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STOCHASTIC METHODS FOR THE NEUTRON TRANSPORT EQUATION II: ALMOST SURE GROWTH

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Abstract The neutron transport equation (NTE) describes the flux of neutrons across a planar cross-section in an inhomogeneous fissile medium when the process of nuclear fission is active. Classical work on the NTE emerges from the applied mathematics literature in the 1950s through the work of R. Dautray and collaborators, [8, 9, 24]. The NTE also has a probabilistic representation through the semigroup of the underlying *physical process* when envisaged as a stochastic process; cf. [8, 22, 23, 25]. More recently, [6] and [18] have continued the probabilistic analysis of the NTE, introducing more recent ideas from the theory of spatial branching processes and quasi-stationary distributions. In this paper, we continue in the same vein and look at a fundamental description of stochastic growth in the supercritical regime. Our main result provides a significant improvement on the last known contribution to growth properties of the physical process in [25], bringing neutron transport theory in line with modern branching process theory such as [15, 13]. An important aspect of the proofs focuses on the use of a skeletal path decomposition, which we derive for general branching particle systems in the new context of non-local branching mechanisms.

1. Introduction. In this article we continue our previous work in [18] and look in more detail at the stochastic analysis of the Markov process that lies behind the Neutron Transport Equation (NTE). We recall that the latter describes the flux, Ψ_t, at time $t \geq 0$, of neutrons across a planar cross-section in an inhomogeneous fissile medium (measured in number of neutrons per cm^2 per second). Neutron flux is described in terms of the configuration variables $(r, \upsilon) \in D \times V$, where $D \subseteq \mathbb{R}^3$ is (in general) a non-empty, smooth, open, bounded and convex domain such that ∂D has zero Lebesgue measure, and V is the velocity space, which is given by $V = \{\upsilon \in \mathbb{R}^3 : \upsilon_{\mathtt{min}} \leq |\upsilon| \leq \upsilon_{\mathtt{max}}\}$, where $0 < \upsilon_{\mathtt{min}} < \upsilon_{\mathtt{max}} < \infty$.

In its backwards form, the NTE is introduced as an integro-differential equation of the form

$$\frac{\partial}{\partial t}\psi_t(r, \upsilon) = \upsilon \cdot \nabla \psi_t(r, \upsilon) - \sigma(r, \upsilon)\psi_t(r, \upsilon)$$

$$(1.1) \qquad + \sigma_{\mathtt{s}}(r, \upsilon) \int_V \psi_t(r, \upsilon')\pi_{\mathtt{s}}(r, \upsilon, \upsilon')\mathrm{d}\upsilon' + \sigma_{\mathtt{f}}(r, \upsilon) \int_V \psi_t(r, \upsilon')\pi_{\mathtt{f}}(r, \upsilon, \upsilon')\mathrm{d}\upsilon',$$

where the five fundamental quantities $\sigma_{\mathtt{s}}$, $\pi_{\mathtt{s}}$, $\sigma_{\mathtt{f}}$, $\pi_{\mathtt{f}}$ and σ (known as *cross-sections* in the

physics literature) are all uniformly bounded and measurable with the following interpretation:

$$\sigma_{\mathtt{s}}(r, \upsilon): \quad \text{the rate at which scattering occurs from incoming velocity } \upsilon,$$
$$\sigma_{\mathtt{f}}(r, \upsilon): \quad \text{the rate at which fission occurs from incoming velocity } \upsilon,$$
$$\sigma(r, \upsilon): \quad \text{the sum of the rates } \sigma_{\mathtt{f}} + \sigma_{\mathtt{s}} \text{ and is known as the total cross section,}$$
$$\pi_{\mathtt{s}}(r, \upsilon, \upsilon'): \quad \text{probability density that an incoming velocity } \upsilon \text{ scatters to an outgoing,}$$
$$\text{probability } \upsilon' \text{ satisfying } \int_V \pi_{\mathtt{s}}(r, \upsilon, \upsilon') \mathrm{d}\upsilon' = 1, \text{ and}$$
$$\pi_{\mathtt{f}}(r, \upsilon, \upsilon'): \quad \text{density of expected neutron yield at velocity } \upsilon' \text{ from fission with}$$
$$\text{incoming velocity } \upsilon \text{ satisfying } \int_V \pi_{\mathtt{f}}(r, \upsilon, \upsilon') \mathrm{d}\upsilon' < \infty.$$

It is also usual to assume the additional boundary conditions

(1.2)
$$\begin{cases} \psi_0(r, \upsilon) = g(r, \upsilon) & \text{for } r \in D, \upsilon \in V, \\ \\ \psi_t(r, \upsilon) = 0 & \text{for } t \geq 0 \text{ and } r \in \partial D, \text{ if } \upsilon \cdot \mathbf{n}_r > 0, \end{cases}$$

where $\mathbf{n}_r$ is the outward facing normal of D at $r \in \partial D$ and $g : D \times V \to [0, \infty)$ is a bounded, measurable function which we will later assume has some additional properties. Roughly speaking, this means that neutrons at the boundary which are travelling in the direction of the exterior of the domain are lost to the system.

We will also work with some of (but not necessarily all of) the following assumptions in our results:

(H1) Cross-sections $\sigma_{\mathtt{s}}$, $\sigma_{\mathtt{f}}$, $\pi_{\mathtt{s}}$ and $\pi_{\mathtt{f}}$ are uniformly bounded away from infinity.

(H2) We have $\sigma_{\mathtt{s}}\pi_{\mathtt{s}} + \sigma_{\mathtt{f}}\pi_{\mathtt{f}} > 0$ on $D \times V \times V$.

(H3) There is an open ball B compactly embedded in D such that $\sigma_{\mathtt{f}}\pi_{\mathtt{f}} > 0$ on $B \times V \times V$.

(H4) Fission offspring are bounded in number by the constant $n_{\max} > 1$.

We note that these assumptions are sufficient but not necessary, and refer the reader to Remark 2.1 in [18] for a discussion of their implications.

1.1. *Rigorous interpretation of the NTE.* As explained in the companion paper [18], the NTE (1.1) is not a meaningful equation in the pointwise sense. Whereas previously (1.1) has been interpreted as an abstract Cauchy process on the $L_2(D \times V)$ space, for probabilistic purposes, the NTE can be better understood in its mild form; see the review discussion in [6]. In particular, the NTE is henceforth understood as the unique bounded solution on bounded intervals of time which satisfy (1.2) and the so-called *mild equation*

(1.3)
$$\psi_t[g](r, \upsilon) = \mathtt{U}_t[g](r, \upsilon) + \int_0^t \mathtt{U}_s[(\mathtt{S} + \mathtt{F})\psi_{t-s}[g]](r, \upsilon) \mathrm{d}s, \qquad t \geq 0, r \in D, \upsilon \in V.$$

for $g \in L_\infty^+(D \times V)$, the space of non-negative functions in $L_\infty(D \times V)$. In (1.3), the advection semigroup is given by

$$(1.4) \qquad \mathtt{U}_t[g](r, v) = g(r + vt, v)\mathbf{1}_{(t < \kappa_{r,v}^D)}, \qquad t \geq 0.$$

where $\kappa_{r,v}^D := \inf\{t > 0 : r + vt \notin D\}$, the scattering operator is given by

$$(1.5) \qquad \mathtt{S}g(r, v) = \sigma_\mathtt{s}(r, v) \int_V g(r, v')\pi_\mathtt{s}(r, v, v')\mathrm{d}v' - \sigma_\mathtt{s}(r, v)g(r, v),$$

and the fission operator is given by

$$(1.6) \qquad \mathtt{F}g(r, v) = \sigma_\mathtt{f}(r, v) \int_V g(r, v')\pi_\mathtt{f}(r, v, v')\mathrm{d}v' - \sigma_\mathtt{f}(r, v)g(r, v),$$

for $r \in D$, $v \in V$ and $g \in L_\infty^+(D \times V)$.

The papers [18] and [6] discuss in further detail how the mild representation relates to the other classical representation of the NTE via an abstract Cauchy problem which has been treated in e.g. [8, 9, 24]. To understand better why the mild equation (1.3) is indeed a suitable representation fo the NTE, we need to understand the probabilistic model of the physical process of nuclear fission.

1.2. *Neutron Branching Process.* Let us recall from [18], the *neutron branching process* (NBP), whose expectation semigroup provides the solution to (1.3). It is modelled as a branching process, which at time $t \geq 0$ is represented by a configuration of particles which are specified via their physical location and velocity in $D \times V$, say $\{(r_i(t), v_i(t)) : i = 1, \ldots, N_t\}$, where N_t is the number of particles alive at time $t \geq 0$. In order to describe the process, we will represent it as a process in the space of finite atomic measures

$$(1.7) \qquad X_t(A) = \sum_{i=1}^{N_t} \delta_{(r_i(t), v_i(t))}(A), \qquad A \in \mathcal{B}(D \times V), \ t \geq 0,$$

where δ is the Dirac measure, defined on $\mathcal{B}(D \times V)$, the Borel subsets of $D \times V$. The evolution of $(X_t, t \geq 0)$ is a stochastic process valued in the space of measures $\mathcal{M}(D \times V) := \{\sum_{i=1}^n \delta_{(r_i, v_i)} : n \in \mathbb{N}, (r_i, v_i) \in D \times V, i = 1, \cdots, n\}$ which evolves randomly as follows.

A particle positioned at r with velocity v will continue to move along the trajectory $r + vt$, until one of the following things happen.

(i) The particle leaves the physical domain D, in which case it is instantaneously killed.
(ii) Independently of all other neutrons, a scattering event occurs when a neutron comes in close proximity to an atomic nucleus and, accordingly, makes an instantaneous change of velocity. For a neutron in the system with position and velocity (r, v), if we write $T_\mathtt{s}$ for the random time that scattering may occur, then independently of any other physical event that may affect the neutron, $\Pr(T_\mathtt{s} > t) = \exp\{-\int_0^t \sigma_\mathtt{s}(r + vs, v)\mathrm{d}s\}$, for $t \geq 0$.

When scattering occurs at space-velocity (r, v), the new velocity is selected in V independently with probability $\pi_\mathtt{s}(r, v, v')\mathrm{d}v'$.
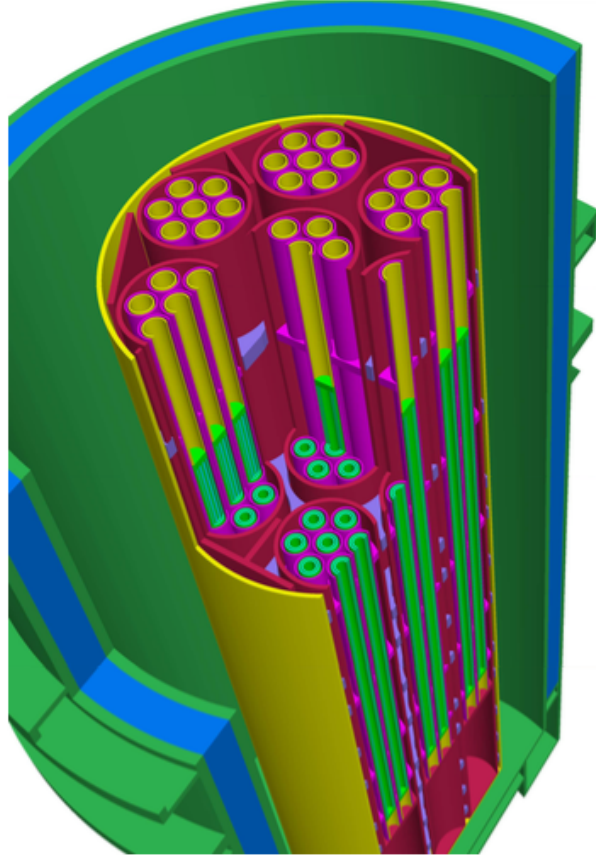
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FigURE 1. *The geometry of a nuclear reactor core representing a physical domain D, on to which the different cross-sectional values of $\sigma_{\mathtt{s}}, \sigma_f, \pi_{\mathtt{s}}, \pi_{\mathtt{f}}$ as mapped as numerical values.*

(iii) Independently of all other neutrons, a fission event occurs when a neutron smashes into an atomic nucleus. For a neutron in the system with initial position and velocity (r, υ), if we write $T_{\mathtt{f}}$ for the random time that fission may occur, then independently of any other physical event that may affect the neutron, $\Pr(T_{\mathtt{f}} > t) = \exp\{-\int_0^t \sigma_{\mathtt{f}}(r + \upsilon s, \upsilon)\mathrm{d}s\}$, for $t \geq 0$.

When fission occurs, the smashing of the atomic nucleus produces lower mass isotopes and releases a random number of neutrons, say $N \geq 0$, which are ejected from the point of impact with randomly distributed, and possibly correlated, velocities, say $\{\upsilon_i : i = 1, \cdots, N\}$. The outgoing velocities are described by the atomic random measure

$$(1.8) \qquad \mathcal{Z}(A) := \sum_{i=1}^{N} \delta_{\upsilon_i}(A), \qquad A \in \mathcal{B}(V).$$

When fission occurs at location $r \in \mathbb{R}^d$ from a particle with incoming velocity $\upsilon \in V$, we denote by $\mathcal{P}_{(r,\upsilon)}$ the law of $\mathcal{Z}$. The probabilities $\mathcal{P}_{(r,\upsilon)}$ are such that, for $\upsilon' \in V$, for

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bounded and measurable $g : V \to [0, \infty)$,

$$(1.9) \qquad \int_V g(v')\pi_{\mathtt{f}}(r, v, v')\mathrm{d}v' = \mathcal{E}_{(r,v)}\left[\int_V g(v')\mathcal{Z}(\mathrm{d}v')\right] =: \mathcal{E}_{(r,v)}[\langle g, \mathcal{Z}\rangle].$$

Note, the possibility that $\Pr(N = 0) > 0$, which will be tantamount to neutron capture (that is, where a neutron slams into a nucleus but no fission results and the neutron is absorbed into the nucleus).

Write $\mathbb{P}_\mu$ for the the law of X when issued from an initial configuration $\mu \in \mathcal{M}(D \times V)$. Coming back to how the physical process relates to the NTE, it was show in [6, 18, 8, 9] that, under the assumptions (H1) and (H2), the unique solution, which is bounded on bounded intervals of time, to (1.3) is given by

$$(1.10) \qquad \psi_t[g](r, v) := \mathbb{E}_{\delta_{(r,v)}}[\langle g, X_t\rangle], \qquad t \geq 0, r \in \bar{D}, v \in V,$$

for $g \in L_\infty^+(D \times V)$. The NBP is thus parameterised by the quantities $\sigma_{\mathtt{s}}, \pi_{\mathtt{s}}, \sigma_{\mathtt{f}}$ and the family of measures $\mathcal{P} = (\mathcal{P}_{(r,v)}, r \in D, v \in V)$ and accordingly we refer to it as a $(\sigma_{\mathtt{s}}, \pi_{\mathtt{s}}, \sigma_{\mathtt{f}}, \mathcal{P})$-NBP. It is associated to the NTE via the relation (1.9), but this association does not uniquely identify the NBP. Nonetheless for a given quadruple $(\sigma_{\mathtt{s}}, \pi_{\mathtt{s}}, \sigma_{\mathtt{f}}, \pi_{\mathtt{f}})$, it was shown in [18] that under the assumptions (H1) and (H3), at least one NBP always exists that can be associated to it via (1.10).

There is, however, a second equation similar to (1.3), which describes the non-linear semi-group of the neutron branching process and which *does* uniquely identify the $(\sigma_{\mathtt{s}}, \pi_{\mathtt{s}}, \sigma_{\mathtt{f}}, \mathcal{P})$-NBP. Write the *branching generator* associated with the physical process by[1]

$$(1.11) \qquad G[g](r, v) = \sigma_{\mathtt{f}}(r, v)\mathcal{E}_{(r,v)}\left[\prod_{j=1}^N g(r, v_j) - g(r, v)\right]$$

for $r \in D$, $v \in V$ and $g \in L_\infty^+(D \times V)$ and define

$$(1.12) \qquad u_t[g](r, v) := \mathbb{E}_{\delta_{(r,v)}}\left[\prod_{i=1}^{N_t} g(r_i(t), v_i(t))\right], \qquad t \geq 0.$$

Formally speaking, by extending the domain in which particles live to include a cemetery state $\{\dagger\}$, corresponding to neutron capture or going neutrons going to the boundary ∂D, we will always work with the convention (cf. [19, 20, 21]) that functions appearing in additive functionals are valued as zero on $\{\dagger\}$, whereas in multiplicative functionals, they are valued as one on $\{\dagger\}$. One may think of this as requiring that empty sums are valued as zero where as empty products are valued as one.

As shown in Section 8 of [18], we can break the expectation over the event of scattering or fission in (1.12) and, appealing to standard manipulations (cf. [6, 18]) we see that, for $g \in L_\infty^+(D \times V)$, which is uniformly bounded by unity,

$$(1.13) \qquad u_t[g] = \hat{\mathsf{U}}_t[g] + \int_0^t \mathsf{U}_s[\mathsf{S}u_{t-s}[g] + G[u_{t-s}[g]]]\mathrm{d}s, \qquad t \geq 0,$$

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[1]Here and elsewhere, an empty product is always understood to be unity.

where

(1.14)
$$\hat{\mathsf{U}}_t[g](r, \upsilon) = g(r + \upsilon(t \wedge \kappa_{r,\upsilon}^D), \upsilon).$$

Under the assumptions (H1), (H2) and (H4), it was also shown in [18] that (1.13) has a unique solution in the space of non-negative functions, which are bounded over bounded intervals of time.

Before moving on to the asymptotics of $(\psi_t, t \geq 0)$, let us make an important note regarding alternative representations of equations (1.3) and (1.13) for later use. In order to do so, let us momentarily introduce what we mean by a neutron random walk (NRW); cf. [18]. A NRW on D, is defined by its scatter rates, $\alpha(r, \upsilon)$, $r \in D, \upsilon \in V$, and scatter probability densities $\pi(r, \upsilon, \upsilon')$, $r \in D, \upsilon, \upsilon' \in V$ where $\int_V \pi(r, \upsilon, \upsilon')\mathrm{d}\upsilon' = 1$ for all $r \in D, \upsilon \in V$. When issued from $r \in D$ with a velocity υ, the NRW will propagate linearly with that velocity until either it exits the domain D, in which case it is killed, or at the random time T_{s} a scattering occurs, where $\Pr(T_{\mathsf{s}} > t) = \exp\{-\int_0^t \alpha(r + \upsilon s, \upsilon)\mathrm{d}s\}$, for $t \geq 0$. When the scattering event occurs in position-velocity configuration (r, υ), a new velocity υ' is selected with probability $\pi(r, \upsilon, \upsilon')\mathrm{d}\upsilon'$. We refer more specifically to the latter as an $\alpha\pi$-NRW.

The linear mild equation (1.3) and its accompanying non-linear mild form (1.13), although consistent with existing literature (cf. [6, 18, 7, 5]) can be equally identified as the unique (in the same sense as mentioned in the previous paragraph) solution to the equations

(1.15)
$$\psi_t[g](r, \upsilon) = \mathsf{Q}_t[g](r, \upsilon) + \int_0^t \mathsf{Q}_s[\mathsf{F}\psi_{t-s}[g]](r, \upsilon)\mathrm{d}s, \qquad t \geq 0, r \in D, \upsilon \in V.$$

and

(1.16)
$$u_t[g] = \hat{\mathsf{Q}}_t[g](r, \upsilon) + \int_0^t \mathsf{Q}_s[G[u_{t-s}[g]](r, \upsilon)\mathrm{d}s, \qquad t \geq 0, r \in D, \upsilon \in V,$$

respectively, where for $g \in L_\infty^+(D \times V)$,

$$\mathsf{Q}_t[g](r, \upsilon) = \mathbf{E}_{(r,\upsilon)}[f(R_t, \Upsilon_t)\mathbf{1}_{(t < \tau^D)}],$$

and

$$\hat{\mathsf{Q}}_t[g](r, \upsilon) = \mathbf{E}_{(r,\upsilon)}[f(R_{t \wedge \tau^D}, \Upsilon_{t \wedge \tau^D})],$$

are the expectation semigroups associated with the $\sigma_{\mathsf{s}}\pi_{\mathsf{s}}$-NRW and $\tau^D = \inf\{t > 0 : R_t \notin D\}$.

1.3. *Lead order asymptotics of the expectation semigroup.* In the accompanying predecessor to this article, [18], a Perron-Frobenius type asymptotic was developed for $(\psi_t, t \geq 0)$. In order to state it we need to introduce another assumption, which is slightly stronger than (H2). To this end, define

(1.17)
$$\alpha(r, \upsilon)\pi(r, \upsilon, \upsilon') = \sigma_{\mathsf{s}}(r, \upsilon)\pi_{\mathsf{s}}(r, \upsilon, \upsilon') + \sigma_{\mathsf{f}}(r, \upsilon)\pi_{\mathsf{f}}(r, \upsilon, \upsilon') \qquad r \in D, \upsilon, \upsilon' \in V.$$

Our new condition is:

(H2)*: We have $\inf_{r \in D, \upsilon, \upsilon' \in V} \alpha(r, \upsilon)\pi(r, \upsilon, \upsilon') > 0.$

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THEOREM 1.1.   *Suppose that (H1) and (H2)* hold. Then, for semigroup $(\psi_t, t \geq 0)$ identified by (1.3), there exists a $\lambda_* \in \mathbb{R}$, a positive[2] right eigenfunction $\varphi \in L_\infty^+(D \times V)$ and a left eigenmeasure which is absolutely continuous with respect to Lebesgue measure on $D \times V$ with density $\tilde{\varphi} \in L_\infty^+(D \times V)$, both having associated eigenvalue $\mathrm{e}^{\lambda_* t}$, and such that φ (resp. $\tilde{\varphi}$) is uniformly (resp. a.e. uniformly) bounded away from zero on each compactly embedded subset of $D \times V$. In particular for all $g \in L_\infty^+(D \times V)$*

$$(1.18) \qquad \langle \tilde{\varphi}, \psi_t[g] \rangle = \mathrm{e}^{\lambda_* t} \langle \tilde{\varphi}, g \rangle \quad (\text{resp. } \psi_t[\varphi] = \mathrm{e}^{\lambda_* t} \varphi) \quad t \geq 0.$$

Moreover, there exists $\varepsilon > 0$ such that, for all $g \in L_\infty^+(D \times V)$,

$$(1.19) \qquad \left\| \mathrm{e}^{-\lambda_* t} \varphi^{-1} \psi_t[g] - \langle \tilde{\varphi}, g \rangle \right\|_\infty = O(\mathrm{e}^{-\varepsilon t}) \text{ as } t \to +\infty.$$

In light of Theorem 1.1, we can categorise the physical process according to the value of λ_*. In particular, when $\lambda_* > 0$ we say the process is supercritical, when $\lambda_* = 0$, the process is critical and when $\lambda_* < 0$, the process is subcritical.

1.4. *Strong law of large numbers at supercriticality.*   The main aim of this article as a continuation of [18] is to understand the almost sure behaviour of the $(\sigma_\mathtt{s}, \pi_\mathtt{s}, \sigma_\mathtt{f}, \mathcal{P})$-NBP in relation to what is, in effect, a statement of mean growth in Theorem 1.1, in the setting that $\lambda_* > 0$. In the aforesaid article, it was noted that

$$(1.20) \qquad W_t := \mathrm{e}^{-\lambda_* t} \frac{\langle \varphi, X_t \rangle}{\langle \varphi, \mu \rangle}, \qquad t \geq 0,$$

is a unit mean martingale under $\mathbb{P}_\mu$, $\mu \in \mathcal{M}(D \times V)$ and, moreover its convergence was studied. Before stating the result regarding the latter, we require one more assumption on the NBP:

(H3)*: There exists a ball B compactly embedded in D such that

$$\inf_{r \in B, v, v' \in V} \sigma_\mathtt{f}(r, v) \pi_\mathtt{f}(r, v, v') > 0.$$

THEOREM 1.2 ([18]).   *For the $(\sigma_\mathtt{s}, \pi_\mathtt{s}, \sigma_\mathtt{f}, \mathcal{P})$-NBP satisfying the assumptions (H1), (H2)*, (H3)* and (H4), the martingale $(W_t, t \geq 0)$ is $L_1(\mathbb{P})$ convergent if and only if $\lambda_* > 0$ (in which case it is also $L_2(\mathbb{P})$ convergent) and otherwise $W_\infty = 0$.*

Note that when $\lambda_* \leq 0$, since $\lim_{t \to 0} W_t = 0$ almost surely, it follows that, for each Ω compactly embedded in $D \times V$, $\lim_{t \to \infty} X_t(\Omega) = 0$. It therefore remains to describe the growth of $X_t(\Omega)$, $t \geq 0$, for $\lambda_* > 0$. This is the main result of this paper, given below. In order to state it, we must introduce the notion of a directionally continuous function on $D \times V$. Such functions are defined as having the property that, for all $r \in D$, $v \in V$,

$$\lim_{t \downarrow 0} g(r + vt, v) = g(r, v).$$

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[2]To be precise, by a positive eigenfunction, we mean a mapping from $D \times V \to (0, \infty)$. This does not prevent it being valued zero on ∂D, as D is an open bounded, convex domain.

THEOREM 1.3. *For all measurable and directionally continuous g on $D \times V$ such that, up to a multiplicative constant, $g \leq \varphi$, under the assumptions of Theorem 1.2,*

$$\lim_{t \to \infty} \mathrm{e}^{-\lambda_* t} \frac{\langle g, X_t \rangle}{\langle \varphi, \mu \rangle} = \langle g, \tilde{\varphi} \rangle W_\infty.$$

$\mathbb{P}_\mu$-*almost surely, for $\mu \in \mathcal{M}(D \times V)$.*

To the best of our knowledge no such results can be found in the existing neutron transport literature. The closest known results are found in the final section of [25] and are significantly weaker than Theorem 1.3.

We can think of Theorem 1.3 as stating a stochastic analogue of (1.19) since the former states that, for $\mu \in \mathcal{M}(D \times V)$,

$$(1.21) \qquad \lim_{t \to \infty} \mathrm{e}^{-\lambda_* t} \frac{\langle \varphi g, X_t \rangle}{\langle \varphi, \mu \rangle} = \langle \tilde{\varphi} \varphi, g \rangle W_\infty.$$

$\mathbb{P}_\mu$-almost surely.

The proof of Theorem 1.3 relies on a fundamental path decomposition, often referred to in the theory of spatial and non-spatial branching processes as a *skeletal decomposition*, see e.g. [13, 15, 2, 10, 26]. The skeletal decomposition is essential in that it identifies an embedded NBP within the original one for which there is no neutron-absorption (neither at ∂D nor into nuclei at collision). This 'thinned down tree' is significantly easier to analyse for technical reasons, but nonetheless provides all the mass in the limit (1.21).

2. Skeletal decomposition. Inspired by [15], we dedicate this section to the proof of a so-called skeletal decomposition, which necessarily requires us to have $\lambda_* > 0$. In very rough terms, for the NBP, we can speak of genealogical lines of descent, meaning neutrons that came from a fission event of a neutron that came from a fission event of a neutron ... and so on, back to one of the initial neutrons a time $t = 0$. If we focus on an individual genealogical line of descent embedded in the NBP, it has a space-velocity trajectory which takes the form of a NRW whose spatial component may or may not hit the boundary of D. Indeed, when the NBP survives for all time (i.e. $\lambda_* > 0$), there must necessarily be some genealogical lines of descent whose spatial trajectories remain in D forever.

The basic idea of the skeletal decomposition is to consider the collection of all surviving genealogical lines of descent and understand their space-velocity dynamics collectively as a process (*the skeleton*). It turns out that the skeleton forms another NBP but with different scatter and fission statistics from the underlying NBP, due to the fact that we are considering genealogical lines of descent which are biased, since they remain in D for all time. For the remaining neutron trajectories that go to the boundary of D, the skeletal decomposition identifies them as immigrants that are thrown off the path of the skeleton.

Below, we develop the statement of the skeletal decomposition. It was brought to our attention by a referee that the proof is robust enough to work in the relatively general setting of a *Markov branching process (MBP) with non-local branching* and hence we first set up

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the notation of a general branching process. It is worthy of note that the motivation for this switch to a general setting is that, for branching particle systems, nothing is known of skeletal decompositions for non-local branching mechanisms; although some results have been identified in the more continuous setting of superprocesses, cf [26], they do not apply to particle systems. Our proof is inspired by the martingale arguments found in [15] which gives a skeletal decomposition for branching Brownian motion in an strip with local branching.

2.1. *The general branching Markov setup.* Until the end of this section (Section 2), unless otherwise mentioned, we will work in the setting of a general MBP, which we will shortly define in more detail. The reader will note that we necessarily choose to overlap our notation for this general setting with that of the NBP. As such, the reader is encouraged to keep in mind the application to the NBP at all times. Additionally, we provide some remarks at the end of this section to illustrate how the general case takes a specific form in the case of the NBP.

Henceforth, $X = (X_t, t \geq 0)$ will be a $(\mathtt{P}, G)$-Markov branching process on a non-empty, open Euclidian domain[3] $E \subseteq \mathbb{R}^d$, where $\mathtt{P} = (\mathtt{P}_t, t \geq 0)$ is a Markov semigroup on E and G is the associated branching generator. More precisely, X is an atomic measure-valued stochastic process (in a similar sense to (1.7)) in which particles move independently according to a copy of the Markov process associated to $\mathtt{P}$ such that, when a particle is positioned at $x \in E$, at the instantaneous spatial rate $\varsigma(x)$, the process will branch and a random number of offspring, say N, are thrown out in positions, say $x_1, \cdots, x_N$ in E, according to some law $\mathcal{P}_x$.

We do not need $\mathtt{P}$ to have the Feller property, and we assume nothing of the boundary conditions on E, in particular, $\mathtt{P}$ need not be conservative. That said, it will prove to be more convenient to introduce a (possible) cemetery state † appended to E, which is to be treated as an absorbing state, and regard $\mathtt{P}$ as conservative. As such,

$$(2.1) \qquad \mathtt{P}_t[f](x) = \mathbf{E}_x[f(\xi_t)] = \mathbf{E}_x[f(\xi_t)\mathbf{1}_{(t<\mathtt{k})}], \qquad x \in E, f \in L_\infty^+(E),$$

where the process ξ, with probabilities $(\mathbf{P}_x, x \in E)$, is the Markov process on $E \cup \{†\}$ with lifetime $\mathtt{k} = \inf\{t > 0 : \xi_t \in †\}$, $L_\infty^+(E)$ is the space of bounded, measurable functions on E and, in this context, we always take $f(†) := 0$.

As such, in a similar spirit to (1.11), we can think of the branching generator, G, as having definition

$$(2.2) \qquad G[f](x) = \varsigma(x)\mathcal{E}_x\left[\prod_{j=1}^{N} f(x_j) - f(x)\right], \qquad x \in E,$$

for $f \in L_\infty^{+,1}(E)$, the space of non-negative measurable functions on E bounded by unity. As previously, we always define the empty product as equal to unity.

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[3]The arguments presented here are robust enough to work with more abstract domains; see for example the set up in [1].

We use $\mathbb{P}_{\delta_x}$ for the law of X issued from a single particle positioned at $x \in E$. In a similar spirit to (1.12), we can introduce the non-linear semigroup of the branching process,

$$(2.3) \qquad u_t[g](x) := \mathbb{E}_{\delta_x}\left[\prod_{i=1}^{N_t} g(x_i(t))\right], \qquad t \geq 0, x \in E, g \in L_\infty^{+,1}(E),$$

where $X_t = \sum_{i=1}^{N_t} \delta_{x_i(t)}$, $t \geq 0$. As before, we define the empty product to be unity, and for consistency, functions, g, appearing in such functionals can be valued on $E \cup \{\dagger\}$ and forced to take the value $g(\dagger) = 1$.

Similarly to the derivation of (1.13) and (1.16), it is straightforward to show that, for such functions, $u_t[g]$ solves the non-linear mild equation

$$(2.4) \qquad u_t[g] = \hat{\mathsf{P}}_t[g](x) + \int_0^t \mathsf{P}_s[G[u_{t-s}[g]](x)\mathrm{d}s, \qquad t \geq 0, x \in E,$$

where we need to adjust P to $\hat{\mathsf{P}}$ to accommodate for the fact that empty products are valued as one, as follows

$$(2.5) \qquad \hat{\mathsf{P}}_t[g](x) = \mathbf{E}_x[g(\xi_{t \wedge \mathbf{k}})], \qquad x \in E.$$

Now, define

$$(2.6) \qquad \zeta := \inf\{t \geq 0 : \langle 1, X_t \rangle = 0\},$$

the time of extinction, and let

$$(2.7) \qquad w(x) := \mathbb{P}_{\delta_x}(\zeta < \infty).$$

Recalling that we need to take as a definition $w(\dagger) = 1$, by conditioning on $\mathcal{F}_t = \sigma(X_s, s \leq t)$, for $t \geq 0$,

$$(2.8) \qquad w(x) = \mathbb{E}_{\delta_x}\left[\prod_{i=1}^{N_t} w(x_i(t))\right].$$

Taking (2.8), (1.12) and (1.13) into account, it is easy to deduce that w also solves

$$(2.9) \qquad w(x) = \hat{\mathsf{P}}_t[w](x) + \int_0^t \mathsf{P}_s\left[G[w]\right](x)\mathrm{d}s, \qquad t \geq 0, x \in E.$$

We will assume:

(M1): $\inf_{x \in E} w(x) > 0$ **and** $w(x) < 1$ **for** $x \in E$.

Beyond this, we assume relatively little about P and G other than:

(M2): ς **is uniformly bounded from above.**

Re-writing (2.9) in the form

$$w(x) = \mathbf{E}_x[w(\xi_{t\wedge\mathbf{k}})] + \mathbf{E}_x\left[\int_0^{t\wedge\mathbf{k}} w(\xi_s)\frac{G[w](\xi_s)}{w(\xi_s)}\mathrm{d}s\right], \qquad t \geq 0,$$

and noting that $\sup_{x\in E} G[w](x)/w(x) < \infty$, we can appeal to the method of exchanging exponential potential for additive potential[4] in e.g. [12, Lemma 1.2, Chapter 4, Part 1], which yields

$$(2.10) \qquad w(x) = \mathbf{E}_x\left[w(\xi_{t\wedge\mathbf{k}})\exp\left(\int_0^{t\wedge\mathbf{k}} \frac{G[w](\xi_s)}{w(\xi_s)}\mathrm{d}s\right)\right], \qquad x \in E, t \geq 0.$$

This identity will turn out to be extremely useful in our analysis, in particular, the equality (2.10) together with the Markov property of ξ implies that the object in the expectation on the right-hand side of (2.10) is a martingale.

Below we give the skeletal decomposition in the form of a theorem. As there is rather a lot of notation, we include a table in the Appendix which the reader may refer to as needed.

THEOREM 2.1 (Skeletal decomposition). *We assume throughout that (M1) and (M2) are in force. Suppose that $\mu = \sum_{i=1}^n \delta_{x_i}$, for $n \in \mathbb{N}$ and $x_1, \cdots, x_n \in E$. Then $(X, \mathbb{P}_\mu)$ is equal in law to*

$$(2.11) \qquad \sum_{i=1}^n \left(\mathtt{B}_i X_t^{i,\updownarrow} + (1 - \mathtt{B}_i)X_t^{i,\downarrow}\right), \qquad t \geq 0,$$

where, for each $i = 1, \ldots, n$, $\mathtt{B}_i$ is an independent Bernoulli random variable with probability of success given by

$$(2.12) \qquad p(x_i) := 1 - w(x_i)$$

and the processes $X^{i,\downarrow}$ and $X^{i,\updownarrow}$ are described as follows.

(i) *For $i = 1, \ldots, n$, the process $X^{i,\downarrow}$ is equal in law to a $(\mathsf{P}^\downarrow, G^\downarrow)$-MBP with probabilities $\mathbb{P}^\downarrow := (\mathbb{P}_{\delta_x}^\downarrow, x \in E)$, defined as follows. In the sense of (2.1), the motion semigroup $\mathsf{P}^\downarrow$ is that of the Markov process ξ with probabilities $(\mathbf{P}_x^\downarrow, x \in E)$, where*

$$(2.13) \qquad \left.\frac{\mathrm{d}\mathbf{P}_x^\downarrow}{\mathrm{d}\mathbf{P}_x}\right|_{\sigma(\xi_s, s\leq t)} = \frac{w(\xi_{t\wedge\mathbf{k}})}{w(x)}\exp\left(\int_0^{t\wedge\mathbf{k}} \frac{G[w](\xi_s)}{w(\xi_s)}\mathrm{d}s\right), \qquad t \geq 0.$$

Moreover, for $x \in E$ and $f \in L_\infty^{+,1}(E)$,

$$(2.14) \qquad G^\downarrow[f] = \varsigma^\downarrow(x)\mathcal{E}_x^\downarrow\left[\prod_{j=1}^N f(x_j) - f(x)\right] = \frac{1}{w}\left[G[fw] - fG[w]\right],$$

---

[4]We will use this trick throughout this paper and consistently refer to it as the 'transfer of the exponential potential to the additive potential' and vice versa in the other direction.

where,

$$(2.15) \qquad \varsigma^\downarrow(x) = \varsigma(x) + \frac{G[w](x)}{w(x)}, \qquad x \in E,$$

and

$$(2.16) \qquad \left. \frac{\mathrm{d}\mathcal{P}_x^\downarrow}{\mathrm{d}\mathcal{P}_x} \right|_{\mathcal{X}} = \frac{\varsigma(x)}{\varsigma^\downarrow(x)w(x)} \prod_{i=1}^N w(x_i).$$

where $\mathcal{X} = \sigma(x_i, i = 1, \cdots, N)$

(ii) *For $i = 1, \ldots, n$, the process $X^{i,\updownarrow}$ is equal in law to a $(\mathrm{P}^\uparrow, G^\uparrow)$-MBP, say $X^\uparrow$, with probabilities $\mathbb{P}^\uparrow := (\mathbb{P}_{\delta_x}^\uparrow, x \in E)$, along the trajectory of which, additional particles are immigrated non-locally in space at branch points that continue to evolve as copies of $X^\downarrow$. The motion semigroup $\mathrm{P}^\uparrow$ corresponds to the Markov process ξ on $E \cup \{\dagger\}$ with probabilities $(\mathbf{P}_x^\uparrow, x \in E)$, where (recalling that p is valued 0 on $\dagger$)*

$$(2.17) \qquad \left. \frac{\mathrm{d}\mathbf{P}_x^\uparrow}{\mathrm{d}\mathbf{P}_x} \right|_{\sigma(\xi_s, s \le t)} = \frac{p(\xi_t)}{p(x)} \exp\left(-\int_0^t \frac{G[w](\xi_s)}{p(\xi_s)} \mathrm{d}s \right), \qquad t \ge 0.$$

The branching mechanism $G^\uparrow$ is given by

$$(2.18) \qquad G^\uparrow[f] = \frac{1}{1-w} \left(G[f(1-w) + w] + \varsigma f(1-w) - G[w] \right).$$

The joint law of the non-local branching together with the instantaneous non-local immigration can be described by the joint generator

$$(2.19) \qquad G^{\updownarrow}[f, g](x) = \varsigma^{\updownarrow}(x) \left(\mathcal{E}_x^{\updownarrow}\left[\prod_{i=1}^{N^\uparrow} f(x_i^\uparrow) \prod_{j=1}^{N^\downarrow} g(x_j^\downarrow) \right] - f(x) \right)$$

for $f, g \in L_\infty^{+,1}(E)$, where $(x_i^\uparrow, i = 1, \cdots, N^\uparrow)$ are the positions and number of offspring of $X^\uparrow$ and $(x_j^\downarrow, j = 1, \cdots, N^\downarrow)$ are the positions and number of simultaneous immigrants, each of which spawns an independent copy of $(X^\downarrow, \mathbb{P}^\downarrow)$ from their respective point of issue. Necessarily $G^\uparrow[f] = G^{\updownarrow}[f, 1]$. The joint branching/immigration rate is given by

$$(2.20) \qquad \varsigma^{\updownarrow}(x) = \mathcal{E}_x \left[\sum_{\substack{I \subseteq \{1, \ldots, N\} \\ |I| \ge 1}} \left(\prod_{i \in I} p(x_i) \prod_{i \in \{1, \ldots, N\} \setminus I} w(x_i) \right) \right], \qquad x \in E.$$

The pairs $(x_i^\uparrow, i = 1, \cdots, N^\uparrow)$ and $(x_j^\downarrow, j = 1, \cdots, N^\downarrow)$ can be seen as an index selection from an independent sample of the non-local branching configuration $((x_k, k = $

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$1, \cdots, N), \mathcal{P}_x)$ albeit under an additional change of measure. As such, in (2.20) we understand the sum to be over all possible subsets of $\{1, \cdots, N\}$, which are then assigned the mark $\uparrow$ (with probabilities p), the unselected indices are given the mark $\downarrow$ (with probabilities w). The aforesaid change of measure takes the form

$$(2.21) \qquad \left. \frac{\mathrm{d}\mathcal{P}_x^{\updownarrow}}{\mathrm{d}\mathcal{P}_x} \right|_{\mathcal{X} \cap A_I} = \mathbf{1}_{A_I} \frac{\varsigma(x)}{\varsigma^{\updownarrow}(x)p(x)} \prod_{i \in I} p(x_i) \prod_{i \in \{1,\dots,N\} \setminus I} w(x_i), \qquad x \in E,$$

where $\mathcal{X} = \sigma(x_i : i = 1, \cdots, N)$ and A_I is the event $\{(x_i^{\uparrow}, i = 1, \cdots, N^{\uparrow}) = (x_i, i \in I)\}$, so that

$$(2.22) \qquad G^{\updownarrow}[f, g](x) = \frac{\varsigma(x)}{p(x)} \mathcal{E}_x \left[\sum_{\substack{I \subseteq \{1,\dots,N\} \\ |I| \geq 1}} \prod_{i \in I} p(x_i) f(x_i) \prod_{i \in \{1,\dots,N\} \setminus I} w(x_i) g(x_i) \right]$$

As alluded to previously, Theorem 2.1 pertains to a classical decomposition of branching trees in which the process (2.11) describes how the MBP divides into the genealogical lines of descent which are 'prolific' (surviving with probability p), in the sense that they create eternal subtrees which never leave the domain, and those which are 'unsuccessful' (dying with probability w), in the sense that they generate subtrees in which all genealogies die out.

REMARK 2.1. It is an easy consequence of Theorem 2.1 that, for $t \geq 0$, the law of $X_t^{\uparrow}$ conditional on $\mathcal{F}_t = \sigma(X_s, s \leq t)$, is equal to that of a Binomial point process with intensity $p(\cdot)X_t(\cdot)$. The latter, written $\mathrm{BinPP}(pX_t)$, is an atomic random measure given by

$$\mathrm{BinPP}(pX_t) = \sum_{i=1}^{N_t} \mathrm{B}_i \delta_{x_i(t)},$$

where (we recall) that $X_t = \sum_{i=1}^{N_t} \delta_{x_i(t)}$, and B_i is a Bernoulli random variable with probability $p(x_i(t))$, $i = 1, \cdots, N_t$.

REMARK 2.2. It is also worthy of special note that the skeleton process $X^{\uparrow}$, has at least one offspring at each branch point. As such, we may also think of the skeleton as a MBP with at least two offspring at each branch point with the motion corresponding to $\mathtt{P}^{\uparrow}$ together with additional discontinuities added in, which correspond to branch points in the $(\mathtt{P}^{\uparrow}, G^{\uparrow})$-MBP with one offspring. More precisely, from the latter point of view, the branching generator can be written as

$$G^{\Uparrow}[f] = \frac{1}{1-w} \left(G[f(1-w) + w] - (1-f)G[w] \right)$$

and the motion would correspond to $\mathtt{P}^{\uparrow}$ with additional continuities given by $G^{\uparrow}[f] - G^{\Uparrow}[f]$.

We prove Theorem 2.1 by splitting it into two parts, Section 2.2 and Section 2.3, which roughly and respectively correspond to parts (i) and (ii) of the statement of the theorem. Before doing so, we need to introduce a little more notation

Let $c_i(t)$ denote the label of a particle $i \in \{1, \ldots, N_t\}$. We label a neutron 'prolific', denoted $c_i(t) = \uparrow$, if it has an infinite genealogical line of descent, and $c_i(t) = \downarrow$, if its line of descent dies out (i.e. 'non-prolific'). We can split the MBP into a tree, consisting of $\uparrow$-labelled neutrons, which is dressed with trees of $\downarrow$-labelled neutrons.

Let $\mathbb{P}^{\updownarrow}_{\delta_x}$ denote the law of the two-labelled process issued from $x \in E$. Then for $t \geq 0$ and $x \in E$ we have the following relationship between $\mathbb{P}^{\updownarrow}$ and $\mathbb{P}$:

$$(2.23) \qquad \left. \frac{\mathrm{d}\mathbb{P}^{\updownarrow}_{\delta_x}}{\mathrm{d}\mathbb{P}_{\delta_x}} \right|_{\mathcal{F}_\infty} = \prod_{i=1}^{N_t} \left(\mathbf{1}_{(c_i(t)=\uparrow)} + \mathbf{1}_{(c_i(t)=\downarrow)} \right) = 1,$$

where $\mathcal{F}_\infty = \sigma\left(\cup_{t \geq 0} \mathcal{F}_t\right)$. Projecting onto $\mathcal{F}_t$, for $t \geq 0$, we have

$$\left. \frac{\mathrm{d}\mathbb{P}^{\updownarrow}_{\delta_x}}{\mathrm{d}\mathbb{P}_{\delta_x}} \right|_{\mathcal{F}_t} = \mathbb{E}_{\delta_x} \left(\prod_{i=1}^{N_t} \left(\mathbf{1}_{(c_i(t)=\uparrow)} + \mathbf{1}_{(c_i(t)=\downarrow)} \right) \middle| \mathcal{F}_t \right)$$

$$= \sum_{I \subseteq \{1, \ldots N_t\}} \prod_{i \in I} \mathbb{P}_{\delta_x}(c_i(t) = \uparrow | \mathcal{F}_t) \prod_{i \in \{1, \ldots, N_t\} \setminus I} \mathbb{P}_{\delta_x}(c_i(t) = \downarrow | \mathcal{F}_t)$$

$$(2.24) \qquad = \sum_{I_t \subseteq \{1, \ldots N_t\}} \prod_{i \in I_t} p(x_i(t)) \prod_{i \in \{1, \ldots, N_t\} \setminus I_t} w(x_i(t)),$$

where we understand the sum to be taken over all subsets of $\{1, \cdots, N_t\}$, each of which is denoted by I_t.

The decomposition in (2.24) indicates the beginning point of how we break up the law of the (P, G)-MBP according to subtrees that are categorised as $\downarrow$ (with probability w) and subtrees that are categorised as $\uparrow$ with $\downarrow$ dressing (with probability p).

2.2. $\downarrow$-trees. Following [15], let us start by characterising the law of genealogical trees populated by the marks $\downarrow$. Thanks to the branching property, it suffices to consider trees which are issued with a single particle with mark $\downarrow$. By definition of the mark $c_\emptyset(0) = \downarrow$, where $\emptyset$ is the initial ancestral particle, this is the same as understanding the law of $(X, \mathbb{P})$ conditioned to become extinct. Indeed, for $A \in \mathcal{F}_t$,

$$\mathbb{P}^{\downarrow}_{\delta_x}(A) := \mathbb{P}^{\updownarrow}_{\delta_x}(A | c_\emptyset(0) = \downarrow)$$

$$= \frac{\mathbb{P}^{\updownarrow}_{\delta_x}(A; c_i = \downarrow, \text{ for each } i = 1, \ldots, N_t)}{\mathbb{P}^{\updownarrow}_{\delta_x}(c_\emptyset(0) = \downarrow)}$$

$$(2.25) \qquad = \frac{\mathbb{E}_{\delta_x}(\mathbf{1}_A \prod_{i=1}^{N_t} w(x_i(t)))}{w(x)}.$$

We are now in a position to characterise the MBP trees which are conditioned to become extinct (equivalently, with genealogical lines of descent which are marked entirely with $\downarrow$).

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Heuristically speaking, the next proposition shows that the conditioning creates a branching particle process in which particles are prone to die out (whether that be due to being killed at the boundary under P, or suppressing offspring). Our proof is partly inspired by Proposition 11 of [15].

PROPOSITION 2.1 ($\downarrow$ Trees). *For initial configurations of the form $\nu = \sum_{i=1}^{n} \delta_{x_i}$, for $n \in \mathbb{N}$ and $x_1, \cdots, x_n \in E$, define the measure $\mathbb{P}_\nu^\downarrow$ via (2.25), i.e.*

$$\mathbb{P}_\nu^\downarrow = \otimes_{i=1}^{n} \mathbb{P}_{\delta_{x_i}}^\downarrow.$$

Then under $\mathbb{P}_\nu^\downarrow$, X is a $(\mathrm{P}^\downarrow, G^\downarrow)$-MBP. That is, the underlying particle movement is given by (2.13), the branching rate is given by $\varsigma^\downarrow$ in (2.15) and the branching mechanism $G^\downarrow$ is given by (2.14). More precisely, if we write

$$(2.26) \qquad u_t^\downarrow[g](x) := \mathbb{E}_{\delta_x}^\updownarrow \left[\prod_{i=1}^{N_t} g(x_i(t)) \,\middle|\, c_\emptyset(0) = \downarrow \right] = \frac{1}{w(x)} u_t[wg](x),$$

then, in the spirit of (1.13), we have, for $g \in L_\infty^{+,1}(E)$,

$$(2.27) \qquad u_t^\downarrow[g](x) = \hat{\mathrm{P}}_t^\downarrow[g](x) + \int_0^t \mathrm{P}_s^\downarrow[G^\downarrow[u_{t-s}^\downarrow[g]](x)\mathrm{d}s, \qquad t \geq 0, x \in E.$$

where $\hat{\mathrm{P}}$ is defined in a similar spirit to (2.5).

PROOF OF PROPOSITION 2.1. First let us show that the change of measure results in a particle process that respects the Markov branching property. In a more general sense, for ν as in the statement of this proposition, (2.25) takes the form

$$\left. \frac{\mathrm{d}\mathbb{P}_\nu^\downarrow}{\mathrm{d}\mathbb{P}_\nu} \right|_{\mathcal{F}_t} = \frac{\prod_{i=1}^{N_t} w(x_i(t))}{\prod_{i=1}^{n} w(x_i)}$$

It is clear from the conditioning that every particle in the resulting process under the new measure $\mathbb{P}_\nu^\downarrow$ must carry the mark $\downarrow$, i.e. be non-prolific, by construction. In the statement of the proposition, we defined, for $g \in L_\infty^{+,1}(E)$,

$$(2.28) \qquad u_t^\downarrow[g](x) = \frac{1}{w} u_t[wg](x)$$

In particular, for $g \in L_\infty^{+,1}(E)$, $x \in E$ and $s, t \geq 0$, note that

$$\mathbb{E}_{\delta_x}^\downarrow \left[\prod_{i=1}^{N_{t+s}} g(x_i(t+s)) \,\middle|\, \mathcal{F}_t \right] = \frac{1}{w(x)} \prod_{i=1}^{N_t} w(x_i(t)) \mathbb{E}_{\delta_x} \left[\frac{\prod_{j=1}^{N_s^i} w(x_j^i(s)) g(x_j^i(s))}{w(x_i(t))} \,\middle|\, \mathcal{F}_t \right]$$

$$(2.29) \qquad \qquad = \frac{1}{w(x)} \prod_{i=1}^{N_t} w(x_i(t)) u_s^\downarrow[g](x_i(t)),$$

where, given $\mathcal{F}_t$, $((x_j^i(t)), j = 1, \cdots, N_s^i)$ are the physical configurations of particles at time $t + s$ that are descendent from particle $i \in N_t$. This ensures Markov branching property. It thus suffices for the remainder of the proof to show that (2.27) holds.

From (2.4) and (2.28) it follows that, for $g \in L_\infty^{+,1}(E)$,

$$(2.30) \qquad u_t^\downarrow[g] = \frac{1}{w}\hat{\mathsf{P}}_t[wg] + \int_0^t \frac{1}{w}\mathsf{P}_s[G[wu_{t-s}^\downarrow[g]]]\mathrm{d}s, \qquad t \geq 0,$$

In the spirit of the derivation of (2.10), we can apply [12, Lemma 1.2, Chapter 4, Part 1] and use (2.13) and (2.30) to get, for $x \in E$,

$$
\begin{aligned}
u_t^\downarrow[g](x) &= \frac{1}{w(x)}\hat{\mathsf{P}}_t[wg](x) + \int_0^t \frac{1}{w(x)}\mathsf{P}_s\left[w\frac{G[wu_{t-s}^\downarrow[g]]}{w}\right](x)\mathrm{d}s \\
&\quad + \int_0^t \frac{1}{w(x)}\mathsf{P}_s\left[\frac{G[w]}{w}wu_{t-s}^\downarrow[g]\right](x)\mathrm{d}s - \int_0^t \frac{1}{w(x)}\mathsf{P}_s\left[\frac{G[w]}{w}wu_{t-s}^\downarrow[g]\right](x)\mathrm{d}s \\
&= \frac{1}{w(x)}\mathbf{E}_x\left[g(\xi_{t\wedge\mathbf{k}})w(\xi_{t\wedge\mathbf{k}})\mathrm{e}^{\int_0^{t\wedge\mathbf{k}}\frac{G[w](\xi_u)}{w(\xi_u)}\mathrm{d}u}\right] \\
&\quad + \frac{1}{w(x)}\mathbf{E}_x\left[\int_0^{t\wedge\mathbf{k}}\frac{G[wu_{t-s}^\downarrow[g]](\xi_s)}{w(\xi_s)}w(\xi_s)\mathrm{e}^{\int_0^s\frac{G[w](\xi_u)}{w(\xi_u)}\mathrm{d}u}\mathrm{d}s\right] \\
&\quad - \frac{1}{w(x)}\mathbf{E}_x\left[\int_0^{t\wedge\mathbf{k}}\frac{G[w](\xi_s)}{w(\xi_s)}u_{t-s}^\downarrow[g](\xi_s)w(\xi_s)\mathrm{e}^{\int_0^s\frac{G[w](\xi_u)}{w(\xi_u)}\mathrm{d}u}\mathrm{d}s\right] \\
&= \hat{\mathsf{P}}_t^\downarrow[g](x) + \int_0^t \mathsf{P}_s^\downarrow\left[\frac{G[wu_{t-s}^\downarrow[g]]}{w}\right](x)\mathrm{d}s - \int_0^t \mathsf{P}_s^\downarrow\left[\frac{G[w]}{w}u_{t-s}^\downarrow[g]\right](x)\mathrm{d}s \\
&= \hat{\mathsf{P}}_t^\downarrow[g](x) + \int_0^t \mathsf{P}_s^\downarrow\left[G^\downarrow[u_{t-s}^\downarrow[g]]\right](x)\mathrm{d}s
\end{aligned}
$$

where we have used the definition (2.14).

It remains to identify the internal structure of $G^\downarrow$. Recalling that $\varsigma^\downarrow = \varsigma + w^{-1}G[w]$, we have, for $f \in L_\infty^{+,1}(E)$,

$$
\begin{aligned}
G^\downarrow[f](x) &= \frac{1}{w(x)}\left[G[fw] - fG[w]\right](x) \\
&= \frac{1}{w(x)}\left[\varsigma(x)\mathcal{E}_x\left[\prod_{i=1}^N f(x_i)w(x_i)\right] - \varsigma(x)f(x)w(x) - fG[w](x)\right] \\
&= \frac{\varsigma(x)}{w(x)}\mathcal{E}_x\left[\prod_{i=1}^N f(x_i)w(x_i)\right] - \left(\varsigma(x) + \frac{G[w]}{w}(x)\right)f(x) \\
&= \varsigma^\downarrow(x)\left(\frac{\varsigma(x)}{\varsigma^\downarrow(x)w(x)}\mathcal{E}_x\left[\prod_{i=1}^N w(x_i)f(x_i)\right] - f(x)\right),
\end{aligned}
$$

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Moreover, recalling the change of measure (2.16), note that, for $x \in E$, $\mathcal{P}_x^{\downarrow}$ is a probability measure on account of the fact that, when we set $f \equiv 1$, recalling again that $\varsigma^{\downarrow} = \varsigma + w^{-1}G[w]$ as well as the definition of G given in (2.2),

$$\mathcal{E}_x \left[\frac{\varsigma(x)}{\varsigma^{\downarrow}(x)w(x)} \prod_{i=1}^{N} w(x_i) \right] = \frac{G[w](x) + \varsigma(x)w(x)}{\varsigma(x) + w^{-1}(x)G[w](x)} \frac{1}{w(x)} = 1$$

as required. $\qquad \square$

We are now read to complete the proof of the skeletal decomposition.

2.3. *Proof of Theorem 2.1.* We may think of $((x_i(t), c_i(t)), i \leq N_t)$, $t \geq 0$, as a two-type branching process. To this end, let us write $N_t^{\uparrow} = \sum_{i=1}^{N_t} \mathbf{1}_{(c_i(t)=\uparrow)}$ and $N^{\downarrow} = N_t - N_t^{\uparrow}$, for $t \geq 0$. Define, for $f, g \in L_{\infty}^{+,1}(E)$,

(2.31) $$u_t^{\updownarrow}[f,g](x) = \mathbb{E}_{\delta_x}^{\updownarrow} \left[\Pi_t[f,g] | c_{\emptyset}(0) = \uparrow \right], \qquad t \geq 0.$$

where, for $t \geq 0$,

$$\Pi_t[f,g] = \prod_{i=1}^{N_t^{\uparrow}} p(x_i^{\uparrow}(t)) f(x_i^{\uparrow}(t)) \prod_{j=1}^{N_t^{\downarrow}} w(x_j^{\downarrow}(t)) g(x_j^{\downarrow}(t)),$$

where
$$(x_i^{\uparrow}(t), i = 1, \cdots, N_t^{\uparrow}) = (x_i(t) \text{ such that } c_i(t) = \uparrow, i \leq N_t)$$

and $(x_i^{\downarrow}(t), i = 1, \cdots, N^{\downarrow})$ is similarly defined. Recalling (2.24), we note that a slightly more elaborate version of (2.25) states that, for $\nu = \sum_{i=1}^{n} \delta_{x_i}$ with $n \geq 1$ and $x_i \in E$, $i = 1, \ldots, n$,

$$\mathbb{E}_{\nu}^{\updownarrow}[\Pi_t[f,g]] = \sum_{I \subseteq \{1,\ldots,n\}} \prod_{i \in I} p(x_i) u_t^{\updownarrow}[f,g](x_i) \prod_{i \in \{1,\ldots,n\} \setminus I} w(x_i) u_t^{\downarrow}[wg](x_i).$$

What this shows is that the change of measure (2.24) (which is of course unity) is equivalent to a Doob h-transform on a two-type branching particle system (i.e. types $\{\uparrow, \downarrow\}$) where we consider the system after disregarding the marks. (Note that, ultimately, this also accommodates for the final statement of Theorem 2.1.) We have already analysed the effect of the Doob h-transform on $\downarrow$-marked particles. It thus remains to understand $u_t^{\updownarrow}[f,g]$ which tells us the dynamics of the typed branching subtrees, $X^{\updownarrow}$, which are rooted with particle s that are $\uparrow$-marked (i.e. the branching processes $X^{\uparrow}$ which are dressed with immigrating trees of the type $X^{\downarrow}$ at fission and scattering).

To this end, we can break the expectation in the definition of $u_t^{\updownarrow}[f,g]$ over the first branching event, noting that until that moment, the initial ancestor is necessarily prolific. We have

(again remembering $p(\dagger) = 0$)

$$u_t^{\updownarrow}[f, g](x)$$

$$= \frac{\mathbb{E}_{\delta_x}\left[\Pi_t[f, g]\mathbf{1}_{(c_\emptyset(0) = \uparrow)}\right]}{\mathbb{P}_{\delta_x}(c_\emptyset(0) = \uparrow)}$$

$$= \frac{1}{p(x)}\mathbf{E}_x\left[p(\xi_t)f(\xi_t)e^{-\int_0^t \varsigma(\xi_u)\mathrm{d}u}\right]$$

$$+ \frac{1}{p(x)}\mathbf{E}_x\left[\int_0^t p(\xi_s)\frac{\varsigma(\xi_s)}{p(\xi_s)}e^{-\int_0^s \varsigma(\xi_u)\mathrm{d}u}\right.$$

$$(2.32) \qquad \left. \mathcal{E}_{\xi_s}\left[\sum_{\substack{I \subseteq \{1,\ldots,N\} \\ |I| \geq 1}} \prod_{i \in I} p(x_i)u_{t-s}^{\updownarrow}[f, g](x_i) \prod_{i \in \{1,\ldots,N\}\setminus I} w(x_i)u_{t-s}^{\downarrow}[g](x_i)\right]\mathrm{d}s\right].$$

To help the reader interpret (2.32) better, we note that the first term on the right-hand side comes from the event that no branching occurs up to time t, in which case the initial ancestor is positioned at ξ_t. Moreover, we have used the fact that $\mathbb{P}_{\delta_x}(c_\emptyset(0) = \uparrow |\mathcal{F}_t) = p(\xi_t)$. The second term is the consequence of a branching event occurring at time $s \in [0, t]$, at which point in time, the initial ancestor is positioned at ξ_s and thus has offspring scattered at $(x_i, i = 1\cdots, N)$ according to $\mathcal{P}_{\xi_s}$. The contribution thereof from time s to t, can be either captured by $u_{t-s}^{\updownarrow}[f, g]$, with probability p, if a given offspring is of type $\uparrow$ (thereby growing a tree of particles marked both $\uparrow$ and $\downarrow$), or captured by $u_{t-s}^{\downarrow}[g]$, with probability w, if a given offspring is of type $\downarrow$ (thereby growing a tree of particles marked only with $\downarrow$). Hence projecting the expectation of $\Pi_t[f, g]\mathbf{1}_{(c_\emptyset = \uparrow)}$ onto the given configuration $(x_i, i = 1\cdots, N)$ at time s, we get the sum inside the expectation with respect to $\mathcal{P}_{\xi_s}$, which caters for all the possible markings of the offspring of the initial ancestor, ensuring that at least one of them is $\uparrow$ (which guarantees $c_\emptyset(0) = \uparrow$). In both expectations, the event of killing is accommodated for the fact that $p(\dagger) = f(\dagger) = \varsigma(\dagger) = 0$.

In order to continue the above calculation, we note from (2.9) that

$$p(x) = \mathbf{E}_x[p(\xi_t)] - \int_0^t \mathbf{E}_x\left[\frac{G[w](\xi_s)}{p(\xi_s)}p(\xi_s)\right]\mathrm{d}s, \qquad t \geq 0.$$

and hence, we can transfer the additive potential to an exponential potential in a similar manner to before (cf. [12, Lemma 1.2, Chapter 4, Part 1])

$$p(x) = \mathbf{E}_x\left[p(\xi_t)\exp\left(-\int_0^t \frac{G[w](\xi_s)}{p(\xi_s)}\mathrm{d}s\right)\right].$$

Next, write

$$(2.33) \qquad \varsigma^{\updownarrow}(x) = \varsigma(x) - \frac{G[w](x)}{p(x)} = \varsigma(x)\frac{\mathcal{E}_x\left[1 - \prod_{i=1}^N w(x_i)\right]}{p(x)}.$$

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and subsequently, for $f, g \in L_\infty^{+,1}(E)$,

$$H^{\updownarrow}[f, g](x) + \varsigma^{\updownarrow}(x)f(x)$$

$$(2.34) \qquad := \frac{\varsigma(x)}{p(x)}\mathcal{E}_x\left[\sum_{\substack{I \subseteq \{1,\dots,N\} \\ |I| \geq 1}} \prod_{i \in I} p(x_i)f(x_i) \prod_{i \in \{1,\dots,N\}\setminus I} w(x_i)g(x_i)\right],$$

for $r \in D$, $v \in D$.

We may now substitute (2.33) and (2.34) into (2.32), then transfer the exponential potential to an additive potential (cf. Lemma 1.2, Chapter 4 in [11]), using (2.17) to get, on E,

$$(2.35) \qquad u_t^{\updownarrow}[f, g] = \mathsf{P}_t^{\uparrow}[f] + \int_0^t \mathsf{P}_s^{\uparrow}\left[H^{\updownarrow}[u_{t-s}^{\updownarrow}[f, g], u_{t-s}^{\downarrow}[g]]\right]\mathrm{d}s, \qquad t \geq 0.$$

(Note, there is no need to define the object $\hat{\mathsf{P}}^{\uparrow}$ in the sprit of (2.5) as the semigroup $\mathsf{P}^{\uparrow}$ is that of a conservative process on E.) This is the semigroup of a two-type MBP in which $\downarrow$-marked particles immigrate off an $\uparrow$-marked MBP. In order to see this more clearly, note that for any $x \in E$,

$$(2.36) \qquad 1 = \mathcal{E}_x\left[\prod_{i=1}^N (p(x_i) + w(x_i))\right] = \mathcal{E}_x\left[\sum_{I \subseteq \{1,\dots,N\}} \prod_{i \in I} p(x_i) \prod_{i \in \{1,\dots,N\}\setminus I} w(x_i)\right],$$

so that

$$\varsigma^{\updownarrow}(x) = \frac{\varsigma(x)}{p(x)}\mathcal{E}_x\left[\sum_{\substack{I \subseteq \{1,\dots,N\} \\ |I| \geq 1}} \prod_{i \in I} p(x_i) \prod_{i \in \{1,\dots,N\}\setminus I} w(x_i)\right]$$

Recalling (2.22), to conclude, we must show that $H^{\updownarrow} = G^{\updownarrow}$. However, this is an easy consequence of (2.21) as soon as we note that

$$\mathcal{E}_x\left[\frac{\varsigma(x)}{\varsigma^{\updownarrow}(x)p(x)} \sum_{\substack{I \subseteq \{1,\dots,N\} \\ |I| \geq 1}} \prod_{i \in I} p(x_i) \prod_{i \in \{1,\dots,N\}\setminus I} w(x_i)\right] = 1,$$

thanks to the definition of $\varsigma^{\updownarrow}$.

To verify (2.18), we need to understand the law of just the $\uparrow$-marked particles that occur at a $\uparrow$-fission point. In other words, we must look at $G^{\uparrow}[f] := G^{\updownarrow}[f, 1]$. We note that, for $r \in D$,

$\upsilon \in V$,

$$G^{\uparrow}[f](x)$$

$$= \frac{\varsigma(x)}{p(x)}\mathcal{E}_x\left[\sum_{\substack{I\subseteq\{1,\dots,N\}\\|I|\geq 1}}\prod_{i\in I}p(x_i)f(x_i)\prod_{i\in\{1,\dots,N\}\setminus I}w(x_i)\right]$$

$$- f(x)\frac{\varsigma(x)}{p(x)}\mathcal{E}_x\left[\sum_{\substack{I\subseteq\{1,\dots,N\}\\|I|\geq 1}}\prod_{i\in I}p(x_i)\prod_{i\in\{1,\dots,N\}\setminus I}w(x_i)\right]$$

$$= \frac{1}{1-w}\left(G[f(1-w)+w]+\varsigma f(1-w)-G[w]\right),$$

as required. $\qquad\square$

2.4. *Remarks on the skeletal decomposition for the NBP.* The case of the skeletal decomposition for the NBP adds an additional layer of intricacy to the general picture given above. In this case, we have $E = D \times V$ with cemetery state $\dagger = \{(r,\upsilon) : r \in \partial D \text{ and } \upsilon \cdot \mathbf{n}_r > 0\}$. It turns out that for the NBP, it is more convenient to view Theorem 2.1 in the spirit of Remark 2.2, i.e. we view the process $X^{\uparrow}$ as a branching process that has at least two offspring at every branching event and whose movement corresponds to advection plus an extra discontinuity, which accounts for a branching event with one offspring.

To make this statement more precise, we first enforce the conditions of Theorem 1.2 in order to ensure (M1) and (M2) are satsfied. Indeed, on account of the inclusion $\{\zeta < \infty\} \subseteq \{W_\infty = 0\}$, we see that $w(x) \leq \mathbb{P}_{\delta_x}(W_\infty = 0)$, $r \in D, \upsilon \in V$. Recalling that W converges both almost surely as well as in $L^1(\mathbb{P})$ to its limit, we have that $\mathbb{P}_{\delta_x}(W_\infty = 0) < 1$ for $r \in D, \upsilon \in V$. This, combined with the fact that every particle may leave the bounded domain D directly without scattering or undergoing fission with positive probability, gives us that

(2.37) $$\mathrm{e}^{-\int_0^{\kappa_{r,\upsilon}^D}\sigma(r+\upsilon s,\upsilon)\mathrm{d}s} < w(r,\upsilon) < 1 \text{ for all } r \in D, \upsilon \in V.$$

Note that the lower bound is uniformly bounded away from 0 thanks to the maximal diameter of D, the minimal velocity $\upsilon_{\mathtt{min}}$ (which, together uniformly upper bound $\kappa_{r,\upsilon}^D$) and the uniformly upper bounded rates of fission and scattering. The upper inequality becomes an equality for $r \in \partial D$ and $\upsilon \cdot \mathbf{n}_r > 0$.

Now, viewing the NBP X as a process with movement $\mathtt{Q}$ and branching generator G, heuristically speaking, we can understand a little better the the motions of $X^{\uparrow}$ and $X^{\downarrow}$ through the action of their generators. By considering only the leading order terms in small time (the process $(X_t, t \geq 0)$ is but a Markov chain), the action of the generator can be see as the result of the limit

(2.38) $$\mathtt{L}f = \lim_{t\downarrow 0}\frac{1}{t}\left(\mathtt{Q}_t[f]-f\right),$$

for suitably smooth f (e.g. continuously differentiable within $L_\infty^+(D\times V)$). It is easy to show, and indeed known (cf. e.g. [6, 9]), that the action of the generator corresponding to Q is given by

$$(2.39)\qquad \mathsf{L}f(r,v) = v\cdot\nabla f(r,v) + \int_V \left(f(r,v') - f(r,v)\right)\sigma_{\mathsf{s}}(r,v)\pi_{\mathsf{s}}(r,v,v')\mathrm{d}v',$$

for $f\in L_\infty^+(D\times V)$ such that ∇f is well defined (here ∇ is assumed to act on the spatial variable r). We emphasise again that, in view of Remark 2.2, this corresponds to motion plus a branching event with one offspring (or scattering).

The change of measure (2.13) induces a generator action given by

$$\mathsf{L}^\downarrow f(r,v) = \frac{1}{w(r,v)}\mathsf{L}(wf)(r,v) + f(r,v)\frac{G[w]}{w}(r,v)$$

$$= v\cdot\nabla f(r,v) + \int_V \left(f(r,v') - f(r,v)\right)\sigma_{\mathsf{s}}(r,v)\frac{w(r,v')}{w(r,v)}\pi_{\mathsf{s}}(r,v,v')\mathrm{d}v'$$

$$+ f(r,v)\left(\frac{\mathsf{L}w}{w} + \frac{G[w]}{w}\right)(r,v)$$

$$(2.40)\qquad = v\cdot\nabla f(r,v) + \int_V \left(f(r,v') - f(r,v)\right)\sigma_{\mathsf{s}}(r,v)\frac{w(r,v')}{w(r,v)}\pi_{\mathsf{s}}(r,v,v')\mathrm{d}v',$$

where the fact that the right-hand side of (2.13) is a martingale will lead to $\mathsf{L}w + G[w] = 0$. In other words, our heuristic reasoning above shows that the motion on the $\downarrow$-marked tree is tantamount to a w-tilting of the scatting kernel. This tilting favours scattering in a direction where extinction becomes more likely, and as such, $\mathsf{L}^\downarrow$ encourages $\downarrow$-marked trees to become extinct 'quickly'.

Similar reasoning shows that the change of measure (2.17) has generator with action

$$\mathsf{L}^\uparrow f(r,v) = \frac{1}{p(r,v)}\mathsf{L}(pf)(r,v) - f(r,v)\frac{G[w]}{p}(r,v)$$

$$= v\cdot\nabla f(r,v) + \int_V \left(f(r,v') - f(r,v)\right)\sigma_{\mathsf{s}}(r,v)\frac{p(r,v')}{p(r,v)}\pi_{\mathsf{s}}(r,v,v')\mathrm{d}v'$$

$$+ f(r,v)\left(\frac{\mathsf{L}p}{p} - \frac{G[w]}{p}\right)(r,v)$$

$$= v\cdot\nabla f(r,v) + \int_V \left(f(r,v') - f(r,v)\right)\sigma_{\mathsf{s}}(r,v)\frac{p(r,v')}{p(r,v)}\pi_{\mathsf{s}}(r,v,v')\mathrm{d}v'$$

$$- f(r,v)\left(\frac{\mathsf{L}w}{p} + \frac{G[w]}{p}\right)(r,v)$$

$$(2.41)\qquad = v\cdot\nabla f(r,v) + \int_V \left(f(r,v') - f(r,v)\right)\sigma_{\mathsf{s}}(r,v)\frac{p(r,v')}{p(r,v)}\pi_{\mathsf{s}}(r,v,v')\mathrm{d}v',$$

for suitably smooth f, where we have again used $\mathsf{L}w + G[w] = 0$. One sees again a p-tilting of the scattering kernel, and hence $L^\uparrow$ rewards scattering in directions that 'enable

survival'. Note, moreover for regions of $D \times V$ for which $p(r, \upsilon)$ can be come arbitrarily small (corresponding to a small probability of survival), the scattering rate also becomes very large, and hence $L^{\uparrow}$ 'urgently' scatters neutrons away from such regions.

2.5. *Remarks on other BBM.* On account of the fact that we have stated Theorem 2.1 for a relatively general MBP with non-local branching, it is worth pointing to the known example of a BBM in a strip that has previously been worked out in detail in [15]. This model has the features that P is that of a Brownian motion with drift μ killed on existing an interval $[0, K]$, so that $L = (1/2)d^2/dx^2 + \mu d/dx$, the branching rate ς is constant (not spatially dependent) and the offspring distribution is concentrated at the point of death of each particle. As such, the generator G in (2.2) takes the simpler form

$$(2.42) \qquad\qquad G[\theta] = \varsigma \mathcal{E} \left(\theta^N - \theta \right)$$

where it suffices to take θ as a number in $(0, 1)$, rather than a function, as there is no spatial dependency. The extinction probability now solves the differential equation

$$\frac{1}{2} \frac{dw}{dx^2} + \mu \frac{dw}{dx} + G[w] = 0 \text{ on } (0, K) \text{ with } w(0) = w(K) = 1.$$

In order for survival to occur with positive probability, it is required that the leading eigenvalue of the mean semigroup associated to the branching process, which is $\lambda_* := (m - 1)\varsigma - \mu^2/2 - \pi^2/2K$, must satisfy $\lambda_* > 0$, where $m = \sum_{k=0}^{\infty} k p_k$ is the mean number of offspring. Note, the mean semigroup is the analogue of (1.10) and the leading eigenvalue plays precisely the role of λ_* in Theorem 1.1 for the NTE.

For the $\downarrow$ process, writing $G^{\downarrow}$ in (2.14) in a similar format to (2.42), it is straightforward to verify that it agrees with the branching mechanism stipulated in analysis of the red tree given in [15].

However, for the $\uparrow$ process, this model also takes the point of view described in Remark 2.2. Indeed, it is straightforward to show that the branching mechanism for the blue tree in [15] agrees with $G^{\Uparrow}$ given in Remark 2.2 and the 'discontinuity' associated with a birth of one offspring is appended to the motion. However, since this model only has local branching and the movement is a Brownian motion, this does not actually change the motion. On the other hand, this choice does affect the overall process $X^{\updownarrow}$ and leads to two types of immigration of red trees onto the blue tree: immigration at branch points and immigration along the trajectory, with the latter immigration occurring at the points corresponding to a 'birth of one offspring'.

When, additionally, the interval $[0, K]$ is replaced by $\mathbb{R}$, the extinction probability w is no longer spatially dependent and is a simple solution of $G[w] = 0$. Assuming that $w \in (0, 1)$, it is easy to see that $L^{\uparrow}$ and $L^{\downarrow}$ are both equal to L and the skeletal decomposition is nothing more than the original skeletal decomposition for Galton–Watson processes (albeit in continuous time) given in the book of Harris [17].

3. SLLN on the skeleton. Our aim is to use the skeletal decomposition of the neutron branching process to prove Theorem 1.3 by first stating and proving the analogous result

for $X^\uparrow$. Hence, in what follows, we will assume (H1), (H2)*, (H3)* and (H4) hold. Before continuing to the proof, let us consider a useful identity. For a suitable $g \in L_\infty(D \times V)$ and $t \geq 0$, we have from Theorem 2.1 (cf. Remark 2.1) that

$$(3.1) \qquad \mathbb{E}^\uparrow_{\delta_{(r,v)}}\left[\langle g, X^\uparrow_t\rangle\right] = \mathbb{E}^\updownarrow_{\delta_{(r,v)}}\left[\langle g, pX_t\rangle\big|c_\emptyset(0) =\uparrow\right] = \frac{1}{p(r,v)}\mathbb{E}_{\delta_{(r,v)}}\left[\langle gp, X_t\rangle\right].$$

We can use this identity to show that λ_* is also an eigenvalue for the linear semigroup of $X^\uparrow$, as well as to compute the associated left and right eigenfunctions (in a similar sense to (1.18)). Our first claim is that the right eigenfunction is given by φ/p. Indeed, for $(r,v) \in D \times V$, due to the above computation,

$$(3.2) \qquad \mathbb{E}^\uparrow_{\delta_{(r,v)}}[\langle \varphi/p, X^\uparrow_t\rangle] = \frac{\mathbb{E}_{\delta_{(r,v)}}[\langle \varphi, X_t\rangle]}{p(r,v)} = \mathrm{e}^{\lambda_* t}\frac{\varphi(r,v)}{p(r,v)}.$$

For the left eigenfunction, again using (3.1), we have

$$(3.3) \qquad \langle \tilde\varphi p, \mathbb{E}^\uparrow_{\delta_\cdot}[\langle g, X^\uparrow_t\rangle]\rangle = \langle \tilde\varphi p, \mathbb{E}_{\delta_\cdot}[\langle gp, X_t\rangle]/p(\cdot)\rangle = \langle \tilde\varphi, \mathbb{E}_{\delta_\cdot}[\langle gp, X_t\rangle]\rangle = \mathrm{e}^{\lambda_* t}\langle \tilde\varphi p, g\rangle.$$

Hence $\tilde\varphi p$ is the corresponding left eigenfunction with eigenvalue $\mathrm{e}^{\lambda_* t}$.

It now follows by similar arguments to those given in [18] that

$$(3.4) \qquad W^\uparrow_t := \mathrm{e}^{-\lambda_* t}\frac{\langle \varphi/p, X^\uparrow_t\rangle}{\langle \varphi/p, \mu\rangle}, \quad t \geq 0,$$

is a positive martingale under $\mathbb{P}^\uparrow_\mu$ for $\mu \in \mathcal{M}(D \times V)$, and hence has a finite limit, which we denote $W^\uparrow_\infty$.

A second useful fact that we will use is the following result.

LEMMA 3.1. *There exists a constant $C \in (0, \infty)$ such that $\sup_{r \in D, v \in V} \varphi(r,v)/p(r,v) < C$.*

PROOF. Let us introduce the family of measures $\mathbb{P}^\varphi := (\mathbb{P}^\varphi_\mu, \mu \in \mathcal{M}(D \times V))$, where

$$(3.5) \qquad \left.\frac{\mathrm{d}\mathbb{P}^\varphi_\mu}{\mathrm{d}\mathbb{P}_\mu}\right|_{\mathcal{F}_t} = W_t, \qquad t \geq 0,$$

We start by noting that, for all $r \in D, v \in V$, $p(r,v) = 1 - \mathbb{P}_{\delta_{(r,v)}}(\zeta < \infty) = \mathbb{P}_{\delta_{(r,v)}}(X \text{ survives})$, where ζ is the lifetime of X defined in (2.6). Taking account of (3.5), we can thus write, with the help of Fatou's Lemma and Jensen's inequality,

$$\begin{aligned}
p(r,v) &= \lim_{t \to \infty} \mathbb{P}_{\delta_{(r,v)}}(t < \zeta) \\
&= \lim_{t \to \infty} \mathbb{E}^\varphi_{\delta_{(r,v)}}\left[\frac{1}{W_t}\mathbf{1}_{(t<\zeta)}\right] \\
&\geq \mathbb{E}^\varphi_{\delta_{(r,v)}}[1/W_\infty] \\
&\geq 1/\mathbb{E}^\varphi_{\delta_{(r,v)}}[W_\infty],
\end{aligned}$$

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where we note that the indicator is dropped in the first inequality as, from Lemma 6.1 in [18], the process $(X, \mathbb{P}^\varphi)$ is immortal.

From equations (10.1) and (10.3) in [18], it has already been shown that

$$(3.6) \qquad \mathbb{E}^\varphi_{\delta_{(r,v)}}[W_\infty] = \lim_{t\to\infty} \mathbb{E}_{\delta_{(r,v)}}[W_t^2] \leq c \int_0^\infty e^{-2\lambda_* t} \frac{\psi_t[1](r,v)}{\varphi(r,v)} dt,$$

for some constant $c \in (0, \infty)$. Taking account of Theorem 1.1, which tells us that

$$\lim_{t\to\infty} e^{-\lambda_* t} \psi_t[1](r,v) = \langle 1, \tilde\varphi \rangle \varphi(r,v) \leq \|\varphi\|_\infty \langle 1, \tilde\varphi \rangle,$$

we deduce that there exists a constant $C \in (0, \infty)$, which does not depend on $(r,v) \in D \times V$, such that

$$p(r,v) \geq \frac{\varphi(r,v)}{C}.$$

The result now follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We are now in a position to state and prove a strong law for the skeleton $X^\uparrow$.

THEOREM 3.1. *For all non-negative and directionally continuous g (in the sense that $\lim_{s\downarrow 0} g(r + vs, v) = g(r,v)$ for all $r \in D$, $v \in V$) such that, for some constant $c > 0$, $g \leq c\varphi/p$,*

$$(3.7) \qquad \lim_{t\to\infty} e^{-\lambda_* t} \langle g, X_t^\uparrow \rangle = \langle g, \tilde\varphi p \rangle \langle \varphi/p, \mu \rangle W_\infty^\uparrow.$$

$\mathbb{P}^\uparrow_\mu$-*almost surely for $\mu \in \mathcal{M}(D \times V)$*

We prove this theorem by breaking it up into several parts. Starting with the following lemma, we first prove that Theorem 3.1 holds along lattice times. Our proofs are principally by techniques that have been used a number of times in the literature, developed by [1, 14, 4, 3] amongst others. Just before we state the next lemma, it will be convenient to quickly introduce the notation $\mathcal{F}_t^\uparrow = \sigma(X_s^\uparrow : s \leq t)$, $t \geq 0$.

LEMMA 3.2. *Fix $\delta > 0$. For non-negative bounded functions $g \in L_\infty^+(D \times V)$, define*

$$(3.8) \qquad U_t = e^{-\lambda_* t} \langle g\varphi/p, X_t^\uparrow \rangle, \quad t \geq 0.$$

Then, for any non-decreasing sequence $(m_n)_{n\geq 0}$ with $m_0 > 0$ and $(r,v) \in D \times V$,

$$(3.9) \qquad \lim_{n\to\infty} |U_{(m_n+n)\delta} - \mathbb{E}^\uparrow[U_{(m_n+n)\delta}|\mathcal{F}_{n\delta}^\uparrow]| = 0, \quad \mathbb{P}^\uparrow_{\delta_{(r,v)}}\text{-}a.s..$$

PROOF. By the Borel-Cantelli lemma, it is sufficient to prove that for each $(r,v) \in D \times V$ and all $\varepsilon > 0$,

$$\sum_{n\geq 1} \mathbb{P}^\uparrow_{\delta_{(r,v)}} \left(|U_{(m_n+n)\delta} - \mathbb{E}[U_{(m_n+n)\delta}|\mathcal{F}_{n\delta}^\uparrow]| > \varepsilon \right) < \infty.$$

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To this end, note that Markov's inequality gives

$$\mathbb{P}^{\uparrow}_{\delta_{(r,\upsilon)}}\left(\left|U_{(m_n+n)\delta} - \mathbb{E}[U_{(m_n+n)\delta}|\mathcal{F}^{\uparrow}_{n\delta}]\right| > \varepsilon\ \right) \le \varepsilon^{-2}\mathbb{E}^{\uparrow}_{\delta_{(r,\upsilon)}}\left(\left|U_{(m_n+n)\delta} - \mathbb{E}[U_{(m_n+n)\delta}|\mathcal{F}^{\uparrow}_{n\delta}]\right|^2\right).$$

Hence, let us consider the term in the conditional expectation on the right-hand side above. First note that

$$(3.10) \qquad U_{(m_n+n)\delta} - \mathbb{E}^{\uparrow}[U_{(m_n+n)\delta}|\mathcal{F}^{\uparrow}_{n\delta}] = \sum_{i=1}^{N_{n\delta}} \mathrm{e}^{-n\delta\lambda_*}(U^{(i)}_{m_n\delta} - \mathbb{E}^{\uparrow}[U^{(i)}_{m_n\delta}|\mathcal{F}^{\uparrow}_{n\delta}]),$$

where, conditional on $\mathcal{F}^{\uparrow}_{n\delta}$, $Z_i = U^{(i)}_{m_n\delta} - \mathbb{E}^{\uparrow}(U^{(i)}_{m_n\delta}|\mathcal{F}^{\uparrow}_{n\delta})$ are independent with $\mathbb{E}[Z_i] = 0$. The formula for the variance of sums of zero mean independent random variables together with the inequality $|a+b|^2 \le 2(|a|^2 + |b|^2)$, we get

$$\mathbb{E}^{\uparrow}(|U_{(m_n+n)\delta} - \mathbb{E}[U_{(m_n+n)\delta}|\mathcal{F}^{\uparrow}_{n\delta}]|^2|\mathcal{F}^{\uparrow}_{n\delta})$$

$$= \sum_{i=1}^{N_{n\delta}} \mathrm{e}^{-2\lambda_*n\delta}\mathbb{E}^{\uparrow}\left[\left|U^{(i)}_{m_n\delta} - \mathbb{E}^{\uparrow}[U^{(i)}_{m_n\delta}|\mathcal{F}^{\uparrow}_{n\delta}]\right|^2\middle|\mathcal{F}^{\uparrow}_{n\delta}\right]$$

$$\le \sum_{i=1}^{N_{n\delta}} \mathrm{e}^{-2\lambda_*n\delta}\mathbb{E}^{\uparrow}\left[4(|U^{(i)}_{m_n\delta}|^2 + |\mathbb{E}^{\uparrow}[U^{(i)}_{m_n\delta}|\mathcal{F}^{\uparrow}_{n\delta}]|^2)|\mathcal{F}^{\uparrow}_{n\delta}\right]$$

$$\le 4\sum_{i=1}^{N_{n\delta}} \mathrm{e}^{-2\lambda_*n\delta}\mathbb{E}^{\uparrow}\left[|U^{(i)}_{m_n\delta}|^2|\mathcal{F}_{n\delta}\right],$$

where we have used Jensen's inequality again in the final inequality. Hence, with $\{(R_i(n\delta), \Upsilon_i(n\delta)) : i = 1, \ldots, N_{n\delta}\}$ describing the configurations of the particles at time $N_{n\delta}$ in $X^{\uparrow}$, we have

$$\sum_{n=1}^{\infty}\mathbb{E}^{\uparrow}\left[|U_{(m_n+n)\delta} - \mathbb{E}^{\uparrow}(U_{(m_n+n)\delta}|\mathcal{F}^{\uparrow}_{n\delta})|^2\right]$$

$$\le 4\sum_{n=1}^{\infty}\mathrm{e}^{-2\lambda_*n\delta}\mathbb{E}^{\uparrow}_{\delta_{(r,\upsilon)}}\left[\sum_{i=1}^{N_{n\delta}}\mathbb{E}^{\uparrow}_{\delta_{(R_i(n\delta),\Upsilon_i(n\delta))}}\left[U^2_{m_n\delta}\right]\right]$$

$$(3.11) \qquad \le 4\|g\|^2_{\infty}\sum_{n=1}^{\infty}\mathrm{e}^{-2\lambda_*n\delta}\mathbb{E}^{\uparrow}_{\delta_{(r,\upsilon)}}\left[\sum_{i=1}^{N_{n\delta}}\frac{\varphi(R_i(n\delta),\Upsilon_i(n\delta))^2}{p(R_i(n\delta),\Upsilon_i(n\delta))^2}\mathbb{E}^{\uparrow}_{\delta_{(R_i(n\delta),\Upsilon_i(n\delta))}}\left[(W^{\uparrow}_{m_n\delta})^2\right]\right],$$

where the final inequality was obtained by noting that, from the definitions of U_t and $W^{\uparrow}_t$, we have

$$\mathbb{E}^{\uparrow}_{\delta_{(r,\upsilon)}}[U_t^2] \le \|g\|^2_{\infty}\frac{\varphi(r,\upsilon)^2}{p(r,\upsilon)^2}\mathbb{E}^{\uparrow}_{\delta_{(r,\upsilon)}}[(W_t^{\uparrow})^2].$$

Due to Theorem 2.1, in particular Remark 2.1, and the calculation leading to (3.1), we have,

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for all $t \geq 0$,

$$
\begin{aligned}
\mathbb{E}_{\delta_{(r,v)}}^{\uparrow}[(W_t^{\uparrow})^2] &= \frac{\mathrm{e}^{-2\lambda_* t}}{(\varphi(r,v)/p(r,v))^2} \mathbb{E}_{\delta_{(r,v)}}^{\uparrow}[\langle \varphi/p, X_t^{\uparrow}\rangle^2] \\
&= \frac{\mathrm{e}^{-2\lambda_* t}}{(\varphi(r,v)/p(r,v))^2} \mathbb{E}_{\delta_{(r,v)}}^{\updownarrow}[\langle \varphi/p, \mathrm{BinPP}(pX_t)\rangle^2 | c_{\emptyset}(0) = \uparrow] \\
&\leq \frac{p(r,v)^2}{\varphi(r,v)^2}\left\{ \mathrm{e}^{-2\lambda_* t}\mathbb{E}_{\delta_{(r,v)}}\left[\langle \varphi^2/p, X_t\rangle\right]/p(r,v) + \mathrm{e}^{-2\lambda_* t}\mathbb{E}_{\delta_{(r,v)}}\left[\langle \varphi, X_t\rangle^2\right]/p(r,v)\right\} \\
&\leq C\left(\frac{\mathrm{e}^{-\lambda_* t}}{\varphi(r,v)} + \mathbb{E}_{\delta_{(r,v)}}\left[W_t^2\right]\right) p(r,v) \\
&\leq C\frac{p(r,v)}{\varphi(r,v)}\left(\varphi(r,v)\mathbb{E}_{\delta_{(r,v)}}\left[W_t^2\right] + 1\right) \quad (3.12)
\end{aligned}
$$

where we have used Lemma 3.1 in the second inequality. From Corollary 5.3 of [18], more precisely from its proof, we know that $\mathbb{E}_{\delta_{(r,v)}}[\sup_{t\geq 0} W_t^2] < \infty$. Hence we have from Doob's maximal inequality that, for each fixed $t \geq 0$,

$$
\begin{aligned}
\mathbb{E}_{\delta_{(r,v)}}^{\uparrow}\left[(W_t^{\uparrow})^2\right] &\leq \mathbb{E}_{\delta_{(r,v)}}^{\uparrow}\left[\sup_{s\geq 0}(W_s^{\uparrow})^2\right] \\
&\leq \limsup_{s\to\infty} 4\mathbb{E}_{\delta_{(r,v)}}^{\uparrow}\left[(W_s^{\uparrow})^2\right] \\
&\leq 4C\frac{p(r,v)}{\varphi(r,v)}\left(\varphi(r,v)\mathbb{E}_{\delta_{(r,v)}}\left[W_{\infty}^2\right] + 1\right) \\
&\leq 4C\frac{p(r,v)}{\varphi(r,v)}(C' + 1) < \infty \quad (3.13)
\end{aligned}
$$

for some constant C' which does not depend on (r,v), where we have used (3.6). (Note (3.13) implies that $W^{\uparrow}$ is an $L_2(\mathbb{P}^{\uparrow})$-convergent martingale.)

Substituting the estimate (3.12) back into (3.11) and making use of the uniform boundedness of φ, we get

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\mathbb{E}^{\uparrow}\left[|U_{(m_n+n)\delta} - \mathbb{E}^{\uparrow}(U_{(m_n+n)\delta}|\mathcal{F}_{n\delta}^{\uparrow})|^2\right] \\
&\qquad\leq K\|g\|_{\infty}^2 \sum_{n=1}^{\infty}\mathrm{e}^{-2\lambda_* n\delta}\mathbb{E}_{\delta_{(r,v)}}^{\uparrow}\left[\sum_{i=1}^{N_{n\delta}}\frac{\varphi(R_i(n\delta), \Upsilon_i(n\delta))}{p(R_i(n\delta), \Upsilon_i(n\delta))}\right], \quad (3.14)
\end{aligned}
$$

for some constant $K \in (0, \infty)$. Now the fact that φ/p is an eigenfunction for the linear semigroup of $X^{\uparrow}$, we get

$$
\begin{aligned}
\sum_{n=1}^{\infty}\mathbb{E}^{\uparrow}\left[|U_{(m_n+n)\delta} - \mathbb{E}^{\uparrow}(U_{(m_n+n)\delta}|\mathcal{F}_{n\delta}^{\uparrow})|^2\right] &\leq K\|g\|_{\infty}^2 \sum_{n\geq 1}\mathrm{e}^{-2\lambda_* n\delta}\mathbb{E}_{\delta_{(r,v)}}^{\uparrow}[\langle \varphi/p, X_{n\delta}^{\uparrow}\rangle] \\
&= K\|g\|_{\infty}^2\frac{\varphi(r,v)}{p(r,v)}\sum_{n\geq 1}\mathrm{e}^{-\lambda_* n\delta} < \infty. \quad (3.15)
\end{aligned}
$$

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Taking expectations one more time with respect to $\mathbb{P}^{\uparrow}_{\delta_{(r,\upsilon)}}$ and appealing to Fubini's Theorem to exchange the sum and expectation on the left hand side of (3.15) completes the proof of the lemma. $\qquad\square$

It is worth noting that a small corollary falls out of the above proof, which will be useful later on.

COROLLARY 3.2. *We have* $\sup_{t\geq 0} W^{\uparrow}_t$ *is square integrable and hence* $W^{\uparrow}$ *converges in* $L_2(\mathbb{P}^{\uparrow})$.

PROOF OF THEOREM 3.1 (LATTICE SEQUENCES). We have already noted that

$$\mathbb{E}^{\uparrow}_{\delta_{(r,\upsilon)}}\left[U_{t+s}\big|\mathcal{F}^{\uparrow}_t\right] = \sum_{i=1}^{N_t} e^{-\lambda_* t} U^{(i)}_s,$$

where, given $\mathcal{F}^{\uparrow}_t$, the $U^{(i)}_s$ are independent and equal in distribution to U_s under $\mathbb{P}^{\uparrow}_{\delta_{(R_i(t),\Upsilon_i(t))}}$ and $\{(R_i(t),\Upsilon_i(t)) : i = 1,\cdots,N_t\}$ describes the configuration of $X^{\uparrow}$ at time $t \geq 0$. Hence, once again using (3.1), the many-to-one formula and the spine change of measure, we have

$$
\begin{aligned}
\mathbb{E}^{\uparrow}_{\delta_{(r,\upsilon)}}\left[U_{t+s}\big|\mathcal{F}^{\uparrow}_t\right] &= \sum_{i=1}^{N_t} e^{-\lambda_* t}\mathbb{E}^{\uparrow}_{\delta_{(R_i(t),\Upsilon_i(t))}}\left[e^{-\lambda_* s}\langle g\varphi/p, X^{\uparrow}_s\rangle\right]\\
&= \sum_{i=1}^{N_t} e^{-\lambda_* t}\frac{\mathbb{E}_{\delta_{(R_i(t),\Upsilon_i(t))}}\left[e^{-\lambda_* s}\langle g\varphi, X_s\rangle\right]}{p(R_i(t),\Upsilon_i(t))}\\
&= \sum_{i=1}^{N_t} e^{-\lambda_* t}\frac{e^{-\lambda_* s}\psi_s[\varphi g](R_i(t),\Upsilon_i(t))}{p(R_i(t),\Upsilon_i(t))}\\
&= \frac{p(r,\upsilon)}{\varphi(r,\upsilon)}\langle g\varphi,\tilde{\varphi}\rangle W^{\uparrow}_t\\
&\quad + \sum_{i=1}^{N_t} e^{-\lambda_* t}\left(e^{-\lambda_* s}\frac{\psi_s[\varphi g](R_i(t),\Upsilon_i(t))}{\varphi(R_i(t),\Upsilon_i(t))} - \langle g\varphi,\tilde{\varphi}\rangle\right)\frac{\varphi(R_i(t),\Upsilon_i(t))}{p(R_i(t),\Upsilon_i(t))}.
\end{aligned}
$$
(3.16)

Appealing to Theorem 1.1, we can pick s sufficiently large so that, for any given $\varepsilon > 0$,

$$\|e^{-\lambda_* s}\varphi^{-1}\psi_s[\varphi g] - \langle\tilde{\varphi},\varphi g\rangle\|_\infty < \varepsilon.$$
(3.17)

Combining this with (3.16) yields

$$\lim_{t\to\infty}\left|\mathbb{E}^{\uparrow}_{\delta_{(r,\upsilon)}}[U_{t+s}|\mathcal{F}^{\uparrow}_t] - W^{\uparrow}_\infty\langle\varphi g,\tilde{\varphi}\rangle\frac{p(r,\upsilon)}{\varphi(r,\upsilon)}\right| = 0.$$
(3.18)

The above combined with the conclusion of Lemma 3.2 gives the conclusion of Theorem 1.3 along lattice sequences. $\qquad\square$

We now make the transition from lattice times to continuous times.

PROOF OF THEOREM 3.1 (FULL SEQUENCE). For $\varepsilon > 0$ and $(r, \upsilon) \in D \times V$, define

$$\Omega_\varepsilon(r, \upsilon) := \left\{ (r', \upsilon') \in D \times V \, : \, g(r', \upsilon') \frac{\varphi(r', \upsilon')}{p(r', \upsilon')} \geq (1 + \varepsilon)^{-1} g(r, \upsilon) \frac{\varphi(r, \upsilon)}{p(r, \upsilon)} \right\}.$$

If we consider the equation (2.9) for the special setting of the NBP, we can decompose it over the first scatter event, rather than the first fission event, from which we will obtain

$$w(r, \upsilon) = \hat{\mathtt{U}}_t[w](r, \upsilon) + \int_0^t \mathtt{U}_s \left[\mathtt{S}w + G[w] \right](r, \upsilon)\mathrm{d}s, \qquad t \geq 0, x \in E,$$

where the semigroup $(\mathtt{U}_t, t \geq 0)$ was defined in (1.4), $(\hat{\mathtt{U}}_t, t \geq 0)$ was defined in (1.14), and the scattering operator $\mathtt{S}$ was defined in (1.5). This implies that, for a given $r \in D$ and $\upsilon \in V$, $w(r + \upsilon t, \upsilon)$, and hence $p(r + \upsilon t, \upsilon)$, are continuous for all t sufficiently small. Similarly noting that $\psi_t[\varphi] = \mathrm{e}^{\lambda_* t}\varphi$, from (1.3), we can also deduce a similar continuity property of φ. Hence, together with the assumed directional continuity of g, for each $r \in D$, $\upsilon \in V$ and $\varepsilon \ll 1$, there exists a δ_ε such that $(r + \upsilon t, \upsilon) \in \Omega_\varepsilon(r, \upsilon)$ for all $t \leq \delta_\varepsilon$.

Next, for each $\delta > 0$ define

$$\Xi^{\delta, \varepsilon}(r, \upsilon) := \mathbf{1}_{\{\mathrm{supp}(X_t^\uparrow) \subset \Omega_\varepsilon(r, \upsilon) \text{ for all } t \in [0, \delta]\}}, \qquad (r, \upsilon) \in D \times V,$$

and let $\eta^{\delta, \varepsilon}(r, \upsilon) = \mathbb{E}_{\delta_{(r, \upsilon)}}^\uparrow[\Xi^{\delta, \varepsilon}(r, \upsilon)] \leq 1$. Appealing to Fatou's Lemma and the continuity properties discussed above, we have, for $\varepsilon \ll 1$,

$$\liminf_{\delta \downarrow 0} \eta^{\delta, \varepsilon}(r, \upsilon) \geq \mathbb{E}_{\delta_{(r, \upsilon)}}^\uparrow\left[\liminf_{\delta \downarrow 0} \mathbf{1}_{\{\mathrm{supp}(X_t^\uparrow) \subset \Omega_\varepsilon(r, \upsilon) \text{ for all } t \in [0, \delta]\}}\right]$$

$$= \mathbb{E}_{\delta_{(r, \upsilon)}}^\uparrow\left[\lim_{\delta \downarrow 0} \mathbf{1}_{\{(r + \upsilon t, \upsilon) \in \Omega_\varepsilon(r, \upsilon) \text{ for all } t \in [0, \delta]\}}\right]$$

$$= 1.$$

Since we can effectively see the skeleton as producing at least two offspring at every fission event (see Remark 2.2)[5], it follows that if $t \in [n\delta, (n + 1)\delta)$ then,

$$\mathrm{e}^{-\lambda_* t}\langle g\varphi/p, X_t^\uparrow \rangle$$

(3.19)
$$\geq \frac{\mathrm{e}^{-\delta}}{(1 + \varepsilon)} \sum_{i=1}^{N_{n\delta}} \mathrm{e}^{-\lambda_* n\delta} g(R_i(n\delta), \Upsilon_i(n\delta)) \frac{\varphi(R_i(n\delta), \Upsilon_i(n\delta))}{p(R_i(n\delta), \Upsilon_i(n\delta))} \Xi^{\delta, \varepsilon}(R_i(n\delta), \Upsilon_i(n\delta)).$$

If we denote the summation on the right-hand side of the above equation by $\tilde{U}_{n\delta}(r, \upsilon)$, and assume that $\mathrm{supp}(g)$ is compactly embedded in D, then we can apply similar arguments to

---

[5]Although a subtle point in the argument, this is fundamentally the reason why the skeletal decomposition is needed and makes the proof much easier than otherwise.

those given in the proof of Lemma 3.2 together with (3.1) to show that

$$\sum_{n=1}^{\infty} \mathbb{E}_{\delta_{(r,\upsilon)}}^{\uparrow} \left[|\tilde{U}_{n\delta} - \mathbb{E}^{\uparrow}[\tilde{U}_{n\delta} | \mathcal{F}_{n\delta}^{\uparrow}]|^2 \right]$$

$$\leq C \sum_{n=1}^{\infty} \mathrm{e}^{-\lambda_* n\delta q} \mathbb{E}_{\delta_{(r,\upsilon)}}^{\uparrow} \left[\sum_{i=1}^{N_{n\delta}} g(R_i(n\delta), \Upsilon_i(n\delta))^2 \frac{\varphi(R_i(n\delta), \Upsilon_i(n\delta))^2}{p(R_i(n\delta), \Upsilon_i(n\delta))^2} \eta^{\delta,\varepsilon}(R_i(n\delta), \Upsilon_i(n\delta)) \right]$$

$$\leq C \sum_{n=1}^{\infty} \mathrm{e}^{-2\lambda_* n\delta} \mathbb{E}_{\delta_{(r,\upsilon)}}^{\uparrow} \left[\langle (g\varphi/p)^2, X_{n\delta}^{\uparrow} \rangle \right]$$

$$\leq \frac{C}{p(r,\upsilon)} \sum_{n=1}^{\infty} \mathrm{e}^{-2\lambda_* n\delta} \mathbb{E}_{\delta_{(r,\upsilon)}} \left[\langle (g\varphi)^2 p^{-1}, X_{n\delta} \rangle \right]$$

(3.20)

$$= \frac{C}{p(r,\upsilon)} \sum_{n=1}^{\infty} \mathrm{e}^{-2\lambda_* n\delta} \psi_{n\delta}[(g\varphi)^2 p^{-1}](r,\upsilon).$$

Note in particular that the compact embedding of the support of g in $D \times V$ together with Lemma 3.1, the fact that $p \leq 1$, φ belongs to $L_\infty^+(D \times V)$ and is bounded away from 0 on compactly embedded subsets of $D \times V$ ensures that $(g\varphi)^2 p^{-1}$ is uniformly bounded away from 0 and ∞ and hence, taking account of the conclusion of Theorem 1.1, the expectation on the right-hand side of (3.20) is finite.

Noting that

$$\mathbb{E}^{\uparrow}[\tilde{U}_{n\delta} | \mathcal{F}_{n\delta}^{\uparrow}] = \mathrm{e}^{-\lambda_* n\delta} \langle g\varphi\eta^{\delta,\varepsilon}/p, X_{n\delta}^{\uparrow} \rangle,$$

the consequence of (3.20), when taken in the light of the Borel-Cantelli Lemma and the already proved limit (3.7) on lattice times, means that, $\mathbb{P}_{\delta_{(r,\upsilon)}}$-almost surely,

$$\liminf_{t\to\infty} \mathrm{e}^{-\lambda_* t} \langle g\varphi/p, X_t^{\uparrow} \rangle \geq \frac{\mathrm{e}^{-\delta}}{1+\varepsilon} \langle g\varphi\eta^{\delta,\varepsilon}/p, \tilde{\varphi}p \rangle W_\infty^{\uparrow} \frac{\varphi(r,\upsilon)}{p(r,\upsilon)}.$$

Letting $\delta \downarrow 0$ with the help of Fatou's Lemma and then $\varepsilon \downarrow 0$ in the above inequality yields

(3.21)
$$\liminf_{t\to\infty} \mathrm{e}^{-\lambda_* t} \langle g\varphi/p, X_t^{\uparrow} \rangle \geq \langle g\varphi, \tilde{\varphi} \rangle W_\infty^{\uparrow} \frac{\varphi(r,\upsilon)}{p(r,\upsilon)},$$

$\mathbb{P}_{\delta_{(r,\upsilon)}}$-almost surely. Now replacing g by hp/φ, ensuring still that the support of h is compactly embedded in $D \times V$, so that hp/φ is uniformly bounded away from 0 and ∞, the lower bound (3.21) yields

(3.22)
$$\liminf_{t\to\infty} \mathrm{e}^{-\lambda_* t} \langle h, X_t^{\uparrow} \rangle \geq \langle h, \tilde{\varphi}p \rangle W_\infty^{\uparrow} \frac{\varphi(r,\upsilon)}{p(r,\upsilon)}.$$

We can push (3.22) a little bit further by removing the requirement that the support of h is compactly embedded in $D \times V$. Indeed, suppose that, for $n \geq 1$, $h_n = h\mathbf{1}_{B_n}$, where $h \leq c\varphi/p$

for some constant $c > 0$ and B_n is an increasing sequence of compactly embedded domains in $D \times V$, such that $\cup_{n \geq 1} B_n = D \times V$. Then (3.22) and together with monotonicity gives us

$$\liminf_{t \to \infty} \mathrm{e}^{-\lambda_* t} \langle h, X_t^{\uparrow} \rangle \geq \lim_{n \to \infty} \liminf_{t \to \infty} \mathrm{e}^{-\lambda_* t} \langle h_n, X_t^{\uparrow} \rangle$$

$$\geq \lim_{n \to \infty} \langle h_n, \tilde{\varphi} p \rangle W_{\infty}^{\uparrow} \frac{\varphi(r,v)}{p(r,v)}$$

(3.23)
$$= \langle h, \tilde{\varphi} p \rangle W_{\infty}^{\uparrow} \frac{\varphi(r,v)}{p(r,v)}$$

$\mathbb{P}_{\delta_{(r,v)}}$-almost surely.

To complete the proof of Theorem 3.1 it now suffices to show that, $\mathbb{P}_{\delta_{(r,v)}}$-almost surely, $\limsup_{t \to \infty} \mathrm{e}^{-\lambda_* t} \langle g, X_t^{\uparrow} \rangle \leq \langle g, \tilde{\varphi} p \rangle W_{\infty}^{\uparrow} \varphi(r,v)/p(r,v)$. To this end note that, for $0 \leq g \leq c\varphi/p$, for some constant $c > 0$ (which, without loss of generality, we may take equal to 1),

$$\limsup_{t \to \infty} \mathrm{e}^{-\lambda_* t} \langle g, X_t^{\uparrow} \rangle = \limsup_{t \to \infty} \left(\frac{\varphi(r,v)}{p(r,v)} W_t^{\uparrow} - \mathrm{e}^{-\lambda_* t} \langle \varphi/p - g, X_t^{\uparrow} \rangle \right)$$

$$= \frac{\varphi(r,v)}{p(r,v)} W_{\infty}^{\uparrow} - \liminf_{t \to \infty} \mathrm{e}^{-\lambda_* t} \langle \varphi/p - g, X_t^{\uparrow} \rangle$$

$$\leq \frac{\varphi(r,v)}{p(r,v)} W_{\infty}^{\uparrow} - \langle \varphi/p - g, \tilde{\varphi} p \rangle \frac{\varphi(r,v)}{p(r,v)} W_{\infty}^{\uparrow}$$

$$= \langle g, \tilde{\varphi} p \rangle W_{\infty}^{\uparrow} \frac{\varphi(r,v)}{p(r,v)},$$

as required, where we have used the normalisation $\langle \varphi, \tilde{\varphi} \rangle = 1$. $\square$

4. Proof of Theorem 1.3. The proof we will give relies on the stochastic embedding of the skeleton process $X^{\uparrow}$ in X together with a measure theoretic trick. It is worth stating the latter in the format of a proposition which is lifted from [16]. (The reader will note that there is a slight variation in the statement as the original version was missing an additional condition.)

PROPOSITION 4.1. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), \mathbb{P})$ be a filtered probability space and define $\mathcal{F}_{\infty} := \sigma(\cup_{i=1}^{\infty} \mathcal{F}_t)$. Suppose $(U_t, t \geq 0)$ is an $\mathcal{F}$-measurable process such that $\sup_{t \geq 0} U_t$ has finite expectation and $\mathbb{E}(U_t | \mathcal{F}_t)$ càdlàg. If*

$$\lim_{t \to \infty} \mathbb{E}(U_t | \mathcal{F}_{\infty}) = Y, \ a.s,$$

then

$$\lim_{t \to \infty} \mathbb{E}(U_t | \mathcal{F}_t) = Y, \ a.s..$$

We will take the quantities in the above proposition from their definition in the context of the physical process of the neutron transport equation. In a similar fashion to the proof of Theorem 3.1, set $U_t = \mathrm{e}^{-\lambda_* t} \langle g, X_t^{\uparrow} \rangle$, for $g \in L_{\infty}^+(D \times V)$, and recall that $(\mathcal{F}_t, t \geq 0)$ is the

filtration generated by the neutron branching process $(X_t, t \geq 0)$. Note that we can easily bound $(U_t, t \geq 0)$ by a multiple of $(W_t^\uparrow, t \geq 0)$ and hence we automatically get that $\sup_{t \geq 0} U_t$ has a second, and hence first, moments thanks to Corollary (3.2). Due to Theorem 3.1 and the fact that $X_t^\uparrow$ is $\mathcal{F}_\infty$-measurable, $U_t = \mathbb{E}(U_t | \mathcal{F}_\infty)$ and hence

$$\lim_{t \to \infty} \mathbb{E}(U_t | \mathcal{F}_\infty) = \langle gp, \tilde{\varphi} \rangle W_\infty^\uparrow \frac{\varphi(r, v)}{p(r, v)}$$

$\mathbb{P}_{\delta_{(r,v)}}$-almost surely, for $r \in D$, $v \in V$.

Using (3.1) (which comes from the skeleton embedding Theorem 2.1, cf. Remark 2.1) as we have in the proof of Theorem 3.1, we get

$$\mathbb{E}(U_t | \mathcal{F}_t) = \mathbb{E}(\mathrm{e}^{-\lambda_* t} \langle g, X_t^\uparrow \rangle | \mathcal{F}_t) = \mathrm{e}^{-\lambda_* t} \langle g, p X_t \rangle.$$

Combining this with Proposition 4.1 yields

$$(4.1) \qquad \lim_{t \to \infty} \mathrm{e}^{-\lambda_* t} \langle g, p X_t \rangle = \langle gp, \tilde{\varphi} \rangle W_\infty^\uparrow \frac{\varphi(r, v)}{p(r, v)},$$

$\mathbb{P}_{\delta_{(r,v)}}$-almost surely. If the support of g is compactly embedded in $D \times V$, then we can replace g by g/p, with the assurance that the latter is uniformly bounded away from 0 and ∞ (cf. Lemma 3.1), and (4.1) gives us

$$(4.2) \qquad \lim_{t \to \infty} \mathrm{e}^{-\lambda_* t} \langle g, X_t \rangle = \langle g, \tilde{\varphi} \rangle W_\infty^\uparrow \frac{\varphi(r, v)}{p(r, v)},$$

$\mathbb{P}_{\delta_{(r,v)}}$-almost surely. We can remove the assumption that the support of g is compactly embedded in $D \times V$ by appealing to similar reasoning as that of the computation in (3.23).

To complete the proof, we need to show that almost surely, $W_\infty^\uparrow / p = W_\infty$. To do so, note that if we take $g = \varphi$ in (4.2), noting that the left-hand side is equal to $\lim_{t \to \infty} W_t \varphi(r, v)$ and $\langle \varphi, \tilde{\varphi} \rangle = 1$, we get the desired result. □

5. Concluding remarks. The proof of Theorem 1.3 above gives a generic approach for branching particle systems which have an identified skeletal decomposition. Indeed, the reasoning is robust and will show in any such situation that the existence of a strong law of large numbers for the skeleton implies almost immediately a strong law of large numbers for the original process into which the skeleton is embedded. As an exercise, the reader is encouraged to consider the setting of a branching Brownian motion in a strip (cf. [15]). Supposing a strong law of large numbers exists on the skeleton there (in that setting it is called the 'blue tree'), then we claim that the the above reasoning applied verbatim will deliver the strong law of large numbers for the branching Brownian motion in a strip.

More generally, we claim that, modulo some minor technical modifications (e.g. taking account of the fact that E may be unbounded), in the general MBP setting of Theorem 2.1, an analogue of Theorem 1.3 may be reconstructed once the following three important components are in hand: (i) An analogue of Theorem 1.1; (ii) A degree of knowledge concerning the

continuity properties of φ and p; (iii) the martingale W has the property $\mathbb{E}_{\delta_x}[\sup_{t\geq 0} W_t^2] < \infty$, for all $x \in E$. Indeed, last of these three may be weakened to γ-integrability of the martingale W, for $\gamma \in (1, 2)$, in which case one may replace many of the estimates in the Borel-Cantelli arguments by γ moment estimates instead of second moment estimates (see e.g. [14] for comparison).

It is also worth pointing out however that the reasoning in the proof of Theorem 1.3 does not so obviously work in the setting of superprocesses with a skeletal decomposition. Indeed a crucial step, which is automatic for branching particle systems, but less obvious for superprocesses, is the point in the argument at which we claim that $U_t = \mathbb{E}(U_t|\mathcal{F}_\infty)$. In the particle system, this statement follows immediately from the fact that $\mathcal{F}_\infty$ carries enough information to construct the marks $\uparrow$ and $\downarrow$ on particles because individual genealogical lines of descent are identifiable. For superprocesses, it is less clear how to choose the filtration $(\mathcal{F}_t, t \geq 0)$ so that the notion of genealogy or otherwise can be used to claim that $X_t^{\uparrow}$, and hence U_t, is $\mathcal{F}_\infty$-measurable.

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Glossary of some commonly used notation
(Th. = Theorem, a. = above, b. = below)

Notation	Description	Introduced
$(\psi_t, t \geq 0)$	Solution to mild NTE/NBP expectation semigroup	(1.10), (1.3)
D and V	Physical and velocity domain	§1
$\sigma_{\mathtt{s}}$, $\sigma_{\mathtt{f}}$ and σ	Scatter, fission and total cross-sections	b. (1.1)
$\pi_{\mathtt{s}}$ and $\pi_{\mathtt{f}}$	Scatter and fission kernels	b. (1.1)
$\mathtt{S}$ and $\mathtt{F}$	Scatter and fission operators	(1.5), (1.6)
$n_{\mathtt{max}}$	Maximum number of neutrons in a fission event	b. (1.9)
λ_*, φ and $\tilde{\varphi}$	Leading eigenvalue, right- and left-eigenfunctions	Th. 1.1
$(W_t, t \geq 0)$	Additive martingale	(1.20)
E and $\dagger$	Domain and cemetery state on which $\mathtt{P}$ and ξ is defined	§2.1
$\mathtt{P}$ and $\hat{\mathtt{P}}$	Particle motion semigroup on E and $E \cup \{\dagger\}$ resp.	§2.1
$\mathtt{L}$	Generator associated to $\mathtt{P}$ in the setting of NBP	(2.39)
$(\xi, \mathbf{P}_x)$	Markov process issued from $x \in E$ whose semigroup is $\mathtt{P}$	(2.1)
$(X, \mathbb{P}_\mu)$	General $(\mathtt{P}, G)$-MBP (and NBP) when issued from μ	§2.1 (and (1.7))
$(u_t, t \geq 0)$	Non-linear semigroup of X (and NBP)	(2.3) (and (1.12))
ζ	Lifetime of X	(2.6)
$\varsigma(x)$	Instantaneous branching rate of X at $x \in E$	§2.1
$\mathcal{P}_x$	Offspring law of X when parent at $x \in E$ (and for NBP)	a. (2.1) (and (1.9))
G	Branching generator (and for NBP)	(2.2) (and (1.11))
$(x_i, i = 1, \cdots, N)$	Position and number of offspring positions of a family in X	§2.1
$w(x)$ (resp. $p(x)$)	Prob. extinction (resp. surivival) when issued from $x \in E$	(2.7) (resp. (2.12))
$(X^\downarrow, \mathbb{P}^\downarrow_\mu)$	MBP conditioned to die out and law when issued from μ	Th. 2.1 (i)
$(u^\downarrow_t, t \geq 0)$	Non-linear semigroup of $X^\downarrow$	(2.27), (2.26)
$\mathtt{P}^\downarrow$ and $\hat{\mathtt{P}}^\downarrow$	Markov semigroup associated to $X^\downarrow$ on E and $E \cup \{\dagger\}$ resp.	Th. 2.1 (i)
$(\xi, \mathbf{P}^\downarrow_x)$	Markov process associated to $\mathtt{P}^\downarrow$ issued from $x \in E$	(2.13)
$\mathtt{L}^\downarrow$	Generator associated to $\mathtt{P}^\downarrow$ in the setting of NBP	(2.40)
$\varsigma^\downarrow(x)$	Instantaneous branching rate of $X^\downarrow$ at $x \in E$	(2.15)
$\mathcal{P}^\downarrow_x$	Offspring law of $X^\downarrow$ when parent at $x \in E$	(2.16)
$G^\downarrow$	Branching generator of $X^\downarrow$	(2.14)
$(x^\downarrow_i, i = 1, \cdots, N^\downarrow)$	Position and number of offspring positions of a family in $X^\downarrow$	Th. 2.1 (ii)
$(X^\uparrow, \mathbb{P}^\uparrow_\mu)$	Skeleton MBP (X conditioned to survive) when issued from μ	Th. 2.1 (ii)
$(X^\updownarrow, \mathbb{P}^\updownarrow_\mu)$	Skeleton $X^\uparrow$ dressed with $X^\downarrow$ trees when issued from μ	Th. 2.1 (ii), (2.24)
$(u^\updownarrow_t, t \geq 0)$	Non-linear semigroup of $X^\updownarrow$	(2.31), (2.35)
$\mathtt{P}^\uparrow$	Markov semigroup associated to $X^\uparrow$	Th. 2.1 (ii)
$(\xi, \mathbf{P}^\uparrow_x)$	Markov process associated to $\mathtt{P}^\uparrow$ issued from $x \in E$	(2.17)
$\mathtt{L}^\uparrow$	Generator associated to $\mathtt{P}^\uparrow$ in the setting of NBP	(2.41)
$\varsigma^\updownarrow(x)$	Instantaneous branching rate of $X^\uparrow$ and $X^\updownarrow$ at $x \in E$	(2.20)
$\mathcal{P}^\updownarrow_x$	Joint $\uparrow$ and $\downarrow$ offspring law of $X^\updownarrow$ when parent at $x \in E$	(2.21)
$G^\uparrow$	Branching generator of $X^\uparrow$	(2.18)
$(x^\uparrow, i = 1, \cdots, N^\uparrow)$	Position and number of offspring positions of a family in $X^\uparrow$	Th. 2.1 (ii)
$G^\updownarrow$	Joint branching generator of $\uparrow$-type and $\downarrow$-type in $X^\updownarrow$	(2.19), (2.22)