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A simple construction of basic polynomials invariant under the Weyl group of the simple finite-dimensional complex Lie algebra

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Abstract. For every simple finite-dimensional complex Lie algebra, I give a simple construction of all (except for the Pfaffian) basic polynomials invariant under the Weyl group. The answer is given in terms of the two basic polynomials of smallest degree.

1 Basics

For necessary information on Lie algebras and Lie groups, see [5]. Recall the notation: let $\mathfrak{g}$ be a simple finite-dimensional complex Lie algebra of rank l, let R_+ (resp. R_-) be the set of its positive (resp. negative) roots, and $\{\alpha_1, \ldots, \alpha_l\}$ the set of simple roots. Let the Weyl group $W_\mathfrak{g}$ of the root system R act in the space $V = \mathbb{R}^l$, let $(-,-)$ be the non-degenerate $W_\mathfrak{g}$-invariant bilinear form on V, such that $(x,y) = \sum_{j=1}^{l} x_j y_j$ for any $x, y \in V$; let h be the Coxeter number; $\delta = \sum_{j=1}^{l} b_j \alpha_j$ the highest root; $b = \max b_j$.

As is known (by abuse of notation we denote the Lie algebra $\mathfrak{sl}(n+1)$ by the symbol of its root system A_n in Cartan's notation, and similarly for the other

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simple Lie algebras),

$$b = \begin{cases} 1 & \text{for } A_l, \\ 2 & \text{for } B_l,\, C_l \text{ and } D_l, \\ 3 & \text{for } G_2 \text{ and } E_6, \\ 4 & \text{for } F_4 \text{ and } E_7, \\ 6 & \text{for } E_8; \end{cases} \qquad h = \begin{cases} l+1 & \text{for } A_l, \\ 2l & \text{for } B_l, \text{ and } C_l, \\ 2l-2 & \text{for } D_l, \\ 12 & \text{for } E_6 \text{ and } F_4, \\ 18 & \text{for } E_7, \\ 30 & \text{for } E_8, \\ 6 & \text{for } G_2. \end{cases} \tag{1}$$

If $\alpha = \sum_{j=1}^{l} n_j \alpha_j \in R_+$, then we define the *height* of α to be $ht(\alpha) = \sum_{j=1}^{l} n_j$.

As it was shown in [1], the algebra of invariant polynomials is generated by l basic homogeneous polynomials of degrees d_j for $j = 1, \ldots, l$ such that $d_1 = 2$, $d_j \leq d_{j+1}$, $d_l = h$.

Note that

$$d_j = \begin{cases} j+1 & \text{for } A_l \\ 2j & \text{for } B_l \text{ and } C_l, \end{cases} \tag{2}$$

$$\{d_j \mid j = 1, \ldots, l-1, l\} = \{2, 4, \ldots, 2(l-1),\ l\}, \text{ as ordered sets, for } D_l. \tag{3}$$

For the exceptional simple Lie algebras, the d_j were first found in [6] using results of [7]:

$$\{d_1, \ldots, d_l\} = \begin{cases} \{2,6\} & \text{for } G_2; \\ \{2,6,8,12\} & \text{for } F_4; \\ \{2,5,6,8,9,12\} & \text{for } E_6; \\ \{2,6,8,10,12,14,18\} & \text{for } E_7; \\ \{2,8,12,14,18,20,24,30\} & \text{for } E_8. \end{cases} \tag{4}$$

Note that the degrees satisfy duality relations

$$d_j + d_{l+1-j} = h + 2. \tag{5}$$

Denote by I_j the invariant polynomial of degree d_j in eq. (2), (3), and (4).

2 Degrees and exponents

Let us remind a characterization of *exponents*, i.e., numbers $m_j := d_j - 1$, given in [2]. Let n_k be the number of roots of height k. Then $n_k - n_{k+1}$ is the number of times k occurs as an exponent of $\mathfrak{g}$.

Note that d_1=2 for all simple Lie algebras. Recall the values of b, see (1). The values of d_2, d_3 and d_4 are given by

Theorem 1. *We have*

$$\begin{aligned} d_2 &= b + 2 && \textit{for all } \mathfrak{g}, \textit{ except } G_2, \\ d_3 &= 2b && \textit{for all exceptional } \mathfrak{g}, \textit{ except } G_2, \\ d_4 &= 2b + 2 && \textit{for } E_6, E_7 \textit{ and } E_8. \end{aligned}$$

The other quantities d_j can be obtained from duality.

The proof follows easily from Table (4) and duality eq. (5) borrowed from any sufficiently thick book (e.g., [5]) and papers [6], [7]. Below we give an independent proof.

Proof. We give the proof only for the most complicated case $\mathfrak{g}= E_8$. For all other cases the proof is analogous.

We enumerate the simple roots of E_8 first along the Dynkin diagram, starting from the longest end of the branch, as the simple roots of A_7, and set

$$\begin{aligned}&(\alpha_8, \alpha_5) = -1,\\ &(\alpha_8, \alpha_k) = 0 \qquad \text{for } k \neq 5, 8,\\ &(\alpha_j, \alpha_j) = 2 \qquad \text{for } j = 1, \dots, 8.\end{aligned}$$

Then, as it is well-known [5], the highest root δ satisfies the conditions:

$$\begin{aligned}&(\alpha_1, \delta) = 1,\\ &(\alpha_j, \delta) = 0 \quad \text{for } j = 2, \dots, 8,\end{aligned}$$

and hence it has the form

$$\delta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8, \qquad ht(\delta) = 29 = h - 1.$$

Let us consider the set of positive roots decreasing in height. We denote

$$\beta_j := \alpha_1 + \alpha_2 + \dots + \alpha_j \ \text{ for } j = 1, 2, \dots, 8.$$

Then we see that

(1) One root for the each height in the interval $[h - 1, h - b]$=$[29, 24]$, namely,

$$\delta,\ \delta - \beta_1,\ \delta - \beta_2,\ \delta - \beta_3,\ \delta - \beta_4,\ \text{and } \delta - \beta_5.$$

(2) Two roots for the each height in the interval $[h-b-1, h-2\,b+2]$=$[23, 20]$, namely,

$$\begin{aligned}&\delta - \beta_6\,, && \delta - (\beta_5 + \alpha_8)\,;\\ &\delta - \beta_7\,, && \delta - (\beta_6 + \alpha_8)\,;\\ &\delta - \beta_8\,, && \delta - (\beta_6 + \alpha_5 + \alpha_8)\,;\\ &\delta - (\beta_8 + \alpha_5)\,, && \delta - (\beta_6 + \alpha_4 + \alpha_5 + \alpha_8)\,.\end{aligned}$$

Note that this is due to the fact that node 5 in the Dynkin diagram for E_8 is the branch node, and $b = 6$.

(3) Three roots for height $h - 2b + 1 = 19$, namely,

$$\delta - (\beta_8 + \alpha_5 + \alpha_6),\quad \delta - (\beta_8 + \alpha_4 + \alpha_5),\quad \delta - (\beta_6 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_8).$$

Hence, from Kostant's characterization of exponents [2] we have

$$m_7 = h - 1 - b = 23, \quad m_6 = h - 2\,b + 1 = 19,$$

and from the duality we have

$$m_2 = b + 1 = 7, \quad m_3 = 2b - 1 = 11.$$

Note that $m_1 = 1$ and $m_8 = 29 = h - 1$. So,

$$d_2 = m_2 + 1 = b + 2 = 8, \quad d_3 = m_3 + 1 = 2b = 12.$$

We analogously obtain $d_4 = 2b + 2 = 14$. □

3 Main Theorem

For an explicit construction of $W_{\mathfrak{g}}$-invariant polynomials, see the papers by Mehta [4] and by Macdonald [3].

Let $\Delta = \sum \frac{\partial^2}{\partial x_j^2}$, and $AI = (\nabla I_2, \nabla I)$ for any polynomial I, where

$$\nabla := (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_l})\,;$$

clearly, $\deg AI = \deg I + b$.

Theorem 2. *All basic $W_{\mathfrak{g}}$-invariant polynomials for all algebras $\mathfrak{g}$, except for D_l and G_2, can be obtained by applying A^k for some k to polynomials I_1 and I_3, where $I_3 = \Delta(AI_2)$.*

More precisely, for the root systems A_l, B_l, C_l, and D_l, we have

$$I_{j+1} = \begin{cases} A^j I_1 & \text{for } j = 0, \ldots, l-1 \quad \text{for the root systems } A_l,\ B_l, \text{ and } C_l, \\ A^j I_1 & \text{for } j = 0, \ldots, l-2 \quad \text{for the root system } D_l, \end{cases}$$

and we obtain all basic polynomials, except for the Pfaffian for D_l.

For the root systems F_4, E_6, E_7 and E_8 the basic invariant polynomials have the form: $A^j I_1$ and $A^k I_3$, where

$$\begin{array}{lll} j = 0, 1 & \text{and } k = 0, 1 & \text{for } F_4; \\ j = 0, 1, 2 & \text{and } k = 0, 1, 2 & \text{for } E_6; \\ j = 0, 1, 2, 3, 4 & \text{and } k = 0, 1 & \text{for } E_7; \\ j = 0, 1, 2, 3 & \text{and } k = 0, 1, 2, 3 & \text{for } E_8. \end{array}$$

Proof. Consider the set of polynomials $A^j I_1$ and $A^k I_3$ for fixed Lie algebra $\mathfrak{g}$ of rank l from the list of Theorem 2. We obtain all l first $W_{\mathfrak{g}}$-invariant polynomials I_l, except for the Pfaffian for D_l. The explicit formulas in the paper [4] imply that these l polynomials are algebraically independent. So we may take these polynomials, and the Pfaffian for a basis of $W_{\mathfrak{g}}$-invariant polynomials. □

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