

Impartial selection with prior information

IOANNIS CARAGIANNIS, Aarhus University, Denmark

GEORGE CHRISTODOULOU, University of Liverpool, United Kingdom

NICOS PROTOPAPAS, University of Liverpool, United Kingdom

We study the problem of *impartial selection*, a topic that lies at the intersection of computational social choice and mechanism design. The goal is to select the most popular individual among a set of community members. The input can be modeled as a directed graph, where each node represents an individual, and a directed edge indicates nomination or approval of a community member to another. An *impartial mechanism* is robust to potential selfish behavior of the individuals and provides appropriate incentives to voters to report their true preferences by ensuring that the chance of a node to become a winner does not depend on its outgoing edges. The goal is to design impartial mechanisms that select a node with an in-degree that is as close as possible to the highest in-degree. We measure the efficiency of such a mechanism by the difference of these in-degrees, known as its *additive approximation*.

Following the success in the design of auction and posted pricing mechanisms with good approximation guarantees for welfare and profit maximization, we study the extent to which prior information on voters' preferences could be useful in the design of efficient deterministic impartial selection mechanisms with good additive approximation guarantees. We consider three models of prior information, which we call the *opinion poll*, the *a priori popularity*, and the *uniform* model. We analyze the performance of a natural selection mechanism that we call *approval voting with default* (AVD) and show that it achieves a $O(\sqrt{n \ln n})$ additive guarantee for opinion poll and a $O(\ln^2 n)$ for a priori popularity inputs, where n is the number of individuals. We consider this polylogarithmic bound as our main technical contribution. We complement this last result by showing that our analysis is close to tight, showing an $\Omega(\ln n)$ lower bound. This holds in the uniform model, which is the simplest among the three models.

1 INTRODUCTION

We study the problem of *impartial selection*, which has recently attracted a lot of attention from a social choice theory and mechanism design point of view. The goal is to select the most popular individual, among a set of community members. The input can be modeled as a directed graph, where each node represents an individual, and a directed edge indicates nomination or approval of a community member to another. A selection mechanism takes a graph as input and returns a single node as the winner. This would be a trivial task from the algorithmic point of view, but the challenge here is that the true preferences of the individuals are private information known only to them. In settings where each individual is simultaneously a voter and a candidate and therefore has also personal interest in becoming a winner, she may manipulate the mechanism and misreport her true preferences if this could increase her chance to win. An *impartial mechanism* is robust to such behavior, and provides appropriate incentives to voters to report their true preferences by ensuring that the chance of a node to become a winner, does not depend on its outgoing edges.

Unfortunately, it is well-known that the obvious selection mechanism that always returns the highest in-degree node as a winner, suffers from possible manipulation, i.e., it is not impartial. The challenge is to design an impartial selection mechanism that selects a winner with an in-degree that approximates well the highest in-degree.

Impartial selection, was introduced independently by Holzman and Moulin [19] and Alon et al. [1]. The former work considered minimum axiomatic properties that impartial selection rules should satisfy, while the latter quantified the efficiency loss with the notion of the approximation ratio, defined as the worst-case ratio of the maximum in-degree over the in-degree of the node which is selected by the mechanism. This line of research concluded with the work of Fischer and Klimm [15], who proposed randomized impartial mechanisms with optimal approximation ratio.

It is well-known [1, 15] that the most challenging nomination profiles in terms of their approximation ratio, for both deterministic and randomized mechanisms, are those with small in-degrees. In particular, all deterministic impartial mechanisms have unbounded ratio, and this can be demonstrated even on inputs with two nodes and of maximum in-degree of 1 (see [1] for an example). Inspired by this crucial observation, Bousquet et al. [8] designed impartial randomized mechanisms that return a nearly optimal node when the maximum in-degree is high enough. Caragiannis et al. [10] went one step further to quantify this effect, by advocating that *additive* approximation may be a more appropriate measure to evaluate impartial mechanisms, and also by providing mechanisms with sublinear additive approximation guarantees.

The most natural and well-studied selection rule is the *approval voting rule* (AV), which has received much attention in social choice theory [22]. In our context, AV always returns the node with the highest in-degree. Unfortunately, as already mentioned, this mechanism is not impartial. The reason is that in case of a tie at the maximum degree, some of the nodes involved in the tie may have incentive to vote non-truthfully. Fortunately, there is a simple fix of this deficiency, which is inspired by the simpler *plurality with default* mechanism by Holzman and Moulin [19]; *in case of a tie, select as winner a predetermined/default node*. We refer to this modified version of AV as *approval voting with default* (AVD). Although (a careful implementation of) this tweak re-establishes impartiality, this modification comes at a cost, as this preselection should be independent of the input graph. Imagine a scenario, where there is a tie between two nodes with the maximum degree. In the unfortunate situation where the default node receives only a small number of votes, this might lead to a poor additive approximation, linear in the number of nodes.

Most of the previous work consider randomized mechanisms, hence the efficiency is measured in expectation. However, in the design of selection mechanisms, determinism is arguably more desirable. Unfortunately, all the known deterministic mechanisms have very poor, linear additive approximation, and it is wide open whether substantially better mechanisms exist.

In this work, we take a different route: we study the extent to which *prior information* on the preferences of the voters could allow the design of deterministic impartial selection mechanisms with good additive approximation guarantees. Our focus is on the analysis of AVD, for which our design choice boils down to an effective choice of the default node, with the help of the prior information. We assume that the preferences are drawn from a probability distribution that is known to the mechanism. We assume throughout *voter independence*,¹ that is, the random choice of preferences for each voter is independent of those of the others.

We propose different models that capture several aspects of the problem. In the *opinion poll* model, we assume that the prior information concerns information about the preferences of different (types of) voters. The designer has access to the probability $p_i(S)$, with which voter i (or all voters of type i) would approve a subset S of candidates. The *a priori popularity* model assumes that the designer has prior information about the popularity of each candidate j , which is summarized by a scalar p_j . We assume that each candidate j receives independently a vote from each voter with probability p_j . As a special case, we also study the uniform model, in which every candidate j has the same popularity $p_j = p$.

Note that these models capture different information scenarios; the former assumes that the designer has access to opinion poll statistics for each (type of) voter, while the latter assumes that the designer has access only to aggregate information about the popularity of a candidate. This aggregation is over the whole population of voters, as the actual information may be sanitized to preserve anonymity of those (types) participated in the poll. Note that popularity may measure other forms of biases over specific individuals. For example, consider the situation in which a PC

¹We should note that, with correlated distributions there is not much one can achieve (see Example 1 in the appendix).

wants to decide the best paper award; then, the a priori popularity of a paper could be a function of the authors' esteem, affiliation, etc.

1.1 Contribution and techniques

Our main focus is the analysis of the AVD mechanism. We begin with the opinion poll model and, as a warm-up, in Section 3, we present a simple mechanism that ignores the edges of the graph and selects as winner a pre-selected node of *maximum expected in-degree*. We call this mechanism the *constant mechanism*, and show that it is $\Theta(\sqrt{n \ln n})$ -additive (Theorem 5, Theorem 6). The AVD mechanism that selects as default the node of highest expected in-degree, can only perform better than the constant mechanism. Our main and most technically involved result shows that this version of the AVD is $O(\ln^2 n)$ -additive in the a priori popularity model (Theorem 7). We complement this result by showing that our analysis is tight, up to a logarithmic factor: even for uniform inputs where all candidates are a priori equally popular, there is a class of instances for which AVD has additive approximation $\Omega(\ln n)$, for any choice of the default node (Theorem 13).

The analysis of the constant mechanism serves multiple purposes. It illustrates that when prior information is available, a low expected additive approximation is achievable even by simple deterministic mechanisms, and by the simplest statistic of the prior, that is the expected in-degree. This is in sharp contrast to the no-prior case, where deterministic mechanisms have a very poor performance for both additive [10] and multiplicative approximation [1]. Second, the analysis of the constant mechanism is quite simple; e.g., the upper bound follows by a simple application of the Hoeffding bound. However, it introduces some of the techniques (such as tail inequalities and reverse Chernoff bounds) that our strongest results in Sections 4 and 5 use. Finally, the upper bound on the expected additive approximation of the constant mechanism serves as a benchmark of efficiency for all impartial mechanisms with priors.

The analysis of AVD is considerably more involved than the analysis of the constant mechanism. Roughly speaking, the important quantity that affects the additive approximation is the difference between the maximum in-degree and the in-degree of the default node when two or more nodes are tied with the highest in-degree, times the probability of this tie. In the a priori popularity model, the in-degree d of a node is a random variable following the binomial probability distribution with parameters n (the number of trials) and p (the probability that a trial is successful). Furthermore, the in-degrees of different nodes are independent. Hence, bounding the probability of a tie at the maximum in-degree is related to (but more demanding than) bounding the probability that two out of many independent binomial random variables take the same maximum value.

Unfortunately, even though problems of this kind have been studied in the literature of applied probability and statistics (e.g., see [9, 12, 13]), the existing results have not been proved useful for our purposes. When the difference between the maximum in-degree and the in-degree of the default node is large, Chernoff bounds can unsurprisingly be used to show that the probability of a tie at maximum is negligible and, hence, the contribution to the expectation of the quantity of interest is negligible as well. The real challenge is when the difference of the two in-degrees is small. In this regime, it turns out that we need sharp bounds on the ratio $\Pr[d = x]/\Pr[d \geq x]$ (also called the *hazard function*) for a binomial random variable d and value x that is close to the expectation $\mu = pn$ of d (i.e., so that $\Pr[d \geq x]$ is only polynomially small in terms of n). As we show in Lemma 12, en route to proving Theorem 7, the ratio $\Pr[d = x]/\Pr[d \geq x]$ is at most $O\left(\sqrt{\frac{\ln n}{\min\{\mu, n-\mu\}}}\right)$ in this case.

We believe that this technical tool can be of independent interest and could find applications elsewhere. The bound is asymptotically tight; its tightness for $p = 1/2$ is exploited in the proof of our logarithmic lower bound (Theorem 13). The exact dependence on the quantity $\sqrt{\min\{\mu, n - \mu\}}$

is very important to achieve a polylogarithmic upper bound on the additive approximation of AVD for every a priori popularity input. In addition to the above crucial idea and Chernoff bounds, our proof in Theorem 13 involves an inverse Chernoff bound to bound *from below* the probability that a binomial random variable is far from its expectation. These statements are less popular than Chernoff bounds but rather standard.

1.2 Further related work

Impartial selection was introduced independently by Alon et al. [1] and Holzman and Moulin [19]. Alon et al. [1] proposed the approximation ratio as the fraction between the highest in-degree and the (expected) in-degree of the winner. They provided a simple, 4-approximate randomized mechanism and noted that no randomized impartial mechanism can achieve an approximation ratio less than 2, even with randomization. If randomization is not allowed, however, the approximation ratio can be arbitrarily large. Later on, Fischer and Klimm [15] introduced a 2-approximation randomized mechanism, closing that gap. Bousquet et al. [8] proposed a randomized mechanism with an arbitrarily close to optimal approximation ratio, provided that the maximum in-degree of the graph is large enough.

Holzman and Moulin [19] considered various mechanisms under the more restricted family of graphs where each node has an out-degree equal to 1. Among others, they proposed the plurality with default mechanism, which can be seen as a version of AVD mechanism, tailored to that family of inputs. They also came up with an important impossibility result regarding the quality of impartial mechanisms: Any deterministic impartial mechanism can guarantee, either to never select 0 in-degree nodes or to always select a unanimously nominated node, but never both. Variations of the problem are studied in [6, 11, 23, 28, 29].

Additive approximation was first studied by Caragiannis et al. [10]. Therein, they propose simple randomized mechanisms with sub-linear additive approximation guarantees. They also show that a specific class of deterministic mechanisms cannot achieve additive approximation less than $n - 1$ (i.e. the worst possible additive approximation), while an equivalent class, in the randomized setting, cannot achieve additive approximation better than $\Omega(\sqrt{n})$. For general deterministic mechanisms however, they only show a lower bound of 2, while no deterministic mechanism is known with additive approximation smaller than $n - 1$. Closing this gap remains a tantalizing open question.

Impartiality is encountered in various domains. In the AI literature, a related application is peer-reviewing [3, 20, 21, 25]. In another direction, Babichenko et al. [4, 5] present impartial mechanisms for the selection of the most influential node in a network. The main difference with our setting is that the influence of a specific node does not depend merely on its in-degree, but also on all the paths leading to that node. Mackenzie [24] analyses the papal conclave through the lens of impartiality.

Our motivation for considering prior information comes from its successful application to auction and posted pricing mechanisms. An excellent survey of related work can be found in [17]. We should also note that there is an interesting connection of the techniques needed for the analysis in the a priori popularity model with the literature on random graphs [7, 16] and, in particular, results regarding the multiplicity of the highest in-degree in $G_{n,p}$ graphs. Unfortunately, such results have a focus on asymptotics: for example, en route to proving bounds on the chromatic number, Erdős and Wilson [14] showed that the maximum degree is unique with probability $1 - o(1)$ in $G_{n,1/2}$ graphs. Instead, for proving our approximation guarantees, we need sharp estimations of the hidden $o(1)$ term. So, such results are not directly applicable to our analysis.

1.3 Roadmap

The rest of the paper is structured as follows. We begin with preliminary definitions and tail inequality statements in Section 2. Section 3 is devoted to the analysis of the constant mechanism in the opinion poll model. Our polylogarithmic additive approximation for AVD is presented in Section 4 and the logarithmic lower bound in Section 5. We conclude with open problems in Section 6. Two additional observations are given in the appendix.

2 PRELIMINARIES

We denote by N the set of individuals (or *agents*). For a set $S \subset N$, we use N_S as an abbreviation of $N \setminus S$ and write $N_{i,\dots,j}$ instead of $N_{\{i,\dots,j\}}$ for simplicity. A *nomination profile* $G = (N, E)$ is a directed graph without self-loops that has the agents of N as nodes. Each directed edge $(i, j) \in E$ represents a nomination from agent i to agent j . Occasionally, we refer to the outgoing edges as *votes*. We define as $x_i = \{(i, j) \in E\}$ the set of outgoing edges from node $i \in N$ and use the tuple $\mathbf{x} = (x_1, \dots, x_n)$ as an alternative representation for G . We use \mathbf{x}_{-i} to denote the graph $(N, E \setminus (\{i\} \times N))$.

Denoting by \mathcal{G} the set of all nomination profiles over the agents of N , a (deterministic) *selection mechanism* is simply a function $f : \mathcal{G} \rightarrow N$ which maps each nomination profile to a single node (the *winner*). A deterministic selection mechanism f is *impartial* when for any agent $i \in N$, any graph $\mathbf{x} \in \mathcal{G}$ and any set x'_i of outgoing edges from node i , it is $f(\mathbf{x}) = i$ if and only if $f(x'_i, \mathbf{x}_{-i}) = i$. In other words, no agent (node) has any incentive to misreport her preferences (its outgoing edges).

We use $d_j(S, \mathbf{x})$ to denote the in-degree of node $j \in N$, taking into account only the incoming edges from nodes of set S , given the profile \mathbf{x} , i.e.,

$$d_j(S, \mathbf{x}) = |\{i \in S : (i, j) \in E\}|.$$

We use the simplified notations $d_j(\mathbf{x})$ when $S = N_j$. $\Delta(\mathbf{x})$ denotes the maximum in-degree of the profile \mathbf{x} , i.e., $\Delta(\mathbf{x}) = \max_{j \in N} d_j(\mathbf{x})$. Following the work of Caragiannis et al. [10], we evaluate the performance of a mechanism f on a nomination profile \mathbf{x} using the additive approximation $\Delta(\mathbf{x}) - d_{f(\mathbf{x})}(\mathbf{x})$, i.e., the difference between the maximum in-degree over all nodes and the in-degree of the winner returned by mechanism f .

We assume that the input is a random nomination profile (among the agents of N), selected according to a probability distribution \mathbf{P} over all such profiles. We assume *voter independence*, which means that the distribution \mathbf{P} is a product $\prod_{i \in N} \mathbf{P}_i$ of independent distributions, where \mathbf{P}_i denotes the distribution according to which node i selects its set of outgoing edges.

We examine a hierarchy of three families of distributions, giving raise to *opinion poll*, *a priori popularity*, and *uniform* instances (or models), respectively:

- In the opinion poll model, each node $i \in N$ selects its set of outgoing edges among all possible edges to nodes of N_i , according to the probability distribution \mathbf{P}_i . Due to voter independence, the in-degree $d_j(\mathbf{x})$ of each node j is equal to the sum $\sum_{i \in N_j} x_{ij}$ of independent Bernoulli random variables, each denoting whether the directed edge from node i to node j exists in the nomination profile ($x_{ij} = 1$) or not ($x_{ij} = 0$). For simplicity of exposition, in our proofs, we consider N to have $n + 1$ agents; then, the in-degree of each node is the sum of n independent random variables.
- The a priori popularity model is the special case of opinion poll where each node j has a popularity $p_j \in [0, 1]$ and the directed edge (i, j) exists in the nomination profile with probability p_j , independently on all other edges. In this case, the in-degree of node j follows the binomial distribution $\mathcal{B}(n, p_j)$, where n denotes the number of trials and p_j is the success probability for each trial.

- We call uniform the special case of the a priori popularity model with $p_j = p$ for every agent j .

We assume that *prior information* about the underlying probability distributions is known in advance. Hence, we examine selection mechanisms that are defined using this information and evaluate them in terms of their expected additive approximation

$$\mathbb{E}_{\mathbf{x} \sim \mathcal{P}}[\Delta(\mathbf{x}) - d_{f(\mathbf{x})}(\mathbf{x})].$$

We use the term α -additive to refer to a selection mechanism with expected additive approximation at most α . Our aim is to design deterministic impartial selection mechanisms that have as low as possible expected additive approximation in any distribution from the above classes. Our positive results apply to opinion poll or to a priori popularity distributions; our proofs of negative results use the simplest uniform ones.

2.1 Tail inequalities

We include some tail bounds here that will be very useful later in our analysis.

LEMMA 1 (Hoeffding [18]). *Let X_1, X_2, \dots, X_n be independent random variables so that $\Pr[a_j \leq X_j \leq b_j] = 1$. Then, the expectation of the random variable $X = \sum_{j=1}^n X_j$ is $\mathbb{E}[X] = \sum_{j=1}^n \mathbb{E}[X_j]$ and, furthermore, for every $v \geq 0$,*

$$\Pr[|X - \mathbb{E}[X]| \geq v] \leq 2 \exp\left(-\frac{2v^2}{\sum_{j=1}^n (b_j - a_j)^2}\right).$$

LEMMA 2 (Chernoff Bounds). *Let $B \sim \mathcal{B}(n, p)$ and $\mu = np$. Then, the following inequalities hold*

- *Let $x \geq \mu$. Then*

$$\Pr[B \geq x] \leq \exp\left(-\frac{(x - \mu)^2 n}{2\mu(n - \mu)}\right) \quad (1)$$

if $\mu \geq n/2$, and

$$\Pr[B \geq x] \leq \exp\left(-\frac{(x - \mu)^2}{3\mu}\right) \quad (2)$$

if $\mu < n/2$ and, furthermore, $x \leq 2\mu$.

- *Let $x \leq \mu$. Then,*

$$\Pr[B \leq x] \leq \exp\left(-\frac{(\mu - x)^2 n}{2\mu(n - \mu)}\right) \quad (3)$$

if $\mu \leq n/2$, and

$$\Pr[B \leq x] \leq \exp\left(-\frac{(\mu - x)^2}{3(n - \mu)}\right) \quad (4)$$

if $\mu > n/2$ and, furthermore, $x \geq 2\mu - n$.

Inequalities (2) and (4) are the standard Chernoff bounds; e.g., see [26]. Inequalities (1) and (3) are due to Okamoto [27]. The following lemma (see [2], Lemma 4.7.2, page 116) indicates that Chernoff bounds are asymptotically tight.

LEMMA 3. *Let $B \sim \mathcal{B}(n, p)$ and $\delta \in [0, 1 - p)$. Then,*

$$\Pr[B \geq n(p + \delta)] \geq \frac{1}{\sqrt{8n(p + \delta)(1 - p - \delta)}} \cdot \left(\left(\frac{p}{p + \delta}\right)^{p + \delta} \left(\frac{1 - p}{1 - p - \delta}\right)^{1 - p - \delta}\right)^n.$$

In particular, we will utilize the following cleaner statement, which follows easily by Lemma 3.

COROLLARY 4 (INVERSE CHERNOFF BOUND). *Let $B \sim \mathcal{B}(n, 1/2)$ and $\delta \in [0, 1/10]$. Then,*

$$\Pr \left[B \geq n \left(\frac{1}{2} + \delta \right) \right] \geq \frac{1}{\sqrt{2n}} \exp(-3\delta^2 n)$$

PROOF. By applying Lemma 3 to the random variable B , we have

$$\begin{aligned} \Pr \left[B \geq n \left(\frac{1}{2} + \delta \right) \right] &\geq \frac{1}{\sqrt{2n}} \left(\left(\frac{1/2}{1/2 + \delta} \right)^{1/2 + \delta} \left(\frac{1/2}{1/2 - \delta} \right)^{1/2 - \delta} \right)^n \\ &= \frac{1}{\sqrt{2n}} \left(\frac{1}{\sqrt{1 - 4\delta^2}} \left(\frac{1/2 - \delta}{1/2 + \delta} \right)^\delta \right)^n \\ &\geq \frac{1}{\sqrt{2n}} \exp \left(\frac{-2\delta^2 - 4\delta^3}{1 - 2\delta} n \right) \geq \frac{1}{\sqrt{2n}} \exp(-3\delta^2 n). \end{aligned}$$

The first inequality follows by Lemma 3 and since $(p + \delta)(1 - p - \delta)$ is at most $1/4$. The second inequality follows by the inequality $e^z \geq 1 + z$ for $z \in \mathbb{R}$ which implies that $\sqrt{1 - 4\delta^2} \leq \exp(-2\delta^2)$ and $\frac{1/2 + \delta}{1/2 - \delta} \leq \exp\left(\frac{4\delta}{1 - 2\delta}\right)$. The third inequality follows since $\delta \leq 1/10$. \square

3 WARMING UP: THE CONSTANT MECHANISM

We first consider a simple mechanism, which we call the *constant* mechanism. This mechanism ignores all edges and awards a particular preselected node, which we call the *default winner* (or default node). The selection of the default winner depends only on the prior. For example, the criterion that we consider here is to select as default winner a node of maximum expected in-degree, i.e.,

$$f_{\text{co}} \in \operatorname{argmax}_{v \in N} \mathbb{E}[d_v(\mathbf{x})].$$

Our first statement is an upper bound on the additive approximation of the constant mechanism; its proof follows by a simple application of the Hoeffding bound (Lemma 1).

THEOREM 5. *For opinion poll inputs, the constant mechanism that uses the maximum expected in-degree node as the default winner has expected additive approximation $O(\sqrt{n \ln n})$.*

PROOF. Recall that, in the opinion poll model, the in-degree of node v is the sum of n independent Bernoulli random variables, i.e., $d_v(\mathbf{x}) = \sum_{u \in N_v} x_{uv}$. Then, a simple application of the Hoeffding bound (Lemma 1) yields

$$\Pr \left[d_v(\mathbf{x}) \geq \mathbb{E}[d_v(\mathbf{x})] + \sqrt{n \ln n} \right] \leq \Pr \left[|d_v(\mathbf{x}) - \mathbb{E}[d_v(\mathbf{x})]| \geq \sqrt{n \ln n} \right] \leq \frac{2}{n^2}.$$

Hence, the probability that some node has in-degree at least $\mathbb{E}[d_{f_{\text{co}}}(\mathbf{x})] + \sqrt{n \ln n}$ is at most the probability that some node v has in-degree at least $\mathbb{E}[d_v(\mathbf{x})] + \sqrt{n \ln n}$. By the inequality above and the union bound, this probability is at most $\frac{2}{n^2} \cdot (n + 1) \leq \frac{3}{n}$. Thus, the expected maximum in-degree is

$$\mathbb{E}[\Delta(\mathbf{x})] \leq \mathbb{E}[d_{f_{\text{co}}}(\mathbf{x})] + \sqrt{n \ln n} + n \cdot \frac{3}{n} \leq \mathbb{E}[d_{f_{\text{co}}}(\mathbf{x})] + 3 + \sqrt{n \ln n},$$

and the expected additive approximation $\mathbb{E}[\Delta(\mathbf{x}) - d_{f_{\text{co}}}(\mathbf{x})]$ is no more than $3 + \sqrt{n \ln n}$. \square

The bound in Theorem 5 is asymptotically tight. The lower bound instances that we use in the proof of the next statement are the simplest ones: uniform instances with $p = 1/2$. Consequently, it holds for any selection of the default winner. The proof exploits the reverse Chernoff bound (Corollary 4).

THEOREM 6. *The constant mechanism has expected additive approximation $\Omega(\sqrt{n \ln n})$, even when applied to uniform inputs.*

PROOF. Consider a uniform prior with $p = 1/2$ over $n + 1$ nodes, where n is large, e.g., $n \geq 80$. Then, the in-degree of any node u is a random variable following the binomial probability distribution $\mathcal{B}(n, 1/2)$. Let $u^* = f_{\text{co}}$ be the node returned by the constant mechanism; clearly, $\mathbb{E}[d_{u^*}(\mathbf{x})] = n/2$. Denote by \mathcal{E} the event that some node different than u^* has in-degree at least $\frac{n}{2} + \sqrt{\frac{n \ln n}{6}}$. By applying Corollary 4 with $\delta = \sqrt{\frac{n \ln n}{6n}}$ (the fact that n is large guarantees that $\delta \leq 1/10$) to the random variable $d_u(\mathbf{x})$, we have

$$\Pr \left[d_u(\mathbf{x}) \geq \frac{n}{2} + \sqrt{\frac{n \ln n}{6}} \right] \geq \frac{1}{n\sqrt{2}},$$

for every node $u \neq u^*$ and, hence,

$$\Pr[\mathcal{E}] \geq 1 - \left(1 - \frac{1}{n\sqrt{2}}\right)^n \geq 1 - e^{-1/\sqrt{2}} \geq \sqrt{2} - 1,$$

where the second inequality follows by the inequality $(1 - r/n)^n \leq e^{-r}$ and the third one by the inequality $e^z \geq 1 + z$ (and, thus, $e^{1/\sqrt{2}} \geq 1 + 1/\sqrt{2}$). We now have

$$\begin{aligned} \mathbb{E}[\Delta(\mathbf{x})] &\geq \mathbb{E}[\max_{u \neq u^*} d_u(\mathbf{x}) \mathbb{1}\{\mathcal{E}\}] + \mathbb{E}[d_{u^*}(\mathbf{x}) \mathbb{1}\{\bar{\mathcal{E}}\}] \geq \left(\frac{n}{2} + \sqrt{\frac{n \ln n}{6}}\right) \Pr[\mathcal{E}] + \mathbb{E}[d_{u^*}(\mathbf{x})] \Pr[\bar{\mathcal{E}}] \\ &= \mathbb{E}[d_{u^*}(\mathbf{x})] + \sqrt{\frac{n \ln n}{6}} \cdot \Pr[\mathcal{E}] \geq \mathbb{E}[d_{u^*}(\mathbf{x})] + \frac{1}{6} \sqrt{n \ln n}, \end{aligned}$$

and the desired lower bound on the expected additive approximation $\mathbb{E}[\Delta(\mathbf{x}) - d_{u^*}(\mathbf{x})]$ follows. \square

4 A PRIORI POPULARITY AND THE AVD MECHANISM

We devote this section to AVD mechanism and its analysis on a priori popularity instances. AVD uses a preselected node t as the default winner. To give a formal definition of the mechanism, we say that a non-default node k *beats* another non-default node j in the nomination profile \mathbf{x} if $d_k(N_{j,k,t}, \mathbf{x}) > d_j(N_{j,k,t}, \mathbf{x})$, i.e., if node k has higher in-degree than node j when ignoring incoming edges from nodes j, k , and the default node t . Node k beats (respectively, is beaten by) the default node t if $d_k(N_{k,t}, \mathbf{x}) > d_t(N_{k,t}, \mathbf{x})$ (respectively, $d_k(N_{k,t}, \mathbf{x}) < d_t(N_{k,t}, \mathbf{x})$). When applied on the nomination profile \mathbf{x} , AVD returns as the winner w the node that beats every other node, or the default node if no node that beats every other node exists.² We remark that the default node is not prohibited to win by beating every other node.

Notice that, by misreporting its outgoing edges, a node cannot affect the set of other nodes it beats. Hence, AVD is clearly impartial. In addition, the above formal definition allows us to observe that the in-degree of the winner returned by AVD is never lower than the in-degree of the default node t . Indeed, when the default node is not the winner, it is beaten by the winner, who has at least as high in-degree. Hence, the upper bound of $\mathcal{O}(\sqrt{n \ln n})$ on the expected additive approximation of the constant mechanism on opinion poll instances carries over to AVD mechanism when the

²The case in which no node beats every other node refines the notion of a tie that we informally used in Section 1.

default node is selected to be a node of highest expected in-degree. In the following, we present a much stronger result that applies specifically to a priori popularity instances.

THEOREM 7. *The AVD mechanism that uses the node of highest expected in-degree as the default node has an expected additive approximation of $O(\ln^2 n)$ when applied on a priori popularity instances.*

Again, for simplicity of notation, in our analysis of AVD, we consider profiles with $n+1$ nodes. We assume that the number of nodes is large, e.g., $n \geq 10^6$ (otherwise, Theorem 7 holds trivially). Let p_k be the popularity of node k and recall that the in-degree $d_k(\mathbf{x})$ of node k is a random variable taking values between 0 and n following the binomial distribution $\mathcal{B}(n, p_k)$, which is also independent on the in-degree of the other nodes. Also, let $\mu_k = \mathbb{E}[d_k(\mathbf{x})] = p_k n$ and $\xi_k = \min\{\mu_k, n - \mu_k\}$.

We first consider the case of $\xi_t < 8200 \ln n$. This means that the expected in-degree of the default node is either very high, i.e., $\mu_t > n - 8200 \ln n$, or very low, i.e., $\mu_t < 8200 \ln n$. When $\mu_t > n - 8200 \ln n$, the expected degree of the winner (be it the default node or not; recall the argument above that compares AVD with the constant mechanism) is more than $n - 8200 \ln n$. As $d_w \leq n$, the expected additive approximation is less than $8200 \ln n$. In the case $\mu_t < 8200 \ln n$, a simple application of a Chernoff bound (i.e., the tail inequality (2) from Lemma 2) yields that $\Pr[d_k(\mathbf{x}) \geq 9000 \ln n] \leq n^{-26}$ and, hence, the expected additive approximation is at most $n^{-26} \cdot n + (1 - n^{-26}) \cdot 9000 \ln n \leq 1 + 9000 \ln n$.

So, in the following, we analyze the AVD mechanism assuming that $\xi_t \geq 8200 \ln n$. Let h be the highest in-degree among all nodes. Denote by A the event that there is no node that beats every other node and the default node is beaten by a non-default node of degree h . The following lemma addresses the simplest case where the event A is not true.

LEMMA 8. $\mathbb{E}[(h - d_w(\mathbf{x})) \mathbb{1}\{\bar{A}\}] \leq 1$.

PROOF. If the event A does not hold, there must either be a node that beats every other node or the default node is not beaten by any node of degree h .

So, first, assume that there is a node w that beats every other node. The lemma follows if this node has degree h . Otherwise, let i be a node of degree h . Since w beats i , we have $d_w(\mathbf{x}) \geq d_w(N_{i,w,t}, \mathbf{x}) \geq d_i(N_{i,w,t}, \mathbf{x}) + 1 \geq d_i(\mathbf{x}) - 1 = h - 1$ if $w \neq t$, and $d_w(\mathbf{x}) \geq d_t(N_{i,t}, \mathbf{x}) \geq d_i(N_{i,t}, \mathbf{x}) + 1 \geq d_i(\mathbf{x}) = h$ if $w = t$.

Now, assume that the default node is not beaten by any node of in-degree h and there is no node that beats every other node. In this case, the winner will be the default node t . The lemma clearly follows if t has in-degree h . Otherwise, since t is not beaten by some node i of degree h , we have $d_t(\mathbf{x}) \geq d_t(N_{i,t}, \mathbf{x}) \geq d_i(N_{i,t}, \mathbf{x}) \geq d_i(\mathbf{x}) - 1 = h - 1$. \square

We will now bound $\mathbb{E}[(h - d_w(\mathbf{x})) \mathbb{1}\{A\}]$; to do so, we will use a structural lemma.

LEMMA 9. *Assume that A is true and let i be a node of highest in-degree h that beats the default node t . Then, there is a node j , different than i and t , that has degree either h , or $h - 1$, or $h - 2$.*

PROOF. Since node i does not beat every other node, there must be some node j that is not beaten by i (clearly, j is different than t). Then, $d_j(\mathbf{x}) \geq d_j(N_{i,j,t}, \mathbf{x}) \geq d_i(N_{i,j,t}, \mathbf{x}) \geq d_i(\mathbf{x}) - 2 = h - 2$. \square

By Lemma 9, we can bound $\mathbb{E}[(h - d_w(\mathbf{x})) \mathbb{1}\{A\}]$ by the expected value of the difference $h - d_t(\mathbf{x})$ for all possible values of the maximum degree h , all possibilities for an agent $i \neq t$ having degree h and an agent $j \neq i, t$ having degree either h or $h - 1$ or $h - 2$, with the degree of the default node ranging from 0 to h and the degree of all other nodes ranging from 0 to h as well. We have

$$\mathbb{E}[(h - d_w(\mathbf{x})) \mathbb{1}\{A\}] \leq \sum_{h=0}^n \sum_{g=0}^h (h - g) \cdot \Pr[d_t(\mathbf{x}) = g] \sum_{i \in N_t} \Pr[d_i(\mathbf{x}) = h]$$

$$\cdot \sum_{j \in N_{i,t}} \Pr[\max\{0, h-2\} \leq d_j(\mathbf{x}) \leq h] \prod_{k \in N_{i,j,t}} \Pr[d_k(\mathbf{x}) \leq h] \quad (5)$$

For every $k \in \{1, \dots, n+1\}$, define the *comfort zone* Z_k of agent k to be the set of integers $\{L_k, \dots, U_k\}$ with the boundaries satisfying $L_k \leq \mu_k \leq U_k$ and being defined as follows. The lower boundary L_k is equal to the highest integer c such that $\Pr[d_k(\mathbf{x}) < c] \leq n^{-5.33}$ or 0 if no such c exists. The upper boundary U_k is equal to the lowest integer c such that $\Pr[d_k(\mathbf{x}) > c] \leq n^{-5.33}$ or n if no such c exists. We use the terms “above Z_k ” and “below Z_k ” to denote the ranges of integers (if any) $\{0, \dots, L_k - 1\}$ and $\{U_k + 1, \dots, n\}$.

Now, by simple properties of the binomial distribution and the fact that node t has maximum expected in-degree, we observe that if h lies above the comfort zone of agent t , it also lies above the comfort zone of agent i and, hence, $\Pr[d_i(\mathbf{x}) = h] \leq n^{-5.33}$. Also, if g lies below the comfort zone Z_t , it holds $\Pr[d_t(\mathbf{x}) = g] \leq n^{-5.33}$. Furthermore, if $h-2$ lies above the comfort zone Z_j , then $\Pr[\max\{0, h-2\} \leq d_j(\mathbf{x}) \leq h] \leq \Pr[d_j(\mathbf{x}) \geq U_j] < n^{-5.33}$ as well. Since, trivially, $h - d_t(\mathbf{x}) \leq n$, the contribution of the at most n^4 terms of the sum in which either h or g does not belong to the comfort zone Z_t or $h-2$ lies above the comfort zone Z_j is at most $n^4 \cdot n \cdot n^{-5.33} < 1$. Hence, equation (5) becomes

$$\begin{aligned} \mathbb{E}[(h - d_w(\mathbf{x})) \mathbb{1}\{A\}] &\leq 1 + \sum_{h=L_t}^{U_t} \sum_{g=L_t}^h (h-g) \cdot \Pr[d_t(\mathbf{x}) = g] \sum_{i \in N_i: h \in Z_i} \Pr[d_i(\mathbf{x}) = h] \\ &\cdot \sum_{j \in N_{i,t}: h-2 \in Z_j} \Pr[\max\{0, h-2\} \leq d_j(\mathbf{x}) \leq h] \prod_{k \in N_{i,j,t}} \Pr[d_k(\mathbf{x}) \leq h] \quad (6) \end{aligned}$$

Our aim in the following is to evaluate each term in the sum at the RHS of (6). To do so, we will need three auxiliary technical lemmas. The proofs of the first two follow easily by applying Chernoff bounds.

LEMMA 10. *For the boundaries of the comfort zone Z_t we have $U_t \leq \mu_t + 4\sqrt{\xi_t \ln n}$ and $L_t \geq \mu_t - 4\sqrt{\xi_t \ln n}$.*

PROOF. If $\mu_t \geq n/2$, by applying the tail inequality (1) from Lemma 2, we get

$$\Pr[d_t(\mathbf{x}) \geq \mu_t + 4\sqrt{\xi_t \ln n}] \leq \exp\left(-\frac{(4\sqrt{\xi_t \ln n})^2 n}{2\xi_t(n - \xi_t)}\right) \leq n^{-8}.$$

If $\mu_t < n/2$, observe that $4\sqrt{\xi_t \ln n} \leq \mu_t$, by our assumption $\xi_t \geq 8200 \ln n$. Hence, by applying the tail inequality (2) from Lemma 2, we get

$$\Pr[d_t(\mathbf{x}) \geq \mu_t + 4\sqrt{\xi_t \ln n}] \leq \exp\left(-\frac{(4\sqrt{\xi_t \ln n})^2}{3\mu_t}\right) \leq n^{-5.33}.$$

The bounds on U_t follows by its definition.

Similarly, if $\mu_t \leq n/2$, by applying the tail inequality (3) from Lemma 2, we get

$$\Pr[d_t(\mathbf{x}) \leq \mu_t - 4\sqrt{\xi_t \ln n}] \leq \exp\left(-\frac{(4\sqrt{\xi_t \ln n})^2 n}{2\xi_t(n - \xi_t)}\right) \leq n^{-8}.$$

If $\mu_t > n/2$, observe that $\mu_t - 4\sqrt{\xi_t \ln n} \geq 2\mu_t - n$, by our assumption $\xi_t = n - \mu_t \geq 8200 \ln n$. Hence, by applying the tail inequality (4) from Lemma 2, we get

$$\Pr[d_t(\mathbf{x}) \leq \mu_t - 4\sqrt{\xi_t \ln n}] \leq \exp\left(-\frac{(4\sqrt{\xi_t \ln n})^2}{3\mu_t}\right) \leq n^{-5.33}.$$

Again, the bound on L_k follows by its definition. \square

We will say that the comfort zones Z_k and $Z_{k'}$ *almost intersect* if $L_{k'} - U_k \leq 2$ or $L_k - U_{k'} \leq 2$. For example, since $h \in Z_t$ and $h - 2 \in Z_j$, the two comfort zones Z_t and Z_j almost intersect.

LEMMA 11. *If two comfort zones Z_k and $Z_{k'}$ almost intersect, then $\frac{3}{4}\mu_k \leq \mu_{k'} \leq \frac{4}{3}\mu_k$ and $\frac{16}{25}\xi_k \leq \xi_{k'} \leq \frac{25}{16}\xi_k$.*

PROOF. Without loss of generality, assume that $\mu_k \leq \mu_{k'}$; the other case is completely symmetric. Then, $L_{k'} - U_k \leq 2$, which, using the facts $\xi_k, \xi_{k'} \leq \mu_{k'}$ and $\mu_{k'} \geq 8200 \ln n$ as well as Lemma 10, implies that

$$\mu_{k'} \leq 2 + \mu_k + 4\sqrt{\xi_k \ln n} + 4\sqrt{\xi_{k'} \ln n} \leq 2 + \mu_k + 8\sqrt{\mu_{k'} \ln n} \leq \mu_k \left(\frac{1}{4100} + 1 \right) + \frac{8\mu_{k'}}{\sqrt{8200}},$$

which clearly implies that $\mu_{k'} \leq \frac{4}{3}\mu_k$.

Also, observe that

$$\begin{aligned} \max\{\xi_{k'}, \xi_k\} - \min\{\xi_{k'}, \xi_k\} &\leq \mu_{k'} - \mu_k \leq 2 + 4\sqrt{\xi_k \ln n} + 4\sqrt{\xi_{k'} \ln n} \\ &\leq 2 + 8\sqrt{\max\{\xi_k, \xi_{k'}\} \ln n} \\ &\leq \left(\frac{1}{4100} + \frac{8}{\sqrt{8200}} \right) \max\{\xi_k, \xi_{k'}\}, \end{aligned}$$

which implies that $\max\{\xi_{k'}, \xi_k\} \leq \frac{25}{16} \min\{\xi_{k'}, \xi_k\}$ as desired. \square

We now prove the most important technical lemma in our analysis.

LEMMA 12. *Let $\ell \in \{0, 1, 2\}$ and h be such that $h \in Z_t$ and $h - 2 \in Z_k$ for some agent k . Then,*

$$\Pr[d_k(\mathbf{x}) = h - \ell] \leq 264e \sqrt{\frac{\ln n}{\xi_t}} \cdot \Pr[d_k(\mathbf{x}) > h].$$

PROOF. By the definition of the binomial distribution, we have

$$\Pr[d_k(\mathbf{x}) = z] = \binom{n}{z} p_k^z (1 - p_k)^{n-z}$$

for every integer z with $0 \leq z \leq n$. Let x be any positive integer with $x \leq \mu_k + 4\sqrt{\xi_k \ln n}$. For every integer $y > x$, we have

$$\begin{aligned} \frac{\Pr[d_k(\mathbf{x}) = x]}{\Pr[d_k(\mathbf{x}) = y]} &= \frac{\binom{n}{x} p_k^x (1 - p_k)^{n-x}}{\binom{n}{y} p_k^y (1 - p_k)^{n-y}} \\ &= \frac{(x+1) \cdot (x+2) \cdot \dots \cdot y}{(n-y+1) \cdot (n-y+2) \cdot \dots \cdot (n-x)} \cdot \frac{(1-p_k)^{y-x}}{p_k^{y-x}} \\ &= \frac{\left(1 + \frac{x-\mu_k+1}{\mu_k}\right) \cdot \left(1 + \frac{x-\mu_k+2}{\mu_k}\right) \cdot \dots \cdot \left(1 + \frac{y-\mu_k}{\mu_k}\right)}{\left(1 - \frac{y-\mu_k-1}{n-\mu_k}\right) \cdot \left(1 - \frac{y-\mu_k}{n-\mu_k}\right) \cdot \dots \cdot \left(1 - \frac{x-\mu_k}{n-\mu_k}\right)} \\ &\leq \frac{\left(1 + \frac{y-\mu_k}{\mu_k}\right)^{y-x}}{\left(1 - \frac{y-\mu_k-1}{n-\mu_k}\right)^{y-x}} \end{aligned}$$

$$\leq \exp\left(\frac{(y - \mu_k)(y - x)}{\mu_k} + \frac{(y - \mu_k + 1)(y - x)}{n - y + 1}\right) \quad (7)$$

The first inequality follows since $x < y$. In the second inequality, we have used the properties $1 + z \leq e^z$ for $z \in \mathbb{R}$ and, consequently, $\frac{1}{1-z} = 1 + \frac{z}{1-z} \leq \exp(\frac{z}{1-z})$ for $z \neq 1$.

We now use inequality (7) to argue that by selecting y such that $y > h - \ell$ and

$$(y - \mu_k + 1)(y - h + \ell) \leq \frac{3}{11}(\xi_t - y + \mu_k), \quad (8)$$

we get

$$\frac{\Pr[d_k(\mathbf{x}) = h - \ell]}{\Pr[d_k(\mathbf{x}) = y]} \leq e. \quad (9)$$

Recall that Z_k and Z_t almost intersect. Hence, we have $\mu_k \geq 3\mu_t/4$ (by Lemma 11), and (8) yields

$$\frac{(y - \mu_k)(y - h + \ell)}{\mu_k} \leq \frac{3}{11} \cdot \frac{\xi_t - y + \mu_k}{\mu_k} \leq \frac{4}{11} \cdot \frac{2\mu_t - y}{\mu_t} \leq \frac{8}{11}. \quad (10)$$

Furthermore, using again (8), and the inequalities $\mu_k \leq \mu_t$ and $\xi_t \leq n - \mu_t$, we get

$$\frac{(y - \mu_k + 1)(y - d + \ell)}{n - y + 1} \leq \frac{3}{11} \cdot \frac{\xi_t - y + \mu_k}{n - y + 1} \leq \frac{3}{11} \cdot \frac{\xi_t - y + \mu_t}{n - y + 1} \leq \frac{3}{11} \cdot \frac{n - y}{n - y + 1} \leq \frac{3}{11}. \quad (11)$$

Inequality (9) now follows by inequalities (7), (10), and (11).

Solving inequality (8), we get that the range of values for y so that (9) is true satisfies

$$h - \ell < y \leq \frac{h - \ell + \mu_k - \frac{14}{11} + \sqrt{(h - \ell - \mu_k)^2 + \frac{16}{11}(h - \ell - \mu_k) + \frac{12}{11}\xi_t + \frac{196}{121}}}{2}.$$

Hence, the number of integer values for y so that $y > h - \ell$ and (8) holds is at least

$$\begin{aligned} & \frac{h - \ell + \mu_k - \frac{14}{11} + \sqrt{(h - \ell - \mu_k)^2 + (d - \ell - \mu_k) + \frac{12}{11}\xi_t + \frac{196}{121}}}{2} - h + \ell - 2 \\ &= \frac{\sqrt{(h - \ell - \mu_k)^2 + (d - \ell - \mu_k) + \frac{12}{11}\xi_t + \frac{196}{121}} - (h - \ell - \mu_k + 96/11)}{2} + 2. \end{aligned} \quad (12)$$

The derivative of the quantity at the RHS of (12) with respect to h is

$$\frac{2(h - \ell - \mu_k) + 1}{4\sqrt{(h - \ell - \mu_k)^2 + (h - \ell - \mu_k) + \frac{12}{11}\xi_t + \frac{196}{121}}} - \frac{1}{2} < 0,$$

i.e., it is decreasing. Since $h - \ell \in Z_k$ and Z_k and Z_t almost intersect, using Lemmas 10 and 11 we have $h - \ell - \mu_k \leq 4\sqrt{\xi_k \ln n} \leq 5\sqrt{\xi_t \ln n}$. Hence, we can bound the RHS of (12) as follows:

$$\begin{aligned} & \frac{\sqrt{(h - \ell - \mu_k)^2 + (d - \ell - \mu_k) + \frac{12}{11}\xi_t + \frac{196}{121}} - (h - \ell - \mu_k + \frac{96}{11})}{2} + 2 \\ &= \frac{(h - \ell - \mu_k)^2 + (d - \ell - \mu_k) + \frac{12}{11}\xi_t + \frac{196}{121} - (h - \ell - \mu_k + \frac{96}{11})^2}{2(\sqrt{(h - \ell - \mu_k)^2 + (d - \ell - \mu_k) + \frac{12}{11}\xi_t + \frac{196}{121}} + (h - \ell - \mu_k + 96/11))} + 2 \\ &= \frac{12\xi_t - 181(h - \ell - \mu_k) - 820}{22\left(\sqrt{(h - \ell - \mu_k)^2 + (d - \ell - \mu_k) + \frac{12}{11}\xi_t + \frac{196}{121}} + h - \ell - \mu_k + 96/11\right)} + 2 \end{aligned}$$

$$\geq \frac{12\xi_t - 905\sqrt{\xi_t \ln n} - 820}{22 \left(\sqrt{25\xi_t \ln n} + 5\sqrt{\xi_t \ln n} + \frac{12}{11}\xi_t + \frac{196}{121} + 5\sqrt{\xi_t \ln n} + \frac{96}{11} \right)} \geq \frac{1}{264} \sqrt{\frac{\xi_t}{\ln n}} + 2.$$

In the second inequality, we have used $905\sqrt{\xi_t \ln n} \leq 10\xi_t$ and $820 \leq \xi_t$ to bound the numerator by ξ_t (recall that $\xi_t \geq 8200 \ln n$), while the parenthesis in the denominator is clearly at most $12\sqrt{\xi_t \ln n}$.

Now, let $\ell \in \{0, 1, 2\}$ and $r = \left\lceil \frac{1}{264} \sqrt{\frac{\xi_t}{\ln n}} \right\rceil + 2$. By the discussion above, for $x = h - \ell$ we have

$$\Pr[d_k(\mathbf{x}) = h - \ell] \leq e \Pr[d_k(\mathbf{x}) = y]$$

for $y = h - \ell + 1, h - \ell + 2, \dots, h - \ell + r + 2$. By summing these inequalities for $y = h + 1, \dots, h + r$, we get

$$r \Pr[d_k(\mathbf{x}) = h - \ell] \leq e \sum_{y=h+1}^r \Pr[d_k(\mathbf{x}) = y] \leq e \Pr[d_k(\mathbf{x}) > h]$$

and, equivalently,

$$\Pr[d_k(\mathbf{x}) = h - \ell] \leq \frac{e}{r} \Pr[d_k(\mathbf{x}) > h] \leq 264e \sqrt{\frac{\ln n}{\xi_t}} \Pr[d_k(\mathbf{x}) > h].$$

The lemma follows. \square

We are ready to complete the proof of Theorem 7. For $h \in Z_t$, using Lemma 10, we have

$$\sum_{h \in Z_t} \sum_{g \in Z_t} (h - g) \Pr[d_t(\mathbf{x}) = g] \leq \sum_{h \in Z_t} 8\sqrt{\xi_t \ln n} \sum_{g \in Z_t} \Pr[d_t(\mathbf{x}) = g] \leq 64\xi_t \ln n. \quad (13)$$

Furthermore, Lemma 12 yields

$$\begin{aligned} & \sum_{i \in N_t} \sum_{j \in N_{i,t}} \Pr[d_i(\mathbf{x}) = h] \Pr[\max\{0, h - 2\} \leq d_j(\mathbf{x}) \leq h] \prod_{k \in N_{i,j,t}} \Pr[d_k(\mathbf{x}) \leq h] \\ & \leq 3 \left(264 \cdot e \sqrt{\frac{\ln n}{\xi_t}} \right)^2 \sum_{i \in N_t} \sum_{j \in N_{i,t}} \Pr[d_i(\mathbf{x}) > h] \Pr[d_j(\mathbf{x}) > h] \prod_{k \in N_{i,j,t}} \Pr[d_k(\mathbf{x}) \leq h] \\ & \leq 209088 \cdot e^2 \cdot \frac{\ln n}{\xi_t}. \end{aligned} \quad (14)$$

since the last double sum is simply the probability that exactly two agents have degree higher than h (and, hence, has value at most 1). Using (13) and (14), equation (6) yields $\mathbb{E}[(h - d_w(\mathbf{x})) \mathbb{1}\{A\}] \leq 1 + 10^8 \cdot \ln^2 n$ and the proof of Theorem 7 is now complete. \square

5 A LOWER BOUND FOR AVD

In this section, we prove the following lower bound for the uniform domain.

THEOREM 13. *When applied on uniform instances with $p = 1/2$, the AVD mechanism has expected additive approximation $\Omega(\ln n)$.*

With uniform instances, the in-degree of each node follows the binomial distribution. In the proof of Theorem 13, we use the random variables B and B' following the distributions $\mathcal{B}(n, 1/2)$ and $\mathcal{B}(n - 1, 1/2)$, respectively. We also assume that n is sufficiently large.

Let U be the lowest integer c such that $\Pr[B > c] \leq \frac{1}{3e^2 n \sqrt{6}}$. Similarly, let L be the lowest integer c such that $\Pr[B > c] < \frac{1}{n\sqrt{2}}$. Consider the following event D :

- The default node has degree at most $n/2$,
- two non-default nodes (called the *potential winners*) have the same in-degree $d \in [L + 1, U]$, without counting the edges between them,
- the remaining non-default nodes (called the *losers*) have in-degree at most $d - 1$.

Then, AVD returns the default node as a winner and the additive approximation is at least $L - n/2$. We will show that $\mathbb{E}[(L - n/2) \mathbb{1}\{D\}]$ is $\Omega(\ln n)$, proving the lemma. In particular, we will use the inequality

$$\mathbb{E}[(L - n/2) \mathbb{1}\{D\}] \geq \frac{1}{2}(L - n/2) \sum_{d=L+1}^U \binom{n}{2} \Pr[B' = d]^2 \cdot \Pr[B \leq d - 1]^{n-2}. \quad (15)$$

The RHS of equation (15) is the product of the lower bound of the additive approximation $L - n/2$ when event D happens, with $1/2$ which is (a lower bound on) the probability that the default node has degree at most $n/2$, and with the probability $\Pr[B' = d]^2$ that the two potential winners have degree d (ignoring the edges between them) and the probability $\Pr[B \leq d - 1]^{n-2}$ that the losers have degree at most $d - 1$, for all the $\binom{n}{2}$ selections of the two potential winners.

We will make use of a series of lemmas to bound the several quantities that appear in the RHS of equation (15).

LEMMA 14. $L \geq \frac{n}{2} + \sqrt{\frac{n \ln n}{6}}$.

PROOF. By applying Corollary 4 to the random variable $B \sim \mathcal{B}(n, 1/2)$ with $\delta = \sqrt{\frac{\ln n}{6n}}$ (observe that $\delta \leq 1/10$ since n is large), we have $\Pr\left[B \geq \frac{n}{2} + \sqrt{\frac{n \ln n}{6}}\right] \geq \frac{1}{n\sqrt{2}}$ for the random variable $B \sim \mathcal{B}(n, 1/2)$. The lemma follows by the definition of L . \square

LEMMA 15. $U \leq \frac{n}{2} + \sqrt{n \ln n}$.

PROOF. A simple application of the Chernoff bound (inequality (1) from Lemma 2) to the binomial random variable $B \sim \mathcal{B}(n, 1/2)$ yields $\Pr\left[B \geq \frac{n}{2} + \sqrt{n \ln n}\right] \leq \frac{1}{n^2} \leq \frac{1}{3e^2\sqrt{6}}$. The lemma then follows by the definition of U . \square

LEMMA 16. *For the random variable $B \sim \mathcal{B}(n, 1/2)$, it holds that*

$$\Pr[B = x] \geq \frac{2L - n}{n} \cdot \Pr[B \geq x],$$

for every integer $x \geq L$.

PROOF. Consider integers x, y with $L \leq x \leq y$. By the definition of the binomial distribution $\mathcal{B}(n, 1/2)$, we have

$$\frac{\Pr[B = y]}{\Pr[B = x]} = \frac{\binom{n}{y}}{\binom{n}{x}} = \frac{x!(n-x)!}{y!(n-y)!} \leq \left(\frac{n-x}{x}\right)^{y-x} \leq \left(\frac{n-L}{L}\right)^{y-x}.$$

Hence,

$$\begin{aligned} \Pr[B \geq x] &= \sum_{y=x}^n \Pr[B = y] \leq \Pr[B = x] \cdot \sum_{y=x}^n \left(\frac{n-L}{L}\right)^{y-x} \\ &\leq \frac{L}{2L-n} \cdot \Pr[B = x] \leq \frac{n}{2L-n} \cdot \Pr[B = x], \end{aligned}$$

and the lemma follows by rearranging. \square

The proof of the next lemma follows a similar roadmap with the proof of Lemma 12 in Section 4 but is considerably simpler. In the proof, we will use the following claim, which will also be useful later. The proof follows easily by the definition of the binomial distribution.

CLAIM 17. *For the random variable $B \sim \mathcal{B}(n, 1/2)$ and integers x and y with $x \leq y$, it holds that*

$$\Pr[B = x] \leq \left(\frac{y}{n-y}\right)^{y-x} \cdot \Pr[B = y].$$

PROOF. By the definition of the binomial distribution $\mathcal{B}(n, 1/2)$, we have

$$\frac{\Pr[B = x]}{\Pr[B = y]} = \frac{\binom{n}{x}}{\binom{n}{y}} = \frac{y!(n-x)!}{x!(n-y)!} \leq \left(\frac{y}{n-y}\right)^{y-x}. \quad \square$$

LEMMA 18. *For the random variable $B \sim \mathcal{B}(n, 1/2)$, it holds that*

$$\Pr[B = U] \leq \frac{8}{3e\sqrt{6}} \frac{\sqrt{\ln n}}{n^{3/2}}.$$

PROOF. By Claim 17, we have

$$\frac{\Pr[B = U]}{\Pr[B = y]} \leq \left(\frac{y}{n-y}\right)^{y-U} \leq \exp\left(\frac{(y-U)(2y-n)}{n-y}\right) \quad (16)$$

for every integer $y > U$. The second inequality follows since $e^z \geq 1 + z$ for $z \in \mathbb{R}$. By selecting y such that

$$\frac{(y-U)(2y-n)}{n-y} \leq 1, \quad (17)$$

we get

$$\frac{\Pr[B = U]}{\Pr[B = y]} \leq e. \quad (18)$$

Solving inequality (17), we get that the range of values for y so that (18) is true satisfies

$$U < y \leq \frac{2U + n - 1 + \sqrt{(2U - n)^2 + 6n - 4U + 1}}{4}.$$

Hence, the number of integer values for y so that $y > U$ and (17) is satisfied is at least

$$\begin{aligned} & \frac{2U + n - 1 + \sqrt{(2U - n)^2 + 6n - 4U + 1}}{4} - U - 1 \\ &= \frac{\sqrt{(2U - n)^2 + 6n - 4U + 1} - 2U + n - 5}{4}. \end{aligned} \quad (19)$$

Now observe that the quantity at the RHS of (19) is non-increasing with respect to U since its derivative

$$\frac{2U - n - 1}{2\sqrt{(2U - n)^2 + 6n - 4U + 1}} - \frac{1}{2}$$

is non-positive. So, we can bound the RHS of (19) from below using the upper bound on U from Lemma 15. We get

$$\frac{\sqrt{(2U - n)^2 + 6n - 4U + 1} - 2U + n - 5}{4}$$

$$\begin{aligned}
&= \frac{16n - 24U - 25}{4 \left(\sqrt{(2U - n)^2 + 6n - 4U + 1} + 2U - n + 5 \right)} \\
&\geq \frac{4n - 24\sqrt{n \ln n} - 25}{4 \left(\sqrt{4n \ln n + 4n - 4\sqrt{n \ln n} + 1} + 2\sqrt{n \ln n} - n + 5 \right)} \\
&\geq \frac{1}{8} \sqrt{\frac{n}{\ln n}}.
\end{aligned}$$

In the last inequality, we have used $24\sqrt{n \ln n} + 25 \leq n$ to lower-bound the numerator by $3n$ and $5 \leq \sqrt{n \ln n}$ to upper-bound the parenthesis in the denominator by $6\sqrt{n \ln n}$.

Now, let $r = \lceil \frac{1}{8} \sqrt{\frac{n}{\ln n}} \rceil$. Multiplying inequality (18) by $1/r$ and summing these inequalities for $y = U + 1, \dots, U + r$, we have

$$\Pr[B = U] \leq \frac{e}{r} \sum_{y=U+1}^{U+r} \Pr[B = y] \leq \frac{e}{r} \cdot \Pr[B > U] \leq \frac{8}{3e\sqrt{6}} \frac{\sqrt{\ln n}}{n^{3/2}},$$

as desired. The last inequality follows by the definition of U . \square

LEMMA 19. $U - L \geq \frac{1}{6} \sqrt{\frac{n}{\ln n}}$.

PROOF. Let $B \sim \mathcal{B}(n, 1/2)$. Using the definition of U and L , we have

$$\begin{aligned}
\frac{2}{3n} &\leq \frac{1}{n\sqrt{2}} - \frac{1}{3e^2 n \sqrt{6}} \leq \Pr[L \leq B \leq U] \\
&\leq \sum_{x=L}^U \left(\frac{U}{n-U} \right)^{U-x} \cdot \Pr[B = U] \\
&\leq \left(\frac{U}{n-U} \right)^{U-L+1} \cdot \frac{n-U}{2U-n} \cdot \frac{8}{3e\sqrt{6}} \frac{\sqrt{\ln n}}{n^{3/2}} \\
&\leq \exp\left(\frac{2U-n}{n-U} (U-L+1) \right) \cdot \frac{n-L}{2L-n} \cdot \frac{8}{3e\sqrt{6}} \frac{\sqrt{\ln n}}{n^{3/2}} \\
&\leq \exp\left(\frac{2U-n}{n-U} (U-L+1) \right) \cdot \frac{2}{3en}. \tag{20}
\end{aligned}$$

The first inequality is obvious, while the second one uses the definition of U and L (recall that $\Pr[B \geq L] \geq \frac{1}{n\sqrt{2}}$ and $\Pr[B > U] \leq \frac{1}{4e^2 n \sqrt{3}}$). The third inequality follows by Claim 17. The fourth inequality follows by Lemma 18, the fifth one follows since $U \geq L$ and by the definition of U , and the sixth one follows by Lemma 14 and the fact $L \geq n/2$.

By Lemma 15 and due to the high value of n , $2n - 3U \geq n/3$. Hence, inequality (20) implies that

$$U - L \geq \frac{2n - 3U}{2U - n} \geq \frac{1}{6} \sqrt{\frac{n}{\ln n}},$$

as desired. \square

LEMMA 20. Let $B \sim \mathcal{B}(n, 1/2)$ and $B' \sim \mathcal{B}(n-1, 1/2)$. For every integer $x \in [L+1, U]$, $\Pr[B' = x] \geq \frac{2}{3} \Pr[B = x]$.

PROOF. By the definition of the binomial distribution, Lemma 15, and the facts that $x \leq U$ and that n is large (the last two imply that $x \leq 2n/3$), we have

$$\Pr[B' = x] = \binom{n-1}{x} 2^{-n+1} = 2 \frac{n-x}{n} \binom{n}{x} 2^{-n} \geq \frac{2}{3} \Pr[B = x]. \quad \square$$

We are now ready to bound $\mathbb{E}[(L - n/2) \mathbb{1}\{D\}]$ from below. Using equation (15), and the lemmas above, we have

$$\begin{aligned} \mathbb{E}(L - n/2) \mathbb{1}\{D\} &\geq \frac{1}{2} \left(L - \frac{n}{2}\right) \cdot \sum_{d=L+1}^U \binom{n}{2} \Pr[B' = d]^2 \Pr[B \leq d-1]^{n-2} \\ &\geq \frac{2}{9} \left(L - \frac{n}{2}\right) \cdot \sum_{d=L+1}^U \binom{n}{2} \Pr[B = d]^2 \Pr[B \leq d-1]^{n-2} \\ &\geq \frac{8}{9n^2} \left(L - \frac{n}{2}\right)^3 \sum_{d=L+1}^U \binom{n}{2} \Pr[B \geq d]^2 \Pr[B \leq d-1]^{n-2} \\ &\geq \frac{8}{9n^2} \left(L - \frac{n}{2}\right)^3 (U-L) \binom{n}{2} \left(\frac{1}{3e^2 n \sqrt{6}}\right)^2 \left(1 - \frac{1}{n\sqrt{2}}\right)^{n-2} \\ &\geq \frac{1}{6561 e^5 \sqrt{6}} \cdot \ln n. \end{aligned}$$

The second inequality follows by Lemma 20. The third inequality follows by Lemma 16. The fourth inequality follows by the definition of L and U . Finally, the fifth inequality follows by Lemma 14, Lemma 19, and the fact $\left(1 - \frac{1}{n\sqrt{2}}\right)^{n-2} \geq 1/e$. Theorem 13 follows. \square

6 OPEN PROBLEMS

Our polylogarithmic upper bound in Section 4 shows that prior information can yield dramatic improvements on the performance of simple impartial selection mechanisms. It also gives hope that AVD could be similarly efficient for the more general opinion poll instances. Unfortunately, this is not true as the following counter-example indicates.

Indeed, starting from a uniform instance with $n+1$ nodes of popularity $p = 1/2$, we add a new copy j' for each node j . Also, for every edge (i, j) realized in the original instance, we add the edge (i, j') . In this way, we construct opinion poll instances with $2(n+1)$ nodes, where no node can ever beat all the other nodes. Hence, in such instances, AVD will behave as the constant mechanism in the original instance and will always return the default node as winner. By applying Theorem 6 we obtain the following negative result for AVD.

THEOREM 21. *When applied on opinion poll instances, the AVD mechanism has expected additive approximation $\Omega(\sqrt{n \ln n})$.*

We should note that the above construction is fragile, in the sense that it exploits a very specific aspect of the mechanism. So, still, the quest of designing deterministic mechanisms that achieve polylogarithmic additive approximation in the opinion poll model is very important and challenging. A starting step could be to restrict our attention to the instances considered in [19], in which every voter approves exactly one other candidate.

Finally, throughout the paper, we have assumed that the prior information is reliable. This should not be expected to be the case in practice. We expect that our results on the constant mechanism still hold if we have a rough estimate of the highest in-degree. Highest accuracy seems to

be necessary to recover our polylogarithmic upper bound though. This issue is also related to the strengths of prior-independent mechanisms (e.g., see Section 4.3 of [17]) and needs to be investigated further.

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A APPENDIX: MULTIPLICATIVE APPROXIMATION AND VOTER CORRELATION

In the following, we briefly justify two main decisions that we have taken. First, we show that knowing the prior cannot help us improve the approximation ratio of 2 that is best possible for worst-case inputs. This explains why we have completely ignored the study of multiplicative approximations when prior information is available.

We extend the approximation ratio ρ of a mechanism f against a prior \mathbf{P} as follows:

$$\rho = \frac{\mathbb{E}_{\mathbf{x} \sim \mathbf{P}}[\Delta(\mathbf{x})]}{\mathbb{E}_{\mathbf{x} \sim \mathbf{P}}[d_{f(\mathbf{x})}(\mathbf{x})]}$$

THEOREM 22. *For every $\epsilon > 0$, no impartial selection mechanism has approximation ratio better than $2 - \epsilon$ against all uniform priors.*

PROOF. Consider uniform instances with two nodes u and v of popularity p . Clearly, $\mathbb{E}[\Delta(\mathbf{x})] = 1 - (1 - p)^2 = 2p - p^2$. We show that for every impartial mechanism f , it holds $\mathbb{E}[d_{f(\mathbf{x})}(\mathbf{x})] \leq p$. The theorem then follows by taking p to be sufficiently small.

Indeed, consider the profile consisting of the two directed edges between u and v and let q_u and q_v be the probabilities that the winner is node u and node v , respectively. Impartiality means that node u is the winner with probability q_u at the profile consisting only of the directed edge from v to u and node v is the winner at the profile consisting only of the directed edge from u to v with probability q_v . Overall,

$$\mathbb{E}[d_{f(\mathbf{x})}(\mathbf{x})] = (q_u + q_v) \cdot p^2 + q_u \cdot p \cdot (1 - p) + q_v \cdot (1 - p) \cdot p \leq (q_u + q_v) \cdot p \leq p.$$

Notice that our argument includes randomized mechanisms that may return no winner with positive probability at some profiles. \square

Second, we show that our assumption about voter independence is crucial since, otherwise, even our most appealing AVD mechanism has linear additive approximation.

Example 1. Consider the following instance with $8k + 2$ nodes partitioned into sets of nodes A and B of $4k$ nodes each and two additional nodes a and b . Node a is approved by no node with probability $1/2$ and all the $4k$ nodes of set A with probability $1/2$ (i.e., there is correlation between the votes in A). Similarly, and independently from the approvals to node a , node b is approved by no node with probability $1/2$ and by all nodes of set B with probability $1/2$. Notice that there is always a tie and hence AVD always selects the default node, which cannot have expected in-degree higher than $2k$. The expected highest in-degree is $4k$ with probability $3/4$ and 0 with probability $1/4$, i.e., an expected highest in-degree of $3k$. Hence, the additive approximation is k , i.e., linear in the number of nodes.