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Solvability and asymptotic properties for an elliptic geophysical fluid flows model in a planar exterior domain*

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Abstract. In this paper, we study the solvability and asymptotic properties of a recently derived gyre model of nonlinear elliptic Schrödinger equation arising from the geophysical fluid flows. The existence theorems and the asymptotic properties for radial positive solutions are established due to space theory and analytical techniques, some special cases and specific examples are also given to describe the applicability of model in gyres of geophysical fluid flows.

Keywords: gyres of geophysical fluid flows, Schrödinger equations, asymptotic properties, radial positive solutions.

1 Introduction

We study in this paper the solvability and asymptotic properties for the following model of nonlinear Schrödinger equation from the study of the gyres of geophysical fluid flows in a planar exterior domain:

$$\Delta v = -f\big(|z|, v, z \cdot \nabla v\big), \quad z \in \Omega, \tag{1}$$

where $\Omega = \{z \in \mathbb{R}^2\colon |z| \geqslant 1\}$, $f : [1, \infty) \times \mathbb{R}^2 \to \mathbb{R}$ is continuous.

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Nonlinear elliptic equations are closely related to many dynamic model arising from physics, biology, geography and applied mathematics [2, 3, 5, 39, 40, 43]. In Eq. (1), when the term $z \cdot \nabla v$ disappears, Constantin and Johnson [5] model the large gyres of geophysical fluid flows by using a shallow-water asymptotic solution of Euler's equation in spherical coordinates

$$\Delta v + 8\omega \frac{1 - |z|^2}{(1 + |z|^2)^3} - \frac{4F(v)}{(1 + |z|^2)^2} = 0, \tag{2}$$

where the Earth's rotation velocity $\omega > 0$ refers to in nondimensional units, the unknown function $v(z, y)$ stands for the stream function, the oceanic vorticity is defined by the function F. In fact, the gyres are generated by the interaction between the wind stress and the effect of the Earth's rotation, thus it is not good approximation to describe large-scale gyres in spherical geometry of the Earth by a β-plane [1]. Therefore, it is reasonable to consider large-scale gyres in the spherical coordinates as the shallow-water approximation. In brief, in spherical coordinates, suppose the polar angle is $\alpha \in [0, \pi)$ and $\alpha - \pi/2$ is the conventional angle of latitude, if $\alpha = 0$, it corresponds to the south pole. Let the azimuthal angle $\beta \in [0, 2\pi)$ be the angle of longitude and Ψ be the vorticity of the underlying motion of the ocean, $v(\alpha, \beta) = -\omega \cos\alpha + \Psi(\alpha, \beta)$ stands for the stream function. The horizontal flow on the spherical earth will form a gyre with the azimuthal and polar velocity components $v_\beta \sin^{-1}\alpha$, v_α, respectively, and the gyre equation for Ψ on the half-line is governed by

$$\frac{1}{\sin^2\alpha} \Psi_{\beta\beta} + \Psi_\alpha \cot\alpha + \Psi_{\alpha\alpha} = F(\Psi - \omega \cos\alpha). \tag{3}$$

From the south pole to the plane of the equator make the stereographic projection

$$\xi = r\mathrm{e}^{\mathrm{i}\phi}, \quad r = \cot\frac{\alpha}{2} = \frac{\sin\alpha}{1 - \cos\alpha},$$

then gyre governed equation (3) is equivalent to the semilinear elliptic equation (2). Thus for given F and ω, one can explore the semilinear Dirichlet boundary value elliptic problem in a given planar region Ω

$$\begin{aligned} &\Delta v + 8\omega \frac{1 - |z|^2}{(1 + |z|^2)^3} - \frac{4F(v)}{(1 + |z|^2)^2} = 0, \quad z \in \Omega, \\ &v = v_0 \quad \text{on } \partial\Omega. \end{aligned} \tag{4}$$

For obtaining the solvability and asymptotic properties of the gyre governed equation, by using the stereographic projection for (4), Chu [3] found that the flow of an arctic gyre, described by (3), can be equivalently formulated as the following second-order ordinary differential equation:

$$v''(z) = \frac{F(v(z))}{\cosh^2 z} - \frac{2\omega \sinh z}{\cosh^3 z}, \quad z > z_0, \tag{5}$$

subject to the asymptotic conditions

$$\lim_{z\to\infty} v(z) = v_0 \quad \text{and} \quad \lim_{z\to\infty} \{v'(z)\cosh z\} = 0,$$

for some constant v_0. In fact, Eq. (5) on the half-line is also related to the second ordinary differential equation

$$\begin{aligned} &v''(z) - k^2 v(z) + f(z, v) = 0, \quad z > 0, \\ &v(0) = 0, \qquad \lim_{z\to+\infty} v(z) = 0. \end{aligned} \tag{6}$$

By Krasnoselskii fixed point theorem on cone compression and expansion, Zima [46] studied the existence of positive solutions for the above boundary value problem when $f : (0, +\infty) \times [0, +\infty) \to [0, +\infty)$ is continuous. In particular, by using suitable growth condition and fixed point theorem, Yin [33], Constantin [4], Ertem and Zafer [8] presented the existence and asymptotic behavior of positive solutions for an extended version of Eq. (6), respectively. For other related work, we refer reader to [17, 18, 35, 36].

In comparison, model (2) was obtained in the case where the vertical motion of the fluid was neglected. However, in the large-scale oceanic motion, saddle points and singular points on the boundary are normally caused by a jet flowing toward or away from the coastal line or the boundary, and near the saddle points and singular points, the vertical motion of the fluid can not be neglected. Thus Eq. (1) is a general form of the geophysical fluid flows model (2) describing the gyre flow of ocean in a larger scale with the vertical motion. Motivated by the existing works, this paper focuses on the existence of radial positive solutions and its asymptotic properties at boundary for the nonlinear Schrödinger equation (1). Our work has some new features, firstly, model (1) promotes and generalizes the gyre flow model proposed by Constantin, which can describe more general gyre flow of ocean in a larger scale; secondly, some appropriate comparative conditions are introduced for overcoming the influence of perpendicular vorticity, thirdly, the asymptotical property of solution at infinity of model (1) performs as nonlinear behaviour, it is well known that the asymptotical property at boundary of nonoscillatory solutions for most equations often behaves like a linear function such as [4, 43], however, the asymptotical property behaving like a nontrivial nonlinear function is also interesting and important, which here can explain the interdependency of the ocean flows in arctic gyres. On the other hand, although the study of the differential equations has a long history [6, 8–14, 16, 19–34, 37–45], the geophysical fluid flows model (2) arising from arctic gyres still presents some specific challenges.

2 Preliminaries and lemmas

In order to establish the existence and asymptotic behavior of radial solutions for the general geophysical fluid flows model (1), we need to transform the model (1) to a more convenient form.

Lemma 1. *The following second-order differential equation is equivalent to Eq.* (1):

$$w''(x) = -e^{2x} f\bigl(e^x, w(x), w'(x)\bigr), \quad x \in [0, +\infty). \tag{7}$$

Proof. Let $v(|z|) = v(r)$, where $r = |z| = \sqrt{z_1^2 + z_2^2}$, $z \in \Omega$. Then one has

$$
\begin{aligned}
\frac{\partial v}{\partial z_1} &= \frac{\mathrm{d}v}{\mathrm{d}r} \cdot \frac{\partial r}{\partial z_1} = \frac{z_1}{r} \cdot v'(r), \\
\frac{\partial v}{\partial z_2} &= \frac{\mathrm{d}v}{\mathrm{d}r} \cdot \frac{\partial r}{\partial z_2} = \frac{z_2}{r} \cdot v'(r), \\
\frac{\partial^2 v}{\partial z_1^2} &= \left(\frac{\mathrm{d}^2 v}{\mathrm{d}r^2} \cdot \frac{\partial r}{\partial z_1}\right)\frac{z_1}{r} + \frac{\mathrm{d}v}{\mathrm{d}r}\left(\frac{1}{r} - \frac{z_1}{r^2} \cdot \frac{\partial r}{\partial z_1}\right) \\
&= \frac{z_1^2}{r^2} \cdot v''(r) + \left(\frac{1}{r} - \frac{z_1^2}{r^3}\right) \cdot v'(r), \\
\frac{\partial^2 v}{\partial z_2^2} &= \frac{z_2^2}{r^2} \cdot v''(r) + \left(\frac{1}{r} - \frac{z_2^2}{r^3}\right) \cdot v'(r).
\end{aligned} \tag{8}
$$

It follows from (8) that

$$
\begin{aligned}
\Delta v &= \frac{\partial^2 v}{\partial z_1^2} + \frac{\partial^2 v}{\partial z_2^2} = v''(r) + \frac{1}{r}v'(r), \\
z \cdot \nabla v &= z_1 \cdot \frac{\partial v}{\partial z_1} + z_2 \cdot \frac{\partial v}{\partial z_2} = rv'(r),
\end{aligned}
$$

which imply

$$
\begin{aligned}
&\Delta v = -f\big(|z|, v, z \cdot \nabla v\big), \quad z \in \Omega \\
&\iff \quad rv''(r) + v'(r) = -f\big(r, v(r), rv'(r)\big), \quad r \geqslant 1.
\end{aligned}
$$

Next, let $x = \ln r$, $w(x) = v(r)$. So we have $r = \mathrm{e}^x$, $w(x) = v(\mathrm{e}^x)$ and

$$
\begin{aligned}
w'(x) &= v'\big(\mathrm{e}^x\big)\mathrm{e}^x = v'(r)r, \\
w''(x) &= v''\big(\mathrm{e}^x\big)\mathrm{e}^{2x} + v'\big(\mathrm{e}^x\big)\mathrm{e}^x = v''(r)r^2 + v'(r)r.
\end{aligned} \tag{9}
$$

Substituting (9) into (7), we get

$$
w''(x) = -\mathrm{e}^{2x} f\big(\mathrm{e}^x, w(x), w'(x)\big), \quad x \geqslant 0. \qquad \square
$$

Remark 1. It follows from Lemma 1 that if $w(x)$ is a positive solution of the second differential equation (7), then $v(|z|) = w(\ln|z|)$ is a radial symmetric solutions of the model (1). Thus in what follows, we only consider Eq. (7).

Now we are concerned with the solvability of Eq. (7) on $[0, +\infty)$. Firstly, let us introduce the following function space:

$$
X = \Big\{w \in C^1\big([0, +\infty), \mathbb{R}\big)\colon \lim_{x \to +\infty} w(x) \text{ and } \lim_{x \to +\infty} \big\{w(x) + xw'(x)\big\} \text{ exist}\Big\},
$$

which is equipped with the usual maximum norm

$$
\|w\| = \max\Big\{\sup_{x \geqslant 0} \big|w(x)\big|, \sup_{x \geqslant 0} \big|w(x) + xw'(x)\big|\Big\}.
$$

Lemma 2. (*See* [33].) *X is a Banach space.*

Lemma 3. (*See* [33].) *Let X be defined as before and $P \subset X$. Then P is relatively compact in X if the following conditions hold*:

(i) *The set P is uniformly bounded in X, i.e., there exists a positive constant M such that for all $w \in P$ and all $x > 0$, $|w(x)| \leqslant M$ and $|w(x) + xw'(x)| \leqslant M$.*

(ii) *P is equicontinuous, i.e., for any $\epsilon > 0$, there exists $\delta > 0$ such that for all $|x_1 - x_2| < \delta$ and all $w \in P$, $|w(x_1) - w(x_2)| < \epsilon$ and $|w(x_1) + x_1 w'(x_1) - w(x_2) - x_2 w'(x_2)| < \epsilon$.*

(iii) *The function from P is equiconvergent, i.e., for any given $\epsilon > 0$, there exists a $T = T(\epsilon) > 0$ such that for all $y, x > T$ and all $w \in P$, $|w(x) - w(y)| < \epsilon$, $|w(x) + xw'(x) - w(y) - yw'(y)| < \epsilon$.*

3 Main results

Before stating our main results, we firstly introduce the following assumptions:

(A1) There exists a continuous function $F : [1, +\infty) \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, which is nondecreasing with respect to the second and third arguments for each $x \in [1, +\infty)$ such that $|f(x, u, v)| \leqslant F(x, |u|, |v|)$.

(A2) There exists a constant $\mu > 0$ such that $\int_1^{+\infty} s^2 F(s, 2\mu \ln s, 2\mu)\, \mathrm{d}s \leqslant \mu$.

Theorem 1. *Suppose that* (A1) *and* (A2) *hold. Then the general geophysical fluid flows model* (1) *has at least a radial solution v_μ with asymptotic properties*

$$v_\mu(z) > 0, \quad z \in \Omega, \quad v_\mu(z) = 0, \quad z \in \partial\overline{\Omega}, \quad \textit{and} \quad \lim_{|z| \to +\infty} \frac{v_\mu(|z|)}{\ln |z|} = \mu.$$

Proof. We start our proof by studying the equivalent form (7) of model (1). To do this, by the Lebesgue dominated convergence theorem, (A2) implies that there exists a positive constant $\epsilon \in (0, \mu)$ such that

$$\int_1^{+\infty} s^2 F\bigl(s, (2\mu - \epsilon) \ln s,\ 2\mu - \epsilon\bigr)\, \mathrm{d}s \leqslant \mu - \epsilon. \tag{10}$$

For the above $\epsilon > 0$, define a subset of X

$$P = \bigl\{ w \in X\colon\ \epsilon \leqslant w(x) \leqslant 2\mu - \epsilon,\ \epsilon \leqslant w(x) + xw'(x) \leqslant 2\mu - \epsilon \bigr\},$$

notice that $\epsilon \in P$, then P is nonempty bounded closed convex set. Now define a nonlinear operator $T : P \to X$ by

$$\begin{aligned}(Tw)(x) = \mu + \frac{1}{t} \int_0^x x\mathrm{e}^{2x} f\bigl(\mathrm{e}^x,\ xw(x),\ w(x) + xw'(x)\bigr)\, \mathrm{d}x \\ + \int_x^{+\infty} \mathrm{e}^{2x} f\bigl(\mathrm{e}^x,\ xw(x),\ w(x) + xw'(x)\bigr)\, \mathrm{d}x, \quad x \geqslant 0.\end{aligned} \tag{11}$$

Now we divide our proof into the following four steps.

Step 1. We firstly prove that $TP \subseteq P$.

In fact, by (10) and the monotonicity of F, for any $x \geqslant 0$ and $w \in P$, we have

$$\begin{aligned}
\big|(Tw)(x) - \mu\big| &= \Bigg|\frac{1}{x}\int_0^x s\mathrm{e}^{2s} f\big(\mathrm{e}^s,\, sw(s),\, w(s) + sw'(s)\big)\,\mathrm{d}s \\
&\quad + \int_x^{+\infty} \mathrm{e}^{2s} f\big(\mathrm{e}^s,\, sw(s),\, w(s) + sw'(s)\big)\,\mathrm{d}s\Bigg| \\
&\leqslant \frac{1}{x}\int_0^x s\mathrm{e}^{2s} F\big(\mathrm{e}^s,\, sw(s),\, w(s) + sw'(s)\big)\,\mathrm{d}s \\
&\quad + \int_x^{+\infty} \mathrm{e}^{2s} F\big(\mathrm{e}^s,\, sw(s),\, w(s) + sw'(s)\big)\,\mathrm{d}s \\
&\leqslant \int_0^{+\infty} \mathrm{e}^{2s} F\big(\mathrm{e}^s,\, sw(s),\, w(s) + sw'(s)\big)\,\mathrm{d}s \\
&\leqslant \int_0^{+\infty} \mathrm{e}^{2s} F\big(\mathrm{e}^s,\, (2\mu - \epsilon)s,\, 2\mu - \epsilon\big)\,\mathrm{d}s \\
&= \int_1^{+\infty} s^2 F\big(s,\, (2\mu - \epsilon)\ln s,\, 2\mu - \epsilon\big)\,\mathrm{d}s \\
&\leqslant \mu - \epsilon,
\end{aligned} \tag{12}$$

which implies that $\epsilon \leqslant (Tw)(x) \leqslant 2\mu - \epsilon$ for any $x \geqslant 0$.

On the other hand, by (11), one has

$$(Tw)'(x) = -\frac{1}{x^2}\int_0^x s\mathrm{e}^{2s} f\big(\mathrm{e}^s, sw(s), w(s) + sw'(s)\big)\,\mathrm{d}s, \quad x \geqslant 0, \tag{13}$$

and

$$(Tw)'(0) = \lim_{x\to 0}(Tw)'(x) = -\frac{1}{2} f\big(1, 0, w(0)\big).$$

Substituting (13) and (11) into the following formula, we have

$$\begin{aligned}
&\big|(Tw)(x) + x(Tw)'(x) - \mu\big| \\
&\quad\leqslant \int_0^{+\infty} \mathrm{e}^{2s} F\big(\mathrm{e}^s,\, (2\mu - \epsilon)s, 2\mu - \epsilon\big)\,\mathrm{d}s = \int_1^{+\infty} s^2 F\big(s, (2\mu - \epsilon)\ln s,\, 2\mu - \epsilon\big)\,\mathrm{d}s \\
&\quad\leqslant \mu - \epsilon,
\end{aligned} \tag{14}$$

which implies that $\epsilon \leqslant (Tw)(x) + x(Tw)'(x) \leqslant 2\mu - \epsilon$ for any $x \geqslant 0$. Thus $TP \subseteq P$ and $T : P \to P$ is well defined.

Step 2. We show that $T(P)$ is relatively compact in X by verifying that $T(P)$ satisfies conditions of (i)–(iii) of Lemma 3. From (12) and (14) we know $T(P)$ is uniformly bounded, i.e., condition (i) holds. In what follows, we prove that $T(P)$ is equicontinuous. Let $\{w\}_{n\geqslant 1}$ is any sequence in P, by L'Hôspital rule and (10), we have

$$\begin{aligned} \lim_{x\to 0} \frac{\int_1^{\mathrm{e}^x} s^2 F(s,\, (2\mu-\epsilon)\ln s,\, 2\mu-\epsilon)\,\mathrm{d}s}{x} &= F(1,\, 0,\, 2\mu-\epsilon), \\ \lim_{x\to \infty} \frac{\int_1^{\mathrm{e}^x} s^2 F(s,\, (2\mu-\epsilon)\ln s,\, 2\mu-\epsilon)\,\mathrm{d}s}{x} &= 0. \end{aligned} \tag{15}$$

Thus according to (10), (13) and (15), one gets

$$\begin{aligned} \bigl|(Tw_n)'(x)\bigr| &= \left| -\frac{1}{x^2} \int_0^x s\mathrm{e}^{2s} f\bigl(\mathrm{e}^s,\, sw_n(s),\, w_n(s)+sw_n'(s)\bigr)\,\mathrm{d}s \right| \\ &\leqslant \frac{1}{x^2} \int_0^x s\mathrm{e}^{2s} \Bigl| F\bigl(\mathrm{e}^s,\, sw_n(s),\, w_n(s)+sw_n'(s)\bigr) \Bigr|\,\mathrm{d}s \\ &\leqslant \frac{1}{x} \int_0^x \mathrm{e}^{2s} \bigl| F\bigl(\mathrm{e}^s,\, (2\mu-\epsilon)s,\, 2\mu-\epsilon\bigr) \bigr|\,\mathrm{d}s \\ &= \frac{1}{x} \int_1^{\mathrm{e}^x} s^2 F\bigl(s,\, (2\mu-\epsilon)\ln s,\, 2\mu-\epsilon\bigr)\,\mathrm{d}s \\ &\leqslant M, \quad x \geqslant 0,\ n \geqslant 1, \end{aligned} \tag{16}$$

for some $M > 0$. From (16) and the mean value theorem one gets that for any $n \geqslant 1$,

$$\bigl|(Tw_n)(x_1) - (Tw_n)(x_2)\bigr| \leqslant M|x_1 - x_2|, \quad x_1, x_2 \geqslant 0.$$

On the other hand, it follows from (10) and (13) that

$$\begin{aligned} &\bigl|(Tw_n)(x_1) + x_1(Tw_n)'(x_1) - (Tw_n)(x_2) - x_2(Tw_n)'(x_2)\bigr| \\ &\quad\leqslant \int_{x_1}^{x_2} \mathrm{e}^{2s} \bigl| F\bigl(\mathrm{e}^s,\, sw_n(s),\, w_n(s)+sw_n'(s)\bigr) \bigr|\,\mathrm{d}s \\ &\quad\leqslant \int_{x_2}^{x_1} \mathrm{e}^{2s} F\bigl(\mathrm{e}^s,\, (2\mu-\epsilon)s,\, 2\mu-\epsilon\bigr)\,\mathrm{d}s \to 0 \quad \text{as } x_1 \to x_2,\ x_1, x_2 \geqslant 0. \end{aligned}$$

Consequently, $\{Tw_n\}_{n\geqslant 1}$ is equicontinuous in X. Thus condition (ii) of Lemma 3 is satisfied.

Now we check condition (iii). In fact, by (12) and (14), we have

$$\begin{aligned}
&\big|(Tw_n)(x)-\mu\big| \\
&\quad\leqslant \frac{1}{x}\int_0^x s\mathrm{e}^{2s}F\big(\mathrm{e}^s,\,(2\mu-\epsilon)s,\,2\mu-\epsilon\big)\,\mathrm{d}s+\int_x^{+\infty}\mathrm{e}^{2s}F\big(\mathrm{e}^s,\,(2\mu-\epsilon)s,\,2\mu-\epsilon\big)\,\mathrm{d}s \\
&\quad\leqslant \int_1^{+\infty}s^2F\big(s,\,(2\mu-\epsilon)\ln s,\,2\mu-\epsilon\big)\,\mathrm{d}s<+\infty \qquad (17)
\end{aligned}$$

and

$$\begin{aligned}
&\big|(Tw_n)(x)+x(Tw_n)'(x)-\mu\big| \\
&\quad\leqslant \int_x^{+\infty}\mathrm{e}^{2s}F\big(\mathrm{e}^s,\,sw_n(s),w_n(s)+sw_n'(s)\big)\,\mathrm{d}s \\
&\quad\leqslant \int_x^{+\infty}\mathrm{e}^{2s}F\big(\mathrm{e}^s,\,(2\mu-\epsilon)s,\,2\mu-\epsilon\big)\,\mathrm{d}s=\int_{\mathrm{e}^x}^{+\infty}s^2F\big(s,\,(2\mu-\epsilon)\ln s,\,2\mu-\epsilon\big)\,\mathrm{d}s \\
&\quad\leqslant \int_1^{+\infty}s^2F\big(s,\,(2\mu-\epsilon)\ln s,\,2\mu-\epsilon\big)\,\mathrm{d}s<+\infty. \qquad (18)
\end{aligned}$$

By (17) and L'Hôspital rule, we get

$$\begin{aligned}
&\lim_{x\to\infty}\frac{\int_0^x s\mathrm{e}^{2s}F(\mathrm{e}^s,\,(2\mu-\epsilon)s,\,2\mu-\epsilon)\,\mathrm{d}s+x\int_x^{+\infty}\mathrm{e}^{2s}F(\mathrm{e}^s,\,(2\mu-\epsilon)s,\,2\mu-\epsilon)\,\mathrm{d}s}{x} \\
&\quad=\lim_{x\to\infty}\int_{\mathrm{e}^x}^{+\infty}s^2F(s,\,(2\mu-\epsilon)\ln s,\,2\mu-\epsilon)\,\mathrm{d}s=0. \qquad (19)
\end{aligned}$$

(17)–(19) imply that for any $\epsilon>0$, there exists $M>0$ such that

$$\big|(Tw_n)(x)-\mu\big|\leqslant\epsilon,\quad \big|(Tw_n)(x)+x(Tw_n)'(x)-\mu\big|<\epsilon,\quad x\geqslant M,\ n\geqslant 1.$$

Thus $\{Tw_n\}_{n\geqslant 1}$ is equiconvergence in X. According to Lemma 3, $T(P)$ is relatively compact in X.

Step 3. We check that $T:P\to P$ is continuous.

It follows from (A2) and the integral absolutely continuity that for any $\epsilon>0$, there exits some $x^*>0$ such that

$$\int_{\mathrm{e}^{x^*}}^{+\infty}s^2F(s,2\mu\ln s,2\mu)\,\mathrm{d}s<\frac{\epsilon}{3}. \qquad (20)$$

On the other hand, by the continuity of f, the function $\mathrm{e}^{2\tau} f(\mathrm{e}^{\tau}, u, v) : [0, x^*] \times [0, (2\mu - \epsilon)x^*] \times [\epsilon, (2\mu - \epsilon)] \to \mathbb{R}$ is uniformly continuous. Thus for the above $\epsilon > 0$, there exists $\delta > 0$ such that

$$\bigl|\mathrm{e}^{2\tau} f\bigl(\mathrm{e}^{\tau}, u_1, v_1\bigr) - \mathrm{e}^{2\tau} f\bigl(\mathrm{e}^{\tau}, u_2, v_2\bigr)\bigr| < \frac{\epsilon}{3x^*} \tag{21}$$

for all $\tau \in [0, x^*]$, $u_1, u_2 \in [0, (2\mu - \epsilon)x^*], v_1, v_2 \in [\epsilon, (2\mu - \epsilon)]$ with $|u_1 - u_2| < \delta$, $|v_1 - v_2| < \delta$, respectively. Notice that for any $w_1, w_2 \in P$ with $\|w_1 - w_2\| < \min\{\delta,\ \delta/x^*\}$, one has

$$\begin{aligned} &\bigl|sw_1(s) - sw_2(s)\bigr| \leqslant x^*\bigl|w_1(s) - w_2(s)\bigr| \leqslant x^*\|w_1 - w_2\| < \delta, \quad s \in [0, x^*], \\ &\bigl|w_1(s) - w_2(s) + sw_1'(s) - sw_2'(s)\bigr| \leqslant \|w_1 - w_2\| < \delta, \quad s \in [0, x^*]. \end{aligned} \tag{22}$$

Similar to (12), for any $w_1, w_2 \in P$ with $\|w_1 - w_2\| < \min\{\delta,\ \delta/x^*\}$, by (20)–(22), we have

$$\begin{aligned} &\bigl|(Tw_1)(x) - (Tw_2)(x)\bigr| \\ &\quad \leqslant \int_0^{x^*} \bigl|\mathrm{e}^{2s} f\bigl(\mathrm{e}^{s},\ sw_1(s),\ w_1(s) + sw_1'(s)\bigr) \\ &\qquad - \mathrm{e}^{2s} f\bigl(\mathrm{e}^{s},\ sw_2(s),\ w_2(s) + sw_2'(s)\bigr)\bigr| \,\mathrm{d}s \\ &\qquad + \int_{x^*}^{\infty} \bigl[\mathrm{e}^{2s}\bigl|f\bigl(\mathrm{e}^{s},\ sw_1(s),\ w_1(s) + sw_1'(s)\bigr)\bigr| \\ &\qquad + \mathrm{e}^{2s}\bigl|f\bigl(\mathrm{e}^{s},\ sw_2(s),\ w_2(s) + sw_2'(s)\bigr)\bigr|\bigr] \,\mathrm{d}s \\ &\quad \leqslant x^* \frac{\epsilon}{3x^*} + 2\int_{x^*}^{\infty} \mathrm{e}^{2s} F\bigl(\mathrm{e}^{s},\ s(2\mu - \epsilon),\ 2\mu - \epsilon\bigr) \,\mathrm{d}s \\ &\quad \leqslant \frac{\epsilon}{3} + 2\int_{\mathrm{e}^{x^*}}^{\infty} s^2 F(s, 2\mu \ln s, 2\mu) \,\mathrm{d}s < \epsilon \end{aligned} \tag{23}$$

and

$$\begin{aligned} &\bigl|(Tw_1)(x) + x(Tw_1)'(x) - (Tw_2)(x) - x(Tw_2)'(x)\bigr| \\ &\quad \leqslant \int_x^{\infty} \bigl|\mathrm{e}^{2s} f\bigl(\mathrm{e}^{s},\ sw_1(s),\ w_1(s) + sw_1'(s)\bigr) \\ &\qquad - \mathrm{e}^{2s} f\bigl(\mathrm{e}^{s},\ sw_2(s),\ w_2(s) + sw_2'(s)\bigr)\bigr| \,\mathrm{d}s \\ &\quad \leqslant \int_x^{\infty} \bigl|\mathrm{e}^{2s} f\bigl(\mathrm{e}^{s},\ sw_1(s),\ w_1(s) + sw_1'(s)\bigr) \\ &\qquad - \mathrm{e}^{2s} f\bigl(\mathrm{e}^{s},\ sw_2(s),\ w_2(s) + sw_2'(s)\bigr)\bigr| \,\mathrm{d}s \\ &\quad < \epsilon. \end{aligned} \tag{24}$$

Hence (23) and (24) yield

$$\begin{aligned}\|Tw_1 - Tw_2\| \leqslant \max\Big\{&\sup_{x\geqslant 0}\big|(Tw_1)(x) - (Tw_2)(x)\big|,\\ &\sup_{x\geqslant 0}\big|(Tw_1)(x) + x(Tw_1)'(x) - (Tw_2)(x) - x(Tw_2)'(x)\big|\Big\}\\ < \epsilon.&\end{aligned}$$

Thus $T : P \to P$ is a continuous operator. According to Schauder's fixed point theorem, T has a fixed point $\overline{w}_\mu$ in P, i.e., $T\overline{w}_\mu = \overline{w}_\mu$.

Step 4. In the end, we consider the solvability and asymptotic properties of Eq. (1). Set $w_\mu(x) = x\overline{w}_\mu(x)$, $x \geqslant 0$, then we have

$$w_\mu(0) = 0, \quad w_\mu(x) > 0, \quad w'_\mu(x) = \overline{w}_\mu(x) + x\overline{w}'_\mu(x) > 0 \quad \text{for } x \geqslant 0,$$

and for $x \geqslant 0$,

$$\begin{aligned}w_\mu(x) = \mu x + \int_0^x x\mathrm{e}^{2x} f\big(\mathrm{e}^x, w_\mu(x), w'_\mu(x)\big)\,\mathrm{d}x\\ + x\int_x^{+\infty} \mathrm{e}^{2x} f\big(\mathrm{e}^x, w_\mu(x), w'_\mu(x)\big)\,\mathrm{d}x.\end{aligned} \tag{25}$$

Differentiating (25) with respect to x, we have Eq. (7), and thus $w_\mu(x)$ is a positive solution of Eq. (7). By Lemma 1, we get that $v_\mu(|z|) = w_\mu(\ln|z|)$, $z \in \Omega$, is a radial solution of Eq. (1). Moreover, in view of (17)–(19), we have

$$v_\mu(z) = 0, \quad z \in \partial\overline{\Omega}, \qquad v_\mu(z) > 0, \quad z \in \Omega,$$

and

$$\begin{aligned}\lim_{|z|\to+\infty}\left(\frac{v_\mu(|z|)}{\ln|z|} - \mu\right) &= \lim_{|z|\to+\infty}\left(\frac{w(\ln|z|)}{\ln|z|} - \mu\right) = \lim_{x\to+\infty}\left(\frac{w_\mu(x)}{x} - \mu\right)\\ &= \lim_{x\to+\infty}\big(v_\mu(x) - \mu\big) = 0.\end{aligned}$$

Thus we complete the proof of Theorem 1. □

Corollary 1. *Assume that the following conditions hold*:

(A3) *There exist two functions* $p, q : [1, +\infty) \to \mathbb{R}^+$ *such that for each* $t \in [1, +\infty)$, $|f(x, u, v)| \leqslant p(x)|u| + q(x)|v|$.

(A4) $\int_1^{+\infty} s^2\big[p(s)\ln s + q(s)\big]\,\mathrm{d}s \leqslant 1/2$.

Then for any $\mu > 0$, *Eq.* (1) *has at least a radial solution* v_μ *with asymptotic properties*

$$v_\mu(z) > 0, \;\; z \in \Omega, \quad v_\mu(z) = 0, \;\; z \in \partial\overline{\Omega}, \quad \textit{and} \quad \lim_{|z|\to+\infty}\frac{v_\mu(|z|)}{\ln|z|} = \mu.$$

Proof. Let $F(x, |u|, |v|) = p(x)|u| + q(x)|v|$, we prove that (A4) implies (A2). In fact, for any fixed $\mu > 0$, we have

$$\int_1^{+\infty} s^2 F(s, 2\mu \ln s, 2\mu)\, \mathrm{d}s = \int_1^{+\infty} s^2 \bigl[2\mu p(s) \ln s + 2\mu q(s)\bigr]\, \mathrm{d}s \leqslant \mu.$$

According to Theorem 1, the conclusion of Corollary 1 holds. □

Corollary 2. *Assume that the following conditions hold*:

(A5) f *satisfies locally Lipschitz condition in* $[1, \infty) \times \mathbb{R}^2$*, i.e., there exist constants* $a, b > 0$ *such that* $|f(x, u_1, v_1) - f(x, u_2, v_2)| \leqslant a|u_1 - u_2| + b|v_1 - v_2|$, $u_1, u_2, v_1, v_2 \in \mathbb{R}$*, for any* $x \in [1, \infty)$.

(A6) $0 < \int_1^{+\infty} s^2[1 + \ln s + |f(s, 0, 0)|]\, \mathrm{d}s \leqslant 1/(2(a + b + 1))$.

Then for any $\mu > 1/2$*, Eq.* (1) *has at least a radial solution* v_μ *with asymptotic properties*

$$v_\mu(z) > 0, \quad z \in \Omega, \qquad v_\mu(z) = 0, \quad z \in \partial\overline{\Omega}, \qquad \text{and} \qquad \lim_{|z| \to +\infty} \frac{v_\mu(|z|)}{\ln |z|} = \mu.$$

Proof. By (A5), for any $x \in [1, \infty)$, we have $|f(x, u, v)| \leqslant a|u| + b|v| + |f(x, 0, 0)|$, $u, v \in \mathbb{R}$. Let $F(x, |u|, |v|) = a|u| + b|v| + |f(x, 0, 0)|$, we prove that (A6) also implies (A2). In fact, for any fixed $\mu > 1/2$, it follows from (A6) that

$$\begin{aligned} \int_1^{+\infty} s^2 F(s, 2\mu \ln s, 2\mu)\, \mathrm{d}s &= \int_1^{+\infty} s^2 \bigl[2\mu a \ln s + 2\mu b + \bigl|f(s, 0, 0)\bigr|\bigr]\, \mathrm{d}s \\ &\leqslant \int_1^{+\infty} s^2 \bigl[2\mu a \ln s + 2\mu b + 2\mu\bigl|f(s, 0, 0)\bigr|\bigr]\, \mathrm{d}s \\ &\leqslant \mu. \end{aligned}$$

According to Theorem 1, the conclusion of Corollary 2 is also true. □

Corollary 3. *Suppose that* (A1) *holds and*

(A2*) *There exists a constant* $\mu > 0$ *such that* $\int_1^{+\infty} s^2 F(s, 2\mu \ln s, 2\mu)\, \mathrm{d}s < +\infty$.

Then there exists $r > 0$ *such that Eq.* (1) *in* $\Omega_{r+1} = \{z \in \mathbb{R}^N\colon\ |z| \geqslant r + 1\}$ *has a radial solution* v_μ *with asymptotic properties*

$$v_\mu(z) = 0, \quad z \in \partial\overline{\Omega}_{r+1}, \qquad v_\mu(z) > 0, \quad z \in \Omega_{r+1}, \qquad \text{and} \qquad \lim_{|z| \to +\infty} \frac{v_\mu(|z|)}{\ln |z|} = \mu.$$

Proof. From (A2*) we take $r > 0$ large enough such that $\int_{r+1}^{+\infty} s^2 F(s, 2\mu \ln s, 2\mu)\,\mathrm{d}s < \mu$. Since Eq. (7) on $[r, +\infty)$ is equivalent to the equation $z'' = \mathrm{e}^{2(x+r)} f(\mathrm{e}^{x+r}, z(x), z(x))$ on $[0, +\infty)$ with $z(x) = w(x + r)$ and

$$\begin{aligned}\int\limits_1^{+\infty} (s+r)F(s+r,\, 2\mu\ln s,\, 2\mu)\,\mathrm{d}s &\leqslant \int\limits_1^{+\infty} (s+r)F\bigl(s+r,\, 2\mu\ln(s+r),\, 2\mu\bigr)\,\mathrm{d}s \\ &\leqslant \int\limits_{r+1}^{+\infty} s^2 F(s, 2\mu\ln s, 2\mu)\,\mathrm{d}s < \mu,\end{aligned}$$

according to Theorem 1, the conclusion of Corollary 3 is valid. □

In the following, we consider the existence of positive solutions and its asymptotic properties for the following separated type geophysical fluid flows model:

$$\Delta v = f\bigl(|z|, v\bigr) + g\bigl(|z|, z \cdot \nabla v\bigr), \quad z \in \Omega, \tag{26}$$

where $f, g : [1, \infty) \times \mathbb{R} \to \mathbb{R}$ are continuous and $\Omega = \{z \in \mathbb{R}^2\colon\ |z| \geqslant 1\}$.

Theorem 2. *Assume that the following conditions hold*:

(B1) *There exist two continuous functions $F_1, G_1 : [1, +\infty) \times \mathbb{R}^+ \to \mathbb{R}^+$, which are nondecreasing with respect to the second independent variable for each $x \in [1, +\infty)$, respectively, such that $|f(x, u)| \leqslant F_1(x, |u|)$, $|g(x, v)| \leqslant G_1(x, |v|)$.*

(B2) *There exists a constant $\mu > 0$ such that $\int_1^{+\infty} s^2 [F_1(s, 2\mu \ln s) + G_1(s, 2\mu)]\,\mathrm{d}s \leqslant \mu$.*

Then Eq. (26) *has a radial solution v_μ, and the following asymptotic properties are valid*:

$$v_\mu(z) = 0,\ \ z \in \partial\overline{\Omega}, \quad v_\mu(z) > 0,\ \ z \in \Omega, \quad \textit{and} \quad \lim_{|z|\to+\infty} \frac{v_\mu(|z|)}{\ln|z|} = \mu.$$

Proof. Let for $x \in [1, +\infty)$ and $u, v \in \mathbb{R}$,

$$\widetilde{f}(x, u, v) = g(x, v) + f(x, u), \quad F\bigl(x, |u|, |v|\bigr) = G_1\bigl(x, |v|\bigr) + F_1\bigl(x, |u|\bigr).$$

Obviously,

$$\bigl|\widetilde{f}(x, u, v)\bigr| \leqslant F\bigl(x, |u|, |v|\bigr), \quad s \in [1, +\infty),\ u, v \in \mathbb{R},$$

and for any $\mu > 0$,

$$\int\limits_1^{+\infty} s^2 F(s, 2\mu\ln s, 2\mu)\,\mathrm{d}s = \int\limits_1^{+\infty} s^2\bigl[F_1(s, 2\mu\ln s) + G_1(s, 2\mu)\bigr]\,\mathrm{d}s \leqslant \mu.$$

Thus according to Theorem 1, the conclusion of Theorem 2 holds. □

Remark 2. The model possesses separated oceanic vorticity which can describe two different vorticities gyres. Theorem 2 is also valid for the special semilinear elliptic equation $\Delta v + f(z, v) + g(|z|)z \cdot \nabla v = 0$ in an exterior domain under assumption (B1).

For (26), if $f(t, u) \equiv 0$ or $g(t, v) \equiv 0$, then it will reduce to the following form in a planar exterior domain:

$$\Delta v = f\bigl(|z|, v\bigr), \quad z \in \Omega, \tag{27}$$

or

$$\Delta v = \bigl(|z|, z \cdot \nabla v\bigr), \quad z \in \Omega, \tag{28}$$

where $f, g : [1, \infty) \times \mathbb{R} \to \mathbb{R}$ are continuous. From Theorem 2 we have the following corollaries.

Corollary 4. *Assume that the following conditions are satisfied*:

(B1*) *There exists a continuous function $F_1 : [1, +\infty) \times \mathbb{R}^+ \to \mathbb{R}^+$, which is nondecreasing with respect to the second independent variable for each $t \in [1, +\infty)$ such that $|f(x, u)| \leqslant F_1(x, |u|)$.*

(B2*) *There exists a constant $\mu > 0$ such that $\int_1^{+\infty} s^2 F_1(s, 2\mu \ln s)\, \mathrm{d}s \leqslant \mu$.*

Then Eq. (27) *has a radial solution v_μ, and the following asymptotic properties hold*:

$$v_\mu(z) = 0, \quad z \in \partial\overline{\Omega}, \qquad v_\mu(z) > 0, \quad z \in \Omega, \qquad \textit{and} \qquad \lim_{|z| \to +\infty} \frac{v_\mu(|z|)}{\ln |z|} = \mu.$$

Corollary 5. *Suppose that the following conditions hold*:

(B1**) *There exists a continuous function $G_1 : [1, +\infty) \times \mathbb{R}^+ \to \mathbb{R}^+$, which is nondecreasing with respect to the second independent variable for each $t \in [1, +\infty)$ such that $|g(x, u)| \leqslant G_1(x, |u|)$.*

(B2**) *There exists a constant $\mu > 0$ such that $\int_1^{+\infty} s^2 G_1(s, 2\mu)\, \mathrm{d}s \leqslant \mu$.*

Then Eq. (28) *has a radial solution v_μ, and the following asymptotic properties hold*:

$$v_\mu(z) = 0, \quad z \in \partial\overline{\Omega}, \qquad v_\mu(z) > 0, \quad z \in \Omega, \qquad \textit{and} \qquad \lim_{|z| \to +\infty} \frac{v_\mu(|z|)}{\ln |z|} = \mu.$$

4 Examples

Example 1. Consider the following gyres model of nonlinear Schrödinger equation for geophysical fluid flows:

$$\Delta v = \frac{v + \sin(z \cdot \nabla v)}{4|z|^4 (1 + \ln |z|)}, \quad z \in \Omega, \tag{29}$$

where $\Omega = \{z \in \mathbb{R}^2\colon\ |z| \geqslant 1\}$.

Let

$$f(x,u,v) = \frac{u+\sin v}{4x^4(1+\ln x)}, \qquad F\bigl(x,|u|,|v|\bigr) = \frac{|u|+|v|}{4x^4(1+\ln x)},$$

$x \in [1,+\infty)$, $u, v \in \mathbb{R}$, then we have

$$\bigl|f(x,u,v)\bigr| \leqslant F\bigl(x,|u|,|v|\bigr), \quad s \in [1,+\infty),\ u,v \in \mathbb{R},$$

and for any $\mu > 0$,

$$\int\limits_1^{+\infty} s^2 F(s, 2\mu \ln s, 2\mu)\,\mathrm{d}s = \int\limits_1^{+\infty} \frac{s(2\mu \ln s + 2\mu)}{4s^3(1+\ln s)}\,\mathrm{d}s < \mu.$$

Thus all conditions of Theorem 1 are verified. From Theorem 1, for any $\mu > 0$, Eq. (29) has a radial solution v_μ with the asymptotic properties

$$v_\mu(z) > 0,\ \ z \in \Omega, \quad v_\mu(z) = 0,\ \ z \in \partial\overline{\Omega}, \quad \text{and} \quad \lim_{|z|\to+\infty} \frac{v_\mu(|z|)}{\ln|z|} = \mu.$$

Example 2. Consider the following gyres model of nonlinear Schrödinger equation for geophysical fluid flows:

$$\Delta v = \frac{(v+\sin|z|)(z\cdot\nabla v)^5}{8|z|^5(\ln|z|+1)(1+(z\cdot\nabla v)^2)}, \quad z \in \Omega, \tag{30}$$

where $\Omega = \{z \in \mathbb{R}^2\colon\ |z| \geqslant 1\}$.

Let

$$f(x,u,v) = \frac{(u+1)v^5}{8(1+v^2)x^5(1+\ln x)}, \qquad F\bigl(x,|u|,|v|\bigr) = \frac{(|u|+1)v^4}{16x^5(1+\ln x)},$$

$x \in [1,+\infty)$, $u, v \in \mathbb{R}$, then we have

$$\bigl|f(x,u,v)\bigr| \leqslant F\bigl(x,|u|,|v|\bigr), \quad s \in [1,+\infty),\ u,v \in \mathbb{R},$$

and for any $\mu \in [1/2, 1)$,

$$\int\limits_1^{+\infty} s^2 F(s, 2\mu \ln s, 2\mu)\,\mathrm{d}s = \int\limits_1^{+\infty} \frac{(2\mu \ln s + 1)(2\mu)^4}{16s^3(1+\ln s)}\,\mathrm{d}s < \mu.$$

Thus all conditions of Theorem 1 are verified. By Theorem 1, for any $\mu \in [1/2, 1)$, Eq. (30) has a radial solution v_μ, and

$$v_\mu(z) > 0,\ \ z \in \Omega, \quad v_\mu(z) = 0,\ \ z \in \partial\overline{\Omega}, \quad \text{and} \quad \lim_{|z|\to+\infty} \frac{v_\mu(|z|)}{\ln|z|} = \mu.$$

Example 3. Let $\Omega = \{z \in \mathbb{R}^2\colon |z| \geqslant 1\}$, consider the following gyres model of geophysical fluid flows:

$$\Delta v = 8\omega \frac{|z|^2 - 1}{|z|^2(1+|z|^2)^3} + \frac{4F(v)}{(1+|z|^2)^2}, \quad z \in \Omega. \tag{31}$$

If there exists a continuous nondecreasing function $F_1 : \mathbb{R}^+ \to \mathbb{R}^+$ and a constant $\mu > 0$ such that

$$\bigl|F(v)\bigr| \leqslant F_1\bigl(|v|\bigr) \quad \text{and} \quad \int\limits_1^{+\infty} \frac{s^2 F_1(2\mu \ln s)}{(1+s^2)^2}\,\mathrm{d}s \leqslant \frac{\mu - \omega}{4}.$$

Then the gyre flow model (2) has a radial solution v_μ, which possesses the following asymptotic properties:

$$v_\mu(z) > 0, \;\; z \in \Omega, \quad v_\mu(z) = 0, \;\; z \in \partial\overline{\Omega}, \quad \text{and} \quad \lim_{|z|\to+\infty} \frac{v_\mu(|z|)}{\ln|z|} = \mu. \tag{32}$$

Let

$$f(x,v) = 8\omega \frac{x^2-1}{x^2(1+x^2)^3} + \frac{4F(v)}{(1+x^2)^2},$$
$$F\bigl(x,|v|\bigr) = 8\omega \frac{x^2-1}{x^2(1+x^2)^3} + \frac{4F_1(|v|)}{(1+x^2)^2},$$

$(x,v) \in [1,+\infty) \times \mathbb{R}$, then for any $(x,v) \in [1,+\infty) \times \mathbb{R}$, we have

$$\begin{aligned}\bigl|f(x,v)\bigr| &\leqslant 8\omega \frac{x^2-1}{x^2(1+x^2)^3} + \frac{4|F(v)|}{(1+x^2)^2} \\ &\leqslant 8\omega \frac{x^2-1}{x^2(1+x^2)^3} + \frac{4F_1(|v|)}{(1+x^2)^2} \\ &\leqslant F\bigl(x,|v|\bigr).\end{aligned}$$

Thus

$$\int\limits_1^{+\infty} s^2 F(s, 2\mu\ln s)\,\mathrm{d}s = \int\limits_1^{+\infty} \biggl[8\omega \frac{s^2-1}{(1+s^2)^3} + \frac{4s^2 F_1(2\mu \ln s)}{(1+s^2)^2}\biggr]\,\mathrm{d}s < \mu.$$

By Corollary 2, the gyre flow model (31) has a radial solution v_μ with the asymptotic properties (32).

Remark 3. Normally, if the oceanic vorticity contribution disappears, i.e, $F \equiv 0$, then the gyre flow model (2) describes that the flow field is irrotational, Example 3 can be applied. But, for long waves in oceanic vorticity, since the averaged vorticity plays dynamically the dominant role (see [7]), the flow behaves like nonzero constant vorticity $F = a$ when $a \leqslant (\mu - \omega)/2$, we still can use Example 3 to govern the behaviour of long waves in oceanic vorticity. In addition, it is reasonable to model Coriolis vorticities [15] by Example 3 when the prime sources of vorticity in the oceans continues to increase in the form of the tidal currents.

Example 4. Consider the semilinear elliptic equation in the domain $\Omega=\{z\in\mathbb{R}^2\colon |z|\geqslant 1\}$

$$\Delta v = f\bigl(|z|, v\bigr) + h\bigl(|z|\bigr)z\cdot\nabla v, \tag{33}$$

where $f\in C([1,+\infty)\times\mathbb{R},\mathbb{R})$, $g\in C([1,+\infty),\mathbb{R})$. Let there exists a continuous function $F_1:[1,+\infty)\times\mathbb{R}^+\to\mathbb{R}^+$ nondecreasing with respect to the second independent variable for each $t\in[1,+\infty)$ such that $|f(x,u)|\leqslant F_1(x,|u|)$, and there exists a constant $\mu>0$ such that $\int_1^{+\infty}s^2[F_1(s,2\mu\ln s)+2\mu|h(s)|]\,\mathrm{d}s\leqslant\mu$. Then Eq. (33) has a radial solution v_μ, which possesses the asymptotic properties

$$v_\mu(z)>0,\quad z\in\Omega,\qquad v_\mu(z)=0,\quad z\in\partial\overline{\Omega},\qquad \lim_{|z|\to+\infty}\frac{v_\mu(|z|)}{\ln|z|}=\mu. \tag{34}$$

Let

$$G_1\bigl(x,|v|\bigr)=\bigl|h(x)\bigr||v|\quad\text{and}\quad\bigl|g(x,v)\bigr|=h(x)v,\qquad x\in[1,+\infty),\ v\in\mathbb{R}.$$

Obviously,

$$\bigl|g(x,v)\bigr|\leqslant G_1\bigl(x,|v|\bigr),\qquad x\in[1,+\infty),\ v\in\mathbb{R},$$

which implies that all conditions of Theorem 1 are verified. From Theorem 2, the Schrödinger equation (33) has a radial solution v_μ with the asymptotic properties (34).

5 Conclusion

In this paper, we study the solvability and asymptotic properties of a recently derived general nonlinear gyre of geophysical fluid flows model. In a general way, there are two factors (the spin vorticity $2\omega\cos\alpha$ and the oceanic vorticity $F(\Psi-\omega\cos\alpha)$) to influence the vorticity of geophysical flows because the earth rotates and ocean motions (for detail, see [3, 5]). The spin vorticity is completely confirmed by the features of planets, however for the motion of the ocean, one may choose the suitable vorticity associated with the movement of the ocean and coupled to the earth's rotation [5] to express a special gyre. In this paper, the oceanic vorticity F is a general nonlinear function, which can be chosen properly according to the actual situation, this makes the model have more extensive application value and significance.

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