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# *On the surface area of scalene cones and other conical bodies*

Leonhard Euler

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**Translator's note:** This paper first appeared in the *Novi Commentarii academiae scientiarum Petropolitanae*, vol. 1, 1750, pp. 3–19, reprinted in the *Opera Omnia*: Series 1, Volume 27, pp. 181–199. Its Eneström number is E133. This translation and the Latin original are available from the Euler Archive. (https://scholarlycommons.pacific.edu/euler-works/133/.) The original page of Euler's figures follows my translation.

All footnotes and bracketed comments are mine. The format of the notation and the display of the equations have been somewhat modernized. Two major exceptions are that I have written $ff$ for $f^2$ in all cases where Euler used the older notation, and I have left the differential where Euler put it in each integral.

I am grateful for the many excellent suggestions (and corrections of my blunders) made by the reviewer. The remaining errors are mine.

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§ 1. Although the nature of the cone has been studied for so long that it would seem there is nothing overlooked on which we might labor, nevertheless, those before us have not proceeded to measure the surface area of cones other than the right cones, in which the axis is normal to the base. The most famous Varignon first brought forth a new argument in the *Miscell. Societatis Regiae Berolinensis Continuatione II*, where he discovered a curved line whose construction depended on the quadrature of the circle, through whose rectification the area of any scalene cone may be accomplished. Joined to that dissertation may be found the addition of the great Leibniz, in which the same task is achieved through the rectification of an algebraic curve. The construction of this curve provides an excellent example of the most profound talent of the author. In truth, an inadvertent error[1] has crept into the solution of this otherwise most wise man, which, as it may easily be corrected, takes nothing away from the

[1]Euler uses the Greek word *sphalma*, written in Latin, perhaps to soften the criticism even further.

excellence of this solution. It expresses the surface area of a scalene cone as a rectangle from right lines given in magnitude by the arc of a curved line. It has been shown by this construction that the arc may be derived from any prior algebraic quantity. Thus it seems indeed to me not unprofitable in setting forth my work if I first will present the surface area of the scalene cone by means of the rectification of algebraic curves of degree six, and then I will give the the surface area of any conoid by algebraic curved lines, and at the same time correct the slip of the great Leibniz.

§ 2. Let the circle $AMB$ be the base of the scalene[2] cone, whose vertex is put in the highest point $V$. To the base plane is dropped the perpendicular, $VD$. Then from the point $D$ through the center of the circle $C$ draw the line $DACB$. Thus the surface of this cone will be generated by straight lines, always passing through $V$, that go around the circumference of the circle $AMB$. The part of this surface corresponding to the arc $AM$ is bound by the arc $AM$ and the two lines from the points $A$ and $M$ to the vertex $V$. It is required to find a plane area equal to this part of the curved figure. Let the radius of the base $AC = BC = a$, the length of the axis $VC = f$, the perpendicular $VD = b$, and the interval $CD = c$. Thus $ff = bb+cc$.[3] Hence the smallest side of the cone is $VA = \sqrt{bb + cc - 2ac + aa}$ and the largest side $VB = \sqrt{bb + cc + 2ac + aa}$. Now given any arc $AM$ let the angle $ACM = u$, so the arc $AM = au$. Its element is $Mm = a\,du$. From the point $M$ is taken the tangent $MQ$, where the perpendicular to it, $DQ$, is taken from $D$. Thus the line $VQ$ will be normal to the tangent $MQ$. Thus if the lines $VM$ and $Vm$ are formed, the area of the [infinitesimal] triangle $MVm = \frac{1}{2} Mm \cdot VQ$. This region will be the differential of the portion of the surface area of the cone $AVM$, which is what we are seeking. Fig.1

§ 3. Therefore since we wish to investigate the length of the perpendicular $VQ$ in terms of the radius $CM$, if that is needed, let us construct the normal $DN$, produced from $D$, which will be parallel to and equal to the tangent $MQ$, and therefore $DQ = MN$. Thus then in the right triangle $DCN$, since the hypotenuse $CD = c$ and angle $DCN = u$, then $CN = c\cos u$, and from this $MN = DQ = c\cos u - a$. Now since the triangle $VDQ$ has its right angle at $D$,

$$VQ = \sqrt{bb + cc\cos^2 u - 2ac\cos u + aa}\,. \quad {}^4$$

[2]We might say oblique. In any event the vertex is not assumed to be directly over the center of the circle.

[3]Euler usually writes $xx$ for $x^2$, etc, as was customary then. He does sometimes use the $x^2$ notation.

[4]Euler writes $cosu^2$ .

From this the area of the elemental [infinitesimal] triangle

$$MVm = \frac{1}{2} Mm \cdot VQ = \frac{1}{2} a\, du \sqrt{bb + (c \cos u - a)^2}.$$

For that reason the conic surface area

$$AVM = \frac{1}{2} a \int du \sqrt{bb + (c \cos u - a)^2}.$$

From this it can be seen that if the cone is a right cone, in which case the interval $CD = c$ vanishes, the surface area of the right cone on the arc $AM$ will reduce to

$$\frac{1}{2} a \int du \sqrt{aa + bb} = \frac{1}{2} au \sqrt{aa + bb}.$$

Thus it is equal to the area of the triangle whose base is $au$, which is equal to the arc $AM$, and whose height is $\sqrt{aa + bb} = VA$, which agrees with the elementary result.

§ 4. From the equation

$$AVM = \frac{1}{2} a \int du \sqrt{bb + (c \cos u - a)^2}$$

the construction of the Varignonian curve follows immediately, from which the rectification of the conic surface can be produced. In fact such a curve may be formed between orthogonal coordinates $p$ and $q$ by taking $dp = b\, du$ and $dq = du(c \cos u - a)$, so the [arc length] element of this curve will be $du\sqrt{bb + (c \cos u - a)^2}$. Hence the arc length of this curve, multiplied by $\frac{1}{2}a$, will produce a rectangle whose area is equal to the surface area of the cone $AVM$. Now this curve will have abscissa $p = bu = \frac{VD \cdot AM}{AC}$ and ordinate $q = c \int du \cos u - au = c \sin u - au$, from which the abscissa $p = \frac{b}{a} \cdot AM$ corresponds to the ordinate $q = QM - AM$. The curve therefore can be easily constructed from the rectification of the circle. It will be clear to the attentive reader that this curve is the same as Varignon discovered.

§5. If we would like to express the surface area of this cone by the quadrature of curves, it can be done without any difficulty in infinitely many ways, either by algebraic or transcendental curves. Indeed the greatest of geometers already anticipated the construction of transcendental problems, which they achieved through the rectifications of curves, primarily algebraic, since it is easier, in practice anyhow, to assign the length of some curves than the area. For this reason, at the same time when this question was raised in the Miscellany of the Royal Society, the renowned Varignon was seen as not a little deserving of merit in that he reduced the quadrature of the surface area of the scalene cone to the rectification of curves, whose construction may be easily obtained from the rectification of the circle. Without a doubt, Leibniz's solution for all surfaces of cones using algebraic curves would have been accounted among the very best, excelling that of Varignon, were it not that due to the error mentioned above it lacks usefulness. Now however, after the most broadly extended method of the quadrature of all curves was discovered by Hermann, reducing it to the rectification of algebraic curves, the goal which Verignon and Leibniz set themselves will be achieved almost without any difficulty.

§ 6. To this end, let us eliminate the transcendental quantity $u$ from the obtained formula

$$\frac{1}{2}a\int du\sqrt{bb+(c\cos u-a)^2}$$

by taking the cosine of the angle $u = z$. Thus, taking from $M$ the perpendicular $MP$ to the diameter, let $CP = az$ and $MP = a\sqrt{1-zz}$, then $du = \frac{-dz}{\sqrt{1-zz}}$, and the desired surface area of the cone

$$AVM = -\frac{1}{2}a\int\frac{dz\sqrt{bb+(cz-a)^2}}{\sqrt{1-zz}}.$$

Now for the algebraic curves, by means of whose rectification this surface area can be measured, let the abscissa be $x$ and the ordinate be $y$, and let $dy = p\,dx$. Thus its element would be $dx\sqrt{1+pp}$. So it is to be proved that the integral

$$\int dx\sqrt{1+pp}$$

depends on the integration formula

$$\int\frac{dz\sqrt{bb+(cz-a)^2}}{\sqrt{1-zz}}.$$

First it is required that $\int p\,dx$ be an algebraic quantity, otherwise the curve would not be algebraic. Therefore since

$$\int p\,dx = px - \int x\,dp,$$

let $\int x\,dp = q$, and then

$$x = \frac{dq}{dp}$$

and

$$y = \int p\,dx = \frac{p\,dq}{dp} - q.$$

Let the arc [length] of this curve be called $s$. Since

$$s = \int dx\sqrt{1+pp},$$

then

$$s = x\sqrt{1+pp} - \int \frac{xp\,dp}{\sqrt{1+pp}}.$$

Thus the rectification of the curve depends on the integration of the formula

$$\int \frac{xp\,dp}{\sqrt{1+pp}},$$

which formula, since $x\,dp = dq$, is changed to

$$\int \frac{p\,dq}{\sqrt{1+pp}},$$

which is further reduced to

$$\frac{pq}{\sqrt{1+pp}} - \int \frac{q\,dp}{(1+pp)^{3/2}}.$$

Thus the arc length of the curve will become

$$s = \frac{dq\sqrt{1+pp}}{dp} - \frac{pq}{\sqrt{1+pp}} + \int \frac{q\,dp}{(1+pp)^{3/2}}.$$

Let it be given that

$$\int \frac{q\,dp}{(1+pp)^{3/2}} = \int \frac{dz\sqrt{bb+(cz-a)^2}}{\sqrt{1-zz}},$$

which would make

$$q = \frac{dz(1+pp)^{3/2}\sqrt{bb+(cz-a)^2}}{dp\sqrt{1-zz}},$$

where for $p$ any algebraic function of the same $z$ may be assumed. Once that is done $q$ will be expressed as an algebraic function of the same $z$, from which, further, the curve sought will be defined in terms of the coordinates $x$ and $y$.

§ 7. Thus having described this curve by means of the coordinates

$$x = \frac{dq}{dp}$$

and

$$y = \frac{p\,dq}{dp} - q,$$

if the arclength of this curve is called $s$, from

$$s = \frac{dq\sqrt{1+pp}}{dp} - \frac{pq}{\sqrt{1+pp}} + \int \frac{q\,dp}{(1+pp)^{3/2}},$$

our formula will be made, from which the portion $AVM$ of the surface area of the cone is determined as

$$\int \frac{dz\sqrt{bb+(cz-a)^2}}{\sqrt{1-zz}} = s - \frac{dq\sqrt{1+pp}}{dp} + \frac{pq}{\sqrt{1+pp}} + Const.$$

The constant is determined thus: when $z$ is set equal to $0$, this formula would vanish. Thus the area

$$\frac{1}{2}a\left(s - \frac{dq\sqrt{1+pp}}{dp} + \frac{pq}{\sqrt{1+pp}} + Const.\right)$$

will equal the portion $EVM$ of the surface of the cone, where indeed the angle $ACE$ is set as a right angle.

§ 8. In order to illustrate the matter with an example, let us set

$$p = \frac{z}{\sqrt{1-zz}}$$

so

$$\sqrt{1+pp} = \frac{1}{\sqrt{1-zz}}$$

and

$$dp = \frac{dz}{(1-zz)^{3/2}};$$

then

$$q = \frac{\sqrt{bb+(cz-a)^2}}{\sqrt{1-zz}},$$

$$\frac{dq}{dp} = \frac{bbz+(c-az)(cz-a)}{\sqrt{bb+(cz-a)^2}} = x,$$

and

$$y = \frac{\big(a(cz-a)-bb\big)\sqrt{1-zz}}{\sqrt{bb+(cz-a)^2}}.$$

From this the portion of the surface of the cone

$$EVM = \frac{1}{2}a\left(s - \frac{c(cz-a)\sqrt{1-zz}}{\sqrt{bb+(cz-a)^2}} + Const.\right).$$

Then this constant needs to be determined by the formula's vanishing when $z$ is set to $0$. In a similar way, by taking any other value for $p$, innumerable algebraic curves will be obtained by whose rectification the surface area of any cone can be expressed.

§ 9. In curved lines of such kind the arc length is proportional to the surface area of the cone, but it is always necessary to increase it or decrease it by some algebraic quantity, so that an expression might be produced that measures the surface area of the cone exactly. In such case, when possible, such curved lines, whose lengths immediately produce the desired result without adding any other quantity, will usually be justly preferred to others. Therefore it will not be unreasonable to select such an algebraic curve, like Varignon's transcendental curve, that would itself measure exactly the surface area of any portion of a cone without adding any other quantity. Since, therefore, the portion $EVM$ would be expressed by the formula

$$\frac{1}{2}a\int\frac{dz\sqrt{bb+(cz-a)^2}}{\sqrt{1-zz}},$$

the algebraic curve should be investigated whose element is

$$\frac{dz\sqrt{bb+(cz-a)^2}}{h\sqrt{1-zz}}.$$

In fact, for this curve, if the arc length corresponding to the quantity $z$ is set equal to $s$, the portion EVM of the surface area of the cone will be equal to $\frac{1}{2}ahs$.

§ 10. Let the coordinates of the desired curve be $x$ and $y$, which are to be expressed by algebraic functions of $z$. Let

$$dx = \frac{dz(m+kz)}{\sqrt{1-z}}$$

and

$$dy = \frac{dz(n+kz)}{\sqrt{1+z}}.$$

Thus by integration will be obtained

$$x = 2m + \frac{4}{3}k - \left(2m + \frac{4}{3}k + \frac{2}{3}kz\right)\sqrt{1-z}$$

$$y = -2n + \frac{4}{3}k - \left(2n - \frac{4}{3}k + \frac{2}{3}kz\right)\sqrt{1+z}$$

Appropriate constants are chosen so that when $z = 0$, which happens at point $E$, both coordinates $x$ and $y$ will vanish. From this is derived the equation

$$\left.\begin{array}{c} +xx - 4mx - \frac{8}{3}kx \\ +yy + 4ny - \frac{8}{3}ky \end{array}\right\} = \left\{\begin{array}{c} 4(n-m)(n+m)z + \frac{8}{3}(n-m)kzz \\ -\frac{8}{3}(n+m)kz - \frac{8}{3}kkzz \end{array}\right),$$

from which the value of $z$ in terms of $x$ and $y$ may be easily determined, which, when substituted into the other equation, will give the algebraic equation between $x$ and $y$, by means of which the nature of the sought curve will be expressed.

§ 11. Since now

$$dx = \frac{(m+kz)dz}{\sqrt{1-z}}$$

and

$$dy = \frac{(n+kz)dz}{\sqrt{1+z}},$$

the element of the curve will be

$$\sqrt{dx^2+dy^2} = dz\sqrt{\frac{m^2+2mk+k^2zz}{1-z} + \frac{n^2+2nkz+k^2zz}{1+z}},$$

or

$$\sqrt{dx^2+dy^2} = \frac{dz\sqrt{\left(\begin{array}{c} +nn - nnz + 2nkz - 2nkzz \\ +mm + mmz + 2mkz + 2mkzz \end{array} + 2k^2z^2\right)}}{\sqrt{1-zz}}.$$

This [element] is set equal to the form

$$\frac{dz\sqrt{(aa+bb-2acz+cczz)}}{h\sqrt{1-zz}},$$

and from the comparison of homogeneous terms these equations result:

$$\begin{aligned} aa + bb &= (nn+mm)hh, \\ 2ac &= (n-m)(n+m)hh - 2(n+m)khh, \\ cc &= 2k^2h^2 - 2(n-m)khh. \end{aligned}$$

From the last of these

$$n - m = k - \frac{cc}{2khh} = \frac{2kkhh - cc}{2khh},$$

which value substituted in the second equation gives:

$$2ac = -\frac{(n+m)(2kkhh + cc)}{2k}.$$

Therefore

$$n + m = \frac{-4ack}{2kkhh + cc},$$

while we already have:

$$n - m = \frac{2kkhh - cc}{2khh}.$$

From these equations both letters $m$ and $n$ may be determined.

§ 12. Therefore there remains that the third unknown $k$ be determined by the first equation. Since the fourth unknown $h$ remains indeterminate, it may be assigned any value whatever. So let us take

$$hh = \frac{cc}{2kk}$$

Then

$$n - m = 0,$$

$$n + m = -\frac{2ak}{c},$$

and therefore

$$m = n = -\frac{ak}{c}.$$

When this substitution is made in the first equation the unknown $k$ is removed from the calculation. Thus it cannot be that $m = n$. For this reason, let us set

$$2kkhh = gcc$$

or

$$hh = \frac{gcc}{2kk}.$$

Then

$$n - m = \frac{(g-1)k}{g}$$

and

$$n + m = \frac{-4ak}{(g+1)c}.$$

From this

$$n = \frac{(g-1)k}{2g} - \frac{2ak}{(g+1)c} = \frac{(gg-1)ck - 4agk}{2g(g+1)c}$$

and

$$m = \frac{-2ak}{(g+1)c} - \frac{(g-1)k}{2g} = \frac{-4agk - (gg-1)ck}{2g(g+1)c}.$$

§ 13. From these values is obtained

$$mm + nn = \frac{16aaggkk + (gg-1)^2 cckk}{2gg(g+1)^2 cc}.$$

Hence from the first equation,

$$aa + bb = (nn + mm)hh,$$

is obtained

$$aa + bb = \frac{16aagg + (gg-1)^2 cc}{4g(g+1)^2}.$$

Let

$$aa + bb = ee$$

and this derived equation will give[5]

$$\begin{aligned} ccg^4 - 4eeg^3 - 2ccgg &- 4eeg + cc \\ + 16aagg& \\ - 8eegg&, \end{aligned}$$

from which the value of $g$ should be obtained.

§14. Although this equation is of fourth order yet, because it is not changed if $\frac{1}{g}$ is put in place of $g$, it can be reduced to the resolution of quadratic equations. The factors $cgg - 2pg + c = 0$ and $cgg - 2qg + c = 0$ may be created, and the product of these equations set equal. There will thus be this result:

$$\begin{aligned} ccg^4 - 2cpg^2 + 2ccgg - 2cpg + cc = \ 0 \\ - 2cqg^3 + 4pqgg - 2cqg \qquad \end{aligned}$$

This form, when compared to the derived equation, gives

$$p + q = \frac{2ee}{c}$$

[5] For us 0 should appear on the right-hand side. He lines up the $gg$ terms to help visualize solving for $g$.

and

$$pq = 4aa - 2ee - cc,$$

from which

$$(p-q)^2 = \frac{4e^4}{ec} - 16aa + 4ee + 4cc$$

and

$$p - q = \frac{2}{c}\sqrt{e^4 - 4aacc + 2ccee + c^4}.$$

Consequently

$$p = \frac{ee + \sqrt{e^4 - 4aacc + 2ccee + c^4}}{c}$$

and

$$q = \frac{ee - \sqrt{e^4 - 4aacc + 2ccee + c^4}}{c}.$$

§ 15. Now with $p$ and $q$ obtained from the equations above the values of $g$ will be given by

$$g = \frac{p \pm \sqrt{pp - cc}}{c}$$

and

$$g = \frac{q \pm \sqrt{qq - cc}}{c}.$$

Thus since we have found four values for the quantity $g$, we will have first

$$hh = \frac{gcc}{2kk}$$

or, taking the quantity $h$ as arbitrary, it will be

$$k = \frac{a}{h}\sqrt{\frac{1}{2}g}.$$

Further, it yields

$$m = \frac{-(g-1)k}{2g} - \frac{2ak}{(g+1)c},$$

$$n = \frac{+(g-1)k}{2g} - \frac{2ak}{(g+1)c}.$$

Finally from the known values of the letters $m$, $n$, and $k$ the desired curve may be described algebraically through the coordinates $x$ and $y$ shown above. Having done that, if the arc of the corresponding quantity $z$ is called $s$ then the surface area of the portion of the cone $EVM = \frac{1}{2}ahs$.

§ 16. Let us take an example. Let the axis $VC$ of the cone make an angle of $60°$ with the base. Drop the perpendicular $VD$ on the periphery so that the base $CD = CA$ and, therefore, $c = a$. From this, $CV = f = 2a$ and $bb = 3aa$, and so $ee = 4aa$, and thus $p = a(4 + \sqrt{21})$ and $q = a(4 - \sqrt{21})$. From this $g = 4 + \sqrt{21} + 2\sqrt{9 + 2\sqrt{21}}$ because the two remaining values are imaginary. Therefore

$$\sqrt{\frac{1}{2}g} = \frac{1}{4}\sqrt{14} + \frac{1}{4}\sqrt{6} + \frac{1}{2}\sqrt{3 + \sqrt{21}}.$$

Let $h = 1$ so

$$k = \frac{a}{4}\left(\sqrt{14} + \sqrt{6} + 2\sqrt{3 + \sqrt{21}}\right).$$

When these irrationalities are further reduced it is found that

$$m = \frac{a}{4}\left(\sqrt{6} - \sqrt{14} - 2\sqrt{3 + \sqrt{21}}\right) \text{ and}$$

$$n = \frac{a}{4}\left(\sqrt{6} - \sqrt{14} - 2\sqrt{3 + \sqrt{21}}\right).$$

From the given values the curve may be described in the coordinates $x$ and $y$ as follows

$$x = \frac{4}{3}k + 2m - \left(2m + \frac{4}{3}k + \frac{2}{3}kz\right)\sqrt{1 - z},$$

$$y = \frac{4}{3}k - 2n + \left(2n - \frac{4}{3}k + \frac{2}{3}kz\right)\sqrt{1 + z}.$$

For this curve, if the arc of the sine of the angle $ECM$, which corresponds to $z$, is set equal to $s$, the surface area of the portion of the cone $EVM = \frac{1}{2}as$.

§ 17. Having taken care of the scalene cones that have circular bases, and where the perpendicular from the vertex dropped to the base plane falls outside the center, now I will consider any cones that are formed when the straight lines through the vertex are led around any curve whatever. Therefore let any figure $AM$ be the base of such a cone, with a point $V$ placed above as its vertex, and let a perpendicular $VD$ be dropped to the base. From $D$ to any point $M$ on the curve $AM$, let there be drawn a segment $DM$; and at $M$ let there be drawn a tangent $MQ$ to the curve onto which a perpendicular $DQ$ may be dropped from $D$. With the known base given, a relation may be assigned between $DM$ and $DQ$. Therefore letting $DM = x$ and $DQ = y$, there will Fig. 2

be an equation between $x$ and $y$. In addition, let us set the altitude of this cone $VD = b$. Taking the element of this curve as $Mm$, if $Dm$ is drawn and from $M$ a perpendicular $Mn$ is dropped to $Dm$, then $mn = dx$, and from $MQ = \sqrt{xx - yy}$. Because by the similarity of the triangles $DMQ$ and $Mmn$,

$$Mn = \frac{y\,dx}{\sqrt{xx - yy}}$$

and

$$Mm = \frac{x\,dx}{\sqrt{xx - yy}}.$$

§ 18. In order to proceed, let a fixed base point $A$ on the curve be taken to be the initial point. The surface area of the portion of the cone $AVM$ will be the integral of the triangular element $MVm$. Therefore to express the small region of this triangle the line $VQ$ is connected, which is normal to the tangent $MQ$, and thus the area of the triangle

$$MVm = \frac{1}{2} Mm \cdot VQ.$$

Indeed, because the triangle $VDQ$ has right angle at $D$,

$$VQ = \sqrt{bb + yy}.$$

Since

$$Mm = \frac{x\,dx}{\sqrt{xx - yy}},$$

the area of the elementary triangle may be taken to be

$$MVm = \frac{x\,dx\sqrt{bb + yy}}{2\sqrt{xx - yy}}.$$

Thus the desired surface area of the portion of the cone

$$AVM = \frac{1}{2}\int \frac{x\,dx\sqrt{bb + yy}}{\sqrt{xx - yy}}.$$

§ 19. The surface of a cone may be portrayed in a most natural way when it is unfolded onto a plane. Consider a cone spread out on a sheet which is drawn between lines $AV$ and $MV$ and base $AM$ in the plane $VAM$. This mixti-linear figure $VAM$ will be equal to the surface of the portion of the cone Fig. 3

$$AVM = \frac{1}{2}\int \frac{x\,dx\sqrt{bb + yy}}{\sqrt{xx - yy}}.$$

Once this figure is set out the tangent $MQ$ is taken at $M$ and to it is dropped the perpendicular $VQ$ from $V$. Since this triangle $VMQ$ is similar to and equal to triangle $VMQ$ in *Fig. 2*, $VM = \sqrt{bb+xx}$, $VQ = \sqrt{bb+yy}$, and $MQ = \sqrt{xx-yy}$. Then after setting up the elemental triangle $MVm$, and taking $Mr$ perpendicular to $Vm$, there will be first

$$Mm = \frac{x\,dx}{\sqrt{xx-yy}},$$

then

$$mr = \frac{x\,dx}{\sqrt{bb+xx}},$$

and

$$Mr = \frac{x\,dx\sqrt{bb+yy}}{\sqrt{(bb+xx)(xx-yy)}}.$$

§20. Let us investigate the construction of this curve from the given base of the cone in *Fig. 2* To this end, let us set the angle $AVM = v$ and the distance $VM = z$, from which immediately $z = \sqrt{bb+xx}$. Then obviously

$$dv = \frac{Mr}{VM} = \frac{x\,dx\sqrt{bb+yy}}{(bb+xx)\sqrt{xx-yy}}.$$

Similarly, in *Fig. 2* let us call angle $ADM = u$ and then

$$du = \frac{Mn}{DM} = \frac{y\,dx}{x\sqrt{xx-yy}}.$$

From this

$$x^4du^2 - xxyydu^2 = y^2dx^2$$

and

$$y^2 = \frac{x^4du^2}{dx^2 + x^2du^2},$$

and therefore

$$\sqrt{bb+yy} = \frac{\sqrt{bb\,dx^2 + (bb+xx)x^2du^2}}{\sqrt{dx^2+x^2du^2}}$$

and

$$\sqrt{xx-yy} = \frac{x\,dx}{\sqrt{dx^2+x^2du^2}}.$$

So it follows that

$$dv = \frac{\sqrt{bbdx^2 + (bb+xx)xx\,du^2}}{bb+xx}.$$

Because either $u$ or $y$ is given by $x$ it is possible to find angle $v$ which will be marked out by the given curve $AM$ around $V$ in the plane, whose area $AVM$ is equal to the surface area of the cone that is sought.

§ 21. Because the determination of the surface area of the cone depends on the integral formula

$$\int \frac{x\,dx\sqrt{bb+yy}}{\sqrt{xx-yy}},$$

this job may be executed easily in innumerable ways by means of the quadrature and rectification of algebraic curves. However, since we are to correct Leibniz's construction, which is the most elegant, it is necessary for us to proceed in our particular way. It is clear, indeed, the great man derived his construction from consideration of lines leading to the given curve under any angle whatsoever. These lines by their combination form a new curve, whose rectification is so straightforward to express that any quadrature may be easily reduced to that point. From this same source the famous Hermann drew his most ingenious method of reducing the quadrature of any curves to the rectification of algebraic curves, which method the famous Johan Bernoulli later put forward lucidly, translated from geometry into pure analysis.

§ 22. Let us take as the given curve that figure $AM$, which was set before as the base of the cone. At each of its points $M$, $m$ take the lines $MS$, $ms$, which by their contacts form a new curve $FSs$, through whose rectification the surface area of the cone may be expressed. Let the arc of the known curve $AM = s$,[6] and let the angle $SMm = v$, [the angle] the line $SM$ makes with the curve $AM$ at the point $M$. With the element obtained, $Mm = ds$ and the angle $smN = v + dv$. From this the point $S$ would be determined, by the concurrence of lines $MS$ and $Ms$. Consider the center of the osculating circle at $Mm$, which is $R$, and let the osculating radius be called $MR = mR = r$, then the angle $MRm = \frac{ds}{r}$. From the lines $RM$, $Rm$ normal to the curve $AM$, it follows that angle $RMS = 90^\circ - v$ and angle $RmS = 90^\circ - v - dv$. Because Fig. 4

$$RoS = MRm + RMS = MSm + RmS,$$

then angle

$$MSm = MRm + RMS - Rms = \frac{ds}{r} + dv.$$

Now in the triangle $MSm$, from the given angles and the small side $Mm = ds$,

$$\frac{\frac{ds}{r} + dv}{ds} = \frac{\sin v}{mS}, \quad {}^{7}$$

[6] He is using $s$ both for arc length and for a point, but the context makes clear which is which.

[7] Euler uses : notation for the ratio; since $\frac{ds}{r} + dv$ is infinitesimal, it is equal to its sine.

or

$$= \frac{[\sin v]}{MS}.$$

Therefore

$$MS = \frac{r\,ds \sin v}{ds + r\,dv},$$

from which formula the construction of curve $FS$ follows.

§ 23. Let the line $MS = z$, then

$$z = \frac{r\,ds \sin v}{ds + r\,dv},$$

and

$$ms = z + dz.$$

Take the normal $mk$ at $m$ on $MS$. Since angle $mMk = v$, then $mk = ds \sin v$ and $Mk = ds \cos v$. Since $Ss = ms - kS = ms - MS + Mk$, $Ss = ds \cos v + dz$. Now $Ss$ is the element of the curve $FS$, from which the length of this curve becomes $FS = \int ds \cos v + z + Const$. In order to determine this constant let it agree with curve $FS$, [taking] point $F$ to the point $A$ on the given curve $AM$, so that the line $AF$ is tangent to the sought curve $FS$ at point $F$. Thus since $MS = z$, it will turn out that $FS = \int ds \cos v + MS - AF$, if indeed the integral $\int ds \cos v$ is taken so that it vanishes when $s = 0$. Once that is done the integral formula $\int ds \cos v$, may be expressed in turn by the rectification of the curve $FS$, and clearly $\int ds \cos v = FS + AF - MS$.

§ 24. To move on, let $D$ be the footprint in the base of the vertex of the cone, that is the point where the perpendicular dropped from the vertex of the cone meets the base. The height of this perpendicular $VD$ was given above as $b$. After drawing the tangent $MQ$ at $M$, to which the perpendicular $DQ$ is dropped from $D$, we called $DM = x$ and $DQ = y$, and the element was

$$Mm = \frac{x\,dx}{\sqrt{xx - yy}},$$

which now let us call $ds$. For this reason, since we found the conical surface area corresponding to the arc $AM$ of the base corresponding to

$$\frac{1}{2} \int \frac{x\,dx \sqrt{bb + yy}}{\sqrt{xx - yy}},$$

that same surface area will be

$$\frac{1}{2} \int ds \sqrt{bb + yy}.$$

If therefore this surface area is expressed by the rectification of the curve $FS$, angle $v$ should be such that the integration of the formula $\int ds \cos v$ leads to the integration of the formula $\int ds\sqrt{bb+yy}$.

§ 25. To this end let us take

$$\cos v = \frac{\sqrt{bb+yy}}{k}.$$

Since the cosine of this angle $v$ can never grow past the magnitude of the radius, which we set as unity, the quantity $k$ should be taken so that $\sqrt{bb+yy}$ can never exceed it. Hence no matter what the maximum value the formula $\sqrt{bb+yy}$ can ever achieve anywhere on the cone, $k$ is assumed to equal or exceed that value. Thus if the angle $v$ has been defined by this method, the surface area of the cone with base curve $AM$ will be obtained as

$$\frac{1}{2}\int ds\sqrt{bb+yy} = \frac{1}{2}k\int ds\cos v;$$

and thus the area may be expressed by the rectangle $\frac{1}{2}k(FS+AF-MS)$. In fact, if the line $MS$ is set up anywhere so that

$$\cos SMm = \frac{\sqrt{bb+yy}}{k}$$

or

$$\sin RMS = \frac{\sqrt{bb+yy}}{k}$$

and from this the curve $FS$ is constructed, then the area of the rectangle $\frac{1}{2}k(AF+FS-MS)$ will be equal to the desired surface area of the cone, the area may therefore be produced by the rectification of the algebraic curve $FS$. Thus wherever the angle $RMS$ and the length $MS$ can be defined algebraically, the curve $FS$ itself will be algebraic.

§ 26. Thus having taken

$$\cos v = \frac{\sqrt{bb+yy}}{k},$$

then

$$\sin v = \frac{\sqrt{kk-bb-yy}}{k}.$$

Differentiating,

$$dv\cos v = \frac{-ydy}{k\sqrt{kk-bb-yy}} = \frac{-y\,dy}{kk\sin v},$$

and thus
$$dv = \frac{-y\,dy}{kk\sin v\cos v}.$$
When the nature of the curve $AM$ is expressed as an equation involving $DM = x$ and $DQ = y$, the osculating radius will be
$$MR = r = \frac{x\,dx}{dy} = \frac{ds\sqrt{xx-yy}}{dy},$$
$$dy = \frac{ds\sqrt{xx-yy}}{r}$$
and thus
$$dv = \frac{-y\,ds\sqrt{xx-yy}}{kkr\sin v\cos v}.$$
Since above we found
$$MS = z = \frac{r\,ds\sin v}{ds + r\,dv},$$
now we will have
$$MS = z = \frac{kkr\sin^2 v\cos v}{kk\sin v\cos v - y\sqrt{xx-yy}}.$$
This expression we will attempt to construct geometrically in the following way.

§ 27. Again let $AM$ be the base of the cone, $D$ the footprint of the vertex, and $M$ any point on the curve at which is drawn the tangent $MQ$ and normal $MK$. Having taken the line $DM$, draw the perpendicular $DQ$ from $D$ to the tangent; and likewise take to the tangent the undetermined line $DC$, in which $DC$ is taken to be the height of the cone, $b$; and take $CQ$ so $CQ = \sqrt{bb+yy}$. Then along the normal to the curve take $MK = k$, over which, having described on that diameter the semicircle $KPM$, let the chord $KP$ be taken equal to $CQ$. If then we draw $MP$, then the sine of the angle $KMP$ will be $\frac{KP}{k} = \cos v$ and segment $MP$ will be on the line $MS$. Let $MR = r$ be taken on the normal to the curve. Since $DQ = y$ and $MQ = \sqrt{xx-yy}$, then Fig. 4 & 5
$$MS = \frac{MK.MR\sin v}{MK - \frac{DQ.MQ}{MK}\sin v\cos v},$$
or
$$MS = \frac{MK\cos v.MR\sin v}{MK\cos v - \frac{DQ.MQ}{MK}\sin v\cos v}.$$
Now since $MK\cos v = KP$ and $MK\sin v = MP$, from $R$ to $MP$ take the perpendicular $RT$, then $MR\sin v = MT$ and

$$MS = \frac{KP \cdot MT}{KP - \frac{DQ \cdot MQ}{MP}} \ .$$

Let

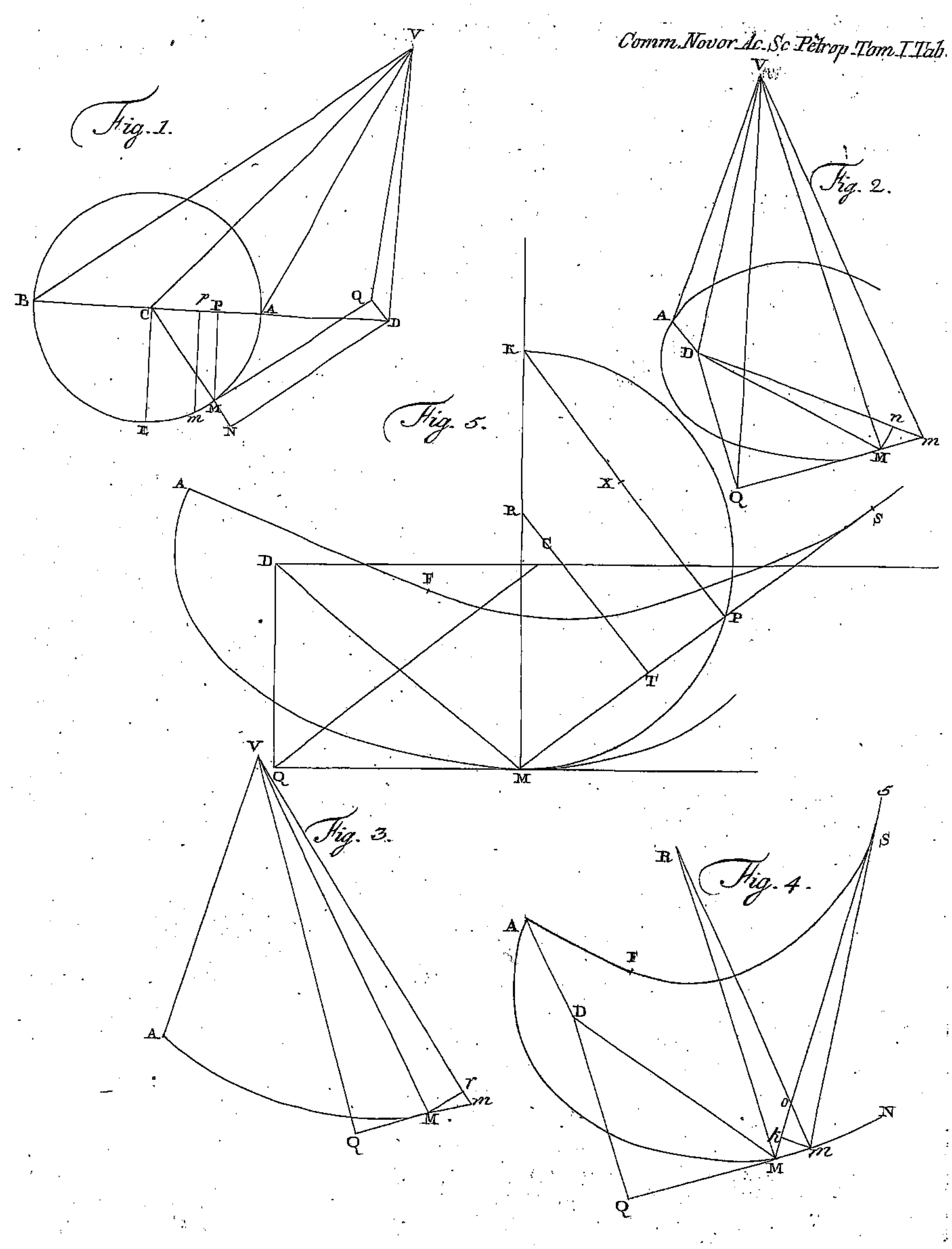
$$PX = \frac{DQ \cdot MQ}{MP},$$

be taken, then

$$MS = \frac{KP \cdot MT}{KX},$$

and the length of the line $MS$ may easily be determined. If this operation is performed for each point $M$, then corresponding points $S$ are determined on the desired curve $FS$. When this [curve] is found the portion of the surface area of the cone of the given arc $AM$ will be equal to the area of the rectangular parallelogram $\frac{1}{2}MK(AF + FS - MS)$.

§ 28. If the curve $AM$ is taken to be a circle, and the point $D$ falls away from the center, then the cone will be the ordinary scalene cone which we first considered. Let the curve $FS$ be constructed according to approach given here. The same curve is produced that the illustrious Leibniz showed how to find in his account mentioned above. From this it is clear that it is not the curve $FS$ stretched along a line, applied to any fixed line, that gives the desired surface area of the cone, but that it is the arc $FS$ itself augmented by the line $AF$ and reduced by the length of the line $MS$. Thus in this way, we have not only emended the Leibnizian construction, which was suitable only for scalene cones, but we also extend it to the cones whose bases are arbitrary figures.




The figures from Euler's paper.