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Abstract. In the present paper, we endow semi De Morgan logic and a family of its axiomatic extensions with proper multi-type display calculi which are sound, complete, conservative, and enjoy cut elimination and subformula property. Our proposal builds on an algebraic analysis of the variety of semi De Morgan algebras, and applies the guidelines of the multi-type methodology in the design of display calculi.

Keywords: Semi De Morgan algebras, Proper display calculus, Multi-type methodology.

Introduction

Semi De Morgan logic, introduced in an algebraic setting by Sankappanavar [31], is a very well known *paraconsistent* logic (indeed, it violates the classical principle $\bot \vdash A$ known as *ex falso quodlibet*, cf. [30]), and is designed to capture the salient features of intuitionistic negation in a paraconsistent setting. Semi De Morgan algebras form a variety of normal distributive lattice expansions (cf. [22, Definition 9]), and are a common abstraction of De Morgan algebras and distributive pseudocomplemented lattices. Besides being studied from a universal-algebraic perspective [4,5,31], semi De Morgan logic has been studied from a duality-theoretic perspective [25], from the perspective of canonical extensions [28], and in proof-theory, where [27] introduces a G3-style sequent calculus in which the negation has also a structural proxy and each connective is introduced via four logical rules, two of which are standard, and the other introduce the given connective under the scope of the structural negation, in the style of [1]. Because of these non-standard rules, this calculus violates the standard subformula property (see also [21] on this specific point).

From a proof-theoretic perspective, the main challenge posed by semi De Morgan logic is that, unlike De Morgan logic, its axiomatization is *not analytic inductive* in the sense of [22, Definition 55] (cf. discussion in Section

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3 and "Appendix A"). This implies that, on the basis of its original axiomatization, semi De Morgan logic is *not properly displayable* in the sense of Wansing [32], given that, as shown in [22], properly displayable logics are exactly those axiomatized by analytic inductive axioms (see also [7] for a purely proof-theoretic characterization of the scope of proper display calculi, and [26] as the seminal reference to this line of research).

To overcome this difficulty, we adopt a methodology, introduced in [9, 11], which has successfully treated an array of logics spanning from linear logic to inquisitive logic [2, 6, 10, 12, 17, 19, 21, 23, 24], each of which—for its own specific reasons—is not properly displayable in the sense of [22, 26]. This method aims at systematically exploiting properties of the *algebraic* semantics of the given logic for *proof-theoretic* purposes: in its most explicit formulation, the base of this method is to find an equivalent presentation of the algebraic semantics of the given logic in terms of some suitable class of *heterogeneous algebras*[1] (cf. Definition 2.2), defined in terms of algebraic (i.e. equational, inequational, quasi-inequational) conditions which turn out to be *analytic inductive*. This class of heterogeneous algebras then forms the algebraic semantics of a *multi-type logic*, i.e. a logic the language of which has more than one sort. Since the class of heterogeneous algebras under consideration is analytic inductive, its associated multi-type logic will be axiomatized by analytic inductive axioms and rules,[2] and hence it can be captured by a *multi-type* proper display calculus. In [20], an analytic *multi-type* calculus for semi De Morgan logic is introduced following the strategy outlined above. This calculus is sound, complete, conservative, and enjoys cut elimination and subformula property. It is worth noticing that *algebraic canonicity* (which uniformly holds for all analytic inductive inequalities in any signature, cf. [8]) guarantees not only the uniform proof of completeness, as is well known from the standard semantic theory, but is also key to uniformly establishing a fundamental *proof-theoretic* property of the resulting calculi, namely their being *conservative* w.r.t. the logics they are intended to capture, i.e. that their proof power exactly matches that of the original logic (cf. Section 5.3).

[1] Heterogeneous algebras [3] are algebras with more than one universe and in which the operations are allowed to span between these.

[2] Recall that the formulation of the core notions and results of the general theory of proper display calculi developed in [22] is parametric, and hence applies to any (single-type or multi-type) signature of normal (distributive) lattice expansions. In "Appendix A", we specialize the definition of analytic inductive inequalities to the (single-type) language of semi De Morgan logic and to its multi-type counterpart introduced in Section 3.

In the present paper, we give a slightly emended version of this calculus (see Footnote 4 for details), and expand on the proofs of its properties. Moreover, we begin to extend the benefits of proper display calculi to semi De Morgan logic; for instance, the possibility to capture *axiomatic extensions* of the basic logic in a *modular* way, by adding structural rules to the basic calculus while guaranteeing the uniform proof of soundness, completeness (relative to the appropriate semantic classes), conservativity, cut elimination and subformula property thanks to the general theory. Specifically, in the present paper we extend the results of [20] to the logics associated with each of the *five subvarieties* of semi De Morgan algebras introduced in [31]. In particular, two of these subvarieties are characterized by axioms which are *not analytic inductive*, but which can again be given equivalent presentations as analytic subclasses of the class of heterogeneous algebras corresponding to semi De Morgan algebras.

Summing up, in the present paper the multi-type methodology has made it possible to modularly introduce proper display calculi for a family of logics which are *not properly displayable*, via an *equivalent presentation* of the *non-analytic* algebraic semantics of these logics as *analytic* classes of heterogeneous algebras. Once established, this equivalent analytic presentation allows for a number of desirable properties to be verified uniformly and straightforwardly thanks to the general theory of (multi-type) properly displayable calculi. However, it is important to stress that achieving this equivalent presentation is *not automatic*, and we presently do not know of general methods by which it can be achieved. Natural questions are then whether we can characterize the non-analytic logics to which this strategy is applicable, or whether general methods can be introduced to establish these equivalent presentations. A possibly helpful observation in relation to the second question is that the equivalence between semi De Morgan algebras (and subvarieties thereof) and their heterogeneous counterparts is very similar to the *term-equivalence* result with which Palma [28] proved that the variety of semi De Morgan algebras is closed under canonical extensions (see Section 2.3 for a discussion of Palma's strategy). Hence we conjecture that the strategy adopted in the present paper can be understood, *mutatis mutandis*, as a term-equivalence result, and hence universal-algebraic results on term equivalence might be usefully exploited to develop uniform strategies to design (multi-type) analytic calculi for logics for which a (single-type) analytic inductive axiomatization is not known.

Structure of the paper. In Section 1, we collect preliminaries on the axioms and rules of semi De Morgan logic and its axiomatic extensions arising from the subvarieties of semi De Morgan algebras introduced in [31], and discuss why the basic axiomatization is not amenable to the standard treatment of display calculi. In Section 2, we define the classes of heterogeneous algebras corresponding to the varieties mentioned above and prove their back-and-forth correspondence. In Section 3, we introduce the multi-type logical language of which the heterogeneous algebras introduced in Section 2 are the canonical algebraic semantics, and define a syntactic translation from the language of semi De Morgan logic to this multi-type language. In Section 4, we introduce the display calculi for semi De Morgan logic and its extensions, and in Section 5, we discuss their soundness, completeness, conservativity, cut elimination and subformula property. We conclude by outlining possible future research in Section 6.

1. Preliminaries

1.1. Semi De Morgan Logic and its Axiomatic Extensions

Fix a denumerable set Atprop of propositional variables, let p denote an element in Atprop. The language $\mathcal{L}$ of semi De Morgan logic over Atprop is defined recursively as follows:

$$A ::= p \mid \top \mid \bot \mid \neg A \mid A \wedge A \mid A \vee A$$

Definition 1.1. Semi De Morgan logic, denoted SM, consists of the following axioms:

$$\bot \vdash A, \quad A \vdash \top, \quad \neg\top \vdash \bot, \quad \top \vdash \neg\bot, \quad A \vdash A, \quad A \wedge B \vdash A, \quad A \wedge B \vdash B,$$
$$A \vdash A \vee B, \quad B \vdash A \vee B, \quad \neg A \vdash \neg\neg\neg A, \quad \neg\neg\neg A \vdash \neg A, \quad \neg A \wedge \neg B \vdash \neg(A \vee B),$$
$$\neg\neg A \wedge \neg\neg B \vdash \neg\neg(A \wedge B), \quad A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$$

and the following rules:

$$\frac{A \vdash B \qquad B \vdash C}{A \vdash C} \qquad \frac{A \vdash B \qquad A \vdash C}{A \vdash B \wedge C} \qquad \frac{A \vdash B \qquad C \vdash B}{A \vee C \vdash B} \qquad \frac{A \vdash B}{\neg B \vdash \neg A}$$

The following table reports the name of each axiomatic extension of SM arising from the subvarieties introduced in [31], its acronym, and its characterizing axiom:

characterizing axioms	*logics*	*axiomatic extensions of* SM
$A \vdash \neg\neg A$	LQM	*lower quasi De Morgan*
$\neg\neg A \vdash A$	UQM	*upper quasi De Morgan*
$\neg A \wedge \neg\neg A \vdash \bot$	DP	*demi pseudo-complemented lattice*
$A \wedge \neg A \vdash \bot$	AP	*almost pseudo-complemented lattice*
$\top \vdash \neg A \vee \neg\neg A$	WS	*weak Stone*

In [22], properly displayable logics (i.e. those logics that can be captured by *proper display calculi* [32, Section 4.1]) have been characterized as exactly those logics which admit a presentation consisting of *analytic inductive* axioms (cf. [22, Definition 55] for a general parametric definition). It is not difficult to verify that the SM-axioms $\neg A \vdash \neg\neg\neg A$, $\neg\neg\neg A \vdash \neg A$ and $\neg\neg A \wedge \neg\neg B \vdash \neg\neg(A \wedge B)$, the UQM-axiom $\neg\neg A \vdash A$, and the DP-axiom $\neg A \wedge \neg\neg A \vdash \bot$ are not analytic inductive (see "Appendix A" for more details). To our knowledge, no equivalent axiomatizations are known for semi De Morgan logic and its extensions using only analytic inductive axioms. In the present paper and in [20], we propose to circumvent this difficulty by introducing an equivalent presentation of semi De Morgan logic and its extensions in a suitable *multi-type* semantic and syntactic environment, for which (multi-type) analytic inductive axiomatizations exist (as in e.g. [2, Definition A.2]). Via this presentation, (multi-type) proper display calculi can be introduced, with the same behaviour and properties (e.g. modularity, uniform and straightforward proofs of soundness, completeness, conservativity, cut elimination and subformula property from the general theory) of single-type proper display calculi.

1.2. The Variety of Semi De Morgan Algebras and its Subvarieties

We recall the definition of the variety of semi De Morgan algebras and the subvarieties of it introduced in [31, Definition 2.2, Definition 2.6]. In what follows, we denote each variety in blackboard bold, and introduce the name of each variety within round brackets, without further explanation. Sometimes we abuse terminology and use the name of the variety as the type of algebra (for instance in expressions such as 'for every $\mathbb{SMA}$ $\mathbb{A}$' used in place of 'for every element $\mathbb{A}$ of the variety $\mathbb{SMA}$').

Definition 1.2. An algebra $\mathbb{A} = (\mathbb{L}, ')$ is a *semi De Morgan algebra* ($\mathbb{SMA}$) if

(S1) $\mathbb{L} = (L, \wedge, \vee, \top, \bot)$ is a bounded distributive lattice;

(S2) $\bot' = \top$ and $\top' = \bot$;

(S3) $(a \vee b)' = a' \wedge b'$ for all $a, b \in L$;

(S4) $(a \wedge b)'' = a'' \wedge b''$ for all $a, b \in L$;

(S5) $a' = a'''$ for any $a \in L$.

In the following table, we report each subvariety of $\mathbb{SMA}$ introduced in [31], its characterizing inequality, and its acronym:

	characterizing (in)equalities	*acronym*	*subvarieties of* $\mathbb{SMA}$ *introduced in [31]*
(S6a)	$a \leq a''$	$\mathbb{LQMA}$	*lower quasi De Morgan*
(S6b)	$a'' \leq a$	$\mathbb{UQMA}$	*upper quasi De Morgan*
(S7)	$a' \wedge a'' = \bot$	$\mathbb{DPL}$	*demi pseudo-complemented lattice*
(S8)	$a \wedge a' = \bot$	$\mathbb{APL}$	*almost pseudo-complemented lattice*
(S9)	$\top = a' \vee a''$	$\mathbb{WSA}$	*weak Stone*

Lemma 1.1. ([31], discussion above Corollary 2.7). $\mathbb{APL} \subseteq \mathbb{DPL}$ *and* $\mathbb{WSA} \subseteq \mathbb{DPL}$*.*

Proof. Clearly, (S7) follows from (S8) by instantiating a as a'. If $\top = a' \vee a''$ then, (S2), (S3) and (S5) imply $\bot = \top' = (a' \vee a'')' = a'' \wedge a''' = a'' \wedge a'$. ■

Definition 1.3. An algebra $(\mathbb{D}, {}^*)$ is a *De Morgan algebra* ($\mathbb{DMA}$) if

(D1) $\mathbb{D} = (D, \cap, \cup, 1, 0)$ is a bounded distributive lattice;

(D2) $0^* = 1$ and $1^* = 0$;

(D3) $(a \cup b)^* = a^* \cap b^*$ for all $a, b \in D$;

(D4) $(a \cap b)^* = a^* \cup b^*$ for all $a, b \in D$;

(D5) $a = a^{**}$ for any $a \in D$.

As is well known, a Boolean algebra ($\mathbb{BA}$) is a $\mathbb{DMA}$ $\mathbb{D}$ satisfying one of the following equations:

(B1) $a \vee a^* = 1$;

(B2) $a \wedge a^* = 0$.

Notation 1. *We let* $\mathsf{S} \in \{\mathsf{SM}, \mathsf{LQM}, \mathsf{UQM}, \mathsf{DP}, \mathsf{AP}, \mathsf{WS}\}$ *and* $\mathbb{V} \in \{\mathbb{SMA}, \mathbb{LQMA}, \mathbb{UQMA}, \mathbb{DPL}, \mathbb{APL}, \mathbb{WSA}\}$ *be defined as indicated in the following table:*

if S *is*	*then* $\mathbb{V}$ *is*
SM	$\mathbb{SMA}$
LQM	$\mathbb{LQMA}$
UQM	$\mathbb{UQMA}$
DP	$\mathbb{DPL}$
AP	$\mathbb{APL}$
WS	$\mathbb{WSA}$

The following theorem can be shown using a routine Lindenbaum-Tarski construction.

THEOREM 1. (Completeness). *The logic* S *is complete with respect to the class of algebras* $\mathbb{V}$.

DEFINITION 1.4. A distributive lattice $\mathbb{A}$ is *perfect* (cf. [16, Definition 2.14]) if $\mathbb{A}$ is complete, completely distributive and completely join-generated by the set $J^{\infty}(\mathbb{A})$ of its completely join-irreducible elements (as well as completely meet-generated by the set $M^{\infty}(\mathbb{A})$ of its completely meet-irreducible elements).

A De Morgan algebra (resp. Boolean algebra) $\mathbb{A}$ as in Definition 1.3 is *perfect* if its lattice reduct is a perfect distributive lattice, and the following distributive laws are valid for every $X \subseteq \mathbb{A}$:

$$(\bigvee X)^* \; = \; \bigwedge X^* \qquad (\bigwedge X)^* \; = \; \bigvee X^*.$$

If $\mathbb{L}_1$ and $\mathbb{L}_2$ are complete lattices, a lattice homomorphism $h : \mathbb{L}_1 \to \mathbb{L}_2$ is *complete* if for each $X \subseteq \mathbb{L}_1$,

$$h(\bigvee X) \; = \; \bigvee h(X) \quad \text{and} \quad h(\bigwedge X) \; = \; \bigwedge h(X).$$

2. Towards a Multi-type Presentation

In the present section, we introduce the algebraic environment which justifies semantically the multi-type approach to the analytic proof theory of semi De Morgan logic and the axiomatic extensions of it listed in Section 1.1. In the next subsection, we define the kernel of an $\mathbb{SMA}$ (resp. $\mathbb{DPL}$) $\mathbb{A}$, and show that it can be endowed with a structure of $\mathbb{DMA}$ (resp. $\mathbb{BA}$). The kernel construction naturally involves the existence of two maps toggling between the kernel of $\mathbb{A}$ and the lattice reduct of $\mathbb{A}$. These are the main components of the heterogeneous algebras which we introduce in Subsection 2.2, and which we show to be equivalent presentations of the original algebras. Based on these facts, we also define the heterogeneous algebras corresponding to the subvarieties of $\mathbb{SMA}$s listed in Section 1.2. In Subsection 2.3, we show that these varieties of heterogeneous algebras are closed under taking canonical extensions, and use this fact to give an alternative proof of the canonicity of $\mathbb{SMA}$ and its five subvarieties.

2.1. The Kernel of a Semi De Morgan Algebra

For any $\mathbb{SMA}$ $\mathbb{A} = (\mathbb{L}, ')$, we let $K{:=}\{a'' \mid a \in L\}$. Then $(K, \leq)$ is a partial order with the order inherited from $\mathbb{L}$. Let $h : L \twoheadrightarrow K$ be defined by the assignment $a \mapsto a''$ for any $a \in L$, and let $e : K \hookrightarrow L$ denote the natural embedding. Hence, by construction, h is a monotone and surjective map, and e is an order embedding; moreover, $eh(a) = a''$, and from axiom (S5) it follows that $h(a) = h(a'')$ for every $a \in L$.

Lemma 2.1. *If $\mathbb{A}$ is an $\mathbb{SMA}$ and K, h, e are as above, then for any $\alpha \in K$,*

$$he(\alpha) \; = \; \alpha \tag{1}$$

Proof. Let $\alpha \in K$, and let $a \in L$ such that $h(a) = \alpha$. Hence,

$$\begin{array}{rcll} he(\alpha) & = & heh(a) & \alpha = h(a) \\ & = & h(a'') & eh(a) = a'' \\ & = & h(a) & h(a) = h(a'') \\ & = & \alpha. & h(a) = \alpha \end{array}$$

■

Definition 2.1. For any $\mathbb{SMA}$ $\mathbb{A} = (\mathbb{L}, ')$, the *kernel* of $\mathbb{A}$ is the algebra $\mathbb{K}_{\mathbb{A}}$:=$(K, \cap, \cup, ^*, 1, 0)$ defined as follows:

(K1) $K := \{a'' \mid a \in L\}$;

(K2) $\alpha \cup \beta := h((e(\alpha) \vee e(\beta))'')$ for all $\alpha, \beta \in K$;

(K3) $\alpha \cap \beta := h(e(\alpha) \wedge e(\beta))$ for all $\alpha, \beta \in K$;

(K4) $1 := h(\top)$;

(K5) $0 := h(\bot)$;

(K6) $\alpha^* := h(e(\alpha)')$ for any $\alpha \in K$.

Throughout the paper, to simplify notation, we abide by the convention that all negative connectives bind more strongly than any other connective, and we omit as many parentheses as we can without generating ambiguous readings. For example, we write $e(h(a)^*)$ in place of $e((h(a))^*)$, and $eh(a)'$ in place of $(eh(a))'$.

Proposition 2.1. *If $\mathbb{A}$ is an $\mathbb{SMA}$ (resp. a $\mathbb{DPL}$), and e, h are as above, then for all $a, b \in L$ and for all $\alpha, \beta \in K$,*

1. $h(a) \cap h(b) = h(a \wedge b) \quad h(a) \cup h(b) = h(a \vee b) \quad h(\top) = 1 \quad h(\bot) = 0.$
2. $e(\alpha) \wedge e(\beta) \; = \; e(\alpha \cap \beta) \qquad e(1) \; = \; \top \qquad e(0) \; = \; \bot.$
3. *$(K, \cap, \cup, 1, 0)$ is a distributive lattice, and $h : \mathbb{A} \twoheadrightarrow \mathbb{K}_{\mathbb{A}}$ is a surjective homomorphism of the lattice reducts of $\mathbb{K}$ and $\mathbb{A}$.*

Proof. 1. For any $a, b \in L$,

$$\begin{array}{rcll} h(a \wedge b) & = & heh(a \wedge b) & \text{Lemma 2.1} \\ & = & h((a \wedge b)'') & eh(a) = a'' \\ & = & h(a'' \wedge b'') & \text{(S4)} \\ & = & h(eh(a) \wedge eh(b)) & eh(a) = a'' \\ & = & h(a) \cap h(b) & \text{(K3)} \end{array}$$

$$\begin{array}{rcll} h(a \vee b) & = & heh(a \vee b) & \text{Lemma 2.1} \\ & = & h((a \vee b)'') & eh(a) = a'' \\ & = & h((a' \wedge b')') & \text{(S3)} \\ & = & h((a''' \wedge b''')') & \text{(S5)} \\ & = & h(a'' \vee b'')'') & \text{(S3)} \\ & = & h((eh(a) \vee eh(b))'') & eh(a) = a'' \\ & = & h(a) \cup h(b) & \text{(K2)} \end{array}$$

The remaining identities hold by (K4) and (K5).

2. For any $\alpha, \beta \in K$,

$$\begin{array}{rcll} e(\alpha \cap \beta) & = & eh(e(\alpha) \wedge e(\beta)) & \text{(K3)} \\ & = & (e(\alpha) \wedge e(\beta))'' & eh(a) = a'' \\ & = & e(\alpha)'' \wedge e(\beta)'' & \text{(S4)} \\ & = & ehe(\alpha) \wedge ehe(\beta) & eh(a) = a'' \\ & = & e(\alpha) \wedge e(\beta) & \text{Lemma 2.1} \end{array}$$

Finally, $e(0) = eh(\bot) = \bot'' = \bot$ and $e(1) = eh(\top) = \top'' = \top$ are straightforward consequences of (K4), (K5) and (S2).

3. Recall that $(K, \leq)$ is a partial order with the order inherited from $\mathbb{L}$. To show that $(K, \cap, \cup, 1, 0)$ is a lattice, we need to show that 1 (resp. 0) is the top (resp. bottom) of $(K, \leq)$, and for any $\alpha, \beta, \gamma \in K$,

   (a) $\alpha \leq \gamma$ and $\beta \leq \gamma$ iff $\alpha \cup \beta \leq \gamma$;
   (b) $\gamma \leq \alpha$ and $\gamma \leq \beta$ iff $\gamma \leq \alpha \cap \beta$.

   Let $\alpha \in K$. Since h is surjective by construction, $\alpha = h(a)$ for some $a \in L$. Hence $\bot \leq a \leq \top$, and since h is monotone by definition, $0 = h(\bot) \leq h(a) \leq h(\top) = 1$, as required. To show (a), assume that $\alpha \leq \gamma$ and $\beta \leq \gamma$. Since e is an order embedding, this is equivalent to $e(\alpha) \leq e(\gamma)$ and $e(\beta) \leq e(\gamma)$, which is equivalent to $e(\alpha) \vee e(\beta) \leq e(\gamma)$. Since h is monotone, this implies $h(e(\alpha) \vee e(\beta)) \leq he(\gamma)$, and so, by Lemma 2.1,

$$\alpha \cup \beta = h(e(\alpha) \vee e(\beta)) \leq he(\gamma) = \gamma.$$

Conversely, assume that $\alpha \cup \beta \leq \gamma$ and let $a, b, c \in L$ s.t. $\alpha = h(a)$, $\beta = h(b)$ and $\gamma = h(c)$. Then by item 2 above, the assumption can be rewritten as $h(a \vee b) = h(a) \cup h(b) \leq h(c)$, hence by the monotonicity of h we get $\alpha = h(a) \leq h(a \vee b) \leq h(c) = \gamma$ and likewise $\beta \leq \gamma$, which finishes the proof of (a). The proof of item (b) is similar and omitted. This finishes the proof that the partial order K inherited from $\mathbb{L}$ is in fact a bounded lattice order, the algebraic structure of which coincides with the corresponding fragment of the 'kernel'-structure. Finally, item 2 implies that h is a surjective lattice homomorphism, and because $\mathbb{L}$ is distributive by assumption, the lattice reduct of $\mathbb{K}$ is the homomorphic image of a distributive lattice, and hence is distributive.

■

PROPOSITION 2.2. *If $\mathbb{A}$ is an $\mathbb{SMA}$, then $\mathbb{K}_{\mathbb{A}}$ is a $\mathbb{DMA}$.*

PROOF. By item 3 of Proposition 2.1, the lattice reduct of $\mathbb{K}_{\mathbb{A}}$ is a bounded distributive lattice. To finish the proof, let us show that $\mathbb{K}_{\mathbb{A}}$ satisfies (D2)-(D5). As to (D2):

$$\begin{array}{rcll} 0^* & = & h(e(0)') & \text{(K6)} \\ & = & h((eh(\bot))') & \text{(K5)} \\ & = & h(\bot''') & eh(a) = a'' \\ & = & h(\bot') & \text{(S5)} \\ & = & h(\top) & \text{(S2)} \\ & = & 1 & \text{(K4)} \end{array} \qquad \begin{array}{rcll} 1^* & = & h(e(1)') & \text{(K6)} \\ & = & h((eh(\top))') & \text{(K4)} \\ & = & h(\top''') & eh(a) = a'' \\ & = & h(\top') & \text{(S5)} \\ & = & h(\bot) & \text{(S2)} \\ & = & 0 & \text{(K4)} \end{array}$$

As to (D3), for any $\alpha, \beta \in K$,

$$\begin{array}{rcll} (\alpha \cup \beta)^* & = & h(e(\alpha \cup \beta)') & \text{(K6)} \\ & = & h((eh((e(\alpha) \vee e(\beta))'')')) & \text{(K2)} \\ & = & h((e(\alpha) \vee e(\beta))''''') & eh(a) = a'' \\ & = & h((e(\alpha)' \wedge e(\beta)')'''') & \text{(S3)} \\ & = & h((e(\alpha)''' \wedge e(\beta)''')'') & \text{(S4)} \\ & = & h((eh(e(\alpha)') \wedge eh(e(\beta)'))'') & eh(a) = a'' \\ & = & h((e(\alpha^*) \wedge e(\beta^*))'') & \text{(K6)} \\ & = & heh(e(\alpha^*) \wedge e(\beta^*)) & eh(a) = a'' \\ & = & h(e(\alpha^*) \wedge e(\beta^*)) & \text{Lemma 2.1} \\ & = & \alpha^* \cap \beta^* & \text{(K3)} \end{array}$$

As to (D4): let $\alpha, \beta \in K$ and let (i) $\alpha = h(a)$ and (ii) $\beta = h(b)$;

$$
\begin{array}{rcll}
(\alpha \cap \beta)^* & = & h(e(\alpha \cap \beta)') & \text{(K6)} \\
& = & h((eh(e(\alpha) \wedge e(\beta)))') & \text{(K3)} \\
& = & h((e(\alpha) \wedge e(\beta))''') & eh(a) = a'' \\
& = & h((e(\alpha)'' \wedge e(\beta)'')') & \text{(S4)} \\
& = & h((e(\alpha)' \vee e(\beta)')'') & \text{(S3)} \\
& = & h((eh(a)' \vee eh(b)')'') & \text{(i) and (ii)} \\
& = & h((a''' \vee b''')'') & eh(a) = a'' \\
& = & h((a''''' \vee b''''')'') & \text{(S5)} \\
& = & h((eh(eh(a)') \vee eh(eh(b)'))'') & eh(a) = a'' \\
& = & h((eh(e(\alpha)') \vee eh(e(\beta)'))'') & \text{(i) and (ii)} \\
& = & h((e(\alpha^*) \vee e(\beta^*))'') & \text{(K6)} \\
& = & \alpha^* \cup \beta^* & \text{(K2)}
\end{array}
$$

As to (D5): let $\alpha \in K$ and let (i) $\alpha = h(a)$;

$$
\begin{array}{rcll}
\alpha^{**} & = & h((eh(e(\alpha)'))') & \text{(K6)} \\
& = & h(e(\alpha)'''') & eh(a) = a'' \\
& = & h((eh(a))'''') & \text{(i)} \\
& = & h(a'''''') & eh(a) = a'' \\
& = & h(a'') & \text{(S5)} \\
& = & heh(a) & eh(a) = a'' \\
& = & h(a) & \text{Lemma 2.1} \\
& = & \alpha & \text{(i)}
\end{array}
$$

■

Corollary 2.1. *If* $\mathbb{A}$ *is a* $\mathbb{DPL}$*, then* $\mathbb{K}_{\mathbb{A}}$ *is a* $\mathbb{BA}$*.*

Proof. By Proposition 2.2, $\mathbb{K}_{\mathbb{A}}$ is a $\mathbb{DMA}$. Hence, it suffices to show that $\mathbb{A}$ satisfies (B1). For any $\alpha \in \mathbb{K}_{\mathbb{A}}$,

$$
\begin{array}{rcll}
\alpha \cap \alpha^* & = & h(e(\alpha) \wedge e(\alpha^*)) & \text{(K3)} \\
& = & h(e(\alpha) \wedge eh(e(\alpha)')) & \text{(K6)} \\
& = & h(e(\alpha) \wedge e(\alpha)''') & eh(a) = a'' \\
& = & h(e(\alpha) \wedge e(\alpha)') & \text{(S5)} \\
& = & heh(e(\alpha) \wedge e(\alpha)') & \text{Lemma 2.1} \\
& = & h((e(\alpha) \wedge e(\alpha)')'') & eh(a) = a'' \\
& = & h(e(\alpha)'' \wedge e(\alpha)''') & \text{(S4)} \\
& = & h(\bot) & \text{(S7)} \\
& = & 0 & \text{(K5)}
\end{array}
$$

■

In what follows, we will drop the subscript of the kernel whenever it does not cause confusion.

2.2. Heterogeneous SMAs as Equivalent Presentations of SMAs

Definition 2.2. A *heterogeneous semi De Morgan algebra* ($\mathbb{HSMA}$) is a tuple $\mathbb{H} = (\mathbb{L}, \mathbb{D}, e, h)$ as in the diagram, satisfying the following conditions:

$$\begin{tikzcd} \mathbb{L} \arrow[rr, bend left, two heads, "h"] & & \mathbb{D} \arrow[ll, bend left, hook, "e"] \arrow[loop right, "*"] \end{tikzcd}$$

(H1) $\mathbb{L} = (L, \wedge, \vee, \bot, \top)$ is a bounded distributive lattice;

(H2a) $\mathbb{D} = (D, \cap, \cup, 0, 1, ^*)$ is a De Morgan algebra;

(H3) $e : \mathbb{D} \hookrightarrow \mathbb{L}$ is an order embedding, and for all $\alpha, \beta \in \mathbb{D}$,

$$e(\alpha) \wedge e(\beta) = e(\alpha \cap \beta) \qquad e(1) = \top \qquad e(0) = \bot.$$

(H4) $h : \mathbb{L} \twoheadrightarrow \mathbb{D}$ is a surjective lattice homomorphism;

(H5) $he(\alpha) = \alpha$ for every $\alpha \in \mathbb{D}$.[3]

Conditions (H3) and (H4) imply that the maps e and h respectively are a semantic normal box-operator and the semantic counterpart of a connective which is both a normal box- and a normal diamond-operator. This fact justifies the properties of the heterogeneous connectives $\Box$ and $\circ$ of the multi-type language introduced in Section 3, interpreted as e and h respectively. Hence, the definition of analytic inductive axioms (cf. Definition 6.2 in "Appendix A"), which is formulated in terms of the logical language of Section 3, naturally applies also to terms and inequalities in the language of heterogeneous algebras. This observation will be useful in Section 2.3, when discussing the closure of classes of heterogeneous algebras under canonical extensions.

We consider the subclasses of $\mathbb{HSMA}$ indicated in Table 1 below. Each of these subclasses will be shown to correspond to one of the subvarieties of $\mathbb{SMA}$ indicated in Section 1.2.

Similarly to the proof of Proposition 1.1, the following proposition is a straightforward consequence of the axioms:

Proposition 2.3. *$\mathbb{HAPL} \subseteq \mathbb{HDPL}$ and $\mathbb{HWSA} \subseteq \mathbb{HDPL}$.*

Notation 2. *Let $\mathbb{HV} \in \{\mathbb{HSMA}, \mathbb{HLQMA}, \mathbb{HUQMA}, \mathbb{HDPL}, \mathbb{HAPL}, \mathbb{HWSA}\}$.*

[3] Condition (H5) implies that h is surjective and e is injective.

Table 1. subclasses of $\mathbb{HSMA}$ and their (in)equational presentations

	characterizing (in)equalities	short name	will be shown to correspond to
(H6a)	$a \leq eh(a)$	$\mathbb{HLQMA}$	$\mathbb{LQMA}$
(H6b)	$eh(a) \leq a$	$\mathbb{HUQMA}$	$\mathbb{UQMA}$
(H2b)	$\mathbb{D}$ is a Boolean algebra	$\mathbb{HDPL}$	$\mathbb{DPL}$
(H7)	$e(h(a)^*) \wedge a = \bot$	$\mathbb{HAPL}$	$\mathbb{APL}$
(H8)	$e(\alpha^*) \vee e(\alpha) = \top$	$\mathbb{HWSA}$	$\mathbb{WSA}$

Definition 2.3. An $\mathbb{HV}$-algebra is *perfect* if:

(PH1) both $\mathbb{L}$ and $\mathbb{D}$ are perfect as a distributive lattice and De Morgan algebra (or Boolean algebra), respectively (see Definition 1.4);

(PH2) e is an order-embedding and is completely meet-preserving;

(PH3) h is a complete lattice homomorphism.

By well known order-theoretic facts, in perfect $\mathbb{HV}$s both $\mathbb{L}$ and $\mathbb{D}$ are endowed with an additional bi-Heyting algebra structure. We let $\rightarrow$ and $\succ\!\!-$ respectively denote the right residual of $\wedge$ and the left residual of $\vee$, and $-\!\!\supset$ and $\supset\!\!-$ respectively denote the right residual of $\cap$ and the left residual of $\cup$. Moreover, e has a left adjoint $e_\ell : \mathbb{L} \to \mathbb{D}$, and h has both a right and a left adjoint $h_\ell : \mathbb{D} \to \mathbb{L}$ and $h_r : \mathbb{D} \to \mathbb{L}$ as in the following diagram.[4]

$$
\begin{tikzcd}
\mathbb{L} \arrow[rr, bend left=40, "h"] \arrow[rr, bend right=60, "e_\ell"'] & & \mathbb{D} \arrow[ll, bend right=70, "h_r"'] \arrow[ll, bend left=10, "h_\ell"] \arrow[ll, bend left=40, "e"] \arrow[loop right, "*"]
\end{tikzcd}
$$

[4]The right and left adjoint of h are in general different maps, and do not need to coincide, as the notation in [20] seemed to suggest.

Definition 2.4. For any $\mathbb{V}$-algebra $\mathbb{A}$, let

$$\mathbb{A}^+ := (\mathbb{L}, \mathbb{K}, e, h),$$

where $\mathbb{L}$ is the lattice-reduct of $\mathbb{A}$ and $\mathbb{K}$ is the kernel of $\mathbb{A}$ (cf. Definition 2.1), and $e : \mathbb{K} \to \mathbb{L}$ and $h : \mathbb{L} \to \mathbb{K}$ are defined as in the beginning of Section 2.1.

Proposition 2.4. *If $\mathbb{A}$ is a $\mathbb{V}$-algebra, then $\mathbb{A}^+$ is an $\mathbb{HV}$-algebra.*

Proof. If $\mathbb{A}$ is an $\mathbb{SMA}$ (resp. $\mathbb{DPL}$), the statement is an immediate consequence of Propositions 2.2 and 2.1. If $\mathbb{A}$ is an $\mathbb{LQMA}$, it suffices to show that $\mathbb{A}_+$ satisfies (H6a), i.e. $a \leq e(h(a))$. This easily follows from (S6a) and H5. Similarly, (H6b) follows from (S6b) and H5 if $\mathbb{A}$ is a $\mathbb{UQMA}$. If $\mathbb{A}$ is an $\mathbb{APL}$, it suffices to show that $\mathbb{A}^+$ satisfies (H7).

$$\begin{array}{lll}
 & e(h(a)^*) \wedge a & \\
= & eh((eh(a))') \wedge a & \text{(K6)} \\
= & a^{'''''} \wedge a & eh(a) = a'' \\
= & a' \wedge a & \text{(S5)} \\
= & \bot & \text{(S8)}
\end{array}$$

If $\mathbb{A}$ is a $\mathbb{WSA}$, it suffices to show that $\mathbb{A}^+$ satisfies (H8).

$$\begin{array}{lll}
 & e(\alpha^*) \vee e(\alpha) & \\
= & eh(e(\alpha)') \vee e(\alpha) & \text{(K6)} \\
= & eh(e(\alpha)') \vee ehe(\alpha) & \text{Lemma 2.1} \\
= & e(\alpha)''' \vee e(\alpha)'' & eh(a) = a'' \\
= & e(\alpha)' \vee e(\alpha)'' & \text{(S5)} \\
= & \top & \text{(S9)}
\end{array}$$

■

Definition 2.5. For any heterogeneous $\mathbb{HV}$-algebra $\mathbb{H} = (\mathbb{L}, \mathbb{D}, e, h)$, let

$$\mathbb{H}_+ := (\mathbb{L}, '),$$

where the map $' : \mathbb{L} \to \mathbb{L}$ is defined by the assignment $a' \mapsto e(h(a)^*)$.

Proposition 2.5. *If $\mathbb{H}$ is an $\mathbb{HSMA}$ ($\mathbb{HDPL}$), then $\mathbb{H}_+$ is an $\mathbb{SMA}$ ($\mathbb{DPL}$).*

Proof. By (H1), $\mathbb{L}$ is a bounded distributive lattice, hence it suffices to show that the operation $'$ satisfies (S2)–(S5) (cf. Definition 1.2).
As to (S2):

$$\begin{array}{lllllll}
 & \bot' & & & & \top' & \\
= & e(h(\bot)^*) & \text{definition of } ' & & = & e(h(\top)^*) & \text{definition of } ' \\
= & e(0^*) & \text{(H4)} & & = & e(1^*) & \text{(H4)} \\
= & e(1) & \text{(H2a)} & & = & e(0) & \text{(H2a)} \\
= & \top & \text{(H3)} & & = & \bot & \text{(H3)}
\end{array}$$

As to (S3):

$$\begin{array}{rcll}(a \vee b)' & = & e(h(a \vee b)^*) & \text{definition of }' \\ & = & e((h(a) \cup h(b))^*) & \text{(H4)} \\ & = & e(h(a)^* \cap h(b)^*) & \text{(H2a)} \\ & = & e(h(a)^*) \wedge e(h(b)^*) & \text{(H3)} \\ & = & a' \wedge b' & \text{definition of }'\end{array}$$

As to (S4):

$$\begin{array}{rcll}(a \wedge b)'' & = & e((he(h(a \wedge b)^*))^*) & \text{definition of }' \\ & = & e(h(a \wedge b)^{**}) & \text{(H5)} \\ & = & eh(a \wedge b) & \text{(H2a)} \\ & = & e(h(a) \cap h(b)) & \text{(H4)} \\ & = & eh(a) \wedge eh(b) & \text{(H3)} \\ & = & e(h(a)^{**}) \wedge e(h(b)^{**}) & \text{(H2a)} \\ & = & e((he(h(a)^*))^*) \wedge e((he(h(b)^*))^*) & \text{(H5)} \\ & = & a'' \wedge b'' & \text{definition of }'\end{array}$$

As to (S5):

$$\begin{array}{rcll}a''' & = & e((he((he(h(a)^*))^*))^*) & \text{definition of }' \\ & = & e(h(a)^{***}) & \text{(H5)} \\ & = & e(h(a)^*) & \text{(H2a)} \\ & = & a' & \text{definition of }'\end{array}$$

This completes the proof that $\mathbb{H}_+$ is an $\mathbb{SDA}$. If $\mathbb{H}$ is an $\mathbb{HDPL}$, we also need to show that $\mathbb{H}_+$ satisfies (S7):

$$\begin{array}{rcll}a' \wedge a'' & = & e(h(a)^*) \wedge e((he(h(a)^*))^*) & \text{definition of }' \\ & = & e(h(a)^*) \wedge e(h(a)^{**}) & \text{(H5)} \\ & = & e(h(a)^*) \wedge eh(a) & \text{(H2a)} \\ & = & e(h(a)^* \cap h(a)) & \text{(H3)} \\ & = & e(0) & \text{(H2a)} \\ & = & \bot & \text{(H3)}\end{array}$$

which completes the proof that $\mathbb{H}_+$ is a $\mathbb{DPL}$. ■

COROLLARY 2.2. *If* $\mathbb{H}$ *is an* $\mathbb{HV}$*-algebra, then* $\mathbb{H}_+$ *is a* $\mathbb{V}$*-algebra.*

PROOF. Proposition 2.5 accounts for the case in which $\mathbb{H}$ is an $\mathbb{HSMA}$. If $\mathbb{H}$ is an $\mathbb{HLQMA}$, it suffices to show that $\mathbb{H}_+$ additionally satisfies (S6a).

$$\begin{array}{rll} & a \leq eh(a) & \text{(H6a)} \\ \text{iff} & a \leq e(h(a)^{**}) & \text{(H2a)} \\ \text{iff} & a \leq e((he(h(a)^*))^*) & \text{(H5)} \\ \text{iff} & a \leq a'' & \text{definition of }'\end{array}$$

If $\mathbb{H}$ is an $\mathbb{HUQMA}$, the argument is dual. If $\mathbb{H}$ is an $\mathbb{HAPL}$, using (H7) it is easy to see that $\mathbb{H}_+$ satisfies (S8). If $\mathbb{H}$ is an $\mathbb{HWSA}$, it suffices to show that $\mathbb{H}_+$ satisfies (S6).

$$\begin{array}{cll} & a' \vee a'' & \\ = & e(h(a)^*) \vee e((he(h(a)^*))^*) & \text{def. of } ' \\ = & e(h(a)^*) \vee e(h(a)^{**}) & \text{Lemma 2.1} \\ = & e(h(a)^*) \vee eh(a) & \text{(H2a)} \\ = & \top & \text{(H8)} \end{array}$$

■

Proposition 2.6. *If* $\mathbb{H} = (\mathbb{L}, \mathbb{D}, e, h)$ *is an* $\mathbb{HSMA}$*, and the map* $' : \mathbb{L} \to \mathbb{L}$ *is defined by the assignment* $a \mapsto e(h(a)^*)$*, then:*

($K2_{\mathbb{D}}$) $\alpha \cup \beta = h((e(\alpha) \vee e(\beta))'')$ *for all* $\alpha, \beta \in \mathbb{D}$*;*

($K3_{\mathbb{D}}$) $\alpha \cap \beta = h(e(\alpha) \wedge e(\beta))$ *for all* $\alpha, \beta \in \mathbb{D}$*;*

($K4_{\mathbb{D}}$) $1 = h(\top)$*;*

($K5_{\mathbb{D}}$) $0 = h(\bot)$*;*

($K6_{\mathbb{D}}$) $\alpha^* = h(e(\alpha)')$*,*

and hence $\mathbb{K}_{\mathbb{H}_+} \cong \mathbb{D}$*.*

Proof. As to $K2_{\mathbb{D}}$,

$$\begin{array}{rcll} h((e(\alpha) \vee e(\beta))'') & = & he((he(h(e(\alpha) \vee e(\beta))^*))^*) & \text{definition of } ' \\ & = & (h(e(\alpha) \vee e(\beta)))^{**} & \text{(H5)} \\ & = & h(e(\alpha) \vee e(\beta)) & \text{(H2a)} \\ & = & he(\alpha) \cup he(\beta) & \text{(H4)} \\ & = & \alpha \cup \beta & \text{(H5)} \end{array}$$

Conditions $K3_{\mathbb{D}}, K4_{\mathbb{D}}$ and $K5_{\mathbb{D}}$ easily follow from H4, H5 and H3, and their proofs are omitted. As to $K6_{\mathbb{D}}$,

$$\begin{array}{rcll} h(e(\alpha)') & = & he((he(\alpha))^*) & \text{definition of } ' \\ & = & \alpha^* & \text{(H5)} \end{array}$$

Let $\mathbb{K} = (K, \cap_k, \cup_k, {}^{*_k}, 0_k, 1_k)$ denote the kernel of $\mathbb{H}_+$ (cf. Definition 2.1). To show that $\mathbb{D}$ and $\mathbb{K}$ are isomorphic, notice that, by definition, the domain of $\mathbb{K}$, seen as a subset of L, is $K{:=}\mathsf{Range}('') = \mathsf{Range}(e(-)^*he(-)^*h) = \mathsf{Range}(e(-)^*(-)^*h) = \mathsf{Range}(eh)$, the penultimate identity being due to (H5), and the last identity being due to the fact that $(-)^*$ is an involution. Since by assumption h is surjective, $K = \mathsf{Range}(eh) = \mathsf{Range}(e)$, and since e is an order embedding, K, regarded as a sub-poset of $\mathbb{L}$, is order-isomorphic to the domain of $\mathbb{D}$ with its lattice order. Let $f : \mathbb{D} \to \mathbb{K}$ denote the order-isomorphism between $\mathbb{D}$ and $\mathbb{K}$. Hence $f(1) = 1_k$ and $f(0) = 0_k$. Define $e_k : \mathbb{K} \hookrightarrow \mathbb{L}$ and $h_k : \mathbb{L} \twoheadrightarrow \mathbb{K}$ as in the beginning of Section 2.1. Thus, $e = e_k \circ f$ and $h_k = f \circ h$. The proof is complete with the verifications in the three items below. For all $\alpha, \beta \in \mathbb{D}$,

1. $f(\alpha) \cap_k f(\beta) = f(\alpha \cap \beta)$,

$$\begin{array}{rcll} f(\alpha) \cap_k f(\beta) & = & h_k(e_k f(\alpha) \wedge e_k f(\beta)) & \text{definition of } \cap_k \\ & = & fh(e_k f(\alpha) \wedge e_k f(\beta)) & h_k = f \circ h \\ & = & fh(e(\alpha) \wedge e(\beta)) & e = e_k \circ f \\ & = & f(\alpha \cap \beta) & (\mathrm{K3}_{\mathbb{D}}) \end{array}$$

2. $f(\alpha) \cup_k f(\beta) = f(\alpha \cap \beta)$,

$$\begin{array}{rcll} f(\alpha) \cup_k f(\beta) & = & h_k((e_k f(\alpha) \vee e_k f(\beta))'') & \text{definition of } \cup_k \\ & = & fh((e_k f(\alpha) \vee e_k f(\beta))'') & h_k = f \circ h \\ & = & fh((e(\alpha) \vee e(\beta))'') & e = e_k \circ f \\ & = & f(\alpha \cup \beta) & (\mathrm{K2}_{\mathbb{D}}) \end{array}$$

3. $f(\alpha)^{*_k} = f(\alpha^*)$,

$$\begin{array}{rcll} (f(\alpha))^{*_k} & = & h_k((e_k f(\alpha))') & \text{definition of } ^{*_k} \\ & = & fh((e_k f(\alpha))') & h_k = f \circ h \\ & = & fh(e(\alpha)') & e = e_k \circ f \\ & = & f(\alpha^*) & (\mathrm{K6}_{\mathbb{D}}) \end{array}$$

■

The next proposition builds on Propositions 2.4 and 2.5, Corollary 2.2, and Proposition 2.6.

PROPOSITION 2.7. *For any $\mathbb{V}$-algebra $\mathbb{A}$ and any $\mathbb{HV}$-algebra $\mathbb{H}$:*

$$\mathbb{A} \cong (\mathbb{A}^+)_+ \quad \textit{and} \quad \mathbb{H} \cong (\mathbb{H}_+)^+.$$

These correspondences restrict appropriately to the relevant classes of perfect algebras and perfect heterogeneous algebras.

PROOF. Let $\mathbb{A} = (\mathbb{L}, ')$ and $(\mathbb{A}^+)_+ = (\mathbb{L}, ^\star)$. It suffices to show that $a' = a^\star$ for every $a \in \mathbb{L}$. By definition, $a^\star = e(h(a)^*)$, where $h : \mathbb{L} \to \mathbb{K}_\mathbb{A}$ is defined by the assignment $a \mapsto a''$, and $e : \mathbb{K}_\mathbb{A} \to \mathbb{L}$ is the natural embedding, and $(-)^* : \mathbb{K}_\mathbb{A} \to \mathbb{K}_\mathbb{A}$ is defined by the assignment $\alpha \mapsto h(e(\alpha)')$. Hence:

$$\begin{array}{rcll} a^\star & = & e(h(a)^*) & \text{definition of } (-)^\star \\ & = & eh(eh(a)') & \text{definition of } (-)^* \\ & = & a''''' & eh(a) = a'' \\ & = & a' & \text{(S5)} \end{array}$$

For the second part, let $\mathbb{H} = (\mathbb{L}, \mathbb{D}, e, h)$ and $(\mathbb{H}_+)^+ = (\mathbb{L}, \mathbb{K}_{\mathbb{H}_+}, e^\star, h^\star)$. By Proposition 2.6, $\mathbb{D}$ and $\mathbb{K}_{\mathbb{H}_+}$ can be identified, and hence $e^\star(\alpha) = e(\alpha)$ for every $\alpha \in \mathbb{D} \cong \mathbb{K}_{\mathbb{H}_+}$. To finish the proof, it remains to be shown that $h^\star(a) = h(a)$ for every $a \in \mathbb{L}$.

$$\begin{array}{rcll} h^\star(a) & = & h^\star(a'') & \text{basic properties of } h \\ & = & h^\star(e(he(h(a)^*)^*)) & \text{definition of } (-)' \text{ in } \mathbb{H}_+ \\ & = & h^\star(e(h(a)^{**})) & \text{Lemma 2.1} \\ & = & h^\star(e(h(a))) & \text{(H2a)} \\ & = & h^\star(e^\star(h(a))) & e^\star(\alpha) = e(\alpha) \\ & = & h(a) & \text{Lemma 2.1} \end{array}$$

■

2.3. Canonical Extensions of Heterogeneous Algebras

As remarked in [24], not only does canonicity provide the main route to achieving uniform *completeness* results for a large class of axiomatic extensions of given logics (semi De Morgan logic in the present case), but is also key for proof-theoretic purposes, as it allows for a general proof strategy to establish that their associated display calculi are *conservative* (cf. Section 5.3). In the present section, we define the canonical extension $\mathbb{H}^\delta$ of any $\mathbb{HSMA}$ $\mathbb{H}$ by instantiating the general definition discussed in [24], and straightforwardly derive the *canonicity* of each class $\mathbb{HV}$ (i.e. its being closed under the canonical extension construction) from the fact that its axiomatization is (analytic) inductive (cf. discussion in "Appendix A"). We will also show that canonical extensions commute with the toggle, also discussed in Section 2.2, between the original varieties and the corresponding classes of heterogeneous algebras, in the sense that, if $\mathbb{A}$ (resp. $\mathbb{H}$) is an $\mathbb{SMA}$ (resp. $\mathbb{HSMA}$), then $\mathbb{A}^\delta \cong (\mathbb{A}^{+\delta})_+$ (resp. $\mathbb{H}^\delta \cong (\mathbb{H}_+{}^\delta)^+$). This makes it possible to give a new proof of the canonicity of $\mathbb{SMA}$ and its subvarieties.

Definition 2.6. If $\mathbb{H} = (\mathbb{L}, \mathbb{D}, e, h)$ is an $\mathbb{HSMA}$, the *canonical extension* of $\mathbb{H}$ is $\mathbb{H}^\delta := (\mathbb{L}^\delta, \mathbb{D}^\delta, e^\pi, h^\delta)$, where $\mathbb{L}^\delta$ and $\mathbb{D}^\delta$ denote the canonical extensions of the distributive lattice $\mathbb{L}$ and of the De Morgan algebra $\mathbb{D}$ respectively, and e^π and h^δ denote the canonical extensions of e and h, respectively.[5]

Proposition 2.8. *If $\mathbb{H}$ is an $\mathbb{HV}$-algebra, then $\mathbb{H}^\delta$ is a perfect $\mathbb{HV}$-algebra.*

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\mathbb{L}^\delta \arrow[rr, two heads, bend left=40, "h^\delta"] \arrow[rr, bend right=60, hook, "e_\ell"'] & & \mathbb{D}^\delta \arrow[ll, hook', bend right=60, "h_r"'] \arrow[ll, bend right=15, "h_\ell"] \arrow[ll, hook, bend left=30, "e^\pi"'] \arrow[loop right, "*^\delta"] \\
\mathbb{L} \arrow[u, hook] \arrow[rr, two heads, bend left=20, "h"] & & \mathbb{D} \arrow[u, hook] \arrow[ll, hook, bend left=20, "e"] \arrow[loop right, "*"]
\end{tikzcd}
$$

Proof. Firstly, $\mathbb{L}^\delta$ and $\mathbb{D}^\delta$ are a perfect distributive lattice and a perfect De Morgan algebra (resp. perfect Boolean algebra) respectively. Secondly, since h is a surjective homomorphism, $h^\delta = h^\pi = h^\sigma$ is a surjective complete lattice homomorphism [15, Theorem 3.7]. Thirdly, since e is finitely meet-preserving, e^π is completely meet-preserving, and it immediately follows from the definition of π-extension that e^π is an order-embedding [15, Corollary 2.25]. The identity $e^\pi(0) = 0$ clearly holds, since $\mathbb{A}$ is a subalgebra of $\mathbb{A}^\delta$. Moreover, $Id_{\mathbb{D}} = h \circ e$ is canonical by [14, Proposition 14], which completes the proof that if $\mathbb{H}$ is an $\mathbb{HSMA}$ (resp. $\mathbb{HDPL}$), then $\mathbb{H}^\delta$ is a perfect $\mathbb{HSMA}$ (resp. perfect $\mathbb{HDPL}$). The proof is completed by observing that (H6a), (H6b), (H7) and (H8) are analytic inductive (cf. Definition 6.2 of "Appendix A", see also discussion after Definition 2.2) and hence canonical (cf. [8, Theorem 7.1]), i.e. their validity is preserved under canonical extensions. ■

[5] See [13] for the relevant definitions. The order-theoretic properties of h and e guarantee that $e^\pi = e^\sigma$ and $h^\pi = h^\sigma$. We let $h^\delta := h^\pi = h^\sigma$. However, we use e^π instead of e^δ to emphasise that we are only using the properties of the π-extension of e.

In [28], Palma studied the canonical extensions of semi De Morgan algebras using insights from the Sahlqvist theory of Distributive Modal Logic. She recognized that not all inequalities in the axiomatization of $\mathbb{SMA}$ are Sahlqvist, and hence the canonicity of $\mathbb{SMA}$ (i.e. its being closed under canonical extensions) cannot be proven via a direct application of Sahlqvist theory. Palma circumvented this problem by showing that the variety $\mathbb{SDMA}$ of the $\{\Diamond, \lhd\}$-free reducts of Distributive Modal Algebras satisfying the following additional axioms:

$$\begin{array}{ccc} \rhd\top \leq \bot & \Box a \leq \rhd\rhd a & \rhd\rhd a \leq \Box a \\ \Box \rhd a \leq \rhd a & \rhd a \leq \Box \rhd a & \end{array}$$

is a term-equivalent presentation of $\mathbb{SMA}$, via the following toggle. If $\mathbb{A} = (\mathbb{L}, ')$ is an $\mathbb{SMA}$, let $\mathbb{A}^\bullet := (\mathbb{L}, \rhd, \Box)$ be such that $\Box$ and $\rhd$ are unary operations respectively defined by the assignments $a \mapsto a''$ and $a \mapsto a'$. For every $\mathbb{S}$ in $\mathbb{SDMA}$, let $\mathbb{S}_\bullet$ be the $\{\Box\}$-free reduct of $\mathbb{S}$. The canonicity of $\mathbb{SDMA}$ immediately follows from the inequalities axiomatizing $\mathbb{SDMA}$ being Sahlqvist[6] and hence canonical. This enables Palma to prove the canonicity of $\mathbb{SMA}$ as a consequence of the Sahlqvist canonicity of $\mathbb{SDMA}$, by showing that $\mathbb{A}^\delta = (\mathbb{A}^{\bullet\delta})_\bullet$ for every $\mathbb{SMA}$ $\mathbb{A}$.
We finish this section by giving a proof of the canonicity of $\mathbb{SMA}$ alternative to Palma's, obtained as a consequence of Proposition 2.8 and the following lemma, and uniformly extending it to every variety $\mathbb{V}$ (cf. Notation 1).

LEMMA 2.2. *If $\mathbb{A} = (\mathbb{L}, ')$ is an $\mathbb{SMA}$, then $\mathbb{A}^\delta = (\mathbb{A}^{+\delta})_+$.*

PROOF. Let $\mathbb{A}^\delta = (\mathbb{L}^\delta, '^\pi)$ and $(\mathbb{A}^{+\delta})_+ = (\mathbb{L}^\delta, ^\star)$, where $u^\star = e^\pi(h^\delta(u)^{*^\delta})$ by definition. Then,

$$\begin{array}{rcll} u^\star & = & e^\pi(h^\delta(u)^{*^\delta}) & \text{definition of } \star \\ & = & e^\pi(-)^{*^\delta} h^\delta(u) & \text{notation} \\ & = & e^\pi(-)^{*^\pi} h^\pi(u) & (-)^\delta = (-)^\pi \\ & = & (e(-)^* h)^\pi(u) & \text{[15, Lemma 3.3, Corollary 2.25]} \\ & = & u'^\pi & \text{proof of Proposition 2.7} \end{array}$$

■

COROLLARY 2.3. *If $\mathbb{A}$ is a $\mathbb{V}$-algebra, then so is $\mathbb{A}^\delta$.*

[6] Notice, however, that the $\mathbb{SDMA}$-axiomatization is *not* analytic (cf. [22, Definition 55]): the branch in the left-hand side term of $\rhd\rhd a \leq \Box a$ is not good.

Proof. If $\mathbb{A}$ is a $\mathbb{V}$-algebra, then $\mathbb{A}^+$ is an $\mathbb{HV}$-algebra (cf. Proposition 2.4). Hence, by Proposition 2.8, $\mathbb{A}^{+\delta}$ is an $\mathbb{HV}$-algebra, therefore $(\mathbb{A}^{+\delta})_+$ is a $\mathbb{V}$-algebra (cf. Corollary 2.2). Then the statement immediately follows from Lemma 2.2. ■

Lemma 2.3. *If* $\mathbb{H} = (\mathbb{L}, \mathbb{D}, e, h)$ *is an* $\mathbb{HSMA}$*, then* $\mathbb{H}^\delta \cong (\mathbb{H}_+{}^\delta)^+$.

Proof. Let $\mathbb{H}^\delta = (\mathbb{L}^\delta, \mathbb{D}^\delta, e^\pi, h^\delta)$. By Proposition 2.7, $\mathbb{H}^\delta \cong ((\mathbb{H}^\delta)_+)^+$ so it is enough to show that $\mathbb{H}^\delta{}_+ \cong \mathbb{H}_+{}^\delta$. By definition, $\mathbb{H}^\delta{}_+ = (\mathbb{L}^\delta, {}^\star)$, where $u^\star = e^\pi(h^\delta(u)^{*\delta})$, and $\mathbb{H}_+{}^\delta = (\mathbb{L}^\delta, {}')$, where $u' = (e(-)^* h)^\pi(u)$. From here on, the proof proceeds exactly as in the proof of Lemma 2.2. ■

3. Multi-type Presentation of Semi De Morgan Logic and its Extensions

In Section 2.2 we showed that $\mathbb{HSMA}$s are equivalent presentations of $\mathbb{SMA}$s. This fact provides a semantic motivation for introducing the multi-type language $\mathcal{L}_{\mathrm{MT}}$, which is naturally interpreted on $\mathbb{HSMA}$s. The language $\mathcal{L}_{\mathrm{MT}}$ consists of terms of types L and D, defined as follows:

$$\begin{array}{lcl} \mathsf{L} \ni A & ::= & p \mid \Box\alpha \mid \top \mid \bot \mid A \wedge A \mid A \vee A \\ \mathsf{D} \ni \alpha & ::= & \circ A \mid 1 \mid 0 \mid {\sim}\alpha \mid \alpha \cup \alpha \mid \alpha \cap \alpha \end{array}$$

The interpretation of $\mathcal{L}_{\mathrm{MT}}$-terms into $\mathbb{HSMA}$s is defined as the easy generalization of the interpretation of propositional languages in universal algebra; namely, the heterogeneous operation e interprets the connective $\Box$, the heterogeneous operation h interprets the connective $\circ$, and L-terms (resp. D-terms) are interpreted in the first (resp. second) domain of $\mathbb{HSMA}$s.

The toggle between single-type algebras and their heterogeneous counterparts (cf. Sections 2.2) is reflected syntactically by the translations $(\cdot)^\tau : \mathcal{L} \to \mathcal{L}_{\mathrm{MT}}$ defined as follows:

$$\begin{array}{rcl} p^\tau & ::= & p \\ \top^\tau & ::= & \top \\ \bot^\tau & ::= & \bot \\ (A \wedge B)^\tau & ::= & A^\tau \wedge B^\tau \\ (A \vee B)^\tau & ::= & A^\tau \vee B^\tau \\ (\neg A)^\tau & ::= & \Box \sim \circ A^\tau \end{array}$$

Recall that $\mathbb{A}^+$ denotes the $\mathbb{HSMA}$-algebra associated with the $\mathbb{SMA}$ $\mathbb{A}$ (cf. Definition 2.4). The following proposition is proved by a routine induction on $\mathcal{L}$-formulas.

Proposition 3.1. *For all $\mathcal{L}$-formulas A and B and any $\mathbb{SMA}$ $\mathbb{A}$,*

$$\mathbb{A} \models A \vdash B \quad \textit{iff} \quad \mathbb{A}^{+} \models A^{\tau} \vdash B^{\tau}.$$

We are now in a position to translate the axioms mentioned in Section 1.1 into $\mathcal{L}_{\mathrm{MT}}$. Below, S2$^\tau$-S5$^\tau$ are the translations of the SM-axioms involving negation, and S6$^\tau$-S9$^\tau$ are the translations of the axioms characterising the axiomatic extensions of SM discussed in Section 1.1.

$$\neg\top \vdash \bot \quad\rightsquigarrow\quad \Box{\sim}\circ\top \vdash \bot \tag{S2a$^\tau$}$$

$$\top \vdash \neg\bot \quad\rightsquigarrow\quad \top \vdash \Box{\sim}\circ\bot \tag{S2b$^\tau$}$$

$$\neg A \wedge \neg B \vdash \neg(A \vee B) \quad\rightsquigarrow\quad \Box{\sim}\circ A^\tau \wedge \Box{\sim}\circ B^\tau \vdash \Box{\sim}\circ(A \vee B)^\tau \tag{S3$^\tau$}$$

$$\neg\neg A \wedge \neg\neg B \vdash \neg\neg(A \wedge B) \quad\rightsquigarrow\quad \Box{\sim}\circ\Box{\sim}\circ A^\tau \wedge \Box{\sim}\circ\Box{\sim}\circ B^\tau \vdash \Box{\sim}\circ\Box{\sim}\circ(A \wedge B)^\tau \tag{S4$^\tau$}$$

$$\neg A \vdash \neg\neg\neg A \quad\rightsquigarrow\quad \Box{\sim}\circ A^\tau \vdash \Box{\sim}\circ\Box{\sim}\circ\Box{\sim}\circ A^\tau \tag{S5a$^\tau$}$$

$$\neg\neg\neg A \vdash \neg A \quad\rightsquigarrow\quad \Box{\sim}\circ\Box{\sim}\circ\Box{\sim}\circ A^\tau \vdash \Box{\sim}\circ A^\tau \tag{S5b$^\tau$}$$

$$A \vdash \neg\neg A \quad\rightsquigarrow\quad A^\tau \vdash \Box{\sim}\circ\Box{\sim}\circ A^\tau \tag{S6a$^\tau$}$$

$$\neg\neg A \vdash A \quad\rightsquigarrow\quad \Box{\sim}\circ\Box{\sim}\circ A^\tau \vdash A^\tau \tag{S6b$^\tau$}$$

$$\neg A \wedge \neg\neg A \vdash \bot \quad\rightsquigarrow\quad \Box{\sim}\circ A^\tau \wedge \Box{\sim}\circ\Box{\sim}\circ A^\tau \vdash \bot \tag{S7$^\tau$}$$

$$A \wedge \neg A \vdash \bot \quad\rightsquigarrow\quad A^\tau \wedge \Box{\sim}\circ A^\tau \vdash \bot \tag{S8$^\tau$}$$

$$\top \vdash \neg A \vee \neg\neg A \quad\rightsquigarrow\quad \top \vdash \Box{\sim}\circ A^\tau \vee \Box{\sim}\circ\Box{\sim}\circ A^\tau \tag{S9$^\tau$}$$

Together with Proposition 2.7, Proposition 3.1 guarantees that each translated axiom is valid on the corresponding class of heterogeneous algebras, as indicated in the following table.

	characterizing axioms	logics	corresponding subclasses of $\mathbb{HSMA}$
(S6a$^\tau$)	$A \vdash \Box \sim \circ \Box \sim \circ A$	HLQM	$\mathbb{HLQMA}$
(S6b$^\tau$)	$\Box \sim \circ \Box \sim \circ A \vdash A$	HUQM	$\mathbb{HUQMA}$
(S7$^\tau$)	$\Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ A \vdash \bot$	HDP	$\mathbb{HDPL}$
(S8$^\tau$)	$A \wedge \Box \sim \circ A \vdash \bot$	HAP	$\mathbb{HAPL}$
(S9$^\tau$)	$\top \vdash \Box \sim \circ A \vee \Box \sim \circ \Box \sim \circ A$	HWS	$\mathbb{HWSA}$

It is easy to see that the translation of a non-analytic inductive axiom in the list above is again not analytic inductive. However, the defining identities of $\mathbb{HSMA}$s (cf. Definition 2.2) and its subclasses are *analytic inductive* (cf. Definition 6.2). Hence, it is from these identities that analytic structural rules can be generated algorithmically with the methodology introduced in [22], which—as we will discuss in Section 5.2—will make it possible to derive the translated axioms listed above in the calculi so obtained.

NOTATION 3. *We let* $\mathsf{HS} \in \{\mathsf{HSM}, \mathsf{HLQM}, \mathsf{HUQM}, \mathsf{HDP}, \mathsf{HAP}, \mathsf{HWS}\}$ *and recall that* $\mathbb{HV} \in \{\mathbb{HSMA}, \mathbb{HLQMA}, \mathbb{HUQMA}, \mathbb{HDPL}, \mathbb{HAPL}, \mathbb{HWSA}\}$. *Then:*

if HS *is*	*then* $\mathbb{HV}$ *is*
HSM	$\mathbb{HSMA}$
HLQM	$\mathbb{HLQMA}$
HUQM	$\mathbb{HUQMA}$
HDP	$\mathbb{HDPL}$
HAP	$\mathbb{HAPL}$
HWS	$\mathbb{HWSA}$

4. Proper Display Calculi for Semi De Morgan Logic and its Extensions

4.1. Language

The language of the calculi introduced in the present section has types L and D, and is built up, as usual, from structural and operational (aka logical) connectives. Notice that some structural connectives do not have an operational counterpart in the logical language considered here (but they do have a semantic interpretation on perfect $\mathbb{HSMA}$s, cf. Definition 2.3); indeed, for economy of presentation, we have included only the logical connectives needed to translate the original language. However, to guarantee the display property, we need to close the logical signature under adjoints and residuals of all connectives at the *structural* level. In the tables below, each structural connective corresponding to a logical connective which belongs to the family $\mathcal{F}$ (resp. $\mathcal{G}$, $\mathcal{H}$) defined in "Appendix A" is denoted by decorating that logical connective with $\hat{}$ (resp. $\check{}$, $\tilde{}$).[7]

[7]For any sequent $x \vdash y$, we define the signed generation trees $+x$ and $-y$ by labelling the root of the generation tree of x (resp. y) with the sign $+$ (resp. $-$), and then propagating the sign to all nodes according to the polarity of the coordinate of the connective assigned to each node. Positive (resp. negative) coordinates propagate the same (resp. opposite) sign to the corresponding child node. Then, a substructure z in $x \vdash y$ is in *precedent* (resp. *succedent*) *position* if the sign of its root node as a subtree of $+x$ or $-y$ is $+$ (resp. $-$). In any derivable sequent, the structural counterparts of $\mathcal{F}$-connectives (resp. $\mathcal{G}$-connectives) will occur only in precedent (resp. succedent) position, whereas structural counterparts of $\mathcal{H}$-connectives occur both in precedent and succedent position.

$$\mathsf{L}\begin{cases} A ::= p \mid \top \mid \bot \mid \Box\alpha \mid A \wedge A \mid A \vee A \\ \\ X ::= A \mid \hat{\top} \mid \check{\bot} \mid \check{\Box}\Gamma \mid \hat{\bullet}_\ell \Gamma \mid \check{\bullet}_r \Gamma \mid X \mathbin{\hat{\wedge}} X \mid X \mathbin{\check{\vee}} X \mid X \mathbin{\hat{\succ\!-}} X \mid X \mathbin{\check{\rightarrow}} X \end{cases}$$

$$\mathsf{D}\begin{cases} \alpha ::= 1 \mid 0 \mid \circ A \mid \sim \alpha \mid \alpha \cap \alpha \mid \alpha \cup \alpha \\ \\ \Gamma ::= \alpha \mid \hat{1} \mid \check{0} \mid \tilde{\circ} X \mid \hat{\blacklozenge} X \mid \tilde{*}\Gamma \mid \Gamma \mathbin{\hat{\cap}} \Gamma \mid \Gamma \mathbin{\check{\cup}} \Gamma \mid \Gamma \mathbin{\hat{\supset\!-}} \Gamma \mid \Gamma \mathbin{\check{-\!\supset}} \Gamma \end{cases}$$

The following synoptic table reports the interpretation of heterogeneous (structural) connectives as operations of perfect $\mathbb{HSMA}$s (cf. Definition 2.3).

$\mathsf{L} \to \mathsf{D}$	$\mathsf{D} \to \mathsf{L}$		$\mathsf{D} \to \mathsf{L}$	$\mathsf{L} \to \mathsf{D}$
$\tilde{\circ}$	$\hat{\bullet}_\ell$	$\check{\bullet}_r$	$\check{\Box}$	$\hat{\blacklozenge}$
h	h_ℓ	h_r	e	e_ℓ

The following synoptic tables report the interpretation of structural connectives as their logical counterparts. The connectives not included in the language presently considered appear between round brackets.

L						D							
$\hat{\top}$	$\hat{\wedge}$	$\hat{\succ\!-}$	$\check{\bot}$	$\check{\vee}$	$\check{\rightarrow}$	$\hat{1}$	$\hat{\cap}$	$\hat{\supset\!-}$	$\check{0}$	$\check{\cup}$	$\check{-\!\supset}$	$\tilde{*}$	
$\top$	$\wedge$	$(\succ\!-)$	$\bot$	$\vee$	$(\rightarrow)$	1	$\cap$	$(\supset\!-)$	0	$\cup$	$(-\!\supset)$	$\sim$	$\sim$

$\mathsf{L} \to \mathsf{D}$		$\mathsf{D} \to \mathsf{L}$	$\mathsf{D} \to \mathsf{L}$	$\mathsf{D} \to \mathsf{L}$	$\mathsf{L} \to \mathsf{D}$
$\tilde{\circ}$		$\hat{\bullet}_\ell$	$\check{\bullet}_r$	$\check{\Box}$	$\hat{\blacklozenge}$
$\circ$	$\circ$	$(\bullet_\ell)$	$(\bullet_r)$	$\Box$	$(\blacklozenge)$

4.2. Multi-type Display Calculi for Semi De Morgan Logic and its Extensions

In what follows, variables X, Y, Z, W denote L-structures, and $\Gamma, \Delta, \Theta, \Pi$ denote D-structures. The proper display calculus $\mathsf{D.HSM}$ for semi De Morgan logic consists of the following rules:

- Identity and cut rules

$$\mathrm{Id}\,\frac{}{p \vdash p} \qquad \frac{X \vdash A \qquad A \vdash Y}{X \vdash Y}\,\mathrm{Cut}_{\mathsf{L}} \qquad \frac{\Gamma \vdash \alpha \qquad \alpha \vdash \Delta}{\Gamma \vdash \Delta}\,\mathrm{Cut}_{\mathsf{D}}$$

- Pure L-type and D-type display rules

$$\mathrm{res}_{\mathsf{L}}\ \begin{array}{c} X \,\hat{\wedge}\, Y \vdash Z \\ \hline\hline Y \vdash X \,\check{\rightarrow}\, Z \end{array} \qquad \begin{array}{c} X \vdash Y \,\check{\vee}\, Z \\ \hline\hline Y \,\hat{\succ\!\!-}\, X \vdash Z \end{array}\ \mathrm{res}_{\mathsf{L}} \qquad \Bigg| \qquad \mathrm{res}_{\mathsf{D}}\ \begin{array}{c} \Gamma \,\hat{\cap}\, \Delta \vdash \Theta \\ \hline\hline \Delta \vdash \Gamma \,\check{-\!\!\supset}\, \Theta \end{array} \qquad \begin{array}{c} \Gamma \vdash \Delta \,\check{\cup}\, \Theta \\ \hline\hline \Delta \,\hat{\supset\!\!-}\, \Gamma \vdash \Theta \end{array}\ \mathrm{res}_{\mathsf{D}}$$

$$\mathrm{adj}_{*}\ \begin{array}{c} \tilde{*}\,\Gamma \vdash \Delta \\ \hline\hline \tilde{*}\,\Delta \vdash \Gamma \end{array} \qquad \begin{array}{c} \Gamma \vdash \tilde{*}\,\Delta \\ \hline\hline \Delta \vdash \tilde{*}\,\Gamma \end{array}\ \mathrm{adj}_{*}$$

- Multi-type display rules

$$\mathrm{adj}_{\mathsf{LD}}\ \begin{array}{c} X \vdash \check{\Box}\,\Gamma \\ \hline\hline \hat{\blacklozenge}\, X \vdash \Gamma \end{array} \qquad \mathrm{adj}_{\mathsf{DL}}\ \begin{array}{c} \tilde{\circ}\, X \vdash \Gamma \\ \hline\hline X \vdash \check{\bullet}_r\,\Gamma \end{array} \qquad \begin{array}{c} \Gamma \vdash \tilde{\circ}\, X \\ \hline\hline \hat{\bullet}_\ell\,\Gamma \vdash X \end{array}\ \mathrm{adj}_{\mathsf{DL}}$$

- Pure L-type and D-type structural rules

$$\hat{\top}\ \begin{array}{c} X \vdash Y \\ \hline\hline X \,\hat{\wedge}\, \hat{\top} \vdash Y \end{array} \qquad \begin{array}{c} X \vdash Y \\ \hline\hline X \vdash Y \,\check{\vee}\, \check{\bot} \end{array}\ \check{\bot} \qquad \Bigg| \qquad \hat{1}\ \begin{array}{c} \Gamma \vdash \Delta \\ \hline\hline \Gamma \,\hat{\cap}\, \hat{1} \vdash \Delta \end{array} \qquad \begin{array}{c} \Gamma \vdash \Delta \\ \hline\hline \Gamma \vdash \Delta \,\check{\cup}\, \check{0} \end{array}\ \check{0}$$

$$\mathrm{E}_{\mathsf{L}}\ \begin{array}{c} X \,\hat{\wedge}\, Y \vdash Z \\ \hline Y \,\hat{\wedge}\, X \vdash Z \end{array} \qquad \begin{array}{c} X \vdash Y \,\check{\vee}\, Z \\ \hline X \vdash Z \,\check{\vee}\, Y \end{array}\ \mathrm{E}_{\mathsf{L}} \qquad \Bigg| \qquad \mathrm{E}_{\mathsf{D}}\ \begin{array}{c} \Gamma \,\hat{\cap}\, \Delta \vdash \Theta \\ \hline \Delta \,\hat{\cap}\, \Gamma \vdash \Theta \end{array} \qquad \begin{array}{c} \Gamma \vdash \Delta \,\check{\cup}\, \Theta \\ \hline \Gamma \vdash \Theta \,\check{\cup}\, \Delta \end{array}\ \mathrm{E}_{\mathsf{D}}$$

$$\mathrm{W}_{\mathsf{L}}\ \begin{array}{c} X \vdash Y \\ \hline X \,\hat{\wedge}\, Z \vdash Y \end{array} \qquad \begin{array}{c} X \vdash Y \\ \hline X \vdash Y \,\check{\vee}\, Z \end{array}\ \mathrm{W}_{\mathsf{L}} \qquad \Bigg| \qquad \mathrm{W}_{\mathsf{D}}\ \begin{array}{c} \Gamma \vdash \Delta \\ \hline \Gamma \,\hat{\cap}\, \Theta \vdash \Delta \end{array} \qquad \begin{array}{c} \Gamma \vdash \Delta \\ \hline \Gamma \vdash \Delta \,\check{\cup}\, \Theta \end{array}\ \mathrm{W}_{\mathsf{D}}$$

$$\mathrm{C}_{\mathsf{L}}\ \begin{array}{c} X \,\hat{\wedge}\, X \vdash Y \\ \hline X \vdash Y \end{array} \qquad \begin{array}{c} X \vdash Y \,\check{\vee}\, Y \\ \hline X \vdash Y \end{array}\ \mathrm{C}_{\mathsf{L}} \qquad \Bigg| \qquad \mathrm{C}_{\mathsf{D}}\ \begin{array}{c} \Gamma \,\hat{\cap}\, \Gamma \vdash \Delta \\ \hline \Gamma \vdash \Delta \end{array} \qquad \begin{array}{c} \Gamma \vdash \Delta \,\check{\cup}\, \Delta \\ \hline \Gamma \vdash \Delta \end{array}\ \mathrm{C}_{\mathsf{D}}$$

$$\mathrm{A}_{\mathsf{L}}\ \begin{array}{c} (X \,\hat{\wedge}\, Y) \,\hat{\wedge}\, Z \vdash W \\ \hline\hline X \,\hat{\wedge}\, (Y \,\hat{\wedge}\, Z) \vdash Z \end{array} \qquad \Bigg| \qquad \mathrm{A}_{\mathsf{D}}\ \begin{array}{c} (\Gamma \,\hat{\cap}\, \Delta) \,\hat{\cap}\, \Theta \vdash \Pi \\ \hline\hline \Gamma \,\hat{\cap}\, (\Delta \,\hat{\cap}\, \Theta) \vdash \Theta \end{array}$$

$$\begin{array}{c} X \vdash (Y \,\check{\vee}\, Z) \,\check{\vee}\, W \\ \hline\hline X \vdash Y \,\check{\vee}\, (Z \,\check{\vee}\, W) \end{array}\ \mathrm{A}_{\mathsf{L}} \qquad \Bigg| \qquad \begin{array}{c} \Gamma \vdash (\Delta \,\check{\cup}\, \Theta) \,\check{\cup}\, \Pi \\ \hline\hline \Gamma \vdash \Delta \,\check{\cup}\, (\Theta \,\check{\cup}\, \Pi) \end{array}\ \mathrm{A}_{\mathsf{D}}$$

$$\begin{array}{c} \Gamma \vdash \Delta \\ \hline\hline \tilde{*}\,\Delta \vdash \tilde{*}\,\Gamma \end{array}\ \mathrm{cont}$$

- Multi-type structural rules

$$\tilde{\circ}\ \begin{array}{c} X \vdash Y \\ \hline \tilde{\circ}\, X \vdash \tilde{\circ}\, Y \end{array} \qquad \begin{array}{c} \hat{\bullet}_\ell\,\Gamma \vdash \check{\bullet}_r\,\Delta \\ \hline \Gamma \vdash \Delta \end{array}\ \check{\bullet} \qquad \hat{\blacklozenge}\,\hat{1}\ \begin{array}{c} \hat{1} \vdash \Gamma \\ \hline \hat{\blacklozenge}\,\hat{\top} \vdash \Gamma \end{array} \qquad \begin{array}{c} X \vdash \check{\Box}\,\check{0} \\ \hline X \vdash \check{\bot} \end{array}\ \check{\Box}\,\check{0} \qquad \begin{array}{c} \Gamma \vdash \tilde{\circ}\,\check{\Box}\,\Delta \\ \hline\hline \Gamma \vdash \Delta \end{array}\ \tilde{\circ}\,\check{\Box}$$

- Pure L-type and D-type operational rules

$$\top\,\frac{\hat{\top} \vdash X}{\top \vdash X} \qquad \frac{}{\hat{\top} \vdash \top}\,\top \qquad\Big|\qquad 1\,\frac{\hat{1} \vdash \Gamma}{1 \vdash \Gamma} \qquad \frac{}{\hat{1} \vdash 1}\,1$$

$$\bot\,\frac{}{\bot \vdash \check{\bot}} \qquad \frac{X \vdash \check{\bot}}{X \vdash \bot}\,\bot \qquad\Big|\qquad 0\,\frac{}{0 \vdash \check{0}} \qquad \frac{\Gamma \vdash \check{0}}{\Gamma \vdash 0}\,0$$

$$\wedge\,\frac{A \hat{\wedge} B \vdash X}{A \wedge B \vdash X} \qquad \frac{X \vdash A \qquad Y \vdash B}{X \hat{\wedge} Y \vdash A \wedge B}\,\wedge \qquad\Big|\qquad \cap\,\frac{\alpha \hat{\cap} \beta \vdash \Gamma}{\alpha \cap \beta \vdash \Gamma} \qquad \frac{\Gamma \vdash \alpha \qquad \Delta \vdash \beta}{\Gamma \hat{\cap} \Delta \vdash \alpha \cap \beta}\,\cap$$

$$\frac{X \vdash A \check{\vee} B}{X \vdash A \vee B}\,\vee \qquad \vee\,\frac{A \vdash X \qquad B \vdash Y}{A \vee B \vdash X \check{\vee} Y} \qquad\Big|\qquad \frac{\Gamma \vdash \alpha \check{\cup} \beta}{\Gamma \vdash \alpha \cup \beta}\,\cup \qquad \cup\,\frac{\alpha \vdash \Gamma \qquad \beta \vdash \Delta}{\alpha \cup \beta \vdash \Gamma \check{\cup} \Delta}$$

- Multi-type operational rules

$$\circ\,\frac{\tilde{\circ} A \vdash \Gamma}{\circ A \vdash \Gamma} \qquad \frac{X \vdash \tilde{\circ} A}{X \vdash \circ A}\,\circ \qquad \Box\,\frac{\alpha \vdash \Gamma}{\Box \alpha \vdash \check{\Box} \Gamma} \qquad \frac{X \vdash \check{\Box} \alpha}{X \vdash \Box \alpha}\,\Box$$

The proper display calculi capturing the (multi-type counterparts of the) axiomatic extensions of SM discussed in Section 1.1 are obtained by adding the analytic structural rules specified in the rightmost column of the following table to the calculus D.HSM introduced above. Each structural rule corresponds to one of the analytic inductive inequalities in Table 1 on page 12 characterizing one of the subclasses of $\mathbb{HSMA}$ as indicated there.

logics	algebras	heterog. algebras	display calculi	characterizing rules
LQM	$\mathbb{LQMA}$	$\mathbb{HLQMA}$	D.HLQM	$\frac{X \vdash Y}{X \vdash \check{\Box}\, \tilde{\circ}\, Y}\,\mathrm{LQM}$
UQM	$\mathbb{UQMA}$	$\mathbb{HUQMA}$	D.HUQM	$\mathrm{UQM}\,\frac{\hat{\bullet}_\ell\ \hat{\blacklozenge} X \vdash Y}{X \vdash Y}$
DP	$\mathbb{DPL}$	$\mathbb{HDPL}$	D.HDP	$\mathrm{res_B}\,\frac{\Gamma \hat{\cap} \Delta \vdash \Sigma}{\Delta \vdash \tilde{\circledast}\, \Gamma \check{\cup} \Sigma}$ (double line)
AP	$\mathbb{APL}$	$\mathbb{HAPL}$	D.HAP	$\frac{X \vdash \check{\Box}\, \tilde{\circledast}\, \tilde{\circ}\, Y}{X \hat{\wedge} Y \vdash \check{\bot}}\,\mathrm{AP}$
WS	$\mathbb{WSA}$	$\mathbb{HWSA}$	D.HWS	$\mathrm{WS}\,\frac{\hat{\blacklozenge} X \vdash \Delta}{\hat{\blacklozenge}\,(\check{\Box}\, \tilde{\circledast}\, \tilde{\circ} X \hat{\succ} \hat{\top}) \vdash \Delta}$

Notation 4. *We let*

$$\mathsf{D.HS} \in \{\mathsf{D.HSM}, \mathsf{D.HLQM}, \mathsf{D.HUQM}, \mathsf{D.HDP}, \mathsf{D.HAP}, \mathsf{D.HWS}\}.$$

5. Properties

As mentioned in the introduction, the design and background theory of proper display calculi guarantee that soundness, completeness, conservativity, cut elimination and subformula property follow immediately from the general theory or require only routine steps for their verification. Specifically for the calculi introduced in the previous section, thanks to a general Belnap-style theorem (cf. [10, Theorem 4.1]) the proof of *cut elimination* only requires the standard verification of the cases in which both cut formulas are principal, and the general condition guaranteeing the *subformula property* to hold is verified by inspection on the rules (cf. Section 5.4). Soundness, completeness and conservativity of proper display calculi for the basic logics of normal distributive lattice expansions in arbitrary signatures have been established in [22, Section 4.2]. The soundness, completeness and conservativity for the specific calculi introduced in the previous section can be uniformly established based on these results thanks to the systematic connections, also established in [22, Sections 5–7], between generalized correspondence theory and the theory of analytic calculi. In particular, the analytic structural rules involving heterogeneous connectives have been generated by running the algorithm ALBA on the inequalities defining the class $\mathbb{HSMA}$ and its subclasses. Hence, the *soundness* of the rules of each calculus w.r.t. its corresponding class of heterogeneous algebras immediately follows from the fact that the algorithm ALBA preserves the validity of its input inequalities on perfect (heterogeneous distributive) lattice expansions in any signature (cf. [8, Theorem 6.1]) In Section 5.1 we illustrate these ideas with a concrete example. The fact that ALBA-generated rules are semantically equivalent to the axioms from which they have been generated provides a uniform route to *completeness*; however, in Section 5.2, we establish completeness by deriving each axiom in the appropriate calculus, since this more specific route allows us to give a concrete illustration of how the calculi work. Finally, in Section 5.3 we prove conservativity by instantiating (modulo translation) the general proof strategy discussed in [22, Section 4.2.3].

5.1. Soundness

In the present subsection, we outline the verification of the soundness of the rules of each calculus introduced in Section 4.2 w.r.t. the semantics of its corresponding class of *perfect* heterogeneous algebras. The first step consists in interpreting structural symbols as their corresponding logical symbols; this makes it possible to interpret sequents as inequalities, and rules as quasi-inequalities. For example, the rules on the left-hand side below are interpreted as the quasi-inequalities on the right-hand side:

$$\tilde{\circ}\ \frac{X \vdash Y}{\tilde{\circ}\, X \vdash \tilde{\circ}\, Y} \qquad \rightsquigarrow \qquad \forall a \forall b[a \leq b \Rightarrow h(a) \leq h(b)]$$

$$\mathrm{ws}\ \frac{\hat{\blacklozenge}\, X \vdash \Delta}{\hat{\blacklozenge}\, (\check{\Box}\, \tilde{\ast}\, \tilde{\circ}\, X \,\hat{\succ}\, \hat{\top}) \vdash \Delta} \qquad \rightsquigarrow \qquad \forall a[e_\ell[e(h(a)^*) \succ \top] \leq e_\ell(a)]$$

The proof of the soundness of each rule then consists in verifying the validity of the quasi-inequality so obtained in the class of perfect heterogeneous algebras to which the given rule belongs. The verification of the soundness of pure-type rules and of the introduction rules following this procedure is routine, and is omitted. The validity of the quasi-inequalities corresponding to structural rules in which variables of two types occur follows straightforwardly from the observation that the quasi-inequality corresponding to each rule is obtained by running the algorithm ALBA (cf. Section 3.4 [22]) on some of the defining inequalities of its corresponding heterogeneous algebras.[8] For instance, the soundness of WS on perfect $\mathbb{HWSA}$s follows from the validity of axiom $(S9^\tau)$ in every $\mathbb{HWSA}$ (cf. discussion in Section 3) and from the soundness of the following ALBA reduction in every $\mathbb{HWSA}$:

$$\begin{array}{lll}
 & \forall a[\top \leq e(h(a)^*) \vee e((he(h(a)^*))^*)] & \\
\text{iff} & \forall a \forall b \forall c[b \leq a \,\&\, c \leq e(h(a)^*) \Rightarrow \top \leq e(h(b)^*) \vee e(h(c)^*)] & \\
\text{iff} & \forall a \forall b \forall c[b \leq a \,\&\, a \leq h_r(e_\ell(c)^*) \Rightarrow \top \leq e(h(b)^*) \vee e(h(c)^*)] & \\
\text{iff} & \forall b \forall c[b \leq h_r(e_\ell(c)^*) \Rightarrow \top \leq e(h(b)^*) \vee e(h(c)^*)] & \\
\text{iff} & \forall b \forall c[b \leq h_r(e_\ell(c)^*) \Rightarrow e(h(c)^*) \succ \top \leq e(h(b)^*)] & \\
\text{iff} & \forall b \forall c[b \leq h_r(e_\ell(c)^*) \Rightarrow b \leq h_r(e_\ell[e(h(c)^*) \succ \top]^*)] & \\
\text{iff} & \forall c[h_r(e_\ell(c)^*) \leq h_r(e_\ell[e(h(c)^*) \succ \top]^*)] & \\
\text{iff} & \forall c[e_\ell(c)^* \leq e_\ell[e(h(c)^*) \succ \top]^*] & h_r \text{ is injective} \\
\text{iff} & \forall c[e_\ell[e(h(c)^*) \succ \top] \leq e_\ell(c)] & {}^* \text{ is injective}
\end{array}$$

5.2. Completeness

In the present subsection, we show that the translations of the axioms and rules of each logic S (cf. Notation 1) are derivable in the corresponding calculus (see table at the end of Section 4.2). Then, the completeness of each calculus w.r.t. its corresponding class $\mathbb{HV}$ immediately follows from the completeness of S w.r.t. $\mathbb{V}$ (cf. Theorem 1) and the equivalence between $\mathbb{V}$ and $\mathbb{HV}$ (cf. Proposition 2.7).

[8]Indeed, as discussed in [2], the soundness of the rewriting rules of ALBA only depends on the order-theoretic properties of the interpretation of the logical connectives and their adjoints and residuals. The fact that some of these maps are not internal operations but have different domains and codomains does not make any substantial difference.

PROPOSITION 5.1. *For every $A \in \mathcal{L}$, the sequent $A^\tau \vdash A^\tau$ is derivable in all display calculi introduced in Section 4.2.*

For the sake of readability, in each derivation below we only label the applications of structural rules and omit the names of the logical rules.

PROOF. By induction on $A \in \mathcal{L}$. The proof of the base cases, i.e. $A{:=}\top$, $A{:=}\bot$ and $A{:=}p$, are straightforward and are omitted.

Inductive cases:

- as to $A{:=}\neg B$,

$$
\begin{prooftree}
\AxiomC{ind.hyp.}
\UnaryInfC{$B^\tau \vdash B^\tau$}
\LeftLabel{$\tilde{\circ}$}
\UnaryInfC{$\tilde{\circ}\, B^\tau \vdash \tilde{\circ}\, B^\tau$}
\UnaryInfC{$\circ\, B^\tau \vdash \tilde{\circ}\, B^\tau$}
\UnaryInfC{$\circ\, B^\tau \vdash \circ\, B^\tau$}
\RightLabel{cont}
\UnaryInfC{$\tilde{\ast} \circ B^\tau \vdash \tilde{\ast} \circ B^\tau$}
\UnaryInfC{$\tilde{\ast} \circ B^\tau \vdash {\sim} \circ B^\tau$}
\UnaryInfC{${\sim} \circ B^\tau \vdash {\sim} \circ B^\tau$}
\UnaryInfC{$\Box \, {\sim} \circ B^\tau \vdash \check{\Box} \, {\sim} \circ B^\tau$}
\UnaryInfC{$\Box \, {\sim} \circ B^\tau \vdash \Box \, {\sim} \circ B^\tau$}
\end{prooftree}
$$

- as to $A{:=}B \vee C$ and $A{:=}B \wedge C$

$$
\begin{prooftree}
\AxiomC{ind.hyp.}
\UnaryInfC{$B^\tau \vdash B^\tau$}
\RightLabel{$\mathrm{W_L}$}
\UnaryInfC{$B^\tau \vdash B^\tau \,\check{\vee}\, C^\tau$}
\AxiomC{ind.hyp}
\UnaryInfC{$C^\tau \vdash C^\tau$}
\RightLabel{$\mathrm{W_L}$}
\UnaryInfC{$C^\tau \vdash C^\tau \,\check{\vee}\, B^\tau$}
\RightLabel{$\mathrm{E_L}$}
\UnaryInfC{$C^\tau \vdash B^\tau \,\check{\vee}\, C^\tau$}
\BinaryInfC{$B^\tau \vee C^\tau \vdash (B^\tau \,\check{\vee}\, C^\tau) \,\check{\vee}\, (B^\tau \,\check{\vee}\, C^\tau)$}
\RightLabel{$\mathrm{C_L}$}
\UnaryInfC{$B^\tau \vee C^\tau \vdash B^\tau \vee C^\tau$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{ind.hyp.}
\UnaryInfC{$B^\tau \vdash B^\tau$}
\LeftLabel{$\mathrm{W_L}$}
\UnaryInfC{$B^\tau \,\hat{\wedge}\, C^\tau \vdash B^\tau$}
\AxiomC{ind.hyp}
\UnaryInfC{$C^\tau \vdash C^\tau$}
\LeftLabel{$\mathrm{W_L}$}
\UnaryInfC{$C^\tau \,\hat{\wedge}\, B^\tau \vdash C^\tau$}
\LeftLabel{$\mathrm{E_L}$}
\UnaryInfC{$B^\tau \,\hat{\wedge}\, C^\tau \vdash C^\tau$}
\BinaryInfC{$(B^\tau \,\hat{\wedge}\, C^\tau) \,\hat{\wedge}\, (B^\tau \,\hat{\wedge}\, C^\tau) \vdash B^\tau \wedge C^\tau$}
\LeftLabel{$\mathrm{C_L}$}
\UnaryInfC{$B^\tau \wedge C^\tau \vdash B^\tau \wedge C^\tau$}
\end{prooftree}
$$

■

PROPOSITION 5.2. *For every $A, B \in \mathcal{L}$, if $A \vdash B$ is derivable in any logic S, then $A^\tau \vdash B^\tau$ is derivable in its corresponding display calculus.*

PROOF. It is enough to prove the statement on the axioms. For the sake of readability, in what follows, we suppress the translation symbol $(\cdot)^\tau$. As to the axioms in SM:

- $\neg\top \vdash \bot \quad\rightsquigarrow\quad \Box \sim \circ \top \vdash \bot$ and $\top \vdash \neg\bot \rightsquigarrow \quad \top \vdash \Box \sim \circ \top$,

$$
\begin{array}{lc}
 & \hat{\top} \vdash \top \\ \hline
\mathrm{W_L} & \hat{\bullet}_\ell\, \tilde{\ast}\, \check{0}\, \hat{\wedge}\, \hat{\top} \vdash \top \\ \hline
\hat{\top} & \hat{\bullet}_\ell\, \tilde{\ast}\, \check{0} \vdash \top \\ \hline
\mathsf{adj_{DL}} & \tilde{\ast}\, \check{0} \vdash \tilde{\circ}\, \top \\ \hline
 & \tilde{\ast}\, \check{0} \vdash \circ\, \top \\ \hline
\mathsf{adj}_\ast & \tilde{\ast} \circ \top \vdash \check{0} \\ \hline
 & \sim \circ \top \vdash \check{0} \\ \hline
 & \Box \sim \circ \top \vdash \check{\Box}\, \check{0} \\ \hline
\check{\Box}\, \check{0} & \Box \sim \circ \top \vdash \check{\bot} \\ \hline
 & \Box \sim \circ \top \vdash \bot
\end{array}
\qquad
\begin{array}{lc}
 & \bot \vdash \check{\bot} \\ \hline
\mathrm{W_L} & \bot \vdash \check{\bot}\, \check{\vee}\, \check{\bullet}_r\, \tilde{\ast}\, \hat{1} \\ \hline
\check{\bot} & \bot \vdash \check{\bullet}_r\, \tilde{\ast}\, \hat{1} \\ \hline
\mathsf{adj_{DL}} & \tilde{\circ}\, \bot \vdash \tilde{\ast}\, \hat{1} \\ \hline
 & \circ \bot \vdash \tilde{\ast}\, \hat{1} \\ \hline
\mathsf{adj}_\ast & \hat{1} \vdash \tilde{\ast} \circ \bot \\ \hline
 & \hat{1} \vdash \sim \circ \bot \\ \hline
\hat{\blacklozenge}\, \hat{\top} & \hat{\blacklozenge}\, \hat{\top} \vdash \sim \circ \bot \\ \hline
\mathsf{adj_{LD}} & \hat{\top} \vdash \check{\Box} \sim \circ \bot \\ \hline
 & \hat{\top} \vdash \Box \sim \circ \bot \\ \hline
 & \top \vdash \Box \sim \circ \bot
\end{array}
$$

- $\neg A \dashv\vdash \neg\neg\neg A \quad\rightsquigarrow\quad \Box \sim \circ A \dashv\vdash \Box \sim \circ \Box \sim \circ \Box \sim \circ A$

$$
\begin{array}{lc}
 & A \vdash A \\ \hline
\tilde{\circ} & \tilde{\circ}\, A \vdash \tilde{\circ}\, A \\ \hline
 & \circ A \vdash \tilde{\circ}\, A \\ \hline
 & \circ A \vdash \circ A \\ \hline
\mathrm{cont} & \tilde{\ast} \circ A \vdash \tilde{\ast} \circ A \\ \hline
 & \tilde{\ast} \circ A \vdash \sim \circ A \\ \hline
\tilde{\circ}\, \check{\Box} & \tilde{\ast} \circ A \vdash \tilde{\circ}\, \check{\Box} \sim \circ A \\ \hline
\mathsf{adj_{DL}} & \hat{\bullet}_\ell\, \tilde{\ast} \circ A \vdash \check{\Box} \sim \circ A \\ \hline
 & \hat{\bullet}_\ell\, \tilde{\ast} \circ A \vdash \Box \sim \circ A \\ \hline
\mathsf{adj_{DL}} & \tilde{\ast} \circ A \vdash \tilde{\circ}\, \Box \sim \circ A \\ \hline
 & \tilde{\ast} \circ A \vdash \circ \Box \sim \circ A \\ \hline
\mathsf{adj}_\ast & \tilde{\ast} \circ \Box \sim \circ A \vdash \circ A \\ \hline
 & \sim \circ \Box \sim \circ A \vdash \circ A \\ \hline
 & \Box \sim \circ \Box \sim \circ A \vdash \check{\Box} \circ A \\ \hline
\tilde{\circ} & \tilde{\circ}\, \Box \sim \circ \Box \sim \circ A \vdash \tilde{\circ}\, \check{\Box} \circ A \\ \hline
 & \circ \Box \sim \circ \Box \sim \circ A \vdash \tilde{\circ}\, \check{\Box} \circ A \\ \hline
\tilde{\circ}\, \check{\Box} & \circ \Box \sim \circ \Box \sim \circ A \vdash \circ A \\ \hline
\mathrm{cont} & \tilde{\ast} \circ A \vdash \tilde{\ast} \circ \Box \sim \circ \Box \sim \circ A \\ \hline
 & \tilde{\ast} \circ A \vdash \sim \circ \Box \sim \circ \Box \sim \circ A \\ \hline
 & \sim \circ A \vdash \sim \circ \Box \sim \circ \Box \sim \circ A \\ \hline
 & \Box \sim \circ A \vdash \check{\Box} \sim \circ \Box \sim \circ \Box \sim \circ A \\ \hline
 & \Box \sim \circ A \vdash \Box \sim \circ \Box \sim \circ \Box \sim \circ A
\end{array}
\qquad
\begin{array}{lc}
 & A \vdash A \\ \hline
\tilde{\circ} & \tilde{\circ}\, A \vdash \tilde{\circ}\, A \\ \hline
 & \circ A \vdash \tilde{\circ}\, A \\ \hline
 & \circ A \vdash \circ A \\ \hline
\mathrm{cont} & \tilde{\ast} \circ A \vdash \tilde{\ast} \circ A \\ \hline
 & \sim \circ A \vdash \tilde{\ast} \circ A \\ \hline
 & \Box \sim \circ A \vdash \check{\Box}\, \tilde{\ast} \circ A \\ \hline
\tilde{\circ} & \tilde{\circ}\, \Box \sim \circ A \vdash \tilde{\circ}\, \check{\Box}\, \tilde{\ast} \circ A \\ \hline
 & \circ \Box \sim \circ A \vdash \tilde{\circ}\, \check{\Box}\, \tilde{\ast} \circ A \\ \hline
\tilde{\circ}\, \check{\Box} & \circ \Box \sim \circ A \vdash \tilde{\ast} \circ A \\ \hline
\mathsf{adj}_\ast & \circ A \vdash \tilde{\ast} \circ \Box \sim \circ A \\ \hline
 & \circ A \vdash \sim \circ \Box \sim \circ A \\ \hline
\tilde{\circ}\, \check{\Box} & \circ A \vdash \tilde{\circ}\, \check{\Box} \sim \circ \Box \sim \circ A \\ \hline
\mathsf{adj_{DL}} & \hat{\bullet}_\ell \circ A \vdash \check{\Box} \sim \circ \Box \sim \circ A \\ \hline
 & \hat{\bullet}_\ell \circ A \vdash \Box \sim \circ \Box \sim \circ A \\ \hline
\mathsf{adj_{DL}} & \circ A \vdash \tilde{\circ}\, \Box \sim \circ \Box \sim \circ A \\ \hline
 & \circ A \vdash \circ \Box \sim \circ \Box \sim \circ A \\ \hline
\mathrm{cont} & \tilde{\ast} \circ \Box \sim \circ \Box \sim \circ A \vdash \tilde{\ast} \circ A \\ \hline
 & \sim \circ \Box \sim \circ \Box \sim \circ A \vdash \tilde{\ast} \circ A \\ \hline
 & \sim \circ \Box \sim \circ \Box \sim \circ A \vdash \sim \circ A \\ \hline
 & \Box \sim \circ \Box \sim \circ \Box \sim \circ A \vdash \check{\Box} \sim \circ A \\ \hline
 & \Box \sim \circ \Box \sim \circ \Box \sim \circ A \vdash \Box \sim \circ A
\end{array}
$$

- $\neg A \wedge \neg B \vdash \neg(A \vee B) \quad \rightsquigarrow \quad \Box \sim \circ A \wedge \Box \sim \circ B \vdash \Box \sim \circ (A \vee B)$,

$$
\begin{array}{lcl}
 & A \vdash A & \\ \hline
 & \tilde{\circ}\, A \vdash \tilde{\circ}\, A & \tilde{\circ} \\ \hline
 & \tilde{\circ}\, A \vdash \circ\, A & \\ \hline
 & \tilde{\ast} \circ A \vdash \tilde{\ast}\, \tilde{\circ}\, A & \mathrm{cont} \\ \hline
 & \sim \circ A \vdash \tilde{\ast}\, \tilde{\circ}\, A & \\ \hline
 & \Box \sim \circ A \vdash \check{\Box}\, \tilde{\ast}\, \tilde{\circ}\, A & \\ \hline
\mathrm{W_L} & \Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ B \vdash \check{\Box}\, \tilde{\ast}\, \tilde{\circ}\, A & \\ \hline
 & \Box \sim \circ A \wedge \Box \sim \circ B \vdash \check{\Box}\, \tilde{\ast}\, \tilde{\circ}\, A & \\ \hline
\mathsf{adj_{LD}} & \hat{\blacklozenge}\,(\Box \sim \circ A \wedge \Box \sim \circ B) \vdash \tilde{\ast}\, \tilde{\circ}\, A & \\ \hline
 & \tilde{\circ}\, A \vdash \tilde{\ast}\; \hat{\blacklozenge}\,(\Box \sim \circ A \wedge \Box \sim \circ B) & \mathsf{adj_{\ast}} \\ \hline
\mathsf{adj_{DL}} & A \vdash \check{\bullet}_r\, \tilde{\ast}\; \hat{\blacklozenge}\,(\Box \sim \circ A \wedge \Box \sim \circ B) &
\end{array}
\qquad
\begin{array}{lcl}
 & B \vdash B & \\ \hline
 & \tilde{\circ}\, B \vdash \tilde{\circ}\, B & \tilde{\circ} \\ \hline
 & \tilde{\circ}\, B \vdash \circ\, B & \\ \hline
 & \tilde{\ast} \circ B \vdash \tilde{\ast}\, \tilde{\circ}\, B & \mathrm{cont} \\ \hline
 & \sim \circ B \vdash \tilde{\ast}\, \tilde{\circ}\, B & \\ \hline
 & \Box \sim \circ B \vdash \check{\Box}\, \tilde{\ast}\, \tilde{\circ}\, B & \\ \hline
\mathrm{W_L} & \Box \sim \circ B \,\hat{\wedge}\, \Box \sim \circ A \vdash \check{\Box}\, \tilde{\ast}\, \tilde{\circ}\, B & \\ \hline
\mathrm{E_L} & \Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ B \vdash \check{\Box}\, \tilde{\ast}\, \tilde{\circ}\, B & \\ \hline
 & \Box \sim \circ A \wedge \Box \sim \circ B \vdash \check{\Box}\, \tilde{\ast}\, \tilde{\circ}\, B & \\ \hline
\mathsf{adj_{LD}} & \hat{\blacklozenge}\,(\Box \sim \circ A \wedge \Box \sim \circ B) \vdash \tilde{\ast}\, \tilde{\circ}\, B & \\ \hline
 & \tilde{\circ}\, B \vdash \tilde{\ast}\; \hat{\blacklozenge}\,(\Box \sim \circ A \wedge \Box \sim \circ B) & \mathsf{adj_{\ast}} \\ \hline
\mathsf{adj_{DL}} & B \vdash \check{\bullet}_r\, \tilde{\ast}\; \hat{\blacklozenge}\,(\Box \sim \circ A \wedge \Box \sim \circ B) &
\end{array}
$$

$$
\begin{array}{lcl}
 & A \vdash \check{\bullet}_r\, \tilde{\ast}\; \hat{\blacklozenge}\,(\Box \sim \circ A \wedge \Box \sim \circ B) \qquad B \vdash \check{\bullet}_r\, \tilde{\ast}\; \hat{\blacklozenge}\,(\Box \sim \circ A \wedge \Box \sim \circ B) & \\ \hline
 & A \vee B \vdash \check{\bullet}_r\, \tilde{\ast}\; \hat{\blacklozenge}\,(\Box \sim \circ A \wedge \Box \sim \circ B) \,\check{\vee}\, \check{\bullet}_r\, \tilde{\ast}\; \hat{\blacklozenge}\,(\Box \sim \circ A \wedge \Box \sim \circ B) & \\ \hline
 & A \vee B \vdash \check{\bullet}_r\, \tilde{\ast}\; \hat{\blacklozenge}\,(\Box \sim \circ A \wedge \Box \sim \circ B) & \mathrm{C_L} \\ \hline
\mathsf{adj_{DL}} & \tilde{\circ}\,(A \vee B) \vdash \tilde{\ast}\; \hat{\blacklozenge}\,(\Box \sim \circ A \wedge \Box \sim \circ B) & \\ \hline
 & \circ\,(A \vee B) \vdash \tilde{\ast}\; \hat{\blacklozenge}\,(\Box \sim \circ A \wedge \Box \sim \circ B) & \\ \hline
 & \hat{\blacklozenge}\,(\Box \sim \circ A \wedge \Box \sim \circ B) \vdash \tilde{\ast} \circ (A \vee B) & \mathsf{adj_{\ast}} \\ \hline
 & \hat{\blacklozenge}\,(\Box \sim \circ A \wedge \Box \sim \circ B) \vdash \sim \circ (A \vee B) & \\ \hline
\mathsf{adj_{LD}} & \Box \sim \circ A \wedge \Box \sim \circ B \vdash \check{\Box} \sim \circ (A \vee B) & \\ \hline
 & \Box \sim \circ A \wedge \Box \sim \circ B \vdash \Box \sim \circ (A \vee B) &
\end{array}
$$

- $\neg\neg A \wedge \neg\neg B \vdash \neg\neg(A \wedge B) \rightsquigarrow \Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B \vdash \Box \sim \circ \Box \sim \circ (A \wedge B)$,

$$
\begin{prooftree}
\AxiomC{$A \vdash A$}
\RightLabel{$\tilde{\circ}$}
\UnaryInfC{$\tilde{\circ} A \vdash \tilde{\circ} A$}
\UnaryInfC{$\circ A \vdash \tilde{\circ} A$}
\RightLabel{cont}
\UnaryInfC{$\tilde{\ast} \tilde{\circ} A \vdash \tilde{\ast} \circ A$}
\UnaryInfC{$\tilde{\ast} \tilde{\circ} A \vdash \sim \circ A$}
\RightLabel{$\tilde{\circ}\check{\Box}$}
\UnaryInfC{$\tilde{\ast} \tilde{\circ} A \vdash \tilde{\circ} \check{\Box} \sim \circ A$}
\RightLabel{$\mathsf{adj}_{\mathsf{DL}}$}
\UnaryInfC{$\hat{\bullet}_\ell \tilde{\ast} \tilde{\circ} A \vdash \check{\Box} \sim \circ A$}
\UnaryInfC{$\hat{\bullet}_\ell \tilde{\ast} \tilde{\circ} A \vdash \Box \sim \circ A$}
\RightLabel{$\mathsf{adj}_{\mathsf{DL}}$}
\UnaryInfC{$\tilde{\ast} \tilde{\circ} A \vdash \tilde{\circ} \Box \sim \circ A$}
\UnaryInfC{$\tilde{\ast} \tilde{\circ} A \vdash \circ \Box \sim \circ A$}
\LeftLabel{$\mathsf{adj}_\ast$}
\UnaryInfC{$\tilde{\ast} \circ \Box \sim \circ A \vdash \tilde{\circ} A$}
\UnaryInfC{$\sim \circ \Box \sim \circ A \vdash \tilde{\circ} A$}
\UnaryInfC{$\Box \sim \circ \Box \sim \circ A \vdash \check{\Box} \tilde{\circ} A$}
\LeftLabel{W_{L}}
\UnaryInfC{$\Box \sim \circ \Box \sim \circ A \mathbin{\hat{\wedge}} \Box \sim \circ \Box \sim \circ B \vdash \check{\Box} \tilde{\circ} A$}
\UnaryInfC{$\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B \vdash \check{\Box} \tilde{\circ} A$}
\LeftLabel{$\mathsf{adj}_{\mathsf{LD}}$}
\UnaryInfC{$\hat{\blacklozenge} (\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \vdash \tilde{\circ} A$}
\RightLabel{$\mathsf{adj}_{\mathsf{DL}}$}
\UnaryInfC{$\hat{\bullet}_\ell \hat{\blacklozenge} (\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \vdash A$}
\AxiomC{$B \vdash B$}
\RightLabel{$\tilde{\circ}$}
\UnaryInfC{$\tilde{\circ} B \vdash \tilde{\circ} B$}
\UnaryInfC{$\circ B \vdash \tilde{\circ} B$}
\RightLabel{cont}
\UnaryInfC{$\tilde{\ast} \tilde{\circ} B \vdash \tilde{\ast} \circ B$}
\UnaryInfC{$\tilde{\ast} \tilde{\circ} B \vdash \sim \circ B$}
\RightLabel{$\tilde{\circ}\check{\Box}$}
\UnaryInfC{$\tilde{\ast} \tilde{\circ} B \vdash \tilde{\circ} \check{\Box} \sim \circ B$}
\RightLabel{$\mathsf{adj}_{\mathsf{DL}}$}
\UnaryInfC{$\hat{\bullet}_\ell \tilde{\ast} \tilde{\circ} B \vdash \check{\Box} \sim \circ B$}
\UnaryInfC{$\hat{\bullet}_\ell \tilde{\ast} \tilde{\circ} B \vdash \Box \sim \circ B$}
\RightLabel{$\mathsf{adj}_{\mathsf{DL}}$}
\UnaryInfC{$\tilde{\ast} \tilde{\circ} B \vdash \tilde{\circ} \Box \sim \circ B$}
\UnaryInfC{$\tilde{\ast} \tilde{\circ} B \vdash \circ \Box \sim \circ B$}
\LeftLabel{$\mathsf{adj}_\ast$}
\UnaryInfC{$\tilde{\ast} \circ \Box \sim \circ B \vdash \tilde{\circ} B$}
\UnaryInfC{$\sim \circ \Box \sim \circ B \vdash \tilde{\circ} B$}
\UnaryInfC{$\Box \sim \circ \Box \sim \circ B \vdash \check{\Box} \tilde{\circ} B$}
\LeftLabel{W_{L}}
\UnaryInfC{$\Box \sim \circ \Box \sim \circ B \mathbin{\hat{\wedge}} \Box \sim \circ \Box \sim \circ A \vdash \check{\Box} \tilde{\circ} B$}
\LeftLabel{E_{L}}
\UnaryInfC{$\Box \sim \circ \Box \sim \circ A \mathbin{\hat{\wedge}} \Box \sim \circ \Box \sim \circ B \vdash \check{\Box} \tilde{\circ} B$}
\UnaryInfC{$\Box \sim \circ \Box \sim \circ B \wedge \Box \sim \circ \Box \sim \circ A \vdash \check{\Box} \tilde{\circ} B$}
\LeftLabel{$\mathsf{adj}_{\mathsf{LD}}$}
\UnaryInfC{$\hat{\blacklozenge} (\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \vdash \tilde{\circ} B$}
\RightLabel{$\mathsf{adj}_{\mathsf{DL}}$}
\UnaryInfC{$\hat{\bullet}_\ell \hat{\blacklozenge} (\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \vdash B$}
\BinaryInfC{$\hat{\bullet}_\ell \hat{\blacklozenge} (\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \mathbin{\hat{\wedge}} \hat{\bullet}_\ell \hat{\blacklozenge} (\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \vdash A \wedge B$}
\LeftLabel{C_{L}}
\UnaryInfC{$\hat{\bullet}_\ell \hat{\blacklozenge} (\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \vdash A \wedge B$}
\RightLabel{$\mathsf{adj}_{\mathsf{DL}}$}
\UnaryInfC{$\hat{\blacklozenge} (\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \vdash \tilde{\circ} (A \wedge B)$}
\UnaryInfC{$\hat{\blacklozenge} (\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \vdash \circ (A \wedge B)$}
\RightLabel{cont}
\UnaryInfC{$\tilde{\ast} \circ (A \wedge B) \vdash \tilde{\ast} \hat{\blacklozenge} (\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B)$}
\UnaryInfC{$\sim \circ (A \wedge B) \vdash \tilde{\ast} \hat{\blacklozenge} (\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B)$}
\UnaryInfC{$\Box \sim \circ (A \wedge B) \vdash \check{\Box} \tilde{\ast} \hat{\blacklozenge} (\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B)$}
\RightLabel{$\tilde{\circ}$}
\UnaryInfC{$\tilde{\circ} \Box \sim \circ (A \wedge B) \vdash \tilde{\circ} \check{\Box} \tilde{\ast} \hat{\blacklozenge} (\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B)$}
\RightLabel{$\tilde{\circ}\check{\Box}$}
\UnaryInfC{$\tilde{\circ} \Box \sim \circ (A \wedge B) \vdash \tilde{\ast} \hat{\blacklozenge} (\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B)$}
\UnaryInfC{$\circ \Box \sim \circ (A \wedge B) \vdash \tilde{\ast} \hat{\blacklozenge} (\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B)$}
\RightLabel{$\mathsf{adj}_\ast$}
\UnaryInfC{$\hat{\blacklozenge} (\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \vdash \tilde{\ast} \circ \Box \sim \circ (A \wedge B)$}
\UnaryInfC{$\hat{\blacklozenge} (\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B) \vdash \sim \circ \Box \sim \circ (A \wedge B)$}
\LeftLabel{$\mathsf{adj}_{\mathsf{LD}}$}
\UnaryInfC{$\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B \vdash \check{\Box} \sim \circ \Box \sim \circ (A \wedge B)$}
\UnaryInfC{$\Box \sim \circ \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ B \vdash \Box \sim \circ \Box \sim \circ (A \wedge B)$}
\end{prooftree}
$$

As to the characterizing axiom of AP:

- $\neg A \wedge A \vdash \bot \quad \rightsquigarrow \quad \Box \sim \circ A \wedge A \vdash \bot$,

$$
\begin{prooftree}
\AxiomC{$A \vdash A$}
\RightLabel{$\tilde{\circ}$}
\UnaryInfC{$\tilde{\circ} A \vdash \tilde{\circ} A$}
\UnaryInfC{$\tilde{\circ} A \vdash \circ A$}
\UnaryInfC{$\sim \circ A \vdash \tilde{\ast} \tilde{\circ} A$}
\UnaryInfC{$\Box \sim \circ A \vdash \check{\Box} \tilde{\ast} \tilde{\circ} A$}
\RightLabel{AP}
\UnaryInfC{$A \mathbin{\hat{\wedge}} \Box \sim \circ A \vdash \check{\bot}$}
\UnaryInfC{$\Box \sim \circ A \vdash A \mathbin{\check{\rightarrow}} \check{\bot}$}
\UnaryInfC{$A \mathbin{\hat{\wedge}} \Box \sim \circ A \vdash \check{\bot}$}
\LeftLabel{E_{L}}
\UnaryInfC{$\Box \sim \circ A \mathbin{\hat{\wedge}} A \vdash \check{\bot}$}
\UnaryInfC{$\Box \sim \circ A \mathbin{\hat{\wedge}} A \vdash \bot$}
\UnaryInfC{$\Box \sim \circ A \wedge A \vdash \bot$}
\end{prooftree}
$$

As to the characterizing axioms of LQM and UQM:

- $A \vdash \neg\neg A \rightsquigarrow A \vdash \Box \sim \circ \Box \sim \circ A$ and $\neg\neg A \vdash A \rightsquigarrow \Box \sim \circ \Box \sim \circ A \vdash A$,

$$
\begin{array}{cl}
A \vdash A & \\ \hline
A \vdash \check{\Box}\, \tilde{\circ}\, A & \text{LQM} \\ \hline
\hat{\blacklozenge} A \vdash \tilde{\circ}\, A & \mathsf{adj}_{\mathsf{LD}} \\ \hline
\hat{\blacklozenge} A \vdash \circ A & \\ \hline
\tilde{\ast} \circ A \vdash \tilde{\ast}\, \hat{\blacklozenge} A & \text{cont} \\ \hline
\sim \circ A \vdash \tilde{\ast}\, \hat{\blacklozenge} A & \\ \hline
\Box \sim \circ A \vdash \check{\Box}\, \tilde{\ast}\, \hat{\blacklozenge} A & \\ \hline
\tilde{\circ}\, \Box \sim \circ A \vdash \tilde{\circ}\, \check{\Box}\, \tilde{\ast}\, \hat{\blacklozenge} A & \tilde{\circ} \\ \hline
\tilde{\circ}\, \Box \sim \circ A \vdash \tilde{\ast}\, \hat{\blacklozenge} A & \tilde{\circ}\, \check{\Box} \\ \hline
\circ \Box \sim \circ A \vdash \tilde{\ast}\, \hat{\blacklozenge} A & \\ \hline
\hat{\blacklozenge} A \vdash \tilde{\ast} \circ \Box \sim \circ A & \mathsf{adj}_{\ast} \\ \hline
\hat{\blacklozenge} A \vdash \sim \circ \Box \sim \circ A & \\ \hline
A \vdash \check{\Box} \sim \circ \Box \sim \circ A & \mathsf{adj}_{\mathsf{LD}} \\ \hline
A \vdash \Box \sim \circ \Box \sim \circ A &
\end{array}
\qquad
\begin{array}{cl}
A \vdash A & \\ \hline
\tilde{\circ}\, A \vdash \tilde{\circ}\, A & \tilde{\circ} \\ \hline
\circ A \vdash \tilde{\circ}\, A & \\ \hline
\tilde{\ast}\, \tilde{\circ}\, A \vdash \tilde{\ast} \circ A & \text{cont} \\ \hline
\tilde{\ast}\, \tilde{\circ}\, A \vdash \sim \circ A & \\ \hline
\tilde{\ast}\, \tilde{\circ}\, A \vdash \tilde{\circ}\, \check{\Box} \sim \circ A & \tilde{\circ}\, \check{\Box} \\ \hline
\hat{\bullet}_{\ell}\, \tilde{\ast}\, \tilde{\circ}\, A \vdash \check{\Box} \sim \circ A & \mathsf{adj}_{\mathsf{DL}} \\ \hline
\hat{\bullet}_{\ell}\, \tilde{\ast}\, \tilde{\circ}\, A \vdash \Box \sim \circ A & \\ \hline
\tilde{\ast}\, \tilde{\circ}\, A \vdash \tilde{\circ}\, \Box \sim \circ A & \mathsf{adj}_{\mathsf{DL}} \\ \hline
\tilde{\ast}\, \tilde{\circ}\, A \vdash \circ \Box \sim \circ A & \\ \hline
\tilde{\ast} \circ \Box \sim \circ A \vdash \tilde{\circ}\, A & \mathsf{adj}_{\ast} \\ \hline
\sim \circ \Box \sim \circ A \vdash \tilde{\circ}\, A & \\ \hline
\Box \sim \circ \Box \sim \circ A \vdash \check{\Box}\, \tilde{\circ}\, A & \\ \hline
\hat{\blacklozenge}\, \Box \sim \circ \Box \sim \circ A \vdash \tilde{\circ}\, A & \mathsf{adj}_{\mathsf{LD}} \\ \hline
\hat{\bullet}_{\ell}\, \hat{\blacklozenge}\, \Box \sim \circ \Box \sim \circ A \vdash A & \mathsf{adj}_{\mathsf{DL}} \\ \hline
\Box \sim \circ \Box \sim \circ A \vdash A & \text{UQM}
\end{array}
$$

As to the characterizing axiom of WS:

- $\top \vdash \neg\neg A \vee \neg A \quad \rightsquigarrow \quad \top \vdash \Box \sim \circ \Box \sim \circ A \vee \Box \sim \circ A$,

$$
\begin{array}{rcl}
 & \vdots & \\
 & \circ A \vdash \circ A & \\ \cline{2-2}
 & \sim \circ A \vdash \tilde{\ast} \circ A & \\ \cline{2-2}
 & \sim \circ A \vdash \sim \circ A & \\ \cline{2-2}
\mathrm{adj}_{\mathsf{LD}} & \Box \sim \circ A \vdash \check{\Box} \sim \circ A & \\ \cline{2-2}
\mathrm{WS} & \hat{\blacklozenge} \Box \sim \circ A \vdash \sim \circ A & \\ \cline{2-2}
\mathrm{adj}_{\mathsf{LD}} & \hat{\blacklozenge} (\check{\Box} \tilde{\ast} \tilde{\circ} \Box \sim \circ A \mathbin{\hat{\succ\!\!-}} \hat{\top}) \vdash \sim \circ A & \\ \cline{2-2}
 & \check{\Box} \tilde{\ast} \tilde{\circ} \Box \sim \circ A \mathbin{\hat{\succ\!\!-}} \hat{\top} \vdash \check{\Box} \sim \circ A & \\ \cline{2-2}
 & \check{\Box} \tilde{\ast} \tilde{\circ} \Box \sim \circ A \mathbin{\hat{\succ\!\!-}} \hat{\top} \vdash \Box \sim \circ A & \mathrm{res}_{\mathsf{L}} \\ \cline{2-2}
 & \hat{\top} \vdash \check{\Box} \tilde{\ast} \tilde{\circ} \Box \sim \circ A \mathbin{\check{\vee}} \Box \sim \circ A & \mathrm{E}_{\mathsf{L}} \\ \cline{2-2}
 & \hat{\top} \vdash \Box \sim \circ A \mathbin{\check{\vee}} \check{\Box} \tilde{\ast} \tilde{\circ} \Box \sim \circ A & \mathrm{res}_{\mathsf{L}} \\ \cline{2-2}
\mathrm{adj}_{\mathsf{LD}} & \Box \sim \circ A \mathbin{\hat{\succ\!\!-}} \hat{\top} \vdash \check{\Box} \tilde{\ast} \tilde{\circ} \Box \sim \circ A & \\ \cline{2-2}
 & \hat{\blacklozenge} (\Box \sim \circ A \mathbin{\hat{\succ\!\!-}} \hat{\top}) \vdash \tilde{\ast} \tilde{\circ} \Box \sim \circ A & \mathrm{adj}_{\ast} \\ \cline{2-2}
 & \tilde{\circ} \Box \sim \circ A \vdash \tilde{\ast}\, \hat{\blacklozenge} (\Box \sim \circ A \mathbin{\hat{\succ\!\!-}} \hat{\top}) & \\ \cline{2-2}
 & \circ \Box \sim \circ A \vdash \tilde{\ast}\, \hat{\blacklozenge} (\Box \sim \circ A \mathbin{\hat{\succ\!\!-}} \hat{\top}) & \mathrm{adj}_{\ast} \\ \cline{2-2}
 & \hat{\blacklozenge} (\Box \sim \circ A \mathbin{\hat{\succ\!\!-}} \hat{\top}) \vdash \tilde{\ast} \circ \Box \sim \circ A & \\ \cline{2-2}
\mathrm{adj}_{\mathsf{LD}} & \hat{\blacklozenge} (\Box \sim \circ A \mathbin{\hat{\succ\!\!-}} \hat{\top}) \vdash \sim \circ \Box \sim \circ A & \\ \cline{2-2}
 & \Box \sim \circ A \mathbin{\hat{\succ\!\!-}} \hat{\top} \vdash \check{\Box} \sim \circ \Box \sim \circ A & \\ \cline{2-2}
 & \Box \sim \circ A \mathbin{\hat{\succ\!\!-}} \hat{\top} \vdash \Box \sim \circ \Box \sim \circ A & \mathrm{res}_{\mathsf{L}} \\ \cline{2-2}
 & \hat{\top} \vdash \Box \sim \circ A \mathbin{\check{\vee}} \Box \sim \circ \Box \sim \circ A & \mathrm{E}_{\mathsf{L}} \\ \cline{2-2}
 & \hat{\top} \vdash \Box \sim \circ \Box \sim \circ A \mathbin{\check{\vee}} \Box \sim \circ A & \\ \cline{2-2}
 & \hat{\top} \vdash \Box \sim \circ \Box \sim \circ A \vee \Box \sim \circ A & \\ \cline{2-2}
 & \top \vdash \Box \sim \circ \Box \sim \circ A \vee \Box \sim \circ A &
\end{array}
$$

As to the characterizing axiom of DP:

- $\neg A \wedge \neg\neg A \vdash \bot \quad \rightsquigarrow \quad \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ A \vdash \bot,$

$$
\begin{array}{rcl}
 & \vdots & \\
 & \sim \circ A \vdash \sim \circ A & \\
 & \overline{} & \tilde{\circ}\check{\Box} \\
 & \sim \circ A \vdash \tilde{\circ}\check{\Box} \sim \circ A & \\
 & \overline{} & \mathrm{adj_{DL}} \\
 & \hat{\bullet}_\ell \sim \circ A \vdash \check{\Box} \sim \circ A & \\
 & \overline{} & \\
 & \hat{\bullet}_\ell \sim \circ A \vdash \Box \sim \circ A & \\
 & \overline{} & \mathrm{adj_{DL}} \\
 & \sim \circ A \vdash \tilde{\circ} \Box \sim \circ A & \\
 & \overline{} & \\
 & \sim \circ A \vdash \circ \Box \sim \circ A & \\
 & \overline{} & \\
 & \Box \sim \circ A \vdash \check{\Box} \circ \Box \sim \circ A & \\
 \mathrm{adj_{LD}} & \overline{} & \\
 & \hat{\blacklozenge} \Box \sim \circ A \vdash \circ \Box \sim \circ A & \\
 & \overline{} & \check{0} \\
 & \hat{\blacklozenge} \Box \sim \circ A \vdash \circ \Box \sim \circ A \,\check{\cup}\, \check{0} & \\
 \mathrm{res_B} & \overline{} & \\
 & \tilde{\ast} \circ \Box \sim \circ A \,\hat{\cap}\, \hat{\blacklozenge} \Box \sim \circ A \vdash \check{0} & \\
 \mathrm{E_L} & \overline{} & \\
 & \hat{\blacklozenge} \Box \sim \circ A \,\hat{\cap}\, \tilde{\ast} \circ \Box \sim \circ A \vdash \check{0} & \\
 \mathrm{res_B} & \overline{} & \\
 & \tilde{\ast} \circ \Box \sim \circ A \vdash \tilde{\ast}\, \hat{\blacklozenge} \Box \sim \circ A \,\check{\cup}\, \check{0} & \\
 & \overline{} & \\
 & \sim \circ \Box \sim \circ A \vdash \tilde{\ast}\, \hat{\blacklozenge} \Box \sim \circ A \,\check{\cup}\, \check{0} & \\
 & \overline{} & \\
 & \Box \sim \circ \Box \sim \circ A \vdash \check{\Box} (\tilde{\ast}\, \hat{\blacklozenge} \Box \sim \circ A \,\check{\cup}\, \check{0}) & \\
 \mathrm{adj_{LD}} & \overline{} & \\
 & \hat{\blacklozenge} \Box \sim \circ \Box \sim \circ A \vdash \tilde{\ast}\, \hat{\blacklozenge} \Box \sim \circ A \,\check{\cup}\, \check{0} & \\
 \mathrm{res_B} & \overline{} & \\
 & \hat{\blacklozenge} \Box \sim \circ A \,\hat{\cap}\, \hat{\blacklozenge} \Box \sim \circ \Box \sim \circ A \vdash \check{0} & \\
 \mathrm{res_D} & \overline{} & \\
 & \hat{\blacklozenge} \Box \sim \circ \Box \sim \circ A \vdash \hat{\blacklozenge} \Box \sim \circ A \,\check{-\!\!\supset}\, \check{0} & \\
 & \overline{} & \\
 & \Box \sim \circ \Box \sim \circ A \vdash \check{\Box} (\hat{\blacklozenge} \Box \sim \circ A \,\check{-\!\!\supset}\, \check{0}) & \\
 \mathrm{W_L} & \overline{} & \\
 & \Box \sim \circ \Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ A \vdash \check{\Box} (\hat{\blacklozenge} \Box \sim \circ A \,\check{-\!\!\supset}\, \check{0}) & \\
 \mathrm{E_L} & \overline{} & \\
 & \Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A \vdash \check{\Box} (\hat{\blacklozenge} \Box \sim \circ A \,\check{-\!\!\supset}\, \check{0}) & \\
 \mathrm{adj_{LD}} & \overline{} & \\
 & \hat{\blacklozenge} (\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A) \vdash \hat{\blacklozenge} \Box \sim \circ A \,\check{-\!\!\supset}\, \check{0} & \\
 \mathrm{res_D} & \overline{} & \\
 & \hat{\blacklozenge} \Box \sim \circ A \,\hat{\cap}\, \hat{\blacklozenge} (\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A) \vdash \check{0} & \\
 \mathrm{E_D} & \overline{} & \\
 & \hat{\blacklozenge} (\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A) \,\hat{\cap}\, \hat{\blacklozenge} \Box \sim \circ A \vdash \check{0} & \\
 \mathrm{res_D} & \overline{} & \\
 & \hat{\blacklozenge} \Box \sim \circ A \vdash \hat{\blacklozenge} (\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A) \,\check{-\!\!\supset}\, \check{0} & \\
 \mathrm{adj_{LD}} & \overline{} & \\
 & \Box \sim \circ A \vdash \check{\Box} (\hat{\blacklozenge} (\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A) \,\check{-\!\!\supset}\, \check{0}) & \\
 \mathrm{W_L} & \overline{} & \\
 & \Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A \vdash \check{\Box} (\hat{\blacklozenge} (\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A) \,\check{-\!\!\supset}\, \check{0}) & \\
 \mathrm{adj_{LD}} & \overline{} & \\
 & \hat{\blacklozenge} (\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A) \vdash \hat{\blacklozenge} (\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A) \,\check{-\!\!\supset}\, \check{0} & \\
 \mathrm{res_D} & \overline{} & \\
 & \hat{\blacklozenge} (\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A) \,\hat{\cap}\, \hat{\blacklozenge} (\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A) \vdash \check{0} & \\
 \mathrm{C_D} & \overline{} & \\
 & \hat{\blacklozenge} (\Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A) \vdash \check{0} & \\
 \mathrm{adj_{LD}} & \overline{} & \\
 & \Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A \vdash \check{\Box} \check{0} & \\
 & \overline{} & \check{\Box}\check{0} \\
 & \Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A \vdash \check{\bot} & \\
 & \overline{} & \\
 & \Box \sim \circ A \,\hat{\wedge}\, \Box \sim \circ \Box \sim \circ A \vdash \bot & \\
 & \overline{} & \\
 & \Box \sim \circ A \wedge \Box \sim \circ \Box \sim \circ A \vdash \bot & \\
\end{array}
$$

■

5.3. Conservativity

To argue that the calculi introduced in Section 4 conservatively capture their respective logics (see Section 1.1), we follow the standard proof strategy discussed in [18,22]. Let S be one of the logics of Definition 1.1, let $\vdash_{\mathsf{S}}$ denote its syntactic consequence relation, and let $\models_{\mathsf{S}}$ (resp. $\models_{\mathsf{HS}}$) denote the semantic consequence relation arising from the class of the perfect (heterogeneous) algebras associated with S. We need to show that, for all $\mathcal{L}$-formulas A and B, if $A^\tau \vdash B^\tau$ is derivable in the display calculus $\mathsf{D.HS}$, then $A \vdash_{\mathsf{S}} B$. This claim can be proved using the following facts: (a) the rules of $\mathsf{D.HS}$ are sound w.r.t. perfect $\mathbb{HV}$-algebras (cf. Section 5.1); (b) S is complete w.r.t. its associated class of algebras (cf. Theorem 1); and (c) $\mathbb{V}$-algebras are equivalently presented as $\mathbb{HV}$-algebras (cf. Section 2.2), so that the semantic consequence relations arising from each type of algebras preserve and reflect the translation (cf. Proposition 3.1). If $A^\tau \vdash B^\tau$ is derivable in $\mathsf{D.HS}$, then by (a), $\models_{\mathsf{HS}} A^\tau \vdash B^\tau$. By (c), this implies that $\models_{\mathsf{S}} A \vdash B$. By (b), this implies that $A \vdash_{\mathsf{S}} B$, as required.

5.4. Cut Elimination and Subformula Property

In the present subsection, we briefly sketch the proof of cut elimination and subformula property for all display calculi introduced in Section 4.2. As discussed earlier on, proper display calculi have been designed so that the cut elimination and subformula property can be inferred from a meta-theorem, following the strategy introduced by Belnap for display calculi. The meta-theorem to which we will appeal was proved in [10, Theorem 4.1].

THEOREM 2. *Cut elimination and subformula property hold for all display calculi introduced in Section 4.2.*

PROOF. All conditions in [10] except C_8' are readily satisfied by inspecting the rules. Condition C_8' requires to check that reduction steps are available for every application of the cut rule in which both cut-formulas are principal, which either remove the original cut altogether or replace it by one or more cuts on formulas of strictly lower complexity. In what follows, we show C_8' for the unary connectives by induction on the complexity of cut formula.

Pure type atomic propositions:

$$\frac{p \vdash p \qquad p \vdash p}{p \vdash p} \quad \rightsquigarrow \quad p \vdash p$$

Pure type constants:

$$\dfrac{\hat{\top} \vdash \top \qquad \dfrac{\stackrel{\vdots\ \pi_1}{\hat{\top} \vdash X}}{\top \vdash X}}{\hat{\top} \vdash X} \quad \rightsquigarrow \quad \stackrel{\vdots\ \pi_1}{\hat{\top} \vdash X}$$

The cases for $\bot$, 1, 0 are standard and similar to the one above.

Pure-type unary connectives:

$$\dfrac{\dfrac{\stackrel{\vdots\ \pi_1}{\Gamma \vdash \tilde{*}\,\alpha}}{\Gamma \vdash \sim \alpha} \qquad \dfrac{\stackrel{\vdots\ \pi_2}{\tilde{*}\,\alpha \vdash \Delta}}{\sim \alpha \vdash \Delta}}{\Gamma \vdash \Delta} \quad \rightsquigarrow \quad \dfrac{\dfrac{\mathrm{adj}_{*}\,\dfrac{\stackrel{\vdots\ \pi_2}{\tilde{*}\,\alpha \vdash \Delta}}{\tilde{*}\,\Delta \vdash \alpha} \qquad \dfrac{\stackrel{\vdots\ \pi_1}{\Gamma \vdash \tilde{*}\,\alpha}}{\alpha \vdash \tilde{*}\,\Gamma}\,\mathrm{adj}_{*}}{\tilde{*}\,\Delta \vdash \tilde{*}\,\Gamma}\,\mathrm{cont}}{\Gamma \vdash \Delta}$$

Pure-type binary connectives:

$$\dfrac{\dfrac{\stackrel{\vdots\ \pi_1}{X \vdash A} \qquad \stackrel{\vdots\ \pi_2}{Y \vdash B}}{X \hat{\wedge} Y \vdash A \wedge B} \qquad \dfrac{\stackrel{\vdots\ \pi_3}{A \hat{\wedge} B \vdash Z}}{A \wedge B \vdash Z}}{X \hat{\wedge} Y \vdash Z} \quad \rightsquigarrow \quad \mathrm{E_L}\,\dfrac{\mathrm{res_L}\,\dfrac{\dfrac{\stackrel{\vdots\ \pi_1}{X \vdash A} \qquad \mathrm{res_L}\,\dfrac{\mathrm{E_L}\,\dfrac{\mathrm{res_L}\,\dfrac{\dfrac{\stackrel{\vdots\ \pi_2}{Y \vdash B} \qquad \mathrm{res_L}\,\dfrac{\stackrel{\vdots\ \pi_3}{A \hat{\wedge} B \vdash Z}}{B \vdash A \check{\rightarrow} Z}}{Y \vdash A \check{\rightarrow} Z}}{A \hat{\wedge} Y \vdash Z}}{Y \hat{\wedge} A \vdash Z}}{A \vdash Y \check{\rightarrow} Z}}{X \vdash Y \check{\rightarrow} Z}}{Y \hat{\wedge} X \vdash Z}}{X \hat{\wedge} Y \vdash Z}$$

The cases for $A \vee B$, $\alpha \cap \beta$, $\alpha \cup \beta$ are standard and similar to the one above.

Multi-type unary connectives:

$$\dfrac{\dfrac{\stackrel{\vdots\ \pi_1}{X \vdash \check{\Box}\,\alpha}}{X \vdash \Box\,\alpha} \qquad \dfrac{\stackrel{\vdots\ \pi_2}{\alpha \vdash \Delta}}{\Box\,\alpha \vdash \check{\Box}\,\Delta}}{X \vdash \check{\Box}\,\Delta} \quad \rightsquigarrow \quad \mathrm{adj_{LD}}\,\dfrac{\dfrac{\mathrm{adj_{LD}}\,\dfrac{\stackrel{\vdots\ \pi_1}{X \vdash \check{\Box}\,\alpha}}{\hat{\blacklozenge}\,X \vdash \alpha} \qquad \stackrel{\vdots\ \pi_2}{\alpha \vdash \Delta}}{\hat{\blacklozenge}\,X \vdash \Delta}}{X \vdash \check{\Box}\,\Delta}$$

$$
\dfrac{\dfrac{\begin{array}{c}\vdots\ \pi_1\\ \Gamma \vdash \tilde{\circ}\, A\end{array}}{\Gamma \vdash \circ\, A} \quad \dfrac{\begin{array}{c}\vdots\ \pi_2\\ \tilde{\circ}\, A \vdash \Delta\end{array}}{\circ\, A \vdash \Delta}}{\Gamma \vdash \Delta}
\quad\rightsquigarrow\quad
\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\ \pi_1\\ \Gamma \vdash \tilde{\circ}\, A\end{array}}{\hat{\bullet}_\ell\, \Gamma \vdash A}\,\mathrm{adj}_{\mathsf{DL}} \quad \mathrm{adj}_{\mathsf{DL}}\,\dfrac{\begin{array}{c}\vdots\ \pi_2\\ \tilde{\circ}\, A \vdash \Delta\end{array}}{A \vdash \check{\bullet}_r\, \Delta}}{\hat{\bullet}_\ell\, \Gamma \vdash \check{\bullet}_r\, \Delta}}{\Gamma \vdash \Delta}\,\bullet
$$

■

6. Conclusions

Contributions We have introduced proper display calculi for the logics arising from the variety of semi De Morgan algebras and the subvarieties of it studied in [31], which are sound, complete, conservative, and with cut elimination and subformula property. We applied the methodology of *multi-type calculi* to circumvent the difficulty posed by the fact that the axiomatization of semi De Morgan logic is not analytic inductive (and no equivalent axiomatizations are known which are analytic inductive). Specifically, we established an equivalence between semi De Morgan algebras and a suitable class of heterogeneous algebras, each of which consisting of a distributive lattice and a De Morgan algebra connected by a pair of heterogeneous maps; unlike semi De Morgan algebras, this class is defined by analytic conditions. This equivalence restricts to the subvarieties considered in [31] in a modular way, so that analytic subclasses of heterogeneous algebras correspond to each subvariety. Hence, the (multi-type) logics naturally arising from these classes of heterogeneous algebras are analytic, and therefore the general theory of proper (multi-type) calculi readily applies, making it possible to generate analytic structural rules corresponding to each axiom, and to readily verify the set of basic properties mentioned above.

Further directions As in the previous cases, the successful application of this methodology rests on the possibility of equivalently representing the algebraic semantics of a given logic (or family thereof) as a class of (heterogeneous) algebras defined by analytic conditions. Despite some commonalities, the properties of such an equivalent representation are unique to each case, and it is presently an open problem to characterize classes of non analytic logics for which such an equivalent analytic representation is guaranteed to exist. The case study treated in the present paper can perhaps offer some insights towards a solution to the general problem, since, interestingly, this equivalence result is very similar to the term-equivalence result with which Palma [28] proved that the variety of semi De Morgan algebras is closed under canonical extensions. So perhaps, insights from the theory of term-equivalence in universal algebra can be usefully exploited for proof-theoretic purposes.

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A. Analytic Inductive Inequalities

In the present section, we specialize the definition of *analytic inductive inequalities* (cf. [22]) to the original language $\mathcal{L}$ of semi De Morgan logic (see Section 1.1) and the multi-type language $\mathcal{L}_{\mathrm{MT}}$, in the types L and D (cf. Section 3), and reported below for the reader's convenience.[9]

$$\begin{aligned}\mathsf{SM} \ni A &::= p \mid \neg A \mid \top \mid \bot \mid A \wedge A \mid A \vee A\\ \mathsf{L} \ni A &::= p \mid \Box\alpha \mid \top \mid \bot \mid A \wedge A \mid A \vee A\\ \mathsf{D} \ni \alpha &::= \circ A \mid 1 \mid 0 \mid \sim\alpha \mid \alpha \cup \alpha \mid \alpha \cap \alpha\end{aligned}$$

We will make use of the following auxiliary definition: an *order-type* over $n \in \mathbb{N}$ is an n-tuple $\epsilon \in \{1, \partial\}^n$. For every order type ϵ, we denote its *opposite* order type by ϵ^∂, that is, $\epsilon^\partial(i) = 1$ iff $\epsilon(i) = \partial$ for every $1 \leq i \leq n$. The connectives of the language above are grouped together into the families $\mathcal{F} := \mathcal{F}_{\mathsf{SM}} \cup \mathcal{F}_{\mathsf{L}} \cup \mathcal{F}_{\mathsf{D}} \cup \mathcal{F}_{\mathrm{MT}}$, $\mathcal{G} := \mathcal{G}_{\mathsf{SM}} \cup \mathcal{G}_{\mathsf{L}} \cup \mathcal{G}_{\mathsf{D}} \cup \mathcal{G}_{\mathrm{MT}}$, and $\mathcal{H} := \mathcal{H}_{\mathsf{SM}} \cup \mathcal{H}_{\mathsf{L}} \cup \mathcal{H}_{\mathsf{D}} \cup \mathcal{H}_{\mathrm{MT}}$ defined as follows:

$$\begin{array}{lll}\mathcal{F}_{\mathsf{SM}} := \{\wedge, \top\} & \mathcal{G}_{\mathsf{SM}} := \{\vee, \bot, \neg\} & \mathcal{H}_{\mathsf{SM}} := \varnothing\\ \mathcal{F}_{\mathsf{L}} := \{\wedge, \top\} & \mathcal{G}_{\mathsf{L}} := \{\vee, \bot\} & \mathcal{H}_{\mathsf{L}} := \varnothing\\ \mathcal{F}_{\mathsf{D}} := \{\cap, 1\} & \mathcal{G}_{\mathsf{D}} := \{\cup, 0\} & \mathcal{H}_{\mathsf{D}} := \{\sim\}\\ \mathcal{F}_{\mathrm{MT}} := \varnothing & \mathcal{G}_{\mathrm{MT}} := \{\Box\} & \mathcal{H}_{\mathrm{MT}} := \{\circ\}\end{array}$$

For any $\ell \in \mathcal{F} \cup \mathcal{G} \cup \mathcal{H}$, we let $n_\ell \in \mathbb{N}$ denote the arity of ℓ, and the order-type ϵ_ℓ on n_ℓ indicates whether the ith coordinate of ℓ is positive ($\epsilon_\ell(i) = 1$) or negative ($\epsilon_\ell(i) = \partial$). The order-theoretic motivation for this partition is that the algebraic interpretations of $\mathcal{F}$-connectives (resp. $\mathcal{G}$-connectives), preserve finite joins (resp. meets) in each positive coordinate and reverse finite meets (resp. joins) in each negative coordinate, while the algebraic interpretations of $\mathcal{H}$-connectives, preserve both finite joins and meets in each

[9]Notice that SM and L share the language of the lattice base and hence we use the same notation.

positive coordinate and reverse both finite meets and joins in each negative coordinate.

For any term $s(p_1, \ldots p_n)$, any order type ϵ over n, and any $1 \leq i \leq n$, an ϵ-*critical node* in a signed generation tree of s is a leaf node $+p_i$ with $\epsilon(i) = 1$ or $-p_i$ with $\epsilon(i) = \partial$. An ϵ-*critical branch* in the tree is a branch ending in an ϵ-critical node. For any term $s(p_1, \ldots p_n)$ and any order type ϵ over n, we say that $+s$ (resp. $-s$) *agrees with* ϵ, and write $\epsilon(+s)$ (resp. $\epsilon(-s)$), if every leaf in the signed generation tree of $+s$ (resp. $-s$) is ϵ-critical. We will also write $+s' \prec *s$ (resp. $-s' \prec *s$) to indicate that the subterm s' inherits the positive (resp. negative) sign from the signed generation tree $*s$. Finally, we will write $\epsilon(s') \prec *s$ (resp. $\epsilon^\partial(s') \prec *s$) to indicate that the signed subtree s', with the sign inherited from $*s$, agrees with ϵ (resp. with ϵ^∂).

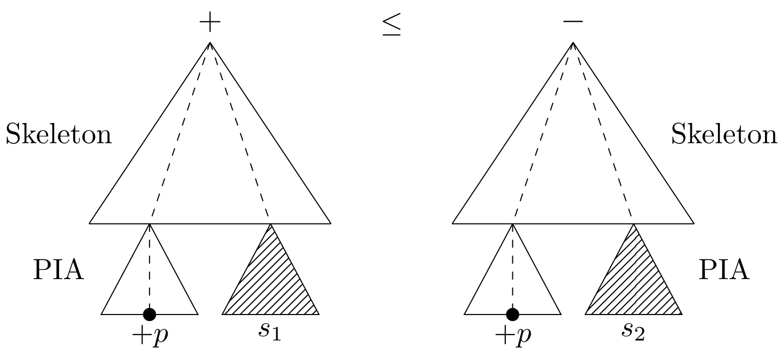
Definition A.1. *(Signed Generation Tree).* The *positive* (resp. *negative*) *generation tree* of any $\mathcal{L}$-term or $\mathcal{L}_{\mathrm{MT}}$-term s is defined by labelling the root node of the generation tree of s with the sign $+$ (resp. $-$), and then propagating the labelling on each remaining node as follows: For any node labelled with $\ell \in \mathcal{F} \cup \mathcal{G} \cup \mathcal{H}$ of arity n_ℓ, and for any $1 \leq i \leq n_\ell$, assign the same (resp. the opposite) sign to its ith child node if $\epsilon_\ell(i) = 1$ (resp. if $\epsilon_\ell(i) = \partial$). Nodes in signed generation trees are *positive* (resp. *negative*) if are signed $+$ (resp. $-$).

Definition 6.1. *(Good branch).* Nodes in signed generation trees will be called Δ-*adjoints*, *syntactically left residual (SLR)*, *syntactically right residual (SRR)*, and *syntactically right adjoint (SRA)*, according to the specification given in Table 2. A branch in a signed generation tree $*s$, with $* \in \{+, -\}$, is called a *good branch* if it is the concatenation of two paths P_1 and P_2, one of which may possibly be of length 0, such that P_1 is a path from the leaf consisting (apart from variable nodes) only of PIA-nodes[10], and P_2 consists (apart from variable nodes) only of Skeleton-nodes.

[10] For an expanded discussion on this definition, see [29, Remark 3.24] and [8, Remark 3.3].

Table 2. Skeleton and PIA nodes

Skeleton	PIA
Δ-adjoints	SRA
$+ \quad \vee \quad \cup$	$+ \quad \wedge \quad \cap \quad \circ \quad \sim \quad \Box \quad \neg$
$- \quad \wedge \quad \cap$	$- \quad \vee \quad \cup \quad \circ \quad \sim$
SLR	SRR
$+ \quad \wedge \quad \cap \quad \circ \quad \sim$	$+ \quad \vee \quad \cup$
$- \quad \vee \quad \cup \quad \circ \quad \sim \quad \Box \quad \neg$	$- \quad \wedge \quad \cap$



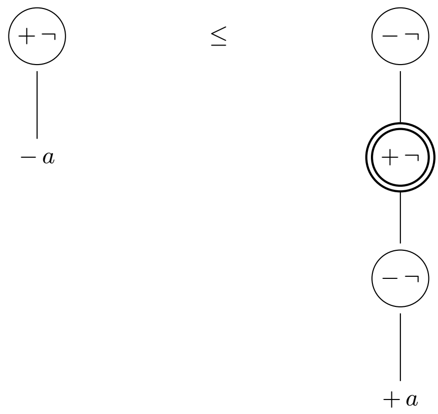
Definition 6.2. *(Analytic inductive inequalities).* For any order type ϵ and any irreflexive and transitive relation $<_\Omega$ on $p_1, \ldots p_n$, the signed generation tree $*s$ ($* \in \{-,+\}$) of an $\mathcal{L}$-term or $\mathcal{L}_{MT}$-term $s(p_1, \ldots p_n)$ is *analytic* (Ω, ϵ)*-inductive* if

1. every branch of $*s$ is good (cf. Definition 6.1);
2. for all $1 \leq i \leq n$, every SRR-node occurring in any ϵ-critical branch with leaf p_i is of the form $\circledast(s, \beta)$ or $\circledast(\beta, s)$, where the critical branch goes through β and
   (a) $\epsilon^\partial(s) \prec *s$ (cf. discussion before Definition 6.1), and
   (b) $p_k <_\Omega p_i$ for every p_k occurring in s and for every $1 \leq k \leq n$.

We will refer to $<_\Omega$ as the *dependency order* on the variables. An inequality $s \leq t$ is *analytic* (Ω, ϵ)*-inductive* if the signed generation trees $+s$ and $-t$ are analytic (Ω, ϵ)-inductive. An inequality $s \leq t$ is *analytic inductive* if is analytic (Ω, ϵ)-inductive for some Ω and ϵ.

In each setting in which they are defined, analytic inductive inequalities are a subclass of inductive inequalities (cf. [22, Definition 16]). In their turn, inductive inequalities are *canonical* (that is, preserved under canonical extensions, as defined in each setting).

It is not difficult to verify that the SM-axioms $\neg A \vdash \neg\neg\neg A$, $\neg\neg\neg A \vdash \neg A$ and $\neg\neg A \wedge \neg\neg B \vdash \neg\neg(A \wedge B)$, the UQM-axiom $\neg\neg A \vdash A$, and the DP-axiom $\neg A \wedge \neg\neg A \vdash \bot$ are not analytic inductive. For instance, consider the following picture, representing the signed generation tree of (the interpretation of) the sequent $\neg A \vdash \neg\neg\neg A$:



In the picture above, the circled variable occurrences are Skeleton nodes and the doubly circled occurrences are PIA nodes (see Table 2). The right-hand side of the inequality is not a good branch because there is a Skeleton node occurring under the scope of a PIA node (see Definition 6.1). Therefore, the (axiom corresponding to) the sequent is not analytic inductive. Similar observations apply to all other the other sequents listed above.

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