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A new generalization of the Pareto distribution and its applications

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ABSTRACT

This paper introduces a new generalization of the Pareto distribution using the Marshall-Olkin generator and the method of alpha power transformation. This new model has several desirable properties appropriate for modelling right skewed data. The Authors demonstrate how the hazard rate function and moments are obtained. Moreover, an estimation for the new model parameters is provided, through the application of the maximum likelihood and maximum product spacings methods, as well as the Bayesian estimation. Approximate confidence intervals are obtained by means of an asymptotic property of the maximum likelihood and maximum product spacings methods, while the Bayes credible intervals are found by using the Monte Carlo Markov Chain method under different loss functions. A simulation analysis is conducted to compare the estimation methods. Finally, the application of the proposed new distribution to three real-data examples is presented and its goodness-of-fit is demonstrated. In addition, comparisons to other models are made in order to prove the efficiency of the distribution in question.

Key words: Marshall-Olkin distribution, alpha power transformation, maximum likelihood estimator, maximum product spacings, Bayes estimation, simulation.

1. Introduction

Marshall-Olkin (MO) is a well-known distribution, which was generated by Marshall and Olkin (1997). The basic idea in this generator is to add a parameter through which the new distribution will be more flexible and will have many good properties. Many authors used MO to generate new lifetime models, for example Jose and Alice (2001, 2005), Ghitany et al. (2005), Ghitany and Kotz (2007), Jose and Uma (2009), Haj Ahmad et al. (2017), Bdair and Haj Ahmad (2019) and Ahmad and Almetwally (2020). The method of alpha power transformation (APT) class is

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a procedure which makes the lifetime distribution more applicable and rich towards real data analysis. It was first introduced by Mahdavi and Kundu (2017). A new generalization appeared in the literature by doing combination between MO and APT, this was first studied by Nassar et al. (2019), and the new family is called "G-family (MOAP-G). It was noticed that the MOAP-G family is analytically tractable and efficient for real data analysis.

The cumulative distribution function (CDF) of MOAP-G random variable *X* is of the form

$$F_{MOAP}(x;\alpha,\theta)=\begin{cases}\frac{\alpha^{G(x)}-1}{(\alpha-1)[\theta+(1-\theta)(\alpha-1)^{-1}(\alpha^{G(x)}-1)]} & ,\alpha>0,\alpha\neq 1\\ G(x) & ,\alpha=1\end{cases} \tag{1}$$

The corresponding probability density function (pdf)

$$f_{MOAP}(x;\alpha,\theta)=\begin{cases}\frac{\theta\log(\alpha)\alpha^{G(x)}g(x)}{(\alpha-1)[\theta+\frac{1-\theta}{\alpha-1}(\alpha^{G(x)}-1)]^2} & ,\alpha>0,\alpha\neq 1\\ g(x) & ,\alpha=1\end{cases} \tag{2}$$

where *G(x)* is the baseline distribution.

In this paper we will consider Pareto distribution with shape parameter λ as a baseline distribution, where the pdf and cdf are respectively as follows:

$$g(x)=\frac{\lambda}{x^{\lambda+1}},x\geq 1 \tag{3}$$

$$G(x)=1-\frac{1}{x^{\lambda}},\ \ x\geq 1 \tag{4}$$

The new generated distribution, namely Marshall-Olkin Alpha Power Pareto (MOAPP), is a lifetime model with three parameters. This distribution has several desirable properties and acts well for modelling right skewed data, it has upside-down bathtub hazard rate and attractive time series representation by which many statistical computations can be easily handled. Real data examples show that MOAPP behaves better than many other generalized Pareto distributions.

The main purpose of this paper is to introduce MOAPP distribution and study some of its statistical properties, which are useful in data modelling. We use statistical inference such as maximum likelihood, maximum product spacings and Bayes estimation methods to perform point estimation. We construct confidence intervals for the unknown parameter as well. A simulation study is conducted to check the performance of the different estimation methods applied in this work This is done by comparing the bias and the mean square error (MSE) for point estimation methods and by using interval length for interval estimation. Finally, we present numerical examples that illustrate the model efficiency.

The rest of this paper is organized as follows: In Section 2 we introduce MOAPP distribution with some of its properties. Classical point estimation methods for the unknown parameters are discussed in Section 3, while in Section 4 the Bayesian estimation method is considered. In Section 5 interval estimation methods are presented. In Section 6 a simulation study and real-life data analysis are conducted and finally conclusions are given in Section 7.

2. Probability Density Function

Let *X* be a continuous random variable with Marshall-Olkin Alpha Power Pareto distribution (MOAPP), then using Eqs. (1) and (2) and assuming that the baseline distribution *G(x)* is Pareto distribution given in Eqs. (3) and (4), we obtain the pdf and CDF of (MOAPP) respectively as

$$f_{MOAPP}(x;\alpha,\theta,\lambda) = \frac{\theta\lambda(log\alpha)\alpha^{1-x^{-\lambda}}}{(\alpha-1)x^{\lambda+1}[\theta+(1-\theta)(\alpha-1)^{-1}\left(\alpha^{1-x^{-\lambda}}-1\right)]^2}, \quad x \geq 1, \alpha \neq 1 \qquad (5)$$

$$F_{MOAPP}(x;\alpha,\theta,\lambda) = \frac{\alpha^{1-x^{-\lambda}}-1}{(\alpha-1)[\theta+(1-\theta)(\alpha-1)^{-1}\left(\alpha^{1-x^{-\lambda}}-1\right)]}, \; x \geq 1, \alpha \neq 1, \qquad (6)$$

In the following subsection we investigate some important properties of MOAPP distribution such as: monotonicity, hazard rate function, series representation, moments and quantiles.

2.1. Monotonicity of MOAPP Distribution

The monotonicity of MOAPP distribution is necessary to be investigated for data modelling, many areas such as medical, industrial, engineering and reliability researches need data modelling for prediction of future values and estimation of some unknown or missing variables; hence, in this section we study the monotonicity of MOAPP distribution. We consider the pdf of MOAPP distribution in Eq. (5), and study the monotonicity of this pdf by using the logarithmic function of its pdf. The following lemma illustrates the behaviour of MOAPP distribution for different parameter values, and Figure (1) shows these cases.

Lemma 1

The pdf of MOAPP distribution is either decreasing when $0 < \theta < 1$, or upside-down bathtub curve that attains its maximum at some point $x_0 \in [1, \infty)$ when $\theta > 1$ and $\lambda > 1$

Proof

Consider the pdf of MOAPP density in Eq. (5), then the derivative of the logarithmic function of pdf with respect to x is

$$\frac{\text{dLog } f_{MOAPP}(x;\alpha,\theta,\lambda)}{dx} = \lambda Log(\alpha)x^{-\lambda-1} - \frac{\lambda+1}{x} - 2\frac{\lambda Log(\alpha)\frac{(1-\theta)}{(\alpha-1)}\alpha^{1-x^{-\lambda}}x^{-\lambda-1}}{\left[\theta+\frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{1-x^{-\lambda}}-1\right)\right]}$$

$$\frac{\text{dLog } f_{MOAPP}(x;\alpha,\theta,\lambda)}{dx} = \frac{S(x)(\lambda Log(\alpha)-(\lambda+1)x^{\lambda})-2\lambda Log(\alpha)}{s(x)x^{\lambda+1}} \qquad (7)$$

where $S(x) = [\frac{(\alpha-1)}{(1-\theta)}\theta\alpha^{x^{-\lambda}-1} + (1-\alpha^{x^{-\lambda}-1})]$. Equating (7) to zero, we obtain the cases:

1- If $0 < \theta < 1$ then $S(x)$ is positive and since $\lambda Log(\alpha) < (\lambda+1)x^{\lambda}$ then the numerator of equation (7) is negative hence the derivative of the logarithmic function of MOAPP is negative, which indicates that the pdf of MOAPP is a decreasing function.
2- If $\theta > 1$ and $\lambda > 1$ then by using Bolzano theorem on the interval [1,∞) there exist a root $x_0 \in [1,\infty)$ of Log f_{MOAPP} hence f_{MOAPP} attains its maximum at x_0.

2.2. Hazard Rate Function

The hazard rate function or failure rate is important in survival analysis and reliability theory. The hazard rate function for MOAPP distribution is of the form

$$\mathrm{h}(x;\alpha,\theta,\lambda) = \frac{\lambda \mathrm{Log}(\alpha) x^{-(\lambda+1)}}{(\alpha^{x^{-\lambda}}-1)(\theta+(1-\theta)(\alpha-1)^{-1}(\alpha^{1-x^{-\lambda}}-1))}, x \geq 1 \quad (8)$$

In order to determine the shape of h(x;α,θ, λ) it is quite enough to determine the shape of log h(x;α,θ, λ), as shown in the following lemma.

Lemma 2

The hazard rate of MOAPP distribution is either decreasing or upside down curve where the curve is skewed to the right.

Proof:

We consider a logarithmic function of the hazard rate given in Eq. (8) and take the first derivative with respect to x so that:

$$\frac{d\log \mathrm{h}(x;\alpha,\theta,\lambda)}{dx} = \frac{-(\lambda+1)\left(\alpha^{x^{-\lambda}}-1\right)w(x) + \lambda\log(\alpha)\,x^{-\lambda}[w(x)\alpha^{x^{-\lambda}} - (1-\theta)\alpha(1-\alpha^{-x^{-\lambda}})]}{x\left(\alpha^{x^{-\lambda}}-1\right)w(x)}$$

where $w(x) = \alpha\left(\theta\left(1-\alpha^{-x^{-\lambda}}\right) + \alpha^{-x^{-\lambda}}\right) - 1$. The hazard rate curve may take several shapes according to different parameter values, so we summarize these cases by:

1- If $\theta > 1$ and $\alpha > 1$ then $\alpha^{-x^{-\lambda}} < 1$ and hence $w(x) < 0$, then $\log \mathrm{h}(x;\alpha,\theta,\lambda)$ attains its maximum at a certain point $h_0 \in (1,\infty)$ so the hazard rate function is increasing on the interval $(1, h_0)$ and is decreasing (h_0,∞).
2- If $0 < \theta < 1$ and $0 < \alpha < 1$ then $\alpha^{-x^{-\lambda}} > 1$ hence $w(x) > 0$ and $\log \mathrm{h}(x;\alpha,\theta,\lambda)$ is decreasing for all values of x, which indicates a decreasing hazard rate where h(1)= $\frac{\lambda \mathrm{Log}(\alpha)}{(\alpha-1)\theta}$, and h(∞)=0.

Figure 2 illustrated the shape of the hazard rate function for some selected parameters' values.

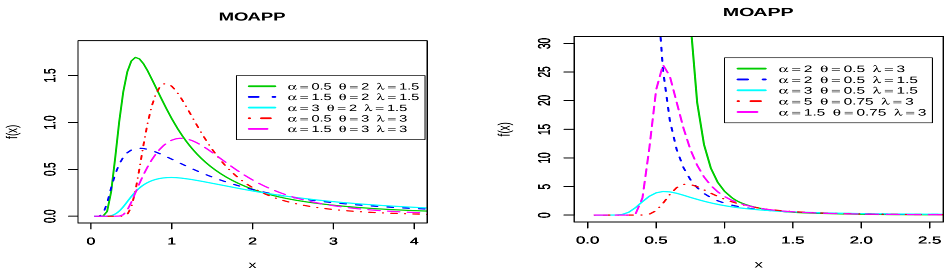

Figure 1. pdf of MOAPP under different values of the parameters

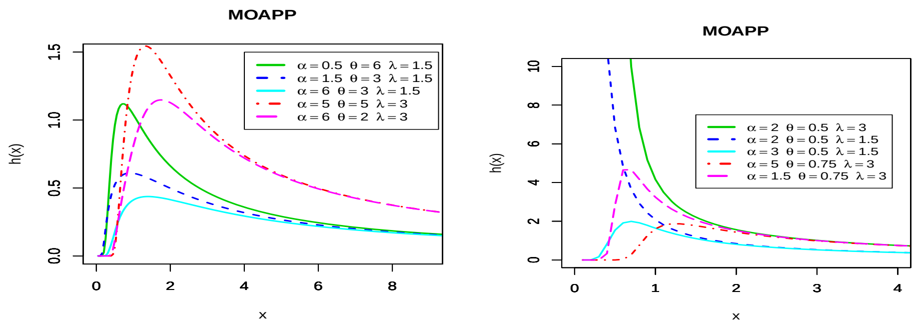

Figure 2. Hazard rate function of MOAPP under different values of the parameters

2.3. Moments

In order to obtain the moments for MOAPP distribution we use series representation for the pdf that is given in Eq. (5). The generalized binomial expansion will be used for this purpose, hence the MOAPP density can be rewritten as:

$$f_{MOAPP}(x) = \sum_{m=0}^{\infty} \wp_m \, \Omega_{m+1}(Y_m) \tag{9}$$

where $\wp_m =$

$$\begin{cases} \sum_{k=0}^{\infty} \sum_{j=0}^{k} (-1)^j (k+1)\theta(1-\theta)^k \binom{k}{j} \alpha^{k-j} \frac{(\log \alpha)^{m+1}(j+1)^m}{(\alpha-1)^{k+1}(m+1)!}, & 0 < \theta < 1 \\ \sum_{k=0}^{\infty} \sum_{j=0}^{k} (-1)^j (k+1)(1-\theta^{-1})^k \binom{k}{j} \frac{(\log \alpha)^{m+1}(k+1-j)^m}{\theta(\alpha-1)^{k+1}(m+1)!} & \theta > 1 \end{cases},$$

and $\Omega_{m+1}(Y_m) = \frac{\lambda\,(m+1)}{y^{\lambda+1}}(1 - \frac{1}{y^{\lambda}})^m$, y≥1, which is the exponentiated-Pareto distribution with two shape parameters (m+1, λ).

Eq. (9) represents the MOAPP family density as a linear combination of exponentiated-Pareto density, hence some mathematical properties can be determined from this representation.

The r^{th} moment MOAPP distribution can be computed from

$$E(X^r) = \sum_{m=0}^{\infty} \wp_m \, \mathrm{E}({Y_m}^r),$$

where $\mathrm{E}({Y_m}^r) = \int_1^\infty y^r \, \Omega_{m+1}(y) dy = (m+1)B(m+1, 1-\frac{r}{\lambda})$, and $B(\alpha, \beta)$ is beta function.

2.4. Quantile function

By inverting Equation (6), we have the quantile of MOAPP distribution as follows:

$$x_q = \left(1 - \frac{1}{\ln(\alpha)} \ln\left(1 + \frac{\theta q(\alpha-1)}{1-q(1-\theta)}\right)\right)^{-1/\lambda} ; 0 < q < 1 \tag{10}$$

3. Classical Point Estimation Methods

In this section we discuss two different classical point estimation methods, namely maximum likelihood estimation and maximum product spacing. Simulation analysis will take place in Section 6 in order to compare between the efficiency of these two methods.

3.1. Maximum Likelihood Estimation

The maximum likelihood estimation (MLE) is used in inferential statistics since it has many attractive properties, such as invariance, consistency and normal approximation properties. It depends basically on maximizing the likelihood function of MOAPP distribution. Let $X_1, X_2, \ldots, X_n$ be a random sample from MOAPP distribution, then the log likelihood function for the vector of parameters $\gamma = (\alpha, \theta, \lambda)$ can be expressed by

$$\ell(\gamma) = nLog[\theta\lambda Log(\alpha)] + \left(n - \sum_{i=1}^{n} {x_i}^{-\lambda}\right) Log(\alpha) - nLog(\alpha-1) - (\lambda+1)\sum_{i=1}^{n} Log x_i - 2\sum_{i=1}^{n} Log[\theta + \frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{1-{x_i}^{-\lambda}} - 1\right)] \tag{11}$$

In order to obtain the MLE of the parameters α, θ and λ it is necessary to find the derivative of equation (11) with respect to α, θ and λ respectively.

$$\frac{\partial \ell(\gamma)}{\partial \alpha} = \frac{n + Log(\alpha)(n - \sum_{i=1}^{n} x_i)}{\alpha Log(\alpha)} - \frac{n}{\alpha-1} - 2\sum_{i=1}^{n} \frac{(1-\theta)\alpha^{-{x_i}^{-\lambda}}[(1-\alpha){x_i}^{-\lambda} + \alpha^{{x_i}^{-\lambda}} - 1]}{(\alpha-1)^2[\theta + \frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{1-{x_i}^{-\lambda}} - 1\right)]}$$

$$\frac{\partial \ell(\gamma)}{\partial \theta} = \frac{n}{\theta} - 2\sum_{i=1}^{n} \frac{1 - (\alpha-1)^{-1}(\alpha^{1-{x_i}^{-\lambda}} - 1)}{[\theta + \frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{1-{x_i}^{-\lambda}} - 1\right)]}$$

$$\frac{\partial \ell(\gamma)}{\partial \lambda} = \frac{n}{\lambda} + 2\sum_{i=1}^{n} \frac{(1-\theta)(\alpha-1)^{-1}\alpha^{1-{x_i}^{-\lambda}}{x_i}^{-\lambda} Log(x_i) Log(\alpha)}{[\theta + \frac{(1-\theta)}{(\alpha-1)}\left(\alpha^{1-{x_i}^{-\lambda}} - 1\right)]}$$

The solution for the above normal equations is not in an explicit form, hence the MLEs can be obtained numerically by using Newton or Newton-Raphson methods.

3.2. Maximum Product Spacings

The Maximum Product Spacings (MPS) method is a new point estimation method that is considered as an alternative to MLE, see Cheng and Amin (1983). This method was recently used by many authors, see, for example Singh et al. (2014), Singh et al. (2016), and Almetwally and Almongy ($2019_{a,\,b}$). It was observed that MPS acts better than MLE in many cases. The MPS is defined as:

$$M = \left(\prod_{i=1}^{n+1} D_i\right)^{\frac{1}{n+1}},$$

where M is defined as the geometric mean of the product spacings function D_i such that

$$D_1 = F(x_1)$$
$$D_i = F(x_i) - F(x_{i-1});\, i = 2, \ldots, n$$
$$D_{n+1} = 1 - F(x_n)$$

It is easy to see that $\sum_{i=1}^{n+1} D_i = 1$. The MPS method is based on the observed ordered sample $x_1 < \cdots < x_n$ from MOAPP distribution, hence the product spacings function is

$$M(\gamma) = \left\{\frac{\alpha^{1-{x_1}^{-\lambda}}-1}{(\alpha-1)u(x_1)}\left(1-\frac{\alpha^{1-{x_n}^{-\lambda}}-1}{(\alpha-1)u(x_n)}\right)\prod_{i=2}^{n}\left[\frac{\alpha^{1-{x_i}^{-\lambda}}-1}{(\alpha-1)u(x_i)}-\frac{\alpha^{1-{x_{i-1}}^{-\lambda}}-1}{(\alpha-1)u(x_{i-1})}\right]\right\}^{\frac{1}{n+1}},$$

where $u(x_i) = \theta + (1-\theta)(\alpha-1)^{-1}\left(\alpha^{1-{x_i}^{-\lambda}}-1\right)$.

The natural logarithm of the product spacings function is

$$lnM(\gamma) = \frac{1}{n+1}\left\{ln\left(\alpha^{1-{x_1}^{-\lambda}}-1\right) - \ln\left((\alpha-1)u(x_1)\right) + \ln\left(1-\frac{\alpha^{1-{x_n}^{-\lambda}}-1}{(\alpha-1)u(x_n)}\right) + \sum_{i=2}^{n} ln\left[\frac{\alpha^{1-{x_i}^{-\lambda}}-1}{(\alpha-1)u(x_i)}-\frac{\alpha^{1-{x_{i-1}}^{-\lambda}}-1}{(\alpha-1)u(x_{i-1})}\right]\right\}. \quad (12)$$

To obtain the normal equations for the unknown parameters, we differentiate Eq. (12) partially with respect to the vector parameter γ and equate them to zero.

$$\frac{dlnM(\gamma)}{d\alpha} = \frac{1}{n+1}\left\{\frac{(1-{x_1}^{-\lambda})\alpha^{-{x_1}^{-\lambda}}}{\alpha^{1-{x_1}^{-\lambda}}-1} - \frac{u(x_1)+(\alpha-1)u_\alpha(x_1)}{(\alpha-1)u(x_1)} + \frac{(\alpha-1)u(x_n)(1-{x_n}^{-\lambda})\alpha^{-{x_n}^{-\lambda}}-(\alpha^{1-{x_n}^{-\lambda}}-1)[(\alpha-1)u_\alpha(x_n)+u(x_n)]}{((\alpha-1)u(x_n))^2-((\alpha-1)u(x_n)\alpha^{1-{x_n}^{-\lambda}}-1)} + \sum_{i=1}^{n}\frac{\frac{(\alpha-1)u(x_i)(1-{x_i}^{-\lambda})\alpha^{-{x_i}^{-\lambda}}-\left(\alpha^{1-{x_i}^{-\lambda}}-1\right)[(\alpha-1)u_\alpha(x_i)+u(x_i)]}{((\alpha-1)u(x_i))^2}-\frac{(\alpha-1)u(x_{i-1})(1-{x_{i-1}}^{-\lambda})\alpha^{1-{x_{i-1}}^{-\lambda}}-(\alpha^{1-{x_{i-1}}^{-\lambda}}-1)[(\alpha-1)u_\alpha(x_{i-1})+u(x_{i-1})}{((\alpha-1)u(x_{i-1}))^2}}{\frac{\alpha^{1-{x_i}^{-\lambda}}-1}{(\alpha-1)u(x_i)}-\frac{\alpha^{1-{x_{i-1}}^{-\lambda}}-1}{(\alpha-1)u(x_{i-1})}}\right\}$$

$$\frac{dlnM(\gamma)}{d\theta}=\frac{1}{n+1}\left\{\frac{u_\theta(x_1)}{u(x_1)}+\frac{(\alpha^{1-x_n{}^{-\lambda}}-1)u_\theta(x_n)}{u(x_n)((\alpha-1)u(x_n)-(\alpha^{1-x_n{}^{-\lambda}}-1))}+\right.$$

$$\left.\sum_{i=2}^{n}\frac{-\frac{\left(\alpha^{1-x_i{}^{-\lambda}}-1\right)u_\theta(x_i)}{\left(u(x_i)\right)^2}+\frac{(\alpha^{1-x_{i-1}{}^{-\lambda}}-1)u_\theta(x_{i-1})}{(u(x_{i-1}))^2}}{\left[\frac{\alpha^{1-x_i{}^{-\lambda}}-1}{u(x_i)}-\frac{\alpha^{1-x_{i-1}{}^{-\lambda}}-1}{u(x_{i-1})}\right]}\right\},$$

$$\frac{dlnM(\gamma)}{d\lambda}=\frac{1}{n+1}\left\{\frac{Log(\alpha x_1)x_1{}^{-\lambda}}{1-\alpha^{x_1{}^{-\lambda}-1}}-\frac{(\alpha-1)u_\lambda(x_1)}{(\alpha-1)u(x_1)}-\frac{u(x_n)\alpha^{1-x_n{}^{-\lambda}}Log(\alpha x_n)x_n{}^{-\lambda}-\left(\alpha^{1-x_n{}^{-\lambda}}-1\right)u_\lambda(x_n)}{u(x_n)((\alpha-1)u(x_n)-(\alpha^{1-x_n{}^{-\lambda}}-1))}+\right.$$

$$\left.\sum_{i=2}^{n}\frac{\frac{u(x_i)\alpha^{1-x_i{}^{-\lambda}}Log(\alpha x_i)x_i{}^{-\lambda}-\left(\alpha^{1-x_i{}^{-\lambda}}-1\right)u_\lambda(x_i)}{u(x_i)((\alpha-1)u(x_i)-(\alpha^{1-x_i{}^{-\lambda}}-1))}-\frac{u(x_{i-1})\alpha^{1-x_{i-1}{}^{-\lambda}}Log(\alpha x_{i-1})x_{i-1}{}^{-\lambda}-\left(\alpha^{1-x_{i-1}{}^{-\lambda}}-1\right)u_\lambda(x_{i-1})}{u(x_{i-1})((\alpha-1)u(x_{i-1})-(\alpha^{1-x_{i-1}{}^{-\lambda}}-1))}}{\frac{\alpha^{1-x_i{}^{-\lambda}}-1}{(\alpha-1)u(x_i)}-\frac{\alpha^{1-x_{i-1}{}^{-\lambda}}-1}{(\alpha-1)u(x_{i-1})}}\right\},\tag{13}$$

where u_α, u_θ and u_λ represent the partial derivative of $u(x_i)$ with respect to α, θ and λ respectively. The estimators of γ can be obtained by solving the above system of nonlinear equations numerically, so the MPS of α, θ and λ are denoted by $\hat{\alpha}_{MP}$, $\hat{\theta}_{MP}$ and $\hat{\lambda}_{MP}$ respectively.

4. Bayesian Estimation Method

In this section we consider the non-classical method of estimation that is Bayes estimates for the unknown parameters α, θ and λ of MOAPP distribution. The quadratic loss and LINEX loss functions are the assumed loss functions.

In Bayesian method, all parameters are random variables with a certain distribution called prior distribution. If prior information is not available which is usually the case, we need to select a prior distribution. Since the selection of prior distribution plays an important role in estimation of the parameters, our choice for the prior of α,θ and λ are the independent gamma distributions, which are $G(a_1, b_1)$, $G(a_2, b_2)$ and $G(a_3, b_3)$ respectively. Thus, the suggested prior for α, θ and λ can be written as:

$$\pi_1(\alpha)\propto\alpha^{a_1-1}e^{-b_1\alpha},\ \pi_2(\theta)\propto\theta^{a_2-1}e^{-b_2\theta},\ \pi_3(\lambda)\propto\lambda^{a_3-1}e^{-b_3\lambda},$$

respectively, where a_1, a_2, a_3, b_1, b_2 and b_3 are the hyper parameters of prior distributions.

The joint prior of α, θ and λ is

$$k(\alpha,\theta,\lambda)\propto\alpha^{a_1-1}\theta^{a_2-1}\lambda^{a_3-1}e^{-b_1\alpha-b_2\theta-b_3\lambda},\ \alpha,\theta,\lambda,a_1,a_2,a_3,b_1,b_2,b_3>0.$$

The joint posterior of α, θ and λ is given by

$$p(\alpha,\theta,\lambda/\underline{x}) \propto L(\underline{x}/\alpha,\theta,\lambda)k(\alpha,\theta,\lambda),$$

where $L(\underline{x}/\alpha,\theta,\lambda)$ is the likelihood function of MOAPP distribution. When substituting the likelihood function $L(\underline{x}/\alpha,\theta,\lambda)$ and the joint prior $k(\alpha,\theta,\lambda)$in the above equation, the joint posterior will be:

$$p\left(\alpha,\theta,\frac{\lambda}{\underline{x}}\right) \propto \alpha^{n-\sum_{i=1}^{n} x_i^{-\lambda}+a_1-1}\theta^{n+a_2-1}\lambda^{n+a_3-1}e^{-b_1\alpha-b_2\theta-b_3\lambda}\prod_{i=1}^{n}\frac{1}{\alpha-1}\frac{Log\alpha}{x_i^{\lambda+1}}\left(\theta+\frac{(1-\theta)}{\alpha-1}\left(\alpha^{1-x_i^{-\lambda}}-1\right)\right)^{-2}$$

$$p(\alpha,\theta,\lambda/\underline{x}) \propto G_{\alpha\backslash\lambda}(n-\textstyle\sum_{i=1}^{n} x_i^{-\lambda}+a_1,b_1)G_\theta(n+a_2,b_2)G_\lambda(n+a_3,b_3)e^{\phi(\alpha,\theta,\lambda)},$$

where $\phi(\alpha,\theta,\lambda)=\sum_{i=1}^{n} ln\frac{1}{\alpha-1}\frac{Log\alpha}{x_i^{\lambda+1}}-2\ln\left(\theta+\frac{(1-\theta)}{\alpha-1}\left(\alpha^{1-x_i^{-\lambda}}-1\right)\right)$

In the case of quadratic loss function, Bayes estimate is the posterior mean, the determination of posterior mean for the purpose of obtaining Bayes estimation of the parameters α, θ and λ, is not easy to obtain unless we use numerical approximation methods.

In the literature, there are several approximation methods available to solve this kind of problem. Here, we consider Monte Carlo Markov Chain (MCMC) approximation method, see Karandikar (2006). This approximation method reduces the ratio of integrals into a whole and produces a single numerical result.

A wide variety of MCMC schemes are available. An important sub-class of MCMC methods are Gibbs sampling and more general Metropolis within Gibbs samplers. Indeed, the MCMC samples may be used to completely summarize the posterior uncertainty about the parameters α, θ and λ, through a kernel estimate of the posterior distribution. This is also true of any function of the parameters.

Therefore, to generate samples from MOAPP distribution, we use the Metropolis-Hastings method (Metropolis et al. (1953) with normal proposal distribution). For details regarding the implementation of the Metropolis-Hasting algorithm, the readers may refer to Robert and Casella (2013) and Almetwally et al. (2018). The full conditional posterior densities of α, θ and λ and the data are given by:

$$\pi(\alpha/\theta,\lambda;\underline{x}) \propto G_{\alpha\backslash\lambda}\left(n-\sum_{i=1}^{n} x_i^{-\lambda}+a_1,b_1\right)e^{\phi(\alpha,\theta,\lambda)}$$

$$\pi(\theta/\alpha,\lambda,\underline{x}) = G_\theta(n+a_2,b_2)e^{-2\ln\left(\theta+\frac{(1-\theta)}{\alpha-1}\left(\alpha^{1-x_i^{-\lambda}}-1\right)\right)} \quad (14)$$

$$\pi(\lambda/\alpha,\theta,\underline{x}) = G_\lambda(n+a_3,b_3)e^{\phi(\alpha,\theta,\lambda)}$$

To apply the Gibbs technique we need the following algorithm:

(1) Start with initial values $(\alpha_0,\theta_0,\lambda_0)$

(2) Use M-H algorithm to generate posterior sample for α, θ and λ from Eq. (14)

(3) Repeat step 2 (T)-times and obtain $(\alpha_1, \theta_1, \lambda_1), \alpha_2, \theta_2, \lambda_2),\ldots,(\alpha_T, \theta_T, \lambda_T)$
(4) After obtaining the posterior sample, the Bayes estimates of α, θ and λ with respect to quadratic loss function are:

$$\hat{\alpha}^{MC} = [E_\pi(\alpha/x)] \approx (\frac{1}{T-T_0}\sum_{i=1}^{T-T_0} \alpha_i)$$

$$\hat{\theta}^{MC} = [E_\pi(\theta\big/x)] \approx (\frac{1}{T-T_0}\sum_{i=1}^{T-T_0} \theta_i)$$

$$\hat{\lambda}^{MC} = [E_\pi(\lambda\big/x)] \approx (\frac{1}{T-T_0}\sum_{i=1}^{T-T_0} \lambda_i)$$

where T_0 is the burn-in-period of Markov Chain.

The Bayes estimates of the unknown parameters α, θ and λ under the LINEX loss function can be calculated through the following equation:

$$\gamma_j = \frac{-1}{\upsilon} ln\left(\sum_{i=1}^{L} \frac{e^{-\upsilon\gamma^{(i)}}}{L}\right),$$

where ν reflects the direction and degree of asymmetry, L is number of periods in the MCMC.

5. Interval Estimation Methods

In this section we consider three methods of approximate confidence intervals for the parameters of MOAPP distribution. Numerical analysis via simulation is used for comparisons between these methods in Section 6.

5.1. Asymptotic confidence Interval for (MLE)

When the sample size is large enough, the normal approximation of the MLE can be used to construct asymptotic confidence intervals for the parameters α, θ and λ. The asymptotic normality of MLE can be stated as $\sqrt{n}(\hat{\gamma}-\gamma) \overset{d}{\rightarrow} N_3(0, I^{-1}(\gamma))$, where γ=(α,θ, λ) is a vector of parameters, $\overset{d}{\rightarrow}$ denotes convergence in distribution and $I(\gamma)$ is the Fisher information matrix

$$I(\gamma) = -\begin{bmatrix} E(\ell_{\alpha\alpha}) & E(\ell_{\alpha\theta}) & E(\ell_{\alpha\lambda}) \\ E(\ell_{\theta\alpha}) & E(\ell_{\theta\theta}) & E(\ell_{\theta\lambda}) \\ E(\ell_{\lambda\alpha}) & E(\ell_{\lambda\theta}) & E(\ell_{\lambda\lambda}) \end{bmatrix}$$

The expected values of the second derivatives can be found by using some integration techniques. Therefore, the (1- ζ) 100% approximate CIs for α, θ and λ are

$$\hat{\alpha} \pm z_{\frac{\zeta}{2}}\sqrt{\upsilon_{11}}, \hat{\theta} \pm z_{\frac{\zeta}{2}}\sqrt{\upsilon_{22}}, \hat{\lambda} \pm z_{\frac{\zeta}{2}}\sqrt{\upsilon_{33}},$$

respectively, where $\upsilon_{11}, \upsilon_{22}, \upsilon_{33}$ are the entries in the main diagonal of the Fisher matrix $I^{-1}(\gamma)$, and $z_{\frac{\zeta}{2}}$ is the ($\frac{\zeta}{2}$) 100% lower percentile of the standard normal distribution.

5.2. Asymptotic Confidence Interval for (MPS)

In this section, we propose the asymptotic confidence intervals using MPS method. As it was mentioned by Cheng and Amin [1979], Ghosh and Jammalamadaka [2001] and Anatolyev and Kosenok [2005], the MPS method also shows asymptotic properties like the maximum likelihood estimator and is asymptotically equivalent to MLE. Therefore, we may propose the asymptotic confidence intervals using MPS. The exact distribution of the MPS cannot be obtained explicitly. Therefore, the asymptotic properties of MPS similar to that of MLE can be used to construct the confidence intervals.

$$J(\gamma) = -\begin{bmatrix} E(M_{\alpha\alpha}) & E(M_{\alpha\theta}) & E(M_{\alpha\lambda}) \\ E(M_{\theta\alpha}) & E(M_{\theta\theta}) & E(M_{\theta\lambda}) \\ E(M_{\lambda\alpha}) & E(M_{\lambda\theta}) & E(M_{\lambda\lambda}) \end{bmatrix}$$

The first derivatives of the product of spacing, i.e. the function M with respect to parameters α, θ and λ, are given by Equation (13), second derivative can be found numerically and hence one can obtain the (1- ζ) 100% asymptotic confidence intervals based on MPS as follows:

$$\hat{\alpha}_{MP} \pm z_{\frac{\zeta}{2}}\sqrt{\omega_{11}},\ \hat{\theta}_{MP} \pm z_{\frac{\zeta}{2}}\sqrt{\omega_{22}}\ ,\ \hat{\lambda}_{MP} \pm z_{\frac{\zeta}{2}}\sqrt{\omega_{33}},$$

where ω_{11}, ω_{22} and ω_{33} are the diagonal entries of the Fisher matrix $J^{-1}(\gamma)$.

5.3. Credible intervals

Using MCMC techniques in Section (4), the Bayes credible intervals of the parameter α, θ and λ can be obtained as follows:

(1) Arrange α_i, θ_i and λ_i; in ascending order as follow $\alpha_{[1]}, \alpha_{[2]}, \dots, \alpha_{[T]}$, $\theta_{[1]}, \theta_{[2]}, \dots, \theta_{[T]}$ and $\lambda_{[1]}, \lambda_{[2]}, \dots, \lambda_{[T]}$

(2) A two-sided (1- ζ) 100% credible intervals for the unknown parameters α, θ and λ are given by

$$(\alpha_{\left[T\frac{\zeta}{2}\right]}, \alpha_{\left[T_{1-\frac{\zeta}{2}}\right]}),\ (\theta_{\left[T\frac{\zeta}{2}\right]}, \theta_{\left[T_{1-\frac{\zeta}{2}}\right]}),\ (\lambda_{\left[T\frac{\zeta}{2}\right]}, \lambda_{\left[T_{1-\frac{\zeta}{2}}\right]}),$$

respectively, where [x] denoted the largest integer less than or equal to x.

6. Simulation Study and Data Analysis

6.1. Simulation Study

In this section, we consider some experimental results that are produced to see the effectiveness of different point and interval estimation methods. We mainly compare different point estimates in terms of mean squared errors (MSE) and bias values. Efficiency of confidence intervals is compared in terms of average interval length (AIL). Based on the generated data, we compute MLE and MPS estimates using the Newton-Raphson method. Further, we compute Bayes estimates using a Monte Carlo simulation and the MH algorithm under both squared error and LINEX loss functions with v=1.5 by using R language.

We start by building our model with generating all simulation controls. In this stage, we must do the following steps in sequence:

Step 1: Suppose the following values for the parameter vector of MOAPP distribution γ=(α,θ, λ), case 1=(0.5, 0.5, 1.5), case 2=(1.5, 0.5, 1.5), case 3=(3,0.5,1.5), case 4=(0.5,1.5, 1.5), case 5 =(1.5,1.5,1.5) and case 6=(3,1.5,1.5), case 7=(0.5,3,1.5), case 8=(1.5,3,1.5), case 9=(3,3,1.5).

Step 2: Choose sample sizes n =30, 70 and 200.

Step 3: Generate the sample random values of MOAPP distribution by using quantile function$X = \left(1 - \frac{1}{\ln(\alpha)} \ln\left(1 + \frac{\theta U(\alpha-1)}{1-U(1-\theta)}\right)\right)^{-1/\lambda}$, where U is a uniform distribution (0, 1).

Step 4: Solve differential equations for each estimation methods, to obtain the estimators of the parameters for MOAPP distribution, so we calculate α, θ, and λ.

Step 5: Repeat this experiment (L-1) times. In each experiment use the same values of the parameters. It is certain that the values of generating random samples are varying from experiment to experiment even though the sample size (n) does not change.

Finally, we have L-values of bias and MSE. We compute the average biases and average MSE's over 10,000 runs. This number of runs will give the accuracy in the order ±0.01 (see Karian and Dudewicz (1998)). The bias of estimator is equal to $\hat{\gamma} - \gamma$, where $\hat{\gamma}$ is the estimated value of γ, and the mean squared error (MSE) of the estimator is MSE=Mean $(\hat{\gamma} - \gamma)^2$.

The simulated results of point estimation methods are presented in Tables (1) to (3), where the MSE and the bias are given in each cell and it can be pointed out that the MPS and Bayesian methods for estimating the unknown parameters of MOAPP distribution are better than the MLE method, where the MSE value is considered for comparison. We summarize the cases as follows:

1- For $0 < \alpha < 1$, the best point estimation method for estimating α is the Bayesian method under LINEX loss function, while for $\alpha > 1$ the best estimation method is the MPS and Baysian under the SE loss function.

2- For $0 < \theta < 1$, the best point estimation method for θ is the Bayesian method under LINEX loss function, while for $\theta > 1$ the best estimation method is the MPS and Bayesian under the SE loss function.
3- For all values of λ the Bayesian under the SE loss function is the best estimation method.
4- The average bias and MSE decrease as the sample size increases. It verifies the consistency properties of all the estimators.
5- The MLE overestimates α, θ and λ for almost all cases except for case 3, where the MLE underestimates α. It is also noticed that when the sample size n=200, the MLE underestimate α for cases 1, 3, 6 and underestimate θ for case 7, see Table (3).
6- MPS and Bayesian estimation sometimes overestimate the parameters and sometimes underestimate them.

Figure 3 shows the three dimension plots of MSE with different parameters values

For confidence interval estimation of MOAPP parameters α, θ and λ, we observe the 95% confidence intervals (L,U) where L represents the lower bound and U is the upper bound of this interval. Three confidence intervals are considered in simulation analysis, i.e. asymptotic confidence intervals of MLE and MPS, also the credible intervals of Bayesian method under SE and LINEX loss functions. The comparison is conducted depending on the average interval length (AIL), hence the smaller the AIL is the better confidence estimate we observe. The results are reported in Tables (4) to (6) below.

Table 1. Bias and MSE for $\alpha, \theta, and\ \lambda$, with n=30

λ	θ	α	n=30	MLE		MPS		SE		LINEX ($\upsilon = 1.5$)	
				Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
1.5	0.5	0.5	α	0.1072	0.1866	-0.0160	0.0818	0.1051	0.0994	0.0310	0.0661
			θ	0.3212	0.4946	0.0540	0.2012	0.0517	0.0536	0.0071	0.0395
			λ	0.3134	0.9250	-0.2585	0.6378	-0.0858	0.1772	-0.1893	0.1832
		1.5	α	0.0975	0.4482	0.0092	0.0306	-0.2242	0.3509	-0.3552	0.4101
			θ	0.2341	0.3511	-0.0125	0.1299	0.1058	0.0816	0.0559	0.0549
			λ	0.2786	0.6504	-0.1677	0.4781	-0.0966	0.1989	-0.1919	0.2103
		3	α	-0.0792	1.1131	-0.0032	0.0421	-0.3287	0.5870	-0.5214	0.8308
			θ	0.2925	0.4977	-0.0030	0.1425	0.1049	0.0894	0.0561	0.0620
			λ	0.2350	0.4980	-0.1624	0.3824	-0.0554	0.1675	-0.1436	0.1748
	1.5	0.5	α	0.3291	0.6917	0.0265	0.1905	0.1369	0.1079	0.0642	0.0693
			θ	0.2813	0.7987	-0.0524	0.3616	-0.1709	0.2413	-0.2738	0.2666
			λ	0.1274	0.4280	-0.2490	0.4257	-0.1173	0.1662	-0.2064	0.1787
		1.5	α	0.4186	1.2902	-0.0008	0.1538	-0.1472	0.2932	-0.2716	0.3255
			θ	0.3391	1.1591	-0.0960	0.5198	-0.1357	0.2656	-0.2441	0.2761
			λ	0.1351	0.2503	-0.1453	0.2163	-0.1746	0.1315	-0.2488	0.1643
		3	α	0.2624	2.2557	0.0121	0.1190	-0.2997	0.5282	-0.4753	0.7364
			θ	0.5562	1.8669	-0.0793	0.6735	-0.1658	0.2256	-0.2638	0.2538
			λ	0.1207	0.2080	-0.1428	0.1813	-0.1191	0.0944	-0.1806	0.1122

Table 1. Bias and MSE for $\alpha, \theta, and\ \lambda$, with n=30 (cont.)

λ	θ	α	**n=30**	MLE		MPS		SE		LINEX ($\upsilon = 1.5$)	
				Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
1.5	3	0.5	α	0.4689	1.2082	0.0355	0.2951	0.1449	0.1438	0.0665	0.0872
			θ	0.1613	0.9882	-0.0498	0.3326	-0.3164	0.4939	-0.4915	0.7154
			λ	0.0897	0.2667	-0.2146	0.2902	-0.0869	0.0977	-0.1582	0.1124
		1.5	α	0.6631	3.0228	-0.1891	0.6353	-0.1309	0.3296	-0.2644	0.3409
			θ	0.4555	2.2871	-0.0747	0.5972	-0.3635	0.5986	-0.5534	0.8297
			λ	0.0844	0.1606	-0.1612	0.1698	-0.1407	0.0853	-0.1967	0.1038

Table 2. Bias and MSE for $\alpha, \theta, and\ \lambda$, with n=70

λ	θ	α	n=70	MLE		MPS		SE		LINEX ($\upsilon = 1.5$)	
				Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
1.5	0.5	0.5	α	0.0336	0.1130	-0.0739	0.0776	0.0250	0.0670	-0.0041	0.0572
			θ	0.2013	0.2553	0.0725	0.1261	0.0400	0.0359	0.0215	0.0304
			λ	0.1153	0.3558	-0.2104	0.3237	-0.0639	0.1000	-0.0990	0.1095
		1.5	α	0.0623	0.2540	0.0138	0.0160	-0.0476	0.1202	-0.0944	0.1319
			θ	0.0885	0.0804	-0.0261	0.0491	0.0456	0.0296	0.0289	0.0255
			λ	0.1287	0.2330	-0.1076	0.2087	-0.0293	0.0771	-0.0618	0.0782
		3	α	-0.0249	0.6191	0.0128	0.0235	-0.0472	0.1116	-0.0870	0.1191
			θ	0.1154	0.1198	-0.0197	0.0518	0.0161	0.0295	0.0005	0.0265
			λ	0.1113	0.1867	-0.0977	0.1677	-0.0338	0.0770	-0.0667	0.0822
	1.5	0.5	α	0.1695	0.3175	-0.0285	0.1061	0.0281	0.0477	0.0027	0.0410
			θ	0.1438	0.3712	0.0043	0.1709	-0.0248	0.0884	-0.0606	0.0921
			λ	0.0368	0.1866	-0.1745	0.2275	-0.0403	0.0616	-0.0687	0.0652
		1.5	α	0.2486	0.6849	-0.0010	0.0723	-0.0557	0.1385	-0.1022	0.1463
			θ	0.1372	0.4181	-0.0742	0.2391	-0.0262	0.0840	-0.0630	0.0839
			λ	0.0646	0.1015	-0.0772	0.0965	-0.0315	0.0444	-0.0537	0.0460
		3	α	0.1262	1.3314	0.0191	0.0541	-0.0835	0.1412	-0.1363	0.1604
			θ	0.2561	0.6573	-0.0634	0.3260	-0.0548	0.0829	-0.0907	0.0905
			λ	0.0609	0.0867	-0.0736	0.0816	-0.0377	0.0371	-0.0590	0.0397
			θ	0.2134	0.7791	-0.0560	0.2755	-0.0493	0.1043	-0.0916	0.1168
			λ	0.0865	0.2809	-0.1488	0.2530	-0.0658	0.0681	-0.0989	0.0762

Table 3. Bias and MSE for $\alpha, \theta, and\ \lambda$, with n=200

λ	θ	α	**n=200**	MLE		MPS		SE		LINEX ($\upsilon = 1.5$)	
				Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
1.5	0.5	0.5	α	-0.0015	0.0775	-0.0905	0.0613	0.0009	0.0125	-0.0042	0.0124
			θ	0.1046	0.0836	0.0714	0.0588	0.0091	0.0066	0.0058	0.0064
			λ	0.0222	0.1137	-0.1368	0.1294	-0.0161	0.0117	-0.0210	0.0119
		1.5	α	0.0142	0.0877	0.0118	0.0068	-0.0160	0.0178	-0.0224	0.0182
			θ	0.0353	0.0227	-0.0186	0.0152	-0.0021	0.0049	-0.0053	0.0049
			λ	0.0528	0.0709	-0.0515	0.0683	-0.0041	0.0117	-0.0087	0.0119
		3	α	-0.0391	0.3088	0.0124	0.0111	-0.0136	0.0120	-0.0189	0.0124
			θ	0.0460	0.0303	-0.0153	0.0155	-0.0028	0.0054	-0.0058	0.0054
			λ	0.0470	0.0581	-0.0453	0.0551	-0.0054	0.0120	-0.0103	0.0120

Table 3. Bias and MSE for $\alpha, \theta, and\ \lambda$, with n=200 (cont.)

λ	θ	α	n=200	MLE		MPS		SE		LINEX (υ = 1.5)	
				Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
1.5	1.5	0.5	α	0.0811	0.1085	-0.0388	0.0462	-0.0007	0.0105	-0.0051	0.0104
			θ	0.1062	0.2893	0.0313	0.0781	-0.0098	0.0133	-0.0153	0.0133
			λ	0.0015	0.0617	-0.0828	0.0750	-0.0065	0.0107	-0.0110	0.0110
		1.5	α	0.2053	0.7433	-0.0043	0.0249	-0.0106	0.0184	-0.0164	0.0189
			θ	0.0514	0.1709	-0.0382	0.0765	-0.0062	0.0175	-0.0120	0.0177
			λ	0.0244	0.0320	-0.0322	0.0321	-0.0068	0.0093	-0.0111	0.0093
		3	α	-0.0239	1.1126	0.0035	0.0187	-0.0147	0.0171	-0.0205	0.0176
			θ	0.1709	0.3354	-0.0348	0.1077	-0.0227	0.0146	-0.0285	0.0151
			λ	0.0272	0.0289	-0.0302	0.0276	-0.0167	0.0074	-0.0207	0.0077
	3	0.5	α	0.0965	0.1246	-0.0138	0.0489	-0.0011	0.0108	-0.0056	0.0106
			θ	-0.0013	0.2440	0.0044	0.0374	-0.0133	0.0136	-0.0191	0.0139
			λ	0.0164	0.0375	-0.0498	0.0434	-0.0005	0.0076	-0.0047	0.0076
		1.5	α	0.0865	0.2790	-0.0494	0.1157	-0.0120	0.0160	-0.0177	0.0163
			θ	0.0794	0.2092	-0.0320	0.1043	-0.0416	0.0195	-0.0484	0.0209
			λ	0.0205	0.0219	-0.0281	0.0209	-0.0158	0.0063	-0.0195	0.0064
			θ	0.0329	0.5907	-0.0334	0.1001	-0.0094	0.0162	-0.0155	0.0166
			λ	0.0312	0.0863	-0.0568	0.0813	-0.0111	0.0129	-0.0162	0.0131

From Tables (4) to (6) we notice that the (AIL) of the credible intervals under SE and LINEX are smaller than the (AIL) of MLE and MPS in most cases except for some restricted ones.

We can summarize the analysis of the confidence interval estimation in the following points:

1. For $0 < \alpha < 1$, the best interval estimate for α is the Bayesian credible interval under SE and LINEX loss functions, while for $\alpha > 1$ the best interval estimation is the asymptotic interval under the MPS method except for cases 8 and 11, where the Bayesian credible interval under LINEX has the smallest AIL.
2. For $\theta < 3$, the best interval estimate for θ is the Bayesian credible interval under LINEX loss function, while for $\theta \geq 3$, the best interval estimation is the asymptotic interval under the MPS method and the Bayesian credible interval under the SE loss function.
3. Bayesian credible interval under the LINEX loss function has the smallest AIL for estimating λ , and hence it can be considered as the best confidence interval of λ. For the case 5, the Bayesian credible interval under the SE loss function is preferable to estimate λ.
4. AIC decreases as the sample size increases.

Table 4. 95% confidence intervals and Average Interval length for $\alpha, \theta, and\ \lambda$, with n=30

λ	θ	α	30	MLE			MPS			SE			LINEX($\upsilon = 1.5$)		
				L	U	AIL	L	U	AIL	L	U	AIL	L	U	AIL
0.5	0.5	0.5	α	-0.2133	1.4278	1.6411	-0.0759	1.0439	1.1198	0.0210	1.1892	1.1683	0.0294	1.0326	1.0033
			θ	-0.4056	2.0480	2.4536	-0.3191	1.4271	1.7462	0.1082	0.9952	0.8870	0.1169	0.8972	0.7803
			λ	0.0303	3.5965	3.5662	-0.2403	2.7233	2.9636	0.6044	2.2240	1.6196	0.5564	2.0650	1.5086
		1.5	α	0.2986	2.8963	2.5976	1.1668	1.8515	0.6847	0.1985	2.3531	2.1546	0.0978	2.1918	2.0940
			θ	-0.3332	1.8014	2.1346	-0.2188	1.1938	1.4126	0.0845	1.1270	1.0425	0.1088	1.0030	0.8942
			λ	0.2944	3.2627	2.9683	0.0168	2.6477	2.6309	0.5480	2.2588	1.7108	0.4897	2.1264	1.6367
		3	α	0.8578	4.9839	4.1260	2.5943	3.3992	0.8049	1.3115	4.0312	2.7196	1.0097	3.9475	2.9379
			θ	-0.4663	2.0514	2.5176	-0.2431	1.2372	1.4803	0.0549	1.1549	1.1000	0.0793	1.0329	0.9535
			λ	0.4303	3.0398	2.6095	0.1675	2.5076	2.3401	0.6478	2.2414	1.5936	0.5849	2.1280	1.5430
1.5	1.5	0.5	α	-0.6687	2.3269	2.9956	-0.3277	1.3808	1.7085	0.0501	1.2236	1.1735	0.0627	1.0658	1.0031
			θ	0.1178	3.4447	3.3268	0.2730	2.6223	2.3493	0.4241	2.2340	1.8099	0.3661	2.0863	1.7202
			λ	0.3690	2.8857	2.5167	0.0684	2.4337	2.3653	0.6155	2.1500	1.5345	0.5688	2.0184	1.4496
		1.5	α	-0.1520	3.9892	4.1412	0.7303	2.2682	1.5379	0.3290	2.3767	2.0476	0.2425	2.2143	1.9718
			θ	-0.1647	3.8429	4.0076	0.0028	2.8053	2.8025	0.3874	2.3412	1.9539	0.3417	2.1701	1.8284
			λ	0.6905	2.5797	1.8892	0.4884	2.2209	1.7325	0.7009	1.9499	1.2489	0.6224	1.8800	1.2576
		3	α	0.3625	6.1622	5.7997	2.3361	3.6882	1.3521	1.3994	4.0013	2.6020	1.1209	3.9286	2.8078
			θ	-0.3910	4.5035	4.8945	-0.1811	3.0224	3.2035	0.4596	2.2088	1.7492	0.3928	2.0797	1.6869
			λ	0.7583	2.4830	1.7247	0.5705	2.1440	1.5735	0.8245	1.9374	1.1129	0.7651	1.8737	1.1086
	3	0.5	α	-0.9806	2.9184	3.8990	-0.5275	1.5985	2.1261	-0.0438	1.3336	1.3774	0.0013	1.1317	1.1304
			θ	1.2377	5.0848	3.8470	1.8234	4.0769	2.2535	1.4506	3.9167	2.4661	1.1560	3.8610	2.7050
			λ	0.5923	2.5870	1.9946	0.3165	2.2544	1.9379	0.8230	2.0032	1.1802	0.7610	1.9227	1.1617
		1.5	α	-0.9886	5.3148	6.3033	-0.2075	2.8293	3.0368	0.2708	2.4675	2.1967	0.2128	2.2585	2.0457
			θ	0.6277	6.2833	5.6557	1.4170	4.4336	3.0166	1.2945	3.9785	2.6840	1.0250	3.8681	2.8431
			λ	0.8161	2.3526	1.5364	0.5950	2.0826	1.4876	0.8565	1.8621	1.0056	0.8019	1.8046	1.0027
		3	α	0.2547	6.8300	6.5753	1.7246	4.0180	2.2933	1.3338	4.0549	2.7210	1.0840	3.9622	2.8781
			θ	-0.0412	7.4860	7.5271	0.4755	4.9025	4.4270	1.4397	3.8171	2.3774	1.1806	3.7464	2.5657
			λ	0.8854	2.2719	1.3865	0.7585	1.9615	1.2030	0.9602	1.8124	0.8522	0.9186	1.7663	0.8477
3			α	-0.3982	7.9265	8.3247	1.7148	4.0117	2.2969	1.4737	4.0449	2.5712	1.2437	3.9261	2.6824
			θ	-0.4460	8.2602	8.7062	0.5164	4.8636	4.3472	1.4178	3.9410	2.5232	1.1647	3.8338	2.6690
			λ	1.7433	4.6034	2.8601	1.5358	3.9054	2.3696	1.9628	3.6126	1.6498	1.8242	3.5302	1.7061
		1.5	α	-1.4085	6.2428	7.6513	-0.1771	2.8018	2.9789	0.2497	2.5138	2.2641	0.2016	2.3069	2.1053
			θ	0.1217	6.8545	6.7328	1.4359	4.4034	2.9675	1.4646	3.7652	2.3006	1.2311	3.6803	2.4492
			λ	1.6064	4.7305	3.1241	1.2692	4.1083	2.8392	1.7903	3.6714	1.8810	1.6402	3.5800	1.9398

Table 5. 95% confidence intervals and Average Interval length for $\alpha, \theta, and\ \lambda$, with n=70

λ	θ	α	**n=70**	MLE			MPS			SE			LINEX ($\upsilon = 1.5$)		
				L	U	AIL	L	U	AIL	L	U	AIL	L	U	AIL
			α	-0.1221	1.1894	1.3115	-0.1005	0.9527	1.0532	0.0188	1.0311	1.0124	0.0262	0.9656	0.9394
		0.5	θ	-0.2076	1.6101	1.8178	-0.1093	1.2543	1.3636	0.1760	0.9040	0.7280	0.1817	0.8613	0.6796
			λ	0.4676	2.7630	2.2954	0.2531	2.3260	2.0729	0.8275	2.0448	1.2173	0.7805	2.0215	1.2410
			α	0.5817	2.5430	1.9613	1.2670	1.7606	0.4936	0.7775	2.1272	1.3497	0.7164	2.0948	1.3784
	0.5	1.5	θ	0.0601	1.1169	1.0568	0.0424	0.9054	0.8630	0.2195	0.8717	0.6522	0.2204	0.8373	0.6170
			λ	0.7164	2.5410	1.8245	0.5217	2.2631	1.7413	0.9281	2.0133	1.0852	0.9025	1.9740	1.0715
			α	1.4329	4.5173	3.0844	2.7130	3.3126	0.5996	2.3029	3.6027	1.2998	2.2568	3.5692	1.3124
		3	θ	-0.0246	1.2553	1.2798	0.0356	0.9250	0.8893	0.1800	0.8521	0.6721	0.1808	0.8201	0.6392
			λ	0.7925	2.4301	1.6376	0.6225	2.1821	1.5596	0.9249	2.0075	1.0826	0.8856	1.9810	1.0954
			α	-0.3842	1.7232	2.1074	-0.1648	1.1079	1.2727	0.1026	0.9535	0.8509	0.1047	0.9007	0.7960
		0.5	θ	0.4829	2.8048	2.3218	0.6937	2.3150	1.6213	0.8930	2.0574	1.1644	0.8552	2.0236	1.1684
			λ	0.6928	2.3809	1.6882	0.4551	2.1960	1.7409	0.9784	1.9410	0.9626	0.9481	1.9144	0.9664
			α	0.2007	3.2964	3.0957	0.9716	2.0263	1.0547	0.7212	2.1673	1.4461	0.6736	2.1221	1.4485
1.5	1.5	1.5	θ	0.3980	2.8763	2.4782	0.4780	2.3736	1.8957	0.9067	2.0409	1.1342	0.8816	1.9925	1.1109
			λ	0.9529	2.1763	1.2234	0.8326	2.0130	1.1804	1.0593	1.8778	0.8185	1.0385	1.8541	0.8156
			α	0.8771	5.3753	4.4982	2.5645	3.4737	0.9093	2.1967	3.6364	1.4397	2.1237	3.6038	1.4801
		3	θ	0.2477	3.2646	3.0169	0.3239	2.5493	2.2254	0.8900	2.0004	1.1105	0.8455	1.9730	1.1275
			λ	0.9959	2.1259	1.1300	0.8850	1.9677	1.0827	1.0909	1.8337	0.7428	1.0669	1.8151	0.7481
			α	-0.3870	1.7949	2.1819	-0.2668	1.2531	1.5200	0.0700	1.0371	0.9671	0.0774	0.9710	0.8936
		0.5	θ	1.9630	4.1461	2.1831	2.3265	3.6698	1.3434	2.2411	3.5523	1.3113	2.1837	3.5206	1.3369
			λ	0.8594	2.2130	1.3536	0.6518	2.0811	1.4293	1.0213	1.9079	0.8866	0.9933	1.8881	0.8948
			α	-0.0243	3.5307	3.5550	0.2681	2.5019	2.2338	0.7207	2.0742	1.3535	0.6664	2.0365	1.3701
	3		θ	1.5904	4.7831	3.1927	1.9043	3.9939	2.0896	2.2433	3.6152	1.3719	2.1870	3.5810	1.3940
		1.5	λ	1.0468	2.0441	0.9973	0.9438	1.9072	0.9634	1.1532	1.7436	0.5904	1.1356	1.7237	0.5881
			θ	1.5340	4.8929	3.3589	1.9207	3.9674	2.0467	2.3237	3.5777	1.2540	2.2615	3.5554	1.2940
			λ	2.0612	4.1119	2.0507	1.9090	3.7934	1.8844	2.4380	3.4304	0.9924	2.3947	3.4075	1.0128

Table 6. 95% confidence intervals and Average Interval length for $\alpha, \theta, and\ \lambda$, with n=200

λ	θ	α	n=200	MLE			MPS			SE			LINEX ($\upsilon = 1.5$)		
				L	U	AIL	L	U	AIL	L	U	AIL	L	U	AIL
			α	-0.0474	1.0444	1.0918	-0.0423	0.8613	0.9036	0.2813	0.7205	0.4392	0.2773	0.7142	0.4369
		0.5	θ	0.0758	1.1333	1.0575	0.1171	1.0257	0.9086	0.3506	0.6677	0.3170	0.3491	0.6625	0.3134
			λ	0.8623	2.1822	1.3198	0.7108	2.0155	1.3046	1.2741	1.6937	0.4196	1.2683	1.6898	0.4216
			α	0.9343	2.0941	1.1598	1.3516	1.6720	0.3204	1.2235	1.7445	0.5210	1.2160	1.7392	0.5232
	0.5	1.5	θ	0.2482	0.8224	0.5742	0.2421	0.7207	0.4786	0.3602	0.6356	0.2754	0.3582	0.6313	0.2731
			λ	1.0410	2.0647	1.0237	0.9461	1.9510	1.0049	1.2832	1.7086	0.4254	1.2777	1.7049	0.4272
			α	1.8739	4.0479	2.1739	2.8074	3.2174	0.4099	2.7727	3.2001	0.4273	2.7659	3.1963	0.4304
		3	θ	0.2169	0.8750	0.6581	0.2425	0.7268	0.4843	0.3527	0.6417	0.2890	0.3508	0.6377	0.2869
			λ	1.0836	2.0104	0.9268	1.0031	1.9064	0.9032	1.2794	1.7098	0.4304	1.2751	1.7044	0.4293
			α	-0.0451	1.2072	1.2524	0.0466	0.8758	0.8292	0.2982	0.7004	0.4022	0.2951	0.6946	0.3996
		0.5	θ	0.5721	2.6402	2.0681	0.9867	2.0760	1.0893	1.2644	1.7159	0.4516	1.2599	1.7095	0.4495
			λ	1.0143	1.9886	0.9742	0.9054	1.9290	1.0236	1.2903	1.6968	0.4065	1.2839	1.6940	0.4101
			α	0.0633	3.3473	3.2840	1.1863	1.8051	0.6189	1.2238	1.7549	0.5311	1.2157	1.7515	0.5358
1.5	1.5	1.5	θ	0.7470	2.3557	1.6086	0.9246	1.9990	1.0744	1.2344	1.7532	0.5188	1.2278	1.7482	0.5204
			λ	1.1770	1.8718	0.6948	1.1222	1.8135	0.6913	1.3046	1.6819	0.3772	1.3006	1.6772	0.3767
			α	0.9082	5.0440	4.1358	2.7353	3.2717	0.5364	2.7297	3.2409	0.5112	2.7223	3.2367	0.5145
		3	θ	0.5858	2.7560	2.1702	0.8252	2.1053	1.2801	1.2441	1.7105	0.4665	1.2366	1.7064	0.4698
			λ	1.1981	1.8563	0.6581	1.1494	1.7902	0.6408	1.3176	1.6489	0.3313	1.3123	1.6464	0.3341
			α	-0.0692	1.2622	1.3314	0.0536	0.9188	0.8651	0.2951	0.7027	0.4076	0.2927	0.6961	0.4034
		0.5	θ	2.0302	3.9673	1.9371	2.6254	3.3834	0.7580	2.7593	3.2140	0.4547	2.7524	3.2095	0.4571
			λ	1.1380	1.8949	0.7569	1.0534	1.8470	0.7936	1.3280	1.6709	0.3429	1.3247	1.6658	0.3410
			α	0.5647	2.6084	2.0437	0.7906	2.1106	1.3200	1.2407	1.7352	0.4946	1.2341	1.7304	0.4963
	3		θ	2.1961	3.9628	1.7668	2.3378	3.5981	1.2603	2.6968	3.2200	0.5232	2.6839	3.2194	0.5355
		1.5	λ	1.2333	1.8076	0.5743	1.1939	1.7498	0.5559	1.3316	1.6369	0.3053	1.3275	1.6334	0.3059
			θ	1.5271	4.5387	3.0116	2.3497	3.5835	1.2338	2.7410	3.2402	0.4993	2.7333	3.2356	0.5023
			λ	2.4584	3.6040	1.1456	2.3951	3.4913	1.0962	2.7670	3.2109	0.4439	2.7608	3.2069	0.4461

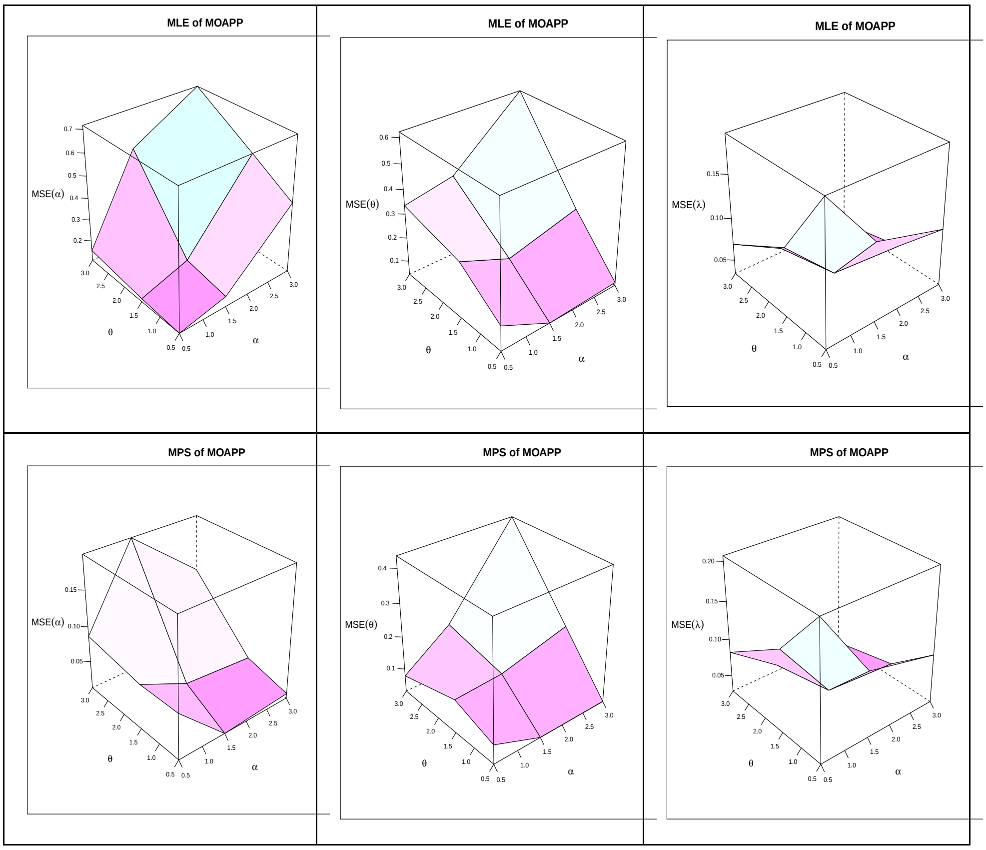

Figure 3. MSE for MLE, MPS and Bayes estimation under SE and LINEX Loss Functions for n= 120.

6.2. Data Analysis

In this section we take three different examples of real-life data set. The MLEs estimates of the parameters are reported in Tables (9), (10) and (11), then the MOAPP model is compared with other special case models like Pareto type 1, generalized Pareto (GP), and alpha power Pareto (APP). This comparison was conducted using Kolmogorov–Smirnov (KS) distance (D) between the fitted and the empirical distribution functions and the corresponding p-values. Also, Akaike information criterion (AIC) such that AIC=-2 L(γ)+2p, where p is the number of parameters in the model and L is the maximized value of the likelihood function for the model. Given a set of candidate models for the data, the preferred model is the one with the minimum AIC value. Bayesian information criterion (BIC) is also used for comparison between models, where BIC can be defined as: BIC=-2 L(γ)+p ln(n), where n is the sample size. As a model selection criterion, the researcher must choose the model with the minimum BIC value. The MLEs of $\alpha, \theta, and\ \lambda$ are computed numerically using the function optimal in R statistical package. The values of the KS statistic with p-values, AIC and BIC are reported in Tables (7), (8) and (9).

The first example is from Lawless (1982). The data set consists of failure times or censoring times for 36 appliances subjected to an automated life test. Failures are mainly classified into 18 different modes, although among 33 observed failures only 7 modes are present and only model 6 and 9 appear more than once. We are mainly interested in the failure mode 9. The data are given below:

Data 1: 1167, 1925, 1990, 2223, 2400, 2471, 2551, 2568, 2694, 3034, 3112, 3214, 3478, 3504, 4329, 176976, 7846.

Table 7. MLE estimation with KS, p-values and different model goodness of fit criterion for data 1

	$\hat{\alpha}_{MLE}$	$\hat{\theta}_{MLE}$	$\hat{\lambda}_{MLE}$	D	P-value	AIC	BIC
P	-	-	0.12231	0.57843	6.34E-06	385.4278	3.86E+02
GP	-	3341.032	0.609273	0.33089	0.03686	334.5826	336.2491
APP	78.74852	-	0.257025	0.48467	0.00034	367.7567	3.69E+02
MOAPP	9.00E+07	4.60E+07	2.57536	0.2088	0.3941	324.6243	327.124

The second example represents survival times of guinea pigs injected with different amount of tubercle bacilli studied by Bjerkedal [1960]. Guinea pigs are subject to high susceptibility of human tuberculosis, which is one of the causes for choosing this species.

Table 8. MLE estimation with KS, p-values and different model goodness of fit criterion for data 2

	$\hat{\alpha}_{MLE}$	$\hat{\theta}_{MLE}$	$\hat{\lambda}_{MLE}$	D	P-value	AIC	BIC
P	-	-	0.199805	0.51093	2.2E-16	1098.51	1.10E+03
GP	-	237.9788	0.393478	0.23142	0.000895	879.3132	883.8665
APP	152.982	-	0.446334	0.40147	1.67E-10	1009.978	1014.531
MOAPP	112100	322998.1	3.011837	0.06837	0.8894	857.2212	864.0512

The third example is from Almetwally et al. (2019). The data set consists of economic data of 31 observations subjected to a GDP growth of Egypt. The data are given below.

Table 9. MLE estimation with KS, p-values and different model goodness of fit criterion for data 3

	$\hat{\alpha}_{MLE}$	$\hat{\theta}_{MLE}$	$\hat{\lambda}_{MLE}$	D	P-value	AIC	BIC
P	-	-	0.6473	0.3956	0.0001	186.7626	188.1966
GP	-	6.8839	-0.6607	0.2910	0.0080	149.8744	152.7423
APP	90.2664	-	1.3679	0.2501	0.0340	160.3021	163.1700
MOAPP	8.5373	153.5946	3.7857	0.0726	0.9927	139.4962	143.7982

When comparing the values of KS statistics of MOAPP and other sub models like Pareto type 1, GP and APP for the two data examples above, we obtain the minimum KS for MOAPP with highest p-values. Also, it can be noticed that the values of AIC and BIC take their minimum values when the distribution is MOAPP. Therefore, this indicates that the MOAPP distribution fits the two sets of data very well and is better than other distributions. This also emphasizes the need of new distributions in managing real-life data. So, in general we can say that the new distribution is superior according to other sub models.

7. Conclusions

In this study we have considered MOAPP distribution which has three unknown parameters. This new distribution proved to be more flexible and more appropriate for monotone and right skewed lifetime data, also its hazard rate function can be either a decreasing or upside-down bathtub curve. We estimate the parameters of MOAPP using MLE, MPS and Bayesian method under SE and LINEX loss functions. It is not possible to compare different methods theoretically, so we have used some simulations to compare different estimators. We have compared different estimators mainly with respect to biases and mean squared errors. Confidence intervals are obtained and are compared numerically in terms of interval lengths. The best method for estimating α and θ is the Bayesian method under the LINEX and SE loss functions depending on the values of α and θ, it is also noticed that the MPS method acts better for estimating α and θ than the MLE method. The Bayesian method under the SE loss function is the best appropriate method for estimating λ. Confidence intervals under the MPS method and the Bayesian credible interval are preferable to confidence intervals under the MLE method. Therefore, we recommend the use of the MPS and Bayes estimation methods for practical purposes. The flexibility of this distribution was illustrated in some applications to real data sets, where the new model proves to better fit data than some other sub models.

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