex structure is fixed and the symplectic form is allowed to vary within a given cohomology class.

In the same paper it is also proven that if one shrinks the class of metrics considered to Riemannian metrics which are compatible with the symplectic form via an integrable almost complex structure then a uniform bound can be found for the first eigenvalue. For the purposes of stating the next theorem we shall call such metrics compatible Kähler metrics.

Theorem 4.9.3 (Theorem 1.2.B. [45]). *Let (M, Ω) be a symplectic manifold of real dimension $2n$ such that $[\Omega] \in H^2(M, \mathbb{Q})$. Then for any compatible Kähler metric g on (M, Ω) it holds that*

$$\lambda_1(M, g) \leq c(n) \left(n + 2 - \frac{(c_1(TM) \cup [\Omega]^{n-1}, [M])}{([\Omega]^n, M)} \right),$$

where $c(n)$ is some constant depending only on n .

By applying the bounds of Kokarev [33] in the argument of Polterovich [45], Theorem 4.9.3 can be generalised to the k -th eigenvalue.

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