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I. INTRODUCTION

Scholars in economics have highlighted a number of game-theoretic contradictions and paradoxes in which individual decision-making in real-world situations is at odds with what is predicted by game theory (Goeree and Holt 2001; Luce and Raiffa 1957; Selten 1978; Rios, Rios and Banks 2009; Rosenthal 1981). In the real world people are often more cooperative than that predicted by the outcome of the Nash equilibrium (Nash 1951) in game theory (Camerer 2003). In this paper we provide a reason for this unreason[1]. We examine a number of probabilistic choice models by which people might form their beliefs to play their strategies in a game theoretic setting in order to propose alternative equilibrium concepts to the Nash equilibrium. In particular, we examine the Blavatskyy model (2011) which satisfies both stochastic dominance and weak stochastic transitivity and compare it to the Returns Based Beliefs (RBB) model (2012). We choose these two models as the primary focus of comparison for our study because they can be derived from a set of behavioral axioms. We then compare and contrast them to other probabilistic choice models of decision making in game theory such as the Quantal Response Equilibrium (QRE) model proposed by McKelvey and Palfrey (1995), the Boundedly Rational Nash Equilibrium (BRNE) model of Chen, Friedman and Thisse (1997), the Random Belief Equilibrium (RBE) proposed by Friedman and Mezzetti (2005) and the Utility Proportional Beliefs (UPB) model proposed by Bach and Perea (2014). Blavatskyy (2011) showed that their model fitted experimental data better than other existing probabilistic models. Loomes et al (2014) showed that the Blavatskyy's claim of empirical fit to data is not as robust as the initial claim. However, these empirical fit estimations were done for a single decision maker facing risky choices. We develop these papers further by testing the model fit when there are two players making decisions under a game theoretic experimental setting. In particular, we test the closeness of fit of the Blavatskyy model and the RBB model, and by incorporating loss aversion, using data from Selten and Chmura (2008) for completely mixed 2×2 games. In doing so, we

[1] We attribute this expression to Herbert Simon (1987) who wrote, 'Sometimes the term rational (or logical) is applied to decision making that is consciously analytic, the term non-rational to decision making that is intuitive and judgmental, and the term irrational to decision making and behavior that responds to the emotions or that deviates from action chosen "rationally". We will be concerned then, with the non-rational and the irrational components of individual decision making and behavior. Our task, you might say, is to discover the reason that underlies unreason.' (Simon, 1987, p.57)

compare and contrast the Blavatskyy and the RBB models with other stationary concepts and discuss when one is better than the others. Finally, the Blavatskyy model was developed in the case of binary choice and we expand it to cover more than two choices.

We make three contributions to the literature. First we outline the underlying assumptions of the different probabilistic choice models and fully explicate them in terms of probabilistic actions, probabilistic beliefs and hence their epistemic characterizations. In doing so, we are able to provide a more complete account of the similarities and differences of these probabilistic choice models and relate them to how agents might make decisions under uncertainty. Second, we test the model predictions using empirical data and show which models perform better under which conditions. In doing so, we provide a more nuanced understanding of the predictions of different types of probabilistic models. There are often no clear winners as it depends on the type of games and the assumptions used to test these models. Third, we extend the Blavatskyy (2010) model which was developed to consider games with two actions to cases where there are three or more actions. In doing so, we are able to show how the results from the Blavatsky model carries over to more complex actions and hence the ability to compare and contrast better with other probabilistic choice models.

Our paper has several theoretical and managerial implications. First, the paper provides the basis for systematically developing probabilistic choice models based on the underlying axioms and being able to compare and contrast them with Nash equilibrium based models. These probabilistic choice models would enable researchers to more clearly uncouple the underlying assumptions behind the Nash equilibrium and hence be able to better understand why the results are similar or different. Such a process of building on the axioms systematically is needed to evaluate the predictions of these models. To use an analogy, the comparisons between the probabilistic choice models do not have the simple flavour of a horse race but a more nuanced understanding of the physiological structure of the horses in providing insights on which model might be better under different circumstances. Finally, the probabilistic choice models could deliver predictions that are opposite to the Nash equilibrium predictions which have been shown in Velu, Iyer and Gair (2012). Therefore, understanding when a model's assumptions might fit the real world would provide policymakers and managers with the tools to make more reliable predictions and hence take optimal actions.

II. CONCEPTUAL OUTLINE

The tendency for people to cooperate in game theoretic settings more than that predicated by Nash equilibrium could be attributed to altruism, genetic relatedness, reputation enhancement and cultural factors among others (Camerer 2003). The strategic interaction in a game theoretic setting has the property of interdependence of payoff whereby each player's payoff depends on what the other player does and vice versa. In such a setting each player needs to form beliefs by assigning probabilities to the possible actions of the other player and to choose a utility-maximizing action. However, the opponent is also doing the same. In the Nash equilibrium the issue of belief formation is resolved by assuming that players will choose actions in equilibrium, i.e., actions such that no single player gains from deviating if others do not deviate. Hence, the players playing best response in the sense of a utility maximizing response to each of the opponent's actions would result in the Nash equilibrium which would then be the basis of forming beliefs in the first place. However, some would argue that the interdependence in beliefs between players could make any possible beliefs or probability assignment feasible which creates strategic uncertainty (Brandenberger 1996; Van Huyck et al 1990). Therefore, an alternative formulation calls upon a decision theoretic approach to belief formation based on a set of axioms to choose one action over another when there is uncertainty. The Blavatskyy and RBB are two such models which we explore further in this paper. The Blavatskyy model uses expected utility as its core theory and assumes a number of axioms. These axioms are completeness, weak stochastic transitivity, continuity, common consequence independence, outcome monotonicity and odds ratio independence. The RBB model may also be derived from expected utility theory via a simple modification of the axiom of common consequence independence. We shall discuss these axioms later in the paper. The Blavatskyy model is based on binary choice calculation of another lottery, called the Greatest Lower Bound (GLB) which is the best lottery dominated by both the original lotteries. The probability that one lottery is chosen over the other is based on the difference between the expected utility of that lottery and the expected utility of the GLB divided by the sum of such a difference for both the lotteries. In the case of the RBB model the probabilistic response follows the Luce (1959) probabilistic choice model whereby it is the expected utility of each choice divided by the sum of expected utilities of all choices.

In the Blavatskyy and RBB models, we propose three modifications to standard Nash equilibrium for finite normal form games. The first modification to the Nash equilibrium is to introduce probabilistic beliefs. The second is to replace best replying by each agent with either the Blavatskyy probabilistic choice or the Luce's (1959) probabilistic choice model in the case of RBB. The third modification is the interpretation of *randomization*. In doing so the models borrow from and build on three previous well established models, namely the Quantal Response Equilibrium (QRE), the Boundedly Rational Nash Equilibrium (BRNE) model and the Random Belief Equilibrium (RBE). The Blavatskyy and RBB models borrow from the QRE and BRNE in formulating probabilistic actions whereby the actions with higher expected payoffs are played with higher probabilities. However, the models differ from the QRE and BRNE in the interpretation of the probabilistic best response as arising not from errors but because of uncertainty as to the actions of the other player. The models also borrow from RBE in formulating probabilistic beliefs but they differ from the RBE whereby beliefs are uncertain and not random. We shall elaborate on the difference between uncertain and random beliefs below. Therefore, both the Blavatskyy and the RBB models have both probabilistic actions and probabilistic beliefs. This is in contrast to other non-Nash equilibrium probabilistic choice models which typically assume either probabilistic actions (such as the QRE and BRNE), or probabilistic beliefs (such as the RBE or the Utility Proportional Beliefs (UPB) model proposed by Bach and Perea, 2014), but not both.

Our first modification to the Nash equilibrium introduces probabilistic beliefs. We assume that the players' beliefs about the choice of other players are probabilistic i.e., the player holds the beliefs that his opponent will choose actions with particular probabilities. We assume that players' probabilistic beliefs are in equilibrium with reference to each other. In Nash equilibrium, each player chooses the action that maximizes their returns subject to the opponent's possible actions and no player can gain by changing their strategy unilaterally. Therefore, in the Nash equilibrium the players form beliefs in such a way as to be consistent with the actions being in equilibrium. However, a player's actions are determined by her beliefs about other players which may depend upon their real-life contexts such as custom or history (Harsanyi 1982). In order to address such issues, scholars of risk analysis and game theory have highlighted the difference between game theoretic and decision analytic solution concepts[2] (Rios, Rios and Banks 2009, pp. 845-849). The Blavatskyy and RBB models use a decision analytic approach where individuals form subjective beliefs based on subjective probabilities[3] over the actions of their opponent. Therefore, consistent with a decision theoretic perspective, players adopt strategies on the basis of their respective subjective beliefs or personal interpretation of probability. The RBB model uses the Luce (1959) probabilistic choice model as the basis for player actions and that subjective beliefs are formed from the assumption that opponents respond to uncertainty in a similar way. The Blavatskyy model assumes the response function as described earlier and assumes that subjective beliefs are formed from the assumption that opponents respond to uncertainty in a similar way.

A natural extension of probabilistic beliefs is probabilistic actions. Therefore, the second modification to the Nash equilibrium is to replace best replying by each agent with a version of Luce's (1959) probabilistic choice model in which players choose strategies with probabilities in proportion to their expected returns – returns-based beliefs (RBB) or the Blavatskyy response choice respectively. Mathematically, the probabilistic actions part of the RBB response is a special case of the BRNE. As the BRNE is a type of QRE, the RBB probabilistic action can also be mathematically interpreted as a QRE. However, the Luce rule also arises naturally in several other contexts. Research has shown that optimization under the notion of stochastic preference implies the Luce rule (Gul, Natenzon and Pesendorfer 2014) and the Luce rule has similarities to a stochastic utility representation[4] (Blavatskyy 2008). Moreover, the RBB model attempts to represent *conscious* decision making in the presence of uncertainty and therefore any probabilistic action must be simply computable. In the RBB version of the Luce model, players choose actions in proportion to

[2] Kadane and Larkey (1982) in their seminal paper propose a decision analytic solution concept over the game theoretic solution concept. Also, see Bono and Wolpert (2009).

[3] Subjective probability is the probability that a person assigns to a possible outcome, or some process based on his own judgment, the likelihood that the outcome will be obtained which then influences the strategies chosen (DeGroot 1975, p. 4; Savage 1954).

[4] Blavatskyy (2008) shows that the choice probabilities have a stochastic utility representation if they can be written as a non-decreasing function of the difference in the expected utilities of the actions. Studies have shown that the stochastic utility representation has properties that are similar to the Luce (1959) choice rule when the random variable has a Gumbel (double exponential) distribution function (e.g., the cumulative distribution function of the logistic distribution) (Yellot 1977). In fact, Yellot derives the logistic QRE solution with parameter $\lambda = 1$ (see Section III A 1).

their expected returns, which could quite conceivably be evaluated by players participating in a game. The interpretation of why a player adopts a mixed strategy rather than choosing the pure strategy that maximizes their returns is that there is uncertainty regarding the conjecture about the choice by the other players[5] (Basu 1990; Brandenberger and Dekel 1987). The use of a consciously probabilistic action model naturally gives rise to priors for the probabilistic beliefs. If a player believes that their opponent thinks like them, then they will form probabilistic beliefs for their opponent according to their own probabilistic action model, i.e., according to the RBB rule. The Blavatskyy model on the other hand captures the essence of the Luce type response function but calculates the incremental returns of a choice relative to an guaranteed minimum lottery (GLB) as a basis for forming the prior. In this way the Blavatskyy and the RBB approaches are able to provide a basis for players to form priors which is in contrast to the QRE, BRNE and RBE which are silent on where the priors might emerge from.

The third modification is the interpretation of *randomization*. In the Blavatskyy and RBB models uncertainty comes from subjective beliefs which gives rise to subjective probabilities. Such subjective beliefs driven by the perceptive and evaluational premises of an individual can create 'strategic uncertainty' regarding the conjecture about the choice of the other player. We define strategic uncertainty as uncertainty concerning the actions and beliefs (and beliefs about the beliefs) of others (Brandenberger 1996). Researchers have argued that strategic uncertainty can arise even when all possible actions and returns are completely specified and are common knowledge (Ellsberg 1959; Van Huyck et al 1990). The rational decision-maker has to form beliefs about the strategy that the other decision-maker will use as a result of strategic uncertainty. Therefore, Blavatskyy and RBB model makes a distinction here between an uncertain player, who makes a conscious decision when faced with uncertain information about their opponent, and a random player, whose interpretation of the game changes subconsciously every time they encounter it. In the RBB model, when

[5] We are not assuming that the opponent is using a randomized strategy. The mixture merely reflects the representation of player 1's belief about player 2. As Wilson (1986, p.47) points out, although it makes little difference to the mathematics, conceptually this distinction between randomization and subjective beliefs to explain the mixed strategies is a pertinent one. This interpretation is also sympathetic to Harsanyi's (1973) purification interpretation of mixed strategy where mixing represents uncertainty in a player's mind about how other players will choose their strategies, rather than deliberate randomization (Morris 2008).

a given player plays a given game against a given opponent, the player forms the same beliefs and has the same actions. These beliefs and actions are probabilistic due to the player's uncertainty about real-life experience such as custom of the opponent. The Blavatskyy model captures the uncertainty through an incremental expected returns type framework while RBB encapsulates this uncertainty through the use of the Luce choice rule. This is different to the random actions/beliefs used in other equilibrium concepts, in which the interpretation is that every time a player plays a game, his actions or beliefs change probabilistically, and the "equilibrium" solution is an ensemble average. This is a subtle distinction, but there is also a mathematical difference — in the Blavatskyy and RBB models, we compute a player's best response to the average action, as opposed to the average of his best response to a particular action. The concept of randomization in Blavatskyy and RBB is not to make the opponent indifferent to their actions as is the basis of Nash equilibrium and other corresponding error driven models such as QRE, BRNE and RBE but it is the result of the strategic uncertainty in the opponents actions.

The efficacy of a particular equilibrium concept to explain experimental data is normally assessed by comparing the observed outcome, i.e., the strategies chosen by each player in each game, to the predictions of the model of interest. The mathematical equivalence of the QRE and RBB means that it is not possible to distinguish between these two models using only data of that form. However, if the players in a game were asked not only to choose a strategy to play but also to state, on a scale of belief, what they thought their opponent was going to do, it would be possible to differentiate the models. The QRE does not provide a mechanism through which players form beliefs, but is based on the notion of errors. A person with given beliefs about their opponent will make a different strategy choice each time, depending on the particular errors they make. By contrast, under the Blavatskyy, RBB and RBE concepts, the players will choose the same particular strategy profile every time they have the same beliefs about their opponent. The Blavatkyy and RBB and RBE can themselves be distinguished by the nature of the beliefs — under the RBE the beliefs change each time the game is played, while under Blavatksyy and RBB the beliefs are also fixed. Data of the form necessary to carry out these tests is not currently available, so we will not be able to do this comparison here. However, it is important to emphasize that the RBB and Blavatskyy can be experimentally distinguished from the other equilibrium concepts if appropriate data is collected when running the experiment.

The next section develops axiomatically the returns-based beliefs model in relation to the Blavatskyy (2011) model and compares these two models with other probabilistic choice models. Section IV tests the applicability of the RBB, Blavatskyy and other probabilistic choice models to empirical data from completely mixed 2×2 games from Selten and Chmura's (2008). Section V extends the Blavatskyy to 3x3 games. Section VI concludes.

III. THE MODEL

In this section we develop the basic elements of a one-shot game and then introduce discrete choice theory based on subjective probabilities. We use below a particular variation of the one-shot game formulation of McKelvey and Palfrey (1995). Let (N, S, π) be a finite game. N denotes the set of players. Each player $i \in N$ has a set of pure strategies, S^i with elements s^i. The set of strategy profiles of all players other than i is $S^{-i} = \otimes_{j \neq i} S^j$ with elements, s^{-i}. The benefit a player i derives from playing a strategy profile $s^i \in S^i$ is $\pi^i_{s^i}; \pi^i = \{\pi^i_{s^i}\}_{s^i \in S^i}$. Each player knows who is in the game, N, and the strategy sets available to each other, $S^i \; \forall \; i \in N$. However, each player is uncertain of the belief structure held by the other player. We discuss below our approach to how players form a reasonable subjective probability belief structure.

In Blavatskyy (2008), the author described a stochastic utility theorem. This showed that there is a unique form for the decision rule for choosing between lotteries when you impose a set of five axioms. We use L_i to denote a lottery (a probability distribution on a set of possible outcomes) and $\Pr(L_1, L_2)$ to denote the probability that the decision maker chooses to play lottery L_1 when presented with a choice between L_1 and L_2. If $\{p^1_i\}$ are the probabilities of the possible outcomes $\{x_1, \ldots, x_n\}$ in lottery L_1 and $\{p^2_i\}$ are the corresponding probabilities in lottery L_2, then we can also define a combined lottery $\alpha L_1 + (1-\alpha)L_2$, for $\alpha \in [0, 1]$, as the lottery with probabilities $\alpha p^1_i + (1-\alpha)p^2_i$. This combined lottery can also be interpreted as the decision maker playing lottery L_1 with probability *alpha* and lottery L_2 with probability $(1-\alpha)$. Using this notation, the five axioms of the stochastic utility theorem are

1. **Completeness**: For any two lotteries L_1, L_2, $\Pr(L_1, L_2) + \Pr(L_2, L_1) = 1$.
2. **Strong Stochastic Transitivity**: For any three lotteries L_1, L_2, L_3, if $\Pr(L_1, L_2) \geq 1/2$ and $\Pr(L_2, L_3) \geq 1/2$ then $\Pr(L1, L_3) \geq$max$\{\Pr(L_1, L_2), \Pr(L_2, L_3)\}$.

3. **Continuity**: For any three lotteries L_1, L_2, L_3, the sets $\{\alpha \in [0,1] | \Pr(\alpha L_1 + (1-\alpha)L_2, L_3) \geq 1/2\}$ and $\{\alpha \in [0,1] | \Pr(\alpha L_1 + (1-\alpha)L_2, L_3) \leq 1/2\}$ are closed.

4. **Common Consequence Independence**: For any four lotteries L_1, L_2, L_3, L_4 and any probability $\alpha \in [0,1]$, $\Pr(\alpha L_1 + (1-\alpha)L_3, \alpha L_2 + (1-\alpha)L_3) = \Pr(\alpha L_1 + (1-\alpha)L_4, \alpha L_2 + (1-\alpha)L_4)$.

5. **Interchangeability**: For any three lotteries L_1, L_2, L_3, if $\Pr(L_1, L_2) = 1/2 = \Pr(L_2, L_1)$ then $\Pr(L_1, L_3) = \Pr(L_2, L_3)$.

A probability assignment $\Pr(L_1, L_2)$ satisfies these axioms if and only if there exists an assignment of real numbers u_i to every outcome x_i and a non-decreasing function $\Psi : \mathbb{R} \to [0,1]$ such that $\Pr(L_1, L_2) = \Psi(\sum u_i p_i^1 - \sum u_i p_i^2)$.

In a game theoretic context, the real numbers $\{u_i\}$ can be interpreted as (subjective) payoffs for each action. Taking the function $\Psi(x)$ to be $1/(1 + \exp(-\lambda x))$ the choice rule reduces to

$$\Pr(L_1, L_2) = \frac{\exp\left(\lambda \sum_{i=1}^n u_i p_i^1\right)}{\exp\left(\lambda \sum_{i=1}^n u_i p_i^1\right) + \exp\left(\lambda \sum_{i=1}^n u_i p_i^2\right)}$$

which is the QRE choice rule.

Axiom 4, that of common consequence independence, states that if the possibility of playing an additional lottery, with fixed possible outcomes and fixed probability of playing, is introduced to both lotteries, then the decision maker should not change their relative weightings of the original two lotteries. This is a logical axiom for linear utilities, but is not completely satisfactory when we consider non-linear utilities. For example, this axiom suggest that given a player given a choice between a payoff of \$100 and \$200 would make the same decision as when presented with a choice between a payoff of \$10100 and \$10200. Experience suggests that in fact the decision might change since in the second case the two payoffs are essentially equal, while in the first case the second payoff is double the first. An appropriate modification to axiom 4 that addresses this concern is

4a **Common Utility Independence**: For any four lotteries L_1, L_2, L_3, L_4, and a player with utility function $U(\cdot)$, if $U(\langle L_1 \rangle) - U(\langle L_2 \rangle) = U(\langle L_3 \rangle) - U(\langle L_4 \rangle)$ then $\Pr(L_1, L_2) = \Pr(L_3, L_4)$, where $\langle L_i \rangle$ denotes the expected monetary payoff from lottery L_i.

For a linear utility function, this axiom is satisfied when the common consequence independence axiom is satisfied. This axiom could also be formulated using the expected utility rather than the utility of the expected payoff, but the above form of axiom 4a leads to a decision rule of the form $\Pr(L_1, L_2) = \Psi(U(\langle L_1\rangle) - U(\langle L_2\rangle))$. Using a logarithmic utility function and taking the function $\Psi(x)$ to be $1/(1+\exp(-\lambda x))$ as before gives

$$\Pr(L_1, L_2) = \frac{(\sum_{i=1}^{n} u_i p_i^1)^\lambda}{(\lambda \sum_{i=1}^{n} u_i p_i^1)^\lambda + (\sum_{i=1}^{n} u_i p_i^2)^\lambda}$$

which is the BRNE choice rule and, for $\lambda = 1$, reduces to the Luce choice rule[6]. It is this decision rule that will form the basis for the returns based beliefs model discussed in this paper.

In a game between multiple players, we assume that each player has a probability distribution over the choices available. This probability distribution, P^i, over the elements S^{-i} is defined such that $P^i(s^{-i})$ is the probability associated with $s^{-i} \in S^{-i}$. We operationalize our model by assuming an expected return framework, so that the expected payoff of the jth pure strategy of player i, given P^i, is as follows[7]:

$$\Pi_j^i(P^i) = \sum_{s^{-i} \in S^{-i}} P^i(s^{-i}) \pi_j^i(s^{-i}) \tag{1}$$

where $\pi_j^i(s^{-i})$ is player i's payoff from choosing a pure strategy j when the other players choose s^{-i} and $P^i(s^{-i})$ is the belief held by player i about the probability the other players will choose s^{-i}. The decision probabilities over pure strategies for player i in turn follow the

[6] Luce showed using probability axioms that if the ratio of probabilities associated with any two decisions is independent of the payoff of any other decisions, then the choice probabilities for decision j of player i can be expressed as a ratio of the expected value of that decision over the total expected value of all decisions: $V_j^i / \sum_k V_k^i$, where V_j^i is the expected value associated with decision j. In addition, Gul, Natenzon and Pesendorfer (2014) show that the Luce rule is the unique random choice rule that admits well-defined ranking of option sets. In particular, the authors show that the Luce rule is the unique random choice rule that admits a context-independent stochastic preference. Moreover, this method of arriving at decision probabilities has been supported by empirical work which provides empirical justification for our approach. In particular, empirical research for paired comparison data supports the Luce (1959) method of arriving at decision probabilities (Abelson and Bradley 1954).

[7] This formulation, similar to the BRNE with a logit distribution, is invariant to linear (i.e., scale) transformations, but not to changes in the payoff origin. On the other hand, the QRE with an extreme value distribution, accommodates negative payoffs and is invariant to changes in the payoff origin but not to linear transformations.

specification outlined above which is proportional to the expected returns as follows:

$$p_j^i = \frac{\Pi_j^i(P^i)}{\sum_{k=1}^m \Pi_k^i(P^i)} \tag{2}$$

This model admits a Nash-like equilibria in belief formation such that the belief probabilities match the decision probabilities for all players. This equilibrium in beliefs can be found by iterating between the expected payoff in equation (1) and the decision probabilities in equation (2). In relation to this, we discuss the epistemic condition for the returns-based beliefs equilibrium in section III A 3.

The returns-based beliefs model can be used to find equilibria for games between any number of players. However, for clarity, the games discussed in the rest of this paper will be games played between 2 players only. We will use u_{ij}^1 and u_{ij}^2 to denote the payoffs to player 1 and 2, respectively, if player 1 plays move i and player 2 plays move j. We will denote a mixed strategy of player 1 by $\mathbf{p} = \{p_i\}$, where p_i is the probability that player 1 chooses $s^i \in S^1$, and denote a mixed strategy of player 2 by $\mathbf{q} = \{q_j\}$, where q_j is the probability that player 2 chooses $s^j \in S^2$. The Luce rule, Eq. (2), defines a mapping $\mathcal{M}_{ij} : \sigma_j \to \sigma_i$ from mixed actions, σ_j, of player j to mixed actions, σ_i, of player i. This leads us to define the notion of a *returns-based beliefs (RBB) equilibrium* as

Definition A returns-based beliefs equilibrium is a pair of mixed actions, $(\mathbf{p},\mathbf{q})$, such that $\mathcal{M}_{12}(\mathbf{q}) = \mathbf{p}$ and $\mathcal{M}_{21}(\mathbf{p}) = \mathbf{q}$. In other words, this is a solution in which both players play the Luce-type response, and each player's belief about their opponent coincides with the actual strategy the opponent adopts.

The properties of RBB equilibrium and the associated proofs are in Velu, Iyer and Gair (2012). One characteristic of the RBB solution is that it assigns positive probability to any strategy that has non-zero expected payoff and this remains true even if a strategy is uniformly dominated by another. As an illustration, suppose that the payoff matrix for the row player takes the form

$$\mathbf{u^1} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}.$$

The second row is uniformly dominated by the first and so any rational player would always play the first choice. However, after assigning probabilities q and $(1-q)$ that the column player chooses left or right, the RBB solution assigns a non-zero probability to the second action of $(2-q)/(5-2q)$, which ranges from 1/3 to 2/5 as q varies from 1 to 0.

One resolution to this problem is to transform the payoff matrices to incorporate loss aversion (see, for example, Selten and Chmura (2008) and Brunner et al. (2010)), by expressing payoffs as differences to the guaranteed payoff. A player can compute their guaranteed payoff by finding the max-min of the elements in their payoff matrix, u^1, i.e., for each possible move, i, they compute their minimal payoff, $min_j u^1_{ij}$, over possible strategies of their opponent, and then finds the maximum, $max_i(min_j u^1_{ij})$. If the maximum occurs for $i = I$, for which the minimum is at $j = J$, then by playing move I the player could guarantee a payoff at least as big as u^1_{IJ}. The payoff matrix may then be transformed by subtracting u^1_{IJ} from each element of the matrix, to represent payoffs relative to the guaranteed payoff. The RBB approach naturally places unfavourable weight on negative payoffs, so this transformation already encodes loss aversion. However, the standard approach is to also multiply all negative entries in the transformed matrix by a multiplier, λ. This multiplier is a measure of the degree of loss aversion of the player and a value of $\lambda = 2$ is typically used. In the preceding example the row player can guarantee a payoff of at least 2 by playing the first row. Subtracting this from all entries and multiplying negative entries by λ gives the transformed matrix

$$\mathbf{u^1} = \begin{pmatrix} 0 & 1 \\ -\lambda & 0 \end{pmatrix}.$$

Evaluating the RBB solution as before, the probability now assigned to the second row is $-\lambda q/(1-(1+\lambda)q) < 0$. This is an invalid probability, but a minor modification to the RBB rule fixes that by replacing the probability of taking action i by $P_i / \sum_j P_j$, where $P_i = \max(\sum u_{ij} q_j, 0)$ and $\{q_j\}$ are the probabilities assigned to the opponent's actions. In other words, we compute the expected payoffs as before, but if we find a negative payoff we replace that payoff by zero, before playing in proportion to those transformed payoffs.

RBB with loss aversion and the probability-zeroing algorithm just described now correctly assigns zero probability to dominated strategies. However, this introduces an arbitrariness in the choice of loss-aversion multiplier and requires the probability-zeroing modification to the Luce rule which means that the equilibrium solution can no longer necessarily be found as the solution of an eigenvalue problem. An alternative approach is to construct a choice rule that satisfies first order stochastic dominance. This was discussed in Blavatskyy (2011) and relies on defining a new lottery, $L_1 \wedge L_2$, that yields a given possible outcome, x_i, from the set of possible outcomes, X, with probability

$$\max\left\{\sum_{y\in X|x\geq y} L_1(y), \sum_{y\in X|x\geq y} L_2(y)\right\} - \max\left\{\sum_{y\in X|x>y} L_1(y), \sum_{y\in X|x>y} L_2(y)\right\}$$

where $L(u)$ denotes the probability that lottery L yields outcome u. This is the greatest lower bound on the lotteries in terms of stochastic dominance. The choice rule then follows from six axioms

1. **Completeness**: For any two lotteries L_1, L_2, $\Pr(L_1, L_2) + \Pr(L_2, L_1) = 1$.
2. **Weak Stochastic Transitivity**: For any three distinct lotteries L_1, L_2, L_3:
   (a) if $\Pr(L_1, L_2) \geq 1/2$ and $\Pr(L_2, L_3) \geq 1/2$, then $\Pr(L1, L_3) \geq 0.5$;
   (b) if $\Pr(L_1, L_2) = 1$ and $\Pr(L_2, L_3) = 1$ then $\Pr(L_1, L_3) = 1$.
3. **Continuity**: For any three lotteries L_1, L_2, L_3, such that $L_3 \neq \alpha L_1 + (1-\alpha)L_2$, the sets $\{\alpha \in [0,1] | \Pr(\alpha L_1 + (1-\alpha)L_2, L_3) \geq 1/2\}$ and $\{\alpha \in [0,1] | \Pr(\alpha L_1 + (1-\alpha)L_2, L_3) \leq 1/2\}$ are closed.
4. **Common Consequence Independence**: For any four lotteries L_1, L_2, L_3, L_4 and any probability $\alpha \in [0,1]$, $\Pr(\alpha L_1 + (1-\alpha)L_3, \alpha L_2 + (1-\alpha)L_3) = \Pr(\alpha L_1 + (1-\alpha)L_4, \alpha L_2 + (1-\alpha)L_4)$.
5. **Outcome monotonicity**: For lotteries L_1 and L_2 such that L_1 yields outcome x_1 with probability 1 and L_2 yields outcome x_1 with probability α and x_2 with probability $(1-\alpha)$, then $\Pr(L_1, L_2) = 0$ or 1.
6. **Odds ratio independence**: For any three lotteries L_1, L_2, L_3 such that $\Pr(L_2, L_1) \neq 0$, $\Pr(L_3, L_1) \neq 0$ and $\Pr(L_3, L_2) \neq 0$

Axioms 1, 3 and 4 coincide with the axioms of Blavatskyy's stochastic utility theorem, but axiom 2 is a weaker form of transitivity than assumed there. Axioms 5 and 6 replace the interchangeability axiom of the stochastic utility theorem. There is a unique choice rule that follows from these axioms, which takes the form

$$\Pr(L_1, L_2) = \frac{\phi(U(L_1) - U(L_1 \wedge L_2))}{\phi(U(L_1) - U(L_1 \wedge L_2)) + \phi(U(L_2) - U(L_1 \wedge L_2))}$$

where $U(L_i)$ is the expected utility function and $\phi : \mathbb{R}_+ \to \mathbb{R}$ is a non-decreasing function with $\phi(0) = 0$.

Taking both U and ϕ to be the identity yields a decision rule that is similar in form to the Luce rule, but payoffs are expressed relative to the greatest lower bound lottery. Returning again to the preceding example, if the opponent plays left/right with probabilities $q/(1-q)$ as before, the player has a choice between the first row lottery, which yields 2 with probability q and 3 with probability $(1-q)$, or the second row lottery, which yields 1 with probability q and 2 with probability $(1-q)$. The greatest lower bound lottery, $L_1 \wedge L_2$ yields payoff 1 with probability

$$\max\{0, q\} - \max\{0, 0\} = q;$$

it yields payoff 2 with probability

$$\max\{q, 1\} - \max\{0, q\} = 1 - q$$

and it yields payoff 3 with probability

$$\max\{1, 1\} - \max\{q, 1\} = 0.$$

In other words, as we might hope, the greatest lower bound lottery coincides with the second row lottery and hence the choice rule described above assigns zero probability to the second row, and probability of 1 to the first. This decision rule therefore naturally avoids the issue of dominated choices, without the need to transform the payoff matrix and introduce loss aversion in a semi-arbitrary way.

We will explore both this alternative decision model and the RBB model in the remainder of this paper. The arguments presented above illustrate that the Blavatskyy rule incorporates loss aversion in a completely natural way, without the need for additional parameters or modifications to the payoff matrices. This is an advantage of the Blavatskyy rule over the RBB model. However, the RBB model has other advantages. One is that RBB naturally extends to games with arbitrarily large choice sets, while the Blavatskyy rule is specifically for binary choice problems. We will discuss how this problem can be addressed in Section V. Another advantage of the RBB model as a concept is that the equilibrium solution can, in general, be derived iteratively. The RBB equilibrium solution can be found by solving for the eigenvalues of the matrix $\mathbf{u_1}\mathbf{u_2}^T$ (see Velu, Iyer and Gair 2012). In the case that all payoffs are positive, this matrix is a positive matrix and so it follows from the

Perron-Frobenius theorem that the largest eigenvalue has corresponding eigenvector with all components strictly positive. This eigenvector represents the RBB equilibrium solution. Since it is the largest eigenvalue, an iterative procedure in which an estimate of the solution, $\mathbf{v}$, is repeatedly updated by multiplying by $\mathbf{u_1}\mathbf{u_2}^T$, will converge to the corresponding eigenvector (if the initial vector is decomposed on the eigenbasis as $\mathbf{v} = \sum a_i \mathbf{e}_i$, then after N iterations the vector is parallel to $\sum a_i \lambda_i^N \mathbf{e}_i$, which is increasingly dominated by the largest $\mathbf{e}_1$ as $N \to \infty$). Although this is only guaranteed to work for the pure RBB solution in the case of strictly positive payoffs, convergence to a solution appears to occur more widely, even when including loss aversion and using the probability-zeroing modification to RBB described earlier. This convergence to equilibrium is important as it is possible to believe that a player would actually carry out such an iteration in their analysis of a game. The Blavatskyy model, on the other hand, does not seem to share this model. Empirically, repeated iteration leads to an oscillation between possible eigenstates. Due to the complexity of evaluating the Blavatskyy solution, it is therefore much harder to imagine that a player computes their optimal strategy under this choice rule, but the player must find themselves in equilibrium by accident.

We next examine the relationship of the RBB and Blavatskyy models to other non-Nash equilibrium models.

A. Relation to other non-Nash equilibrium models

In this section, we discuss how the RBB and Blavatskyy models borrow from and build on various other equilibrium concepts. In Section III A 1, we describe the RBB and Blavatskyy models for probabilistic actions and compare it mathematically and philosophically to the QRE and BRNE. In Section III A 2 we discuss the probabilistic beliefs component of the model, comparing it to the RBE. In Section III A 3 we will discuss epistemic conditions that lead to an RBB equilibrium. For completeness, in Appendix B, we also show how the RBB, QRE and BRNE models can all be derived from information entropy maximization.

1. Probabilistic actions

In the treatment of probabilistic actions, the RBB model is mathematically equivalent to the BRNE model with rationality parameter $\mu = 1$. The BRNE model is a type of QRE and so the RBB model can also be seen as a special case of the QRE, albeit different to the most commonly used form of the QRE, the logistic QRE. The interpretation of the RBB model is different, however, since we assume the probabilistic actions arise from uncertainty about the actions of the opponent. The RBB probabilistic action can be obtained in a way that parallels the QRE derivation as follows. We suppose that a player has some belief about the likely action of his opponent, represented by a mixed strategy profile $\mathbf{q}$. The player can then compute his expected return from each strategy i, $\Pi^i = \sum_j u_{ij} q^j$. We take this return to be the raw payoff of that strategy, but the expected utility could be used instead. If the player is certain about his opponent's actions, he would choose strategy I satisfying $\{I : \Pi^I \geq \Pi^j \; \forall \, j \neq I\}$. However, the player may be uncertain as to whether his assessment of his opponent is correct and therefore hedges his bets. This can be represented by supposing the player treats his expected payoff as $P_i(1+\epsilon_i)$ where $\boldsymbol{\epsilon} = \{\epsilon_1, \epsilon_2, \cdots, \epsilon_N\}$ follows a given probability distribution $p(\boldsymbol{\epsilon})$ and then adopts the mixed strategy

$$p_i = \int_{\boldsymbol{\epsilon}: P_i(1+\epsilon_i) \geq P_j(1+\epsilon_j) \, \forall j \neq i} p(\boldsymbol{\epsilon}) \, \mathrm{d}^N \epsilon \tag{3}$$

i.e., he chooses his strategy in proportion to the probability that this strategy will yield the largest payoff. If $1/(1+\epsilon_i)$ follows an exponential distribution for each i

$$p(\boldsymbol{\epsilon}) = \prod_{i=1}^{N} \left[(1+\epsilon_i)^{-2} \, \mathrm{e}^{-\frac{1}{1+\epsilon_i}} \right] \tag{4}$$

we obtain the RBB solution

$$p_i = \frac{P^i}{\sum_{j=1}^{N} P^j}. \tag{5}$$

If we instead took $1/(1+\epsilon_i)^\mu$ to follow the exponential distribution, we would obtain the BRNE with rationality parameter μ and taking $\epsilon_i P^i$, the absolute uncertainty[8], to follow a Gumbel (double exponential) distribution, $p(y \leq Y) = \exp(-\exp(-\lambda Y))$ we obtain the

[8] We note that the logistic QRE is distinct from the BRNE or RBB in the sense that the payoff-independent distribution that gives rise to the QRE is a distribution on the absolute error, $\epsilon_i P^i$, while for the BRNE/RBB it is a distribution on the fractional error $(1+\epsilon_i)$.

logistic QRE[9] with parameter λ (Yellott 1977). The mathematical equivalence of the RBB and QRE is thus clear[10], however, in the QRE and BRNE the modifications to the payoffs are regarded as errors. The QRE can be viewed as an application of stochastic choice theory to strategic games or as a generalization of the Nash equilibrium that allows noisy optimization (Haile, Hortacsu and Kosenok 2008). The "noise" or error the decision maker makes could be interpreted as either unmodeled costs of information processing (McKelvey and Palfrey 1995) or as unmodeled determinants of utility from any particular strategy (Chen, Friedman and Thisse 1997). In addition, in the QRE, the actions are random rather than uncertain — the assumption is that each time a player plays a game they make an error drawn from the specified distribution and then best responds to the modified payoffs. The equilibrium mixed strategy arises as an average over many realizations of the game. In the RBB model the modifications to the payoff represent the player's conscious uncertainty about the likely action of his opponent. The mixed strategy arises when the player weighs up his options based on his subjective beliefs about the other players and he consciously arrives at the same mixed strategy each time he is presented with the game[11].

The probabilistic action described above follows after a player has made an assessment of the expected utility of their different actions. These utilities are derived from a belief about the likely actions of their opponents. If a player believes that their opponents think like themselves, they will form these beliefs according to their own probabilistic action model, i.e., that their opponents will make a returns-based response.

The paper by McKelvey and Palfry (1995, p. 31) introducing the QRE discusses the

[9] We note that in the QRE the range of the error parameter is therefore $\epsilon_i P^i \in [-\infty, \infty]$ and therefore it allows for the possibility of negative estimated payoffs. In the BRNE/RBB $(1 + \epsilon_i) \in [0, \infty]$ and so negative estimated payoffs only arise for $P^i < 0$.

[10] An alternative way to model a player acting under uncertainty would be to model the opponent's strategy as $\mathbf{q} \rightarrow \mathbf{q} + \boldsymbol{\epsilon}$ and specify a distribution on the uncertainty in the strategy, $\boldsymbol{\epsilon}$. This could be done in a similar way to the hyperprior treatment of the expected strategy, $\mathbf{q}$. This would not give rise to a QRE, BRNE or RBB like model since the uncertainty parameters will mix the payoffs of the various strategies. We will not consider this further in the current paper, but have instead presented the preceding derivation of the RBB model to highlight the mathematical connection to other equilibrium concepts.

[11] Our model based upon returns-based beliefs is also different to other alternative models that provide an explanation for non-Nash equilibrium outcomes such as the level-k models (Costa-Gomes and Crawford 2006) and cognitive hierarchy (CH) model (Camerer, Ho and Chong 2004). The level-k and CH models assume that decision makers are heterogenous in their level of sophistication with respect to their strategic thinking. The returns-based beliefs model differs from the level-k and the CH models in that the former does not assume heterogeneity in the levels of sophistication in thinking by decision makers.

concept of heterogeneity across players but does not develop the concept further. We do so here with the RBB model in order to understand the subtle implications. The Blavatskyy model does the same but assumes that the response will follow the Blavatskyy rule. Heterogeneity across players have significant implications for the outcome of the game in the context of error based models such as the QRE and the BRNE compared to strategic uncertainty driven models such as RBB and Blavatskyy. In order to illustrate this, let us assume that in a two player, two strategy based model, we have one player playing with an error distribution that represents a double exponential distribution and the other plays with an error distribution that represents a Gumbel distribution. In this set-up the question of mutual knowledge to arrive at an equilibrium becomes key. On the one hand, in the error based models such as the QRE and BRNE, people are making mistakes, so presumably they do not know that they are making mistakes. This is because if they did know that they were making mistakes then they would not be making them. Therefore, to assume that they are making different forms of errors and each knows about the others' method of making errors i.e., mutual knowledge does not appear to be reasonable. Therefore, the concept of arriving at an equilibrium outcome in a one shot game, although mathematically possible, does appear to contradict common sense. On the other hand, in strategic uncertainty driven models such as RBB and Blavatskyy, the players are consciously hedging their bets and so it is perfectly reasonable for them to assume a different model for themselves and their opponent i.e., heterogeneity in beliefs with mutual knowledge. In this case mutual knowledge of the different axioms upon which players will be responding to the uncertainty makes sense and hence, the corresponding equilibrium can be logically derived mathematically and such an equilibrium conforms to common sense. The philosophy behind the strategic uncertainty based models such as RBB and Blavatskyy thus leads naturally to probabilistic beliefs, i.e., a mixed-strategy belief profile, which we discuss in the next subsection.

2. Probabilistic beliefs

The RBB and Blavatskyy models also builds on the RBE model by including probabilistic beliefs. In the QRE and BRNE, the players only have probabilistic actions and not probabilistic beliefs, while in the RBE the players only have probabilistic beliefs and not probabilistic actions. RBB and Blavatskyy models includes both. In the RBE the beliefs are considered random, not uncertain — each time a game is played, the player forms a belief about which pure strategies his opponents will play and responds to that. The equilibrium solution arises as the ensemble average of the response over many realizations of the game and in equilibrium the beliefs are statistically consistent in that the expected choice of each player coincides with the focus of the belief distributions of the other players about the first player's choice. In the RBB and Blavatskyy models we treat beliefs as uncertain not random — the mixed strategy profile that a player believes his opponents will play is a conscious assessment of the relative likelihood of different strategies and the player then responds to that. This assessment will be based on historical experience, for instance the desire to want to cooperate in order to maximize the payoff to the players. In addition, as described at the end of the preceding section, in the RBB model the probabilistic action model gives rise to a natural focus for the belief distribution as the RBB equilibrium solution, itself a mixed strategy profile. In this paper we will not explore how the players might reach beliefs that deviate from the profile, but in the philosophy of RBB this process should be assumed to be intrinsic to the player, based on a conscious assessment of the relative probabilities that their opponents will play particular mixed strategies. The player then responds to this mixed belief profile. The RBB and Blavatskyy models is thus a response to an average over a distribution, while in the RBE the equilibrium is the average over a distribution of a response. In the RBE, a player will always adopt the same mixed strategy playing a given game against a given opponent, while in the RBE players make a different assessment of their opponent each time the game is played. The RBB model of course also differs from the RBE in how the player responds to their beliefs — in the RBE the player chooses the payoff-maximizing strategy assuming his opponent plays as believed, while in the RBB the player responds probabilistically in proportion to the expected return/Greatest Lower Bound Blavatskyy rule to allow for uncertainty in his assessment of his opponent.

The Utility Proportional Beliefs (UPBs) model introduced by Bach and Perea (2014) shares the notion of uncertain rather than random beliefs. In that model, beliefs are assigned in proportion to expected payoffs. The UPBs model does not make an explicit statement about probabilistic actions, but does have an equilibrium solution in which all players have UPBs, which would be equivalent to adding a probabilistic action model in which the action probabilities take the same form as the UPB belief probabilities. The UPB model includes an arbitrary parameter, λ, and can be shown to reduce the the RBB model for a certain choice of this parameter. In Bach and Perea (2014), they advocate using a particular choice of λ, the maximum allowed value, but this does not appear to give very good results when compared to experimental data. Further discussion of the UPB model can be found in Appendix D 1.

3. An epistemic characterization of the returns-based beliefs

The RBB and Blavatskyy equilibriums are an equilibrium in both beliefs and in actions. In the RBB and Blavatskyy models equilibriums, conjectures coincide with actions: what players do is the same as what the other player believes they will do. Hence, in that sense, the conjectures as well as the actions are in equilibrium. In this section we compare the RBB/Blavatskyy models to the Nash equilibrium and other equilibrium concepts in terms of the epistemic conditions for equilibrium.

Traditional solution concepts in game theory such as the Nash equilibrium assume common knowledge of rationality or at least a high degree of mutual knowledge of rationality. Aumann and Brandenburger (1995) show that for a two player game mutual knowledge of rationality is sufficient for Nash equilibrium. Mutual knowledge implies that player 1 only needs to know what player 2 believes about his strategy (which is rational in the sense of maximizing returns) and vice versa for player 2. Therefore, in a two player game there is no need for higher orders of beliefs for the outcome to be a Nash equilibrium.

We can apply the same reasoning of mutual knowledge to the RBB and Blavatskyy equilibriums. In a two player game, for the players to be in an RBB equilibrium we need each player to follow the Luce rule (players identify with rationality based on the Luce rule), to believe that their opponent is Luce type and that the strategy that they believe their opponent thinks they will adopt coincides with the actual strategy they adopt. A similar reasoning would apply for the Blavatskyy model but following the Greatest Lower Bound Blavatskyy response rule. However, the mutual knowledge equilibrium implies that the players do not have to know they are in the RBB/Blavatskyy equilibriums, but they do have to be in it "accidentally". The question which the mutual knowledge formulation does not address is how the players reach their belief about what strategy their opponent assumes about them. Presumably, for player 1 to believe that player 2 thinks that he, player 1, will play the RBB/Blavatskyy equilibrium strategy, then he must have gone through the iterative process in his head to reach that conclusion. So, whereas mathematically the different orders of beliefs are decoupled and we only need the mutual knowledge condition to hold, in practice the first order state of beliefs would be a product of iteration on higher order beliefs. In short, the mutual knowledge equilibrium requires the players to be in a particular belief state but does not explain how they reached that state or where their priors came from. The non-Nash equilibrium probabilistic choice models discussed earlier such as the QRE, BRNE or RBE do not provide a theory of where the priors come from. This issue is neatly summarized by Chen, Friedman and Thisse (1997, p. 38-39) in their discussion of the BRNE: *Where do the choice probabilities come from? Admittedly, this is not clear; the probability of choosing a particular pure strategy depends on the subconscious utility attached to that strategy as compared with the subconscious utility attached to the other pure strategies. Meanwhile, the subconscious utility of a strategy cannot be computed without knowing the mixed strategies of the other players.* On the other hand, since the Luce/Blavatskyy rule comes from a set of axioms on how an individual would choose between choices in an uncertain environment (Blavatskyy 2008; Gul, Natenzon and Pesendorfer 2014, Luce 1959), the RBB/Blavatskyy provides a theory of the formulation of priors[12].

IV. EXPERIMENTAL 2×2 GAMES

In order to test the applicability of the RBB/Blavatskyy model in an empirical setting we will look at data from Selten and Chmura (2008). They presented empirical results for 12 games, each played between two players and with two possible moves for each player. The games are asymmetric, games 1-6 are constant sum games, but games 7-12 are non-constant sum. The payoffs for all twelve games are shown in Figure 1. The constant sum and non-constant sum games form pairs (e.g., Game 1 with Game 7, Game 2 with Game 8 and so on). As noted by Selten and Chmura (2008, p. 950), ‘the non-constant sum game in a pair is derived from the constant sum game in the pair by adding the same constant to

[12] When there is an inter-linked belief structure whereby player 1 plays the Luce/Blavatskyy rule because he thinks player 2 beliefs that player 1 is a Luce/Blavatskyy type and vice versa (whereby player 2 plays the Luce/Blavatskyy rule because he thinks that player 1 thinks that player 2 is Luce type), there needs to be common knowledge of beliefs for the RBB/Blavatskyy equilibrium. The higher order belief structure requires a basis upon which the players derive their priors.

player 1's payoff in the column for R and player 2's payoff in the row for U'.

In this Section we will apply the RBB and Blavatskyy models to the Selten and Chmura (2008) data. We will compare their performance to that of several other equilibrium concepts, including Nash and QRE which have been previously applied to explain this data set, and UPB, RBE and BRNE which have not. These comparisons demonstrate that the RBB and Blavatskyy models are as good as existing concepts at explaining this data and are therefore worthy of further study. We are not attempting to explain the data better than previous ideas, which have already been seen to be adequate, but provide an alternative explanation of the mechanism by which players might think about strategic decision making.

In comparing the models to the data, for concepts such as the QRE which attempt to represent uncertainty in actions, it is logical to compare the observed actions to the actions predicted by the models. For the RBB and Blavatskyy models, the players respond in a definite way to uncertain moves of their opponents and the equilibrium solution represents an equilibrium in beliefs. In such models, it is reasonable both to compare predicted to observed *actions* or predicted to observed *beliefs*, i.e., given the specified model of how a player responds when they hold certain beliefs, we compute the beliefs that they must have held in order to respond in the way that they were observed to respond and compare those beliefs, $\mathbf{q}^{\mathrm{b}}$, to the predicted equilibrium solution, $\mathbf{q}^{\mathrm{rbbe}}$ [13]. This is justified further in Appendix A. We will compare the RBB and Blavatskyy solutions to the data in this way, while comparing observed and predicted actions for the other equilibrium concepts [14].

[13] Note that it is possible to reformulate some of the other equilibrium concepts, for instance the QRE and Nash solutions, in terms of a hyperprior on the belief probabilities of one player about the other. Such models are worth further investigation but will not be considered further here.

[14] By comparing the models in these two distinct ways, we are treating them asymmetrically in our analysis. If the RBB solutions are assessed by comparing actions not beliefs, their performance is not as impressive as the results below indicate, although it is still good. However, we think the asymmetric treatment is justified qualitatively by the different interpretation of the equilibria concepts. Moreover, the purpose of this comparison is not to say that the RBB and Blavatskyy models are superior to all other equilibrium concepts, but that they perform sufficiently well that they should be regarded as viable equilibrium models which have certain desirable properties.

Game 1

(10,8)	(0,18)
(9,9)	(10,8)

Game 7

(10,12)	(4,22)
(9,9)	(14,8)

Game 2

(9,4)	(0,13)
(6,7)	(8,5)

Game 8

(9,7)	(3,16)
(6,7)	(11,5)

Game 3

(8,6)	(0,14)
(7,7)	(10,4)

Game 9

(8,9)	(3,17)
(7,7)	(13,4)

Game 4

(7,4)	(0,11)
(5,6)	(9,2)

Game 10

(7,6)	(2,13)
(5,6)	(11,2)

Game 5

(7,2)	(0,9)
(4,5)	(8,1)

Game 11

(7,4)	(2,11)
(4,5)	(10,1)

Game 6

(7,1)	(1,7)
(3,5)	(8,0)

Game 12

(7,3)	(3,9)
(3,5)	(10,0)

FIG. 1: Pay-offs for the 12 games in Selten and Chmura (2008). In each cell an entry (x, y) indicates a payoff of x for the row player and y for the column player.

A. Loss aversion

Before analyzing the data from the Selten games, we first transform the pay-off matrices to include loss-aversion (consistent with the Impulse Balance Equilibrium in Selten and Chmura 2008, p. 947). It was demonstrated by Brunner et. al (2010) that the explanatory power of the various models compared by Selten and Chmura (2008) improves significantly once loss aversion is included before computing the equilibrium solutions. We include loss aversion by subtracting the guaranteed payoff from each entry in the matrix and then multiplying negative entries by a multiplier, as described earlier. In the following, we will show results using both $\lambda = 1$ and $\lambda = 2$ as the loss aversion multiplier.

After carrying out this transformation, some of the initially positive pay-offs become negative. Our previous results no longer guarantee the existence of the RBB equilibrium solution under those circumstances. However, it is possible to prove for 2×2, constant-sum games, that an RBB equilibrium exists for any choice of the loss multiplier used when transforming the matrices to include loss-aversion. This proof is provided in Appendix C. In the non-constant sum case it is more difficult to prove generic existence of the RBB equilibrium solutions, although such solutions do exist for all the games considered by Selten and Chmura (2008). Two of the games, numbers 8 and 10, have a singular behavior that arises from the equality of pay-offs for some moves, which leads to a column of zeros in the transformed matrix. However, modifying the payoffs by a small amount, $O(\epsilon)$ say, requiring an internal RBB equilibrium solution to exist and then taking the limit $\epsilon \to 0$, leads to a unique solution. For Game 8 this is $p = 2\lambda_2/(9 + 2\lambda_2)$, $q = (6\lambda_2 + 27\lambda_1)/(10\lambda_2 + 27\lambda_1 + 27)$ where p is the probability of player 1 choosing the first row, q is the probability of player 2 choosing the first column and λ_i is the loss-aversion multipler used to transform the payoff matrix of player i. For Game 10 the equilibrium solution is $p = 4\lambda_2/(7 + 4\lambda_2)$, $q = (24\lambda_2 + 21\lambda_1)/(24\lambda_2 + 21\lambda_1 + 14)$.

B. Comparison to other equilibrium concepts

In Selten and Chmura (2008), the observations described here were compared to the predictions of several equilibrium concepts — the Nash equilibrium, the QRE, the Action Sampling equilibrium, the Payoff Sampling equilibrium and the Impulse Balance equilibrium. In the original analysis, it was found that the Impulse Balance equilibrium was a much better predictor of behavior. However, the impulse balance approach naturally incorporates loss-aversion and it was subsequently demonstrated by Brunner et al. (2010) that if the analysis was repeated on the transformed game including loss aversion with $\lambda = 2$, the equilibrium concepts other than Nash are able to explain the data as well as the impulse balance results. Here, we extend this comparison to five additional equilibrium concepts — the RBB and Blavatskyy models discussed here, plus the Utility Proportional Beliefs model (Bach and Perea 2014, see Appendix D 1 for more details), the Random Belief Equilibrium (Friedman and Mezzetti 2005, see Appendix D 2) and the Boundedly Rationale Nash

Equilibrium (Chen, Friedman and Thisse 1997, see Appendix D 3). We will also report results for the Nash equilibrium as a benchmark, and for the QRE including loss aversion as representative of the five alternatives to Nash considered in Selten and Chmura (2008).

To compare these various equilibrium concepts we evaluate their goodness of fits, by first computing the squared error in the equilibrium prediction for each data point, i.e., $(p_X - p_{obs})^2 + (q_X - q_{obs})^2$, where (p_X, q_X) are the equilibrium model predictions for "up" and "left". For the RBB and Blavatskyy models we compare the belief probabilities to the equilibrium predictions, as above, while for the other concepts we compare the move probabilities to the predictions. As argued in Brunner et al. (2010), several of the equilibrium concepts have free parameters and it is appropriate to tune these to give the best match to the data. We evaluate the QRE with $\lambda = 1.05$, as in Brunner et al. (2010), the RBE with $\epsilon = 0.703$ and the BRNE with $\mu = 7.58$. The latter values were found by minimizing the total square error over choices of the tuning parameter. The Nash, Blavatskyy and RBB equilibria do not have free parameters to tune. The UPB equilibrium does have a tunable parameter, but in Bach and Perea (2014) they advocate using the maximum possible value for that parameter, as discussed in Appendix D 1.

In Figure 2 we compare the observed data to the various equilibrium concepts for Game 5 of Selten and Chmura (2008). The data is represented in three different ways — as the raw observed values, as the corresponding RBB belief probabilities (i.e., the beliefs the individual must have held to act as they did, *if* they are following the Luce choice rule) and as the corresponding Blavatskyy belief probabilities (i.e., the beliefs the individual must have held to act as they did, *if* they are following the Blavatskyy choice rule). In addition we plot the equilibrium solutions for all seven of the equilibrium concepts that we compare here. This Figure makes clear what we will conclude in the more careful analysis below. The UPB and RBE concepts lie well outside the distribution of the observed data, but the other concepts all lie within it and so have fairly comparable performance. The fact that, for the same game, the observed data has some variability, is another reason for considering multiple different equilibrium concepts. The variability between the better concepts is comparable to the variability in the observed data.

Selten and Chmura (2008) compared the equilibrium concepts using the Wilcoxon signed-rank matched-pairs test. However, as pointed out in Brunner et al. (2010), this is a parametric test which makes distributional assumptions about the data which may not apply

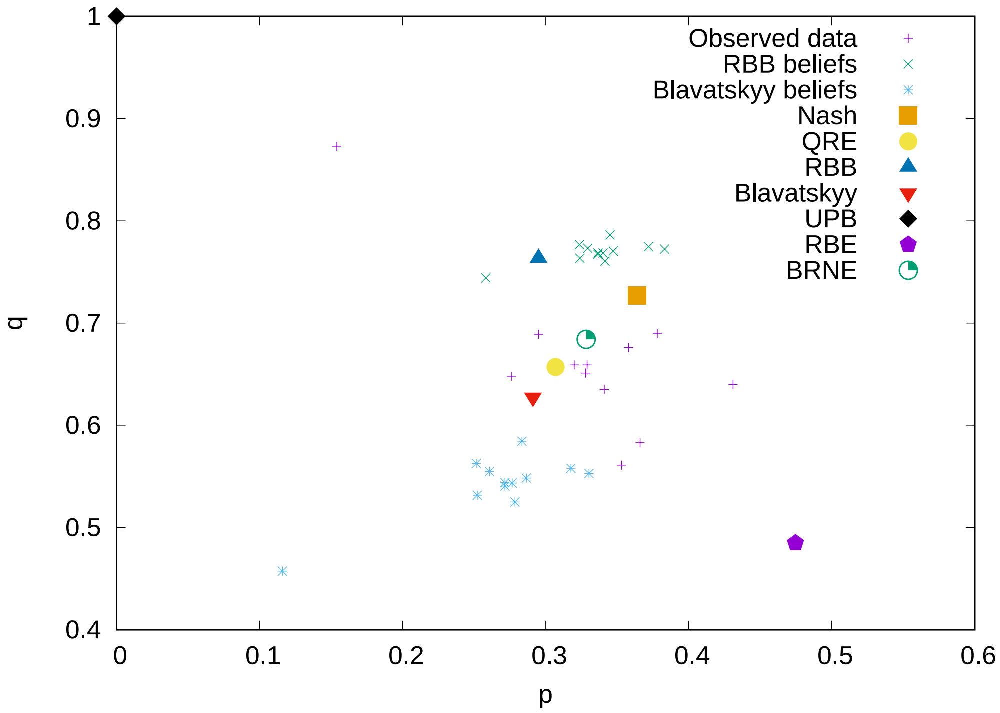

FIG. 2: Comparison of the observed experimental data (labelled "Observed data") to the equilibrium solutions for the various equilibrium concepts (labelled "Nash" etc. according to the name of the concept) for Game 5 from Selten and Chmura (2008). We also show the RBB and Blavatskyy "beliefs", i.e., the belief that the player must have held in order to play as observed, if they follow the RBB (Luce) or Blavatskyy choice rules.

here. Brunner et al. (2010) instead used the non-parametric Kolmogorov-Smirnov and robust rank-order tests to compare models. Selten, Chmura and Goerg (2011) then noted that the Kolmogorov-Smirnov test is not a matched-pairs test and advocated using the Fisher-Pitman test, which is a non-parametric test for matched pair comparisons. We carry out comparisons using the Fisher-Pitman test, although we have also repeated the analysis with both the Wilcoxon and Kolmogorov-Smirnov tests and found consistent conclusions.

In Table I we show p-values comparing the mean squared errors between the models using the Fisher-Pitman test. Where a significant difference was found, we also did a one-sided test on the same data to see which model predicted significantly smaller errors. The results of these one-sided tests are also shown in the table. For these tests we use only data from Games 1-6, 7, 9, 11 and 12 due to the issue in computing the RBB equilibrium solution for Games 8 and 10. Overall, the UPB and RBE models are quite poor at explaining this data set and in fact do not explain the data as well as the Nash equilibrium. One explanation for the poor performance of UPB is the choice to fix the tuning parameter λ to its maximum possible value rather than choosing it to minimize the squared error. Another reason is the fact that in the UPB model actions are chosen as pure strategies that maximise the expected utility computed from the equilibrium beliefs. For RBE, the lack of flexibility in the model predictions is probably responsible (see Appendix D 2). The performance of BRNE is quite comparable to that of the Nash equilibrium, which is unsurprising since the tuned value of the parameter, $\mu = 7.58$, is large and BRNE tends to Nash in the limit $\mu \to \infty$. BRNE also tends to RBB in the limit $\mu = 1$. However, the RBB model is compared to the data by comparing belief probabilities and so the apparent improvement of RBB over BRNE is due to that choice. The Blavatskyy model outperforms all of the concepts except RBB. These results are based on comparing beliefs not actions, as in the RBB case. However, when the Blavatskyy solution is compared by actions, it still has performance comparable (but slightly inferior to) the QRE, and hence is still somewhat better than UPB, RBE, BRNE and Nash. It achieves this performance without having a tuning parameter or the need to include loss aversion (which is incorporated naturally in the model). Finally, the RBB model appears to have the best performance on this data set, although this is at least partially down to the choice of comparing beliefs rather than actions for this concept. If the RBB model is compared by comparing action probabilities it still outperforms the UPB and RBE concepts, but not the others, showing performance comparable to, but slightly worse than, the BRNE. These results suggest that both the Blavatskyy model and the RBB concept are just as good at explaining this particular data set as some of the other equilibrium concepts in the literature, and are therefore models worthy of further study. The performance of the Blavatskyy model is particularly encouraging as it is achieves this performance without the need for any tuning of parameters.

V. EXTENSION OF BLAVATSKYY TO 3X3 GAMES

We have seen above that the Blavatskyy model performs favourably compared to other equilibrium concepts and it is able to do this without the need to specify a tunable parameter, as in models such as the QRE, or including loss aversion, which is necessary for the RBB and the impulse balance equilibrium. The absence of tunable parameters puts it on a par

	Nash	QRE	RBB	Blavatskyy	UPB	RBE
	0.1% (g: 0.1%)	0.1% (g: 0.1%)				
RBB	0.1% (g: 0.1%)	0.1% (g: 0.1%)				
	0.1% (g: 0.1%)	2% (g: 1%)				
	0.1% (g: 0.1%)	0.1% (g: 0.1%)	10% (l: 5%)			
Blavatskyy	0.1% (g: 0.1%)	0.1% (g: 0.1%)	n.s.			
	0.5% (g: 0.2%)	0.1% (g: 0.1%)	10% (l: 5%)			
UPB	0.1% (l: 0.1%)	0.1% (l: 0.1%)	0.1% (l: 0.1%)	0.1% (l: 0.1%)		
($\lambda = \lambda_{\max}$)	0.1% (l: 0.1%)	0.1% (l: 0.1%)	0.1% (l: 0.1%)	0.1% (l: 0.1%)		
	0.1% (l: 0.1%)	0.1% (l: 0.1%)	0.1% (l: 0.1%)	0.1% (l: 0.1%)		
RBE	0.1% (l: 0.1%)	0.1% (l: 0.1%)	0.1% (l: 0.1%)	0.1% (l: 0.1%)	0.1% (g: 0.1%)	
($\epsilon = 0.703$)	0.1% (l: 0.1%)	0.1% (l: 0.1%)	0.1% (l: 0.1%)	0.1% (l: 0.1%)	0.1% (g: 0.1%)	
	0.1% (l: 0.1%)	0.1% (l: 0.1%)	0.1% (l: 0.1%)	0.1% (l: 0.1%)	0.1% (g: 0.1%)	
BRNE	0.1% (g: 0.1%)	n.s.	0.1% (l: 0.1%)	0.1% (l: 0.1%)	0.1% (g: 0.1%)	0.1% (g: 0.1%)
($\mu = 7.58$)	0.1% (g: 0.1%)	0.5% (l: 0.5%)	0.1% (l: 0.1%)	0.1% (l: 0.1%)	0.1% (g: 0.1%)	0.1% (g: 0.1%)
	2% (g: 1=0.5%)	1% (g: 0.5%)	5% (l: 2%)	10% (l: 5%)	0.1% (g: 0.1%)	0.1% (g: 0.1%)

TABLE I: P-values in favour of column concepts, using the Fisher-Pitman test (rounded to the next higher level among 0.1%, 0.2%, 0.5%, 1%, 2%, 5% and 10%). Where results are significant for the two-tailed test, we quote the significance of the one-sided test in brackets, with "l" indicating the mean of the error distribution for the column value is significantly lower than that of the the row value, and "g" indicating it is significantly greater. In each entry of the table, the first row uses data from Games 1-6, 7, 9, 11 and 12; the middle row uses data from Games 1-6 only; the last row uses data from Games 7, 9, 11 and 12 only. The comparison between QRE and Nash is omitted since it was included in Brunner et al. (2010), but for completeness the significance levels are 0.1%, 0.1% and 2% respectively, with QRE having a significantly lower mean in each case (significances 0.1%, 0.1% and 1%).

with the impulse balance equilibrium and its performance is quite similar, but it is able to achieve that performance without any transformation to the payoffs matrices. It is therefore a model clearly worthy of further study. However, the Blavatskyy model is a binary choice model and it does not naturally extend to situations in which the player has more than two choices. We want to assign probabilities $\{p_i\}$ that the player will make choice i for each i. If such probabilities exist, then when faced with a binary choice between choice 1 and choice 2, the player will choose 1 with probability $p_{12} = p_1/(p_1+p_2)$. However, the inverse calculation cannot necessarily be done, i.e., given a set of probabilities $\{p_{ij}\}$ for every possible binary decision problem, there is not necessarily a set of $\{p_i\}$'s that satisfy $p_{ij} = p_i/(p_i + p_j)$. This is most easily seen by considering a case where there are three choices, A, B and C, with binary choice probabilities $p(A, B) = 0.5$, $p(A, C) = 0.6$ and $p(B, C) = 0.7$. The assignment $p(A, B) = 0.5$ is compatible only with $p_A = p_B$, but that is incompatible with the assignment $p(A, C) \neq p(B, C)$.

To extend the Blavatskyy model to arbitrary decision problems, we want to make an assignment of probabilities to each action that gets "as close as possible" to the Blavtskyy model for each binary choice problem and reduces exactly to the Blavtskyy model whenever that is possible. There are a few possible approaches to this, which rely on the assumption of weak stochastic transitivity that provides a unique preference ordering among the choices

1. Use the pairwise probability assignments to construct the unique ordering of choices, $c_{i_1} < c_{i_2} < c_{i_3} < \ldots$. Assign the least preferred choice, c_{i_1}, a weight of 1, then assign the next choice, c_{i_2}, a weight of $p(c_{i_2}, c_{i_1})/p(c_{i_1}, c_{i_2})$, assign the next choice a weight of $p(c_{i_2}, c_{i_1})/p(c_{i_1}, c_{i_2}) \times p(c_{i_3}, c_{i_2})/p(c_{i_2}, c_{i_3})$ and so on. After all weights have been assigned, renormalise to give probabilities that sum to unity. This preserves the pairwise probabilities for neighbouring choices in the ordering, but not necessarily for other pairs of choices.

2. Assign probabilities, p_i, to the various choices in order to minimize the squared error $\sum_{i\,j:j>i}(p(i, j) - p_i/(p_i + p_j))^2$ subject to the constraints $\{p_i \geq 0 \forall i\}$ and $p_{i_j} \leq p_{i_k}$ when $j < k$, where $\{i_j\}$ denotes the ordered list of preferences as in the previous item. This approach does not necessarily preserve any pairwise choice probabilities, but assigns equal weight to all pairwise comparisons.

3. Assign initial weights of 1 to all choices. Choose an initial ordering of choices, O_1,

to look at choices. Take the first element in O_1 and then choose a second random ordering, O_2, of the remaining elements. Following the sequence O_2, multiply the weights of the elements by $p(c_2, c_1)/p(c_1, c_2)$ where c_1 is the current choice from O_1 and c_2 is the current choice from O_2. If this rescaling of the weights would change the ordering of two elements then set the weight equal to the element that it would swap order with. Then move on to the second element of O_1 and repeat, but do not adjust the weights of the elements of O_1 previously considered. At the end, renormalise the weights to give probabilities that sum to unity. Repeat this procedure for all possible choices of orderings at the two stages and average the results (or average over a large number of realisations). This procedure is a combination of the previous two, in that it uses pairwise probabilities directly to construct weights, but also tries to give equal weight to all pairs.

In all three cases, if an assignment of probabilities exists that is consistent with the pairwise probabilities then the algorithms will return this. The third algorithm described above is somewhat complex to implement and will probably not give results that are significantly different to the second algorithm, so we do not consider this approach further here, but it may warrant further investigation.

To illustrate the application of this extension of the Blavatskyy model we will consider a number of 3×3 two-player games. We consider eleven games, divided into a set of three and a set of eight. The first three games are given in Figure 3, while the rest are in Figure 4. The 3x3 games were chosen in order to represent variation to the equilibrium outcomes in order to make comparisons between the different game theoretic models. [15] For all of these games we computed the equilibrium solutions using the first two methods described above for extending the Blavatskyy solution, and also using RBB with and without loss aversion. We took the loss aversion multiplier to be $\lambda = 2$, but the equilibrium solution is independent of this choice.

For the first three games, A1–A3, the equilibrium solutions are the same for all four equilibrium concepts. For games A1 and A2, the equilibrium solution is $\mathbf{p} = \mathbf{q} = (1/3, 1/3, 1/3)$,

[15] **Game 1-4** taken from http://www.eprisner.de/MAT109/Mixedb.html, **Game 5** taken from http://economics.fundamentalfinance.com/game-theory/nash-equilibrium.php, **Game 6** taken from Perea (2012), **Game 7-9** taken from to fill in, **Game 10-11** taken from Risse (2000, p. 366), adapted from Kreps (1990, p.32)

Game A1

(0,0)	(1,-1)	(-1,1)
(-1,1)	(0,0)	(1,-1)
(1,-1)	(-1,1)	(0,0)

Game A2

(5,5)	(6,5)	(5,6)
(5,6)	(5,5)	(6,5)
(6,5)	(5,6)	(5,5)

Game A3

(4,4)	(4,4)	(0,0)
(4,4)	(4,4)	(0,0)
(0,0)	(0,0)	(1,1)

FIG. 3: Pay-offs for three 3×3 two-player games. As in Figure 1, in each cell an entry (x, y) indicates a payoff of x for the row player and y for the column player. For these games, both versions of the RBB solution and both methods for extending the Blavatskyy equilibrium concept lead to the same equilibrium solution.

while for game A3 there are two equilibrium solutions, one at $\mathbf{p} = \mathbf{q} = (1/2, 1/2, 0)$ and one at $\mathbf{p} = \mathbf{q} = (0, 0, 1)$. For the other eight games, $B1 - -B8$, there is a variation in the equilibrium solution between concepts and the equilibrium solution are given in Table II. We are able to obtain equilibrium solutions in all cases using both methods for extending the Blavatskyy solution. These are different to the RBB equilibrium solution whether or not loss aversion is included in the latter. The two different methods for extending Blavatskyy also tend to give different results, although in all cases the two solutions are quite similar. Whether the extended Blavatskyy method is any better at explaining experimental data than the RBB or other equilibrium concepts requires experimental studies with these or similar games that are beyond the scope of the current paper. However, it is clear that this is an interesting equilibrium concept that warrants further study.

VI. CONCLUSION

Scholars in economics have long been interested in the empirical testing of game-theoretic anomalies. A sizeable experimental literature shows that there is a dissonance between the theory and empirics of games in a strategic setting where defection is predicted by theory

Game B1

(0,0)	(2,-1)	(-1,1)
(-2,2)	(0,0)	(1,-1)
(1,-1)	(-1,1)	(0,0)

Game B5

(1,11)	(7,5)	(3,9)
(4,8)	(4,8)	(4,8)
(8,4)	(2,10)	(3,9)

Game B2

(5,4)	(7,4)	(5,6)
(4,7)	(5,5)	(6,5)
(6,5)	(5,6)	(5,5)

Game B6

(7,5)	(8,4)	(9,3)
(5,7)	(11,1)	(9,13)
(3,9)	(1,11)	(10,2)

Game B3

(1,1)	(2,0)	(3,3)
(0,2)	(3,3)	(0,0)
(0,3)	(0,0)	(10,10)

Game B7

(4,8)	(7,5)	(10,2)
(7,5)	(7,5)	(7,5)
(11,1)	(7,5)	(6,6)

Game B4

(300, 300)	(250, 350)	(300,300)
(350,250)	(300,300)	(350,250)
(300,300)	(250, 350)	(300,300)

Game B8

(4,6)	(5,4)	(0,0)
(5,7)	(4,8)	(0,0)
(0,0)	(0,0)	(1,1)

FIG. 4: Pay-offs for eight additional 3×3 two-player games that we use to illustrate the extension of the Blavatskyy method to more than two choices. As in Figure 1, in each cell an entry (x, y) indicates a payoff of x for the row player and y for the column player. Foer these games, the Blavatskyy and RBB concepts yield differing equilibirum solutions.

to be the optimal outcome over cooperation. Players in experimental settings cooperate far more than the theory would predict.

In this paper, we provide a reason for this unreason and explain this anomaly by discussing two probabilistic choice models, namely the Blavatskyy model and the RBB model and compare and contrast them to other probabilitic choice models. The Blavatskyy and the RBB models treat beliefs rather than strategies as the primary concept. In so doing, they assume that the players' beliefs and actions are in equilibrium. The two models combine a decision analytic solution concept where individuals form subjective probabilities over the actions of the individual's opponent and then choose a mixed strategy profile over the

Game	RBB (no loss aversion)		RBB with loss av. ($\lambda = 2$)		Ext. Blavatskyy (method 1)		Ext. Blavatskyy (method 2)	
	p	**q**	**p**	**q**	**p**	**q**	**p**	**q**
B1	No solution		(0.371, 0.273, 0.355)	(0.371, 0.273, 0.355)	(0.315, 0.224, 0.461)	(0.315, 0.224, 0.461)	(0.304, 0.224, 0.472)	(0.304, 0.224, 0.472)
B2	(0.352, 0.312, 0.335)	(0.352, 0.312, 0.335)	(0.299, 0.071, 0.631)	(0.299, 0.071, 0.631)	(0.383, 0.178, 0.439)	(0.383, 0.178, 0.439)	(0.362, 0.170, 0.467)	(0.362, 0.170, 0.467)
B3	(0.248, 0.000, 0.752)	(0.232, 0.000, 0.787)	(0.248, 0.000, 0.752)	(0.232, 0.000, 0.787)	(0.125, 0.000, 0.875)	(0.500, 0.000, 0.500)	(0.125, 0.000, 0.875)	(0.500, 0.000, 0.500)
B4	(0.315, 0.371, 0.315)	(0.315, 0.371, 0.315)	(0.000, 1.000, 0.000)	(0.000, 1.000, 0.000)	(0.000, 1.000, 0.000)	(0.000, 1.000, 0.000)	(0.000, 1.000, 0.000)	(0.000, 1.000, 0.000)
B5	(0.308, 0.336, 0.356)	(0.313, 0.325, 0.362)	No solution		(0.240, 0.505, 0.255)	(0.132, 0.182, 0.686)	(0.262, 0.451, 0.287)	(0.163, 0.221, 0.616)
B6	(0.402, 0.399, 0.199)	(0.485, 0.309, 0.206)	No solution		(1.000, 0.000, 0.000)	(1.000, 0.000, 0.000)	(1.000, 0.000, 0.000)	(1.000, 0.000, 0.000)
B7	(0.318, 0.319, 0.363)	(0.324, 0.359, 0.317)	No solution		(0.326, 0.213, 0.461)	(0.173, 0.562, 0.265)	(0.318, 0.198, 0.484)	(0.198, 0.500, 0.302)
B8	(0.498, 0.502, 0.000)	(0.520, 0.480, 0.000)	(0.498, 0.502, 0.000)	(0.520, 0.480, 0.000)	(0.000, 0.000, 1.000)	(0.000, 0.000, 1.000)	(0.000, 0.000, 1.000)	(0.000, 0.000, 1.000)
	(0.000, 0.000, 1.000)	(0.000, 0.000, 1.000)	(0.000, 0.000, 1.000)	(0.000, 0.000, 1.000)				

TABLE II: Equilibrium solutions for the eight games shown in Figure 4, computed using both the RBB and extended Blavatskyy models. Results are shown for RBB without and two choices of the loss aversion parameter (first three columns) and using the two different approaches to extend Blavatskyy to more than two choices.

actions. The models are different to the Nash equilibrium as we (1) introduce probabilistic beliefs; (2) replace best replying by each player with a probabilistic choice based on a set of axioms; and (3) treat the actions/beliefs as uncertain rather than random. In doing so, we are able to distinguish both the Blavatskyy and the RBB equilibrium with other non-Nash equilibrium concepts such as the QRE, BRNE, RBE and UPB. Hence, we hypothesize that when agents form subjective beliefs about the strategies of the other players then there might be opportunities for profitable deviation from the Nash equilibrium strategies when the other player is expected to deviate from playing the Nash strategy.

We test the closeness of fit of these models using data from Selten and Chmura (2008) for completely mixed 2×2 games. In particular, we show that the RBB model is able to explain the empirical data from completely mixed 2×2 games once loss aversion is included, with small deviations about the equilibrium solution. We show that the RBB equilibrium is generally as good as explaining the data compared to other stationary concepts discussed in Selten and Chmura (2008). In particular, we show that UPB and RBE are quite poor at explaining the data compared to Nash equilibrium. BRNE is comparable to Nash equilibrium in explaining the data. BRNE is also comparable to BRBB in the limiting case. The Blavatskyy model outperforms all models except RBB when comparing beliefs and not actions. The Blavatskyy model comparing actions is comparable to QRE but better than UPB, RBE, BRNE and Nash. RBB based on beliefs appears to have the best performance. RBB based on actions outperforms UPB and RBE but not others, showing comparble but slighly worse performance than BRNE. In contrast to the other equilibrium concepts discussed in the paper, the RBB and Blavatskyy equilibrium solutions are not based on optimising a parameter in order to obtain the best fit to the data. Assessing the RBB equilibrium involves comparing the beliefs that need to be held by the player against the predicted belief of the model.

We also show how the Blavatskyy model developed for Binary choice could be extended to more than two actions and discuss the various methods to solve for equilibrium. We also discuss the pros and cons of each method.

We make three contributions to the literature. First we outline the underlying assumptions of the different probabilistic choice models and fully explicate them in terms of probabilistic actions, probabilistic beliefs and hence their respective epistemic characterizations. Second, we test the models using empirical data and show their respective strengths and weaknesses in prediction. Third, we extend the Blavatskyy (2010) model to cases where there are more than two actions.

Our paper has a number of theoretical and managerial implications. From a theoretical perspective, the paper provides the axiomatic principles for the probabilistic choice models. Such an axiomatic basis for the models provide a more sound footing to develop future models and to be able to compare and contrast them with Nash equilibrium based models. From a managerial perspective, probabilistic model predictions could be very different and sometimes overturns the conventional logic predicted by Nash equilibrium models. For example, Velu, Iyer and Gair (2012) show that their probabilistic choice model provides support for the thesis that dominant firms invest more in research and development (R&D) within an asymmetric mixed-strategy game which is in contrast to Nash equilibrium models that predict less dominant firms would invest more. Hence, managers would need to understand when some of the assumptions of the models might most closely resemble the real world context in order to help them make more reliable predictions and hence, take optimal actions compared to merely basing the decision making on Nash equilibrium based models. Our results do have significant implications for policy formualtion, for example, for the formulation of competition or tax policies.

It is important to acknowledge that there are some limitations of our two primary models, namely the RBB and the Blavatskyy. First, the RBB model does not allow for negative returns in its basic set up. However, we provide one possibility to overcome this shortcoming through the zeroing algorithm discussed earlier. Second, the Luce rule suffers from the well known regularity of the duplicates problem outlined by Debreu, whereby the addition of duplicates to the choice set might alter the probability of choosing them in such a way that is inconsistent (Gul, Natenzon and Pesendorfer 2014). However, Gul, Natenzon and Pesendorfer (2014) propose a modification of the Luce rule whereby the attributes weighted by their value overcomes the duplicates problem. Third, the Luce rule might violate first order stochastic dominance (see Blavatskyy 2011). However, the RBB model is easy to compute and can be seen to be used by players in strategic situations. Blavatskyy's (2011) model of probabilistic choice satisfies first order stochastic dominance. However, the computation of the equilibrium especially beyod binary choice is extremely complex. Identifying these caveats provides avenues for further refinement of the RBB equilibrium concept as well as the Blavatskyy model.

Acknowledging these limitations, we argue that the Blavatskyy and RBB models are useful in game theory because they allow us to reconcile game theoretic predictions and empirical experimental results. The sensitivity of the equilibrium to changes in payoff is an important aspect of these models that accords more with reality than the conventional Nash equilibrium. Moreover, even if Blavatskyy and RBB do provide predictions in some situations, there may be other avenues for modeling non-Nash equilibrium behavior, which when combined with our model may provide additional insights. Therefore, we hope that this study provides a helpful prelude for better understanding the impulses which ultimately govern human behavior.

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ONLINE APPENDIX

Appendix A: Comparing the RBB and Blavatskyy models to observed data

Under the Luce (1959) rule, the probability of move i for player 1 is just $p_i \propto u^1_{ij} q_j$, where q_j is the player's belief about player 2's strategy. In the RBB equilibrium, q_j is determined from p_i via a similar equation. However, if player 1 is observed to play the strategy $p_i^{\rm obs}$, then this equation can be inverted to give $q_j^{\rm b} \propto (u^1)^{-1}_{jk} p_k^{\rm obs}$, which is the belief that player 1 must have had about the strategy of player 2 in order to lead them to play as they did. For any results observed in an experiment, it is possible either to compare the observed actions to the predictions of the RBB equilibrium solution, or to compare the the belief probabilities $q_j^{\rm b}$ that would lead to the observed behavior, $p_i^{\rm obs}$, computed in this way. If the person is playing a pure RBB equilibrium strategy, then $q_j^{\rm b} = q_j^{\rm rbbe}$.

For two-move games, there is only one probability, which we denote $q = q_1$, as $q_2 = 1 - q$. In Figure 5 we show $q^{\rm b}$ versus $q^{\rm rbbe}$ and $p^{\rm b}$ versus $p^{\rm rbbe}$ for all of the data in the Selten and Chmura games. The two plots correspond to transformed games with $\lambda = 1$ and $\lambda = 2$ respectively. We see that there is a tight correlation between the beliefs that players hold and the beliefs predicted in the RBB equilibrium, which reinforces the idea that the RBB equilibrium is a good model for describing choice behavior in games. However, there are some deviations. These might arise because the player is ambiguous as to the cooperative stance of the other player and therefore the likelihood that their opponent will play the RBB equilibrium solution.

When comparing the RBB and Blavatskyy equilibrium solutions to the data, we will

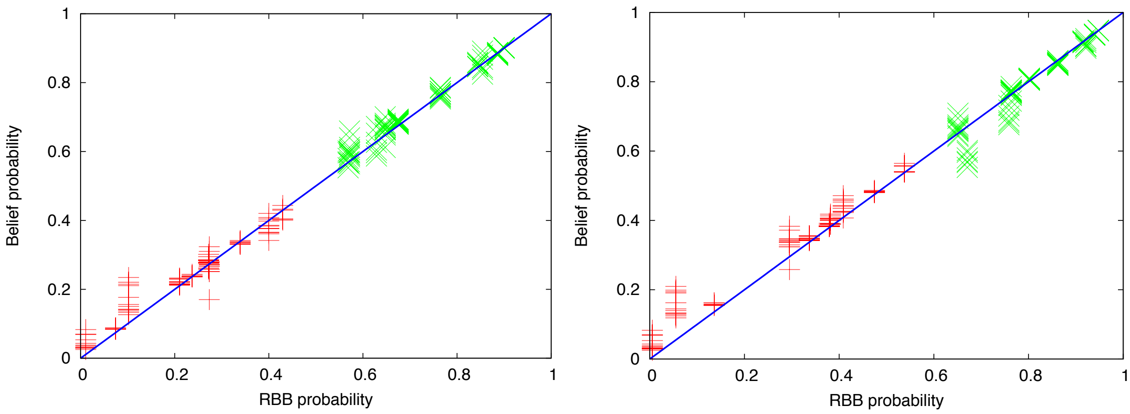

FIG. 5: Belief probability (y axis) versus RBB equilibrium probability (x axis) for all data in Selten and Chmura. The red plusses are values for p, the probability for the row player, while the green crosses are for q, the probability of the column player. The line indicates $q^{\mathrm{b}} = q^{\mathrm{rbbe}}$ to guide the eye. The left panel uses a multiplier $\lambda = 1$ to transform the game, while the right panel uses $\lambda = 2$.

compare the belief probabilities inferred from the observed data to the equilibrium solutions, rather than comparing the observed actions to the equilibrium solution.

Appendix B: Entropy maximization derivation of Luce rule based response

An alternative way to derive the Luce rule based response is through information entropy maximization (Miyano 2001). The information entropy of a particular mixed strategy $\{p_i\}$ is given by

$$S = -\sum p_i \ln(p_i). \tag{B1}$$

The probabilities are subject to the constraint $\sum p_i = 1$. If we assume that the player holds some belief about their opponent's strategy and can therefore calculate their expected utilities, u_i, from each strategy i, we can also impose a constraint that there is a fixed total utility $U = p_i u_i$. Maximizing the total entropy subject to these constraints, we need to extremize

$$-\sum p_i \ln(p_i) - \lambda_1 \sum p_i - \lambda_2 \sum p_i u_i \tag{B2}$$

where λ_1 and λ_2 are Lagrange multipliers that enforce the constraints. The maximum must satisfy

$$-\ln(p_i) - (1 + \lambda_1) - \lambda_2 u_i = 0 \qquad \forall i. \tag{B3}$$

Which gives the Luce-type rule

$$p_i = \frac{\exp(-\lambda_2 u_i)}{\sum_j \exp(-\lambda_2 u_j)}. \tag{B4}$$

If we take the utility to be the expected payoff, we obtain a logistic QRE-type response with parameter $-\lambda_2$. Taking a logarithmic utility function $u_i = \ln P^i$ we obtain a BRNE-type response[16] with rationality parameter $-\lambda_2$ and hence the RBB is found in the case $\lambda_2 = -1$. Other solutions obeying the Luce-type rule above will follow from other specifications of the utility function. Bernoulli used a logarithmic utility function to represent risk aversion in his solution of the St Petersburg paradox (Bernoulli 1738). Holt and Laury (2002) also demonstrated that Luce-type rules naturally incorporate a degree of risk aversion. Thus, another interpretation of the RBB model is as the action of a risk-averse entropy maximizing player. The demonstration above shows that many different Luce-type rules can be obtained from this prescription. As discussed earlier, we take the simplest, in which players play in proportion to their expected returns, since it is conceivable that a player in a game could be computing their expected returns when assessing their best strategy.

Appendix C: RBB equilibrium solution for two move, constant-sum games

In the first six of the Selten games, the form of the matrix is greatly simplified since there are only two moves for each player, and the games are constant sum. We show here that games of this form always have a unique RBB equilibrium solution. First, we reorder the moves for the players such that the guaranteed payoff is in the top left hand corner of the payoff matrix for player 1. If the constant sum is T then the payoff matrices take the form

$$u^1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad u^2 = \begin{pmatrix} T-a & T-b \\ T-c & T-d \end{pmatrix} \tag{C1}$$

with $a < b$ and either $c < a < d$ or $d < a < c$ since a is the guaranteed payoff. If $c < d$ then the two elements in the first column of player 2's payoff matrix are bigger than

[16] Note that we have not completely derived the QRE or BRNE model, since the response is a function of the expected utility, which depends on the player's subjective belief about their opponent. The response is QRE-like or BRNE-like, since the mathematical form is the same, although further conditions must be imposed on the opponent's action to arrive at the full QRE or BRNE solution.

the corresponding elements in the second column (due to the fact it is a constant sum game), and so the second column is a dominated strategy. There could be an incentive to play a dominated strategy if the players would mutually benefit from such cooperation. However, when $c < d$, the dominance in these games is of a more extreme form such that $\min(T-a, T-c) > \max(T-b, T-d)$ so the worst payoff to player 2 from choosing the left hand column is better than the best payoff to player 2 from the right hand column. It is clear that in such circumstances the second column would never be played and so we can restrict to games with no such extremely dominated strategies[17].

Therefore we need $d < a < c$, and there are two cases: (i) $c < b$; (ii) $c > b$. The transformed payoffs are then

$$u^1 = \begin{pmatrix} 0 & b-a \\ c-a & \lambda_1(d-a) \end{pmatrix},\ u^2 = \begin{pmatrix} c-a & \lambda_2(c-b) \\ 0 & c-d \end{pmatrix} \tag{C2}$$

Case(i)

$$u^1 = \begin{pmatrix} 0 & b-a \\ c-a & \lambda_1(d-a) \end{pmatrix},\ u^2 = \begin{pmatrix} b-a & 0 \\ \lambda_2(b-c) & b-d \end{pmatrix} \tag{C3}$$

Case (ii)

where λ_1, λ_2 are the loss-aversion multipliers, which we allow to be different for the two players.

The RBB equilibrium solution for player 1 is an eigenvector of the matrix $u^1(u^2)^T$. In case (i) there is an RBB equilibrium eigenstate

$$\begin{aligned} \text{Eigenvalue: } \mu_{(i)} &= \frac{1}{2}\sqrt{\left(\lambda_1(d-a)(c-d)+\lambda_2(b-a)(c-b)\right)^2 + 4(b-a)(c-d)(c-a)^2} \\ &\quad -\frac{1}{2}\left(\lambda_1(a-d)(c-d)+\lambda_2(b-a)(b-c)\right), \\ \text{Eigenvector: } \mathbf{p} &\propto \begin{pmatrix} (b-a)(c-d) \\ \mu_{(i)} - \lambda_2(b-a)(c-b) \end{pmatrix} \\ \mathbf{q} &\propto \begin{pmatrix} (b-a)(c-a)(c-d) \\ \mu_{(i)}(c-d) \end{pmatrix} \end{aligned}$$

[17] Formally if the payoff to player i from strategy $s^i \in S^i$ when his opponents play $s^{-i} \in S^{-i}$ is $u(s^i, s^{-i})$ then $p^i \in S^i$ dominates $q^i \in S^i$ if $u(p^i, s^{-i}) \geq u(q^i, s^{-i})\ \ \forall\, s^{-i} \in S^{-i}$, while $p^i \in S^i$ has this extreme dominance over $q^i \in S^i$ if $\min\left\{u(p^i, s^{-i}); s^{-i} \in S^{-i}\right\} \geq \max\left\{u(q^i, s^{-i}); s^{-i} \in S^{-i}\right\}$.

and in case (ii) there is an RBB equilibrium eigenstate

$$\begin{aligned}
\text{Eigenvalue: } \mu_{(ii)} &= \frac{1}{2}\sqrt{\left(\lambda_1(d-a)(b-d)+\lambda_2(c-a)(b-c)\right)^2+4(b-a)^2(b-d)(c-a)} \\
&\quad -\frac{1}{2}\left(\lambda_1(a-d)(b-d)+\lambda_2(c-a)(c-b)\right), \\
\text{Eigenvector: } \mathbf{p} &\propto \begin{pmatrix} (b-a)(b-d) \\ \mu_{(ii)} \end{pmatrix} \\
\mathbf{q} &\propto \begin{pmatrix} (b-a)^2(b-d)-\lambda_2\mu_{(ii)}(c-b) \\ (b-d)\mu_{(ii)} \end{pmatrix}
\end{aligned}$$

In the above $\mathbf{p}$ represents the eigenvalue of $u^1(u^2)^T$ which is the RBB equilibrium probability for player 1, while $\mathbf{q} = (u^2)^T\mathbf{p}$ is the corresponding RBB equilibrium probability for player 2. To be a valid RBB equilibrium solution, the eigenvalues must be positive and both $\mathbf{p}$ and $\mathbf{q}$ must lie in the positive quadrant. The preceding solutions satisfy these requirements for any choice of λ_1 and λ_2.

Appendix D: Description of other equilibrium concepts included in study

1. Utility Proportional Beliefs

The UPB model of Bach and Perea (2014) describes the choice process as one of players forming beliefs about the actions of their opponents. The belief structure is of the form

$$b_i(t_i)(c_j|t_j) - b_i(t_i)(c_j'|t_j) = \frac{\lambda_{ij}}{\overline{u}_j - \underline{u}_j}\left[u_j(c_j,t_j) - u_j(c_j',t_j)\right]. \tag{D1}$$

Here t_i denotes the type of the player and $b_i(t_i)(c_j|t_j)$ is the belief that a player of type i has that their opponent will make choice c_j given their belief that their opponent is of type t_j. These beliefs sum to 1 and can be interpreted as the probability that the player of type i assigns to their opponent making choice c_j. The quantity $u_j(c_j, t_j)$ is the expected utility of a player of type t_j from choice c_j, given their beliefs about their opponents' choices

$$u_i(c_i, t_i) = \sum_{c_{-i}} b_i(t_i)(c_{-i})U_i(c_i, c_{-i})$$

where $U_i(c_i, c_{-i})$ is the utility to player i form the choice combination (c_i, c_{-i}). Finally, $\overline{u}_i$ and $\underline{u}_i$ denote the maximum and minimum possible utilities player i can obtain in the game.

The multiplier λ_{ij} is a free parameter. There is a maximum value that λ_{ij} can take to ensure all belief probabilities are positive. For a given set of $u_j(c_j, t_j)$'s (which means for a fixed specification of the opponent's beliefs) the minimum belief probability of player i is

$$(b_i(t_i)(c_j, t_j))_{\min} = \frac{1}{|C_j|} + \frac{\lambda_{ij}}{\overline{u}_j - \underline{u}_j} \left(\min_{c_j^*} u_j(c_j^*, t_j) - u_j^{\mathrm{av}}(t_j) \right)$$

where $|C_j|$ denotes the size of the choice set for player j and $u_j^{\mathrm{av}}(t_j) = \sum_{C_j} u(c_j, t_j)/|C_j|$. Hence

$$\lambda_{ij}^{\max}(b_j) = \frac{|C_j|(\overline{u}_j - \underline{u}_j)}{\left(u_j^{\mathrm{av}}(t_j) - \min_{c_j^*} u_j(c_j^*, t_j)\right)}.$$

This quantity is a function of $u_j(c_j, t_j)$, which in turn depends on the beliefs the opponents hold, but λ_{ij} should be a constant. Therefore we follow a minimax procedure and take a minimum of these maximum values

$$\lambda_{ij}^{\max} = \min_{\{b_j\}} \lambda_{ij}^{\max}(b_j).$$

Bach and Perea (2014) suggest setting the λ_{ij}'s to these maximum values when evaluating the UPB. A UPB equilibrium is an equilibrium in beliefs, i.e., when the belief probabilities used to by each player to evaluate their opponent's utilities and hence determine their own belief probabilities coincide with the belief probabilities computed by each player. Players are then assumed to take actions which maximise their expected utilities given their belief probabilities, so in the UPB model beliefs and actions do not coincide. Bach and Perea (2014) show that there is a unique UPB solution in two player games which is obtained by iteratively eliminating utility-disproportional beliefs. They also discuss 'level-k' solutions in which the elimination of utility-disproportional beliefs is done only k times, for which the final solution is not guaranteed to be a singleton.

The form of the UPB model is similar to that of RBB. Setting the UPB utility multiplier to

$$\lambda_{ij} = \frac{(\overline{u}_j - \underline{u}_j)}{\sum_{C_j} u_j(c_j, t_j)}$$

in Eq. (D1) we obtain

$$b_i(t_i)(c_j|t_j) - b_i(t_i)(c'_j|t_j) = \frac{u_j(c_j|t_j) - u_j(c'_j|t_j)}{\sum_{C_j} u_j(c_j, t_j)}$$

which coincides with the RBB solution given in Eq. (2). We note, however, that with this assignment the multiplier λ_{ij} is no longer a constant for a particular game, but depends on the utilities $u_j(c_j, t_j)$, which depend on the particular assignment of belief probabilities to the player's opponents. If the λ_{ij}'s are set to the λ_{ij}'s given by the above formula for utilities evaluated at the final equilibrium solution, then the UPB and RBB equilibrium solutions would coincide. However, this choice of λ_{ij} might exceed the maximum allowed and hence might be inconsistent with an iterative solution that converges to the equilibrium. In addition, this does not explain how the players reach their particular λ_{ij} assignments. There is also a philosophical difference between the two models. In the RBB model, Eq. (2) represents a players own action probabilities, while in the UPB model Eq. (D1) represents the opponent's assignment of probabilities to their actions. In the RBB equilibrium there is an equilibrium in beliefs, but then the assumed (mixed strategy) actions coincide with those beliefs. In the UPB equilibrium there is only equilibrium in beliefs, and the subsequent utility-maximising pure strategy actions do not coincide with those beliefs.

In assessing the UPB model on the Selten and Chmura games here we assume that there are just two player types, a row player and a column player. We set the λ_{ij}'s to their minimax values as described above and we assume infinite iterative elimination of utility-disproportional beliefs so the UPB solution is unique. Finally, we assume that the action of each player is utility maximising given the equilibrium belief probabilities, so the actions are pure strategies. The assumed values for the λ_{ij}'s and resulting UPB equilibrium beliefs and actions are given in Table III. As can be seen from the results in Table I, the UPB model applied in this way does not reproduce the experimental results particularly well. Inclusion of loss aversion prior to applying the UPB model did not make a significant difference to the quality of fit. The performance was slightly improved by comparing the belief probabilities predicted by UPB to the observed actions, but even then the performance was still the worst of the various equilibrium concepts. Our interpretation of the UPB solution is consistent with the presentation in Bach and Perea (2014), but there may be alternative interpretations which would provide a better fit to the data.

2. Random Belief Equilibrium

The Random Belief Equilibrium (RBE) model was proposed in Friedman and Mezzetti (2005). In this model, each player, denoted i, is assumed to have an associated belief measure $\mu_\epsilon^i(\sigma_{-i}^f)$, which is a probability distribution on the actions of their opponents. This

Game	$\lambda^{\max}_{\text{col,row}}$	$\lambda^{\max}_{\text{row,col}}$	$b_{\text{col}}(\text{up})$	$b_{\text{row}}(\text{left})$	Action (col)	Action (row)
1	0.1	0.1	0.232	0.422	right	down
2	0.125	0.111	0.296	0.430	right	down
3	0.1	0.125	0.274	0.499	right	down
4	0.111	0.143	0.324	0.531	left	down
5	0.125	0.143	0.390	0.480	right	down
6	0.143	0.167	0.419	0.533	left	down
7	0.1	0.1	0.232	0.422	right	down
8	0.125	0.111	0.296	0.430	right	down
9	0.1	0.125	0.274	0.499	right	down
10	0.111	0.143	0.324	0.531	left	down
11	0.125	0.143	0.351	0.510	left	down
12	0.143	0.167	0.419	0.533	left	down

TABLE III: UPB equilibrium solutions for the 12 two player games summarised in Figure 1. The columns in this table give the game number, the two assumed λ_{ij}'s (determined as the maximum allowed values), the belief probabilities for the UPB equilibrium solution and finally the corresponding utility-maximising actions.

belief measure is parameterised by a focus, σ^f_{-i}, characterising the mean of the distribution, and additional parameters, ϵ, characterising its variance. In any game, the players draw a random sample from their belief measure and best reply to that strategy and so their actions are characterised by the expectation of the best reply over this belief measure

$$\psi_{\mu^i_\epsilon}\left(\sigma^f_{-i}\right) = \int b_i(\sigma^R_{-i})\mathrm{d}\mu^i_\epsilon \tag{D2}$$

in which $b_i(\sigma^R_{-i})$ denotes the best reply of player i to the mixed strategy σ^R_{-i}. This expression is formally only valid when the best reply is single valued except on a set of measure zero, which is the case for the experimental comparison included here. However, Friedman and Mezzetti (2005) also present a generalisation to cases where the best response can be multi-valued. The equilibrium solution is when the expected best reply coincides with the focus of the belief measure, $\psi_{\mu_\epsilon}\left(\sigma^f\right) = \sigma^f$.

For the comparison to experimental results presented in the original RBE paper, the authors assumed a Dirichlet distribution with focus equal to the mean, and a common variance, ϵ^2, for all players. We adopt the same model here and choose the value of ϵ^2 to minimize the total squared error. For 2x2 games the Dirichlet distribution reduces to a Beta distribution, and the best reply function takes a simple form. If the payoffs to player 1 are

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and we write $\sigma_2 = (q, 1-q)^T$, then $b_i(\sigma_2) =$ "up" if $(a+d-b-c)q > (d-b)$ or "down" otherwise (with a measure zero non-unique response when equality holds). Denoting the transition point by q_{crit}, with p_{crit} for the column player defined similarly, the RBE solution defined by Eq. (D2) depends only on these two numbers and ϵ^2. For the Selten and Chmura (2008) experimental games, Games 1 and 7 have the same p_{crit} and q_{crit} and hence the same RBE solution. This is also true for games 2 and 8, 3 and 9 etc. This property of the model is partially responsible for the comparatively poor performance of RBE in explaining the experimental results. More flexibility in the RBE could be achieved by modifying the assumed form of the belief measure, but we do not consider this here.

3. Boundedly Rational Nash Equilibrium

The Boundedly Rational Nash Equilibrium (BRNE) was proposed in Chen, Friedman and Thisse (1997). It results in actions of the form

$$p_i = \frac{u_i(\mathbf{q})^\mu}{\sum_j u_j(\mathbf{q}^\mu}$$

where $\mathbf{q}$ is the assumed strategy of the opponent, $u_i(\mathbf{q})$ is the utility to the player from choice i, given the assumed strategy of their opponent, and μ is the rationality parameter. The BRNE admits an equilibrium, where actions and beliefs coincide. The parameter μ is a free parameter which can be tuned to improve the fit of the model to any given data set. In the limit $\mu \to \infty$ the player places all weight on the action that maximises their expected utility, and so the BRNE tends to the Nash equilibrium. In the limit $\mu = 1$, the BRNE reduces to the same form as the Luce rule in Eq. (2) and hence coincides with the RBB equilibrium, assuming utilities and payoffs coincide.

In the analysis of the experimental results presented in Section IV we evaluated the BRNE assuming that the utility $u_i(\mathbf{q})$ was equal to the expected payoff. For non-integer

μ, quantities of the form u_i^μ are complex if $u_i < 0$, which could arise when including loss aversion. For this reason, results for the analysis of BRNE are reported without the inclusion of loss aversion, although we did verify that the quality of fit was not significantly improved when loss aversion was incorporated.