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A study about SIF estimation using XFEM

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Abstract. The numerical approach to fracture mechanics has been a challenge in recent decades, especially for the aeronautical and naval industry. In general, these problems involve discontinuities, singularities and high gradients, becoming impracticable for being addressed by the classical finite element method (FEM) where one faces significant precision losses and excessive remesh in the vicinity at the crack tip. In order to overcome these difficulties, the eXtended Finite Element Method (XFEM), has been developed to enhance the modeling of discontinuities such as jumps, singularities and other non-smooth features. XFEM is a numerical technique that extends the classic FEM, enriching the approximation space with information coming from asymptotic and other solutions knowledge of the physics of the problem. The main objective of this work is to investigate the features of the application of XFEM for LEFM problems in plane elasticity problems by numerical calculation the stress intensity factor (SIF) using an approximate approach to obtain the *J* integral and then compare the stress intensity factors with those available in the literature. For classic cases numerical results are close to the solutions available in the literature.

Keywords: Singularities, Fem, XFem, Stress intensity factor.

1 Introduction

The classical FEM depend on the local approximation of polynomials functions, Melenk and Babuška [1]. However, for jumps, kinks, or singularities in the solution within elements, polynomials have poor approximation properties. Consequently, the accuracy of a standard FEM is, in general, quite poor for problems involving arbitrary discontinuities. Several methods have been developed with the ability to introduce special characteristics of the solution into the approximation space, such as the partition of unity FEM (PU-FEM) by Babuška and Melenk [2], the generalized finite element method (GFEM) by Strouboulis, Babuška, and Copps [3] and Strouboulis, Copps, and Babuška [4], and the XFEM by Belytschko and Black [5] and Moës, Dolbow, and Belytschko [6]. According to Belytschko, Gracie and Ventura [7] the XFEM and GFEM are umbilical methods, being the name GFEM employed by the Texas school and the name XFEM by the Northwestern school. The XFEM is a numerical technique based on the FEM and the partition of unity method (PUM). It extends the classical FEM approach by enriching the approximation space in order to improve the ability to capture jumps, kinks, or singularities. For instance, in the FEM the displacement jump near the crack tip is basically captured by refining the mesh locally making the number of degrees of freedom (dofs) increase drastically. Moës, Dolbow and Belytschko [6], presented an improvement of a technique for modelling cracks in the FE framework. A standard displacement-based approximation is enriched near a crack by incorporating both discontinuous fields and near tip asymptotic fields using a PUM. The XFEM methodology enriches the nodes of the elements in which the crack crosses and thus we can represent the crack independently of the mesh, so it is not necessary to remesh at each step of the growth of the crack and does not have to be coinciding with the edge of the elements. In order to evaluate SIF for various fracture problems, many researchers relied on numerical methods, such as path-independent J-integral by Rice [8] and with Griffith [9] in a Griffith energy method. By assuming the LEFM, the stress, strain and displacement fields can be determined using the concept of SIF near the crack tip region. In this work, the XFEM method is used to numerical calculation of the stress intensity factor of a standard specimen where the concept of the interaction

integral, as presented in Khoei [10], is employed for calculating the SIFs of mixed-mode fracture problem.

2 The eXtended finite element method

The key idea of XFEM is to enrich the basic functions of FEM, eq. (1), to represent fracture behavior. Enrichment is defined only in elements where discontinuity is present and thus new dofs are added to the element that contains the displacement jump at the crack interface and element in which the crack tip is present.

The XFEM displacement approximation functions can be written as

$$u(x) = \sum_{I \in \mathcal{N}} N_I(x)\tilde{u}_I + \sum_{J \in \mathcal{N}^{dis}} N_J(x)(H(x) - H(x_J))\tilde{d}_J + \sum_{K \in \mathcal{N}^{tip}} N_K(x) \sum_{\alpha=1}^{4} (F_\alpha(x) - F_\alpha(x_K))\tilde{b}_{\alpha K}. \tag{1}$$

The dofs in terms of displacement are u_i, d_j and $b_{\alpha K}$ where the last two are dofs associated with the Heaviside function $H(x)$ which represents the displacement jump at the crack interface and four singular functions $F_\alpha(x)$ which represent the behavior at the crack tip. $\mathcal{N}$ is a set of dofs associated with the standard FEM, $\mathcal{N}^{dis}$ refers to the set of nodes whose support is totally divided by the crack, as seen in Figure 2, nodes indicated by circles, and $\mathcal{N}^{tip}$ refers to the set of nodes whose support contains the tip of the crack, nodes indicated by squares refers elements that have discontinuity.

In eq. (2) the first term of enrichment is the Heaviside function which is defined as:

$$H(x) = \begin{cases} +1, \textit{ when x is above of crack} \\ -1, \textit{ when x is below of crack} \end{cases}. \tag{2}$$

The last enrichment represents the singularity at the crack tip. Here, $F_\alpha(x)$ are functions chosen to build the expressions of the asymptotic expansion of the displacement field in LEFM, see Belytschko & Black [5]. The four asymptotic functions are

$$F_1 = \sqrt{r}sin\left(\frac{\theta}{2}\right),\ F_2 = \sqrt{r}cos\left(\frac{\theta}{2}\right),\ F_3 = \sqrt{r}\sin\left(\frac{\theta}{2}\right)\sin(\theta) \ \text{ and } \ F_4 = \sqrt{r}\cos\left(\frac{\theta}{2}\right)\sin(\theta). \tag{3}$$

where r and θ are polar coordinates in relation to crack tip. Finally, is possible to rewrite the XFEM approximation functions as

$$u(x) = N^{std}(x)\tilde{u} + N^{hev}(x)\tilde{d} + N^{tip}(x)\tilde{b} \equiv N^{std}(x)\tilde{u} + N^{enr}(x)\tilde{a}. \tag{4}$$

Where $N^{std}(x) = N(x)$ is classic shape function of FEM and $N^{enr}(x) = \left[N^{hev}(x),\ N^{Tip}(x)\right]$ is Matrix of enrichment functions. According to the approximate displacement field, the corresponding deformation vector in terms of normal and enriched nodal values can be written as

$$\varepsilon(x) = \boldsymbol{B}^{std}(x)\tilde{u} + \boldsymbol{B}^{hev}(x)\tilde{d} + \boldsymbol{B}^{tip}(x)\tilde{b} \equiv \boldsymbol{B}^{std}(x)\tilde{u} + \boldsymbol{B}^{enr}(x)\tilde{a}. \tag{5}$$

where

$$\boldsymbol{B}^{std} = \boldsymbol{L}N^{std}(x),\ \boldsymbol{B}^{hev} = \boldsymbol{L}N^{hev}(x),\ \boldsymbol{B}^{tip} = \boldsymbol{L}N^{tip}(x) \ \text{ and } \ \boldsymbol{L} = \begin{bmatrix} 0 & \frac{d}{dy} & \frac{d}{dx} \\ \frac{d}{dx} & 0 & \frac{d}{dy} \end{bmatrix}^T.$$

2.1 The governing equation of a body with a crack

According to Khoei [10], to derive a governing equation from the mechanics of solids of a fractured domain using XFEM was considered a 2D domain cut by a discontinuity Γ_d. The strong form of the equilibrium equation of a cracked body can be expressed as

$$\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0} \tag{6}$$

in which ∇ is the gradient operator, $\boldsymbol{\sigma}$ is the Cauchy stress tensor and $\boldsymbol{b}$ is the body force applied to the system.

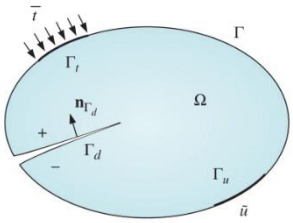

Figure 1 - Fractured domain geometry involving discontinuity Γ_d. [Adapted from Khoei, 2014]

Assuming that the material has an isotropic linear elastic behavior, with the stress field being defined as $\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$, where $\mathbf{D}$ is a constitutive material tensor.

The essential and natural boundary conditions are applied to the problem:

$$\text{Essential } \{u = \bar{u} \text{ em } \Gamma_u \qquad \text{Natural } \begin{cases} \boldsymbol{\sigma}\boldsymbol{n}_\Gamma = \boldsymbol{t} & \text{on } \Gamma_t \\ \boldsymbol{\sigma}\boldsymbol{n}_{\Gamma_d} = \boldsymbol{0} & \text{on } \Gamma_d \end{cases}$$

Where $\bar{u}$ is prescribed displacement, t is prescribed traction vector, $\boldsymbol{n}_\Gamma$ - normal vector directed out of the Γ and $\boldsymbol{n}_{\Gamma_d}$ - normal vector to discontinuity Γ_d.

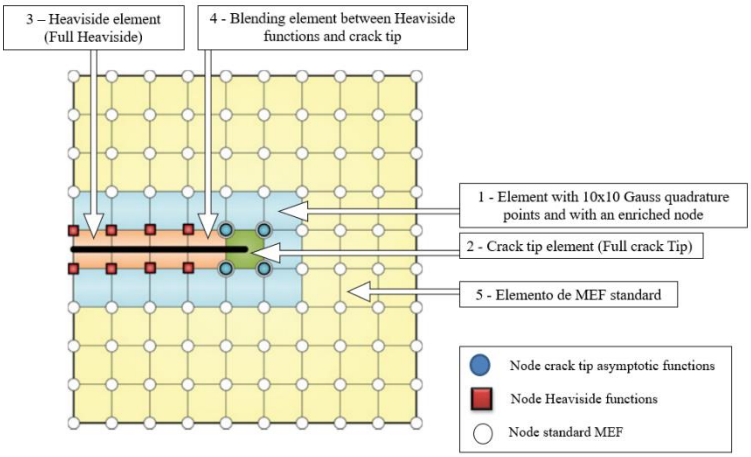

Figure 2 - Mesh with all domain and enriched nodes around the crack. [Adapted from Khoei, 2014]

Using eq. (4) and eq. (6), and applying the Galerkin procedure

$$\int_\Omega \delta\boldsymbol{u}(x,t)\left(\nabla\cdot\ \boldsymbol{\sigma} + \boldsymbol{b}\right) d\Omega = 0. \tag{7}$$

It is possible to write the following system of linear equations

$$\mathbf{K}\boldsymbol{u} = \boldsymbol{f} \quad \text{or} \quad K_{ij}u_i = f_i. \tag{8}$$

where $\mathbf{K}$ is the stiffness matrix, $\boldsymbol{u}$ is the dofs vector for the classical and enriched nodes and $\boldsymbol{f}$ is the external vector forces.

2.2 XFEM discretization

The global matrix $\mathbf{K}$ and the vectors $\boldsymbol{f}$ and $\boldsymbol{u}$ are obtained by assembling the matrix and the vectors of each element and are defined as

$$\mathbf{K} = \boldsymbol{B}^T \boldsymbol{D}\boldsymbol{B} \equiv \begin{Bmatrix} (\boldsymbol{B}_u)^T \\ (\boldsymbol{B}_d)^T \\ (\boldsymbol{B}_b)^T \end{Bmatrix} \boldsymbol{D} \begin{Bmatrix} \boldsymbol{B}_u & \boldsymbol{B}_d & \boldsymbol{B}_b \end{Bmatrix}. \tag{9}$$

where $\mathbf{K}$ is

$$\mathbf{K} = \begin{bmatrix} \int_\Omega (\boldsymbol{B}_u)^T \boldsymbol{D}\boldsymbol{B}_u d\Omega & \int_\Omega (\boldsymbol{B}_u)^T \boldsymbol{D}\boldsymbol{B}_d d\Omega & \int_\Omega (\boldsymbol{B}_u)^T \boldsymbol{D}\boldsymbol{B}_b d\Omega \\ \int_\Omega (\boldsymbol{B}_d)^T \boldsymbol{D}\boldsymbol{B}_u d\Omega & \int_\Omega (\boldsymbol{B}_d)^T \boldsymbol{D}\boldsymbol{B}_d d\Omega & \int_\Omega (\boldsymbol{B}_d)^T \boldsymbol{D}\boldsymbol{B}_b d\Omega \\ \int_\Omega (\boldsymbol{B}_b)^T \boldsymbol{D}\boldsymbol{B}_u d\Omega & \int_\Omega (\boldsymbol{B}_b)^T \boldsymbol{D}\boldsymbol{B}_d d\Omega & \int_\Omega (\boldsymbol{B}_b)^T \boldsymbol{D}\boldsymbol{B}_b d\Omega \end{bmatrix}. \tag{10}$$

and $\boldsymbol{f}$ is given by

$$\boldsymbol{f} = \begin{Bmatrix} \int_{\Gamma^t} (N_u)^T t d\Gamma + \int_\Omega (N_u)^T b d\Omega \\ \int_{\Gamma^t} (N_d)^T t d\Gamma + \int_\Omega (N_d)^T b d\Omega \\ \int_{\Gamma^t} (N_b)^T t d\Gamma + \int_\Omega (N_b)^T b d\Omega \end{Bmatrix}. \tag{11}$$

Here, the subscripts are u for the normal displacements of FEM, d for displacements due to Heaviside enrichment, and b for displacements due to enrichments that describe the behavior in the region of the crack tip. A more compact form of Eq. (9), Eq. (10), and Eq. (11) was proposed by Ru, Zhao and Yin in [11].

3 *J* integral

By assuming a behavior in LEFM, the stress, strain and displacement fields can be determined using the concept of SIF near the crack tip region. Therefore, it is important to accurately evaluate SIFs for FEM analysis in LEFM. Irwin and Wit [12] showed that for a linear elastic material, the quantity $G = J$. The *J*-integral technique was originally defined to calculate the G in crack problems using a local coordinate system of the crack tip (x_1, x_2), see Khoei [10]

$$J = \int_\Gamma \left[W\delta_{1j} - \sigma_{ij} \frac{\partial u_i}{\partial x_1} \right] n_j d\Gamma. \tag{12}$$

By superimposing two states in mode I, one writes

$$J^{(1+2)} = \int_\Gamma \left[\frac{1}{2} \left(\sigma_{ij}^{(1)} + \sigma_{ij}^{(2)} \right) \left(\varepsilon_{ij}^{(1)} + \varepsilon_{ij}^{(2)} \right) \delta_{1j} - \left(\sigma_{ij}^{(1)} + \sigma_{ij}^{(2)} \right) \frac{\partial \left(u_i^{(1)} + u_i^{(2)} \right)}{\partial x_1} \right] \mathrm{n}_j d\Gamma. \tag{13}$$

in which (1) - Indicates the pure or current state and (2) - Indicates the auxiliary state. The *J* integral for pure state 1 and an auxiliary state 2 can be separated, appearing a new term

$$J^{(1+2)} = J^{(1)} + J^{(2)} + I^{(1,2)}. \tag{14}$$

where $\mathrm{I}^{(1,2)}$ is an interaction term

$$I^{(1+2)} = \int_\Gamma \left[W^{(1,2)} \delta_{1j} - \sigma_{ij}^{(1)} \frac{\partial u_i^{(2)}}{\partial x_1} - \sigma_{ij}^{(2)} \frac{\partial u_i^{(1)}}{\partial x_1} \right] \mathrm{n}_j d\Gamma. \tag{15}$$

The deformation energy density *W* is defined by

$$W^{(1,2)} = \sigma_{ij}^{(1)} \varepsilon_{ij}^{(2)} = \sigma_{ij}^{(2)} \varepsilon_{ij}^{(1)}. \tag{16}$$

Since two states are being superimposed on each other, one can write

$$J^{(1+2)} = J^{(1)} + J^{(2)} + \frac{2\left(K_I{}^{(1)} K_I{}^{(2)} + K_{II}{}^{(1)} K_{II}{}^{(2)}\right)}{E_{eff}}. \tag{17}$$

result in

$$I^{(1+2)} = \frac{2\left(K_I{}^{(1)} K_I{}^{(2)} + K_{II}{}^{(1)} K_{II}{}^{(2)}\right)}{E'}. \tag{18}$$

The SIF for the current state can be found by separating the two fracture modes, that is

$$K_I{}^{(2)} = 1 \quad \text{and} \quad K_{II}{}^{(2)} = 0. \tag{19}$$

so $K_I{}^{(1)}$ can be found

$$K_I{}^{(1)} = \frac{I^{(1,\text{mode I})} E'}{2}. \tag{20}$$

A similar procedure can be done for $K_{II}{}^{(1)}$. The *J* integral, eq. (12), can be directly evaluated along the FEM mesh contour. This contour can usually be evaluated by passing through the Gauss integration points along the element, where the stresses can be calculated more accurately. However, the practical implementation of this technique rarely shows path independence and the results become dependent on the mesh. In Li et al. [13], the *J* integral was calculated by transforming the boundary integral to an equivalent area integral. The area form of the *J* integral is defined as

$$J = \int_A \left[\sigma_{ij} \frac{\partial u_i}{\partial x_1} - W \delta_{1j} \right] \frac{\partial q}{\partial x_j} dA. \tag{21}$$

where *q* is a defined "weighting function" on the integration domain. The integration domain must be selected in such a way that it is sufficiently close to a complex crack pattern, it must be simple to implement in a fully automatic simulation procedure and finally, it must be consistent with geometry and contour limitation in complex contours and problems with cracks, Figure 3. The function *q* has the value of 1 (unit) in the region of the crack tip and disappears on the external side of the prescribed contour for calculating the *J* integral, as seen in Figure 4. Based on the *J* integral defined in eq. (21), the new interaction integral can be defined as

$$I^{(1+2)} = \int_A \left[\sigma_{ij}{}^{(1)} \frac{\partial u_i{}^{(2)}}{\partial x_1} - \sigma_{ij}{}^{(2)} \frac{\partial u_i{}^{(1)}}{\partial x_1} - W^{(1,2)} \delta_{1j} \right] \frac{\partial q}{\partial x_j} dA. \tag{22}$$

The mode I of SIF, K_I, can finally be obtained according to eq. (20) by assuming the auxiliary state (2) as pure mode I and/or pure mode II as asymptotic field, evaluating the displacement and stress fields in the area of the crack tip and by replacing these variables in eq. (22).

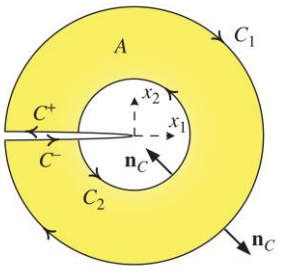

Figure 3 - The domain of J integral for the calculation of mixed mode SIFs. (Adapted from Khoei [10])

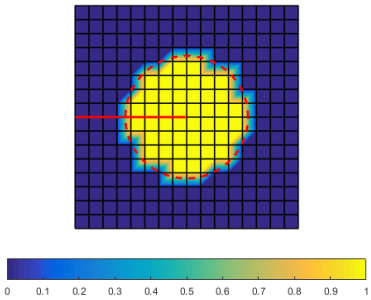

Figure 4 – Map of the selection of domain nodes using the q weight function.

4 Numerical case

4.1 Calculation of the Stress intensity factor of a standard specimen

For the case of Figure 5 the exact solution for K_I is given by

$$K_I = \frac{P}{B\sqrt{W}} Y\left(\frac{a}{W}\right). \tag{23}$$

where

$$Y\left(\frac{a}{W}\right) = \frac{\left(2+\frac{a}{W}\right)\left[0.886+4.64\frac{a}{W}-13.32\left(\frac{a}{W}\right)^2+14.72\left(\frac{a}{W}\right)^3-5.6\left(\frac{a}{W}\right)^4\right]}{\left(1-\frac{a}{W}\right)^{\frac{3}{2}}}. \tag{24}$$

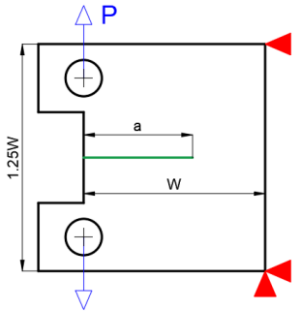

Figure 5 – Standard specimen with crack submitted to uniaxial loading.

For this case, the following dimensions were considered for the problem: a = 30mm, W = 50mm, B = 18mm (thickness), and a load P = 2000N was applied in the region shown in Figure 5. Also, the analysis was performed using plane strain state. The results for a mesh refining are presented in Table 1

Table 1 - Standard specimen results.

Elements	K_I $\left[\text{MPa}\sqrt{\text{mm}}\right]$	Radius J-integral [mm]	K_I exact $\left[\text{MPa}\sqrt{\text{mm}}\right]$	1/h [mm^{-1}]	Error [%]
576	209.71			0.61	2.28
852	211.08	6	214.6	0.75	1.64
1455	212.42			0.95	1.02

2304	214.03		1.16	0.27

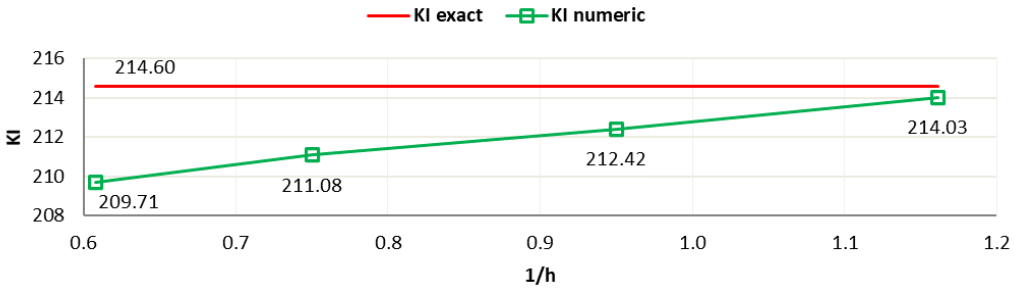

Figure 6 - Convergence graph for K_I of the standard specimen.

The convergence results obtained for K_I are presented in Figure 6. The red line is the exact value for the K_I of the problem while the green line is the convergence for mesh refining. Note that there is a fairly clear convergence curve for the problem.

5 Conclusions

The enrichment functions are introduced in the XFEM to simulate cracks and the singular displacement field around the crack tip, so the interaction integral is employed to calculate the SIF of mixed mode fracture problems. The proposed method of analysis does not require that the crack geometry be aligned with the edge of the element, thus providing good flexibility and versatility in modeling.

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