Northumbria Research Link

UniversityLibrary

Article

On Laplacian Eigenvalues of the Zero-Divisor Graph Associated to the Ring of Integers Modulo n

Bilal A. Rather [1], Shariefuddin Pirzada [1,*], Tariq A. Naikoo [2] and Yilun Shang [3,*]

[1] Department of Mathematics, University of Kashmir, Srinagar 190006, India; bilalahmadrr@gmail.com
[2] Department of Mathematics, Islamia College of Science and Commerce, Srinagar 190003, India; tariqnaikoo@rediffmail.com
[3] Department of Computer and Information Sciences, Northumbria University, Newcastle NE1 8ST, UK
[*] Correspondence: pirzadasd@kashmiruniversity.ac.in (S.P.); yilun.shang@northumbria.ac.uk (Y.S.)

Abstract: Given a commutative ring R with identity $1 \neq 0$, let the set $Z(R)$ denote the set of zero-divisors and let $Z^*(R) = Z(R) \setminus \{0\}$ be the set of non-zero zero-divisors of R. The zero-divisor graph of R, denoted by $\Gamma(R)$, is a simple graph whose vertex set is $Z^*(R)$ and each pair of vertices in $Z^*(R)$ are adjacent when their product is 0. In this article, we find the structure and Laplacian spectrum of the zero-divisor graphs $\Gamma(\mathbb{Z}_n)$ for $n = p^{N_1} q^{N_2}$, where $p < q$ are primes and N_1, N_2 are positive integers.

Keywords: laplacian matrix; zero-divisor graph; integers modulo ring; gaussian integer ring; Eulers's totient function

1. Introduction

All graphs considered in the present article are connected, undirected, simple and finite. A graph is denoted by $G = G(V(G), E(G))$, where $V(G)$ is the vertex set and $E(G)$ is the edge set of G. The *order* and the *size* of G are the cardinalities of $V(G)$ and $E(G)$, respectively. The *neighborhood* of a vertex v, denoted by $N(v)$, is the set of vertices of G adjacent to v. The degree of v, denoted by d_v, is the cardinality of $N(v)$. A graph G is called *r-regular* if degree of every vertex is r. The adjacency matrix $A(G) = (a_{ij})$ of G is a square matrix of order n, whose (i, j)-entry is 1, if v_i and v_j are adjacent and is 0, otherwise. Let $D(G) = Diag(d_1, d_2, \ldots, d_n)$ be the diagonal matrix, where d_i are the degrees of the vertices of G. The matrix $L(G) = D(G) - A(G)$ is the Laplacian matrix and its eigenvalues with multiplicities is known as the Laplacian spectrum of G. This matrix is real symmetric and positive semi-definite matrix, so the eigenvalues can be ordered as $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. Also, we note that each row (column) sum is zero, so 0 is the Laplacian eigenvalue of G. Furthermore, it is well known that the Laplacian eigenvalue μ_{n-1} is positive if and only if G is connected and is known as the algebraic connectivity of G. More about the matrix $L(G)$ can be seen in [1,2].

Let R be a commutative ring with non-zero identity. An element $x \in R$, $x \neq 0$, is known as the zero-divisor of R if we can find $y \in R$, $y \neq 0$, such that $xy = 0$. Beck [3] introduced the concept of the zero-divisor graphs of commutative rings and included 0 in the definition. He was mainly interested in colorings of these rings. Later Anderson and Livingston [4] modified the definition of the zero-divisor graphs by excluding 0 of the ring in the zero-divisor set and defined the edges between two non-zero zero-divisors if and only if their product is zero. The adjacency, the Laplacian, the signless Laplacian, distance Laplacian and the signless Laplacian spectral analysis can be seen in [5–11]. More literature about zero-divisor graphs can be seen in [4,12–14] and the references therein.

In G, $x \sim y$ denotes that the vertices x and y are adjacent and xy denotes an edge. We use the standard notation, for K_n and $K_{a,b}$ respectively denote the complete graph and the complete bipartite graph. Other undefined notations and terminology can be seen in [1,15].

The remaining part of the paper is organized as follows. In Section 2, we present some preliminaries and investigate the structure of $\Gamma(\mathbb{Z}_{p^{N_1}q^{N_2}})$ and discuss some of its graph invariants. In Section 3, we obtain the Laplacian eigenvalues of $\Gamma(\mathbb{Z}_{p^{N_1}q^{N_2}})$, for $n = p^{N_1}q^{N_2}$, where p and q are primes. We deduce several consequences from these results, which include the determination of the eigenvalues of the graphs $\Gamma(\mathbb{Z}_{p^{2m}})$, $\Gamma(\mathbb{Z}_{2^m}[i])$ (zero-divisor graph of Gaussian integers modulo 2^m), $\Gamma(\mathbb{Z}_{p^{2m+1}})$, $\Gamma(\mathbb{Z}_{pq})$ and $\Gamma(\mathbb{Z}_{pqr})$. At the end of the article, we give the conclusion and discussion for possible further work.

2. Structure of the Zero-Divisor Graph $\Gamma(\mathbb{Z}_{p^{N_1}q^{N_2}})$

We begin with the following definition.

Definition 1. *Let G be a graph of order n with vertex set $\{1, 2, \ldots, k\}$ and G_i be disjoint graphs of order n_i, $1 \leq i \leq k$. The graph $G[G_1, G_2, \ldots, G_n]$ is formed by taking the graphs $G_1, G_2, \ldots, G_n$ and joining each vertex of G_i to every vertex of G_j whenever i and j are adjacent in G.*

This graph operation is known by different names in the literature, such as *G*-join, generalized composition, generalized join, joined union, and here we follow the latter name.

Let n be a positive integer and let $\tau(n)$ denote the number of positive factors of n. Please note that $d|n$ denotes d divides n. The *Euler's totient function* or *Euler's phi function*, denoted by $\phi(n)$, is the number of positive integers less or equal to n and relatively prime to n. We say that n is in *canonical decomposition* if $n = p_1^{n_1} p_2^{n_2} \ldots p_l^{n_l}$, where $l, n_1, n_2, \ldots, n_l$ are positive integers and $p_1, p_2, \ldots, p_l$ are distinct primes.

The following observations will be used in the sequel.

Lemma 1 ([16]). *If n is in canonical decomposition $p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r}$, then*

$$\tau(n) = (n_1 + 1)(n_2 + 1) \ldots (n_r + 1)$$

Theorem 1 ([16]). *The Euler's totient function ϕ satisfies the following.*

(i) *ϕ is multiplicative, i.e., $\phi(pq) = \phi(p)\phi(q)$, whenever p and q are relatively prime.*
(ii) *$\sum\limits_{d|n} \phi(d) = n$.*
(iii) *For prime p, $\sum\limits_{i=1}^{l} \phi(p^l) = p^l - 1$.*

For positive integer n, $\mathbb{Z}_n$ represents the set of congruence classes $\{\overline{0}, \overline{1}, \ldots, \overline{n-1}\}$ of integer modulo n. The ring of *Gaussian integers modulo n*, denoted by $\mathbb{Z}_n[i]$, is represented by $\mathbb{Z}_n[i] = \{\overline{a} + i\overline{b} : \overline{a}, \overline{b} \in \mathbb{Z}_n\}$

An integer d dividing n is a proper divisor of n if and only if $1 < d < n$. Let Y_n be the simple graph with vertex set as the proper divisor set $\{d_1, d_2, \ldots, d_t\}$ of n, where two vertices are adjacent provided $d_i d_j$ is a multiple of n. Evidently, this graph is a connected graph [5]. If $p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r}$ is the canonical decomposition of n, by Lemma 1, it follows that the order of Y_n is given by

$$|V(Y_n)| = (n_1 + 1)(n_2 + 1) \ldots (n_r + 1) - 2.$$

For $1 \leq i \leq t$, let $A_{d_i} = \{r \in \mathbb{Z}_n : (r, n) = d_i\}$, where (r, n) is the greatest common divisor of r and n. We observe that $A_{d_i} \cap A_{d_j} = \phi$, when $i \neq j$, so, the sets $A_{d_1}, A_{d_2}, \ldots, A_{d_t}$ are pairwise disjoint and partitions the vertex set of $\Gamma(\mathbb{Z}_n)$ as $V(\Gamma(\mathbb{Z}_n)) = A_{d_1} \cup A_{d_2} \cup \cdots \cup A_{d_t}$. From the definition of A_{d_i}, a vertex of A_{d_i} is adjacent [5] to the vertex of A_{d_j} in $\Gamma(\mathbb{Z}_n)$ provided that $n|d_i d_j$, for $i, j \in \{1, 2, \ldots, t\}$. The cardinality of A_{d_i} is given as follows.

Lemma 2 ([11]). *For a divisor d of n, the cardinality of the set A_d is equal to $|A_d| = \phi\left(\frac{n}{d_i}\right)$.*

We note that that the induced subgraphs $\Gamma(A_{d_i})$ of $\Gamma(\mathbb{Z}_n)$ are either cliques or null graphs, as can be seen below [5].

Lemma 3. *For the positive integer n and its proper d_i, the following hold.*

(i) *If $i \in \{1, 2, \ldots, t\}$, then the subgraph $\Gamma(A_{d_i})$ of $\Gamma(\mathbb{Z}_n)$ on A_{d_i} is either the complete graph $K_{\phi\left(\frac{n}{d_i}\right)}$ or its complement $\overline{K}_{\phi\left(\frac{n}{d_i}\right)}$. Also, $\Gamma(A_{d_i})$ is $K_{\phi\left(\frac{n}{d_i}\right)}$ provided d_i^2 is a multiple of n.*

(ii) *For distinct i, j in $\{1, 2, \ldots, t\}$, a vertex of A_{d_i} is adjacent to all A_{d_j} or none of the vertices in A_{d_j}.*

(iii) *For distinct i, j in $\{1, 2, \ldots, t\}$, a vertex of A_{d_i} is adjacent to a vertex of A_{d_j} in $\Gamma(\mathbb{Z}_n)$ provided $d_i d_j$ is a multiple of n.*

The graph formed in part (iii) of Lemma 3 is known as $\mathcal{G}(A(d_i))$ graph. Clearly, $\Gamma(\mathbb{Z}_n)$ can be expressed as a joined union of complete graphs and empty graphs.

Lemma 4. *[5] For the induced subgraph $\Gamma(A_{d_i})$ of $\Gamma(\mathbb{Z}_n)$ on vertices A_{d_i} for $1 \leq i \leq t$, the zero-divisor graph $\Gamma(\mathbb{Z}_n) = Y_n[\Gamma(A_{d_1}), \Gamma(A_{d_2}), \ldots, \Gamma(A_{d_t})]$.*

For a commutative ring R with non-zero identity $1 \neq 0$, and $a \in R$, the *annihilator* of a, denoted by $ann(a)$, is the set of those elements of R that annihilates a, and we write $ann(a) = \{b \in R : ab = 0\}$. Define a relation on R by $a \sim b$ whenever $ann(a) = ann(b)$. Obviously, $ann(a) = ann(a)$ and if $ann(a) = ann(b)$ then $ann(b) = ann(a)$ implying that $\sim$ is symmetric relation. Also, if $ann(a) = ann(b)$ and $ann(b) = ann(c)$, then $\sim$ is transitive and is an equivalence relation on R which partitions R into equivalence classes. Furthermore, $[a]$ represents the class of $a \in R$, that is, $[a] = \{b \in R : ann(a) = ann(b)\}$.

The *compressed zero-divisor graph* of a commutative ring R, denoted by $\Gamma_E(R)$, is the undirected, simple graph with the vertex set $Z(R_E) - \{[0]\} = R_E - \{[0], [1]\}$ and is defined by $R_E = \{[a] : a \in R\}$, where $[a] = \{b \in R : ann(a) = ann(b)\}$ and the two vertices $[a]$ and $[b]$ are adjacent provided $[x][y] = [0] = [xy]$. This graph was first defined in [17] and their properties for $\mathbb{Z}_{p^n}$ were investigated in [13].

For example, consider $\mathbb{Z}_{12}$ with non-zero zero-divisor set $\{2, 3, 4, 6, 8, 9, 10\}$. The annihilators of this set are

$$ann(2) = \{6\}, \ ann(3) = \{4, 8\}, \ ann(4) = \{3, 6, 9\}, \ ann(6) = \{2, 4, 6, 8, 10\},$$
$$ann(8) = \{3, 6, 9\}, \ ann(9) = \{4, 8\}, \ ann(10) = \{6\}.$$

The compressed zero-divisor graph $\mathbb{Z}_{12}$ with the vertex set $\{[2], [3], [4], [6]\}$ and the proper divisor graph of $\mathbb{Z}_{12}$ with the vertex set $\{2, 3, 4, 6\}$ are shown in Figure 1

Figure 1. The Compressed zero-divisor graph, the graph $\mathcal{G}(A(d_i))$ and the proper divisor graph of $\mathbb{Z}_{12}$.

In case of $R = \mathbb{Z}_n$, we observe that the vertex sets of R_E, $\mathcal{G}(A(d_i))$ and Y_n are in one-one correspondence.

Proposition 1. *If $\mathbb{Z}_n$ is the finite commutative ring, then $\Gamma_E(R) \cong \mathcal{G}(A(d_i)) \cong Y_n$.*

Now, we find the structure of $\Gamma(\mathbb{Z}_n)$, for $n = p^{N_1} q^{N_2}$, where p and q, $p < q$, are primes. This generalizes the results obtained in [13]. We prove the cases when N_1 and N_2, $N_1 \leq N_2$, are positive even integers, N_1 and N_2 are positive odd integers and the other possible cases can be similarly proved.

Theorem 2. *Let* $\Gamma(\mathbb{Z}_n)$ *be the zero-divisor graph of order N, where* $n = p^{N_1}q^{N_2}$ *and* $N_1 = 2m_1 \leq 2m_2 = N_2$. *Then*

$$
\begin{aligned}
\Gamma(\mathbb{Z}_n) = \Upsilon_n\Big[&\overline{K}_{\phi(p^{N_1-1}q^{N_2})}, \ldots, \overline{K}_{\phi(p^{m_1}q^{N_2})}, \ldots, \overline{K}_{\phi(q^{N_2})}, \overline{K}_{\phi(p^{N_1}q^{N_2-1})}, \ldots, \overline{K}_{\phi(p^{N_1}q^{m_2})}, \\
&\ldots, \overline{K}_{\phi(p^{N_1})}, \overline{K}_{\phi(p^{N_1-1}q^{N_2-1})}, \ldots, \overline{K}_{\phi(p^{N_1-1}q^{m_2})}, \ldots, \overline{K}_{\phi(p^{N_1-1})}, \ldots, \\
&\overline{K}_{\phi(p^{m_1}q^{N_2-1})}, \ldots, K_{\phi(p^{m_1}q^{m_2-1})}, K_{\phi(p^{m_1}q^{m_2})}, \ldots, K_{\phi(p^{m_1})}, \ldots, \overline{K}_{\phi(q^{N_2-1})}, \ldots, \\
&\overline{K}_{\phi(q^{m_2-1})}, K_{\phi(q^{m_2})}, \ldots, K_{\phi(q)}\Big].
\end{aligned}
\tag{1}
$$

Proof. Let $n = p^{N_1}q^{N_2}$, where p and q, $2 < p < q$, are primes and N_1 and N_2, $2 \leq N_1 = 2m_1 \leq 2m_2 = N_2$, are positive even integers. The proper divisors of n are

$$
\begin{aligned}
\Big\{&p, p^2, \ldots, p^{m_1}, \ldots, p^{N_1}, q, q^2, \ldots, q^{m_2}, \ldots, q^{N_2}, pq, pq^2, \ldots, pq^{m_2}, \ldots, pq^{N_2}, \ldots, p^{m_1}q, p^{m_1}q^2, \ldots, \\
&p^{m_1}q^{m_2-1}, p^{m_1}q^{m_2}, \ldots, p^{m_1}q^{N_2}, \ldots, p^{N_1}q, p^{N_1}q^2, \ldots, p^{N_1}q^{m_2-1}, p^{N_1}q^{m_2}, \ldots, p^{N_1}q^{N_2-1}\Big\}.
\end{aligned}
$$

By Lemma 1, order of Υ_n is $(N_1 + 1)(N_2 + 1) - 2 = N_1 N_2 + N_1 + N_2 - 1$. From the definition of Υ_n, we have

$$
\begin{aligned}
&p \sim p^{N_1-1}q^{N_2}, \ p^2 \sim p^{N_1-2}q^{N_2}, p^{N_1-1}q^{N_2}, \ldots, p^{m_1} \sim p^{m_1}q^{N_2}, p^{m_1+1}q^{N_2}, \ldots, p^{N_1-1}q^{N_2}, \\
&\ldots p^{N_1} \sim q^{N_2}, pq^{N_2}, p^2q^{N_2}, \ldots, p^{m_1}q^{N_2}, \ldots, p^{N_1-1}q^{N_2},
\end{aligned}
$$

which in iteration form can be read as

$$
p^i \sim p^j q^{N_2}, \ i + j \geq N_1, \text{ for } i = 1, 2, \ldots, N_1.
$$

Arguing as above, other adjacency relations are

$$
\begin{aligned}
q^i &\sim p^{N_1}q^j, \ i + j \geq N_2, \text{ for } i = 1, 2, \ldots, N_2, \\
pq^i &\sim p^k q^j, \ i + j \geq N_2, \text{ for } i = 1, 2, \ldots, N_2 \text{ and } k \geq 2m_1 - 1, \\
&\vdots \\
p^{m_1}q^i &\sim p^k q^j, \ i + j \geq N_2, \text{ for } i = 1, 2, \ldots, N_2 \text{ and } k \geq m_1, \\
&\vdots \\
p^{N_1}q^i &\sim p^k q^j, \ i + j \geq N_2, \text{ for } i = 1, 2, \ldots, N_2 - 1 \text{ and } k \geq 0.
\end{aligned}
$$

For $i = 1, 2, \ldots, N_1$, $j = 1, 2, \ldots, N_2$ and $k = 1, 2, \ldots, N_2 - 1$, by Lemma 2, the cardinalities of A_{d_i} are

$$
\begin{aligned}
|A_{p^i}| = \phi(p^{N_1-i}q^{N_2}), |A_{q^j}| = \phi(p^{N_1}q^{N_2-j}), |A_{pq^j}| = \phi(p^{N_1-1}q^{N_2-j}), \ldots, |A_{p^{m_1}q^j}| = \phi(p^{m_1}q^{N_2-j}), \\
\ldots, |A_{p^{N_1-1}q^j}| = \phi(pq^{N_2-j}), |A_{p^{N_1}q^k}| = \phi(q^{N_2-k}).
\end{aligned}
$$

Also, by Lemma 3, the induced graphs $\Gamma\left(A_{d_{p^i}}\right)$ are

$$
G_i = \begin{cases}
\Gamma\left(A_{d_{p^i}}\right) = \overline{K}_{\phi(p^{N_1-i}q^{N_2})}, & 1 \leq i \leq N_1, \\
\Gamma\left(A_{d_{q^j}}\right) = \overline{K}_{\phi(p^{N_1}q^{N_2-j})}, & 1 \leq j \leq N_2, \\
\Gamma\left(A_{d_{p^i q^j}}\right) = \overline{K}_{\phi(p^{N_1-i}q^{N_2-j})}, & 1 \leq i \leq m_1 - 1 \text{ and } 1 \leq j \leq N_2 \\
& \text{or } m_1 \leq i \leq N_1 \text{ and } 1 \leq j \leq m_2 - 1, \\
\Gamma\left(A_{d_{p^i q^j}}\right) = K_{\phi(p^{N_1-i}q^{N_2-j})}, & m_1 \leq i \leq N_1 \text{ and } m_2 \leq j \leq N_2,
\end{cases}
\tag{2}
$$

where we avoid $\Gamma\left(A_{d_{p^{N_1}q^{N_2}}}\right)$ corresponding to the proper divisor $p^{N_1}q^{N_2}$. Lastly, by Lemma 4, the structure of the zero-divisor graph of $\Gamma(\mathbb{Z}_n)$ is as in Equation (1). This completes the proof. $\square$

In Theorem 2, taking $N_2 = 0$, we have the following consequence.

Corollary 1. *If $\Gamma(\mathbb{Z}_n)$ is the zero-divisor graph of order N, where $n = p^{2m}$, then*

$$\Gamma(\mathbb{Z}_n) = Y_n\left[\overline{K}_{\phi(p^{2m-1})}, \overline{K}_{\phi(p^{2m-2})}, \ldots, \overline{K}_{\phi(p^{m+1})}, K_{\phi(p^m)}, K_{\phi(p^{m-1})}, \ldots, K_{\phi(p^2)}, K_{\phi(p)}\right]. \quad (3)$$

Proof. The proper divisors of $n = p^{2m}$ are $\{p, p^2, \ldots, p^{m-1}, p^m, p^{m+1}, \ldots, p^{2m-1}\}$. In $Y_{p^{2m}}$, vertex p^i is adjacent to vertex p^j if and only if $i + j \geq 2m$ with $1 \leq i \leq 2m - 1$ and to avoid loops, we assume $i \neq j$. Also, n does not divide $(p^i)^2$, for $i = 1, 2, \ldots, m - 1$, so $G_i = \overline{K}_{\phi(p^{2m-i})}$ and n divides $(p^i)^2$, for $i = m, m+1, \ldots, 2m - 2, 2m - 1$, and thus $G_i = K_{\phi(p^{2m-i})}$. Thus, Equation (3) follows. $\square$

Another consequence gives the diameter of $\Gamma(\mathbb{Z}_{p^{2m_1}q^{2m_2}})$.

Corollary 2. *The diameter of $\Gamma(\mathbb{Z}_n)$ is 3 for $n = p^{2m_1}q^{2m_2}$, and is 2 if $m_2 = 0$.*

Proof. In the proof of Theorem 2, we observe that $p^i \sim q^j$ if and only if $i = j = n$, otherwise $p^i \sim p^k q^n$, $i + k \geq n$ and $q^j \sim p^n q^h$, $j + h \geq n$. Lastly, $p^k q^n \sim p^n q^h$, $k \geq 1, h \geq 1$. This implies that $d(p^i, q^j) = 3$, if $1 \leq i, j \leq n - 1$ in Y_n. Similarly, from Corollary 1, distance in $Y_{p^{2m}}$ is at most 2. $\square$

The following consequence gives the clique number of $\Gamma(\mathbb{Z}_{p^{2m_1}q^{2m_2}})$.

Corollary 3. *The clique number of $\Gamma(\mathbb{Z}_n)$ is*

$$\omega(\Gamma(\mathbb{Z}_n)) = \begin{cases} p^{m_1}q^{m_2} - 1 & \text{if } n = p^{2m_1}q^{2m_2} \\ p^{m_1} - 1 & \text{if } n = p^{2m}. \end{cases}$$

Proof. By the definition of $Y_{p^{2m_1}q^{2m_2}}$, we can easily see that $p^i q^j$, $i \geq m_1$, $j \geq m_2$ are the vertices of the clique of $Y_{p^{2m_1}q^{2m_2}}$ and the number of such vertices is

$$\underbrace{m_2 + 1 + m_2 + 1 + \cdots + m_2 + 1}_{m_1} + m_2 = m_1(m_2 + 1) + m_2.$$

By Lemma 3, $\Gamma(A_{d_i})$ is $K_{\phi(\frac{n}{d_i})}$ if and only if n divides d_i^2, so that the clique number of $\Gamma(\mathbb{Z}_{p^{N_1}q^{N_2}})$ is

$$|\Gamma(A_{p^{m_1}q^{m_2}})| + |\Gamma(A_{p^{m_1}q^{m_2-1}})| + \cdots + |\Gamma(A_{p^{m_1}q})| + |\Gamma(A_{p^{m_1}})|$$
$$+ |\Gamma(A_{p^{m_1-1}q^{m_2}})| + |\Gamma(A_{p^{m_1-1}q^{m_2-1}})| + \cdots + |\Gamma(A_{p^{m_1-1}q})| + |\Gamma(A_{p^{m_1-1}})|$$
$$\vdots$$
$$+ |\Gamma(A_{pq^{m_2}})| + |\Gamma(A_{pq^{m_2-1}})| + \cdots + |\Gamma(A_{pq^2})| + |\Gamma(A_{pq})|$$
$$+ |\Gamma(A_{q^{m_2}})| + |\Gamma(A_{q^{m_2-1}})| + \cdots + |\Gamma(A_{q^2})| + |\Gamma(A_q)|$$
$$= \phi(p^{m_1}q^{m_2}) + \phi(p^{m_1}q^{m_2-1}) + \cdots + \phi(p^{m_1}q) + \phi(p^{m_1})$$
$$+ \phi(p^{m_1-1}q^{m_2}) + \phi(p^{m_1-1}q^{m_2-1}) + \cdots + \phi(p^{m_1-1}q) + \phi(p^{m_1-1})$$
$$\vdots$$
$$+ \phi(pq^{m_2}) + \phi(pq^{m_2-1}) + \cdots + \phi(pq) + \phi(p)$$
$$+ \phi(q^{m_2}) + \phi(q^{m_2-1}) + \cdots + \phi(q^2) + \phi(q)$$
$$= \phi(p^{m_1})q^{m_2} + \phi(p^{m_1-1})q^{m_2} + \cdots + \phi(p)q^{m_2} + q^{m_2} - 1 = p^{m_1}q^{m_2} - 1.$$

If $m_2 = 0$, then by definition of $Y_{p^{2m}}$, the vertices p^i, $i \geq m$, form the clique in it and its size is m. Thus, the sum cardinality of the cardinalities $\Gamma(A_{p^i})$, $i \geq m$, is the clique size of $\Gamma(\mathbb{Z}_{p^{2m}})$. Using Lemma 1, we have

$$cl(\Gamma(A_{p^i})) = |\Gamma(A_{p^m})| + |\Gamma(A_{p^{m+1}})| + \cdots + |\Gamma(A_{p^{2m-1}})|$$
$$= \phi(p^m) + \phi(p^{m-1}) + \cdots + \phi(p) = p^m - 1.$$

□

The following result gives the structure of $\Gamma(\mathbb{Z}_{p^{N_1}q^{N_2}})$, where N_1 and N_2 are both odd.

Theorem 3. *Let $\Gamma(\mathbb{Z}_n)$ be the zero-divisor graph of order N where $n = p^{N_1}q^{N_2}$ and $N_1 = 2m_1 + 1 \leq 2m_2 + 1 = N_2$. Then*

$$\Gamma(\mathbb{Z}_n) = Y_n \Big[\overline{K}_{\phi(p^{2m_1}q^{N_2})}, \ldots, \overline{K}_{\phi(p^{m_1}q^{N_2})}, \ldots, \overline{K}_{\phi(q^{N_2})}, \overline{K}_{\phi(p^{N_1}q^{2m_2})}, \ldots, \overline{K}_{\phi(p^{N_1}q^{m_2})},$$
$$\ldots, \overline{K}_{\phi(p^{N_1})}, \overline{K}_{\phi(p^{2m_1}q^{2m_2})}, \ldots, \overline{K}_{\phi(p^{2m_1}q^{m_2})}, \ldots, \overline{K}_{\phi(p^{2m_1})}, \ldots,$$
$$\overline{K}_{\phi(p^{m_1}q^{2m_2})}, \ldots, K_{\phi(p^{m_1}q^{m_2})}, K_{\phi(p^{m_1}q^{m_2-1})}, \ldots, K_{\phi(p^{m_1})}, \ldots, \overline{K}_{\phi(q^{2m_2})}, \ldots,$$
$$\overline{K}_{\phi(q^{m_2})}, K_{\phi(q^{m_2-1})}, \ldots, K_{\phi(q)} \Big]. \tag{4}$$

Proof. Let $n = p^{N_1}q^{N_2}$, where p and q, $2 < p < q$, are primes and N_1 and N_2, $2 \leq N_1 = 2m_1 + 1 \leq 2m_2 + 1 = N_2$, are positive even integers. Then the proper divisors of n are

$$\Big\{ p, p^2, \ldots, p^{m_1+1}, \ldots, p^{2m_1+1}, q, q^2, \ldots, q^{m_2+1}, \ldots, q^{2m_2+}, pq, pq^2, \ldots, pq^{m_2+1}, \ldots, pq^{2m_2+1}, \ldots,$$
$$p^{m_1+1}q, p^{m_1+1}q^2, \ldots, p^{m_1+1}q^{m_2}, p^{m_1+1}q^{m_2+1}, \ldots, p^{m_1+1}q^{2m_2+1}, \ldots, p^{2m_1+1}q, p^{2m_1+1}q^2, \ldots,$$
$$p^{2m_1+1}q^{m_2}, p^{2m_1+1}q^{m_2+1}, \ldots, p^{2m_1+1}q^{2m_2} \Big\}.$$

Therefore, by the definition of Y_n, we have

$$p^i \sim p^j q^{2m_2+1}, \ i+j \geq 2m_1+1, \ \text{for } i = 1, 2, \ldots, 2m_1+1,$$

$$q^i \sim p^{2m_1} q^j, \ i+j \geq 2m_2+1, \ \text{for } i = 1, 2, \ldots, 2m_2+1,$$

$$pq^i \sim p^k q^j, \ i+j \geq 2m_2+1, \ \text{for } i = 1, 2, \ldots, 2m_2+1 \text{ and } k \geq 2m_1,$$

$$\vdots$$

$$p^{m_1+1} q^i \sim p^k q^j, \ i+j \geq 2m_2+1, \ \text{for } i = 1, 2, \ldots, 2m_2+1 \text{ and } k \geq m_1,$$

$$\vdots$$

$$p^{2m_1+1} q^i \sim p^k q^j, \ i+j \geq 2m_2+1, \ \text{for } i = 1, 2, \ldots, 2m_2 \text{ and } k \geq 0.$$

By Lemma 2, for $i = 1, 2, \ldots, 2m_1+1$, $j = 1, 2, \ldots, 2m_2+1$ and $k = 1, 2, \ldots, 2m_2$, the cardinalities of A_{d_i} are

$$|A_{p^i}| = \phi(p^{2m_1+1-i} q^{2m_2+1}), |A_{q^j}| = \phi(p^{2m_1+1} q^{2m_2+1-j}), |A_{pq^j}| = \phi(p^{2m_1} q^{2m_2+1-j}), \ldots,$$

$$|A_{p^{m_1+1} q^j}| = \phi(p^{m_1} q^{2m_2+1-j}), \ldots, |A_{p^{2m_1} q^j}| = \phi(pq^{2m_2+1-j}), |A_{p^{2m_1+1} q^k}| = \phi(q^{2m_2+1-k}).$$

Also, by Lemma 3, the induced graphs $\Gamma\left(A_{d_{p^i}}\right)$ are

$$G_i = \begin{cases} \Gamma\left(A_{d_{p^i}}\right) = \overline{K}_{\phi(p^{2m_1+1-i} q^{2m_2+1})}, & 1 \leq i \leq 2m_1+1, \\ \Gamma\left(A_{d_{q^j}}\right) = \overline{K}_{\phi(p^{2m_1+1} q^{2m_2+1-j})}, & 1 \leq j \leq 2m_2+1, \\ \Gamma\left(A_{d_{p^i q^j}}\right) = \overline{K}_{\phi(p^{2m_1+1-i} q^{2m_2+1-j})}, & 1 \leq i \leq m_1 \text{ and } 1 \leq j \leq 2m_2+1 \\ & \text{or } 1 \leq i \leq 2m_1+1 \text{ and } 1 \leq j \leq m_2, \\ \Gamma\left(A_{d_{p^i q^j}}\right) = K_{\phi(p^{2m_1+1-i} q^{2m_2+1-j})}, & m_1+1 \leq i \leq 2m_1+1 \text{ and } m_2+1 \leq j \leq N_2, \end{cases} \tag{5}$$

where we avoid $\Gamma\left(A_{d_{p^{2m_1+1} q^{2m_2+1}}}\right)$ corresponding to proper divisor $p^{2m_1+1} q^{2m_2+1}$. Therefore, by Lemma 4, the structure of zero-divisor graph of $\Gamma(\mathbb{Z}_n)$ is

$$\Gamma(\mathbb{Z}_n) = Y_n \Big[\overline{K}_{\phi(p^{2m_1} q^{N_2})}, \ldots, \overline{K}_{\phi(p^{m_1} q^{N_2})}, \ldots, \overline{K}_{\phi(q^{N_2})}, \overline{K}_{\phi(p^{N_1} q^{2m_2})}, \ldots, \overline{K}_{\phi(p^{N_1} q^{m_2})},$$

$$\ldots, \overline{K}_{\phi(p^{N_1})}, \overline{K}_{\phi(p^{2m_1} q^{2m_2})}, \ldots, \overline{K}_{\phi(p^{2m_1} q^{m_2})}, \ldots, \overline{K}_{\phi(p^{2m_1})}, \ldots,$$

$$\overline{K}_{\phi(p^{m_1} q^{2m_2})}, \ldots, K_{\phi(p^{m_1} q^{m_2})}, K_{\phi(p^{m_1} q^{m_2-1})}, \ldots, K_{\phi(p^{m_1})}, \ldots, \overline{K}_{\phi(q^{2m_2})}, \ldots,$$

$$\overline{K}_{\phi(q^{m_2})}, K_{\phi(q^{m_2-1})}, \ldots, K_{\phi(q)} \Big]. $$

$\square$

If $N_2 = 0$, in Theorem 3, we have the following consequence.

Corollary 4. *Let $\Gamma(\mathbb{Z}_n)$ be the zero-divisor graph of order N, where $n = p^{2m+1}$. Then*

$$\Gamma(\mathbb{Z}_n) = Y_n \Big[\overline{K}_{\phi(p^{2m})}, \overline{K}_{\phi(p^{2m-1})}, \ldots, \overline{K}_{\phi(p^{m+1})}, K_{\phi(p^m)}, K_{\phi(p^{m-1})}, \ldots, K_{\phi(p^2)}, K_{\phi(p)} \Big].$$

Proof. The proper divisors of $n = p^{2m+1}$ are $\{p, p^2, \ldots, p^m, p^{m+1}, p^{m+2}, \ldots, p^{2m}\}$. In the graph $Y_{p^{2m+1}}$, the vertex p^i is adjacent to the vertex p^j if and only if $i+j \geq 2m+1$ with $1 \leq i \leq 2m$ and to avoid loops we assume $i \neq j$. Also, n does not divide $(p^i)^2$, for $i = 1, 2, \ldots, m$, this implies that $G_i = \overline{K}_{\phi(p^{2m+1-i})}$ and n divides $(p^i)^2$, for $i = m+1, m+2, \ldots, 2m-1, 2m$, so that $G_i = K_{\phi(p^{2m+1-i})}$. Now result follows. $\square$

Other graph invariants of $\Gamma(\mathbb{Z}_{p^{N_1} q^{N_2}})$, like automorphism group, chromatic number, domination number, independence number, matching number can similarly be investigated.

3. Laplacian Eigenvalues of the Zero-Divisor Graph $\Gamma(\mathbb{Z}_n)$

Consider an $n \times n$ matrix

$$
M = \begin{pmatrix}
\mathcal{A}_{1,1} & \mathcal{A}_{1,2} & \cdots & \mathcal{A}_{1,l} \\
\mathcal{A}_{2,1} & \mathcal{A}_{2,2} & \cdots & \mathcal{A}_{2,l} \\
\vdots & \vdots & \cdots & \vdots \\
\mathcal{A}_{l,1} & \mathcal{A}_{l,2} & \cdots & \mathcal{A}_{l,l}
\end{pmatrix},
$$

whose rows and columns are partitioned according to a partition $P = \{P_1, P_2, \ldots, P_l\}$ of the set $X = \{1, 2, \ldots, n\}$. The quotient matrix Q is a matrix of order l whose (i, j)th entry is the average row sums of the blocks $\mathcal{A}_{i,j}$ of M. If each block $\mathcal{A}_{i,j}$ has constant row (column) sum, then the partition P is called *equitable* and the matrix Q is known as *equitable quotient matrix*. In general, the spectrum of Q is contained in the spectrum of M, equality holds in case of the equitable partition [1].

The following lemma gives a different method of finding determinant (det) of a matrix.

Lemma 5 ([18]). *Let A_1, A_2, A_3 and A_4 be respectively $p \times p$, $p \times q$, $q \times p$ and $q \times q$ matrices with A_1 and A_4 invertible. Then*

$$
det \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} = det(A_1) det(A_4 - A_3 A_1^{-1} A_2)
$$

$$
= det(A_4) det(A_1 - A_2 A_4^{-1} A_3),
$$

where $A_4 - A_3 A_1^{-1} A_2$ and $A_1 - A_2 A_4^{-1} A_3$ are known as the Schur complement of A_1 and A_4, respectively.

The following result gives the Laplacian spectrum of $G[G_1, \ldots, G_n]$ in terms of the Laplacian spectrum of G_i's and the eigenvalues of the quotient matrix.

Theorem 4. *Let G be a graph of order n and let G_i be regular graphs of order n_i with Laplacian eigenvalues $\mu_{i1} \geq \mu_{i2} \geq \ldots \geq \mu_{in_i}$, where $i = 1, 2, \ldots, n$. Then the Laplacian eigenvalues of $G[G_1, \ldots, G_n]$ are the eigenvalues $\alpha_i + \mu_{ik}(G_i)$ for $i = 1, \ldots, n$ and $k = 2, 3, \ldots, n_i$, where $\alpha_i = \sum\limits_{v_j \in N_G(v_i)} n_i$ is the sum of the cardinalities of the graphs $G_j, j \neq i$, which corresponds to the neighbors of vertex $v_i \in G$ and n eigenvalues of the following matrix*

$$
\mathcal{Q} = \begin{pmatrix}
\alpha_1 & -\psi_{12} & \cdots & -\psi_{1n} \\
-\psi_{21} & \alpha_2 & \cdots & -\psi_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
-\psi_{n1} & -\psi_{n2} & \cdots & \alpha_n
\end{pmatrix}, \tag{6}
$$

where for $i \neq j$, $\psi_{ij} = n_j$, if $v_i \sim v_j$, while as $\psi_{ij} = 0$, if $v_i \nsim v_j$.

An equivalent statement of Theorem 4 can be seen in [19], so we omit the proof here.

Usually it is difficult to obtain the Laplacian eigenvalues of graphs in general. So, researchers attempt to get the Laplacian eigenvalues of particular class of graphs. It is important to mention that the structure of the zero-graphs associated with $\mathbb{Z}_n$ for $n = p^{N_1} q^{N_2}$ has not been obtained earlier. Therefore, it becomes essential to write graphs in some known structure and obtain their Laplacian spectrum.

Now, we will find the Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$, for $n = p^{N_1} q^{N_2}$, where p and q, $p < q$, are primes. This generalizes the results obtained in [5] and that too by using different technique. We prove the case when N_1 and N_2, $N_1 \leq N_2$, are positive even integers and the odd case can be similarly proved.

Theorem 5. *Let $\Gamma(\mathbb{Z}_n)$ be the zero-divisor graph of order N, where $n = p^{N_1}q^{N_2}$ and $N_1 = 2m_1 \leq 2m_2 = N_2$. The Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ consists of the eigenvalues*

$$\left\{ (p^i - 1)^{[\phi(p^{N_1-i}q^{N_2})-1]}, (q^j - 1)^{[\phi(p^{N_1}q^{N_2-j})-1]}, (pq^j - 1)^{[\phi(p^{N_1-1}q^{N_2-j})-1]}, \ldots, (p^{m_1}q^k - 1)^{[\phi(p^{m_1}q^{N_2-k})-1]} \right.$$

$$\left. (p^{m_1}q^l - 1)^{[\phi(p^{m_1}q^{N_2-l})-1]}, \ldots, (p^{2m_1}q^k - 1)^{[\phi(q^{N_2-k})-1]}, (p^{2m_1}q^t - 1)^{[\phi(q^{N_2-t})-1]}, \right\}$$

where $i = 1, 2, \ldots, m_1, \ldots, N_1$, $j = 1, 2, \ldots, N_2$, $k = 1, 2, \ldots, m_2 - 1$, $l = m_2, \ldots, 2m_2$ and $t = m_2, \ldots, 2m_2 - 1$. The remaining Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$ are the eigenvalues of the matrix given in (6).

Proof. By using Theorems 1 and 4, the value of α_i's are

$$\alpha_1 = \phi(p) = p - 1, \alpha_2 = \phi(p) + \phi(p^2) = p^2 - 1,$$

$$\vdots$$

$$\alpha_{m_1} = \phi(p^{m_1}) + \phi(p^{m_1-1}) + \cdots + \phi(p) = p^{m_1} - 1,$$

$$\vdots$$

$$\alpha_{N_1} = \phi(p^{N_1}) + \phi(p^{N_1-1}) + \cdots + \phi(p) = p^{N_1} - 1,$$

that is,

$$\alpha_i = p^i - 1, \text{ for } i = 1, 2, \ldots, N_1.$$

For $i \geq m_2$ and $j \geq m_2$, we note that $\Gamma(A_{p^iq^j})$ as vertex of Y_n are adjacent to itself, so we add and subtract cardinalities of such type of $\Gamma(A_{p^iq^j})$'s so that α_i's are easy to calculate. Now, as above other α_i's are given by

$$\alpha_i = q^j - 1, \text{ for } i = N_1 + 1, \ldots, N_1 + N_2, \text{ and } j = 1, 2, \ldots, m_2, \ldots, N_1,$$
$$\alpha_i = pq^j - 1 \text{ for } i = N_1 + N_2 + 1, \ldots, N_1 + 2N_2 \text{ and } j = 1, 2, \ldots, m_2, \ldots, N_1,$$

$$\vdots$$

$$\alpha_i = p^{m_1}q^j - 1, \text{ for } i = N_1 + m_1 N_2 + 1, \ldots, N_1 + m_1 N_2 + m_2 - 1 \text{ and } j = 1, 2, \ldots, m_2 - 1,$$
$$\alpha_i = p^{m_1}q^j - 1 - \phi(p^{m_1}q^j), \text{ for } i = N_1 + m_1 N_2, \ldots, N_1 + (m_1 + 1)N_2 \text{ and } j = m_2, \ldots, N_2,$$

$$\vdots$$

$$\alpha_i = p^{N_1}q^j - 1, \text{ for } i = N_1 + N_1 N_2 + 1, \ldots, N_1 + N_1 N_2 + m_2 - 1 \text{ and } j = 1, 2, \ldots, m_2 - 1,$$
$$\alpha_i = p^{N_1}q^j - 1 - \phi(q^{N_2-j}), \text{ for } i = N_1 + N_1 N_2 + m_2, \ldots, N_1 + N_1 N_2 + N_2 - 1$$
$$\text{and } j = m_2, \ldots, N_2 - 1.$$

By using Theorem 4, Equation (2) and the fact that Laplacian spectrum of $\overline{K}_\omega$ is $\{0^{[\omega]}\}$, we have

$$\alpha_i + \lambda_{ik}(G_i) = \alpha_i + \lambda_{ik}(\overline{K}_{\phi(p^{N_1-i}q^{N_2})}) = \alpha_i = p^i - 1, \text{ for } i = 1, 2, \ldots, N_1.$$

Thus, for $i = 1, 2, \ldots, N_1$, we see that $p^i - 1$ is the Laplacian eigenvalue of $\Gamma(\mathbb{Z}_n)$ with multiplicity $\phi(p^{N_1-i}q^{N_2}) - 1$.

Now, following similar steps, it is easy to see that

$$(q^j - 1)^{[\phi(p^{N_1}q^{N_2-j})-1]}, (pq^j - 1)^{[\phi(p^{N_1-1}q^{N_2-j})-1]}, \ldots, (p^{m_1}q^k - 1)^{[\phi(p^{m_1}q^{N_2-k})-1]}$$

are also the Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$. Again, by Equation (2), $G_i = K_{\phi(p^{N_1-i}q^{N_2-j})}$, when $i \geq m_1$ and $j \geq m_2$ and Laplacian spectrum of K_ω is $\{0, \omega^{\omega-1}\}$, so

$$\alpha_i + \lambda_{ik}(G_i) = p^{m_1}q^l - 1 - \phi(p^{m_1}q^l) + \phi(p^{m_1}q^l) = p^{m_1}q^l - 1$$

is the Laplacian eigenvalue of $\Gamma(\mathbb{Z}_n)$ with multiplicity $\phi(p^{m_1}q^l) - 1$, where $l = m_2, \ldots, N_2$. Similarly, for $k = 1, 2, \ldots, m_2 - 1$ and $t = m_2, \ldots, N_2 - 1$, we see that $p^{N_1}q^k - 1$ and $p^{N_1}q^t - 1$ are also the Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$ with multiplicities $\phi(q^{N_2-t}) - 1$ and $\phi(q^{N_2-k}) - 1$, respectively. The other Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$ are the zeros of the characteristic polynomial of the quotient matrix (6). □

If we put $m_2 = 0$ in Theorem 5, it reduces to the following result [5] with a different technique.

Corollary 5. *If $n = p^{2m}$ for some positive integer $m \geq 2$, then the Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$ are*

$$\Big\{0, (p-1)^{[\phi(p^{2m-1})]}, (p^2-1)^{[\phi(p^{2m-2})1]}, \ldots, (p^{m-1}-1)^{[\phi(p^{m+1})]}, (p^m-1)^{[\phi(p^m)-1]},$$

$$(p^{m+1}-1)^{[\phi(p^{m-1})]}, \ldots, (p^{2m-2}-1)^{[\phi(p^2)]}, (p^{2m-1}-1)^{[\phi(p)]}\Big\}.$$

Proof. Using Corollary 1, $p \sim p^{2m-1}$ implies that $\alpha_1 = \phi(p)$. However, in general, we see that $\alpha_i = p^i - 1$, for $i = 1, 2, \ldots, m-1$, where from Theorem 1, we have used $\sum_{i=1}^{l} p^i = p^l - 1$. As $p^m \sim p^m$, so we add and subtract cardinality of $\Gamma(A_{p^m})$ and thus α_m is given as

$$\begin{aligned}\alpha_m &= \phi(p^{m-1}) + \phi(p^{m-2}) + \cdots + \phi(p^2) + \phi(p) \\ &= \phi(p^m) + \phi(p^{m-1}) + \phi(p^{m-2}) + \cdots + \phi(p^2) + \phi(p) - \phi(p^m) \\ &= p^m - 1 - \phi(p^m).\end{aligned}$$

Likewise, for $i = m+1, \ldots, 2m-2, 2m-1$, it is clear that

$$\alpha_i = \sum_{j=1}^{i} \phi(p^j) - \phi(p^{2m-i}) = p^i - 1 - \phi(p^{2m-i}).$$

For $i = 1, 2, \ldots, m-1$, clearly $\alpha_i = p^i - 1$ are the Laplacian eigenvalue of $\Gamma(\mathbb{Z}_n)$ with multiplicity $\phi(p^{2m-i}) - 1$. Also, from Theorem 4 and for $i = m, m+1, \ldots, 2m-1$, we see that

$$\alpha_i + \mu_{ik}(G_i) = p^i - 1 - \phi(p^{2m-i}) + \mu_{ik}(K_{\phi(p^{2m-i})}) = p^i - 1$$

are also Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$ with multiplicities $\phi(p^{2m-i}) - 1$. The remaining Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$ are given by the following quotient matrix

$$\mathcal{Q} = \begin{pmatrix} A_{m-1 \times m-1} & B_{m-1 \times m} \\ C_{m \times m-1} & D_{m \times m} \end{pmatrix}$$

where $A = diag(p-1, p^2-1, \ldots, p^{m-1}-1)$,

$$B = \begin{pmatrix} 0 & 0 & \cdots & 0 & -\phi(p) \\ 0 & 0 & \cdots & -\phi(p^2) & -\phi(p) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & -\phi(p^{m-1}) & \cdots & -\phi(p^2) & -\phi(p) \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & -\phi(p^{m+1}) \\ \vdots & \vdots & \cdots & \vdots \\ 0 & -\phi(p^{2m-2}) & \cdots & -\phi(p^{m+1}) \\ -\phi(p^{2m-1}) & -\phi(p^{2m-2}) & \cdots & -\phi(p^{m+1}) \end{pmatrix}$$

$$\text{and } D = \begin{pmatrix} p^m - 1 - \phi(p^m) & -\phi(p^{m-1}) & \cdots & -\phi(p^2) & -\phi(p) \\ -\phi(p^m) & p^{m-1} - 1 - \phi(p^{m-1}) & \cdots & -\phi(p^2) & -\phi(p) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\phi(p^m) & -\phi(p^{m-1}) & \cdots & p^{2m-2} - 1 - \phi(p^2) & -\phi(p) \\ -\phi(p^m) & -\phi(p^{m-1}) & \cdots & -\phi(p^2) & p^{2m-1} - 1 - \phi(p) \end{pmatrix}.$$

Applying Lemma 5, we have

$$det(xI - \mathcal{Q}) = det(xI - A)det((xI - D) - C(xI - A)^{-1}B). \tag{7}$$

By evaluating Equation (7), we can verify that

$$\left\{ 0, p - 1, p^2 - 1, \dots, p^{m-1} - 1, p^{m+1} - 1, \dots, p^{2m-2} - 1, p^{2m-1} - 1 \right\}$$

are the remaining Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$. We note that all the Laplacian eigenvalues of quotient matrix $\mathcal{Q}$ are repeated with the eigenvalues obtained by $\alpha_i + \mu_{ik}(G_i)$ except $p^m - 1$. □

As $\Gamma(\mathbb{Z}_{2^m}[i]) \cong \Gamma(\mathbb{Z}_{2^{2m}})$, so for $p = 2$ in Corollary 5, we get the following.

Corollary 6. *The Laplacian eigenvalues of the zero-divisor graph $\Gamma(\mathbb{Z}_{2^m}[i])$ of Gaussian integers modulo 2^m is*

$$\left\{ 0, 1^{[\phi(2^{2m-1})]}, 2^{[\phi(2^{2m-2})1]}, \dots, (2^{m-1} - 1)^{[\phi(2^{m+1})]}, (2^m - 1)^{[\phi(2^m)-1]}, \right.$$
$$\left. (2^{m+1} - 1)^{[\phi(2^{m-1})]}, \dots, (2^{2m-2} - 1)^2, (2^{2m-1} - 1) \right\}.$$

If $m_1 = 1$ and $m_2 = 0$ in Theorem 5, we have $\Gamma(\mathbb{Z}_{p^2}) = K_{\phi(p)}$ and its Laplacian spectrum is given by the following observation.

Corollary 7. *If $n = p^2$, then the Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ is*

$$\left\{ 0, (p - 1)^{[p-2]} \right\}.$$

The following result gives the Laplacian spectrum of $\Gamma(\mathbb{Z}_{p^{N_1}q^{N_2}})$, when both N_1 and N_2 are odd. Its proof is similar to that of Theorem 5.

Theorem 6. *Let $\Gamma(\mathbb{Z}_n)$ be the zero-divisor graph of order N, where $n = p^{N_1}q^{N_2}$ and $N_1 = 2m_1 + 1 \leq 2m_2 + 1 = N_2$. The Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ consists of the eigenvalues*

$$\left\{ (p^i - 1)^{[\phi(p^{N_1-i}q^{N_2})-1]}, (q^j - 1)^{[\phi(p^{N_1}q^{N_2-j})-1]}, (pq^j - 1)^{[\phi(p^{N_1-1}q^{N_2-j})-1]}, \dots, (p^{m_1+1}q^k - 1)^{[\phi(p^{m_1}q^{N_2-j})-1]} \right.$$
$$\left. , \dots, (p^{2m_1+1}q^k - 1)^{[\phi(q^{N_2-k})-1]} \right\},$$

where $i = 1, 2, \dots, m_1, \dots, N_1$, $j = 1, 2, \dots, N_2$ and $k = 1, 2, \dots, 2m_2$. The remaining Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$ are the eigenvalues of the matrix given in (6).

In particular, if $q = 1$ in Theorem 6, we have the following result of [5].

Corollary 8. *If $n = p^{2m+1}$ for some positive integer $m \geq 2$, then the Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ is*

$$\left\{ 0, (p-1)^{[\phi(p^{2m})]}, (p^2-1)^{[\phi(p^{2m-1})1]}, \ldots, (p^{m-1}-1)^{[\phi(p^{m+2})]}, (p^m-1)^{[\phi(p^{m+1})-1]}, \right.$$
$$\left. (p^{m+1}-1)^{[\phi(p^m)]}, \ldots, (p^{2m-1}-1)^{[\phi(p^2)]}, (p^{2m}-1)^{[\phi(p)]} \right\}.$$

If $m_1 = m_2 = 0$, then $n = pq$. Therefore, by Lemmas 3 and 4, we have

$$\Gamma(\mathbb{Z}_{pq}) = Y_{pq}[\Gamma(A_p), \Gamma(A_q)] = K_2[\overline{K}_{\phi(p)}, \overline{K}_{\phi(q)}] = \overline{K}_{\phi(p)} \triangledown \overline{K}_{\phi(q)} = K_{\phi(p),\phi(q)}. \quad (8)$$

The next consequence of Theorem 6 gives the Laplacian spectrum of the complete bipartite graph $\Gamma(\mathbb{Z}_{pq})$.

Corollary 9. *The Laplacian spectrum of $\Gamma(\mathbb{Z}_{pq})$ is*

$$\left\{ 0, (q-1)^{[p-2]}, (p-1)^{[q-2]}, p+q-2 \right\}.$$

For $m = 1$ and $q = 1$ in Theorem 6, we have the following observation for $\Gamma(\mathbb{Z}_{p^3})$.

Corollary 10. *If $n = p^3$, then the Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ is*

$$\left\{ 0, (p-1)^{[p^2-p-1]}, (p^2-1)^{[p-2]} \right\}.$$

Proof. As the proper divisors of n are p and p^2, so Y_n is $K_2 : p \sim p^2$. By Lemma 4, we have

$$\Gamma(\mathbb{Z}_{p^3}) = Y_{p^3}[\Gamma(A_p), \Gamma(A_{p^2})] = K_2[\overline{K}_{\phi(p^2)}, \overline{K}_{\phi(p)}] = \overline{K}_{p(p-1)} \triangledown K_{p-1}.$$

That is, $\Gamma(\mathbb{Z}_{p^3})$ is a complete split graph of order $p^2 - 1$, with independence number $p(p-1)$. Therefore, by Theorem 4, we have $(\alpha_1, \alpha_2) = (p-1, p^2-p)$, and

$$Q(K_2) = \begin{pmatrix} p-1 & -\phi(p) \\ -\phi(p^2) & p^2-p \end{pmatrix}. \quad (9)$$

As $G_1 = \overline{K}_{p(p-1)}$, so the Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ consists of the eigenvalue $\alpha_1 = p - 1$ with multiplicity $p(p-1) - 1$, the eigenvalue $\alpha_2 + \mu_{2k}(K_{p-1}) = p^2 - p + p - 1 = p^2 - 1$ with multiplicity $p - 2$ and the other two Laplacian eigenvalues are the eigenvalues of matrix (9). □

Now, consider the case when one of N_i's is even and other is odd, say N_1 is even and N_2 is odd or N_1 is odd and N_2 is even. In the following result, first case is given and the second case can be treated similarly.

Theorem 7. *Let $\Gamma(\mathbb{Z}_n)$ be the zero-divisor graph of order N, where $n = p^{N_1} q^{N_2}$ and $m_1 < m_2$ so that $N_1 = 2m_1 < 2m_2 + 1 = N_2$. The Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ consists of the eigenvalues*

$$\left\{ (p^i-1)^{[\phi(p^{N_1-i}q^{N_2})-1]}, (q^j-1)^{[\phi(p^{N_1}q^{N_2-j})-1]}, (pq^j-1)^{[\phi(p^{N_1-1}q^{N_2-j})-1]}, \ldots, (p^{m_1}q^j-1)^{[\phi(p^{m_1}q^{N_2-j})-1]} \right.$$
$$\left. , \ldots, (p^{N_1}q^k-1)^{[\phi(q^{N_2-k})-1]} \right\},$$

where $i = 1, 2, \ldots, m_1, \ldots, N_1$, $j = 1, 2, \ldots, N_2$ and $k = 1, 2, \ldots, 2m_2$. The remaining Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$ are the eigenvalues of the matrix given in (6).

Proof. For $n = p^{N_1}q^{N_2}$, with p and q, being primes and $2 \leq N_1 = 2m_1 < 2m_2 + 1 = N_2$. The proper divisor set of n is

$$\Big\{ p, p^2, \ldots, p^{m_1}, \ldots, p^{N_1}, q, q^2, \ldots, q^{m_2+1}, \ldots, q^{N_2}, pq, pq^2, \ldots, pq^{m_2+1}, \ldots, pq^{N_2}, \ldots, p^{m_1}q, p^{m_1}q^2,$$

$$\ldots, p^{m_1}q^{m_2}, p^{m_1}q^{m_2+1}, \ldots, p^{m_1}q^{N_2}, \ldots, p^{N_1}q, p^{N_1}q^2, \ldots, p^{N_1}q^{m_2}, p^{N_1}q^{m_2+1}, \ldots, p^{N_1}q^{N_2-1} \Big\}.$$

Now by the definition of Y_n, the adjacency relations are

$$p^i \sim p^j q^{N_2}, \ i + j \geq N_1, \ \text{for } i = 1, 2, \ldots, N_1$$

$$q^i \sim p^{N_1} q^j, \ i + j \geq N_2, \ \text{for } i = 1, 2, \ldots, N_2,$$

$$pq^i \sim p^k q^j, \ i + j \geq N_2, \ \text{for } i = 1, 2, \ldots, N_2 \text{ and } k \geq 2m_1 - 1,$$

$$\vdots$$

$$p^{m_1}q^i \sim p^k q^j, \ i + j \geq N_2, \ \text{for } i = 1, 2, \ldots, N_2 \text{ and } k \geq m_1$$

$$\vdots$$

$$p^{M_1}q^i \sim p^k q^j, \ i + j \geq N_2, \ \text{for } i = 1, 2, \ldots, N_2 - 1 \text{ and } k \geq 0.$$

Also, the cardinalities of A_{d_i}'s are

$$|A_{p^i}| = \phi(p^{N_1-i}q^{N_2}), \ |A_{q^j}| = \phi(p^{N_1}q^{N_2-j}), \ |A_{pq^j}| = \phi(p^{N_1-1}q^{N_2-j}), \ldots, \ |A_{p^{m_1}q^j}| = \phi(p^{m_1}q^{N_2-j}),$$

$$\ldots, |A_{p^{N_1-1}q^j}| = \phi(pq^{N_2-j}), |A_{p^{N_1}q^k}| = \phi(q^{N_2-k}),$$

where $i = 1, 2, \ldots, N_1$, $j = 1, 2, \ldots, N_2$ and $k = 1, 2, \ldots, N_2 - 1$. Further by using Lemma 3, we have

$$G_i = \begin{cases} \Gamma(A_{d_{p^i}}) = \overline{K}_{\phi(p^{N_1-i}q^{N_2})}, & 1 \leq i \leq N_1, \\ \Gamma(A_{d_{q^j}}) = \overline{K}_{\phi(p^{N_1}q^{N_2-j})}, & 1 \leq j \leq N_2, \\ \Gamma(A_{d_{p^iq^j}}) = \overline{K}_{\phi(p^{N_1-i}q^{N_2-j})}, & 1 \leq i \leq m_1 - 1 \text{ and } 1 \leq j \leq N_2 \\ & \text{or } 1 \leq i \leq N_1 \text{ and } 1 \leq j \leq m_2, \\ \Gamma(A_{d_{p^iq^j}}) = K_{\phi(p^{N_1-i}q^{N_2-j})}, & m_1 \leq i \leq N_1 \text{ and } m_2 \leq j \leq N_2. \end{cases} \quad (10)$$

Thus, by Lemma 4, the joined union of $\Gamma(\mathbb{Z}_n)$ is

$$\Gamma(\mathbb{Z}_n) = Y_n \Big[\overline{K}_{\phi(p^{N_1-1}q^{N_2})}, \ldots, \overline{K}_{\phi(p^{m_1}q^{N_2})}, \ldots, \overline{K}_{\phi(q^{N_2})}, \overline{K}_{\phi(p^{N_1}q^{N_2-1})}, \ldots, \overline{K}_{\phi(p^{N_1}q^{m_2})}, \ldots, \overline{K}_{\phi(p^{N_1})},$$

$$\overline{K}_{\phi(p^{N_1-1}q^{N_2-1})}, \ldots, \overline{K}_{\phi(p^{N_1-1}q^{m_2})}, \ldots, \overline{K}_{\phi(p^{N_1-1})}, \ldots, \overline{K}_{\phi(p^{m_1}q^{N_2-1})}, \ldots, K_{\phi(p^{m_1}q^{m_2-11})},$$

$$K_{\phi(p^{m_1}q^{m_2})}, \ldots, K_{\phi(p^{m_1})}, \ldots, K_{\phi(q^{N_2-1})}, \ldots, K_{\phi(q^{m_2-1})}, K_{\phi(q^{m_2})}, \ldots, K_{\phi(q)} \Big].$$

By using Theorems 1 and 4, the value of α_i's are

$\alpha_i = p^i - 1$, for $i = 1, 2, \ldots, N_1$

$\alpha_i = q^j - 1$, for $i = N_1 + 1, \ldots, N_1 + N_2$ and $j = 1, 2, \ldots, m_2 + 1, \ldots, N_2,$

$\alpha_i = pq^j - 1$ for $i = N_1 + N_2 + 1, \ldots, N_1 + 2N_2$ and $j = 1, 2, \ldots, m_2 + 1, \ldots, N_2,$

$\vdots$

$\alpha_i = p^{m_1} q^j - 1$, for $i = N_1 + m_1 N_2 + 1, \ldots, N_1 + m_1 N_2 + m_2 - 1$ and $j = 1, 2, \ldots, m_2,$

$\alpha_i = p^{m_1} q^j - 1 - \phi(p^{m_1} q^j)$, for $i = N_1 + m_1 N_2, \ldots, N_1 + (m_1 + 1)N_2$ and $j = m_2 + 1, \ldots, N_2,$

$\vdots$

$\alpha_i = p^{N_1} q^j - 1$, for $i = N_1 + N_1 N_2 + 1, \ldots, N_1 + N_1 N_2 + m_2$ and $j = 1, 2, \ldots, m_2,$

$\alpha_i = p^{N_1} q^j - 1 - \phi(q^{N_2 - j})$, for $i = N_1 + N_1 N_2 + m_2 + 1, \ldots, N_1 + N_1 N_2 + N_2 - 1$

and $j = m_2 + 1, \ldots, 2m_2.$

Again, applying Theorem 4 and using Equation (10), we see that

$$\alpha_1 + \mu_{1k}(G_1) = \alpha_1 + 0 = p - 1$$

is the Laplacian eigenvalue of $\Gamma(\mathbb{Z}_n)$ with multiplicity $\phi(p^{2m_1 - 1} q^{2m_2 + 1}) - 1$. Similarly, the other Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$ are as in the statement. $\square$

Next, we find the Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$ when n is the product of three primes.

Theorem 8. *The Laplacian spectrum of* $\Gamma(\mathbb{Z}_{pqr})$ *consists of the eigenvalues*

$$\left\{ (p-1)^{[\phi(qr)-1]}, (q-1)^{[\phi(p1)-1]}, (r-1)^{[\phi(pq)-1]}, (pq-1)^{[\phi(r)-1]}, (pr-1)^{[\phi(q)-1]}, (qr-1)^{[\phi(p)-1]} \right\}.$$

Proof. Let $n = pqr$. Then p, q, r, pq, pr and qr are the proper divisors of n and Y_n is the graph $G_6 : q \sim pr \sim pq \sim r$, $pr \sim qr \sim p$ and $pq \sim qr$, i.e., Y_n is a unicyclic graph with pendent vertices at each vertex of cycle as shown in Figure 2. Ordering the vertices by increasing divisor sequence and applying Lemma 4, we have

$$\Gamma(\mathbb{Z}_{30}) = Y_{30}[\overline{K}_8, \overline{K}_4, \overline{K}_{24}, \overline{K}_4, \overline{K}_2, \overline{K}_1].$$

By Theorem 4, value of α_i's are

$\alpha_1 = \phi(p) = p - 1, \alpha_2 = \phi(q) = q - 1, \alpha_3 = \phi(r) = r - 1, \alpha_4 = \phi(pq) + \phi(p) + \phi(q) = pq - 1,$
$\alpha_5 = \phi(pr) + \phi(p) + \phi(r) = pr - 1, \alpha_6 = \phi(qr) + \phi(q) + \phi(r) = qr - 1.$

Since each of G_i is a null graph, so the Laplacian eigenvalues of $\Gamma(\mathbb{Z}_{pqr})$ are $p - 1$ with multiplicity $\phi(qr) - 1$, $q - 1$ with multiplicity $\phi(pr) - 1$, $r - 1$ with multiplicity $\phi(pq) - 1$, $pq - 1$ with multiplicity $\phi(r) - 1$, $pr - 1$ with multiplicity $\phi(q) - 1$, and $qr - 1$ with multiplicity $\phi(p) - 1$. The remaining Laplacian eigenvalues of $\Gamma(\mathbb{Z}_{pqr})$ are the eigenvalues of the following matrix

$$\begin{pmatrix} \phi(p) & 0 & 0 & 0 & 0 & -\phi(p) \\ 0 & \phi(q) & 0 & 0 & -\phi(q) & 0 \\ 0 & 0 & \phi(r) & -\phi(q) & 0 & 0 \\ 0 & 0 & -\phi(pq) & pq-1 & -\phi(q) & -\phi(p) \\ 0 & -\phi(pr) & 0 & -\phi(r) & pr-1 & -\phi(p) \\ -\phi(qr) & 0 & 0 & -\phi(r) & -\phi(q) & qr-1 \end{pmatrix}.$$

$\square$

Theorem 8 can be generalized for arbitrary product of distinct primes. Although it is hard to find the Laplacian spectra of $\Gamma(\mathbb{Z}_n)$ with canonical decomposition of n, it is interesting and can explore various properties of $\mathbb{Z}_n$ and the structure of its associated zero-divisor graph. The spectral study of zero-divisor graphs of rings may open research work as in the case of Cayley graphs.
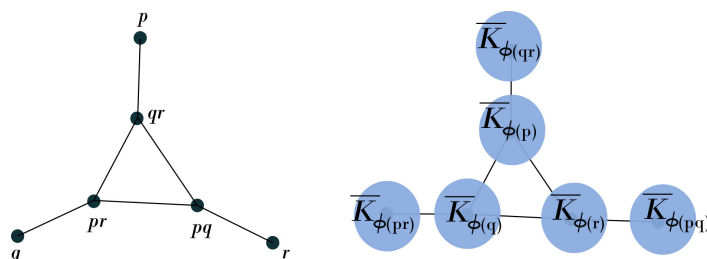

Figure 2. Proper divisor graph Υ_{pqr} and zero-divisor graph $\Gamma(\mathbb{Z}_{pqr})$.

4. Conclusions and Comments

Let $\mathbb{M}_n(\mathbb{C})$ be the set of all square matrices of order n with complex entries. The trace norm of a matrix $M \in \mathbb{M}_n(\mathbb{C})$ is defined as $\|M\|_* = \sum_{i=1}^{n} \sigma_i(M)$, where $\sigma_1(M) \geq \sigma_2(M) \geq \cdots \geq \sigma_n(M)$ are the singular values of M (that is the square roots of the eigenvalues of MM^*, where M^* is the complex conjugate of M). In case of symmetric matrices, the singular values coincide with the absolute values of the eigenvalues, i.e., if $\sigma_i(M)$ are the singular values and $\lambda_i(M)$, $i = 1, 2, \ldots, n$, are the eigenvalues of M, then $\sigma_i(M) = |\lambda_i(M)|$. Thus, the sum of the absolute values of eigenvalues of the matrix $L(\Gamma(Z_n)) - \frac{2m}{n} I_n$ is the trace norm of $L(\Gamma(Z_n)) - \frac{2m}{n} I_n$, where I_n is the identity matrix of order n. It is an interesting problem in Matrix theory, to determine among a given class of matrices the matrix (or the matrices) which attain the maximum value and the minimum value for the trace norm. The trace norm of matrices associated with the graphs and digraphs are extensively studied. For some recent papers in this direction see [20,21] and the references therein.

In spectral graph theory, the trace norm is studied under the name graph energy. Gutman and Zhou [22] defined the *Laplacian energy* of G as

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|.$$

Using the fact that $\sum_{1=i}^{n-1} \mu_i = 2m$, from [23], we have

$$LE(G) = 2 \left(\sum_{i=1}^{\sigma} \mu_i - \sigma \overline{d} \right) = 2 \max_{1 \leq k \leq n} \left(\sum_{1=i}^{k} \mu_i - k \overline{d} \right), \tag{11}$$

where σ is the number of Laplacian eigenvalues greater than or equal to the average degree $\overline{d}$. We note that $\sum_{1=i}^{k} \mu_i$ is actually the *Ky Fan k-norm*, which for positive semi-definite matrices is the sum of k largest eigenvalues. The parameter σ is an active component of the present research and some work mostly on trees can be found in the literature [24]. In fact, it is shown in [25] that the Laplacian energy has remarkable chemical applications beyond the molecular orbital theory of conjugated molecules. For some recent works on Laplacian energy and related results, we refer to [2,26–28] and the references therein.

In case of $n = p^2$, $n = p^3$ and $n = pq$, the trace norm of $\mathcal{L} = L(\Gamma(Z_n)) - \frac{2m}{n} I_n$ are

$$2(p-2), \quad \frac{2(2p+q^2-pq-2)}{p+q-2}, \quad \text{and} \quad \frac{2(2p^3-4p^2-p+3)}{p^3-1}.$$

Similarly, Laplacian energy of $\Gamma(\mathbb{Z}_n))$ can be discussed for other values of n and various upper bounds and lower bounds can be obtained.

As zero-divisor graph of $\mathbb{Z}_n$ has been written in terms of the joined union, where components are either cliques or their complements, but, in general the zero-divisor graphs of ring R cannot be expressed as the joined union of graphs. So, their spectral analysis becomes difficult. No general method is yet available in discussing the spectra of zero-divisor graphs of rings such as $\mathbb{Z}_n[i], \mathbb{Z}_p \times \mathbb{Z}_q$, $(p \neq q), \mathbb{Z}_p[i] \times \mathbb{Z}_q[i]$, $(p \neq q)$ and many other zero-divisor graphs associated with commutative as well as non-commutative rings. Also relating spectral properties with the graph invariants such as connectivity, chromatic number, matching number and other parameters are very interesting problems.

Author Contributions: Investigation, B.A.R., S.P., T.A.N. and Y.S.; Writing—original draft, B.A.R., S.P., T.A.N. and Y.S.; Writing—review and editing, B.A.R., S.P., T.A.N. and Y.S. All authors have read and agreed to the published version of the manuscript.

Informed Consent Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Cvetković, D.M.; Rowlison, P.; Simić, S. *An Introduction to Theory of Graph Spectra Spectra of Graphs, Theory and Application*; Lonndon Math. S. Student Text, 75; Cambridge University Press: Cambridge, UK, 2010.
2. Pirzada, S.; Ganie, H.A. On the Laplacian eigenvalues of a graph and Laplacian energy. *Linear Algebra Appl.* **2015**, *486*, 454–468.
3. Beck, I. Coloring of a commutative rings. *J. Algebra* **1988**, *116*, 208–226.
4. Anderson, D.F.; Livingston, P.S. The zero-divisor graph of a commutative ring. *J. Algebra* **1999**, *217*, 434–447.
5. Chattopadhyay, S.; Patra, K.L.; Sahoo, B.K. Laplacian eigenvalues of the zero-divisor graph of the ring $\mathbb{Z}_n$. *Linear Algebra Appl.* **2020**, *584*, 267–286.
6. Magi, P.M.; Jose, S.M.; Kishore, A. Adjacency matrix and eigenvalues of the zero-divisor graph $\Gamma(\mathbb{Z}_n)$. *J. Math. Comput. Sci.* **2020**, *10*, 1285–1297.
7. Pirzada, S.; Rather, B.A.; Aijaz, M.; Chishti, T.A. On distance signless Laplacian spectrum of graphs and spectrum of zero divisor graphs of $\mathbb{Z}_n$. *Linear Multilinear Algebra* **2020**, doi:10.1080/03081087.2020.1838425.
8. Pirzada, S.; Rather, B.A.; Shaban, R.U.M. On signless Laplacian spectrum of zero divisor graphs of the ring $\mathbb{Z}_n$. *Korean J. Math.* **2021**, in print.
9. Pirzada, S.; Rather, B.A.; Chishti, T.A. On distance Laplacian spectrum of zero divisor graphs $\mathbb{Z}_n$. *Carpathian Math. Publ.* **2021**, in print.
10. Shang, Y. Lower bounds for the Estrada index using mixing time and Laplacian spectrum. *Rocky Mt. J. Math.* **2013**, *43*, 2009–2016.
11. Young, M. Adjacency matrices of zero-divisor graphs of integer modulo n. *Involve* **2015**, *8*, 753–761.
12. Anderson, D.F.; Weber, D. The zero-divisor graph of a commutative ring without identity. *Int. Elect. J. Algebra* **2018**, *223*, 176–202.
13. Pirzada, S.; Aijaz, M.; Imran, M. On zero-divisor graphs of the ring $\mathbb{Z}_n$. *Afr. Math.* **2020**, *31*, 727–737.
14. Shang, Y. A note on the commutativity of prime near-rings. *Algebra Colloq.* **2015**, *22*, 361–366.
15. Pirzada, S. *An Introduction to Graph Theory*; Orient BlackSwan: Hyderabad, India, 2012.
16. Koshy, T. *Elementary Number Theory with Applications*, 2nd ed.; Academic Press: Cambridge, MA, USA, 2007.
17. Spiroff, S.; Wickham, C. A zero-divisor graph determined by equivalence classes of zero-divisors. *Common. Algebra* **2011**, *39*, 2338–2348.
18. Horn, R.; Johnson, C. *Matrix Analysis*; Cambridge University Press: Cambridge, UK, 1985.
19. Cardoso, D.M.; Freitas, M.A.D.; Martins, E.N.; Robbiano, M. Spectra of graphs obtained by a generalization of the join of graph operations. *Discret. Math.* **2013**, *313*, 733–741.
20. Monsalve, J.; Rada, J. Oriented bipartite graphs with minimal trace norm. *Linear Multilinear Algebra* **2019**, *67*, 1121–1131.
21. Nikiforov, V. The trace norm of r-partite graphs and matrices. *C. R. Acad. Sci. Paris Ser. I* **2015**, *353*, 471–475.
22. Gutman, I.; Zhou, B. Laplacian energy of a graph. *Linear Algebra Appl.* **2006**, *414*, 29–37.
23. Fritscher, E.; Hoppen, C.; Rocha, I.; Trevisan, V. On the sum of the Laplacian eigenvalues of a tree. *Linear Algebra Appl.* **2011**, *435*, 371–399.
24. Das, K.C.; Mojallal, S.A. Open problems on σ-invariant. *Taiwan. J. Math.* **2019**, *23*, 1041–1059.
25. Radenkovic, S.; Gutman, I. Total electron energy and Laplacian energy: How far the analog goes? *J. Serb. Chem. Soc.* **2007**, *72*, 1343–1350.

26. Alhevaz, A.; Baghipur, M.; Das, K.C.; Shang, Y. Sharp bounds on (generalized) distance energy of graphs. *Mathematics* **2020**, *8*, 426.

27. Ganie, H.A.; Pirzada, S.; Rather, B.A.; Trevisan, V. Further developments on Brouwer's conjecture for the sum of Laplacian eigenvalues of graphs. *Linear Algebra Appl.* **2020**, *588*, 1–18.

28. Ganie, H.A.; Pirzada, S.; Rather, B.A.; Shaban, R. On Laplacian eigenvalues of graphs and Brouwer's conjecture. *J. Ramanujan Math. Soc.* **2021**, *36*, 1–9.