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Daniel J. Curtin
Northern Kentucky University, curtin@nku.edu

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The Scalene Cone: Euler, Varignon, and Leibniz

Daniel J. Curtin, Northern Kentucky University (Emeritus)
148 Brentwood Place, Fort Thomas, KY 41075
curtin@nku.edu

Abstract

The work translated alongside this article is Euler's contribution to the problem of the scalene cone, in which he discusses the earlier work of Varignon and Leibniz. He improves Varignon's solution and corrects Leibniz's more ambitious solution. Here I will discuss the problem, provide extensive notes on Euler's paper, and lay out its relation to the other two. For ease of reference translations of the papers of Varignon and Leibniz are appended, with the Latin originals. Euler's version in Latin is available at https://scholarlycommons.pacific.edu/euler-works/133

1 The Problem

1.1 The History of the Problem

Finding the surface area of a right cone, a cone with circular base and vertex directly over the center of the circle, was known from antiquity, certainly by Archimedes.

The problem of finding the surface area of a scalene (or oblique) cone, one with circular base but vertex not necessarily directly over the center of the circle, seems to have been solved first by Pierre Varignon. In 1727 he published his result in the *Miscellanea Berolinensia* [3]. He called his paper a *schediasma*, a Latin version of a word that in Greek means a caprice, or whimsy. Varignon probably meant a sketch or rough draft. In fact, the mathematics in his paper is fully formed, but the layout and referencing are not carefully done. He solved the problem by giving a curve whose arc length at certain points gives the surface area of the corresponding part of the cone.

In the same volume, in fact in the article immediately following Varignon's, Gottfried Leibniz [4] gave his suggested improvements. Varignon's curve was transcendental, implicitly using trigonometric functions. Leibniz gave an algebraic curve to solve the problem and outlined how to extend his solution to base curves other than the circle. The two papers share a single diagram, printed in between them.

In his introductory paragraph, Leibniz asserted that Gilles de Roberval told Leibniz he had a solution, but never published it, but Leibniz noted that he could not find such work among Roberval's papers, after his death.

A bit later, Euler took up the problem, publishing his solution in 1750 [2]. In his introduction, he referred to the two earlier papers by Varignon and Leibniz. He remarked that Leibniz's solution was excellent, but involved a slight error. Euler used a Greek word in Latin form, *sphalma* meaning a misstep or stumble, perhaps to soften the criticism of a mathematician he greatly admired. Euler gave his own version of Varignon's solution, then extended it in several ways. He explained where Leibniz went wrong, then corrected the misstep.

1.2 The Differential Approach

A cone is defined by starting with a circle in the plane and a vertex V that is above the plane but not necessarily located directly over the center of the circle. A point on the circle A is selected and line segment VA is connected. Then the cone is defined by following points M around the circle as VM sweeps out any section AVM of the cone that is desired. This dynamic definition allows infinitesimal methods to be used. For all three authors, a solution to the problem is a curve whose arc length gives the areas of the sectors of the cone. For either the arc length problem or the surface area problem, the integrals that result are rarely integrable in simple terms, since they both involve the element of the arc length $ds = \sqrt{dx^2 + dy^2}$ or something similar.

2 The Right Cone

To illustrate the idea let's do the surface area of the right cone. We don't need infinitesimal arguments here. Let C be the center of the circle and r the radius [Figure 1.] Consider the segment AVM of the cone, with slant height l and central angle $u = \angle MCA$. In the plane the circular arc AM has length ru. Then AVM can be unfolded to a plane figure $A'V'M'$ that is part of a circle of radius $l = A'V'$, with arc $A'M'$ of length ru. The central angle of $A'V'M'$ is ru/l, so the area of $A'V'M'$is

$$\frac{1}{2}l^2\frac{ru}{l} = \frac{1}{2}lru = \frac{lr}{2}u \ .$$

Now l and r are constants, so as a function of u this is a curve, in fact a line, whose arc length for any given point that is parametrized by a particular value u gives the areas of that circular sector.

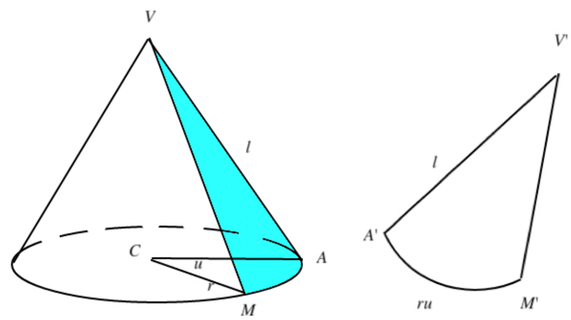


Figure 1: Surface area of a right cone of radius a and slant height l.

3 The Scalene Cone with Circular Base

If the vertex is off center things get quite a bit more complicated. This was first solved by Varignon [3]. Then Leibniz [4], using a different approach, claimed to solve it with an algebraic curve. He also claimed his approach extends to the case of an arbitrary base curve. Euler points out that this is not quite correct and removes the error, as we will see later.

The fundamental idea in all cases is to look for another point m, close to M. The area of the region MVm is then the building block for the surface area. Taking m infinitesimally close to M makes MVm what they called the *element* of the surface area, for us the integrand of the integral that would give the surface area. For Euler and the others, solving the problem meant finding a curve whose arc length element is this element of surface area.

4 Notes on Euler's Paper

In what follows, I will outline what Euler does in various paragraphs, while leaving the reader to see much of the details in the work itself. I also try to include things I wish I had seen the first time I read this paper, in hopes they may prove helpful.

In [1], I go into much more detail about paragraphs 1–16, but note that, as usual, Euler himself does a good job of explaining his own ideas.

4.1 Paragraphs 2–4

Euler first sets out to reproduce Varignon's curve [Figure 2]. He uses the central

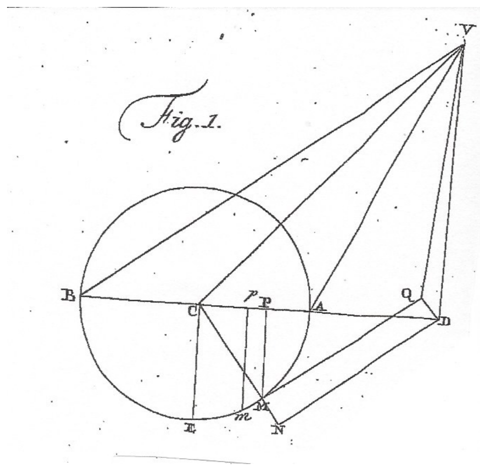


Figure 2: Euler's figure of the scalene cone.

angle of the circle $u = \angle ACM$ as the parameter for his solution. He defines three constants: a, the radius of the circle; b, the height of the cone (VD); and c, the distance from the center of the circle to the foot of the perpendicular from V to the plane of the base ($c = CD$). He then derives the area as

$$AVM = \frac{1}{2}a\int du\sqrt{b^2 + (c\cos u - a)^2}.$$

Euler generally writes bb for b^2, etc. In this commentary I will use the more modern notation, since they mean the same thing. I have not moved his du, etc., to the modern location, since I believe we understand the du quite differently from Euler. I left the notation in my translation as close to Euler's as possible.

In paragraph 4 he concludes the curve is given, in what we would call parametric form:

$$p = bu,$$
$$q = c\sin u - au.$$

He then says, "It will be clear to the attentive reader that this curve is the same as Varignon discovered." It took me a bit of work, but I was able to verify this equivalence; see section 5 below.

4.2 Paragraph 5

Euler does not stop here. As usual he works to get as much out of the idea as possible. He notes that this solution involves transcendental functions, while

Leibniz has found (almost correctly) an algebraic solution, though by a different method that Euler discusses later.

4.3 Paragraphs 6 & 7

Euler then proceeds to substitute $z = \cos u$ and thus to generate not only one algebraic solution but in fact a whole family of algebraic solutions.

4.4 Paragraphs 8–15

Euler works out several examples, also detailed in [1]. He always finds a parametrized curve, never a numerical answer.

4.5 Paragraph 16

Euler solves the case in which the perpendicular from vertex V passes through the point A on the circle and where the angle of the axis VC of the cone to the base is $60°$. He derives parametric equations for a curve that solves the problem, then, almost as an afterthought, notes that the actual area of the part of the cone swept out from E to M will be $\frac{1}{2}az$, where a is the radius of the base circle and $z = \arcsin \angle ECM$, the angle ECM being the complement of the angle $u = \angle ACM$ used in determining the curve.

5 Euler's Solution Is the Same as Varignon's

This section is for those interested in Varignon's paper [3], which is translated in Appendix 1.

At the end of Euler's paragraph 4 he gives a parametrized curve whose arc length (multiplied by $\frac{1}{2}a$) is the surface area of the scalene cone. The curve is given by abscissa $p = bu$ and ordinate $q = c \sin u - au$. He then asserts, "It will be clear to the attentive reader this is the same curve as Varignon discovered." Let's see how.

I will use mostly Varignon's notation. For the radius of the circle, Varignon uses r, Euler a. For the distance from the center of the circle to the foot of the perpendicular from the vertex, Varignon has $r + a$, Euler c. For the distance from the vertex to the plane of the circle both use b. Euler uses u for the generating angle $\angle DCM$. Varignon uses no letter for his angle $\angle AOH$, but I will use Euler's notation u. Further, for the arc swept out on the circle through the angle u, Varignon uses AH, Euler AM. Either of these is equal to ru.

Thus in Varignon's notation (plus u) Euler's curve becomes

$$p = bu,$$
$$q = (r+a)\sin u - ru.$$

Varignon gives his curve, beginning in the part entitled Proof, with a free constant m. For us the relevant parts are as follows, first in the form Varignon wrote them, using the letters from his diagram:

$$AE = \frac{BC \cdot AH}{2m} = \frac{b}{2m}AH,$$
$$EP = \frac{AO \cdot AH + OC \cdot GH}{2m} = \frac{rAH + (r+a)y}{2m}.$$

Here if we suppose p is AE and $q = EP$, then try $2m = r$, we get

$$p = AE = \frac{b}{a}AH,$$
$$q = EP = \frac{rAH + (r+a)y}{r}.$$

Further, looking at Varignon's diagram, we see

$$\sin u = \frac{-y}{r}.$$

With $AH = ru$ this gives us

$$EP = \frac{r^2u + (r+a)r(-\sin u)}{r}$$
$$= ru - (r+a)\sin u.$$

Thus we correct our guess to make $q = -EP$, which makes no difference to the length.

Finally, as mentioned above, Euler had factored $a/2$ out of his integral for the surface area before deriving p and q, but Varignon's curve can account for this by adjusting the value of the constant m to be $\sqrt{a}$.

6 Leibniz's Solution Perfected

For the remainder of the paper, Euler takes up the general problem of a cone with any curve in the plane whatsoever as base [Figure 3]. After setting this up in paragraphs 17–20, he returns to Leibniz's idea and in paragraphs 21–28 corrects his solution.

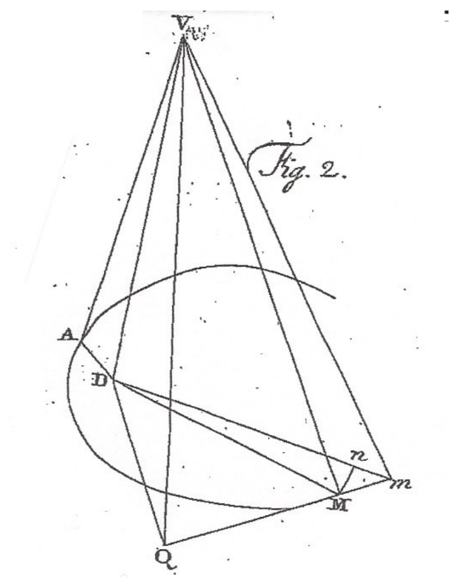


Figure 3: The cone with arbitrary curve as base.

6.1 Paragraphs 17 & 18

Euler no longer has the central angle available, so he takes any point M on the curve and the foot D of the perpendicular from the vertex to the plane, and he sets $x = DM$. He then takes the tangent line to the curve at M and identifies the point Q on that line for which DQ is perpendicular to the tangent. He sets $y = DQ$ and notes that "there will be an equation between x and y."

We'll follow the general trend of the argument. The details are not hard to read in the text.

As before, he considers a point m on the curve infinitely near M. The triangle MVm is the element of the surface area of the cone. Recall that $b = VD$ is the height of the cone. Since VQ is the height of this triangle,

$$MVm = \frac{1}{2}Mm \cdot VQ.$$

He shows that

$$MVm = \frac{1}{2}\frac{xdx\sqrt{b^2+y^2}}{\sqrt{x^2-y^2}},$$

so

$$AVM = \frac{1}{2}\int \frac{xdx\sqrt{b^2+y^2}}{\sqrt{x^2-y^2}}.$$

The last equation gives an integral for the area desired.

6.2 Paragraph 19

Euler unfolds the cone onto the plane [Figure 4], as we do with the right cone, but here he approaches the triangle MVm in a different way. He uses the perpendicular, Mr from M to Vm, which would be the height of the triangle relative to Vm, and derives the distances Mr and mr for later use.

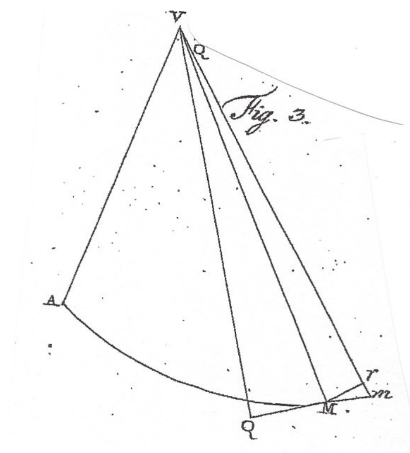


Figure 4: The section of the same cone spread out on the plane.

6.3 Paragraph 20

Next Euler returns to the cone in space [Figure 3], and he assumes the existence of the solution curve AFS. He will derive properties of the point S and eventually show how to find it. Leibniz had based his argument on selecting an angle, essentially $\angle SMN$ to lead to his solution curve FS. In fact, Euler's Fig. 4 [Figure 5] resembles the figure for Leibniz's paper [4].

To see how the area can be defined using an angle, Euler defines $v = \angle AVM$. He also defines $u = \angle ADM$. Then since VM and Vr are radii of an infinitesimal circle centered at V and $dx = Mr$ is an arc of the circle,

$$dv = \frac{Mr}{VM}.$$

By a similar argument, using D as the center,

$$du = \frac{Mn}{DM}.$$

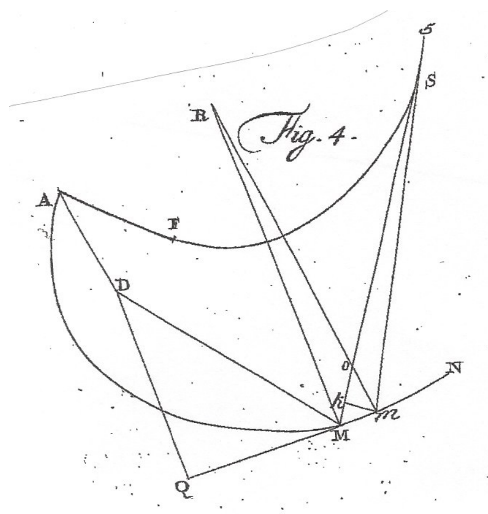


Figure 5: In the base plane.

Using previous calculations for Mr, VM, Mn, and DM from paragraphs 17–19, Euler obtains

$$dv = \frac{xdx\sqrt{b^2+y^2}}{(b^2+x^2)\sqrt{x^2-y^2}}$$

and

$$du = \frac{ydx}{x\sqrt{x^2-y^2}}.$$

He solves the second for y, then substitutes into the first, which yields

$$dv = \frac{\sqrt{b^2dx^2+(b^2+x^2)x^2du^2}}{\sqrt{b^2+x^2}}.$$

Thus, since u depends on x, the angle v can be found by integrating, and it then can be used to find the surface area AVM.

6.4 Paragraph 21

Euler now returns to the basic integral needed to solve the problem,

$$\int \frac{xdx\sqrt{b^2+y^2}}{\sqrt{x^2-y^2}},$$

and he notes that it can be solved in many different ways. Since Euler's goal is to correct Leibniz's approach, which he describes as "the most elegant," he sets out to follow Leibniz's idea.

6.5 Paragraph 22

Euler begins to trace Leibniz's procedure. As noted above, Euler assumes that the desired curve FS is given, then deduces what is needed to show its existence and nature.

For each point M on the base curve AM, there is the segment MS to S on the rectifying curve. For a point m infinitely near M, the corresponding segment ms should be such that S is infinitely near s. The arc length of the base curve is given as $s = AM$. (Yes, Euler uses the same letter for two different things, but the context always makes it clear which is which.) Let $v = \angle SMm$, i.e., the angle of MS to the tangent at M. Then $ds = Mm$ and $\angle smN = v + dv$. He defines the osculating circle to AM at M whose center is R and whose radius is $r = MR$. Then after a bit of work he obtains

$$MS = \frac{rds \sin v}{ds + rdv},$$

from which he claims the construction of the curve FS follows, as we shall see. Here F is the point on the curve corresponding to $M = A$.

6.6 Paragraph 23

Euler sets $z = MS$, so $ms = z + dz$. Using a bit of right angle trigonometry he shows that the element of arc length of the resolving curve $Ss = ds \cos v + dz$. Integrating with respect to z he obtains

$$FS = \int ds \cos v + z + C.$$

We have $z = MS$, and when $s = 0$, $MS = AF$. Assuming $\int ds \cos v$ is 0 for $s = 0$, we have

$$\int ds \cos v = FS - MS + AF.$$

We need to look more closely at this integral.

6.7 Paragraph 24

From previous work,

$$Mm = ds = \frac{xdx}{\sqrt{x^2 - y^2}},$$

and the surface area AVM may be written in two ways:

$$\frac{1}{2}\int ds \cos v = \frac{1}{2}\int \frac{xdx\sqrt{b^2 + y^2}}{\sqrt{x^2 - y^2}}.$$

Thus we want the angle v to be such that $\int ds \cos v$ leads to $\int ds\sqrt{b^2+y^2}$.

6.8 Paragraph 25

Euler now obtains v from $VQ = \sqrt{b^2+y^2}$ by taking a constant k larger than, or equal to, the largest possible value of VQ along the base curve, then setting

$$\cos v = \frac{\sqrt{b^2+y^2}}{k}.$$

Thus the surface area is given by

$$\frac{1}{2}\int ds\sqrt{b^2+y^2} = \frac{1}{2}k\int ds\cos v.$$

Geometrically this is

$$\frac{1}{2}k(FS + AF - MS).$$

6.9 Paragraph 26

Recall that r is the radius of the osculating circle to AM at M. After a bit of work Euler obtains

$$MS = z = \frac{k^2 r \sin^2 v \cos v}{k^2 \sin v \cos v - y\sqrt{x^2-y^2}}.$$

6.10 Paragraph 27

Euler uses this last characterization of MS to show how to derive MS geometrically. With the angle and distance in hand, the point S is found. Doing this for each M gives the rectifying curve.

To this end [Figure 5] he takes MK perpendicular to the tangent MQ so that $MK = k$. Then he forms the semicircle KPM with diameter MK, where P is the point on the semicircle such that $KP = \sqrt{b^2+y^2}$. (Euler does this geometrically.) Then he takes R on MK such that $MR = r$, the radius of the osculating circle. Next he takes MT perpendicular to MP at T. He marks X on KP so that

$$PX = \frac{DQ \cdot MQ}{MP}.$$

Finally,

$$MS = \frac{KP \cdot MT}{KX}.$$

Euler states that the distance MS can be found easily for each M, so S is found and the resolving curve is known.

6.11 Paragraph 28

To end, Euler returns to the case where AM is a circle. The curve FS constructed above is the same curve that Leibniz derived [4]. Euler notes however that the solution is not FS, as Leibniz said, but rather $FS - MS + AF$. He ends with modest triumph, "Thus in this way, we have not only emended the Leibnizian construction, which was suitable only for scalene cones, but we also extend it to the cones whose bases are arbitrary figures."

References

[1] Curtin, Daniel J. *Euler's Work on the Surface Area of Scalene Cones.* In *Research in History and Philosophy of Mathematics: The CSHPM 2017 Annual Meeting in Toronto, Ontario* (Proceedings of the Canadian Society for History and Philosophy of Mathematics), edited by Maria Zack and Dirk Schlimm, pp. 59–67. New York: Birkhäuser, 2018.

[2] Euler, Leonhard. *De superficie conorum scalenorum, aliorumque corporum conicorum.* In *Novi Commentarii academiae scientiarum Petropolitanae* 1, 1750, pp. 3–19.

[3] Varignon, Pierre. *Schediasma de Dimensione Superficiei Coni ad basim circularem obliqui, ope longitudinis Curvæ, cujus constructio à sola Circuli quadratura pendet.* In *Miscellanea Berolinensia, ad incrementum scientiarum ex scriptis Societati Regiæ Scientiarum exhibitis edita* III, 1727, pp. 280–284.

[4] Leibniz, Gottfried Wilhelm. *Additio G. G. L. Ostendens Explanationem superficiei conoïdalis cujuscunque; & speciatim explantionem superficiei Coni scaleni, ita ut ipsi vel ejus portioni cuicunque exhibeatur rectangulum æquale, interventu extensionis in rectam curvæ, per Geometriam ordinariam construendæ.* In *Miscellanea Berolinensia, ad incrementum scientiarum ex scriptis Societati Regiæ Scientiarum exhibitis edita* III, 1727, pp. 285–287.

Appendix 1

Pierre Varignon's sketch, on the dimensions of the surface of a cone that is oblique to a circular base, by means of the length of a curve whose construction depends only on the quadrature of the circle.[1]

Now for many previous centuries the surfaces of cones whose bases are circular and standing upright [i.e., a right cone] have been known to have a rectangular area that depends on the quadrature of the circle. Not so for the oblique case, where I knew no one who had determined their surface area. Here is the curve that provides their areas by its length and whose construction depends on nothing other than the quadrature of the circle.

I. [See the diagram at the end.] Let the cone of any obliquity you like be *AHDZAB*, with base the circle *AHDZA*, vertex *B*, and height *BC* perpendicular at *C* to the plane of the base *AHDZA*. From that point *C* through the center *O* of the base, let the diameter *DA* be produced indefinitely in the direction of *S*. Then on the diameter *AD* let there be the bases of two ordinates as close to each other as you like, *GH*and *gh*, from whose extremities *G*, *H*, *h* to the vertex *B* are drawn the lines *GB*, *HB*, and *hB*. Finally from the point *h* let there be drawn *hk* perpendicular to *HB* at *k*,[2] so from the center *B* through *h* there will be the indefinitely small circular little arc *hk*.

II. Now since *BC* is normal to the base plane *AHDZA* (**I.**) it follows that planes *ABC* and *AHDZA* are perpendicular to one another. Thus, since *HG* is indeed perpendicular to their common section *AD* (**I.**), it will also be perpendicular to the plane *ABC* and thus also to the line *GB*.

[1] Varignon, Pierre, *Schediasma de Dimensione Superficiei Coni ad basim circularem obliqui, ope longitudinis Curvæ, cujus constructio à sola Circuli quadratura pendet*, *Miscellanea Berolinensia, ad incrementum scientiarum ex scriptis Societati Regiæ Scientiarum exhibitis edita* III, 1727, pp. 280–284.

Translator's note: I have modernized the notation and formatting somewhat. Those interested in details of the history of mathematical notation should look at the original. All footnotes and bracketed comments are mine.

[2] The point k is not labelled, but lies directly over the point labeled K in the figure. The point K is not used in this paper, but it is in the Leibniz paper that follows it.

Therefore setting $AO = r$, $AG = x$, $GH = y$, $CD = a$, $BC = b$,[3]

$$\begin{aligned} BH &= \sqrt{HG^2 + GB^2} \\ &= \sqrt{HG^2 + BC^2 + GC^2} \\ &= \sqrt{y^2 + b^2 + (2r + a - x)^2} \\ &= \sqrt{y^2 + b^2 + 4r^2 + 4ar + a^2 - 4rx - 2ax + x^2}. \end{aligned}$$

From the circle $AHDZA$ comes $y^2 = 2rx - x^2$ [so we have further:]

$$[BH] = \sqrt{4r^2 + 4ar + a^2 + b^2 - 2rx - 2ax}. \qquad \text{[Eq. 1]}$$

Let c be [such that]

$$\begin{aligned} c^2 &= 4r^2 + 4ar + a^2 + b^2 \\ &= (2r + a)^2 + b^2 \\ &= AC^2 + BC^2 \\ &= AB^2 \end{aligned}$$

and also

$$f = r + a = OC.$$

[From this and Eq. 1]

$$[BH] = \sqrt{c^2 - 2fx}\ . \qquad \text{[Eq. 2]}$$

[Since $Hk = -d\,BH$,] this will make[4]

$$Hk = \frac{fdx}{\sqrt{c^2 - 2fx}}$$

positive because of the alternate increasing and decreasing of AG (x) and BH.

Then

$$\begin{aligned} hk &= \sqrt{Hh^2 - Hk^2} \\ &= \sqrt{dy^2 + dx^2 - dx^2 \frac{f^2}{c^2 - 2fx}}\ . \end{aligned}$$

[3]Varignon displays this as $BH = V(HGq. + GBq.)$ using a large V for $\surd$ and q. to abbreviate *quadratus*. As was customary at the time, Varignon writes yy for y^2, etc. I will use modern notation. Further, Varignon writes this string of equations inline. I display them in the modern style.

[4]Varignon has HK, but I believe K is a typo for k in the next two displayed equations.

Now with the equation of the base, $y^2 = 2rx - x^2$, this gives

$$\pm dy = \frac{rdx - xdx}{\sqrt{2rx - x^2}}.$$

So

$$\begin{aligned}[hk] &= \sqrt{dx^2 \frac{(r-x)^2}{2rx - x^2} + dx^2 - dx^2 \frac{f^2}{c^2 - 2fx}} \\ &= dx\sqrt{\frac{r^2}{2rx - x^2} - \frac{f^2}{c^2 - 2fx}} \\ &= dx\sqrt{\frac{c^2r^2 - 2fr^2x - 2f^2rx + f^2x^2}{(2rx - x^2)(c^2 - 2fx)}}.\end{aligned}$$

Now setting $g = r + f = 2r + a = AC$,

$$[hk] = dx\sqrt{\frac{c^2r^2 - 2fgrx + f^2x^2}{(2rx - x^2)(c^2 - 2fx)}}\ . \qquad \text{[Eq. 3]}$$

Thus [from Eq. 2 and Eq. 3]

$$HBh\left(\frac{HB \cdot hk}{2}\right) = dx\frac{\sqrt{c^2r^2 - 2fgrx + f^2x^2}}{2\sqrt{2rx - x^2}}$$

will be the sought element of the surface of the cone, from whose summation [integral] would come the surface at hand, with the help of the length of the following curve.

III. Let the semiperimeter of the circle AHD be rolled out in a line AQ equal to it and perpendicular to the diameter AD. Of course the unrolling of the first point D describes a curve DQ. Then taking any constant m you please, draw from Q a line RQ, perpendicular to AQ, such that

$$\frac{RQ}{AQ} = \frac{BC}{2m}.$$

So

$$RQ = \frac{AQ \cdot BC}{2m} = \frac{AHD \cdot BC}{2m}.$$

Then the rectangle QS is completed and its side RS produced to F so that

$$\frac{SF}{AS} = \frac{AO}{BC}.$$

Then
$$SF = \frac{AS \cdot AO}{BC} = \frac{RQ \cdot AO}{BC} = \frac{AQ \cdot AO}{2m} = \frac{AHD \cdot AO}{2m}.$$
Next after drawing the lines AR and AF and taking the indeterminate line (on AQ) $AL = AH$, the arc rolled out on it by AL,[5] describe LM parallel to QR. From the point M (where the line [i.e., LM] meets AR) make the indefinite line MP parallel to AQ so that it meets the lines AS, AF at E, N [respectively]. Finally for MP make it that

$$NP = \frac{OC \cdot GH}{2m},$$

and so on, everywhere.

I say that this is the curve APF passing through each of the points P whose product $m \cdot AP$ of any arc AP by the constant m, whatever it may be, is equal to the corresponding part AHB of the surface of the cone that is sought.

Proof

1.[6] Since (by construction) there is

$$\frac{2m}{BC} = \frac{AQ}{RQ} = \frac{AL}{LM} = \frac{AH}{AE},$$

then
$$AE = \frac{BC \cdot AH}{2m}, \qquad \text{[Eq. 4]}$$

and so the parallel element is

$$\delta p = \frac{BC \cdot Hh}{2m}$$

by the notation in **II**. Since the circle $AHDZA$ yields

$$Hh = \frac{AO \cdot Gg}{HG}, \qquad \text{[Eq. 5]}$$

then
$$[\delta p] = \frac{br\,dx}{2m\sqrt{2rx - x^2}}.$$

[5] My correction: Varignon has HL instead of AL.

[6] Varignon restarts the sequence of Roman numerals. For clarity, I have changed this second series of steps to Hindu-Arabic numerals.

2. Further, (by construction) not only is

$$\begin{aligned}\frac{BC}{AO} &= \frac{AS}{SF} \\ &= \frac{AE}{EN} \quad \text{(I.)} \\ &= \frac{\frac{BC\cdot AH}{2m}}{EN}. \qquad\qquad \text{[from Eq. 4]}\end{aligned}$$

Now with[7]

$$[EN] = \frac{AO \cdot AH}{2m}.$$

and also

$$NP = \frac{OC \cdot GH}{2m},$$

then

$$EP = \frac{AO \cdot AH + OC \cdot GH}{2m}.$$

Next (taking HV perpendicular to gh at V) its element will be

$$\delta P = \frac{AO \cdot Hh + OC \cdot Vh}{2m}$$

(Then from the previous section [Eq. 5], using the equation $y^2 = 2rx - x^2$, with $f = r + a$,

$$\begin{aligned}[\delta P] &= \frac{r^2 dx}{2m\sqrt{2rx - x^2}} + \frac{(r+a)(rdx - xdx)}{2m\sqrt{2rx - x^2}} \\ &= \frac{dx(2r^2 + ar - rx - ax)}{2m\sqrt{2rx - x^2}} \\ &= \frac{(2r^2 + ar - fx)dx}{2m\sqrt{2rx - x^2}}.\end{aligned}$$

3. Therefore there will be the element pP [of the arc length] of the curve APF, namely

$$\begin{aligned}\sqrt{\delta p^2 + \delta P^2} &= \frac{dx\sqrt{b^2r^2 + 4r^4 + 4ar^3 - 4fr^2x + a^2r^2 - 2farx + f^2x^2}}{2m\sqrt{2rx - x^2}} \\ &= \frac{dx\sqrt{c^2r^2 - 2fgrx + f^2x^2}}{2m\sqrt{2rx - x^2}} \quad \text{(since } b^2 + 4r^2 + 4ar + a^2 = c^2\text{)} \\ &= \frac{HBh}{m}.\end{aligned}$$

[7] I correct a printing error in the original.

Therefore $HBh = m \cdot Pp$, whose integral, or such a summation, is the sought surface of the part of the cone, $AHB = m \cdot AP$, and so on for other parts of this surface obtained in this way. Thus the integral extends so that the surface area of this oblique cone $= 2m \cdot APF$. *Quod erat Dem.*

IV. From this measure of the surface of an oblique-angled cone that of the common type, given by a right angle, also follows. Now if the cone $AHDZAB$ be a right cone, then certainly its height BC coincides with BO – for this reason making both [segments] equal and normal to the base of the cone, which would nullify OC. Then (from 2) the curve APF has ordinate

$$EP = \frac{AO \cdot AH + OC \cdot CG}{2m},$$

reducing to

$$\frac{AO \cdot AH}{2m} = EN,$$

while the abcissa (see 1) remains the same:

$$AE = \frac{BC \cdot AH}{2m}.$$

The curve *APF* would be changed into the right line *ANF*, and its arc *AP* changed into *AN*. From this in the right cone, it would follow that

$$m \cdot AN = m \cdot AP = AHB. \qquad \text{(from 3)}$$

And then

$$\begin{aligned} AN &= \sqrt{AE^2 + EN^2} \qquad \text{(from 3)} \\ &= \frac{\sqrt{(BC \cdot AH)^2 + (AO \cdot AH)^2}}{2m} \\ &= \frac{AH\sqrt{BC^2 + AO^2}}{2m}. \end{aligned}$$

(Then because $BO = BC$ and is perpendicular to the plane $AHDZA$ and thus also to the radius $OH = AO$,)

$$\begin{aligned} [AN] &= \frac{AH \cdot \sqrt{BO^2 + OH^2}}{2m} \\ &= \frac{AH \cdot BH}{2m}. \end{aligned}$$

Thus in this case
$$\frac{AH \cdot BH}{2} = AHB,$$
or the portion AHB of the right convex cone will be equal to the product
$$\frac{AH \cdot BH}{2}$$
from half of the side BH times the corresponding base arc AH. Thus the entire surface of such a convex cone taken together will be equal to the product of half the side and the whole circuit of the base, which geometers, by other reasoning, had observed long before now.

The measure of the surface area of the same right-angle cone also follows immediately from the oblique-angle element
$$HBh = \frac{dx\sqrt{c^2r^2 - 2fgrx + f^2x^2}}{2\sqrt{2rx - x^2}}, \qquad \text{(from 3)}$$
in which $c = AB$ and $f = OC$. Now since the surface of the rectangular cone forces $AB = BH$ and $OC = 0$, or $c = BH$ and $f = 0$, the oblique-angle element becomes, in the right-angle case,
$$\begin{aligned} HBh &= \frac{dx\sqrt{r^2BH^2}}{2\sqrt{2rx - x^2}} \\ &= \frac{BH}{2}\frac{rdx}{\sqrt{2rx - x^2}} \\ &= \frac{BH}{2}\frac{OH \cdot HV}{HG} \\ &= \frac{BH \cdot Hh}{2}, \end{aligned}$$
and after integrating, or summing, it will give (and quickly) the surface of the right-angle cone
$$ABH = \frac{BH \cdot AH}{2},$$
etc., as above.

I will not dwell upon this any longer, but only wanted it known how great is the affinity between the preceding case of the oblique-angle cone and the common right-angle cone.

V. This part looks at the curve APF (**III.**) that yields the measure of the surface of the oblique-angle cone.

1) It establishes the angle ADB at F across from A with the ordinate SF and the angle ABC at A across from S with the axis AS. Then the side AB of the oblique cone $AHDZAB$ is normal to this curve at A, which, if it is right, also has its side AB normal at the point A to the line ANF which thus has been added to the curve.

2) Then the curve will be such that the largest of its ordinates EP corresponds to

$$AG = \frac{AC \cdot AO}{OC},$$

which will always be larger than AO and less than AD.

3) The total segment of that curve, $ANFPA$, and any desired indeterminate part $ANPpA$, can be completely rectified.[8] This is possible from the known construction of the quadrature of the circle $AHDZA$. Yet you would not say the areas $FPAS$, $PpAE$ are both quadrable, since the size of the triangles $FNAS$, NAE, though rectilinear, depend on the quadrature of this circle.

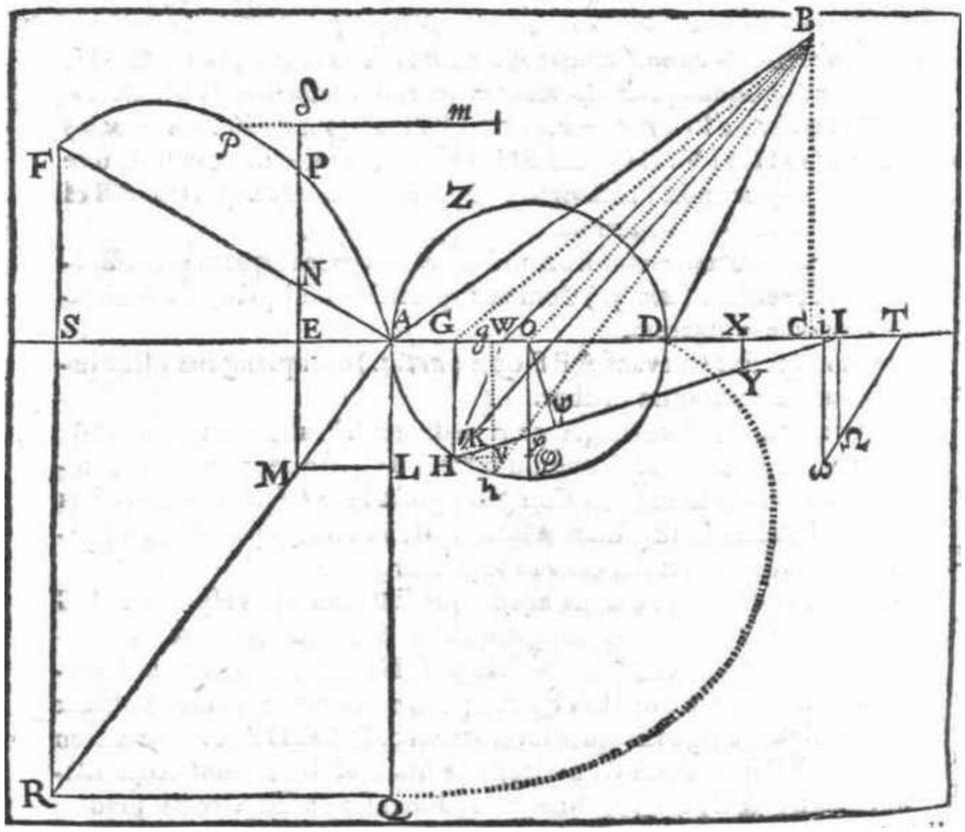
4) However, the length of APF and any part of it AP is measured by the quadrature of the curve placed between two parallel asymptotes with the distance between them the diameter AD of the base of the cone, and [the length] is expressed by the equation

$$z = \frac{\sqrt{c^2r^2 - 2fgrx + f^2x^2}}{2\sqrt{2rx - x^2}}$$

whose coordinates, normal to each other, are x, z.

It may be noted that if the perpendicular BC to the base plane $AHDZA$ of the oblique cone $AHDZAB$, which here [in the diagram] falls outside of the base, were to fall inside, over OC and with the same point D, nothing different from the preceding would occur, except the change of the sign of the quantity a or DC, and with the removal of this term the same construction would remain and the use of the above curve APF.

[8]He says *quadrari,* to be found by making a square.



The figure from Varignon's paper, which is also used by Leibniz's addendum, immediately following in the same volume of the *Miscellanea*.

XIX.

PETRI VARIGNONIS Schediaſma de Dimenſione Superficiei Coni ad baſim circularem obliqui, ope longitudinis Curvæ, cujus conſtructio à ſola Circuli quadratura pendet.

JAm à multis retrò ſæculis notum eſt conorum, baſibus circularibus recte inſiſtentium ſuperficies, menſuram in rectangulis habere, à quadratura circuli pendentem: At non item in obliquangulis, quorum ſuperficies qui definierit, neminem novi. Ecce curva quæ horum menſuram ſua longitudine præbet, cujusque conſtructio non aliunde quam à quadratura circuli pendet.

I. Sit obliquitatis cujusvis Conus A H D Z A B, baſeos circularis AHDZA, verticis B, altitudinis B C perpendicularis in C ad planum baſeos AHDZA; ex quo puncto C per baſeos centrum O ſit diameter ejus DA indefinite producta verſus S; Hanc ad diametrum AD ſint baſeos ordinatæ duæ, invicem quam proximæ quævis GH, gh, ex quarum extremitatibus G, H, h per coni verticem B, ducantur rectæ GB, HB, hB. Tandem ex puncto h fingatur hk perpendicularis ad HB in k, ſive ex centro B per h ſit circularis arculus indefinite parvus hk.

II. Jam cum ſit (*art. 1.*) BC normalis ad planum baſios AHDZA, liquet, plana ABC & AHDZA ad invicem eſſe perpendicularia; atque

que adeo cum HG ſit quoque (*art. 1.*) perpendicularis ad communem eorum ſectionem AD, erit pariter ea perpendicularis ad planum ABC, & conſequenter etiam ad rectam GB.

Ergo poſitis $AO = r$, $AG = x$, $GH = y$, $CD = a$, $BC = b$, erit $BH = V(HGq. + GBq.) = V(HGq. + BCq. + GCq.) = V(yy + bb + (2r + a - x)\,q.) = V(yy + bb + 4rr + 4ar + aa - 4rx - 2ax + xx)$ (circulo AHDZA exigente $yy = 2rx - xx$) $= V(4rr + 4ar + aa + bb - 2rx - 2ax)$ [ſint $cc = 4rr + 4ar + aa + bb = (2r + a)\,qu. + bb = ACq. + BCq. = ABq.$ & $f = r + a = OC$] $= V(cc - 2fx)$. Unde fiet $HK = fdx : V(cc - 2fx)$ poſitiva, propter alternatim creſcentes aut decreſcentes AG (x) & BH. Ergo $hk = V(Hh\,qu. - HK\,qu.) = V(dydy + dxdx - dxdx\,(ff :, cc - 2fx))$ [baſeos æquatione $yy = 2rx - xx$ dante $\pm dy = rdx - xdx, : V(2rx - xx)$] $= V(dxdx\,((r - x)\,qu. :, 2rx - xx) + dxdx - dxdx\,(ff :, cc - 2fx)) = dx\,V((rr :, 2rx - xx) - (ff :, cc - 2fx)) = dx\,V(ccrr - 2frrx - 2ffrx + ffxx, :, (2rx - xx)(cc - 2fx))$ $=$ [ſit $g = r + f = 2r + a = AC$] $dx\,V(ccrr - 2fgrx + ffxx, :, (2rx - xx, cc - 2fx))$. Igitur ſuperficiei conicæ quæſitæ elementum erit HBh [$HB.hk : 2$] $= dx\,V(ccrr - 2fgrx + ffxx) : 2V(2rx - xx)$ qualium ſummatio ſeu ſuperficies iſthæc habetur, ope longitudinis Curvæ ſequentis.

III. Fingatur ſemiperipheria circularis AHD evolvi in rectam ipſi æqualem AQ perpendicularem ad diametrum ejus AD, primo ſcilicet evolutionis ejus puncto D deſcribente curvam DQ. Dein ad perpendiculum ipſi AQ (ſumta quavis conſtante m) ponatur in Q recta RQ ita, ut ſit $RQ : AQ = BC : 2m$. Sive $RQ = AQ.BC : 2m = AHD.BC : 2m$; perfectoque rectangulo QS, latus ejus RS producatur verſus F, donec ſit $SF : AS = AO : BC$. Sive $SF = AS.AO : BC = RQ.AO : BC = AQ.AO : 2m = AHD.AO : 2m$. Tum ductis rectis AR, AF, indeterminateque ſumpta (in AQ) recta $AL = AH$ arcui in ipſam evoluto, per HL, fiat LM parallela ad QR; & ex puncto M (occurſu ejus cum AR) item fiat indefinita MP parallela ad AQ, quæque occurrat in E, N, rectis AS, AF. Tandem in illa MP ſumatur $NP = OC.GH : 2m$; & ſic ubique.

N n Dico

Dico eam esse Curvam APF per omnia ejusmodi puncta P transeuntem, quæ productum $m.AP$ ex arcu quovis ejus AP per constantem m, ubique reddat, æquale respondenti portioni AHB conicæ superficiei quæsitæ.

Demonst. I. Siquidem (*constr.*) ubique est $2m : BC = AQ : RQ = AL : LM = AH : AE$. est etiam ubique $AE = BC.AH : 2m$, & ipsi parallelum elementum ejus $\delta p = BC.Hh : 2m$ (juxta nomina art. 2. & circulum AHDZA exigentem $Hh = AO.Gg : HG) = brdx : 2m V(2rx - xx)$

II. Pariter cum sit (*constr.*) non modo $BC : AO = AS : SF = AE : EN$ (*num.1.*) $= (BC.AH : 2m) : EN = AO.AH : 2m$. Sed etiam $NP = OC.GH : 2m$; erit $EP = AO.AH + OC.GH, : 2m$, & elementum ejus (ductâ HV perpendiculari in V ad gh) $\delta P = AO.Hh + OC.Vh, : 2m$ (ex præmissis nominibus & ex æquatione $yy = 2rx - xx$ art. 2.) $= (r.rdx : 2m V(2rx - xx)) + (r + a)(rdx - xdx) : 2m V(2rx - xx) = dx(2rr + ar - rx - ax) : 2m V(2rx - xx)$ [art. 2. dante $f = r + a$] $= 2rr + ar - fx, dx : 2m V(2rx - xx)$

III. Ergo (*num.1.2.*) erit Curvæ APF elementum PP vel $V(\delta P$ qu. $+ \delta P$ qu.$) = dx V(bbrr + 4r^4 + 4ar^3 - 4frrx + aarr - 2farx + ffxx) : 2m V(2rx - xx)$ (Propter $bb + 4rr + 4ar + aa = cc$) $= dx V(ccrr - 2fgrx + ffxx) : 2m V(2rx - xx)$ [art.2.] $= HBh : m$. Igitur erit $HBh = m.Pp$, cujus integrale, seu qualium summatio est conicæ superficiei quæsitæ portio $AHB = m.AP$; & sic de ceteris ejusdem superficiei portionibus hac ratione sumptis: Unde patet integram hanc Coni obliqui superficiem esse $= 2m$ APF. Quod erat Dem.

IV. Ex hac obliquangulæ conicæ superficiei mensura fluit ultrò vulgaris ea, quæ rectangulæ tribuitur. Nam si Conus AHDZAB fieret rectus, is nimirum, cujus altitudo BC, conversa in BO, hac ratione factam ipsi æqualem & normalem ad Coni basim, aboleret OC; tunc Curvæ APF ordinatis EP (*art.3. num.2.*) $= AO.AH + OC.CG, : 2m$ evadentibus $= AO.AH : 2m = EN$ interim dum abscissæ manerent eædem AE (*art.3.num.1.*) $= BC.AH : 2m$, Curva hæc APF mutaretur in rectam ANF, & ejus arcus AP in AN: Unde hoc in Cono rectangulo foret $m.AN = m.AP$ (*art.3. num.3.*) $= AHB$. Atqui $AN =$

V(AEq.

V(AEqu + ENqu) (*art. 3. num. 1. 2.*) = V(BC. AH) q. + (AO. AH) q.) : *2m* = AH v(BCq. + AOq.) : *2m* (Siquidem hic eſt BO = BC & perpendicularis ad planum AHDZA, indeque ad radium OH = AO) = AH. V(BOq. + OHq.) : *2m* = AH. BH : *2m*. Ergo hoc in eodem caſu erit AH. BH : *2* = AHB, ſive Coni recti convexæ ſuperficiei portio AHB æqualis erit producto AH. BH : *2* ex dimidio latere BH per reſpondentem baſeos arcum AH : Unde tota hujuſmodi Coni ſuperficies convexa pariter erit æqualis producto ex dimidio latere per totum baſeos ambitum ; quod Geometris alia ratione jam pridem notum eſt.

Eadem Conicæ ſuperficiei rectangulæ menſura ſtatim fluit etiam ex obliquangulæ elemento HBh (*art. 2.*) = $dx\,V(ccrr - 2fgrx + ffxx) : 2V(2rx - xx)$ in quo ſunt (*art. 2.*) c = AB, & f = OC. Nam cum rectangula ſuperficies conica exigat AB = BH & OC = o, ſeu c = BH, & $f = o$; iſtud obliquangulæ elementum in rectangula fiet HBh = dx $V(rr \times$ BHq.) : : $2V(2rx - xx)$ = (BH : *2*) ($rdx : V(2rx - xx)$) = (BH : *2*) (OH. HV : HG) = BH. Hh : *2* ; atque integrando, ſive ſummando erit (ut mox) rectangulæ ſuperficiei Conicæ portio ABH = BH. AH : 2, &c. ut ſuperius.

Porro his tamdiu non immoror, niſi ut magis pateat quanta ſit affinitas inter præcedentem (*art. 3.*) Conicæ ſuperficiei obliquangulæ menſuram & rectangulæ vulgarem.

V. Quod ſpectat Curvam APF quæ (*art. 3.*) obliquangulæ illius ſuperficiei Conicæ menſuram præbet.

1. Ea conſtituit in F verſus A cum ordinata SF angulum = ADB & in A cum axe AS verſus S. angulum = ABC; atque ita hanc ad Curvam normale eſt in A Latus AB Coni obliqui AHDZAB; qui, ſi rectus eſſet, pariter haberet Latus ſuum AB normale in eodem puncto A ad rectam ANF in quam tunc illa curva mutaretur.

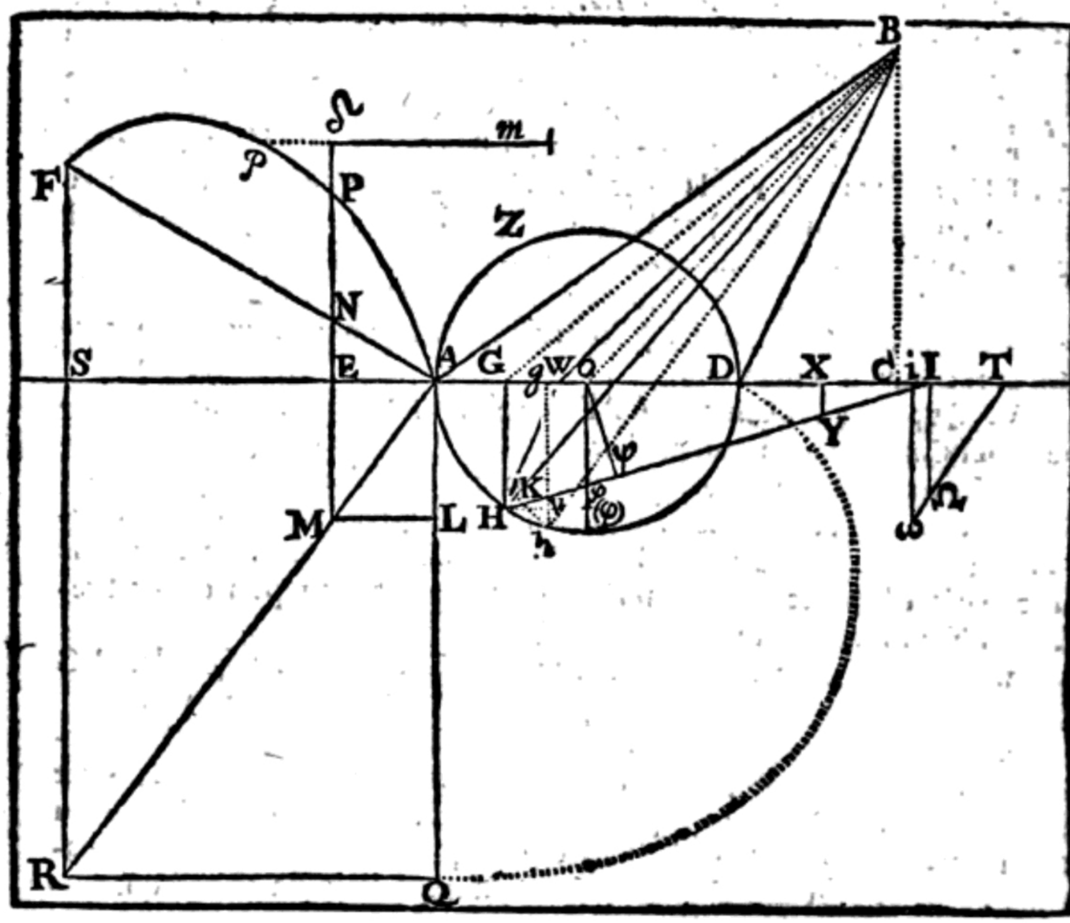
2. Dum ea curva eſt, ordinatarum ejus EP maximæ reſpondet AG = AC. AO : OC ſemper major quam AO, & minor quam AD.

3. Ejuſdem Curvæ ſegmentum totum ANFPA, ac etiam indeterminata quævis ejus portio ANPpA, poſſunt abſolute quadrari, Licet ſint conſtructionis exigentis quadraturam circuli AHDZA. Ne tamen dixeris areas FPAS, PpAE, pariter eſſe quadrabiles ; cum triangularium FNAS, NAE, etſi rectilinearum, dimenſio ab iſta circuli quadratura pendeat.

Nn 2 4. Lon-

4. Longitudo autem APF, & pars ejus quævis AP mensuram habent à Quadratura Curvæ positæ duas inter asymptotos parallelas distantes ab invicem quantitate diametri AD baseos Coni, & expressæ per æquationem $z = V(ccrr - 2fgrx + ffxx) : 2V(2rx - xx)$ cujus coordinatæ ad invicem normales sunt x, z.

Notandum si perpendicularis BC ad planum baseos AHDZA Coni obliqui AHDZAB, quæ hic cadit extra basim istam, caderet intra, super OC, aut in ipsum punctum D; nihil inde varietatis præcedentibus eventurum esse, præter mutationem signi quantitatis a vel DC, aut hujus quantitatis extinctionem, manentibus semper iisdem constructione & usu curvæ superioris APF.



Addi-

Appendix 2

An Addition by G.G.L. Setting forth an explanation of the surface area of any conoidal surface whatsoever; and a step-by-step explanation of the surface area of a scalene cone, so as this or that portion whatsoever is presented equal to a rectangle, by means of the length of a right curve constructed by ordinary geometry.[1]

The surface area of a right cone was already laid out by Archimedes and it is equal to a given circle. It is certain that the ancients did not arrive at the surface area of a scalene cone. Ægidius [Gilles] de Roberval told me when I was young that he had found a solution for it, but he did not tell me how he had managed it, nor was I able to find anything about it among his papers. In fact now with the same method that appeared in the previous sketch [Appendix 1], we will easily be able to give plane figures, constructed by ordinary geometry, that are equal to the surface area not only of scalene cones, but also of conoids generated by lines BH (see the *Figure* above) through a fixed point B taken above and passed along any curve Hh[2] whatsoever in the plane AHC below.

Now let BC be normal to the plane below, and therefore to the line AC, whose directrix HG is applied normal to the curve Hh and meeting AC at G. It is clear that[3]

$$BG = \sqrt{BC^2 + CG^2}\,,$$

which is normal to HG, and thus

$$BH = \sqrt{BG^2 + GH^2}\,.$$

Now GH is given from AG by the nature of the curve, and BC and AC are given. Then CG is the sum or difference of CA and AG. Further, BH is also given from AG. Therefore also is given the ratio of the momentary increment,

[1] Leibniz, Gottfried Wilhelm, *Additio G. G. L. Ostendens Explanationem superficiei conoïdalis cujuscunque; & speciatim explantionem superficiei Coni scaleni, ita ut ipsi vel ejus portioni cuicunque exhibeatur rectangulum æquale, interventu extensionis in rectam curvæ, per Geometriam ordinariam construendæ, Miscellanea Berolinensia ad incrementum scientiarum ex scriptis Societati Regiæ Scientiarum exhibitis edita* III, 1727, pp. 285–287.

Translator's note: This paper appears right after Varignon's paper (Appendix 1) on the scalene cone. They share a diagram, reproduced at the end of Appendix 1. I have modernized the notation and formatting somewhat. Those interested in details of the history of mathematical notation should look at the original. All footnotes and bracketed comments are mine.

[2] The text has $H(H)$, but clearly Hh is meant

[3] The equation in the text is typeset as $BG = V(BC \cdot BC + CG \cdot CG)$.

or element, of the same BH and certainly of HK itself to HV, the element of AG. (Where $[K]$ is placed down [i.e., directly below k] from the normal hk from h to BH.)

Now if the square of HK is subtracted from the square of Hh there remains the square of hK, the element of the arc length of the curve. This line is set in ratio to HV, the increment of AG, and this ratio is called r/a, where a is any constant whatsoever assumed once arbitrarily, and r is determined from AG. Then it will be that[4]

$$hK = \frac{r}{a}\, dAG.$$

But half of the rectangle BH by hK is equal to the triangle BHh, which is the element of the surface area of the conoid. So if the line $(r/2a)BH$ (which will clearly be to BH as r to $2a$) is applied to dAG, that is to Gg, or if GH is always summed from G in the direction of H, then a curvilinear figure will be made whose element is $(r/2a)BH\, dAG$, the same as the element of the surface of the conoid.

Thus the appropriate part of this figure summed will be equal to the corresponding part of the surface of the conoid, both lines being derived from the point above and the base curve. Hence the surface of any conoid can be laid out by ordinary geometry giving a plane figure equal to it, if the base of the conoid AHh is a figure from ordinary geometry.

Now Archimedes presented the surface of a right cone, or a part of it, as equal to a circle, or a part of a circle. Then the area of a circle, or a part of it, may easily be turned into a rectilinear figure through the length[5] of a particular curve (an arc of a circle, in fact). From this the desired [curve] was found in a most elegant way by that most renowned man, Pierre Varignon, as laid out immediately above [Appendix 1], whereby the plane figure equal to the surface area of the scalene cone can be measured by the length of a certain curve.

Now the curve that he sets forth is not ordinary, but transcendental, which it might not have been if it did not require the quadrature [*tetragonismo*] of the circle or the rectification of the arc of the circle for its construction. I prefer to seek an ordinary curve that would provide the same [result]. Here is how I proceeded. In the same way as before, let GW be taken from G towards D, which will be to AG as CD is to the radius DO and let BW be connected, which will yield

$$BH = \frac{\sqrt{c^2r^2 - 2rfgx + f^2x^2}}{r}.$$

[4]The r and a here are not those of Varignon's paper. Lebniz uses $(r : a)$ where I have r/a, and similarly elsewhere.

[5]I use the word "length"; Leibniz says "by extension in a line."

[From Varignon's paper: $OA = r$ is the radius of the circle, $CD = a$, $AB = c$, $OC = f = r + a$, $AC = g = r + f$. The expression follows by a relatively straightforward calculation from $BW = \sqrt{CW^2 + BC^2} = \sqrt{(OW + OC)^2 + BC^2}$, using $BC^2 = c^2 - g^2$ and $OW = r - AG - GW = r - (1 + a/r)x$.]

Then I place A to be the farthest point of the diameter AD from B. HI is led from the point H to the line AC so that, when $O\psi$ is led from O normal to HI, [$O\psi$] is to the radius HO as half of BW is to the constant line, which is larger than half of BA. From the same line HI, $O\phi$ is taken from the center O perpendicular to CO, and continued on the same side as H to make ϕ [on HI]. Likewise all this is done to points h and g, as BW was drawn, so on to hi, and $O\phi$ is made to (ϕ) [on hi].

Now $O\phi$, $O(\phi)$, etc., are equal to $I\Omega$, $i\omega$, etc., the ordinates applied normal to OI, Oi, etc., that make the curve $\Omega\omega$, whose tangent ωT meets AC at T. Finally along AC from I is taken IX to the side where IH [and] ih approach each other. Then IX is to IO as TI is to TO.

From any such point X found in this way the normal XY is taken, running to the corresponding position Y on the given line HI. This will be the applied ordinate of a new curve $Y(Y)$. Then with the elementary arc between point Y and (Y), and with the intercept defined by the process given above, and with the above-mentioned constant that is no less than $\frac{1}{2}BA$, the included rectangle will be equal to the triangle BHh, which is, of course, the element of the surface of the scalene cone. Thus by the evolution or extension of the appropriate curve Y (Y) in a line, if the thread rolled out is led along using the same constant, a rectangle will be produced equal to $ABDHA$, the surface of half the scalene cone, or any portion of it (made up, of course, of half of the line from B and the arc of the base).

Additio G. G. L.

Oſtendens Explanationem ſuperficiei conoïdalis cujuscunque; & ſpeciatim explanationem ſuperficiei Coni ſcaleni, ita ut ipſi vel ejus portioni cuicunque exhibeatur rectangulum æquale, interventu extenſionis in rectam curvæ, per Geometriam ordinariam conſtruendæ.

Superficies Coni recti jam ab Archimede explanata eſt; & æqualis ei circulus aſſignatus. Coni Scaleni ſuperficiem veteres, quod conſtet, non attigêre. Ægidius Robervallius, me juvene ajebat, ejus explanationem ſibi eſſe notam, ſed qualem habuerit non dixit, nihilque ea de re inter ejus ſchedas repertum accepi. Equidem hodie ad eum modum quo proceſſum eſt *in Schediaſmate præcedenti*, poſſumus facile dare figuras planas easque per Geometriam ordinariam conſtruendas, non tantùm quæ æquentur ſuperficiebus coni Scaleni, ſed etiam cujusque Conoeidis generati recta BH (vide eandem quæ ſupra *Figuram*) per punctum B, conſtans, in ſublimi transeunte, & circumlata per curvam quamcunque H (H) in plano ſubjecto AHC poſitam. Nam cum BC ſit normalis ad planum ſubjectum, adeoque & ad rectam AC, quæ ſit directrix ipſarum HG normaliter applicatarum ex curva Hh, & ipſi AC occurrentium in G; patet BG = $\sqrt{}$(BC . BC + CG . CG) fore normalem ad HG; atque adeo BH fore $\sqrt{}$(BG . BG + GH . GH). Datur autem GH ex AG per naturam curvæ; & BC, AC ſunt datæ, & CG eſt ſumma vel differentia ipſarum CA, AG; habetur ergo & BH ex AG. Ergo & ratio datur incrementi momentanei ſeu elementi ipſarum BH; nempe ipſius HK (ſi ponatur ex h demiſſa in BH normalis hk) ad HV elementum ipſarum AG. Quod ſi jam quadratum ipſius HK detrahatur à quadrato ipſius Hh, elementi arcus curvæ, ſupererit quadratum ipſius hK, cujus rectæ adeò etiam dabitur ratio ad HV, incrementum ipſius AG; quæ ratio vocetur $r : a$, poſitâ a

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conſtante quacunque pro arbitrio ſemel aſſumta, & r determinatâ ex AG. Erit ergo $hK = (r:a)\, dAG$. Sed dimidium rectanguli ex BH in hK eſt æquale triangulo BHh, id eſt elemento ſuperficiei Conoidis, ergo ſi recta $(r:2a)$ BH (nempe quæ ſit ad BH ut r ad $2a$) accommodetur ad dAG, id eſt, ad Gg, ſeu ſumatur ſemper in ipſa GH ex G verſus H; fiet figura curvilinea, cujus elementum erit $(r:2a)\, BH \,.\, dAG$ idem cum elemento ſuperficiei conoidis. Itaque ipſius hujus Figuræ portio ritè ſumta æquabitur reſpondenti ſuperficiei Conoeidis portioni, binis rectis ex puncto in ſublimi, & arcu baſis comprehenſæ; & proinde quævis ſuperficies Conoïdalis explanari poteſt, adhibita figura plana ei æquali, per Geometriam ordinariam conſtruenda, ſi ipſa AHh baſis Conoidis, ſit figura Geometriæ ordinariæ.

Sed quia Archimedes Circulum, vel Circuli portiones, ſuperficiei coni recti vel ejus portionibus æquales exhibuit; Circuli autem vel ejus portionum area vel converſio in figuram rectilineam facillime habetur per extenſionem curvæ (nempe ipſius arcus Circularis) in rectam: hinc à Viro Celeberrimo Petro Varignone quæſitus eſt inventusque modus elegans paulo ante poſitus, quo figura plana æqualis ſuperficiei coni ſcaleni etiam menſurari poteſt, per extenſionem curvæ alicujus in rectam. Cum vero, curva illa quam exhibet non ſit ordinaria, ſed transcendens, quamvis non niſi tetragoniſmo Circuli ſeu rectificatione arcus Circularis ad conſtructionem indigeat; placuit mihi quærere curvam ordinariam quæ idem præſtet, ſeu cujus rectificatione exhibeatur ſuperficiei Coni Scaleni vel ejus portioni cuivis æqualis figura rectilinea. Hoc ita ſum conſecutus:

Iisdem quæ antea poſitis, à puncto G verſus D ſumatur GW, quæ ſit ad AG ut CD ad radium DO & jungatur BW, quæ erit $V(ccrr - 2rfgx + ffxx):r$. Pono autem A eſſe extremum diametri AD remotius à B. Ex puncto H ad rectam AC ducatur HI ita ut ex O ducta ad HI normalis Oψ ſit ad HO radium ut dimidia BW ad rectam conſtantem, quæ ſit major dimidia BA; Et recta HI ipſam Oφ ex centro O perpendiculariter ad CO eductam ad partes H, ſecet in φ. Eadem omnia fiant ad puncta h & g, ut ducatur BW, ducatur & hi, quæ ipſam Oφ ſecet in (φ). Jam ipſis Oφ, O(φ), &c. æquales IΩ, $i\omega$, &c. ordinatim

tim applicentur normaliter ad OI, Oi, &c. ut fiat curva $\Omega\omega$, cujus tangens ΩT occurrat ipſi AC in T. Tandem in AC ex I ſumatur IX ad partes ad quas convergunt IH, *ih*, quæ IX ſit ad IO ut TI ad TO; & ex puncto X quovis, hoc modo reperto, normaliter educta XY, occurrens reſpondenti rectæ poſitione datæ HI in Y, erit ordinatim applicata curvæ novæ Y (Y) cujus ſub arcu elementari inter puncta Y & (Y) præſcripta ratione determinanda intercepto, & ſub conſtante ſupradicta, non minore quàm $\frac{1}{2}$BA, comprehenſum rectangulum æquabitur triangulo BHh, nempe elemento ſuperficiei coni ſcaleni: atque adeo evolutione ſeu extenſione debita curvæ Y. (Y) in rectam, filo evoluto ducto in ipſam conſtantem ſupradictam, exhibebitur rectangulum æquale ipſi ABDHA ſuperficiei dimidiæ coni Scaleni vel ejus portioni (rectis ſcilicet binis ex B & arcu baſis comprehenſæ) cuicunque.

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JOH. WILH. WAGNERI
Annotationes ad Eclipſin Solis, 1724. D. 22. Maji.

MAxima in obſcuratione aër colorem ſimulabat cineritium, & lux diei ſuperſtes crepera erat, pallida vel glauca, legendæ tamen adhuc ſcripturæ, & rebus procul mediocriter diſcernendis ſufficiens.

Stellæ, præter Venerem ſeu heſperum, nullæ apparebant, licet Spectatores, viſu quorum plurimi pollebant, in cœlum hinc inde oculorum aciem direxerint. Aër & erat magis, ac ante & poſt, frigidiuſculus. ros non cadebat. hirundines in volatu ſuo, ad terram plus tamen demiſſæ, perſeverabant.

Circa