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Abstract

This paper continues the investigation of the logic of competing theories (be they scientific, social, political etc.) initiated in [4]. We introduce a many-valued, multi-type modal language which we endow with relational semantics based on enriched reflexive graphs, inspired by Ploščica's representation of general lattices. We axiomatize the resulting many-valued, non-distributive modal logic of these structures and prove a completeness theorem. We illustrate the application of this logic through a case study in which we model competition among interacting political promises and social demands within an arena of political parties social groups.
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1. Introduction

This paper is a continuation of the investigation into competing theories started in [4]. Its technical contributions are rooted in the generalized Sahlqvist canonicity and correspondence for *normal lattice-based logics* [10,9], i.e. non-classical propositional logics for which the distributive laws between ∧ and ∨ do not need to hold. Via algebraic and duality-theoretic techniques, these logics, and non-distributive normal modal logics in particular, have been endowed with complete relational semantics based on *formal contexts* [18] and *reflexive graphs* [3,5]. These semantic structures have a well developed theory, both algebraic and proof-theoretic [20,13,15] and model-theoretic [11], and have facilitated new insights on possible interpretations and use of lattice-based modal logics.

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In particular, via formal context semantics, in [7], the basic non-distributive modal logic and some of its axiomatic extensions are interpreted as *epistemic logics of categories and concepts*, and in [8], the corresponding 'common knowledge'-type construction is used to give an epistemic-logical formalization of the notion of *prototype* of a category; in [6,21], formal context semantics for non-distributive modal logic is proposed as an encompassing framework for the integration of rough set theory [25] and formal concept analysis [18], and in this context, the basic non-distributive modal logic is interpreted as the logic of *rough concepts*; via graph-based semantics, in [5], the same logic is interpreted as the logic of *informational entropy*, i.e. an inherent boundary to knowability due e.g. to perceptual, theoretical, evidential or linguistic limits, and in [4], *many-valued* graph-based semantics is introduced for non-distributive normal modal logic, and its potential is explored as a formal framework for modelling *competing theories* in the empirical sciences.

Both in the crisp and in the many-valued setting, in the graphs (Z, E) on which the relational structures are based, the relation E is interpreted as an *indiscernibility* relation, which makes the present approach similar to that of approximation spaces in rough set theory [25]. However, the key difference is that, rather than generating modal operators which associate any subset of Z with its definable E-approximations, E generates a complete lattice in which the distributivity laws do not need to hold. This lattice is defined as the concept lattice of the formal context (Z, Z, E^c) arising from the graph (Z, E). In the approach proposed in [5,4] and followed in the present paper, concepts are not understood as definable approximations of predicates, but rather they represent 'all there is to know', i.e. the theoretical horizon to knowability, given the inherent boundary encoded into E. Interestingly, E is required to be reflexive but in general neither transitive nor symmetric, which is in line with what observed in the literature in psychology (cf. [27,24]) and business science [17].

In this paper, we start exploring a semantic setting for non-distributive modal logics that is not only *many-valued*, as the setting of [4] is, but unlike [4] is also *multi-type*. The main motivation and starting point of the present contribution is to introduce a formal environment in which to analyse the similarities between the competition among *political theories* (both in their institutional incarnations as political parties, and in their social incarnations as social blocks or groups) and the competition between scientific theories as treated in [4].

In [4], scientific theories are identified with the sets of their relevant variables (e.g. mass, speed, position are relevant variables for gravitation theory); hypotheses formulated in the background of a given theory X establish connections between variables in X and are captured as formulas which can be tested (i.e. evaluated) on different databases (i.e. states of the domain Z of a graph-based model), with a greater or lesser degree of *confidence* in the outcome of the test (captured in the truth-value in the many-valued semantics). Since databases themselves are built according to a given theory ("observations are theory-laden"), the degree of confidence in the outcome of tests is formulated in terms of how compatible the background theory of the given hypothesis is with the theory according to which the given database has been built. Theories compete in the arena of databases by their key hypotheses being tested on different databases. Then the criteria establishing whether theory X outcompetes theory Y need to assign different weights to the performances of hypotheses on databases that have high compatibility with the theories to which each hypothesis pertains, and to the performances of the same hypotheses on databases with low compatibility. In the present paper, we propose the following analogies:

Scientific theories	⤳	Socio-political theories
Variables	⤳	Issues
Hypotheses	⤳	Promises / Demands

Here the "issues" could be things like distribution of wealth, access to education and progressive taxation in the context of e.g. a socialist theory. The main difference between the competition of scientific theories outlined above and that of socio-political theories is that competition among the latter plays out not on a single arena but on at least *two arenas* simultaneously: that is, political parties (incarnating socio-political theories) compete with each other by testing how well their promises (phrased in terms of issues) score on different social groups, while at the same time, social groups (also incarnating socio-political theories) compete with each other by testing how well their demands score on political parties. The double-sidedness of this situation calls for a *multi-type* formal framework, both in respect to the language and the models. However, there is another interesting similarity between the socio-political case and the scientific case: as discussed above, the fact that databases are theory-laden results in different degrees of confidence in the outcomes of tests of different hypotheses, depending on the degree of compatibility between

their underlying theories; likewise, the fact that each social group has an underlying theory (captured by the set of issues which are relevant to that social group) results in different degrees of confidence when the promises of different political parties are tested on different social groups, which again depends on the degree of compatibility between their underlying theories. Conversely and symmetrically, the fact that each political party has an underlying theory results in different degrees of confidence when the demands of different social groups are tested on different political parties, which again depends on the degree of compatibility between their underlying theories.

The paper is organized as follows: Section 2 collects some preliminaries on multi-type non-distributive modal logic and the many-valued enriched formal contexts and many-valued graphs upon which its semantics will be built. Section 3 introduces the notion of a many-valued heterogeneous frame – structures we will use to model sets of political parties and social groups and the relations among them. In Section 4 we define the formal semantics of our logic in terms of many-valued heterogeneous models and formulate a completeness theorem, the proof of which is given in Appendix A. Section 5 presents a case study in which we use a many-valued heterogeneous model to capture and reason about a scenario loosely inspired by the British socio-political scene. We conclude in Section 7 by discussing some potential applications of our work and outlining some further research directions.

2. Preliminaries

This section collects and modifies material from [5, Section 2.1], [6, Section 7.2], and [4, Section 3].

2.1. Multi-type nondistributive modal logic

Let Prop be a (countable or finite) set of proposition variables. The language $\mathscr{L}_{\mathrm{MT}}$ of the *multi-type nondistributive modal logic* has terms of types SD, PP defined as follows:

$$\mathsf{SD} \ni \sigma ::= \bot \mid \top \mid p \mid \sigma \wedge \sigma \mid \sigma \vee \sigma \mid \Diamond\pi,$$

$$\mathsf{PP} \ni \pi ::= \bot \mid \top \mid p \mid \pi \wedge \pi \mid \pi \vee \pi \mid \Diamond\sigma,$$

where $p \in$ Prop. Intuitively, we create two copies of the same language, one in which formulas are intended as *social demands* σ and one as *political promises* π. The two types are connected via heterogeneous modal operators, transforming social demands into political promises and vice versa. The *term-algebra* of this language is an example of *heterogeneous algebra*, a notion introduced by Birkhoff and Lipson [2] naturally extending notions and results from universal algebra to a context in which algebras have more than one domain and operations can be defined not only within one and the same domain, but also between different domains.

Definition 2.1. A *normal heterogeneous* $\mathscr{L}_{\mathrm{MT}}$-*algebra* is a tuple $(\mathbb{L}_S, \mathbb{L}_P, \Diamond, \Diamond)$ such that $\mathbb{L}_S$ and $\mathbb{L}_P$ are lattices (intended to interpret formulas of type SD and PP, respectively), and $\Diamond : \mathbb{L}_S \to \mathbb{L}_P$ and $\Diamond : \mathbb{L}_P \to \mathbb{L}_S$ are normal (i.e. $\bot$-preserving and $\vee$-preserving) modal operators.

The *basic multi-type normal* $\mathscr{L}_{\mathrm{MT}}$-*logic* is a set **L** of *type-uniform* $\mathscr{L}_{\mathrm{MT}}$-sequents $\varphi \vdash \psi$ (i.e. sequents with $\varphi, \psi \in$ SD or $\varphi, \psi \in$ PP), containing the following axioms:

$$
\begin{array}{llll}
p \vdash p, & \bot \vdash p, & p \vdash \top, & \\
p \vdash p \vee q, & q \vdash p \vee q, & p \wedge q \vdash p, & p \wedge q \vdash q, \\
\Diamond\bot \vdash \bot, & \Diamond(\pi_1 \vee \pi_2) \vdash \Diamond\pi_1 \vee \Diamond\pi_2 & & \\
\Diamond\bot \vdash \bot, & \Diamond(\sigma_1 \vee \sigma_2) \vdash \Diamond\sigma_1 \vee \Diamond\sigma_2 & &
\end{array}
$$

and closed under the following inference rules:

$$
\frac{\varphi \vdash \chi \quad \chi \vdash \psi}{\varphi \vdash \psi} \qquad \frac{\varphi \vdash \psi}{\varphi(\chi/p) \vdash \psi(\chi/p)} \qquad \frac{\chi \vdash \varphi \quad \chi \vdash \psi}{\chi \vdash \varphi \wedge \psi} \qquad \frac{\varphi \vdash \chi \quad \psi \vdash \chi}{\varphi \vee \psi \vdash \chi}
$$

$$
\frac{\pi_1 \vdash \pi_2}{\Diamond\pi_1 \vdash \Diamond\pi_2} \qquad \frac{\sigma_1 \vdash \sigma_2}{\Diamond\sigma_1 \vdash \Diamond\sigma_2}
$$

An $\mathscr{L}_{\mathrm{MT}}$-*logic* is any extension of **L** with type-uniform $\mathscr{L}_{\mathrm{MT}}$-sequents $\varphi \vdash \psi$. The next proposition can be shown via a routine Lindenbaum Tarski argument.

Proposition 2.2. *The basic logic* **L** *is sound and complete w.r.t. the class of heterogeneous* $\mathscr{L}_{\mathrm{MT}}$-*algebras.*

2.2. Many-valued enriched formal contexts

A formal context, as used in formal concept analysis [18], consists of a set of objects, a set of features, and a relation between these two sets relating objects to the features they posses. Building on the work of Bělohlávek [1], in this section we introduce a many-valued version of a formal context, also enriched with additional relations. These structures will be an essential ingredient in the semantics of our logic, which we will formally introduce in Sections 3 and 4, below. Throughout this paper, we let $\mathbf{A} = (D, 1, 0, \vee, \wedge, \otimes, \rightarrow)$ denote an arbitrary but fixed complete frame-distributive (finite meets distribute over arbitrary joins) and dually frame-distributive (finite joins distribute over arbitrary meets), commutative and associative residuated lattice (understood as the algebra of truth-values). For every set W, an $\mathbf{A}$-*valued subset* (or $\mathbf{A}$-*subset*) of W is a map $u : W \rightarrow \mathbf{A}$. We let $\mathbf{A}^W$ denote the set of all $\mathbf{A}$-subsets. Clearly, $\mathbf{A}^W$ inherits the algebraic structure of $\mathbf{A}$ by defining the operations and the order pointwise. The $\mathbf{A}$-*subsethood* relation between elements of $\mathbf{A}^W$ is the map $S_W : \mathbf{A}^W \times \mathbf{A}^W \rightarrow \mathbf{A}$ defined as $S_W(f, g) := \bigwedge_{z \in W}(f(z) \rightarrow g(z))$. For every $\alpha \in \mathbf{A}$, let $\{\alpha/w\} : W \rightarrow \mathbf{A}$ be defined by $v \mapsto \alpha$ if $v = w$ and $v \mapsto \perp^{\mathbf{A}}$ if $v \neq w$. Then, for every $f \in \mathbf{A}^W$,

$$f = \bigvee_{w \in W} \{f(w)/w\}. \tag{1}$$

When $u, v : W \rightarrow \mathbf{A}$ and $u \leq v$ w.r.t. the pointwise order, we write $u \subseteq v$. An $\mathbf{A}$-*valued relation* (or $\mathbf{A}$-*relation*) is a map $R : U \times W \rightarrow \mathbf{A}$. Two-valued relations can be regarded as $\mathbf{A}$-relations. In particular for any set Z, we let $\Delta_Z : Z \times Z \rightarrow \mathbf{A}$ be defined by $\Delta_Z(z, z') = \top$ if $z = z'$ and $\Delta_Z(z, z') = \perp$ if $z \neq z'$. An $\mathbf{A}$-relation $R : Z \times Z \rightarrow \mathbf{A}$ is *reflexive* if $\Delta_Z \subseteq R$. Any $\mathbf{A}$-valued relation $R : U \times W \rightarrow \mathbf{A}$ induces maps $R^{(0)}[-] : \mathbf{A}^W \rightarrow \mathbf{A}^U$ and $R^{(1)}[-] : \mathbf{A}^U \rightarrow \mathbf{A}^W$ defined as follows: for every $f : U \rightarrow \mathbf{A}$ and every $u : W \rightarrow \mathbf{A}$,[1]

$$R^{(1)}[f] : \quad W \rightarrow \mathbf{A}$$
$$x \mapsto \bigwedge_{a \in U}(f(a) \rightarrow R(a, x))$$

$$R^{(0)}[u] : \quad U \rightarrow \mathbf{A}$$
$$a \mapsto \bigwedge_{x \in W}(u(x) \rightarrow R(a, x))$$

A *formal* $\mathbf{A}$-*context*[2] or $\mathbf{A}$-*polarity* (cf. [1]) is a structure $\mathbb{P} = (A, X, I)$ such that A and X are sets and $I : A \times X \rightarrow \mathbf{A}$. Any formal $\mathbf{A}$-context induces maps $(\cdot)^{\uparrow} : \mathbf{A}^A \rightarrow \mathbf{A}^X$ and $(\cdot)^{\downarrow} : \mathbf{A}^X \rightarrow \mathbf{A}^A$ given by $(\cdot)^{\uparrow} = I^{(1)}[\cdot]$ and $(\cdot)^{\downarrow} = I^{(0)}[\cdot]$. These maps are such that, for every $f \in \mathbf{A}^A$ and every $u \in \mathbf{A}^X$,

$$S_A(f, u^{\downarrow}) = S_X(u, f^{\uparrow}),$$

that is, the pair of maps $(\cdot)^{\uparrow}$ and $(\cdot)^{\downarrow}$ form an $\mathbf{A}$-*Galois connection*. In [1, Lemma 5], it is shown that every $\mathbf{A}$-Galois connection arises from some formal $\mathbf{A}$-context. A *formal* $\mathbf{A}$-*concept* of $\mathbb{P}$ is a pair $(f, u) \in \mathbf{A}^A \times \mathbf{A}^X$ such that $f^{\uparrow} = u$ and $u^{\downarrow} = f$. It follows immediately from this definition that if (f, u) is a formal $\mathbf{A}$-concept, then $f^{\uparrow\downarrow} = f$ and $u^{\downarrow\uparrow} = u$, that is, f and u are *stable*. The set of formal $\mathbf{A}$-concepts can be partially ordered as follows:

---

[1] Note that, in the two-valued case, f and u are crisp sets and $R^{(1)}[f]$ is simply the set of those elements of W to which *all* elements of f are related, while $R^{(0)}[u]$ is the set of all elements of U which are related to all elements of u.

[2] In the crisp setting, a *formal context* [18], or *polarity*, is a structure $\mathbb{P} = (A, X, I)$ such that A and X are sets, and $I \subseteq A \times X$ is a binary relation. Every such $\mathbb{P}$ induces maps $(\cdot)^{\uparrow} : \mathscr{P}(A) \rightarrow \mathscr{P}(X)$ and $(\cdot)^{\downarrow} : \mathscr{P}(X) \rightarrow \mathscr{P}(A)$, respectively defined by the assignments $B^{\uparrow} := I^{(1)}[B]$ and $Y^{\downarrow} := I^{(0)}[Y]$. A *formal concept* of $\mathbb{P}$ is a pair $c = (\llbracket c \rrbracket, (\! [c] \!))$ such that $\llbracket c \rrbracket \subseteq A$, $(\! [c] \!) \subseteq X$, and $\llbracket c \rrbracket^{\uparrow} = (\! [c] \!)$ and $(\! [c] \!)^{\downarrow} = \llbracket c \rrbracket$. The set $L(\mathbb{P})$ of the formal concepts of $\mathbb{P}$ can be partially ordered as follows: for any $c, d \in L(\mathbb{P})$,

$$c \leq d \quad \text{iff} \quad \llbracket c \rrbracket \subseteq \llbracket d \rrbracket \quad \text{iff} \quad (\! [d] \!) \subseteq (\! [c] \!).$$

With this order, $L(\mathbb{P})$ is a complete lattice, the *concept lattice* $\mathbb{P}^+$ of $\mathbb{P}$. Any complete lattice $\mathbb{L}$ is isomorphic to the concept lattice $\mathbb{P}^+$ of some polarity $\mathbb{P}$.

$$(f, u) \leq (g, v) \quad \text{iff} \quad f \subseteq g \quad \text{iff} \quad v \subseteq u.$$

Ordered in this way, the set of the formal **A**-concepts of $\mathbb{P}$ is a complete lattice, which we denote $\mathbb{P}^+$.

An *enriched formal* **A**-*context* (cf. [6, Section 7.2]) is a structure $\mathbb{F} = (\mathbb{P}, R_\square, R_\lozenge)$ such that $\mathbb{P} = (A, X, I)$ is a formal **A**-context and $R_\square : A \times X \to \mathbf{A}$ and $R_\lozenge : X \times A \to \mathbf{A}$ are *I-compatible*, i.e. $R_\square^{(0)}[\{\alpha/x\}]$, $R_\square^{(1)}[\{\alpha/a\}]$, $R_\lozenge^{(0)}[\{\alpha/a\}]$ and $R_\lozenge^{(1)}[\{\alpha/x\}]$ are stable for every $\alpha \in \mathbf{A}$, $a \in A$ and $x \in X$. The *complex algebra* of an enriched formal **A**-context $\mathbb{F} = (\mathbb{P}, R_\square, R_\lozenge)$ is the algebra $\mathbb{F}^+ = (\mathbb{P}^+, [R_\square], \langle R_\lozenge \rangle)$ where $[R_\square], \langle R_\lozenge \rangle : \mathbb{P}^+ \to \mathbb{P}^+$ are defined by the following assignments: for every $c = (\llbracket c \rrbracket, (\!(c)\!)) \in \mathbb{P}^+$,

$$[R_\square]c = (R_\square^{(0)}[(\!(c)\!)], (R_\square^{(0)}[(\!(c)\!)])^\uparrow)$$
$$\langle R_\lozenge \rangle c = ((R_\lozenge^{(0)}[\llbracket c \rrbracket])^\downarrow, R_\lozenge^{(0)}[\llbracket c \rrbracket]).$$

Lemma 2.3. *(cf. [6, Lemma 15]) If $\mathbb{F} = (\mathbb{X}, R_\square, R_\lozenge)$ is an enriched formal* **A**-*context, $\mathbb{F}^+ = (\mathbb{X}^+, [R_\square], \langle R_\lozenge \rangle)$ is a complete normal lattice expansion such that $[R_\square]$ is completely meet-preserving and $\langle R_\lozenge \rangle$ is completely join-preserving.*

2.3. Many-valued graphs

Graph-based models for non-distributive logics are closely connected with the topological structures used by Ploščica [26] to represent general lattices, see also [16,14]. In [5] these models are studied with the TiRS conditions of [14, Section 2] dropped, and many-valued graph-based models are introduced in [4].

A reflexive **A**-*graph* is a structure $\mathbb{X} = (Z, E)$ such that Z is a nonempty set and E is a *reflexive* **A**-*relation*, i.e. $E : Z \times Z \to \mathbf{A}$ and $R(z, z) = \top^\mathbf{A}$ for every $z \in Z$. From now on, we will assume that all **A**-graphs we consider are reflexive even when we drop the adjective.

In what follows, for any set S we let $S_A := \mathbf{A} \times S$ and $S_X := S$. Any $R : S \times W \to \mathbf{A}$ admits the following liftings[3]:

$$I_R : \quad S_A \times W_X \to \mathbf{A}$$
$$((\alpha, s), w) \mapsto R(s, w) \to \alpha$$
$$J_R : \quad S_X \times W_A \to \mathbf{A}$$
$$(s, (\alpha, w)) \mapsto R(s, w) \to \alpha$$

Applying these constructions to **A**-graphs we get:

Definition 2.4. For any reflexive **A**-graph $\mathbb{X} = (Z, E)$, the formal **A**-context associated with $\mathbb{X}$ is

$$\mathbb{P}_\mathbb{X} := (Z_A, Z_X, I_E),$$

where $Z_A := \mathbf{A} \times Z$ and $Z_X := Z$, and $I_E : Z_A \times Z_X \to \mathbf{A}$ is defined by $I_E((\alpha, z), z') = E(z, z') \to \alpha$. We let $\mathbb{X}^+ := \mathbb{P}_\mathbb{X}^+$.

For all $f : \mathbf{A} \times Z \to \mathbf{A}$, and $u : Z \to \mathbf{A}$, we let

$$u^{[0]} = E^{[0]}[u] : \quad \mathbf{A} \times Z \to \mathbf{A}$$
$$(\alpha, z) \mapsto I_E^{(0)}[u](\alpha, z) = u^\downarrow(\alpha, z)$$
$$f^{[1]} = E^{[1]}[f] : \quad Z \to \mathbf{A}$$
$$z \mapsto I_E^{(1)}[f](z) = f^\uparrow(z)$$

where the maps[4] $f^\uparrow : Z \to \mathbf{A}$ and $u^\downarrow : \mathbf{A} \times Z \to \mathbf{A}$ are respectively defined by the assignments

---

[3] In the crisp case, S_A consists of two copies of S, W_X is simply W, and I_R is a relation between these two sets which relates every element of the '1-copy' of S to every element of W, and relates an element s of the '0-copy' of S to an element $w \in W$ iff R does *not* relate s to w.

[4] We will abbreviate $E^{[0]}[u]$ and $E^{[1]}[f]$ as $u^{[0]}$ and $f^{[1]}$, respectively, for each u, f as above, and write $u^{[01]}$ and $f^{[10]}$ for $(u^{[0]})^{[1]}$ and $(f^{[1]})^{[0]}$, respectively. Then $u^{[0]} = I_E^{(0)}[u] = u^\downarrow$ and $f^{[1]} = I_E^{(1)}[f] = f^\uparrow$, where the maps $(\cdot)^\downarrow$ and $(\cdot)^\uparrow$ are those associated with the polarity $\mathbb{P}_\mathbb{X}$.

$$z \mapsto \bigwedge_{(\alpha, z') \in Z_A} [f(\alpha, z') \to (E(z', z) \to \alpha)]$$

$$(\alpha, z) \mapsto \bigwedge_{z' \in Z_X} [u(z') \to (E(z, z') \to \alpha)].$$

Hence, for any $z \in Z$ and $\alpha \in \mathbf{A}$,

$$E^{[0]}[u](\alpha, z) := \bigwedge_{z' \in Z_X} [u(z') \to (E(z, z') \to \alpha)]$$
$$E^{[1]}[f](z) := \bigwedge_{(\alpha, z') \in Z_A} [f(\alpha, z') \to (E(z', z) \to \alpha)].$$

3. Many-valued heterogeneous frames

We will now define the many-valued relational structures upon which the semantics of $\mathscr{L}_{\mathrm{MT}}$ will be built in the next section. Intuitively, these structures consist of a set Z^P of political parties together with an $\mathbf{A}$-relation E_P of similarity among them, together forming a reflexive $\mathbf{A}$-graph $\mathbb{X}_P$; a second reflexive $\mathbf{A}$-graph, $\mathbb{X}_S$, consisting of a set of social groups Z^S and an $\mathbf{A}$-relation E_S encoding similarity among them. These two reflexive $\mathbf{A}$-graphs are then linked by two further $\mathbf{A}$-relations: $R_\Diamond$ encodes the affinity that social groups have for political parties, while $R_\diamond$ records the affinities in the opposite direction. The relations $R_\Diamond$ and $R_\diamond$ are required to satisfy certain compatibility conditions, the purpose of which is to ensure that the operations $R_\Diamond^{[0]}$, $R_\Diamond^{[1]}$, $R_\diamond^{[0]}$ and $R_\diamond^{[0]}$ which are introduced below and upon which the semantics of the modalities $\Diamond$ and $\diamond$ is built, produce stable sets when applied to stable sets.

We formulate the appropriate notion of complex algebra to accompany this type of frame. As in classical modal logic, the carrier of the complex algebra consists of all subsets of the frame that can be legitimately assigned as valuations to proposition letters: in the classical case all subsets of the domain are legitimate targets of a valuation and hence the complex algebra is the Boolean algebra of all subsets together with operations corresponding to the modalities; in the present setting the complex algebra is the lattice of all stable sets together with the operations interpreting the modalities.

Definition 3.1. If $\mathscr{L}_{\mathrm{MT}}$ denotes the multi-type language defined in Section 2.1, a *many-valued graph-based $\mathscr{L}_{\mathrm{MT}}$-frame* (abbreviated as *heterogeneous $\mathbf{A}$-frame*) is a structure $\mathbb{G} = (\mathbb{X}_S, \mathbb{X}_P, R_\diamond, R_\Diamond)$ such that $\mathbb{X}_S = (Z^S, E_S)$ and $\mathbb{X}_P = (Z^P, E_P)$ are reflexive $\mathbf{A}$-graphs, and $R_\diamond : Z^S \times Z^P \to \mathbf{A}$ and $R_\Diamond : Z^P \times Z^S \to \mathbf{A}$ satisfy the following *compatibility* conditions[5]: for any $z \in Z^S$, $z' \in Z^P$ and $\alpha, \beta \in \mathbf{A}$,

$$\begin{aligned}
(R_\diamond^{[0]}[\{\beta/(\alpha, z')\}])^{[01]} &\subseteq R_\diamond^{[0]}[\{\beta/(\alpha, z')\}] \\
(R_\diamond^{[1]}[\{\beta/z'\}])^{[10]} &\subseteq R_\diamond^{[1]}[\{\beta/z'\}] \\
(R_\Diamond^{[0]}[\{\beta/(\alpha, z)\}])^{[01]} &\subseteq R_\Diamond^{[0]}[\{\beta/(\alpha, z)\}] \\
(R_\Diamond^{[1]}[\{\beta/z\}])^{[10]} &\subseteq R_\Diamond^{[1]}[\{\beta/z\}],
\end{aligned}$$

where for all $f : Z_A^S \to \mathbf{A}$ and $u : Z_X^P \to \mathbf{A}$,

$$R_\diamond^{[0]}[f]: \quad Z_X^P \to \mathbf{A}$$
$$z \mapsto J_{R_\diamond}^{(0)}[f](z)$$

$$R_\diamond^{[1]}[u]: \quad Z_A^S \to \mathbf{A}$$
$$(\alpha, z) \mapsto J_{R_\diamond}^{(1)}[u](\alpha, z),$$

and for all $f : Z_A^P \to \mathbf{A}$ and $u : Z_X^S \to \mathbf{A}$,

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[5] In what follows, we drop the indices whenever a property, notion or construction applies verbatim to both domains or when disambiguation can be achieved with other means. For instance, symbols such as $(\cdot)^{[0]}$ and $(\cdot)^{[1]}$ will never occur with indices, since the type of the argument is enough to disambiguate them.

$$R_\Diamond^{[0]}[f]: \quad Z_X^S \to \mathbf{A}$$
$$z \mapsto J_{R_\Diamond}^{(0)}[f](z)$$

$$R_\Diamond^{[1]}[u]: \quad Z_A^P \to \mathbf{A}$$
$$(\alpha, z) \mapsto J_{R_\Diamond}^{(1)}[u](\alpha, z).$$

Hence, for any $z \in Z^P$, $w \in Z^S$ and $\alpha \in \mathbf{A}$,

$$R_\Diamond^{[0]}[f](z) := \bigwedge\nolimits_{(\alpha,z') \in Z_A^S} [f(\alpha, z') \to (R_\Diamond(z, z') \to \alpha)]$$
$$R_\Diamond^{[1]}[u](\alpha, w) := \bigwedge\nolimits_{z' \in Z_X^P} [u(z') \to (R_\Diamond(z', w) \to \alpha)],$$

and for any $z \in Z^S$, $w \in Z^P$ and $\alpha \in \mathbf{A}$,

$$R_\Diamond^{[0]}[f](z) := \bigwedge\nolimits_{(\alpha,z') \in Z_A^P} [f(\alpha, z') \to (R_\Diamond(z, z') \to \alpha)]$$
$$R_\Diamond^{[1]}[u](\alpha, w) := \bigwedge\nolimits_{z' \in Z_X^S} [u(z') \to (R_\Diamond(z', w) \to \alpha)].$$

The *complex algebra* of a heterogeneous $\mathbf{A}$-frame $\mathbb{G}$ as above is the heterogeneous algebra $\mathbb{G}^+ = (\mathbb{X}_S^+, \mathbb{X}_P^+, \langle R_\Diamond \rangle,$ $\langle R_\Diamond \rangle)$, where $\mathbb{X}_S^+ := \mathbb{P}_{\mathbb{X}_S}^+$ and $\mathbb{X}_P^+ := \mathbb{P}_{\mathbb{X}_P}^+$ (cf. Definition 2.4), and $\langle R_\Diamond \rangle : \mathbb{X}_P^+ \to \mathbb{X}_S^+$ and $\langle R_\Diamond \rangle : \mathbb{X}_S^+ \to \mathbb{X}_P^+$ are heterogeneous operations of $\mathbb{G}^+$ defined as follows: for every $c = ([\![c]\!], (\![c]\!]) \in \mathbb{X}_S^+$ and $d = ([\![d]\!], (\![d]\!]) \in \mathbb{X}_P^+$,

$$\langle R_\Diamond \rangle c = ((R_\Diamond^{[0]}[[\![c]\!]])^{[0]}, R_\Diamond^{[0]}[[\![c]\!]])$$
$$\langle R_\Diamond \rangle d = ((R_\Diamond^{[0]}[[\![d]\!]])^{[0]}, R_\Diamond^{[0]}[[\![d]\!]]).$$

With a proof analogous to that of Lemma 2.3, one can readily show that

Lemma 3.2. *If* $\mathbb{G} = (\mathbb{X}_S, \mathbb{X}_P, R_\Diamond, R_\Diamond)$ *is a heterogeneous* $\mathbf{A}$-*frame,* $\mathbb{G}^+ = (\mathbb{X}_S^+, \mathbb{X}_P^+, \langle R_\Diamond \rangle, \langle R_\Diamond \rangle)$ *is such that* $\mathbb{X}_S^+$ *and* $\mathbb{X}_P^+$ *are complete lattices, and* $\langle R_\Diamond \rangle$ *and* $\langle R_\Diamond \rangle$ *are completely join-preserving.*

4. Many-valued heterogeneous models

We are now finally ready to formulate the semantics of $\mathscr{L}_{\mathrm{MT}}$. We begin by adding valuations to the frames introduced above in Section 3 to obtain models.

Definition 4.1. A *graph-based* $\mathbf{A}$-*model* of $\mathscr{L}_{\mathrm{MT}}$ is a tuple $\mathbb{M} = (\mathbb{G}, V)$ such that $\mathbb{G} = (\mathbb{X}_S, \mathbb{X}_P, R_\Diamond, R_\Diamond)$ is a heterogeneous $\mathbf{A}$-frame, and $V : \mathscr{L} \to \mathbb{G}^+$ is a homomorphism of heterogeneous algebras, i.e. V is a pair of homomorphisms[6] $V_S : \mathsf{SD} \to \mathbb{X}_P^+$ and $V_P : \mathsf{PP} \to \mathbb{X}_S^+$. For every $\varphi \in \mathscr{L}_{\mathrm{MT}}$, let $V(\varphi) := ([\![\varphi]\!], (\![\varphi]\!])$, where $[\![\varphi]\!] : \mathbf{A} \times Z \to \mathbf{A}$ and $(\![\varphi]\!] : Z \to \mathbf{A}$, with Z being the domain of the appropriate type, are s.t. $[\![\varphi]\!]^{[1]} = (\![\varphi]\!]$ and $(\![\varphi]\!]^{[0]} = [\![\varphi]\!]$. Hence:

$$V(p) = ([\![p]\!], (\![p]\!])$$
$$V(\top) = (1^{\mathbf{A}^{Z_A}}, (1^{\mathbf{A}^{Z_A}})^{[1]})$$
$$V(\bot) = ((1^{\mathbf{A}^{Z_X}})^{[0]}, 1^{\mathbf{A}^{Z_X}})$$
$$V(\varphi \wedge \psi) = ([\![\varphi]\!] \wedge [\![\psi]\!], ([\![\varphi]\!] \wedge [\![\psi]\!])^{[1]})$$
$$V(\varphi \vee \psi) = (((\![\varphi]\!] \wedge (\![\psi]\!])^{[0]}, (\![\varphi]\!] \wedge (\![\psi]\!])$$

---

[6] Notice the inversion: formulas of type SD (social demands) are evaluated (tested) on the P-side of the model, i.e. on political parties, and conversely, political promises are evaluated on social groups. Hence, the complex algebra $\mathbb{G}^+ = (\mathbb{X}_S^+, \mathbb{X}_P^+, \langle R_\Diamond \rangle, \langle R_\Diamond \rangle)$ of the underlying frame of a model is a heterogeneous $\mathscr{L}_{\mathrm{MT}}$-algebra in the sense that $\mathbb{L}_P := \mathbb{X}_S^+$ and $\mathbb{L}_S := \mathbb{X}_P^+$, and moreover $\Diamond := \langle R_\Diamond \rangle : \mathbb{L}_P \to \mathbb{L}_S$ and $\Diamond := \langle R_\Diamond \rangle : \mathbb{L}_S \to \mathbb{L}_P$.

$$V(\Diamond\sigma) = ((R_{\Diamond}^{[0]}[[\![\sigma]\!]])^{[0]}, R_{\Diamond}^{[0]}[[\![\sigma]\!]]).$$

$$V(\Diamond\pi) = ((R_{\Diamond}^{[0]}[[\![\pi]\!]])^{[0]}, R_{\Diamond}^{[0]}[[\![\pi]\!]]).$$

Valuations induce α-*support relations* between value-state pairs and formulas of the appropriate type for each $\alpha \in \mathbf{A}$ (in symbols: $\mathbb{M}, (\beta, z) \Vdash^{\alpha} \varphi$), and α-*refutation relations* between states of models and formulas for each $\alpha \in \mathbf{A}$ (in symbols: $\mathbb{M}, z \succ^{\alpha} \varphi$) such that for every $\varphi \in \mathscr{L}_{\mathrm{MT}}$, all $z \in Z$ and all $\beta \in \mathbf{A}$,

$$\mathbb{M}, (\beta, z) \Vdash^{\alpha} \varphi \quad \text{iff} \quad \alpha \le [\![\varphi]\!](\beta, z),$$

$$\mathbb{M}, z \succ^{\alpha} \varphi \quad \text{iff} \quad \alpha \le ([\![\varphi]\!])(z).$$

This can be equivalently expressed as follows:

$$\mathbb{M}, (\beta, z) \Vdash^{\alpha} p \quad \text{iff} \quad \alpha \le [\![p]\!](\beta, z);$$

$$\mathbb{M}, (\beta, z) \Vdash^{\alpha} \top \quad \text{iff} \quad \alpha \le 1^{\mathbf{A}^{Z_A}}(\beta, z) \text{ i.e. always;}$$

$$\mathbb{M}, (\beta, z) \Vdash^{\alpha} \bot \quad \text{iff} \quad \alpha \le (1^{\mathbf{A}^{Z_X}})^{[0]}(\beta, z)$$
$$= \bigwedge_{z' \in Z_X}[1^{\mathbf{A}^{Z_X}}(z') \to (E(z, z') \to \beta)]$$
$$= \bigwedge_{z' \in Z_X}[E(z, z') \to \beta]$$
$$= \beta;$$

$$\mathbb{M}, (\beta, z) \Vdash^{\alpha} \varphi \wedge \psi \quad \text{iff} \quad \mathbb{M}, (\beta, z) \Vdash^{\alpha} \varphi \text{ and } \mathbb{M}, (\beta, z) \Vdash^{\alpha} \psi;$$

$$\mathbb{M}, (\beta, z) \Vdash^{\alpha} \varphi \vee \psi \quad \text{iff} \quad \alpha \le (([\![\varphi]\!]) \wedge ([\![\psi]\!]))^{[0]}(\beta, z)$$
$$= \bigwedge_{z' \in Z_X}[(([\![\varphi]\!]) \wedge ([\![\psi]\!]))(z') \to (E(z, z')$$
$$\to \beta)];$$

$$\mathbb{M}, (\beta, z) \Vdash^{\alpha} \Diamond\sigma \quad \text{iff} \quad \alpha \le ((R_{\Diamond}^{[0]}[[\![\sigma]\!]])^{[0]})(\beta, z)$$
$$= \bigwedge_{z' \in Z_X^S}[R_{\Diamond}^{[0]}[[\![\sigma]\!]](z') \to (E_S(z, z') \to \beta)];$$

$$\mathbb{M}, (\beta, z) \Vdash^{\alpha} \Diamond\pi \quad \text{iff} \quad \alpha \le ((R_{\Diamond}^{[0]}[[\![\pi]\!]])^{[0]})(\beta, z)$$
$$= \bigwedge_{z' \in Z_X^P}[R_{\Diamond}^{[0]}[[\![\pi]\!]](z') \to (E_P(z, z') \to \beta)];$$

$$\mathbb{M}, z \succ^{\alpha} p \quad \text{iff} \quad \alpha \le ([\![p]\!])(z);$$

$$\mathbb{M}, z \succ^{\alpha} \bot \quad \text{iff} \quad \alpha \le 1^{\mathbf{A}^{Z_X}}(z) \text{ i.e. always;}$$

$$\mathbb{M}, z \succ^{\alpha} \top \quad \text{iff} \quad \alpha \le (1^{\mathbf{A}^{Z_A}})^{[1]}(z)$$
$$= \bigwedge_{(\beta, z') \in Z_A}[1(\beta, z') \to (E(z', z) \to \beta)]$$
$$= \bigwedge_{(\beta, z') \in Z_A}[E(z', z) \to \beta]$$
$$= \beta;$$

$$\mathbb{M}, z \succ^{\alpha} \varphi \vee \psi \quad \text{iff} \quad \mathbb{M}, z \succ^{\alpha} \varphi \text{ and } \mathbb{M}, z \succ^{\alpha} \psi;$$

$$\mathbb{M}, z \succ^{\alpha} \varphi \wedge \psi \quad \text{iff} \quad \alpha \le ([\![\varphi]\!] \wedge [\![\psi]\!])^{[1]}(z)$$
$$= \bigwedge_{(\beta, z') \in Z_A}[([\![\varphi]\!] \wedge [\![\psi]\!])(\beta, z') \to (E(z', z)$$
$$\to \beta)];$$

$$\mathbb{M}, z \succ^{\alpha} \Diamond\sigma \quad \text{iff} \quad \alpha \le (R_{\Diamond}^{[0]}[[\![\sigma]\!]])(z)$$
$$= \bigwedge_{(\beta, z') \in Z_A^P}[[\![\sigma]\!](\beta, z') \to (R_{\Diamond}(z, z') \to \beta)];$$

$$\mathbb{M}, z \succ^{\alpha} \Diamond\pi \quad \text{iff} \quad \alpha \le (R_{\Diamond}^{[0]}[[\![\pi]\!]])(z)$$
$$= \bigwedge_{(\beta, z') \in Z_A^S}[[\![\pi]\!](\beta, z') \to (R_{\Diamond}(z, z') \to \beta)].$$

Definition 4.2. A type-uniform sequent $\varphi \vdash \psi$ is *true* in a model $\mathbb{M} = (\mathbb{G}, V)$ (notation: $\mathbb{M} \models \varphi \vdash \psi$) if $[\![\varphi]\!] \subseteq [\![\psi]\!]$, or equivalently, if $([\![\psi]\!]) \subseteq ([\![\varphi]\!])$. A type-uniform sequent $\varphi \vdash \psi$ is *valid on a graph-based frame* $\mathbb{G}$ (notation: $\mathbb{G} \models \varphi \vdash \psi$) if $\varphi \vdash \psi$ is true in every model $\mathbb{M} = (\mathbb{G}, V)$ based on $\mathbb{G}$.

Remark 4.3. As remarked in [4], it is not difficult to see that for all stable valuations, if $p \in$ Prop and $\beta, \beta' \in \mathbf{A}$ such that $\beta \leq \beta'$, then $[\![p]\!](\beta, z) \leq [\![p]\!](\beta', z)$ for every $z \in Z$, and one can readily verify that this condition extends compositionally to every $\varphi \in \mathscr{L}$.

The basic multi-type logic **L** which we introduced in Section 2.1 is complete with respect to the class of all graph-based **A**-models of $\mathscr{L}_{\mathrm{MT}}$. The proof of the following theorem is given in Appendix A, where it is listed as Theorem A.8:

Theorem 4.4. *The basic multi-type normal* $\mathscr{L}_{\mathrm{MT}}$-*logic* **L** *is sound and complete w.r.t. the class of graph-based* **A**-*frames.*

5. Case study: the socio-political arena

Let Var be a nonempty set of variables (intended to represent *topics* or *issues*, as in e.g. [19]). As was done in [4], for the purpose of this analysis, a socio-political theory is characterized by (and here identified with) a certain subset $X \subseteq$ Var of issues which are *relevant* to the given theory. The heterogeneous **A**-frames considered in the present section are structures $\mathbb{G} = (\mathbb{X}_S, \mathbb{X}_P, R_\diamond, R_\diamondsuit)$ (cf. Definition 3.1) such that $\mathbb{X}_S := (Z^S, E_S)$ with $Z^S := \{z_{X_i} \mid 0 \leq i \leq n\}$ where $X_i \subseteq$ Var for each $1 \leq i \leq n$ and $\mathbb{X}_P := (Z^P, E_P)$ with $Z^P := \{z_{X_j} \mid 0 \leq i \leq m\}$, again with $X_j \subseteq$ Var for each $1 \leq j \leq m$. The elements of the sets Z^S and Z^P stand for *social groups* and *political parties*, respectively. The set of variables indexing each social group in Z^S (resp. each political party in Z^P) stand for the issues considered relevant by that social group or political party. Sometimes, it can be useful to encode the positive or negative orientation of the group/party towards each relevant issue by assigning a *sign* ($+$ or $-$) to each element of the indexing set X_i or X_j.[7] In this context, SD-formulas (resp. PP-formulas) can be thought of as *social demands* (resp. *political promises*) which will be 'tested' (i.e. will be assigned truth-degrees) at states of Z_P (resp. Z_S), i.e. at political parties (resp. social groups) in models based on these frames. Notice the inversion: SD-formulas will be evaluated at $\mathbb{X}_P$-states, and PP-formulas at $\mathbb{X}_S$-states. This truth value assignment of formulas at states is then meant to act as a proxy for the support (or interest) of the given social group in the given political promise, and of the support (or interest) of the political party in the given social demand, with higher truth values indicating higher levels of support/interest.

The **A**-relation $E_S : Z^S \times Z^S \to \mathbf{A}$ (resp. $E_P : Z^P \times Z^P \to \mathbf{A}$) encodes a graded notion of *similarity* between social groups (resp. political parties). This idea can be concretely implemented e.g. by letting $E(z_{X_1}, z_{X_2})$ record the percentage of variables of z_{X_1} that also occur in z_{X_2}, i.e. by taking $E(z_{X_1}, z_{X_2}) = |X_1 \cap X_2| \div |X_1|$, possibly modulo identification of similar issues.[8] As remarked in [4], a relation defined accordingly will be reflexive (i.e. $E(z, z) = 1$ for every $z \in Z$) but does not need to be symmetric or transitive; moreover, it is not required to record the positive or negative attitudes vis-à-vis an issue, so as to not exclude the possibility that social groups (resp. parties) with directly opposing views on a large percentage of issues have a high similarity degree.

The **A**-relation $R_\diamond : Z^P \times Z^S \to \mathbf{A}$ (resp. $R_\diamondsuit : Z^S \times Z^P \to \mathbf{A}$) encodes the extent to which a political party (resp. social group) has *affinity* with a social group (resp. political party). One would expect that such a measure should be based on the extent to which the political party (resp. social group) perceives its issues to be issues of the social group (resp. political party). This idea can be concretely implemented e.g. as follows: Let $X_P \subseteq$ Var (resp. $X_S \subseteq$ Var) be the set of all issues of political parties (resp. social groups). Encode the extent to which a political party z_{X_j}

---

[7] Of course, a sign is often not enough to achieve a full disambiguation; however, for the sake of the example below, what will matter is whether a given party and social group assign the same or opposite sign to a given issue relevant to both.

[8] In this paper, we are not actually committing to a specific definition of the similarity relations, although the one above naturally arises from the present formal framework and will be employed in the case study we present.

recognizes each of its issues in an issue of a social group, using a *recognition function* $f_{z_{X_j}} : X_j \times X_S \to [0, 1]$. We then set $R_\diamond(z_{X_j}, z_{X_i}) = \sum \{f(x, y) \mid (x, y) \in X_j \times X_i\} \div |\{(x, y) \in X_j \times X_i \mid f(x, y) \neq 0\}|$. Recognition functions for social groups and the resulting definition of $R_\diamond$ are analogous. More nuanced realisations might, among other considerations, also include a weighting to account for the relative importance of issues to the political parties or groups. Notice that we are *not* requiring, because it would be implausible, that $R_\diamond(z_{X_j}, z_{X_i}) = R_\diamond(z_{X_i}, z_{X_j})$ for all z_{X_j} and z_{X_i}. Below, we give a more concrete illustration of this environment by means of an example loosely inspired by the British socio-political scene.

Let $\mathsf{Var} := \{st, o, lt, ap, ft, cr, it, fs, h, at, s\}$ be the set of issues, where the intended meaning of each variable is indicated below:

st	lower income **t**ax on **s**alaries
o	foreigners **o**ut
lt	lower **t**axes on income generated from **l**and
ap	preservation of **a**ristocratic **p**rivileges
ft	lower **f**inancial transactions **t**ax
cl	harmonization of European **c**orporation **l**aw
it	progressive **i**ncome **t**ax
fs	higher tax on **f**oreign **s**tocks flotation on the London stock exchange
h	fox **h**unting
at	lower **t**ax on **a**gricultural sector
s	national **s**overeignty
ur	reduced rights for **u**nion **r**epresentatives in factories
ds	tax **d**eductions for **s**avings of lower income workers
pd	return to the **p**re-**d**ecimal currency system

Let $Z_S := \{z_F, z_D, z_B\}$ and $Z_P := \{z_L, z_C, z_X\}$, where

z_F	**F**actory workers in Manchester	$F := \{+st, +o\}$
z_D	Extended family of **D**uke of Westminster	$D := \{+lt, +ap\}$
z_B	London City **B**ankers	$B := \{+ft, -cl\}$
z_L	**L**abour party	$L := \{+it, +fs, -h\}$
z_C	**C**onservative party	$C := \{+at, +h, +ur\}$
z_X	Bre**x**it party	$X := \{+s, +ds, +pd\}$

To calculate the similarity between political parties, we need to compare their positions in terms of issues. However, since there is ostensibly little overlap in the issues *as formulated*, we will impose an equivalence relation $\sim_P$ to abstract the broad *kinds of issues* and use that for the comparison. Suppose the equivalence classes of $\sim_P$ are given by grouping income tax issues together and 'heritage issues' together, while keeping other issues separate, as follows:

$$\{\{it, at, ds\}, \{fs\}, \{h, pd\}, \{s\}, \{ur\}\}.$$

In the same way, to calculate the similarity between social groups, we impose an equivalence relation $\sim_S$ on their issues, equating tax issues while distinguishing other issues:

$$\{\{st, lt, ft\}, \{o\}, \{ap\}, \{cl\}\}.$$

The equivalence class of an issue i under $\sim_P$ (respectively, $\sim_S$) is denoted by $[i]_P$ (respectively, $[i]_S$).

For the purposes of this case study, we choose to take as truth-value algebra the 11-element Łukasiewicz chain **A** with domain $\{0, 0.1, 0.2, \ldots, 1\}$. The reasons for this choice are discussed in Remark 5.1, below. The similarity relations between Z^S and Z^P takes values in **A** as indicated in the following diagram:
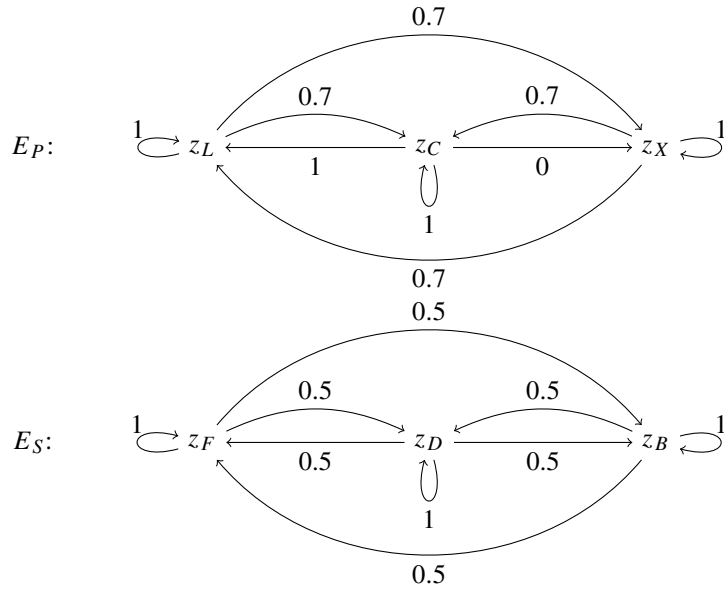
The values of these relations are calculated according to the formula given above, with rounding as necessary. For example, $E_S(z_F, z_D) = |\{[st]_S, [o]_S\} \cap \{[lt]_S, [h]_S\}| \div |\{[st]_S, [o]_S\}| = |\{[st]_S\}| \div |\{[st]_S, [o]_S\}| = 0.5$. In order to define the relation $R_\Diamond : Z^S \times Z^P \to \mathbf{A}$, we will use the following recognition functions:

f_F	fs	it	h	at	s	ur	ds	pd
st	0	0.9	0	0.2	0	0	0.2	0
o	0.5	0	0	0	0.4	0	0	0

f_D	fs	it	h	at	s	ur	ds	pd
lt	0	0.2	0	0.6	0	0	0	0
ap	0	0	0.8	0	0.2	0	0	0

f_B	fs	it	h	at	s	ur	ds	pd
ft	0	0	0	0.4	0	0	0.3	0
cl	0.3	0	0	0	0.5	0	0	0

This enables us to calculate, e.g. $R_\Diamond(z_F, z_L) = [(f_F(st, it) + f_F(o, fs)) \div |\{f_F(st, it), f_F(o, st)\}|] = (0.9 + 0.5) \div 2 = 0.7$. The complete relation $R_\Diamond(z_F, z_L)$ is depicted on the following figure:
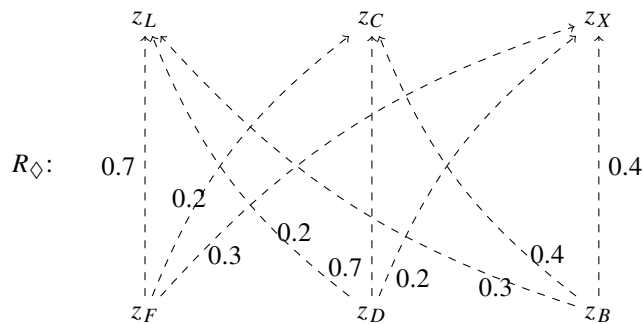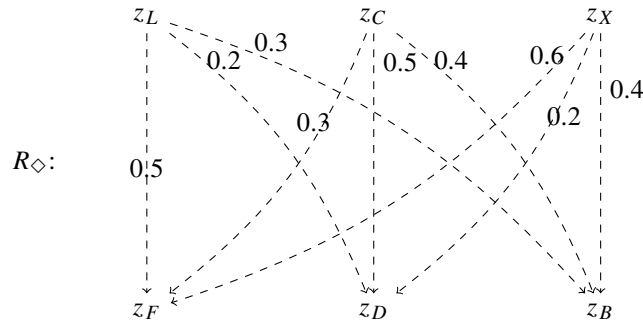

The recognition function of the political parties are given by:

f_L	st	o	lt	ap	ft	cl
fs	0	0.2	0	0	0	0.3
it	0.8	0	0.2	0	0	0
h	0	0	0	0	0	0

f_C	st	o	lt	ap	ft	cl
at	0.3	0	0.4	0	0	0
h	0	0	0	0.6	0	0
ur	0	0	0	0	0	0.4

f_X	st	o	lt	ap	ft	cl
s	0	0.8	0	0.2	0	0.5
ds	0.4	0	0	0	0.3	0
pd	0	0	0	0.2	0	0

Based on these recognition functions, the relation $R_\diamond : Z^P \times Z^S \to \mathbf{A}$ is calculated, and given the figure below:

Notice that e.g. z_L has a *lower* degree of affinity to z_F than z_F has to z_L; this difference is due to the asymmetry between the way in which the Labour party recognises its issues in the issues of the factory workers and the way in which factory workers recognise theirs in those of the Labour party.

Let $\sigma_F, \sigma_D, \sigma_B \in \mathsf{SD}$ and $\pi_L, \pi_C, \pi_U \in \mathsf{PP}$ respectively represent the following social demands and political promises:

$\sigma_F(st, o)$	lower taxes on salaries by cutting social benefits for foreigners
$\sigma_D(lt, ap)$	right for the house of lords to veto laws on grounds of national interest
$\sigma_B(ft, cl)$	UK companies are allowed to pay the salaries of their employees only through UK banks
$\pi_L(it, fs, h)$	tax-money used to enforce fox hunting ban
$\pi_C(at, h)$	reducing the use of tax-money for enforcing fox hunting ban
$\pi_X(s)$	increasing national sovereignty

Each demand (resp. promise) is phrased in terms of some of the issues relevant to a social group (resp. political party). Each PP-formula is 'tested' on *social situations* $(\beta, z) \in \mathbf{A} \times Z^S$ and each SD-formula on *political situations* $(\beta, z) \in \mathbf{A} \times Z^P$, and the outcome of these 'tests' is encoded into interpretation maps for each $\sigma \in \{\sigma_F, \sigma_D, \sigma_B\}$ and each $\pi \in \{\pi_L, \pi_C, \pi_X\}$ of the following types:

$$\llbracket \pi \rrbracket : \mathbf{A} \times Z^S \to \mathbf{A} \quad (\!| \pi |\!) : Z^S \to \mathbf{A}$$
$$\llbracket \sigma \rrbracket : \mathbf{A} \times Z^P \to \mathbf{A} \quad (\!| \sigma |\!) : Z^P \to \mathbf{A}$$

where as usual, for each formula φ, the $\mathbf{A}$-set $\llbracket \varphi \rrbracket : \mathbf{A} \times Z \to \mathbf{A}$ indicates the extent to which φ is supported on each situation of the appropriate type, and $(\!| \varphi |\!) : Z \to \mathbf{A}$ the extent to which it is rejected at each state of the appropriate type. In the setting of [4], the intended interpretation of β is the flexibility in translating/operationalizing data to variables. In analogy with this interpretation, we propose that when political promises are evaluated in situations (β, z) where z is a social group, β captures the maximum degree of flexibility in how voting (polling) is translated into the expression of the will of the group z. This degree might include or take into account e.g. the representativity of the sample, but also how rigorously the rules governing the test (e.g. eligibility criteria) are enforced, voter turnout, features of the electoral system like proportional representation vs first-past-the-post, etc. When social demands are evaluated in situation (β, z) where z is a political party, β captures the maximum degree of flexibility in the outcome of the "test" is interpreted as the reflecting the position of the party on that demand. Under a high β value, one would be allowed to assign high significance to e.g. informal consultations among members of the party, while lower β values would require higher standards of evidence, e.g. official policy documents of formal declarations following a party congress.

Let us represent **A**-sets $[\![\varphi]\!] : \mathbf{A} \times Z \to \mathbf{A}$, for $Z \in \{Z^S, Z^P\}$, in tables with rows labelled by **A**-elements and columns by Z-elements. Moreover, we represent $(\![\varphi]\!) : Z \to \mathbf{A}$ as a triple (α, β, γ) where $(\alpha, \beta, \gamma) = ((\![\varphi]\!)(z_F), (\![\varphi]\!)(z_D), (\![\varphi]\!)(z_B))$ if $Z = Z^S$, and $(\alpha, \beta, \gamma) = ((\![\varphi]\!)(z_L), (\![\varphi]\!)(z_C), (\![\varphi]\!)(z_X))$ if $Z = Z^P$. Then, using this notation, suppose that the interpretation of the political promises π_L results in the following outcome:

$[\![\pi_L]\!]$	z_F	z_D	z_B
0.0	0.6	0.1	0.2
0.1	0.7	0.2	0.3
0.2	0.8	0.3	0.4
0.3	0.9	0.4	0.5
0.4	1.0	0.5	0.6
0.5	1.0	0.6	0.7
0.6	1.0	0.7	0.8
0.7	1.0	0.8	0.9
0.8	1.0	0.9	1.0
0.9	1.0	1.0	1.0
1.0	1.0	1.0	1.0

$(\![\pi_L]\!) = (0.4, 0.9, 0.8)$.

A noticeable feature of the table for $[\![\pi_L]\!]$ is that, from the second row onwards, the value of any entry is always exactly 0.1 greater than the entry one row above in the same column. This is no coincidence, and will be the case if the truth value algebra **A** is any finite subalgebra of the standard Łukasiewics algebra. One can verify this by noting that, for any $\varphi \in \mathscr{L}_{\mathrm{MT}}$, since $[\![\varphi]\!]$ is Galois-closed, it can be recovered from $(\![\varphi]\!)$ as follows:

$$
[\![\varphi]\!](\beta, z) = \bigwedge_{z' \in Z_X} [(\![\varphi]\!)(z') \to (E(z, z') \to \beta)]
$$

$$
= \bigwedge_{z' \in Z_X} [((\![\varphi]\!)(z') \otimes (E(z, z')) \to \beta)]
$$

$$
= \bigvee_{z' \in Z_X} [(\![\varphi]\!)(z') \otimes (E(z, z'))] \to \beta
$$

$$
= \min \left\{ 1, 1 - \left(\bigvee_{z' \in Z_X} [(\![\varphi]\!)(z') \otimes (E(z, z'))] \right) + \beta \right\}
$$

As a result, we will represent the values of the other political promises and social demands more compactly, by giving only the first row of the table in each case. So, suppose that the interpretation of the political promises π_C and π_X results in the following outcomes:

$[\![\pi_C]\!]$	z_F	z_D	z_B
0.0	0.2	0.7	0.7
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$[\![\pi_X]\!]$	z_F	z_D	z_B
0.0	0.6	0.2	0.4
$\vdots$	$\vdots$	$\vdots$	$\vdots$

with $(\![\pi_C]\!) = (0.8, 0.3, 0.3)$ and $(\![\pi_X]\!) = (0.4, 0.8, 0.6)$. Suppose further that the interpretation of the social demand above results in the following outcomes:

$[\![\sigma_F]\!]$	z_L	z_C	z_X
0.0	0.6	0.3	0.6
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$[\![\sigma_D]\!]$	z_L	z_C	z_X
0.0	0.3	0.6	0.3
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$[\![\sigma_B]\!]$	z_L	z_C	z_X
0.0	0.3	0.6	0.6
$\vdots$	$\vdots$	$\vdots$	$\vdots$

with $(\![\sigma_F]\!) = (0.4, 0.7, 0.4)$, $(\![\sigma_D]\!) = (0.7, 0.4, 0.7)$ and $(\![\sigma_B]\!) = (0.7, 0.4, 0.4)$.

We are now in a position to compute the extensions of the SD-formulas $\Diamond \pi_L$, $\Diamond \pi_C \Diamond \pi_U$, and of the PP-formulas $\Diamond \sigma_F$, $\Diamond \sigma_D$, $\Diamond \sigma_B$. We will only consider and interpret two examples. It can be verified that:

$[\![\Diamond\sigma_D]\!]$	z_F	z_D	z_B
0.0	0.0	0.3	0.0
$\vdots$	$\vdots$	$\vdots$	$\vdots$

$[\![\Diamond\pi_C]\!]$	z_L	z_C	z_X
0.0	0.0	0.2	0.1
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Recall that $[\![\pi_C]\!]$ measured the reaction from the three social groups to the promise by the Conservative Party to reduce the use of tax money to enforce the fox hunting ban. This was supported by both the Duke's social group and the bankers to the extent 0.7 (with $\beta = 0$), presumably since both groups seek lower taxes and, in the first case, also presumably since fox hunting is part of British aristocratic culture. The factory workers' support was low, at 0.2. Now $[\![\Diamond\pi_C]\!]$ represents the response of the three political parties to this information: notice that it is proportional to the extent to which the target demographics respond to the promise and the extent to which they are targeted by the parties. For example, given their main target demographics, namely the Duke's social group and the bankers, the Tories are more favourably inclined towards their own promise than is Labour, whose main target demographic, the factory workers, don't respond very well to this promise. The Brexit party lies in between, since there is a mixed response from their main target demographics, namely the factory workers and bankers.

Turning to $[\![\Diamond\sigma_D]\!]$, recall that $[\![\sigma_D]\!]$ represents the degree of support the three political parties give (e.g. as measured by statements of members of the party, policy documents etc.) to the social group of the Duke's demand for veto powers for the house of lords. The value of $[\![\Diamond\sigma_D]\!]$ in turn represents the response of the three social groups to this support expressed by the political parties. Neither the factory workers nor the bankers show any enthusiasm, while the Duke's social group shows a more positive response, since their party of choice gives a fair measure of support to their demand.

A few remarks on some technical aspects of this case study are in order.

Remark 5.1. For the purposes of this case study, we chose to take as truth-value algebra the 11-element Łukasiewicz chain **A** with domain $\{0, 0.1, 0.2, \ldots, 1\}$. This choice is made, firstly, because it gives a convenient and manageable range of truth values which is yet large enough to accommodate the distinctions and gradations needed to tell our story. In this regard we could of course have worked equally well with the 11-element Gödel chain, but the Łukasiewicz implication is in a sense more nuanced than the Heyting implication ($\alpha \rightarrow \beta = \beta$ whenever, $\beta < \alpha$ in a Gödel chain, whereas the value of the corresponding Łukasiewicz implication also depends on the specific value of α) which is the second reason for our choice. In general the choice of truth value algebra is up to the modeller who can use any residuated lattice appropriate to the situation under consideration, subject the constraints specified in Section 2.2.

Remark 5.2. The role of the value β in evaluating political promises in social situations and social demands in political situations has both a mathematical basis in the duality upon which our semantics is built and a natural role in modelling the type of socio-political situations considered in this case study. Since the situations are, mathematically speaking, symmetrical, we will only consider the evaluation of political promises in social situations. As already discussed above, for a political promise π, the **A**-set $[\![\pi]\!] : \mathbf{A} \times Z^S \rightarrow \mathbf{A}$ indicates the extent to which π is supported in each 'social situation' (β, z) where β is a truth value and z a social group. The necessity to evaluate promises at such truth value-social group pairs, rather than simply on social groups, arises mathematically from the duality between normal heterogeneous $\mathscr{L}_{MT}$-algebras and many-valued heterogeneous models – this, albeit in a single-sorted environment, is discussed at length in [6, Section 7], while [10, Section 2] offers more general considerations on obtaining relation semantics for algebraically captured logics via dual-characterization. These mathematical underpinnings aside, the truth value β is naturally interpreted as a degree of liberality in ascribing support to a promise π. Indeed, since $[\![\varphi]\!](\beta, z) = \bigwedge_{z' \in Z_X}[([\![\varphi]\!])(z') \rightarrow (E(z, z') \rightarrow \beta)]$, it is clear that higher values of β translate into higher degrees of ascribed support. In practice, a low value for β might be associated with an attribution of support based on surveys conducted in accordance with best practice norms including rigorous sampling methods and statistical practices etc. Higher values of β can be associated with less rigorous methods that could allow for more bias in favour of attributing support to the promise.

Remark 5.3. If our framework is to be used for modelling and reasoning about real-world situations, one needs a way of translating empirical data into truth values assigned to political promises and social demands evaluated at truth value-social group pairs and truth-value-political party pairs, respectively. Of the various ways in which this could be done, we describe what is probably the most natural one, by way of example. Suppose we want to assign truth values

to Labour's promise π_L of using tax money to enforce the fox hunting ban, and that surveys of the three social groups are conducted in order to gauge their support. Based on the survey of the factory workers, it is estimated that 80% of them are in favour of this promise. Moreover, by evaluating the methodology employed to do the survey, it is judged that there is a 20% chance that the support is overstated. These two facts translate into a truth value assignment of 0.8 to $[\![\pi_L]\!]$ at the pair $(0.2, z_F)$ and, based on the discussion above about how truth values are dependent on β-values, this can be extrapolated to an assignment at pairs (β, z_F) for all other β-values, indeed to the very assignment used in the case study. The truth values at other pairs are then similarly determined by looking at the surveys of the Duke's family and of the Bankers, respectively, and taking into account the chances that these surveys overstate the support.

6. Epilogue

We suggest that several interesting analogies can be drawn between competition of theories in the empirical sciences (cf. [4]) and competition of social groups and political parties embodying socio-political theories; these analogies can be drawn thanks to the general formal framework adopted both in [4] and in the present paper, which we have illustrated with the case study discussed in the previous section.

In [4], the competition of scientific theories (identified with sets of relevant variables) plays out in the arena of (a given graph of) databases, each of which is built according to a different theory, and therefore has different degrees of similarity to other databases in the graph. Theories X and Y compete by having their respective (key) hypotheses $\varphi(X)$ and $\psi(Y)$ tested on all the databases of the given graph; each of these databases will be more or less suitable to test a given hypothesis. Hence, a clear-cut case in which X outcompetes Y is if, while each hypothesis is expected to score well on its 'home-ground' (i.e. on the databases built in accordance with the theory in the variables of which the given hypothesis is formulated, or maximally similar to those), the performances of $\varphi(X)$ on the databases that are not its own 'home-ground' are better than the performances of $\psi(Y)$ on the databases that are not its own 'home-ground'.

Likewise, political competition between parties plays out in the arena of (a given graph of) social groups, each of which has its own 'social theory' (represented as the set of issues relevant to that social group), and therefore has different degrees of similarity to other social groups in the graph. Parties X and Y compete by having their respective (key) promises $\pi_1(X)$ and $\pi_2(Y)$ tested on all the social groups of the given graph. While each of these social groups will be more or less receptive or supportive of a given promise, it is to be expected that a promise will score well on its 'home-ground', i.e. on the social groups with strong affinity to the party and its position, and typically less well social groups that are not its own 'home-ground'. Hence, one clear-cut case in which X outcompetes Y would be when the performance of $\pi_1(X)$ on the social groups that are not its own 'home-ground' is better than the performance of $\pi_2(Y)$ on the social groups that are not its own 'home-ground'.

The socio-political competition has a further interesting twist, given by the possibility of simultaneously representing the competition between social groups playing out in the arena of political parties. Again, a winner of this competition is a social group the demands of which are 'listened to' by a wider audience of political parties than their 'home-ground'.

Key to the possibility of winning on 'away-ground' is a mechanism that is well known in the practice of science, and consists in the possibility of retrieving the values of variables that are not as such represented in the database by using "translations" of the values of other variables as *proxies*. We have proposed that certain unexpected socio-political alignments can be better understood in terms of an analogous mechanism in which issues that figure in the program of a political party can be translated into issues that figure on the agenda of social group, and vice versa.

Finally, although stylised and simplified, this framework offers the possibility to analyse two competitive processes playing out at the same time, thereby paving the way to the possibility of formulating and answering a whole different range of formal questions about socio-political dynamics.

7. Conclusions

In this paper, we have introduced a many-valued semantic environment for a *multi-type* modal language based on the logic of general (i.e. not necessarily distributive) lattices. We have proved soundness and completeness for the basic logic, and, by means of a case study, we have illustrated the potential of this framework as a tool for the formal analysis of socio-political competition.

More concretely, we suggest that this might be (the basis of) a theoretical framework that could be used by political analysts to structure their study and conceptualization of socio-political dynamics. Within increasingly complex political landscapes, it becomes difficult for voters to make informed political decisions. This can be mitigated by automated tools which aid voters in matching their issues and positions with the policies and promises of political parties. A good example is the Dutch website https://www.kieskompas.nl which offers such tools to voters and where, in turn, the data submitted by users can be used by political parties to align their policies to the wishes of the social groups which they target. The logical framework proposed in this paper makes the alignment of social demands and political positions central to its analysis, and so may serve as a basis for such tools. For example, particular socio-political landscapes can be modelled using heterogeneous **A**-models, after which the demands of a voter, a social group or lobby group can be encoded as a formula which is then model checked on the available political parties in the model to determine the best match. Conversely, similar strategies may also be used by politicians and their campaign managers to align their agendas with those of their constituencies. Abstracting away from particular situations, (extensions of) the deductive system which we introduced and proved complete, can be used to derive 'laws of socio-political interaction'.

We conclude by discussing some further questions arising from this preliminary exploration.

Expanding the language with fixed points. Building on [12], 'dual common knowledge' formulas such as $\mu X. \Diamond\Diamond(X \wedge \pi)$ and $\mu X. \Diamond\Diamond(X \wedge \sigma)$ can be understood as describing the convergence of ongoing processes of interaction between social groups and political parties. It would be interesting to use the expressive power of (multi-type) lattice-based fixed-point logic to describe and reason about these phenomena.

Towards an analysis of the dynamics of socio-political competition. Related to the previous point, the framework introduced in this paper lends itself to the formal analysis of the dynamics triggered by the interplay of social groups and political parties, a theme on which recent research in political science has focused (cf. e.g. [23,28]). This direction would address questions relative e.g. to the emergence of political parties in response to issues which are relevant to certain social groups, or to the emergence of novel social group identities by effect of certain political alignments. The framework also offers new connections to the dynamics of market categories [29] and the appreciation of new members of old and new categories [22] by different audiences.

Acknowledgements

Appendix A. Completeness

For the sake of uniformity with previous settings (cf. e.g. [6, Section 7.2]) in this section, we work with graph-based frames $\mathbb{G} = (\mathbb{X}_S, \mathbb{X}_P, R_\Diamond, R_\Diamond)$ the associated complex algebras of which are different from those of Definition 3.1. That is, for the sake of this section, for every graph $\mathbb{X} = (E, Z)$, we define its associated formal context $\mathbb{P}_\mathbb{X} := (Z_A, Z_X, I_E)$ by setting $Z_A := Z$, $Z_X := \mathbf{A} \times Z$ and $I_E : Z_A \times Z_X \to \mathbf{A}$ be defined by the assignment $(z, (\alpha, z')) \mapsto E(z, z') \to \alpha$.

For any lattice $\mathbb{L}$, an **A**-*filter* is an **A**-subset of $\mathbb{L}$, i.e. a map $f : \mathbb{L} \to \mathbf{A}$, which is both $\wedge$- and $\top$-preserving, i.e. $f(\top) = 1$ and $f(a \wedge b) = f(a) \wedge f(b)$ for any $a, b \in \mathbb{L}$. Intuitively, the $\wedge$-preservation encodes a many-valued version of closure under $\wedge$ of filters. An **A**-filter is *proper* if it is also $\bot$-preserving, i.e. $f(\bot) = 0$. Dually, an **A**-*ideal* is a map $i : \mathbb{L} \to \mathbf{A}$ which is both $\vee$- and $\bot$-reversing, i.e. $i(\bot) = \top$ and $i(a \vee b) = i(a) \wedge i(b)$ for any $a, b \in \mathbb{L}$, and is *proper* if in addition $i(\top) = 0$. The *complement* of a (proper) **A**-ideal is a map $u : \mathbb{L} \to \mathbf{A}$ which is both $\vee$- and $\bot$-preserving, i.e. $u(\bot) = 0$ and $u(a \vee b) = u(a) \vee u(b)$ for any $a, b \in \mathbb{L}$ (and in addition $u(\top) = 1$). Intuitively, $u(a)$ encodes the extent to which a does not belong to the ideal of which u is the many-valued complement. We let $\mathsf{F_A}(\mathbb{L})$, $\mathsf{I_A}(\mathbb{L})$ and $\mathsf{C_A}(\mathbb{L})$ respectively denote the set of proper **A**-filters, proper **A**-ideals, and the complements of proper **A**-ideals of $\mathbb{L}$. For any heterogeneous $\mathscr{L}_{\mathrm{MT}}$-algebra $(\mathbb{L}_S, \mathbb{L}_P, \Diamond, \Diamond)$ (cf. Definition 2.1) and all **A**-subsets $k : \mathbb{L}_P \to \mathbf{A}$ and $h : \mathbb{L}_S \to \mathbf{A}$, let $k^{-\Diamond} : \mathbb{L}_S \to \mathbf{A}$ and $h^{-\Diamond} : \mathbb{L}_P \to \mathbf{A}$ be defined as $k^{-\Diamond}(s) = \bigvee\{k(p) \mid \Diamond p \le s\}$ and $h^{-\Diamond}(p) = \bigvee\{h(s) \mid \Diamond s \le p\}$, respectively. Then, by definition, $k(p) \le k^{-\Diamond}(\Diamond p)$ and $h(s) \le h^{-\Diamond}(\Diamond s)$ for every $p \in \mathbb{L}_P$ and $s \in \mathbb{L}_S$. Let $(\mathbf{SD}, \mathbf{PP}, \Diamond, \Diamond)$ be the Lindenbaum–Tarski heterogeneous algebra associated with **L**.

Lemma A.1.

1. *If $f : \mathbb{L}_P \to \mathbf{A}$ is an $\mathbf{A}$-filter, then so is $f^{-\Diamond}$.*
2. *If $g : \mathbb{L}_S \to \mathbf{A}$ is an $\mathbf{A}$-filter, then so is $g^{-\Diamond}$.*
3. *If $f : \mathbf{PP} \to \mathbf{A}$ is a proper $\mathbf{A}$-filter, then so is $f^{-\Diamond}$.*
4. *If $g : \mathbf{SD} \to \mathbf{A}$ is a proper $\mathbf{A}$-filter, then so is $g^{-\Diamond}$.*
5. *If $\pi_1, \pi_2 \in \mathbf{PP}$, then $\pi_1 \vee \pi_2 = \top$ implies that $\pi_1 = \top$ or $\pi_2 = \top$.*
6. *If $\sigma_1, \sigma_2 \in \mathbf{SD}$, then $\sigma_1 \vee \sigma_2 = \top$ implies that $\sigma_1 = \top$ or $\sigma_2 = \top$.*

Proof. 1. For all $s, t \in \mathbb{L}_S$,

$$
\begin{aligned}
f^{-\Diamond}(\top) &= \bigvee \{ f(p) \mid \Diamond p \leq \top \} \\
&= \bigvee \{ f(p) \mid p \in \mathbb{L} \} \\
&= f(\top) \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
& f^{-\Diamond}(s) \wedge f^{-\Diamond}(t) \\
&= \bigvee \{ f(p_1) \mid \Diamond p_1 \leq s \} \wedge \bigvee \{ f(p_2) \mid \Diamond p_2 \leq t \} \\
&= \bigvee \{ f(p_1) \wedge f(p_2) \mid \Diamond p_1 \leq s \text{ and } \Diamond p_2 \leq b \} \qquad \text{frame-distributivity} \\
&= \bigvee \{ f(p_1 \wedge p_2) \mid \Diamond p_1 \leq s \text{ and } \Diamond p_2 \leq t \} \qquad f \text{ is an } \mathbf{A}\text{-filter} \\
&= \bigvee \{ f(p) \mid \Diamond p \leq s \text{ and } \Diamond p \leq t \} \qquad (*) \\
&= \bigvee \{ f(p) \mid \Diamond p \leq s \wedge t \} \\
&= f^{-\Diamond}(s \wedge t),
\end{aligned}
$$

the equivalence marked with $(*)$ being due to the fact that $\Diamond(p_1 \wedge p_2) \leq \Diamond p_1 \wedge \Diamond p_2$.

3. Let $f : \mathbf{PP} \to \mathbf{A}$ be a proper $\mathbf{A}$-filter. $f^{-\Diamond}(\bot) = \bigvee \{ f([\pi]) \mid [\Diamond \pi] \leq [\bot] \} = \bigvee \{ f([\pi]) \mid \Diamond \pi \vdash \bot \} = \bigvee \{ f([\pi]) \mid \pi \vdash \bot \} = f([\bot]) = 0$. The crucial inequality is the third to last, which holds since $\Diamond \pi \vdash \bot$ iff $\pi \vdash \bot$. The right to left implication can be easily derived in $\mathbf{L}$. For the sake of the left to right implication we appeal to the completeness of $\mathbf{L}$ with respect to the class of all heterogeneous $\mathscr{L}_{\mathrm{MT}}$-algebras (cf. Proposition 2.2, see [10] for the general case) and reason contrapositively. Suppose $\pi \nvdash \bot$. Then, by Proposition 2.2, there is a normal heterogeneous $\mathscr{L}_{\mathrm{MT}}$-algebra $\mathbb{H} = (\mathbb{L}_S, \mathbb{L}_P, \Diamond, \Diamond)$ and assignment h of atomic propositions such that $h(\pi) \neq 0$. Now consider the heterogeneous algebra $(\mathbb{L}_S, \mathbb{L}'_P, \Diamond, \Diamond)$ obtained from $(\mathbb{L}_S, \mathbb{L}_P, \Diamond, \Diamond)$ by adding a new least element $0'$ to $\mathbb{L}_P$ and extending the $\Diamond$-operation by declaring $\Diamond 0' = 0'$. We keep the assignment h unchanged. It is easy to check that $(\mathbb{L}_S, \mathbb{L}'_P, \Diamond, \Diamond)$ is a normal heterogeneous $\mathscr{L}_{\mathrm{MT}}$-algebra, and that $h(\Diamond \pi) \geq 0 > 0'$ and hence $\Diamond \pi \nvdash \bot$.

Items 2 and 4 are proven by arguments analogous to the ones above.

5. Suppose, by contraposition, that $\top \nvdash \pi_1$ and $\top \nvdash \pi_2$. By the completeness theorem to which we have appealed in the proof of item 2, there are heterogeneous algebras $\mathbb{H}_1 = (\mathbb{L}_1^S, \mathbb{L}_1^P, \Diamond_1, \Diamond_1)$ and $\mathbb{H}_2 = (\mathbb{L}_2^S, \mathbb{L}_2^P, \Diamond_2, \Diamond_2)$ and corresponding assignments v_i on $\mathbb{H}_i$ such that $v_1(\pi_1) \neq \top^{\mathbb{L}_1^P}$ and $v_2(\pi_2) \neq \top^{\mathbb{L}_2^P}$. Consider the algebra $\mathbb{H}' = (\mathbb{L}_1^S \times \mathbb{L}_2^S, \mathbb{L}^{P'}, \Diamond', \Diamond')$, where $\mathbb{L}^{P'}$ is obtained by adding a new top element $\top'$ to $\mathbb{L}_1^P \times \mathbb{L}_2^P$, defining the operation $\Diamond'$ by the same assignment of $\Diamond^{\mathbb{H}_1 \times \mathbb{H}_2}$ on $\mathbb{L}_1^P \times \mathbb{L}_2^P$ and mapping $\top'$ to $(\Diamond \top)^{\mathbb{L}_1^S \times \mathbb{L}_2^S}$, and the operation $\Diamond'$ by the same assignment of $\Diamond^{\mathbb{H}_1 \times \mathbb{H}_2}$ on $\mathbb{L}_1^S \times \mathbb{L}_2^S$. The monotonicity of $\Diamond'$ and normality (i.e. finite join-preservation) of $\Diamond'$ follow immediately by construction. The normality (i.e. finite join-preservation) of $\Diamond'$ is verified by cases: if $a \vee b \neq \top'$, then it immediately follows from the normality of $\Diamond^{\mathbb{H}_1 \times \mathbb{H}_2}$. If $a \vee b = \top'$, then by construction, either $a = \top'$ or $b = \top'$ (i.e. $\top'$ is join-irreducible), and hence, the join-preservation of $\Diamond'$ is a consequence of its monotonicity. Consider the valuation $v' : \mathsf{Prop} \to \mathbb{H}'$ defined by the assignment $p \mapsto e(v_1(p), v_2(p))$, where $e : \mathbb{H}_1 \times \mathbb{H}_2 \to \mathbb{H}'$ is the natural embedding.

Let us show, for all $\chi \in \mathbf{PP}$, that if $(v_1(\chi), v_2(\chi)) \neq \top^{\mathbb{L}_1^P \times \mathbb{L}_2^P}$, then $v'(\chi) \neq \top'$. We proceed by induction on χ. The cases for atomic propositions and conjunction are immediate. The case for $\chi := \pi'_1 \vee \pi'_2$ uses the join-irreducibility of $\top'$. When $\chi := \Diamond \sigma$, then $v'(\Diamond \sigma) = \Diamond' v'(\sigma) \neq \top'$, since, by construction, $\top'$ is not in the range of $\Diamond'$.

Clearly, $v_1(\pi_1) \neq \top^{\mathbb{L}_1^P}$ and $v_2(\pi_2) \neq \top^{\mathbb{L}_2^P}$ imply that $(v_1(\pi_1), v_2(\pi_1)) \neq \top^{\mathbb{L}_1^P \times \mathbb{L}_2^P}$ and $(v_1(\pi_2), v_2(\pi_2)) \neq \top^{\mathbb{L}_1^P \times \mathbb{L}_2^P}$. So, by the above claim, $v'(\pi_1) \neq \top'$ and $v'(\pi_2) \neq \top'$, and hence, since $\top'$ is join-irreducible, $v'(\varphi \vee \psi) \neq \top'$.

The proof of item 6 is analogous to the one above. $\square$

Lemma A.2. *For any* $f \in \mathsf{F_A}(\mathbb{L}_P)$ *and* $v \in \mathsf{C_A}(\mathbb{L}_S)$ *and* $g \in \mathsf{F_A}(\mathbb{L}_S)$ *and* $u \in \mathsf{C_A}(\mathbb{L}_P)$,

1. $\bigwedge_{s \in \mathbb{L}_S}(f^{-\Diamond}(s) \to v(s)) = \bigwedge_{p \in \mathbb{L}_P}(f(p) \to v(\Diamond p))$;
2. $\bigwedge_{p \in \mathbb{L}_P}(g^{-\Diamond}(p) \to u(p)) = \bigwedge_{s \in \mathbb{L}_S}(g(s) \to u(\Diamond s))$.

Proof. 1. The fact that $f(p) \leq f^{-\Diamond}(\Diamond p)$ implies that $f^{-\Diamond}(\Diamond p) \to v(\Diamond p) \leq f(p) \to v(\Diamond p)$ for every $p \in \mathbb{L}_P$, which is enough to show that $\bigwedge_{s \in \mathbb{L}_S}(f^{-\Diamond}(s) \to v(s)) \leq \bigwedge_{p \in \mathbb{L}_P}(f(p) \to v(\Diamond p))$. Conversely, to show that

$$\bigwedge_{p \in \mathbb{L}_P}(f(p) \to v(\Diamond p)) \leq \bigwedge_{s \in \mathbb{L}_S}(f^{-\Diamond}(s) \to v(s)),$$

we have to show that, for every $s \in \mathbb{L}_S$,

$$\bigwedge_{p \in \mathbb{L}_P}(f(p) \to v(\Diamond p)) \leq f^{-\Diamond}(s) \to v(s),$$

i.e. by definition of $f^{-\Diamond}(s)$ and the fact that $\to$ is completely join-reversing in its first coordinate,

$$\bigwedge_{p \in \mathbb{L}_P}(f(p) \to v(\Diamond p)) \leq \bigwedge_{\Diamond q \leq s}(f(q) \to v(s)).$$

Hence, let $q \in \mathbb{L}_P$ such that $\Diamond q \leq s$, and let us show that

$$\bigwedge_{p \in \mathbb{L}_P}(f(p) \to v(\Diamond p)) \leq f(q) \to v(s).$$

Since v is $\vee$-preserving, hence order-preserving, $\Diamond q \leq s$ implies $v(\Diamond q) \leq v(s)$, hence

$$\bigwedge_{p \in \mathbb{L}_P}(f(p) \to v(\Diamond p)) \leq f(q) \to v(\Diamond q) \leq f(q) \to v(s),$$

as required. The proof of the second item is analogous and omitted. $\square$

Definition A.3. Let $(\mathbf{SD}, \mathbf{PP}, \Diamond, \Diamond)$ be the Lindenbaum-Tarski heterogeneous algebra of $\mathscr{L}_{\mathrm{MT}}$-formulas.[9] The *canonical* graph-based **A**-*frame* is the structure $\mathbb{G} = (\mathbb{X}_S, \mathbb{X}_P, R_\Diamond, R_\Diamond)$ defined as follows[10]:

$$Z^S := \left\{(f, u) \in \mathsf{F_A}(\mathbf{PP}) \times \mathsf{C_A}(\mathbf{PP}) \mid \bigwedge_{\pi \in \mathbf{PP}}(f(\pi) \to u(\pi)) = 1\right\}.$$

$$Z^P := \left\{(g, v) \in \mathsf{F_A}(\mathbf{SD}) \times \mathsf{C_A}(\mathbf{SD}) \mid \bigwedge_{\sigma \in \mathbf{SD}}(g(\sigma) \to v(\sigma)) = 1\right\}.$$

For any $z \in Z^S$ (resp. $z \in Z^P$) as above, we let f_z and u_z (resp. g_z and v_z) denote the first and second coordinate of z, respectively. Then $E_P : Z^P \times Z^P \to \mathbf{A}$, $E_S : Z^S \times Z^S \to \mathbf{A}$, $R_\Diamond : Z^S \times Z^P \to \mathbf{A}$ and $R_\Diamond : Z^P \times Z^S \to \mathbf{A}$ are defined as follows:

$$E_S(z, z') := \bigwedge_{\pi \in \mathbf{PP}}(f_z(\pi) \to u_{z'}(\pi));$$

---

[9] In the remainder of this section, we abuse notation and identify formulas with their equivalence class in $(\mathbf{SD}, \mathbf{PP}, \Diamond, \Diamond)$. Also, notice the inversion: states in Z^S (resp. Z^P) are built out of structures from $\mathbf{PP}$ (resp. $\mathbf{SD}$).

[10] Recall that for any set W, the **A**-*subsethood* relation between elements of **A**-subsets of W is the map $S_W : \mathbf{A}^W \times \mathbf{A}^W \to \mathbf{A}$ defined as $S_W(f, g) := \bigwedge_{w \in W}(f(w) \to g(w))$. If $S_W(f, g) = 1$ we also write $f \subseteq g$.

$$E_P(z, z') := \bigwedge_{\sigma \in \mathbf{SD}} (g_z(\sigma) \to v_{z'}(\sigma));$$

$$R_\Diamond(z = (g_z, v_z), z' = (f_{z'}, u_{z'})) := \bigwedge_{\sigma \in \mathbf{SD}} (f_{z'}^{-\Diamond}(\sigma) \to v_z(\pi)) = \bigwedge_{\pi \in \mathbf{PP}} (f_{z'}(\pi) \to v_z(\Diamond \pi));$$

$$R_\lozenge(z, z') := \bigwedge_{\pi \in \mathbf{PP}} (g_{z'}^{-\lozenge}(\pi) \to u_z(\pi)) = \bigwedge_{\sigma \in \mathbf{SD}} (g_{z'}(\sigma) \to u_z(\lozenge \sigma)).$$

Lemma A.4. *The structure* $\mathbb{G}$ *of Definition A.3 is a graph-based* **A***-frame, in the sense specified at the beginning of the present section.*

Proof. We need to show that $R_\Diamond$ and $R_\lozenge$ satisfy the following compatibility conditions: for every $z \in Z^P$ and all $\alpha, \beta \in \mathbf{A}$,

$$(R_\Diamond^{[1]}[\{\beta/(\alpha, z)\}])^{[10]} \subseteq R_\Diamond^{[1]}[\{\beta/(\alpha, z)\}]$$
$$(R_\lozenge^{[0]}[\{\beta/z\}])^{[01]} \subseteq R_\lozenge^{[0]}[\{\beta/z\}],$$

and for every $z \in Z^S$ and all $\alpha, \beta \in \mathbf{A}$,

$$(R_\lozenge^{[1]}[\{\beta/(\alpha, z)\}])^{[10]} \subseteq R_\lozenge^{[1]}[\{\beta/(\alpha, z)\}]$$
$$(R_\Diamond^{[0]}[\{\beta/z\}])^{[01]} \subseteq R_\Diamond^{[0]}[\{\beta/z\}].$$

Let us show the fourth inclusion above. By definition, for any $(\alpha, w) \in Z_X^P$,

$$R_\Diamond^{[0]}[\{\beta/z\}](\alpha, w) = \bigwedge_{z' \in Z_A^S}[\{\beta/z\}(z') \to (R_\Diamond(w, z') \to \alpha)]$$
$$= \beta \to (R_\Diamond(w, z) \to \alpha)$$

$$(R_\Diamond^{[0]}[\{\beta/z\}])^{[01]}(\alpha, w) = \bigwedge_{z' \in Z_A^P}[(R_\Diamond^{[0]}[\{\beta/z\}])^{[0]}(z') \to (E_P(z', w) \to \alpha)],$$

and hence it is enough to find some $z' \in Z_A^P$ such that

$$(R_\Diamond^{[0]}[\{\beta/z\}])^{[0]}(z') \to (E_P(z', w) \to \alpha) \leq \beta \to (R_\Diamond(w, z) \to \alpha),$$

i.e.

$$\left(\bigwedge_{(\gamma, z'') \in Z_X^P} [\beta \to (R_\Diamond(z'', z) \to \gamma)] \to (E_P(z', z'') \to \gamma) \right) \to (E_P(z', w) \to \alpha) \leq \beta \to (R_\Diamond(w, z) \to \alpha). \quad (2)$$

Let $z' \in Z_P^A$ such that $g_{z'} = f_z^{-\Diamond}$ (cf. Lemma A.1). Then

$$E_P(z', w) = \bigwedge_{\sigma \in \mathbf{SD}}(f_z^{-\Diamond}(\sigma) \to v_w(\sigma)) = R_\Diamond(w, z),$$

and likewise $E_P(z', z'') = R_\Diamond(z'', z)$. Therefore, for this choice of z', inequality (2) can be rewritten as follows:

$$\left(\bigwedge_{(\gamma, z'') \in Z_X^P} [\beta \to (R_\Diamond(z'', z) \to \gamma)] \to (R_\Diamond(z'', z) \to \gamma) \right) \to (R_\Diamond(w, z) \to \alpha) \leq \beta \to (R_\Diamond(w, z) \to \alpha)$$

The inequality above is true if

$$\beta \leq \bigwedge_{(\gamma, z'') \in Z_X} [\beta \to (R_\Diamond(z'', z) \to \gamma)] \to (R_\Diamond(z'', z) \to \gamma),$$

i.e. if for every $(\gamma, z'') \in Z_X^P$,

$$\beta \leq [\beta \to (R_\Diamond(z'', z) \to \gamma)] \to (R_\Diamond(z'', z) \to \gamma),$$

which is an instance of a tautology in residuated lattices.

Let $z \in Z^P$ and $\alpha, \beta \in \mathbf{A}$ and let us show that $(R_\Diamond^{[1]}[\{\beta/(\alpha, z)\}])^{[10]} \subseteq R_\Diamond^{[1]}[\{\beta/(\alpha, z)\}]$. By definition, for every $w \in Z_A^S$,

$$R_\Diamond^{[1]}[\{\beta/(\alpha, z)\}](w) = \bigwedge\nolimits_{(\gamma, z') \in Z_X^P} [\{\beta/(\alpha, z)\}(\gamma, z') \to (R_\Diamond(z', w) \to \gamma)]$$
$$= \beta \to (R_\Diamond(z, w) \to \alpha)$$

$$(R_\Diamond^{[1]}[\{\beta/(\alpha, z)\}])^{[10]}(w) = \bigwedge\nolimits_{(\gamma, z') \in Z_X^S} [(R_\Diamond^{[1]}[\{\beta/(\alpha, z)\}])^{[1]}(\gamma, z') \to (E_S(w, z') \to \gamma)].$$

Hence it is enough to find some $(\gamma, z') \in Z_X^S$ such that

$$(R_\Diamond^{[1]}[\{\beta/(\alpha, z)\}])^{[1]}(\gamma, z') \to (E_S(w, z') \to \gamma) \leq \beta \to (R_\Diamond(z, w) \to \alpha),$$

i.e.

$$\left(\bigwedge_{z'' \in Z_A^S} (\beta \to (R_\Diamond(z, z'') \to \alpha)) \to (E_S(z'', z') \to \gamma) \right) \to (E_S(w, z') \to \gamma) \leq \beta \to (R_\Diamond(z, w) \to \alpha). \tag{3}$$

Let $\gamma := \beta$, and $z' = (f_{z'}, u_{z'}) \in Z^S$ such that $u_{z'} : \mathbf{PP} \to \mathbf{A}$ is defined by the assignment

$$u_{z'}(\pi) = \begin{cases} 1 & \text{if } \top \vdash \pi \\ v_z(\Diamond \pi) & \text{otherwise.} \end{cases}$$

By construction, $u_{z'}$ maps $\top$ to 1 and $\bot$ to 0; moreover, using Lemma A.1.5, it can be readily verified that $u_{z'}$ is $\vee$-preserving. Then, by Lemma A.2,

$$E_S(z'', z') := \bigwedge_{\pi \in \mathbf{PP}} (f_{z''}(\pi) \to u_{z'}(\pi)) = \bigwedge_{\pi \in \mathbf{PP}} (f_{z''}(\pi) \to v_z(\Diamond \pi)) = \bigwedge_{\sigma \in \mathbf{SD}} (f_{z''}^{-\Diamond}(\sigma) \to v_z(\pi)) = R_\Diamond(z, z''),$$

and likewise $E(w, z') = R_\Diamond(z, w)$. Therefore, for this choice of z', inequality (3) can be rewritten as follows:

$$\left(\bigwedge_{z'' \in Z_A^S} (\beta \to (R_\Diamond(z, z'') \to \alpha)) \to (R_\Diamond(z, z'') \to \gamma) \right) \to (R_\Diamond(z, w) \to \gamma) \leq \beta \to (R_\Diamond(z, w) \to \alpha),$$

which is shown to be true by the same argument as the one concluding the verification of the previous inclusion. The remaining inclusions are verified with analogous arguments to those above (using Lemma A.1.6), and their proofs are omitted.   □

Definition A.5. Let $(\mathbf{SD}, \mathbf{PP}, \Diamond, \Diamond)$ be the Lindenbaum-Tarski heterogeneous algebra of $\mathscr{L}_{\mathrm{MT}}$-formulas. The *canonical graph-based* $\mathbf{A}$-*model* is the structure $\mathbb{M} = (\mathbb{G}, V)$ such that $\mathbb{G}$ is the canonical graph-based $\mathbf{A}$-frame of Definition A.3, and if $p \in \mathsf{Prop}$, then $V_S : \mathsf{Prop} \to \mathbb{X}_P^+$ and $V_P : \mathsf{Prop} \to \mathbb{X}_S^+$ are such that:

1. $V_S(p) = (\llbracket p \rrbracket_S, (\!| p |\!)_S)$ with $\llbracket p \rrbracket_S : Z_A^P \to \mathbf{A}$ and $(\!| p |\!)_S : Z_X^P \to \mathbf{A}$ defined by $z \mapsto g_z(p)$ and $(\alpha, z) \mapsto v_z(p) \to \alpha$, respectively;
2. $V_P(p) = (\llbracket p \rrbracket_P, (\!| p |\!)_P)$ with $\llbracket p \rrbracket_P : Z_A^S \to \mathbf{A}$ and $(\!| p |\!)_P : Z_X^S \to \mathbf{A}$ defined by $z \mapsto f_z(p)$ and $(\alpha, z) \mapsto u_z(p) \to \alpha$, respectively.

Lemma A.6. *The structure* $\mathbb{G}$ *of Definition A.5 is a graph-based* $\mathbf{A}$*-model.*

Proof. It is enough to show that for any $p \in \mathsf{Prop}$,

1. $\llbracket p \rrbracket_P^{[1]} = (\!| p |\!)_P$ and $\llbracket p \rrbracket_P = (\!| p |\!)_P^{[0]}$.

2. $[\![p]\!]_S^{[1]} = (\![p]\!)_S$ and $[\![p]\!]_S = (\![p]\!)_S^{[0]}$, and

We only show 1. To show that $(\![p]\!)_P(\alpha, z) \leq [\![p]\!]_P^{[1]}(\alpha, z)$ for any $(\alpha, z) \in Z_X^S$, by definition, we need to show that

$$u_z(p) \to \alpha \leq \bigwedge_{z' \in Z_A^S} ([\![p]\!]_P(z') \to (E_S(z', z) \to \alpha)),$$

i.e. that for every $z' \in Z_A^S$,

$$u_z(p) \to \alpha \leq [\![p]\!]_P(z') \to (E_S(z', z) \to \alpha).$$

By definition, the inequality above is equivalent to

$$u_z(p) \to \alpha \leq f_{z'}(p) \to \left(\bigwedge_{\pi \in \mathbf{PP}} (f_{z'}(\pi) \to u_z(\pi)) \to \alpha \right).$$

Since $\bigwedge_{\pi \in \mathbf{PP}}(f_{z'}(\pi) \to u_z(\pi)) \leq f_{z'}(p) \to u_z(p)$ and $\to$ is order-reversing in its first coordinate, it is enough to show that

$$u_z(p) \to \alpha \leq f_{z'}(p) \to [(f_{z'}(p) \to u_z(p)) \to \alpha].$$

By residuation the inequality above is equivalent to

$$u_z(p) \to \alpha \leq [f_{z'}(p) \otimes (f_{z'}(p) \to u_z(p))] \to \alpha,$$

which is equivalent to

$$[f_{z'}(p) \otimes (f_{z'}(p) \to u_z(p)] \otimes [u_z(p) \to \alpha] \leq \alpha,$$

which is the instance of a tautology in residuated lattices. Conversely, to show that $[\![p]\!]_P^{[1]}(\alpha, z) \leq (\![p]\!)_P(\alpha, z)$, i.e.

$$\bigwedge_{z' \in Z_A^S} ([\![p]\!]_P(z') \to (E_S(z', z) \to \alpha)) \leq u_z(p) \to \alpha,$$

it is enough to show that

$$[\![p]\!]_P(z') \to (E_S(z', z) \to \alpha)) \leq u_z(p) \to \alpha \tag{4}$$

for some $z' \in Z^S$. Let $z' := (f_p, u)$ such that $u : \mathbf{PP} \to \mathbf{A}$ is the constant map 1, and $f_p : \mathbf{PP} \to \mathbf{A}$ is defined by the assignment

$$f_p(\pi) = \begin{cases} 1 & \text{if } p \vdash \pi \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $E_S(z', z) = \bigwedge_{\pi \in \mathbf{PP}}(f_p(\pi) \to u_z(\pi)) = \bigwedge_{p \vdash \pi} u_z(\pi) = u_z(p)$, the last identity holding since u_z is order-preserving. Therefore, $[\![p]\!]_P(z') \to (E_S(z', z) \to \alpha)) = f_p(p) \to (u_z(p) \to \alpha) = 1 \to (u_z(p) \to \alpha) = u_z(p) \to \alpha$, which shows (4).

By adjunction, the inequality $(\![p]\!)_P \leq [\![p]\!]_P^{[1]}$ proven above implies that $[\![p]\!]_P \leq (\![p]\!)_P^{[0]}$. Hence, to show that $[\![p]\!]_P = (\![p]\!)_P^{[0]}$, it is enough to show that $(\![p]\!)_P^{[0]}(z) \leq [\![p]\!]_P(z)$ for every $z \in Z^S$, i.e.

$$\bigwedge_{(\alpha, z') \in Z_X^S} (\![p]\!)_P(\alpha, z') \to (E_S(z, z') \to \alpha) \leq f_z(p),$$

and to show the inequality above, it is enough to show that

$$(\![p]\!)_P(\alpha, z') \to (E_S(z, z') \to \alpha) \leq f_z(p) \tag{5}$$

for some $(\alpha, z') \in Z_X^S$. Let $\alpha := f_z(p)$ and $z' := (f_{z'}, u_p)$ be such that $u_{z'} = u_p : \mathbf{PP} \to \mathbf{A}$ is defined by the following assignment:

$$u_p(\pi) = \begin{cases} 0 & \text{if } \pi \vdash \bot \\ f_z(p) & \text{if } \pi \vdash p \text{ and } \pi \nvdash \bot \\ 1 & \text{if } \pi \nvdash p. \end{cases}$$

By construction, $u_{z'}$ is $\vee$-, $\bot$- and $\top$-preserving. Moreover, $([p])_P(\alpha, z') = u_{z'}(p) \to \alpha = f_z(p) \to f_z(p) = 1$, and $E_S(z, z') = \bigwedge_{\pi \in \mathbf{PP}}(f_z(\pi) \to u_{z'}(\pi)) = \bigwedge_{\pi \vdash p}(f_z(\pi) \to f_z(p)) = 1$. Hence, the left-hand side of (5) can be equivalently rewritten as $1 \to (1 \to f_z(p)) = f_z(p)$, which shows (5) and concludes the proof. $\square$

Lemma A.7 *(Truth Lemma). For every $\pi \in \mathsf{PP}$ and every $\sigma \in \mathsf{SD}$,*

1. *the maps $[\![\pi]\!]_P : Z_A^S \to \mathbf{A}$ and $([\pi])_P : Z_X^S \to \mathbf{A}$ coincide with those defined by the assignments $z \mapsto f_z(\pi)$ and $(\alpha, z) \mapsto u_z(\pi) \to \alpha$, respectively.*
2. *the maps $[\![\sigma]\!]_S : Z_A^P \to \mathbf{A}$ and $([\sigma])_S : Z_X^P \to \mathbf{A}$ coincide with those defined by the assignments $z \mapsto g_z(\sigma)$ and $(\alpha, z) \mapsto v_z(\sigma) \to \alpha$, respectively.*

Proof. We proceed by simultaneous induction on π and σ. If $\pi := p \in \mathsf{Prop}$ (resp. $\sigma := p \in \mathsf{Prop}$), the statement follows immediately from Definition A.5.

If $\pi := \top$, then $[\![\top]\!]_P(z) = 1 = f_z(\top)$ since $\mathbf{A}$-filters are $\top$-preserving. Moreover,

$$\begin{aligned} ([\top])_P(\alpha, z) &= [\![\top]\!]_P^{[1]}(\alpha, z) \\ &= \bigwedge\nolimits_{z' \in Z_A^S}[[\![\top]\!](z') \to (E_S(z', z) \to \alpha)] \\ &= \bigwedge\nolimits_{z' \in Z_A^S}[f_{z'}(\top) \to (E_S(z', z) \to \alpha)] \\ &= \bigwedge\nolimits_{z' \in Z_A^S}[E_S(z', z) \to \alpha]. \end{aligned}$$

So, to show that $u_z(\top) \to \alpha \leq ([\top])_P(\alpha, z)$, we need to show that for every $z' \in Z_A^S$,

$$u_z(\top) \to \alpha \leq E_S(z', z) \to \alpha,$$

and for this, it is enough to show that

$$\bigwedge_{\pi' \in \mathbf{PP}} [f_{z'}(\pi') \to u_z(\pi')] \leq u_z(\top),$$

which is true, since by definition, $u_z(\top) = 1$. To show that $([\top])_P(\alpha, z) \leq u_z(\top) \to \alpha$, i.e. that

$$\bigwedge_{z' \in Z_A^S} [E_S(z', z) \to \alpha] \leq u_z(\top) \to \alpha,$$

it is enough to find some $z' \in Z^S$ such that $E_S(z', z) \to \alpha \leq u_z(\top) \to \alpha$. Let $z' := (f_\top, u)$ such that $u : \mathbf{PP} \to \mathbf{A}$ maps $\top$ to 1 and every other element of $\mathbf{PP}$ to 0, and $f_\top : \mathbf{PP} \to \mathbf{A}$ is defined by the assignment

$$f_\top(\pi') = \begin{cases} 1 & \text{if } \top \vdash \pi' \\ 0 & \text{otherwise.} \end{cases}$$

By definition, $E_S(z', z) = \bigwedge_{\pi' \in \mathbf{PP}}[f_{z'}(\pi') \to u_z(\pi')] = \bigwedge_{\top \vdash \pi'}[1 \to u_z(\pi')] = \bigwedge_{\top \vdash \pi'} u_z(\pi') \geq u_z(\top)$, the last inequality being due to the fact that u_z is order-preserving. Hence, $E_S(z', z) \to \alpha \leq u_z(\top) \to \alpha$, as required. The case in which $\sigma := \top$ is analogous to the one above, and its proof is omitted.

If $\pi := \bot$, then $([\bot])_P(\alpha, z) = 1 = u_z(\bot) \to \alpha$ since complements of $\mathbf{A}$-ideals are $\bot$-preserving. Let us show that $[\![\bot]\!]_P(z) = f_z(\bot)$. The inequality $f_z(\bot) \leq [\![\bot]\!]_P(z)$ follows immediately from the fact that f_z is a proper $\mathbf{A}$-filter and hence $f_z(\bot) = 0$. To show that $[\![\bot]\!]_P(z) \leq f_z(\bot)$, by definition $[\![\bot]\!]_P(z) = ([\bot])^{[0]}(z) = \bigwedge_{(\alpha, z') \in Z_X^S}[(u_{z'}(\bot) \to \alpha) \to (E_S(z, z') \to \alpha)]$, hence, it is enough to find some $(\alpha, z') \in Z_X^S$ such that

$$(u_{z'}(\bot) \to \alpha) \to (E_S(z, z') \to \alpha) \leq f_z(\bot). \tag{6}$$

Let $\alpha := f_z(\bot)$ and let $z' := (f_\top, u_\bot)$ such that $f_\top : \mathbf{PP} \to \mathbf{A}$ is defined as indicated above in the base case for $\pi := \top$, and $u_\bot : \mathbf{PP} \to \mathbf{A}$ is defined by the assignment

$$u_\perp(\pi') = \begin{cases} 0 & \text{if } \pi' \vdash \perp \\ 1 & \text{if } \pi' \nvdash \perp. \end{cases}$$

By definition and since f_z is order-preserving and $\perp$-preserving, $E_S(z, z') = \bigwedge_{\pi' \in \mathbf{PP}}[f_z(\pi') \to u_\perp(\pi')] = 1$. Hence, (6) can be rewritten as follows:

$$(f_z(\perp) \to f_z(\perp)) \to f_z(\perp) \le f_z(\perp),$$

which is true since $f_z(\perp) \to f_z(\perp) = 1$ and $1 \to f_z(\perp) = f_z(\perp)$. The case in which $\sigma := \perp$ is analogous to the one above, and its proof is omitted.

If $\pi := \pi_1 \wedge \pi_2$, then $[\![\pi_1 \wedge \pi_2]\!]_P(z) = ([\![\pi_1]\!]_P \wedge [\![\pi_2]\!])_P(z) = [\![\pi_1]\!]_P(z) \wedge [\![\pi_2]\!]_P(z) = f_z(\pi_1) \wedge f_z(\pi_2) = f_z(\pi_1 \wedge \pi_2)$. Let us show that $([\![\pi_1 \wedge \pi_2]\!])_P(\alpha, z) = u_z(\pi_1 \wedge \pi_2) \to \alpha$. By definition,

$$\begin{aligned}
&([\![\pi_1 \wedge \pi_2]\!])_P(\alpha, z) \\
&= [\![\pi_1 \wedge \pi_2]\!]_P^{[1]}(\alpha, z) \\
&= \bigwedge_{z' \in Z_A^S} [[\![\pi_1 \wedge \pi_2]\!]_P(z') \to (E_S(z', z) \to \alpha)] \\
&= \bigwedge_{z' \in Z_A^S} [f_{z'}(\pi_1 \wedge \pi_2) \to (E_S(z', z) \to \alpha)].
\end{aligned}$$

Hence, to show that $u_z(\pi_1 \wedge \pi_2) \to \alpha \le ([\![\pi_1 \wedge \pi_2]\!])_P(\alpha, z)$, we need to show that for every $z' \in Z_A^S$,

$$u_z(\pi_1 \wedge \pi_2) \to \alpha \le f_{z'}(\pi_1 \wedge \pi_2) \to (E_S(z', z) \to \alpha).$$

Since by definition $E_S(z', z) = \bigwedge_{\pi' \in \mathbf{PP}}[f_{z'}(\pi') \to u_z(\pi')] \le f_{z'}(\pi_1 \wedge \pi_2) \to u_z(\pi_1 \wedge \pi_2)$ and $\to$ is order-reversing in the first coordinate and order-preserving in the second one, it is enough to show that for every $z' \in Z_A^S$,

$$u_z(\pi_1 \wedge \pi_2) \to \alpha \le f_{z'}(\pi_1 \wedge \pi_2) \to ((f_{z'}(\pi_1 \wedge \pi_2) \to u_z(\pi_1 \wedge \pi_2)) \to \alpha).$$

By residuation, the above inequality is equivalent to

$$u_z(\pi_1 \wedge \pi_2) \to \alpha \le [f_{z'}(\pi_1 \wedge \pi_2) \otimes (f_{z'}(\pi_1 \wedge \pi_2) \to u_z(\pi_1 \wedge \pi_2))] \to \alpha.$$

The above inequality is true if

$$f_{z'}(\pi_1 \wedge \pi_2) \otimes (f_{z'}(\pi_1 \wedge \pi_2) \to u_z(\pi_1 \wedge \pi_2)) \le u_z(\pi_1 \wedge \pi_2),$$

which is an instance of a tautology in residuated lattices.

To show that $([\![\pi_1 \wedge \pi_2]\!])_P(\alpha, z) \le u_z(\pi_1 \wedge \pi_2) \to \alpha$, it is enough to find some $z' \in Z_A^S$ such that

$$f_{z'}(\pi_1 \wedge \pi_2) \to (E_S(z', z) \to \alpha) \le u_z(\pi_1 \wedge \pi_2) \to \alpha.$$

Let $z' := (f_{\pi_1 \wedge \pi_2}, u_\perp)$ such that $u_\perp : \mathbf{PP} \to \mathbf{A}$ is defined as indicated above in the base case for $\pi := \perp$, and $f_{\pi_1 \wedge \pi_2} : \mathbf{PP} \to \mathbf{A}$ is defined by the assignment

$$f_{\pi_1 \wedge \pi_2}(\pi') = \begin{cases} 1 & \text{if } \pi_1 \wedge \pi_2 \vdash \pi' \\ 0 & \text{otherwise.} \end{cases}$$

For $z' := z$, since $f_{z'}(\pi_1 \wedge \pi_2) = 1$ and $1 \to (E_S(z', z) \to \alpha) = E_S(z', z) \to \alpha$, the inequality above becomes

$$E_S(z', z) \to \alpha \le u_z(\pi_1 \wedge \pi_2) \to \alpha,$$

to verify which, it is enough to show that $u_z(\pi_1 \wedge \pi_2) \le E_S(z', z)$. Indeed, by definition, $E_S(z', z) = \bigwedge_{\pi' \in \mathbf{PP}}[f_{z'}(\pi') \to u_z(\pi')] = \bigwedge_{\pi_1 \wedge \pi_2 \vdash \pi'}[1 \to u_z(\pi')] = \bigwedge_{\pi_1 \wedge \pi_2 \vdash \pi'} u_z(\pi') \ge u_z(\pi_1 \wedge \pi_2)$, the last inequality being due to the fact that u_z is order-preserving. The case in which $\sigma := \sigma_1 \wedge \sigma_2$ is analogous to the one above, and its proof is omitted.

If $\pi := \pi_1 \vee \pi_2$, then $([\![\pi_1 \vee \pi_2]\!])_P(\alpha, z) = (([\![\pi_1]\!])_P \wedge ([\![\pi_2]\!])_P)(\alpha, z) = ([\![\pi_1]\!])_P(\alpha, z) \wedge ([\![\pi_2]\!])_P(\alpha, z) = (u_z(\pi_1) \to \alpha) \wedge (u_z(\pi_2) \to \alpha) = (u_z(\pi_1) \vee u_z(\pi_2)) \to \alpha) = u_z(\pi_1 \vee \pi_2) \to \alpha$. Let us show that $[\![\pi_1 \vee \pi_2]\!]_P(z) = f_z(\pi_1 \vee \pi_2)$. By definition,

$$[\![\pi_1 \vee \pi_2]\!]_P(z) = (\![\pi_1 \vee \pi_2]\!)_P^{[0]}(z)$$
$$= \bigwedge\nolimits_{(\alpha,z') \in Z_X^S} [(\![\pi_1 \vee \pi_2]\!)_P(\alpha, z') \to (E_S(z, z') \to \alpha)]$$
$$= \bigwedge\nolimits_{(\alpha,z') \in Z_X^S} [(u_{z'}(\pi_1 \vee \pi_2) \to \alpha) \to (E_S(z, z') \to \alpha)].$$

Hence, to show that $f_z(\pi_1 \vee \pi_2) \leq [\![\pi_1 \vee \pi_2]\!](z)$, we need to show that for every $(\alpha, z') \in Z_X^S$,

$$f_z(\pi_1 \vee \pi_2) \leq (u_{z'}(\pi_1 \vee \pi_2) \to \alpha) \to (E_S(z, z') \to \alpha).$$

Since by definition $E_S(z, z') = \bigwedge_{\pi' \in \mathbf{PP}}[f_z(\pi') \to u_{z'}(\pi')] \leq f_z(\pi_1 \vee \pi_2) \to u_{z'}(\pi_1 \vee \pi_2)$ and $\to$ is order-reversing in the first coordinate and order-preserving in the second one, it is enough to show that for every $(\alpha, z') \in Z_X^S$,

$$f_z(\pi_1 \vee \pi_2) \leq (u_{z'}(\pi_1 \vee \pi_2) \to \alpha) \to ((f_z(\pi_1 \vee \pi_2) \to u_{z'}(\pi_1 \vee \pi_2)) \to \alpha).$$

By residuation, associativity and commutativity of $\otimes$, the inequality above is equivalent to

$$f_z(\pi_1 \vee \pi_2) \otimes (f_z(\pi_1 \vee \pi_2) \to u_{z'}(\pi_1 \vee \pi_2)) \otimes (u_{z'}(\pi_1 \vee \pi_2) \to \alpha) \leq \alpha,$$

which is a tautology in residuated lattices.

To show that $[\![\pi_1 \vee \pi_2]\!]_P(z) \leq f_z(\pi_1 \vee \pi_2)$, it is enough to find some $(\alpha, z') \in Z_X^S$ such that

$$(u_{z'}(\pi_1 \vee \pi_2) \to \alpha) \to (E_S(z', z) \to \alpha) \leq f_z(\pi_1 \vee \pi_2). \tag{7}$$

Let $\alpha := f_z(\pi_1 \vee \pi_2)$ and let $z' := (f_\top, u_{\pi_1 \vee \pi_2})$ such that $f_\top : \mathbf{PP} \to \mathbf{A}$ is defined as indicated above in the base case for $\pi := \top$, and $u_{\pi_1 \vee \pi_2} : \mathbf{PP} \to \mathbf{A}$ is defined by the assignment

$$u_{\pi_1 \vee \pi_2}(\pi') = \begin{cases} 0 & \text{if } \pi' \vdash \bot \\ f_z(\pi_1 \vee \pi_2) & \text{if } \pi' \nvdash \bot \text{ and } \pi' \vdash \pi_1 \vee \pi_2 \\ 1 & \text{if } \pi' \nvdash \pi_1 \vee \pi_2. \end{cases}$$

By definition and since f_z is order-preserving and proper, $E_S(z, z') = \bigwedge_{\pi' \in \mathbf{PP}}[f_z(\pi') \to u_{\pi_1 \vee \pi_2}(\psi)] = \bigwedge_{\bot \nvdash \pi' \vdash \pi_1 \vee \pi_2}[f_z(\pi') \to f_z(\pi_1 \vee \pi_2)] = 1$. Hence, (7) can be rewritten as follows:

$$(f_z(\pi_1 \vee \pi_2) \to f_z(\pi_1 \vee \pi_2)) \to f_z(\pi_1 \vee \pi_2) \leq f_z(\pi_1 \vee \pi_2),$$

which is true since $f_z(\pi_1 \vee \pi_2) \to f_z(\pi_1 \vee \pi_2) = 1$ and $1 \to f_z(\pi_1 \vee \pi_2) = f_z(\pi_1 \vee \pi_2)$. The case in which $\sigma := \sigma_1 \vee \sigma_2$ is analogous to the one above, and its proof is omitted.

If $\sigma := \Diamond \pi$, let us show that $(\![\Diamond \pi]\!)_S(\alpha, z) = v_z(\Diamond \psi) \to \alpha$ for any $(\alpha, z) \in Z_X^P$. By definition,

$$(\![\Diamond \pi]\!)_S(\alpha, z) = R_\Diamond^{[0]}[[\![\pi]\!]_P](\alpha, z)$$
$$= \bigwedge\nolimits_{z' \in Z_A^S}[[\![\pi]\!]_P(z') \to (R_\Diamond(z, z') \to \alpha)]$$
$$= \bigwedge\nolimits_{z' \in Z_A^S}[f_{z'}(\pi) \to (R_\Diamond(z, z') \to \alpha)].$$

Hence, to show that $v_z(\Diamond \pi) \to \alpha \leq (\![\Diamond \pi]\!)_S(\alpha, z)$, we need to show that for every $z' \in Z_A^S$,

$$v_z(\Diamond \pi) \to \alpha \leq f_{z'}(\pi) \to (R_\Diamond(z, z') \to \alpha).$$

By definition and Lemma A.2, $R_\Diamond(z, z') = \bigwedge_{\pi' \in \mathbf{PP}}(f_{z'}(\pi') \to v_z(\Diamond \pi')) \leq f_{z'}(\pi) \to v_z(\Diamond \pi)$, and since $\to$ is order-reversing in the first coordinate and order-preserving in the second one, it is enough to show that for every $z' \in Z_A^S$,

$$v_z(\Diamond \pi) \to \alpha \leq f_{z'}(\pi) \to ((f_{z'}(\pi) \to v_z(\Diamond \pi)) \to \alpha).$$

By residuation, associativity and commutativity of $\otimes$, the inequality above is equivalent to

$$[f_{z'}(\pi) \otimes (f_{z'}(\pi) \to v_z(\Diamond \pi))] \otimes (v_z(\Diamond \pi) \to \alpha) \leq \alpha,$$

which is a tautology in residuated lattices.

To show that $(\![\Diamond \pi]\!)_S(\alpha, z) \leq v_z(\Diamond \pi) \to \alpha$, it is enough to find some $z' \in Z_A^S$ such that

$$f_{z'}(\pi) \to (R_\Diamond(z, z') \to \alpha) \leq v_z(\Diamond \pi) \to \alpha. \tag{8}$$

Let $z' := (f_\pi, u_\perp)$ such that $u_\perp : \mathbf{PP} \to \mathbf{A}$ is defined as indicated above in the base case for $\pi := \perp$, and $f_\pi : \mathbf{PP} \to \mathbf{A}$ is defined by the assignment

$$f_\pi(\pi') = \begin{cases} 1 & \text{if } \pi \vdash \pi' \\ 0 & \text{otherwise.} \end{cases}$$

By definition and Lemma A.2,

$$\begin{aligned} R_\Diamond(z, z') &= \bigwedge_{\pi' \in \mathbf{PP}} (f_{z'}(\pi') \to v_z(\Diamond \pi')) \\ &= \bigwedge_{\pi \vdash \pi'} v_z(\Diamond \pi') \\ &\geq v_z(\Diamond \pi), \end{aligned}$$

the last inequality being due to the fact that v_z and $\Diamond$ are order-preserving. Since $\to$ is order reversing in the first coordinate and order-preserving in the second one, to show (8) it is enough to show that

$$f_{z'}(\pi) \to (v_z(\Diamond \pi) \to \alpha) \leq v_z(\Diamond \pi) \to \alpha.$$

This immediately follows from the fact that, by construction, $f_{z'}(\pi) = 1$.

Let us show that $[\![\Diamond \pi]\!]_S(z) = g_z(\Diamond \pi)$ for every $z \in Z_A^P$. By definition,

$$\begin{aligned} [\![\Diamond \pi]\!]_S(z) &= (\![\Diamond \pi]\!)_S^{[0]}(z) \\ &= \bigwedge_{(\alpha, z') \in Z_X^P} [(\![\Diamond \pi]\!)(\alpha, z') \to (E_P(z, z') \to \alpha)] \\ &= \bigwedge_{(\alpha, z') \in Z_X^P} [(v_{z'}(\Diamond \pi) \to \alpha) \to (E_P(z, z') \to \alpha)]. \end{aligned}$$

Hence, to show that $g_z(\Diamond \pi) \leq [\![\Diamond \pi]\!]_S(z)$, we need to show that for every $(\alpha, z') \in Z_X^P$,

$$g_z(\Diamond \pi) \leq (v_{z'}(\Diamond \pi) \to \alpha) \to (E_P(z, z') \to \alpha).$$

Since by definition $E_P(z, z') = \bigwedge_{\sigma' \in \mathbf{SD}} [g_z(\sigma') \to v_{z'}(\sigma')] \leq g_z(\Diamond \pi) \to v_{z'}(\Diamond \pi)$ and $\to$ is order-reversing in the first coordinate and order-preserving in the second one, it is enough to show that for every $(\alpha, z') \in Z_X^P$,

$$g_z(\Diamond \pi) \leq (v_{z'}(\Diamond \pi) \to \alpha) \to ((g_z(\Diamond \pi) \to v_{z'}(\Diamond \pi)) \to \alpha).$$

By residuation, associativity and commutativity of $\otimes$, the inequality above is equivalent to

$$[g_z(\Diamond \pi) \otimes (g_z(\Diamond \pi) \to v_{z'}(\Diamond \pi))] \otimes (v_{z'}(\Diamond \pi) \to \alpha) \leq \alpha,$$

which is a tautology in residuated lattices.

To show that $[\![\Diamond \pi]\!]_S(z) \leq g_z(\Diamond \pi)$, it is enough to find some $(\alpha, z') \in Z_X^P$ such that

$$(u_{z'}(\Diamond \pi) \to \alpha) \to (E_P(z, z') \to \alpha) \leq g_z(\Diamond \pi). \tag{9}$$

Let $\alpha := g_z(\Diamond \pi)$ and let $z' := (g_\top, v_{\Diamond \pi})$ such that $g_\top : \mathbf{SD} \to \mathbf{A}$ maps $\top$ to 1 and every other element of $\mathbf{SD}$ to 0, and $v_{\Diamond \pi} : \mathbf{SD} \to \mathbf{A}$ is defined by the assignment

$$v_{\Diamond \pi}(\sigma') = \begin{cases} 0 & \text{if } \sigma' \vdash \perp \\ g_z(\Diamond \pi) & \text{if } \sigma' \nvdash \perp \text{ and } \sigma' \vdash \Diamond \pi \\ 1 & \text{if } \sigma' \nvdash \Diamond \pi. \end{cases}$$

By definition and since g_z is order-preserving and proper, $E_P(z, z') = \bigwedge_{\sigma' \in \mathbf{SD}} [g_z(\sigma') \to v_{\Diamond \pi}(\sigma')] = \bigwedge_{\sigma' \vdash \Diamond \pi} [g_z(\sigma') \to g_z(\Diamond \psi)] = 1$. Hence, (9) can be rewritten as follows:

$$(g_z(\Diamond \pi) \to g_z(\Diamond \pi)) \to g_z(\Diamond \pi) \leq g_z(\Diamond \varphi),$$

which is true since $g_z(\Diamond \pi) \to g_z(\Diamond \pi) = 1$ and $1 \to g_z(\Diamond \pi) = g_z(\Diamond \pi)$. The case in which $\pi := \Diamond \sigma$ is analogous to the one above, and its proof is omitted. $\square$

Theorem A.8. *The basic multi-type normal $\mathcal{L}_{\mathrm{MT}}$-logic* **L** *is sound and complete w.r.t. the class of graph-based* **A**-*frames.*

Proof. Consider a type-uniform $\mathscr{L}_{\mathrm{MT}}$-sequent $\varphi \vdash \psi$ that is not derivable in **L**. Consider the proper filter f_φ and complement of proper ideal u_ψ given by

$$f_\varphi(\chi) = \begin{cases} 1 & \text{if } \varphi \vdash \chi \\ 0 & \text{if } \varphi \nvdash \chi \end{cases}$$

and

$$u_\psi(\chi) = \begin{cases} 0 & \text{if } \chi \vdash \psi \\ 1 & \text{if } \chi \nvdash \psi. \end{cases}$$

Then $\bigwedge_\chi (f_\varphi(\chi) \to u_\psi(\chi)) = 1$, for else there would have to be a formula χ_0 such that $f_\varphi(\chi_0) = 1$ and $u_\psi(\chi_0) = 0$, which would mean that $\varphi \vdash \chi_0$ and $\chi_0 \vdash \psi$ and hence that $\varphi \vdash \psi$, in contradiction with the assumption that $\varphi \vdash \psi$ is not derivable. It follows that (f_φ, u_ψ) is a state (of the appropriate type) in the canonical model $\mathbb{M}$. By the Truth Lemma, $[\![\varphi]\!](z) = f_\varphi(\varphi) = 1$, and moreover

$$[\![\psi]\!]^{[0]}(z)$$
$$= \bigwedge_{(\alpha, z') \in Z_X} [\![\psi]\!](\alpha, z') \to (E(z, z') \to \alpha)$$
$$\leq [\![\psi]\!](0, z) \to (E(z, z) \to 0)$$
$$= (u_\psi(\psi) \to 0) \to (E(z, z) \to 0)$$
$$= (0 \to 0) \to (1 \to 0)$$
$$= 0,$$

which proves the claim. $\quad\square$

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