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ESTIMATION OF SPACECRAFT ATTITUDE MOTION AND VIBRATIONAL MODES USING SIMULTANEOUS DUAL-LATITUDE GROUND-BASED DATA

By

Zachary William Henry

A Thesis Submitted to the Faculty of Embry-Riddle Aeronautical University

In Partial Fulfillment of the Requirements for the Degree of

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This Thesis was prepared under the direction of the candidate's Thesis Committee Chair, Dr. Troy Henderson, Department of Aerospace Engineering, and has been approved by the members of the Thesis Committee. It was submitted to the Office of the Senior Vice President for Academic Affairs and Provost, and was accepted in the partial fulfillment of the requirements for the Degree of Master of Science in Aerospace Engineering.

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ABSTRACT

Cutting-edge Space Situational Awareness (SSA) research calls for improved methods for rapidly characterizing resident space objects. In this thesis, this will take the form of speeding up convergence of spacecraft attitude estimates, and of a non-model-based approach to the detection of vibrational modes. Because attitude observability from photometric data is angle-based, dual-site simultaneous photometric observations of a resident space object are predicted to improve the convergence speed and steady-state error of spacecraft attitude state estimation from ground-based sensor data. Additionally, it is predicted that by adding polarimetric data to the measurements, the speed of convergence and steady-state error will be reduced further. This thesis models satellite motion and measurements from ground-based sensors for dual-latitude simultaneous light curve simulation, then develops a data fusion process to combine photometric, astrometric, and polarimetric data from both sites in order to more quickly estimate the attitude of an RSO. The Fractional Fourier Transform shows promise as a non-model-based approach to the detection of input vibrational frequencies from the degree of linear polarization. The main results are that dual-site observation geometry is conducive to slight improvements of attitude filter performance, and the addition of polarimetric data to the measurements yields much improved performance over both the single-site and dual-site cases.

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NOMENCLATURE

В	=	body-fixed frame
BRDF	=	bidirectional reflectance distribution function
C	=	earth-centered observer-fixed frame
C_{sum}	=	visible-light flux from Sun
C_{IJ}	=	direction cosine matrix from frame J to frame I
d	=	fraction of reflected diffuse light, distance from object to observer
DCM	=	direction cosine matrix
\mathbf{e}_k	=	measurement prediction error
E	=	nadir-pointing satellite co-moving frame
EOM	=	equations of motion
$f(\omega, t)$	=	time derivatives of angular velocities from EOM
$F(c,\lambda)$	=	Fresnel reflectance at $c=\cos\theta$ and wavelength λ
FIM	=	Fisher information matrix
F_{obs}	=	reflected radiation flux
G	=	process noise covariance matrix
GOES	=	geostationary operational environmental satellite
Н	=	angle bisector of L and V
$\begin{bmatrix} J I_I \end{bmatrix}$	=	mass moment of inertia of object I in frame J
I_i	=	incident light intensity
I_r	=	reflected light intensity
K_k	=	Kalman gain
[K]	=	stiffness matrix for lumped-parameter solar panel vibration model
L	=	system Lagrangian
L	=	vector pointing from facet of RSO to light source
М	=	visual magnitude

[M]	=	mass matrix for lumped-parameter solar panel vibration model
n	=	satellite mean motion
n_i	=	input to reflectance model controlling specular lobe shape
n_j	=	input to reflectance model controlling specular lobe shape
N	=	earth-centered inertial frame
Ν	=	surface normal vector of RSO facet
Р	=	solar panel coordinate frame
P_k^-	=	predicted state estimation covariance
P_k^+	=	updated state estimation covariance
P_k^{ee}	=	measurement error covariance
P_k^{xy}	=	state-measurement cross-correlation matrix
Q	=	state process noise matrix; 0° and 90° linear polarization state
Q	=	vector of generalized forces
Q_i	=	i^{th} component of generalized force vector
q	=	vector of generalized coordinates
q_i	=	i^{th} component of generalized coordinate vector
r	=	arbitrary position vector
\mathbf{r}_1	=	ECIF position vector of site 1
\mathbf{r}_2	=	ECIF position vector of site 2
\mathbf{r}_{E}	=	ECIF position vector of RSO
R	=	bidirectional reflectance; measurement process noise matrix
R_a	=	bidirectional reflectance of ambient light
R_d	=	bidirectional reflectance of diffuse light
R_s	=	bidirectional reflectance of specular light
RSO	=	resident space object
s	=	fraction of reflected specular light
S	=	Stokes vector

SVST	=	satellite visualization and signature tool
T	=	total kinetic energy
$[ilde{u}]$	=	cross-product matrix of arbitrary vector u (see eq. 2.6)
U	=	$\pm45^{\circ}$ linear polarization state
V	=	total potential energy; circular polarization state
v(t)	=	zero-mean gaussian white measurement noise
\mathbf{v}_{0f}	=	velocity of stationary component of fuel
\mathbf{v}_{f}	=	velocity of sloshing component of fuel
\mathbf{v}_{0o}	=	velocity of stationary component of oxidizer
\mathbf{v}_o	=	velocity of sloshing component of oxidizer
V	=	vector pointing from facet of RSO to viewer
$\mathbf{w}(t)$	=	zero-mean gaussian white process noise vector
$W_f(t,\omega)$	=	Wigner-Ville energy distribution function
X	=	state vector
$\hat{\mathbf{x}}_k^-$	=	pre-update state vector estimate
$\hat{\mathbf{x}}_k^+$	=	post-update state vector estimate
x	=	coordinate distance along solar panel neutral axis \hat{p}_1
$\tilde{\mathbf{y}}$	=	measurement vector
$\hat{\mathbf{y}}_k^-$	=	predicted measurement vector
y	=	displacement of solar panel at point x from neutral axis
α	=	right ascension; angle between H and N
β	=	4×1 quaternion vector
$\gamma_{i,k}$	=	measurement prediction from sigma point
δ	=	declination
$\delta \mathbf{p}_{i,k}$	=	error GRP
$\delta \beta_{i,k}$	=	error quaternion
θ	=	angle between two observatory sites, half-angle between ${\bf L}$ and ${\bf V}$

component of body angular velocity normal to solar panel surface
local sidereal time
BRDF; density
range vector from site 1 to RSO
diffuse BRDF
specular BRDF
standard deviation of GRP process noise
standard deviation of visual magnitude process noise

- σ_{ω} = standard deviation of angular velocity process noise
- τ = torque due to solar panel vibration
- ϕ = shape of assumed modes for solar panel
- Φ = principal rotation angle
- $\chi_{i,k}$ = sigma point

 $\dot{\theta}$

Θ

ρ

 ρ_1

 ρ_d

 ρ_s

 σ_{grp}

 σ_{mv}

=

=

=

=

=

=

=

=

 ${}^{I}\omega_{J}$ = angular velocity vector of frame J with respect to frame I

1. Introduction and Background

Whereas one might assume the field of aerospace engineering is confined primarily to the design nuances of spacecraft and aircraft, there are actually many difficulties presented in the field which are less than obvious. Consider, for example, the growing number of spacecraft in terrestrial orbit. As of this writing (December, 2020), there are 3,355 active satellites in the publicly available NORAD catalog, Kelso (2020). This does not paint the full picture, however, since this includes only the operational spacecraft. If extended to all types of objects in Earth orbit (excluding the Moon, of course), there are 19,851 valid two-line element (TLE) sets and 1,372 lost TLE sets. To take things a step further, SpaceX has plans to launch a total of 42,000 satellites as part of its Starlink space-based internet service (C. Henry, 2019). Knowing all of this, it becomes increasingly obvious that we could be presented with a significant problem when launching and operating satellites.

To provide some additional motivation, consider the cost to develop, produce, and operate a satellite. For example, the average cost of a GOES meteorological satellite in fiscal-year 2000 was around \$84 million (Wertz & Larson, 1999), or close to \$125 million in fiscal-year 2020 when adjusted for inflation. This does not even include the cost of the launch vehicle, which could in some cases double the cost. When dealing with such high stakes, it follows that it is important to understand the potential hazards that could be created if an anomalous resident space object (RSO) imposes on any assets. What naturally comes with understanding an object is a need to understand its dynamics, which are influenced not only by the shape and size of the object, but also by external perturbing forces like gravitational fields of other bodies and solar wind, as well as by the internally generated disturbances. Some of these disturbances could include torques due to vibration of solar panels, sloshing of propellant, reaction wheel momentum dumping, impulsive attitude maneuvers, etc. When the only information available is in the form of time, azimuth, elevation, and some unresolved characteristics of the reflected light, capturing the effects of these internal torques and then correlating them to their source proves to be a difficult problem to solve, and is currently an interesting and important topic of study in the broader fields of space situational awareness (SSA), or more recently space domain awareness (SDA), and this is the problem to be addressed by this thesis in some fashion.

There has been a significant amount of development in the capabilities in the last twenty or so years of not only the hardware and software, but also in the creative usage of available information. It turns out that there is a surprising amount of useful information contained within even the minute changes in brightness of space objects. The analysis of an object's brightness as a function of time is known as *photometry* (Roth, 2009), and it has been applied to the characterization and classifications of stars, planets, and even asteroids (Kaasalainen & Torppa, 2001). The latter is of particular interest, since asteroids are relatively small and dim, much like an RSO in orbit around the Earth. The techniques applied to the characterization of these rocks has been extended into the realm of SSA, and the brightness data may be associated and correlated with other sources of data to improve confidence in both the estimated and inferred characteristics of an RSO. This unification of multiple data sources for refining estimates falls under *data fusion* (Mahler, 2004), the theory upon which a large part of SSA has been built.

To ground-based observers, the available sources of data in SSA for object characterization are reflected light in the form of total brightness, polarization states, reflected spectrum, Doppler shift, etc. (Hapke, 2012), right ascension, declination, and range, the last of which can often be available only when using a source of active illumination such as a laser (i.e. using a laser range finder). Unfortunately, pointing a laser at an object can interfere with sensitive instrumentation and as such should only be used with permission of the owner, lest it be considered an act of war. Thus in the case of unknown objects, we are limited to the object's position on the celestial sphere and to the reflected light from passive sources of illumination, primarily being the Sun and possibly the reflected light from the Moon or the Earth. Of note, light which is reflected off some surface will become polarized, the degree of which is an additional piece of useful (albeit difficult and/or costly to obtain) information. Right ascension and declination will come from the combination of geographic location of the observation platforms and surrounding stars in the background image, which may be compared and then matched to templates (or plates) of known stars in a process known as *astrometry* (Roth, 2009). All of these sources of data can, when combined, yield a diverse repertoire of information about an object of interest.

To unify all of these data sources and then draw conclusions about an object, it is necessary to have an understanding of some of the physical processes which govern the motion of the object and the noise in the measurements. Of primary interest to the characterization of the vehicle in this thesis are going to be the oscillatory motion of a solar panel and of propellants (as explained further in Chapter 2 and Chapter 3), since these are some of the more interesting contributors as discovered during preliminary research with advisors. The torques induced by these vibrations will have an effect on the vehicle pointing direction, or *attitude* (Hughes, 2004), and as such estimation of attitude motion can potentially provide insight into the internal dynamics and henceforth allow for inferences to be made about the types of propellants being used. Information about the attitude will be contained in the brightness and polarization states, since the irradiance measurements will come from reflected (i.e. scattered) light (Whittaker, Linares, & Crassidis, 2013). The models used to predict the reflection of light were first developed for computer graphics, but have been repurposed to serve as astronomical and engineering tools.

There has already been some investigation into light-curve-based characterization of RSOs, but this has, to the author's knowledge, been confined primarily to measurements taken from a single site and has scarcely included polarimetric data. Hence, the key results to be shown in this thesis are that the addition of a second observation site for attitude filtering will yield slightly improved speed of convergence and steady-state error, and that

the addition of polarimetric data yields a significant improvement over photometric data alone whether taken from a single site or a dual site. Additionally, to detect the vibrational modes for more complete vehicle characterization and to infer additional information, the Fractional Fourier Transform (FrFT) will be applied to the degree of linear polarization for detection of oscillation frequencies.

This thesis is organized in the following way. Chapter 2 will proceed with a review of the current literature in photometric attitude estimation, followed by a review of the fundamental concepts of mechanics which are to be applied, including general methods of analytical dynamics, some aspects of fluid and solid mechanics, and a concise review of some necessary orbital mechanics. After this, an overview of photometry, reflectance modeling, and polarimetry will be provided. Finally, a description of the process for turning images into useful data will be given, and the chapter will then conclude with a review of the GOES-R mission, of which GOES-16 is to serve as the primary object of study. Chapter 3 will begin with development of a simplified shape model of GOES-16 and of the relevant reference frames. Following this, the system Lagrangian will be derived including the effects of slosh and solar panel motion and the method for obtaining the equations of motion from the Lagrangian is described. Next, the complete model uniting measurements and assumed dynamics is defined. To conclude this chapter is a review of unscented Kalman filtering and an overview of the simulation test cases. Chapter 4 discusses the results of simulating the complex attitude motion of the satellite, then Chapter 5 covers first the vibrational mode detection results followed by the filtering results. Finally, Chapter 6 discusses the implications of the prior chapters and offers suggestions for future work.

2. Theory

With the background and motivation for dual-latitude photometric estimation of attitude, the problem must now be described in greater detail. This chapter will begin with a review of the current photometric attitude estimation literature and an outline of the proposed improvements. Following this, a review of the necessary fundamentals of mechanics will be presented, covering the basics of Lagrangians, attitude dynamics, Eulerian mechanics (AKA rigid body mechanics), low-gravity fluid mechanics, Euler-Bernoulli beam vibration, and finally a dense discussion of orbital mechanics. After this, the details of reflectance modeling and photometric, polarimetric, and astrometric data acquisition through imaging will be explained and their connection to spacecraft attitude states will be defined. Finally, there is a short review of reference systems and coordinate transformations, followed by a brief description of the GOES-R series of geostationary weather satellites.

2.1. Attitude Estimation From Photometry

For nearly two decades, light curves have been used in astronomy to estimate the shapes and sizes of asteroids (Kaasalainen & Torppa, 2001; Kaasalainen, Torppa, & Muinonen, 2001). The general technique has been to first assume some dynamical model relating the motion of the asteroid within its own frame, then relate this to the variations in brightness. When the rotation and light scattering properties are known, there exists a unique shape which will create any given light curve. Extending the estimated parameters to include the rotation and scattering properties was, in this instance, a matter of performing a grid search over the possible values until finding a solution to match the measurements. This is largely possible because asteroids are convex bodies and the algorithms used converge robustly toward the same unique solution when parameters are correct. While there have likely been many refinements in this algorithm since then which

are applicable to asteroids, this technique has been taken and adapted to estimation of the shape, size, and rotation rates of RSOs.

By adapting the work of Crassidis (2003) in developing an unscented filter for attitude estimation, it has been shown possible to use light curve inversion to estimate attitude motion when the shape is well known (C. Wetterer & Jah, 2009). In a similar fashion to the complete asteroid light curve inverse problem, this work has been extended to include surface parameters, geometry, and other features by fusion of astrometric and photometric data (Jah & Madler, 2007; Linares, Crassidis, Jah, & Kim, 2010; C. J. Wetterer, Chow, Crassidis, Linares, & Jah, 2013). There can be difficulty in detecting the synodic variations in brightness for high altitude RSOs, so shape-dependent analysis methods may be preferable since they do not require detection of synodic variations (Hall & Kervin, 2014). There have been methods developed for determining the albedo, shape, and size of high-altitude RSOs based only on temporal photometry (Hall, Calef, Knox, Bolden, & Kervin, 2007; Hejduk, Cowardin, & Stansbery, 2012), so these parameters will be assumed known to a level of certainty which allows for a simple shape model of the satellite. Hejduk (2007) showed that deep-space orbital debris objects have different photometric properties from other spacecraft types, so it can be assumed that an object of interest is known to be an operational satellite.

While the majority of the aforementioned papers show a great deal of promise, a number of attitude observability studies have been performed and make the outlook appear more grim. First, the chosen model for reflectance will have a significant impact on the brightness predictions (Subbarao & Henderson, 2019). This can be problematic, since determining with certainty which model is "best" for any given task is not a simple task. For the sake of development, the most commonly used model in SSA applications will be adopted for this thesis, namely that of Ashikhmin and Shirley (2000). Using this reflectance model as a performance baseline, Hinks, Linares, and Crassidis (2013) showed

that the observability of local disturbances depends on the projected attitude angles of the object itself.

To begin discussion of improvements, define the Fisher Information Matrix (FIM) as,

$$FIM = E\left\{ \left[\frac{\partial}{\partial \mathbf{x}} \ln p(\mathbf{y}|\mathbf{x}) \right]^T \left[\frac{\partial}{\partial \mathbf{x}} \ln p(\mathbf{y}|\mathbf{x}) \right] \right\}$$
(2.1)

where x denotes an arbitrary vector of system states to be estimated, and y denotes a vector of measurements which depend on x. The FIM serves as a means of quantifying the total amount of information about the system states contained in a collection of measurements, and its inverse is the Cramèr-Rao lower bound on the state estimation error covariance Q (Crassidis & Junkins, 2012):

$$Q = E\left\{ (\mathbf{x} - \mathbf{\hat{x}})(\mathbf{x} - \mathbf{\hat{x}})^T \right\} \ge FIM^{-1}$$
(2.2)

The above inequality, known as the Cramér-Rao rule, suggests from an intuitive standpoint that a higher value of ||FIM|| will yield a lower bound on estimation covariance, and thus improve the observability of the attitude. The primary source of this intuition is offered in the following lemma and proof.

Lemma. The inverse of a real-valued and positive definite matrix αA , where $\alpha > 1$ is a real scalar, will have a lower norm than the inverse of the matrix A.

Proof. Define $A : A \in \mathcal{R}^{n \times n}$, A > 0 and $\alpha : \alpha \in \mathcal{R}$, $\alpha > 1$. Then,

$$(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$$

Taking the norm of both sides,

$$\|(\alpha A)^{-1}\| = \frac{1}{|\alpha|} \|A^{-1}\|$$

but since $\alpha > 1$,

$$\frac{1}{|\alpha|} \|A^{-1}\| = \frac{1}{\alpha} \|A^{-1}\|$$

It is self-evident now that $\frac{1}{\alpha} \|A^{-1}\| < \|A^{-1}\|$ since $\frac{1}{\alpha} < 1$. QED

The dependence of ||FIM|| on attitude is shown more clearly in Figure 2.1, where the log of the spectrum of FIM becomes large in two regions: first where the attitude perturbation is small, and second where the attitude approaches the edges of the attitude range which reflects light to the observer. On the edges, however, measurements may be difficult since these correspond to extremely dim values of apparent magnitude as seen in Figure 2.2.



Figure 2.1 Information Magnitude as a Function of Attitude (Hinks et al., 2013, p. 7).

Of note, this attitude information study was confined to measurements from a single site, and thus the information matrix allows for only the given regions. There has been some work which suggests that the convergence of orbit determination filters can be sped up using dual-site observations, as shown in Figure 2.3 (Z. W. Henry, Vavala, Zuehlke, Henderson, & Grage, 2020). While there has been some work done on simultaneous



Figure 2.2 Apparent Brightness as a Function of Attitude (Hinks et al., 2013, p. 4).

dual-site brightness observations (Fulcoly, Kalamaroff, & Chun, 2012; Gasdia, Barjatya, & Bilardi, 2017), it has not yet included extensive analysis of improved observability of attitude states or of state estimate convergence speed. Knowing all of this, the problem to be investigated in the present work is the improved observability of attitude motion from dual-latitude simultaneous observations. Should the attitude states become significantly more visible to the analyst, multi-site observations could become essential to the future of spacecraft characterization.

An additional measurement which could increase ||FIM|| may be the polarization states of the reflected light, since the degree of polarization is not completely dependent on the total amount of reflected light and there are additional effects to consider. There is currently research being performed which investigates both spacecraft seismology and attitude/material estimation by using polarimetric remote sensing techniques. These papers are relatively new and there is a significant amount of research to be done in these areas. For example, Watson and Hart (2017); Watson, Hart, Hilton, Codona, and Pereira (2018) were able to reconstruct audio signals from surface acoustics of a metal object which were detected using only an optical polarimeter. This is promising, since it has been



Figure 2.3 Percent error in estimated position for single-site and dual-site filtering (Z. W. Henry, Vavala, et al., 2020).

suggested that photometry may be insufficient for detecting these small-amplitude seismic activities (Z. W. Henry, Udrea, et al., 2020). In further support of polarimetric investigation, Dianetti and Crassidis (2019) showed a promising result that surface materials may be accurately and reliably determined from polarized light curves. Given this aforementioned work, it is worth investigating the effects of polarimetric remote sensing on satellite attitude observability and seismic detection.

Northern hemisphere dual-latitude observation geometry is shown in Figure 2.4, where, from the Earth center, \mathbf{r}_1 and \mathbf{r}_2 point to the northern and southern observer, respectively. Note that the RSO position vector \mathbf{r}_E and the range vectors ρ_1 and ρ_2 which point from each site to the RSO are not necessarily in the same plane as \mathbf{r}_1 and \mathbf{r}_2 . Such an



Figure 2.4 Dual-latitude observation geometry (Z. W. Henry, Vavala, et al., 2020).

observation geometry could extend the range of observable states from brightness alone, forming a basis for this thesis. It is predicted that, given some angle γ_3 between ρ_1 and ρ_2 , there will be an increased attitude observability range. This will be tested by developing a mathematical model of measurements and satellite dynamics, followed by simulation of observations of an object perturbed by solar panel vibrations and propellant slosh. Finally, this will be compared to data collected on the GOES 16 satellite using a real telescope located atop the MicaPlex building at the Daytona Beach campus of Embry-Riddle.

2.2. Fundamentals of Mechanics

This section will provide an overview of the fundamentals of mechanics which are required for understanding the derivation of the satellite attitude equations of motion. It thus offers no novel results, but its contents are presented due to their importance for understanding the later developments. The most important results from many sources of general mechanics information will be presented in this chapter (Goldstein, 1980; Hughes, 2004; Lanczos, 1970; Meirovitch, 2003; Schaub & Junkins, 2018). The latter reference is particularly useful for mechanics as it pertains to space vehicles, and Hughes is a specialized text for spacecraft attitude dynamics. The rest are more general, but their contents are useful and insightful nonetheless.

There are two primary goals of solving problems in dynamics: to describe the motion of a body or system of bodies, and to predict the motion. Both of these can be difficult to achieve when systems become complex, so several systematic methods of setting up and solving for motion have been developed throughout history. The best known is the method of Sir Isaac Newton due to the ingenious description of motion in terms of three universal laws, most importantly that motion is the result of unbalanced external forces. Although Newton's method for solving dynamics problems is intuitive for small systems, it can become rapidly complex when considering many objects whose motion depends not only on each other but also on the externally applied forces. Thus arose the need for a more holistic approach to the problems of mechanics which was eventually met by Joseph-Louis Lagrange in his generalization of mechanics to arbitrary systems.

In Lagrange's formulation of dynamics, he derived using the methods of the calculus of variations a scalar quantity dependent on the kinetic energy and potential energy of the system as a whole. This came to be known as the Lagrangian, which will be denoted as L. It is defined as,

$$L = T - V \tag{2.3}$$

where T and V are respectively the total kinetic and potential energy of the entire system. Additionally, Lagrange showed that from L, the i^{th} equation of motion can be computed directly as,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = Q_i \tag{2.4}$$

where q_i are generalized coordinates and Q_i are generalized forces derived from nonconservative external influences. This method provides a benefit over Newton's vector method since there is no need to consider the internal forces between particles, it requires kinematics only on the level of velocity, and it involves only the scalar-valued function L. There are some situations for which Newton's method is superior, but it becomes exceedingly tedious when a system becomes more complex. Of interest in this study is the attitude motion of a spacecraft as a result of several internally-generated torques, so the methods of Lagrange will be used to set up the dynamical model.

While Equation 2.4 is an incredibly useful tool, it is not always the best approach to solving the problems of dynamics in its basic form. In the case of attitude dynamics, where one might wish to express the motion in terms of attitude quaternions (i.e. Euler parameters), it can be more efficient to make use of the so-called "quasi-coordinates" which arise when selecting how to represent the velocity or acceleration before the

displacement. For this formulation, Equation 2.4 takes the form (Meirovitch, 2003),

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \omega}\right) + \left[\tilde{\omega}\right]\frac{\partial T}{\partial \omega} = \mathbf{Q}$$
(2.5)

where ω may be integrated to get the quasi-coordinates and $[\tilde{\omega}]$ is the skew-symmetric cross-product matrix:

$$[\tilde{\omega}] \equiv \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}.$$
 (2.6)

In the case of attitude mechanics, the components of ω refer to the angular velocities of the body as described in the body-fixed frame. It is numerically convenient to express the attitude in terms of quaternions (i.e. Euler parameters) since they are non-singular in their description of attitude. For a given rotation angle Φ about some principal axis of rotation \hat{e} , the quaternion vector is defined as (Schaub & Junkins, 2018),

$$\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} \cos(\Phi/2) \\ e_1 \sin(\Phi/2) \\ e_2 \sin(\Phi/2) \\ e_3 \sin(\Phi/2) \end{bmatrix}$$
(2.7)

subject to the constraint,

$$\beta_0^2 + \beta_1^2 + \beta_2^2 + \beta_3^2 = 1 \tag{2.8}$$

First defining the following matrix,

$$B(\beta) = \begin{bmatrix} -\beta_1 & -\beta_2 & -\beta_3 \\ \beta_0 & -\beta_3 & \beta_2 \\ \beta_3 & \beta_0 & -\beta_1 \\ -\beta_2 & \beta_1 & \beta_0 \end{bmatrix}$$
(2.9)

the quaternions are related to the body-frame angular velocities by the kinematic equation,

$$\dot{\beta} = \frac{1}{2}B(\beta)\omega \tag{2.10}$$

An assumption which will be made for several components of this system is that there is no relative motion between the mass elements of a continuous body, i.e. they will be assumed to be rigid bodies. This assumption can be made for the main body of the spacecraft and to some extent for the solar panels. When the vibrations are small (as they often are), large-scale motion of the solar panels of a body can be predicted under a rigid-body assumption, and the effects of vibrations may be applied as torques which act on the spacecraft as a whole. The kinetic energy of an infinitesimal mass element of any body is,

$$dT = \frac{1}{2}\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}dm \tag{2.11}$$

which leads to a closed-form expression for the kinetic energy of the entire body:

$$T = \frac{1}{2} \int_{B} \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} dm \qquad (2.12)$$

where $\dot{\mathbf{r}}$ represents the velocity of each mass element dm in an inertial frame N. By the transport theorem, this can be expressed as (Schaub and Junkins, 2018),

$${}^{N}\dot{\mathbf{r}} = {}^{B}\dot{\mathbf{r}} + {}^{N}\omega_{B} \times \mathbf{r}$$
(2.13)

where the superscripts denote the reference frames in which each vector is described, and the subscript *B* denotes the body being described. Because each mass element of a rigid body does not move with respect to its body-fixed frame, ${}^{B}\dot{\mathbf{r}} = 0$ and thus *T* takes the form,

$$T = \frac{1}{2} \int_{B} ({}^{N}\omega_{B} \times {}^{N}\mathbf{r}) \cdot ({}^{N}\omega_{B} \times {}^{N}\mathbf{r}) dm$$
(2.14)

This may also be written as,

$$T = \frac{1}{2} \int_{B}{}^{N} \omega_{B}^{T} \left[{}^{N} \tilde{\mathbf{r}}\right]^{T} \left[{}^{N} \tilde{\mathbf{r}}\right]^{N} \omega_{B} dm \qquad (2.15)$$

where $[\tilde{\mathbf{r}}]$ is the skew-symmetric cross-product matrix corresponding to \mathbf{r} . The absence of relative motion between the mass elements in the body-fixed frame also implies that ${}^{N}\omega_{B}$ is constant across the body, so the kinetic energy simply takes the form,

$$T = \frac{1}{2}{}^{N}\omega_{B}^{T} \left(\int_{B} \left[{}^{N}\tilde{\mathbf{r}} \right]^{T} \left[{}^{N}\tilde{\mathbf{r}} \right] dm \right) {}^{N}\omega_{B} \equiv \frac{1}{2}{}^{N}\omega_{B}^{T} \left[{}^{N}I_{B} \right] {}^{N}\omega_{B}$$
(2.16)

Here, the integral term defines the mass moment of inertia for the body. This may also be computed from the center of mass of the body at \mathbf{r}_{cg} by applying the parallel axis theorem:

$$\begin{bmatrix} {}^{N}I_{B} \end{bmatrix} = \begin{bmatrix} {}^{B}I_{B} \end{bmatrix} + M[\tilde{\mathbf{r}}_{cg}][\tilde{\mathbf{r}}_{cg}]^{T}$$
(2.17)

where M is the total mass of the body. From here, it is a simple task to obtain the kinetic energy of any rigid body. By combining Equations 2.16 and 2.17, the kinetic energy of any rigid body B is as follows:

$$T_B = \frac{1}{2}{}^N \omega_B^T \left[\left[{}^B I_B \right] + M [\tilde{\mathbf{r}}_{cg}] [\tilde{\mathbf{r}}_{cg}]^T \right] {}^N \omega_B$$
(2.18)

2.3. Mechanics of Slosh

Before explaining the energy contributions of slosh to the Lagrangian, it will be useful to explain what is meant by the term "slosh" in the case of space vehicles. While the key results for slosh modeling will be presented here, a more thorough treatment of propellant slosh is contained in the NASA monograph entitled *The Dynamic Behavior of Liquids in Moving Containers* (Abramson, 1966) or its updated version (Dodge, 2000). In microgravity, the "sloshing" of propellant most frequently refers not to large-amplitude motion of the fluid center of mass, but rather to the small-scale motion of the its free surface. This motion is highly nonlinear and there has yet to be any closed-form solution for it, but a number of approximations have entered the literature and become standard for estimating the effects of propellant slosh. One such approximation is that of a mass spring dampener (MSD) lumped-parameter system with a portion of the propellant mass assumed stationary, and the fluid assumed to be composed of several linear MSD systems, as shown in Figure 2.5. For the sake of space, the details for calculating the model parameters will not be presented here and the results alone are instead shown in Table 2.1.



Figure 2.5 Mass-spring-dampener approximation model for propellant slosh (Dodge, 2000, p. 44).

Table 2.1

Parameter	Value
Spring Constant, K	$m_{liq}\left(\frac{g}{1.19h}\right)\left(\tanh 3.68\frac{h}{d}\right)^2$
Slosh mass, m	$m_{liq}\left(\frac{d}{4.4h}\right) \tanh 3.68\frac{h}{d}$
Stationary Mass, m_0	$m_l i q - m$
Distance from m to surface, l	$\frac{d}{3.68} \tanh 3.68 \frac{h}{d}$
Distance from m_0 to surface, l_0	$rac{m_{liq}}{m_0}\left(rac{h}{d}-rac{d^2}{8h} ight)-lrac{m}{m_0}$

Model Parameters for computing slosh motion in a cylindrical tank (Abramson, 1966).

This model assumes that the Bond number–a measure of the relative importance of inertial versus capillary and surface tension forces–is sufficiently low that the fluid will remain fairly stationary and essentially be "stuck" to the sides of the tank. The Bond number is defined to be,

$$Bo = \frac{\rho g r^2}{\sigma} \tag{2.19}$$

where ρ is the fluid density, g is the net acceleration of the spacecraft, r is the characteristic radius, and σ is the surface tension. In general, there will be some critical Bond number Bo_{crit} at which the fluid interface becomes unstable and there is a large shift in the fluid center of mass. The computation of Bo_{crit} depends greatly on the properties of the fluid and on the shape of the tank, but can be expected to be between 1 and 3 for a variety of conditions (Dodge, 2000). Because g will certainly be very low for an orbiting satellite under nominal conditions, it will be assumed that $Bo \equiv Bo_{crit}$ for the purposes of simulation and the bulk of the fluid acceleration will be assumed due to surface tension forces such that Equation 2.19 may be rearranged to be,

$$g = Bo_{crit} \frac{\sigma}{\rho r^2} \tag{2.20}$$

To justify the use of Equation 2.20 for the restorative acceleration, the Cassini spacecraft propellant tanks were roughly shaped and assumed 40% then 70% full. The

slosh frequencies for nitrogen tetraoxide (NTO) in the oxidizer tank and monomethylhydrazine (MMH) in the fuel tank were calculated. These values were then compared to those predicted by Enright and Wong (1994) and those actually measured by Lee and Stupik (2015). These are presented in Tables 2.2 and 2.3, and it is clear that the use of Equation 2.20 is justifiable for creating a "ballpark" estimate of the slosh for the purposes of this system.

Table 2.2

Comparison of predicted slosh frequency to prior results for Cassini spacecraft at 40% fill.

	Equation 2.20	Enright and Wong	Lee and Stupik
NTO	2.914 mHz	2.9 mHz	2.81 mHz
MMH	4.512 mHz	4.5 mHz	4.36 mHz

Table 2.3

Comparison of predicted slosh frequency to prior results for Cassini spacecraft at 70% fill.

	Equation 2.20	Enright and Wong	Lee and Stupik
NTO	3.081 mHz	3.2 mHz	3.30 mHz
MMH	4.771 mHz	5.1 mHz	5.12 mHz

2.4. Euler-Bernoulli Beams

The contributions of the solar panel vibrations to the attitude motion of the spacecraft can be approximated using an Euler-Bernoulli beam. Euler-Bernoulli beam theory arises as a simplification of continuum mechanics which applies to small deflections of a beam subject only to lateral loads. It is applicable here because generally a solar panel will not have large-amplitude vibrations, and the shape of the panel largely confines any deflections to the direction normal to its surface. See Junkins and Kim (1993) and Meirovitch (1967) for more information. To arrive at this, we note that for a rotating hub with a flexible body attached to it that (Schaub & Junkins, 2018),

$$T_{panel} = \frac{1}{2} I_{hub} \dot{\theta}^2 + \frac{1}{2} \int_{r_0}^{L} \rho A (\dot{y} + x\dot{\theta})^2 dx$$
(2.21)

$$V_{panel} = \frac{1}{2} \int_{r_0}^{L} EI(y'')^2 dx$$
 (2.22)



Figure 2.6 Euler-Bernoulli beam on a rotating hub (Junkins and Kim, 1993, p. 157).

where x is the coordinate along the panel neutral axis \hat{p}_1 , y = y(x, t) is the displacement of the panel from that axis in the \hat{p}_3 direction, and $\dot{\theta}$ is the component of hub rotation normal to the surface of the solar panel. Because the solar panel of GOES 16 resembles a cantilevered beam, the appropriate boundary conditions are,

$$y(r_0, t) = 0$$
 $y'(r_0, t) = 0$
 $y''(L, t) = 0$ $y'''(L, t) = 0$

By assuming a quadratic lumped-parameter approximation of mode shapes $\phi(x)$ with n modes and that y(x,t) is expressable as,

$$y(x,t) = \sum_{k=1}^{n} \phi_k(x) q_k(t),$$
(2.23)

the equation of motion for the solar panel becomes,

$$[M]\ddot{\mathbf{q}} + \left([K] - \dot{\theta}^2(-2[H] - [M])\right)\mathbf{q} = -\ddot{\theta}\mathbf{N}$$
(2.24)

where \mathbf{q} is a vector of time functions to be solved for, and first defining,

$$\psi_{ij} = -\frac{1}{2} \int_0^x \phi'_i \phi'_j dz,$$
(2.25)

the matrices [M], [H], and [K] are square with elements defined as,

$$M_{ij} = \int_0^L \rho \phi_i \phi_j dx \tag{2.26}$$

$$K_{ij} = \int_0^L EI\phi_i''\phi_j''dx \tag{2.27}$$

$$H_{ij} = \int_{0}^{L} \rho(R+x)\psi_{ij}dx$$
 (2.28)

and N is a vector with components,

$$N_i = \int_0^L (R+x)\phi_i dx.$$
 (2.29)

Now by assuming the vibrational mode shape to be,

$$\phi_k(x) = \left(\frac{x}{L}\right)^{k+1},\tag{2.30}$$

Equations 2.26-2.29 become,

$$M_{ij} = \frac{\rho L}{i+j+3} \tag{2.31}$$

$$K_{ij} = EI \frac{ij(i+1)(j+1)}{L^3(i+j-1)}$$
(2.32)

$$H_{ij} = -\frac{\rho(i+1)(j+1)L}{4(i+j+1)}$$
(2.33)

$$N_i = \frac{L^2}{i+3} \tag{2.34}$$
These matrices may now be computed for any order n when the dimensions and material properties of the solar panel are known.

2.5. Orbital Mechanics

Because the orbital motion and attitude motion are to be assumed decoupled, the orbital motion can be simulated by direct numerical integration of the equation of orbital motion. By defining the inertial position vector of the satellite in Cartesian coordinates as,

$$\mathbf{r} = \begin{bmatrix} x & y & z \end{bmatrix}^T \tag{2.35}$$

the equation of motion may be written as,

$$\ddot{\mathbf{r}} = -\frac{\mu_{\oplus}}{||\mathbf{r}||^3}\mathbf{r} + \mathbf{a}_d \tag{2.36}$$

where $\mu_{\oplus} = 398,600.44 \text{ km}^3 \text{s}^{-2}$ is the gravitational parameter and \mathbf{a}_d is a vector of disturbance terms, which may include things such as the effects of Earth's oblateness, and solar radiation pressure. If these two are considered, the disturbance term will take the form,

$$\mathbf{a}_d = \mathbf{a}_{J_2} + \mathbf{a}_{srp} \tag{2.37}$$

where \mathbf{a}_{J_2} represents the oblateness effects computed for J_2 . Higher-order oblateness terms $(J_3, J_4, ...)$ are left out since their contribution is several orders of magnitude lower than the already tiny effects from J_2 . With only this disturbance term, \mathbf{a}_{J_2} becomes,

$$\mathbf{a}_{J_{2}} = -\frac{3}{2} J_{2} \mu_{\oplus} \frac{R_{\oplus}^{2}}{||\mathbf{r}||^{4}} \begin{bmatrix} 1 - 5 \frac{xz^{2}}{||\mathbf{r}||^{3}} \\ 1 - 5 \frac{yz^{2}}{||\mathbf{r}||^{3}} \\ 3 - 5 \frac{z^{3}}{||\mathbf{r}||^{3}} \end{bmatrix}$$
(2.38)

 $R_{\oplus} = 6378.1$ km is the mean equatorial radius for Earth. While the J_2 perturbation does play a role, the largest perturbative contributor at geostationary altitude is that of solar

radiation pressure, which can be written in the form,

$$\mathbf{a}_{srp} = -\frac{p_{srp}c_r A_{\odot}}{m} \frac{\mathbf{r}_{\odot/sat}}{||\mathbf{r}_{\odot/sat}||}$$
(2.39)

where $p_{srp} = 4.57 \times 10^{-6} \text{ N/m}^2$ is the solar pressure per unit area, $c_r \in [0, 2]$ is the effective reflectivity of the spacecraft, A_{\odot} is the exposed area to the Sun, and $\mathbf{r}_{\odot/sat}$ is the vector from the satellite to the Sun. In practice, c_r is difficult to determine since it varies with time and material properties. A value of 0 represents full transmission of the light, 1 represents a perfect black body where all light is absorbed, and a value of 2 represents a perfectly reflective surface.

2.6. Photometry and Reflectance Modeling

When observing any kind of object in space–human-made or not–with which no direct communication exists, the only available source of information about it which is obtainable from ground-based observation is the light radiated or reflected by it. There are several techniques of data collection and analysis which can be used to draw conclusions, but a common method is to measure the total radiation intensity over some wavelength region–e.g. visible light, I-band, etc. Referred to as photometry, this discipline has, in a broad sense, existed since the Greek astronomer Hipparchus first introduced the concept of stellar magnitudes in the second century BC. While his original work was more or less subjective guesswork, the now logarithmically defined stellar magnitude scale has continued to exist through today as the method of describing the relative intensity of light emitted by stars, and has been extended to include objects such as asteroids and RSOs.

When analyzing photometric data, there is not always a known object for comparison, so generally the object is compared to some assumed-unchanging object in the image, be it a star or even the background radiation. The magnitude difference ΔM between two objects in some wavelength window is given by (Roth, 2009),

$$\Delta M = M_1 - M_2 = -2.5 \log_{10} \frac{\Phi_1}{\Phi_2}$$
(2.40)

where Φ_1 and Φ_2 represent the total radiation fluxes of the two objects in the given time window. In the case of RSOs, most light which is detected by the photometer is reflected directly from the Sun which has a known visual magnitude $M_{sun} = -26.74$ (Williams, 2018) and visible-light flux $C_{sun} = 455 \frac{W}{m^2}$, leading to an expression for the magnitude of the RSO in terms of F_{obs} , which the fraction of light reflected off an object as seen by the observer (Dianetti, Weisman, & Crassidis, 2018),

$$M = M_{sun} - 2.5 \log_{10} \frac{F_{obs}}{C_{sun}}$$
(2.41)

When a body has several facets from which it may reflect light as is the case for essentially every human-made object, F_{obs} becomes a sum over all n_f facets such that the magnitude is given by,

$$M = M_{sun} - 2.5 \log_{10} \sum_{k=1}^{n_f} \frac{F_{obs,k}}{C_{sun}}$$
(2.42)

Computation of each $F_{obs,k}$ is a science in and of itself, and the reflectance models in use were by and large developed in the computer graphics community. Before proceeding with the model to be used in this thesis, the model developed by Cook and Torrance (1982) along with its associated notation will be reviewed due to its relative simplicity in order to develop an intuition for reflectance. Following this, the reflectance model to be applied will be described in detail. The geometry of reflection is shown in Figure 2.7 along with the unit vectors along the surface normal of a facet **N**, along the direction toward the incident light source **L**, along the direction toward the observer **V**, and the angle bisector of **V** and **L**, **H**. The angles α and θ are given by $\cos \alpha = \mathbf{H} \cdot \mathbf{N}$ and $\cos \theta = \mathbf{H} \cdot \mathbf{V} = \mathbf{H} \cdot \mathbf{L}$. For development, note that if each facet has a unit normal vector described in the body frame ^B**N**_k, this vector may be rotated to the inertial frame by the transformation,

$${}^{N}\mathbf{N}_{k} = C_{NB}(\beta)^{B}\mathbf{N}_{k} \tag{2.43}$$

where $C_{NB}(\beta)$ is the transformation matrix from frame *B* to frame *N* as a function of the attitude quaternion:

$$C_{NB}(\beta) = \begin{bmatrix} \beta_0^2 + \beta_1^2 - \beta_2^2 - \beta_3^2 & 2(\beta_1\beta_2 + \beta_0\beta_3) & 2(\beta_1\beta_3 - \beta_0\beta_2) \\ 2(\beta_1\beta_2 - \beta_0\beta_3) & \beta_0^2 - \beta_1^2 + \beta_2^2 - \beta_3^2 & 2(\beta_2\beta_3 + \beta_0\beta_1) \\ 2(\beta_1\beta_3 + \beta_0\beta_2) & 2(\beta_2\beta_3 - \beta_0\beta_1) & \beta_0^2 - \beta_1^2 - \beta_2^2 + \beta_3^2 \end{bmatrix}$$
(2.44)

For ease of notation, ${}^{N}N_{k}$ will simply be denoted by N and the reflectance model will be for a single facet.



Figure 2.7 Geometry of reflection (Cook & Torrance, 1982, p. 9).

Defining now the solid angle $d\omega_i$ as the projected area of the light source divided by the square of the distance to the source, the reflected intensity reaching the viewer from each light source is

$$I_r = RI_i (\mathbf{N} \cdot \mathbf{L}) d\omega_i \tag{2.45}$$

where I_i is the incident light intensity and R is the bidirectional reflectance, defined to be the ratio of reflected intensity in a given direction to the energy of the incident light. Rmay generally be expressed as a linear combination of two components, the specular reflectance R_s and the diffuse reflectance R_d :

$$R = sR_s + dR_d \tag{2.46}$$

where s + d = 1. The specular reflectance represents the light which is reflected off the surface of the material, whereas the diffuse reflectance represents either the light which first penetrates beneath the material and scatters before emerging or the light from multiple surface reflections which occur for significantly rough surfaces. In the case of electrical conductors such as metals, there is essentially zero depth penetration due to reemission of electromagnetic waves caused by near-surface electron excitation. This means that reflection effectively occurs at the surface and the diffuse component may be assumed null when the surface roughness is sufficiently low, such that $R = R_s$.

In addition to direct illumination from individual sources, background lighting may cause illumination of the object of interest (e.g. starlight or Earthshine). This is called ambient illumination, and may often be assumed uniform over the hemisphere of illuminating angles. The reflected intensity due to ambient illumination is,

$$I_{ra} = R_a I_{ia} f \tag{2.47}$$

where,

$$f = \frac{1}{\pi} \int_{S} (\mathbf{N} \cdot \mathbf{L}) d\omega_i$$
 (2.48)

is the fraction of the illuminating hemisphere which is not blocked by nearby objects. The domain of integration is the unblocked portion. Thus, the reflected intensity is,

$$I_r = R_a I_{ia} f + \sum_k I_{ik} (\mathbf{N} \cdot \mathbf{L}_k) d\omega_{ik} R_s$$
(2.49)

which accounts for multiple direct illumination sources as well as the ambient illumination. In this particular case, the Sun will be the only source of light to be considered so that the summation has only a single term and the reflected intensity becomes,

$$I_r = R_a I_{ia} f + I_i (\mathbf{N} \cdot \mathbf{L}) d\omega_i R_s \tag{2.50}$$

While this approach is intuitively satisfying, what it has in ease of understanding it lacks in ease of implementation. In order to shape the specular lobe, it requires the surfaces of the spacecraft to have some degree of roughness attributed to them, which in turn requires more complex and expensive modeling and simulation. For this reason, to find an expression for $F_{obs,k}$ in Equation 2.42, a different approach to the above will be employed since it is not only more straight forward to implemene, but also because it has been used many times before and has shown to predict measured brightnesses fairly well (Z. W. Henry, Vavala, et al., 2020). First, the Fresnel reflectance F is given by the Fresnel equation for unpolarized incident light (Schlick, 1994):

$$F(c) = \frac{1}{2} \frac{(a-c)^2 + b^2}{(a+c)^2 + b^2} \left[1 + \frac{(a+c-1/c)^2 + b^2}{(a-c+1/c)^2 + b^2} \right]$$
(2.51)

where $c \equiv \cos \theta = \mathbf{H} \cdot \mathbf{V}$, and a and b are given by,

$$a^{2} = \frac{1}{2} \left(\sqrt{(n^{2} - k^{2} + c^{2} - 1)^{2} + 4n^{2}k^{2}} + n^{2} - k^{2} + u^{2} - 1 \right)$$
$$b^{2} = \frac{1}{2} \left(\sqrt{(n^{2} - k^{2} + c^{2} - 1)^{2} + 4n^{2}k^{2}} - n^{2} + k^{2} - u^{2} + 1 \right)$$

Because the distribution of n and k are seldom known for all wavelengths and are frequently available only for select values in the middle of the visible light spectrum, the angular dependence of F at some wavelength λ may be approximated as,

$$F(c,\lambda) = F(1,\lambda) + (1 - F(1,\lambda))\frac{\bar{F}(c) - F(1,\lambda)}{1 - F_0}$$
(2.52)

where $\overline{F}(c)$ is computed using Equation 2.51 by choosing experimentally determined values of n and k at a given wavelength, and $F(1, \lambda)$ is the Fresnel factor or reflectance at normal incidence for the wavelength of interest. Noting that the main difference in Equation 2.52 is when c arrives at 1, Schlick made the further approximation,

$$F(c,\lambda) = F(1,\lambda) + (1 - F(1,\lambda))(1 - c)^5$$
(2.53)

so that F only depends on $F(1, \lambda)$. This may then be used in a bidirectional reflectance distribution function (BRDF), which in a similar fashion to R may be written as a sum of both a specular and a diffuse term:

$$\rho = \rho_s + \rho_d \tag{2.54}$$

The BRDF model to be used is that developed by Ashikhmin and Shirley (2000), since it has been used extensively for attitude estimation (Hinks et al., 2013; Subbarao & Henderson, 2019) and has been shown to be the best fit for measured BRDFs (Ngan, Durand, & Matusik, 2005). The specular term for this BRDF model is given by,

$$\rho_s(\mathbf{L}, \mathbf{V}) = \frac{\sqrt{(n_i + 1)(n_j + 1)}}{8\pi} \frac{(\mathbf{N} \cdot \mathbf{H})^z}{(\mathbf{N} \cdot \mathbf{V}) + (\mathbf{N} \cdot \mathbf{L}) - (\mathbf{N} \cdot \mathbf{V})(\mathbf{N} \cdot \mathbf{L})} F(c, \lambda)$$
(2.55)

where,

$$z = \frac{n_i (\mathbf{H} \cdot \mathbf{i})^2 + n_j (\mathbf{H} \cdot \mathbf{j})^2}{1 - (\mathbf{H} \cdot \mathbf{N})^2}$$
(2.56)

The vectors **i** and **j** are unit vectors parallel to the facet surface, and form an orthonormal basis with **N**. The terms n_i and n_j are user inputs to the reflectance model which control the shape of the specular lobe. When $n_i \neq n_j$, the model can give an appearance of a "brushed" surface, as in Figure 2.8. In contrast to the diffuse term in the previous development referring to light which transmits through the surface before being reemitted, the diffuse term here refers to light which is reflected equally in all directions. The diffuse part of the BRDF is given by,

$$\rho_d(\mathbf{L}, \mathbf{V}) = \frac{28R_d}{23\pi} (1 - R_s) \left(1 - \left(1 - \frac{\mathbf{N} \cdot \mathbf{L}}{2}\right)^5 \right) \left(1 - \left(1 - \frac{\mathbf{N} \cdot \mathbf{V}}{2}\right)^5 \right)$$
(2.57)

With both of these terms, $\rho(\mathbf{L},\mathbf{V})$ may be used to compute $F_{obs,k}$:

$$F_{obs,k} = \rho \frac{C_{sun} (\mathbf{L} \cdot \mathbf{N}_k) A_k (\mathbf{V} \cdot \mathbf{N}_k)}{d^2}$$
(2.58)

where A_k is the total area of the k^{th} facet and d is the distance from the object to the observer. Thus, the visual magnitude of the RSO is given by,

$$M_v = -26.74 - 2.5 \log_{10} \sum_{k=1}^{n_f} \rho \frac{(\mathbf{L} \cdot \mathbf{N}_k) (\mathbf{V} \cdot \mathbf{N}_k)}{d^2} A_k$$
(2.59)

where n_f is the total number of facets for the spacecraft model.

In case it is not immediately apparent to the reader, a clear benefit to this approach over that of Cook and Torrance is that the shape and size of the specular lobe (i.e. the "blob" of glint) is controlled entirely by the two parameters n_i and n_j , as opposed to a requirement of surface generation using hundreds or thousands of randomly oriented facets on each spacecraft face. For one thing, this would clearly increase the total number of required computations. For another, it complicates the reshaping of the specular lobe. Thus, there is a significant benefit to the implementation process for the Ashikhmin and Shirley model over that of Cook and Torrance.



Figure 2.8 Metallic spheres for various values of n_i (horizontal) and n_j (vertical) (Ashikhmin & Shirley, 2000, p. 27).

2.7. Polarimetric Modeling

A further extension of the information from reflected light can be found in its polarization, since light which is reflected off a surface may often become polarized. To begin this discussion, there is first a review of the basic wave-like description of electromagnetic energy. Much of this review comes from Schott (2009) and Hapke (2012), and those texts should be consulted for more detailed descriptions. The field strength at any location along the propagation direction z and time t at a particular wavelength λ may be expressed as,

$$\varepsilon_x(z,t,\lambda) = \varepsilon_{0x} \sin\left(\omega t - 2\pi \frac{z}{\lambda} + \phi_x\right)$$
 (2.60a)

$$\varepsilon_y(z,t,\lambda) = \varepsilon_{0y} \sin\left(\omega t - 2\pi \frac{z}{\lambda} + \phi_y\right)$$
 (2.60b)

where ϕ_x and ϕ_y are the phase shift parameters, x is in the direction perpendicular to the plane of propagation, y in the direction parallel to the plane of propagation but normal to the direction of propagation, and ε_{0x} and ε_{0y} are the maximum amplitudes for each of the

directions. By letting $\phi = \phi_y - \phi_x$ represent the phase shift between the x and y components, the electromagnetic vector traces out a *polarization ellipse* which can be implicitly described by the following equation:

$$\frac{\varepsilon_x^2}{\varepsilon_{0x}^2} + \frac{\varepsilon_y^2}{\varepsilon_{0y}^2} - \frac{2\varepsilon_x\varepsilon_y}{\varepsilon_{0x}\varepsilon_{0y}}\cos\phi = \sin\phi$$
(2.61)



Figure 2.9 Polarization ellipse.

The polarization ellipse, depicted in Figure 2.9, has its major axis oriented at an angle ψ , where,

$$\tan 2\psi = \frac{2\varepsilon_{0x}\varepsilon_{0y}}{\varepsilon_{0x}^2 - \varepsilon_{0y}^2}$$
(2.62)

A convenient and concise way of representing the polarization states is using the *Stokes vector*, which is defined as,

$$\mathbf{S} = \begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix} = \begin{bmatrix} \varepsilon_{0x}^2 + \varepsilon_{0y}^2 \\ \varepsilon_{0x}^2 - \varepsilon_{0y}^2 \\ 2\varepsilon_{0x}\varepsilon_{0y}\cos\phi \\ 2\varepsilon_{0x}\varepsilon_{0y}\sin\phi \end{bmatrix}$$
(2.63)

where I denotes the total intensity, Q denotes the 0° and 90° linear polarization states, U denotes the $+45^{\circ}$ and -45° polarization states, and V denotes the circular polarization. Intuitively, the linear polarization states describe the dominance of one orientation over another and the circular polarization state–which arises due to partial absorption of electromagnetic energy before reemission–describes the relative phase shift between the orientations. For metals, which have primarily specular reflection off many randomly oriented surfaces, this effect is largely negligible and the x and y components can be assumed to have the same phase angle.

The Stokes vector is frequently normalized to the intensity such that,

$$\mathbf{S} = \begin{bmatrix} 1\\ Q/I\\ U/I\\ V/I \end{bmatrix}$$
(2.64)

$$R(\theta_r) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\theta_r & -\sin 2\theta_r & 0 \\ 0 & \sin 2\theta_r & \cos 2\theta_r & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
(2.65)

The Stokes vector of the reflected light is then given by,

$$\mathbf{S}_r = R(\theta_r) M \mathbf{S}_i \tag{2.66}$$

where where M is the Mueller matrix of the surface (Chang et al., 2002). The Mueller matrix is a means of quantifying the polarimetric effects of the interface between two mediums with different indices of refraction, and for Fresnel reflectance of an incident beam it is given by,

$$M = \frac{1}{2} \begin{bmatrix} R_{\perp} + R_{\parallel} & R_{\perp} - R_{\parallel} & 0 & 0 \\ R_{\perp} - R_{\parallel} & R_{\perp} + R_{\parallel} & 0 & 0 \\ 0 & 0 & 2Re(r_{\perp}r_{\parallel}^{*}) & 2Im(r_{\perp}r_{\parallel}^{*}) \\ 0 & 0 & -2Im(r_{\perp}r_{\parallel}^{*}) & 2Re(r_{\perp}r_{\parallel}^{*}) \end{bmatrix}$$
(2.67)

where $R_{\perp} = |r_{\perp}|^2$ and $R_{\parallel} = |r_{\parallel}^2|$ are the reflectance coefficients corresponding to the perpendicular and parallel components of reflection. Given the complex coefficient of refraction $n = n_r + n_i i$, these reflectances are,

$$R_{\perp} = \frac{\left[\cos\theta - G_{1}\right]^{2} + G_{2}^{2}}{\left[\cos\theta + G_{1}\right]^{2} + G_{2}^{2}}$$
(2.68)

$$R_{\parallel} = \frac{\left[n^2 \cos \theta - G_1\right]^2 + \left[2n_r n_i \cos \theta - G_2\right]^2}{\left[n^2 \cos \theta + G_1\right]^2 + \left[2n_r n_i \cos \theta + G_2\right]^2}$$
(2.69)

where,

$$G_1^2 = \frac{1}{2} \left\{ \left[n^2 - \sin^2 \theta \right] + \left[\left(n^2 - \sin^2 \theta \right)^2 + 4n_r^2 n_i^2 \right]^{1/2} \right\}$$
(2.70)

and,

$$G_2^2 = \frac{1}{2} \left\{ -\left[n^2 - \sin^2\theta\right] + \left[\left(n^2 - \sin^2\theta\right)^2 + 4n_r^2 n_i^2\right]^{1/2} \right\}$$
(2.71)

In the context of SSA, the primary source of illumination will be the Sun, whose emitted light is mostly randomly polarized (i.e. non-polarized). This indicates that the incident Stokes vector on an RSO surface will be given by,

$$\mathbf{S}_{i} = \begin{bmatrix} 1\\0\\0\\0 \end{bmatrix}$$
(2.72)

and therefore, by carrying out the operation in Equation 2.66, the Stokes vector reflected off some surface toward a polarimeter is given by,

$$\mathbf{S}_{r} = \frac{1}{2} \begin{bmatrix} R_{\parallel} + R_{\perp} \\ (R_{\parallel} - R_{\perp}) \cos 2\theta_{r} \\ (R_{\parallel} - R_{\perp}) \sin 2\theta_{r} \\ 0 \end{bmatrix}$$
(2.73)

Because Equation 2.73 considers reflection off a single surface, some further consideration must be made to predict the "total" Stokes vector which considers every surface. First, assume it is of the form,

$$\mathbf{S}_{r,total} = \sum_{k=1}^{n_f} w_k \mathbf{S}_{r,k}$$
(2.74)

where w_k denotes some unknown weight for the reflected Stokes vector from each surface, and n_f once again denotes the total number of faces. Noting that the Stokes vector is normalized to the total intensity, the weight corresponding to face k may be assumed to be given by the ratio of the reflected intensity from face k to the total measured intensity of light from the BRDF. Under this assumption, from the flux portion of Equation 2.59, we arrive at,

$$w_{k} = \frac{\rho_{k} \frac{(\mathbf{L} \cdot \mathbf{N}_{k})(\mathbf{V} \cdot \mathbf{N}_{k})}{d^{2}} A_{k}}{\sum_{j=1}^{n_{f}} \rho_{j} \frac{(\mathbf{L} \cdot \mathbf{N}_{j})(\mathbf{V} \cdot \mathbf{N}_{j})}{d^{2}} A_{j}} = \frac{\rho_{k} (\mathbf{L} \cdot \mathbf{N}_{k})(\mathbf{V} \cdot \mathbf{N}_{k}) A_{k}}{\sum_{j=1}^{n_{f}} \rho_{j} (\mathbf{L} \cdot \mathbf{N}_{j})(\mathbf{V} \cdot \mathbf{N}_{j}) A_{j}}$$
(2.75)

Thus, the process of computing the Stokes vector incident on the polarimeter is summarized as follows:

- 1. Find θ_r for each surface from $\mathbf{N}_k \cdot \mathbf{V}$
- 2. Compute R_{\parallel} and R_{\perp} given $n = n_r + n_i i$ for some material
- 3. Compute $S_{r,k}$ for each surface
- 4. Find appropriate weights w_k based on flux from each respective surface
- 5. Weight and sum all contributions to get $S_{r,total}$

This of course does not account for instrumental polarization and noise, which must often be modeled and corrected for (Snik & Keller, 2013). For the purposes of this thesis, it will be assumed that these effects are accounted for and that the system has a polarimetric sensitivity on the order of 10^{-4} .

2.8. From Images to Astrometric and Photometric Data

Fundamentally, the task of collecting and processing on-sky images of a known object into useful data is fairly straight-forward. Because the object is in this case geostationary,

the electro-optical tracking system may be turned off once the object is centered. From here, the image integration time and gain are adjusted until an appropriate signal-to-noise ratio (SNR) is achieved. Once this is accomplished, an image set may be collected for as long as weather permits. To get from the image set to a set of useful numbers, a few things must be noted.



Figure 2.10 Sample frame used to identify objects and platesolve (Z. W. Henry, Vavala, et al., 2020).

First, the stars are practically stationary in the J2000 inertial frame. This means that any RSO is going to move in a significantly different direction and speed to the stars. The second thing to note is that there are most likely to be more stars in an image than there are RSOs. This is perhaps obvious, but is nonetheless fundamental to the correlation of data between frames. Third, the geocentric RA and DEC of any star is going to be effectively the same as the topocentric RA and DEC. Because plate solved astrometric data is output in the geocentric reference frame, this means that the measured RA and DEC of the RSO is in actuality going to be in the topocentric frame.

The first two points allow for an RSO to be classified as any object in the frame whose change in position relative to the previous frame differs greatly from that of the stars. To draw an analogy, this is akin to the blink comparator, a classical astronomical tool which

has been used to discover asteroids and other planetary bodies. A sample image containing GOES-16 and two other objects is shown in Figure 2.10. GOES-16 was identified in this image as the bottom-most object using the orbit visualization tool on Celestrak (Kelso, 2020). The other two objects are the Brazilian geosynchronous satellites SGDC (top) and Star One C3 (middle).



Figure 2.11 Digital aperture and sky background annulus around GOES-16 (Z. W. Henry, Vavala, et al., 2020).

To measure the visual magnitude of the RSO, the standard method of digital photometry is employed; comparing the object to a known star in the image after plate-solving. This means summing the total number of analog-to-digital units (ADU) of both the object and of the star, and the brightness of the object is then,

$$M_{v,rso} = M_{v,ref} - 2.5 \log_{10} \frac{\text{ADU}_{rso}}{\text{ADU}_{ref}}$$
(2.76)

where $M_{v,rso}$ is the visual magnitude of the RSO and $M_{v,ref}$ is the visual magnitude of the reference star. To define the edges of an object, for the sake of accurate and consistent counting, the point-spread function (PSF) is assumed to be Gaussian. The digital aperture radius is then selected to have three times the full-width half maximum (FWHM). Next,

sky background is subtracted by defining an inner and outer annulus at four and five times the FWHM, respectively. This region is assumed to be representative of the sky behind the object for the purpose of background subtraction. A sample image of the digital aperture around GOES-16 is shown in Figure 2.11. For details on RSO identification, inter-frame correlation, and the remainder of the algorithm, see Z. W. Henry, Vavala, et al. (2020).

2.9. Reference Frames and Coordinate Transformations

For convenience, the frames may sometimes be expressed in vectrix notation, as in the following example:

$$A \equiv \begin{bmatrix} \hat{a}_1 & \hat{a}_2 & \hat{a}_3 \end{bmatrix}^T \tag{2.77}$$

where \hat{a}_i are the orthonormal unit vectors of frame A, spanning \mathcal{R}^3 . For details on the properties of vectrices, refer either to Hughes (2004) or Shuster (1993). Following this convention, the direction cosine matrix (DCM) to rotate any frame I to any frame J will be denoted C_{JI} and is defined as,

$$C_{JI} = I \cdot J^{T} = \begin{bmatrix} \hat{i}_{1} \cdot \hat{j}_{1} & \hat{i}_{1} \cdot \hat{j}_{2} & \hat{i}_{1} \cdot \hat{j}_{3} \\ \hat{i}_{2} \cdot \hat{j}_{1} & \hat{i}_{2} \cdot \hat{j}_{2} & \hat{i}_{2} \cdot \hat{j}_{3} \\ \hat{i}_{3} \cdot \hat{j}_{1} & \hat{i}_{3} \cdot \hat{j}_{2} & \hat{i}_{3} \cdot \hat{j}_{3} \end{bmatrix}$$
(2.78)

For example, the DCM to rotate from frame C to frame N (which will be useful later) is,

$$C_{NC} = C \cdot N^{T} = \begin{bmatrix} \hat{c}_{1} \cdot \hat{n}_{1} & \hat{c}_{1} \cdot \hat{n}_{2} & \hat{c}_{1} \cdot \hat{n}_{3} \\ \hat{c}_{2} \cdot \hat{n}_{1} & \hat{c}_{2} \cdot \hat{n}_{2} & \hat{c}_{2} \cdot \hat{n}_{3} \\ \hat{c}_{3} \cdot \hat{n}_{1} & \hat{c}_{3} \cdot \hat{n}_{2} & \hat{c}_{3} \cdot \hat{n}_{3} \end{bmatrix} = \begin{bmatrix} \cos \Theta & \sin \Theta & 0 \\ -\sin \Theta & \cos \Theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
(2.79)

A well-known and important property of the DCM is that the DCM to rotate from frame I to frame J is the same as the DCM to rotate from frame I to some intermittent frame K premultiplied by the DCM from frame K to frame J:

$$C_{JI} = C_{JK} C_{KI} \tag{2.80}$$

This will allow for easy rotation of, say, the solar panel frame into the inertial frame via the following DCM:

$$C_{NP} = C_{NE}C_{EB}C_{BP} \tag{2.81}$$

since each of these intermittent frames differ from each other by only a single angle.

2.10. Fractional Fourier Transform

A tool which has been revived and forgotten several times over the past century is Fractional calculus, which generalizes the derivatives and integrals of functions to arbitrary degree (Oldham & Spainer, 2006). The integer degrees (e.g. first or second derivative) are, in the fractional sense, special cases of this more general formulation of calculus. The theory is somewhat complex, but there have been many applications discovered in the past few decades which are applicable in image processing, control design, and signal processing. One of the useful tools which arose from the generalized methods of fractional calculus is the fractional Fourier transform (FrFT), which has allowed for lower cost implementation of non-model-based signal processing and filtering (Sejdić, Djurović, & Stanković, 2011).

To make a long story short, the FrFT is a generalization of the Fourier transform to an arbitrary degree α . It is common to let,

$$\alpha = \frac{a\pi}{2} \tag{2.82}$$

and then define,

$$A_{\alpha} \equiv \frac{\exp -i\pi \operatorname{sgn}(\sin \alpha)/4 + i\alpha/2}{|\sin \alpha|^{1/2}}$$
(2.83)

so that the FrFT of a signal f(t), denoted $\mathcal{F}^a \{f\}$, can be expressed in terms of 0 < |a| < 2:

$$\mathcal{F}^{a}\left\{f\right\} = \int_{-\infty}^{\infty} B_{a}(t,\tau)f(\tau)d\tau$$
(2.84)

where,

$$B_a(t,\tau) = A_\alpha \exp\left[i\pi(t^2 \cot\alpha - 2t\tau \csc\alpha + \tau^2 \cot\alpha\right]$$
(2.85)

Of note, the above definition simplifies the ordinary Fourier transform when a = 1 and the inverse Fourier transform when a = -1.



Figure 2.12 Rotation in time-frequency domain (Almeida, 1994, p. 3085).

The benefit to using a Fourier transform of fractional order is that the FrFT of a function corresponds to a rotation of its Wigner-Ville distribution function (WVDF) in the time-frequency domain to dependence on a new coordinate system (u, v), as in Figure 2.12. Hence, it provides time and frequency information about a signal. The Wigner-Ville distribution of a function f is defined as,

$$W_f(t,\omega) = \int_{-\infty}^{\infty} f(t+\tau/2) f^*(t-\tau/2) e^{-2\pi\omega\tau i}$$
(2.86)

where ω denotes the frequency, *i* is the complex variable, and f^* denotes the complex conjugate of *f*. After applying the FrFT to *f*, the WVDF in the rotated coordinate system is given by,

$$W_{f_a}(u,v) = W_f(t\cos\alpha - \omega\sin\alpha, t\sin\alpha + \omega\cos\alpha)$$
(2.87)

This can be advantageous in that a partial rotation can be used to maximize the amplitude of a signal in the frequency domain. Additionally, depending on the nature of the noise, band-pass filters may be applied to the function after partial rotations in the time-frequency domain to eliminate effects which are not exclusively frequency-dependent (Kutay, Ozaktas, Arikan, & Onural, 1997). This can be seen in Figure 2.13. The algorithm used for approximating the FrFT in terms of the fast-fourier transform and the Hermite-Gaussian functions is described in Ozaktas, Arikan, Kutay, and Bozdagi (1996).



Figure 2.13 Noise separation in the α th domain (Kutay et al., 1997, p. 1130).

2.11. GOES-R Series Mission

The Geostationary Operational Environmental Satellites (GOES) R-series satellites are Earth monitoring platforms maintained and operated by the National Oceanic and Atmospheric Administration (NOAA), providing advanced imagery and atmospheric measurements of Earth's Western Hemisphere, real-time mapping of lightning activity, and improved monitoring of solar activity and space weather (*GOES-R Series Mission*, n.d.). These satellites are good subjects for observation for a number of reasons, first and foremost being that they are fairly large objects and are thus relatively easy to spot. Because they must maintain geostationary earth orbit (GEO), there must be propellant onboard and frequent impulsive stationkeeping maneuvers will be required. Additionally, each of them is equipped with a large solar panel which will provide an interesting source of vibrations. Finally, there is a wealth of information available about them in a databook which was prepared for NASA and made publicly available (*GOES-R Series Data Book*, 2019). Figure 2.14 depicts the fully deployed GOES-R satellite.



Figure 2.14 Fully deployed GOES-R Satellite (GOES-R Series Data Book, 2019, p. 2-1).

3. Methodology

This chapter collects the important results from Chapter 2 and applies them to the problem of dynamical modeling, simulation, and attitude estimation of GOES-16. It begins with a definition of important coordinate frames, then proceeds to a derivation of the Lagrangian which characterizes the complex "hidden" attitude motion of the satellite. Following this, there is a discussion of the simulation of the equations of motion. Next is a summary of the important measurements to be included in the Kalman filter. Then, the unscented Kalman filter process is described in detail as it pertains to this problem, and the simulation test cases and methods for vibrational mode detection are described. Finally, there is a brief description of the physical experiment setup which was used for tuning the reflectance model parameters.

3.1. Reference Frame Definitions

Before the kinetic and potential energies can be derived, the reference frames must be defined. The inertial reference frame denoted N is the Earth-centered J2000 frame, for which the \hat{n}_1 vector points toward the vernal equinox and \hat{n}_3 points along the Earth's axis of rotation. Measured with respect to this frame is the observer frame C, defined such that the observer position vector **r** has components in the \hat{c}_1 and \hat{c}_3 directions only. C is offset from N by the current local sidereal time Θ , and \hat{c}_3 is in the same direction as \hat{n}_3 . Following the spacecraft are the co-moving local vertical, local horizontal frame-denoted E-which is defined such that \hat{e}_1 points toward nadir and \hat{e}_3 points along the orbit normal. It will be assumed that the Advanced Baseline Imager is affixed to this frame. For a visual depiction of these frames, refer to Figure 3.1. The bus frame-denoted B-which is a body-fixed frame with \hat{b}_1 pointing along \hat{e}_1 ; and the frame P which is fixed with respect to the solar panel such that \hat{p}_1 is normal to the bus surface and pointed along $-\hat{e}_2$. The mass displacements for propellant slosh are to be considered in the B frame since they will be constrained only to motion parallel to the \hat{b}_2 - \hat{b}_3 plane. For a visual depiction of these frames in relation to *E*, refer to Figure 3.2.



Figure 3.1 ECIF Depiction of Inertial Frame N, Observer Frame C, and Satellite LVLH Frame E.



Figure 3.2 Simplified model of GOES Satellite, depicting body frame B, LVLH frame E, and panel frame P. Not to scale.

3.2. Derivation of Lagrangian

Having reviewed the fundamental concepts of mechanics and established the pertinent coordinate frames, the Lagrangian may be derived. For this system it will be expressed as,

$$L = T_{body} + T_{panel} + T_{slosh} - (V_{panel} + V_{slosh})$$
(3.1)

where T_{body} , T_{panel} , and T_{slosh} are respectively the kinetic energy of the dual-spin body, the solar panel, and the propellant.

As mentioned in the previous chapter, there are two primary sources of vibration are of interest for this research: those caused by propellant slosh and those caused by solar panel flexing. This section will first develop the background information required for understanding the dynamics of these phenomena and their relationship to the motion of a spacecraft. The spacecraft will be simplified and approximated to be a two-box dual-spin rigid body with a flexible solar panel and two compartmented mass-spring systems representing the propellant tanks.

3.2.1 Rigid Body Components

Firstly, because the body of the spacecraft is assumed rigid, the kinetic energy of the nadir-pointing camera is, from the previous chapter,

$$T_c = \frac{1}{2}^N \omega_E^T \left[I_c \right]^N \omega_E \tag{3.2}$$

where ${}^{N}\omega_{E}$ is the angular rotation rate of E with respect to N, and $[I_{c}]$ is its moment of inertia. Similarly, the kinetic energy of the bus is,

$$T_b = \frac{1}{2}^N \omega_B^T \left[I_b \right]^N \omega_B \tag{3.3}$$

where,

$${}^{N}\omega_{B} = {}^{N}\omega_{E} + C_{NE}{}^{E}\omega_{B} \tag{3.4}$$

Then, including parallel axis theorem, the total kinetic energy of the spacecraft body is simply,

$$T_{body} = \frac{1}{2}^{N} \omega_{E}^{T} \left(\left[I_{c} \right] + m_{c} \left[\tilde{r}_{c} \right]^{T} \right)^{N} \omega_{E} + \frac{1}{2}^{N} \omega_{B}^{T} \left(\left[I_{b} \right] + m_{b} \left[\tilde{r}_{b} \right]^{T} \right)^{N} \omega_{B}$$
(3.5)

Finally, the kinetic energy of the solar panel will assume that the solar panel is rigid to capture the broader motions, and the torque caused by it will later be described using the methods of Junkins and Kim (1993). The angular velocity of the solar panel is,

$${}^{N}\omega_{p} = {}^{N}\omega_{b} + C_{NB}{}^{B}\omega_{p} \tag{3.6}$$

and thus the kinetic energy of the panel is,

$$T_{panel} = {}^{N}\omega_{p}^{T} \left(\left[I_{p} \right] + m_{p} \left[\tilde{r}_{p} \right]^{T} \right) {}^{N}\omega_{p}$$

$$(3.7)$$

3.2.2 Contribution of Fuel Slosh

To derive the kinetic energy of the sloshing fluid, the fluid will first be assumed stuck to the end of the propellant tanks at the side of the spacecraft pointed away from the Earth. For fluid motion as in the case of $Bo < Bo_{crit}$, the sloshing will be approximated—as previously mentioned—using a mass-spring-dampener model as depicted in Figures 3.3 and 3.4. The stationary masses will be denoted by M, the slosh masses by m, spring constants as k/2, and the x and y displacements for each mass are denoted as such. To distinguish between the fuel and oxidizer, the subscripts f and o will be used.

By modeling the motion with point masses, the total kinetic energy contributed by sloshing is given by,

$$T_{s} = \frac{1}{2}m_{o}\mathbf{v}_{o} \cdot \mathbf{v}_{o} + \frac{1}{2}m_{0o}\mathbf{v}_{0o} \cdot \mathbf{v}_{0o} + \frac{1}{2}m_{f}\mathbf{v}_{f} \cdot \mathbf{v}_{f} + \frac{1}{2}m_{0f}\mathbf{v}_{0f} \cdot \mathbf{v}_{0f}$$
(3.8)



Figure 3.3 Profile view of fuel (left) and oxidizer (right) mass spring dampener model.



Figure 3.4 Top-down view of fuel and oxidizer mass spring dampener model.

where the velocities of the stationary masses are,

$$\mathbf{v}_{0o} = {}^{N}\omega_{b} \times (-h_{0o}\hat{b}_{1} + r_{o}\hat{b}_{2})$$
(3.9a)

$$\mathbf{v}_{0f} = {}^{N}\omega_{b} \times (-h_{0f}\hat{b}_{1} - r_{f}\hat{b}_{2})$$
(3.9b)

and the velocities of the slosh masses are,

$$\mathbf{v}_o = {}^N \omega_b \times (h_o \hat{b}_1 + (x_o + r_o) \hat{b}_2 + y_o \hat{b}_3) + \dot{x}_o \hat{b}_2 + \dot{y}_o \hat{b}_3$$
(3.10a)

$$\mathbf{v}_f = {}^N \omega_b \times (h_f \hat{b}_1 + (x_f - r_f) \hat{b}_2 + y_f \hat{b}_3) + \dot{x}_f \hat{b}_2 + \dot{y}_f \hat{b}_3$$
(3.10b)

Using the notation of Hughes (2004), the operations may be carried out for the oxidizer as follows,

$$\mathbf{v}_{0o} = [{}^{N}\tilde{\omega}_{b}]C_{NB}(-h_{0o}\hat{b}_{1} + r_{o}\hat{b}_{2})$$
(3.11)

and

$$\mathbf{v}_o = [{}^N \tilde{\omega}_b] C_{NB} (h_o \hat{b}_1 + (r_o + x_o) \hat{b}_2 + y_o \hat{b}_3) + C_{NB} (\dot{x}_o \hat{b}_2 + \dot{y}_o \hat{b}_3)$$
(3.12)

and similarly for the fuel. The total potential energy of the slosh masses is of the same form as that for a typical mass-spring system:

$$V_s = V_o + V_f = \frac{1}{2}k_o(x_o^2 + y_o^2) + \frac{1}{2}k_f(x_f^2 + y_f^2)$$
(3.13)

where k_o and k_f represent the spring stiffness constants, which may be computed as described in Chapter 2.3.

3.2.3 Solar Panel Vibrations

To figure the effects of the vibration of solar panels on the attitude motion of the satellite, the reaction force on the spacecraft body will be calculated by assuming the panel is an Euler-Bernoulli beam as noted in Section 2.4, then applied as a generalized force under a coordinate transformation. To substitute these equations into Equation 2.4, the equations of motion for the panel are related to the body by the torque τ generated as a reaction force at the point of contact between the panel and the body:

$$\tau + \int_{r_0}^{L} \rho A x (\ddot{y} + x\ddot{\theta}) dx = 0$$
(3.14)

$$\rho A(\ddot{y} + x\ddot{\theta}) + EIy'''' = 0 \tag{3.15}$$

For this system,

$$\dot{\theta} = {}^{N}\omega_{B} \cdot \hat{p}_{2} \tag{3.16}$$

and thus $\hat{\theta}$ comes directly from the equations of motion. From this, y(x,t) can be computed using the lumped-parameter development from Section 2.4 and numerically integrated. The torque τ will then be considered a generalized force, i.e. a component of the vector **Q** which may be substituted into the equations of motion.

For computation of the lumped-parameter model parameters, some information is needed about the solar panel. GOES 16 uses 6,720 Spectrolab Ultra Triple Junction (UTJ) photovoltaic cells arranged to fit an array of five panels which are 135.7 cm × 392.3 cm each (*GOES-R Series Data Book*, 2019). The substrate is composed of 140 μ m of germanium, and the panels have a total thickness around 800 μ m and a density of 84 mg/cm² (Spectrolab, n.d.). Using these parameters as a starting point, the simulation can be tuned until the fundamental vibration frequency is about 0.25 Hz, which is the frequency which is observed in reality for the GOES-R satellites (Chapel et al., 2014).

3.3. Simulation of Spacecraft Motion

While there is a small degree of coupling between orbital motion and attitude motion, the dynamics can be treated separately for the purposes of this work since the measurement of these effects is not a driving factor behind it. Thus, a review of the orbital dynamics will be provided, followed by a description of the "true" attitude dynamics of the spacecraft due to the cumulative effects of the motions described thus far in this chapter which would be hidden to the engineers and scientists in an observatory.

3.3.1 Orbital Motion

Since the orbital motion is assumed to be decoupled from attitude motion, and it is therefore simulated without consideration of the attitude. Recall from Chapter 3 that the

equation of orbital motion is,

$$\ddot{\mathbf{r}} = -\frac{\mu_{\oplus}}{||\mathbf{r}||^3}\mathbf{r} + \mathbf{a}_d \tag{3.17}$$

where the disturbance term can include any number of external perturbing accelerations. The initial conditions for this will come from the public TLE files available from the Celestrak website Kelso (2020), propagated using a combination of methods described by Bate, Mueller, White, and Saylor (2020) and Vallado (2013).

3.3.2 Attitude Motion

By combining the results of previous sections, the complete Lagrangian for this system may be concisely expressed as,

$$L = \frac{1}{2}^{N} \omega_{E}^{T} \left([I_{c}] + m_{c} [\tilde{r}_{c}]^{T} \right)^{N} \omega_{E} + \frac{1}{2}^{N} \omega_{B}^{T} \left([I_{b}] + m_{b} [\tilde{r}_{b}]^{T} \right)^{N} \omega_{B} + \frac{1}{2}^{N} \omega_{p}^{T} \left([I_{p}] + m_{p} [\tilde{r}_{p}]^{T} \right)^{N} \omega_{p} + \frac{1}{2} m_{o} \mathbf{v}_{o} \cdot \mathbf{v}_{o} + \frac{1}{2} m_{0o} \mathbf{v}_{0o} \cdot \mathbf{v}_{0o} + \frac{1}{2} m_{f} \mathbf{v}_{f} \cdot \mathbf{v}_{f} + \frac{1}{2} m_{0f} \mathbf{v}_{0f} \cdot \mathbf{v}_{0f} - \frac{1}{2} k_{o} (x_{o}^{2} + y_{o}^{2}) - \frac{1}{2} k_{f} (x_{f}^{2} + y_{f}^{2})$$
(3.18)

which, by substituting the expressions for \mathbf{v}_o , \mathbf{v}_{0o} , \mathbf{v}_f , and \mathbf{v}_{0f} , is fully expanded to be,

$$L = \frac{1}{2}^{N} \omega_{E}^{T} \left(\left[I_{c} \right] + m_{c} \left[\tilde{r}_{c} \right] \left[\tilde{r}_{c} \right]^{T} \right)^{N} \omega_{E} + \frac{1}{2}^{N} \omega_{B}^{T} \left(\left[I_{b} \right] + m_{b} \left[\tilde{r}_{b} \right] \left[\tilde{r}_{b} \right]^{T} \right)^{N} \omega_{B} \right. \\ \left. + \frac{1}{2}^{N} \omega_{p}^{T} \left(\left[I_{p} \right] + m_{p} \left[\tilde{r}_{p} \right] \left[\tilde{r}_{p} \right]^{T} \right)^{N} \omega_{p} - \frac{1}{2} k_{o} (x_{o}^{2} + y_{o}^{2}) - \frac{1}{2} k_{f} (x_{f}^{2} + y_{f}^{2}) \right. \\ \left. + \frac{1}{2} m_{o} \left[\left(h_{o} \hat{b}_{1}^{T} + (r_{o} + x_{o}) \hat{b}_{2}^{T} + y_{o} \hat{b}_{3}^{T} \right) \left[{}^{N} \tilde{\omega}_{b} \right]^{T} \left[{}^{N} \tilde{\omega}_{b} \right] \left(h_{o} \hat{b}_{1} + (r_{o} + x_{o}) \hat{b}_{2} + y_{o} \hat{b}_{3}^{T} \right) \\ \left. + \left(h_{o} \hat{b}_{1}^{T} + (r_{o} + x_{o}) \hat{b}_{2}^{T} + y_{o} \hat{b}_{3}^{T} \right) \left[{}^{N} \tilde{\omega}_{b} \right]^{T} \left(\dot{x}_{o} \hat{b}_{2} + \dot{y}_{o} \hat{b}_{3} \right) \\ \left. + \left(h_{o} \hat{b}_{1}^{T} + (r_{o} + r_{o}) \hat{b}_{2}^{T} + y_{o} \hat{b}_{3}^{T} \right) \left[{}^{N} \tilde{\omega}_{b} \right]^{T} \left[\left(\dot{x}_{o} \hat{b}_{2} + \dot{y}_{o} \hat{b}_{3} \right) + \dot{x}_{o}^{2} + \dot{y}_{o}^{2} \right] \\ \left. + \frac{1}{2} m_{f} \left[\left(h_{f} \hat{b}_{1}^{T} + (x_{f} - r_{f}) \hat{b}_{2}^{T} + y_{f} \hat{b}_{3}^{T} \right) \left[{}^{N} \tilde{\omega}_{b} \right]^{T} \left[{}^{N} \tilde{\omega}_{b} \right]^{T} \left[{}^{N} \tilde{\omega}_{b} \right]^{T} \left(\dot{x}_{f} \hat{b}_{2} + \dot{y}_{f} \hat{b}_{3} \right) + \dot{x}_{f}^{2} + \dot{y}_{f}^{2} \right] \\ \left. + \left(h_{f} \hat{b}_{1}^{T} + (x_{f} - r_{f}) \hat{b}_{2}^{T} + y_{f} \hat{b}_{3}^{T} \right) \left[{}^{N} \tilde{\omega}_{b} \right]^{T} \left[\left(\dot{x}_{f} \hat{b}_{2} + \dot{y}_{f} \hat{b}_{3} \right) + \dot{x}_{f}^{2} + \dot{y}_{f}^{2} \right] \\ \left. + \frac{1}{2} m_{0o} \left(-h_{0o} \hat{b}_{1}^{T} + r_{o} \hat{b}_{2}^{T} \right) \left[{}^{N} \tilde{\omega}_{b} \right]^{T} \left[{}^{N} \tilde{\omega}_{b} \right] \left(-h_{0o} \hat{b}_{1} + r_{o} \hat{b}_{2} \right) \\ \left. + \frac{1}{2} m_{0f} \left(-h_{0f} \hat{b}_{1}^{T} - r_{f} \hat{b}_{2}^{T} \right) \left[{}^{N} \tilde{\omega}_{b} \right]^{T} \left[{}^{N} \tilde{\omega}_{b} \right] \left(-h_{0f} \hat{b}_{1} - r_{f} \hat{b}_{2} \right)$$

$$(3.19)$$

Due to the high complexity of the Lagrangian of this system, substituting it into Equation 2.4 would require extensive and unnecessary work. Additionally, the resulting equations of motion are likely to occupy a large amount of space. For the sake of preserving space, the 45 derivatives required to compute the equations of motion will not be done by hand or fully displayed in the body of this thesis. Instead, they will be carried out in part by using the symbolic toolbox in MATLAB. To see the code used for this purpose, refer to Appendix A. The *Lagrange* script was downloaded from the MathWorks file exchange (Ivanovich, 2020).

While the entirety of the calculations will not be shown, it is useful to provide an example of the process. Thus, for illustrative purposes, the inertial matrices were assumed to have small integer values in all elements. After running the code with these and manipulating its output as necessary, the first equation of motion for angular velocity is,

$$\begin{split} &[m_{0f}r_{f}^{2} + m_{0o}r_{o}^{2} + m_{o}(y_{o}^{2} + (x_{o} + r_{o})^{2}) + m_{f}(y_{f}^{2} + (x_{f} - r_{f})^{2}) + 25/2]\dot{\omega}_{1} \\ &+ (h_{0o}m_{0o}r_{o} - h_{0f}m_{0f}r_{f} - h_{f}m_{f}(x_{f} - r_{f}) - h_{o}m_{o}(x_{o} + r_{o}) + 5)\dot{\omega}_{2} \\ &+ (5/2 - h_{o}m_{o}y_{o} - h_{f}m_{f}y_{f})\dot{\omega}_{3} \\ &= g_{1}(\mathbf{z}, t) + m_{o}[y_{o}(r_{o} + x_{o})(\omega_{2}^{2} - \omega_{3}^{2}) + h_{o}y_{o}\omega_{1}\omega_{2} - h_{o}(r_{o} + x_{o})\omega_{1}\omega_{3} \\ &+ (y_{o}^{2} - (r_{o} + x_{o})^{2})\omega_{2}\omega_{3} - \frac{h_{o}}{2}(\dot{x}_{o}\omega_{2} + \dot{y}_{o}\omega_{3})] + m_{f}[y_{f}(x_{f} - r_{f})(\omega_{2}^{2} - \omega_{3}^{2}) \\ &+ h_{f}y_{f}\omega_{1}\omega_{2} - h_{f}(x_{f} - r_{f})\omega_{1}\omega_{3} + (y_{f}^{2} - (x_{f} - r_{f})^{2})\omega_{2}\omega_{3} - \frac{h_{f}}{2}(\dot{x}_{f}\omega_{2} + \dot{y}_{f}\omega_{3})] \\ &+ m_{0f}(-\omega_{2}\omega_{3}r_{f}^{2} - h_{0f}\omega_{1}\omega_{3}r_{f}) + m_{0o}(-\omega_{2}\omega_{3}r_{o}^{2} + h_{0o}\omega_{1}\omega_{3}r_{o}) \\ &- 2[m_{o}(y_{o}\dot{y}_{o} + \dot{x}_{o}(x_{o} + r_{o})) + m_{f}(y_{f}\dot{y}_{f} + \dot{x}_{f}(x_{f} - r_{f}))]\omega_{1} + (h_{f}m_{f}\dot{x}_{f} + h_{o}m_{o}\dot{x}_{o})\omega_{2} \\ &+ (-h_{o}m_{o}\dot{y}_{o} + h_{f}m_{f}\dot{y}_{f})\omega_{3} \end{split}$$

$$(3.20)$$

where $g_1(\mathbf{z}, t)$ is the first element of the solar panel torque vector, to be denoted by \mathbf{g} . The rest of the angular velocity equations follow a similar form: several inertial elements multiplied by angular accelerations equated to a nonlinear function of the states. Therefore, by defining the state vector to be,

$$\mathbf{z} = \begin{bmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 & x_f & y_f & x_o & y_o & \omega_1 & \omega_2 & \omega_3 & \dot{x}_f & \dot{y}_f & \dot{x}_o & \dot{y}_o \end{bmatrix}^T$$

the 15 equations will altogether take the form:

$$\dot{\mathbf{z}} = A(\mathbf{z}) \left[(\mathbf{g}(\mathbf{z}, t) + \mathbf{f}(\mathbf{z})) : \mathbf{z}, \mathbf{f}, \mathbf{g} \in \mathcal{R}^{15}, \right.$$
$$A(\mathbf{z}) \in \mathcal{R}^{15 \times 15}$$

where $A(\mathbf{z})$ is a square matrix which will be defined by dividing it into the following blocks:

$$A(\mathbf{z}) = \begin{bmatrix} 0_{4\times7} & \frac{1}{2}B(\beta) & 0_{4\times4} \\ 0_{4\times7} & 0_{4\times3} & I_{4\times4} \\ 0_{7\times7} & F_{7\times7}^{-1} \end{bmatrix}$$
(3.21)

The elements of the 7×7 matrix *F* are the coefficients of the $\dot{\omega}$ terms and \ddot{x} terms from the equations of motion for angular velocity and slosh mass acceleration. Hence, from the example in Equation 3.20,

$$F_{11} = m_{0f}r_f^2 + m_{0o}r_o^2 + m_o(y_o^2 + (x_o + r_o)^2) + m_f(y_f^2 + (x_f - r_f)^2) + 25/2$$

$$F_{12} = h_{0o}m_{0o}r_o - h_{0f}m_{0f}r_f - h_fm_f(x_f - r_f) - h_om_o(x_o + r_o) + 5$$

$$F_{13} = 5/2 - h_om_oy_o - h_fm_fy_f$$

Next, the vector **g** is,

$$\mathbf{g}(\mathbf{z},t) = \begin{bmatrix} 0_{8\times 1} \\ C_{NP}{}^{P}\tau(t) \\ 0_{4\times 1} \end{bmatrix}$$
(3.22)

where ${}^{P}\tau(t)$ is torque computed from the solar panel vibrations and C_{NP} is the DCM rotating the solar panel into the inertial frame, given by,

$$C_{BP} = \begin{bmatrix} 0 & \cos\varphi & \sin\varphi \\ -1 & 0 & 0 \\ 0 & \sin\varphi & -\cos\varphi \end{bmatrix}$$
(3.23)

where φ is the angle between the axes \hat{b}_1 and \hat{p}_3 . This angle will be computed "on tracks" to be the angle between nadir and **L**, the direction to the Sun from the RSO. Finally, the vector **f** will be the non-acceleration or torque terms from the equations of motion. For example, $f_1(\mathbf{z})$ is the expression which remains on the right side of Equation 3.20 after subtracting $g_1(\mathbf{z}, t)$. Thus, the attitude history can be simulated by numerical integration of the following nonlinear state-space equation:

$$\dot{\mathbf{z}} = A(\mathbf{z}) \left[\mathbf{g}(\mathbf{z}, t) + \mathbf{f}(\mathbf{z}) \right]$$
(3.24)

3.4. Assumed Truth Model

While all of the above is useful for generating a somewhat realistic data set for generation of simulated measurements, it is unlikely in practice that all dynamical states will be fully accounted for when collecting data to characterize the vehicle attitude motion. Hence, there is, in practice, some "truth" model with parameters and states to be estimated which are assumed to adequately represent the system. This section will review the fundamental observations for photometric measurements by quickly recapping the important equations from Section 2.6, and will introduce the fundamental observations of astrometric measurements. The sections will then conclude by unifying these measurements with an "*assumed true*" dynamical and measurement state space model to be later applied in the filtering algorithm.

3.4.1 Photometric Measurement Model

This section will concisely summarize the key results in photometric measurement modeling from Section 2.6. First, by letting $c = \mathbf{H} \cdot \mathbf{V}$, the model of bidirectional reflectance is given by,

$$F(c,\lambda) = F(1,\lambda) + (1 - F(1,\lambda))(1 - c)^{5}$$
(3.25)

where $F(1, \lambda)$ is the experimentally determined or assumed reflectance and normal incidence at the light wavelength λ . The bidirectional reflectance distribution function is given by the sum of a specular term and a diffuse term:

$$\rho = \rho_s + \rho_d \tag{3.26}$$

where,

$$\rho_s(\mathbf{L}, \mathbf{V}) = \frac{\sqrt{(n_i + 1)(n_j + 1)}}{8\pi} \frac{(\mathbf{N}_k \cdot \mathbf{H})^z}{(\mathbf{N}_k \cdot \mathbf{V}) + (\mathbf{N}_k \cdot \mathbf{L}) - (\mathbf{N}_k \cdot \mathbf{V})(\mathbf{N}_k \cdot \mathbf{L})} F(c, \lambda) \quad (3.27)$$

and

$$\rho_d(\mathbf{L}, \mathbf{V}) = \frac{28R_d}{23\pi} (1 - R_s) \left(1 - \left(1 - \frac{\mathbf{N}_k \cdot \mathbf{L}}{2}\right)^5 \right) \left(1 - \left(1 - \frac{\mathbf{N}_k \cdot \mathbf{V}}{2}\right)^5 \right)$$
(3.28)

The exponent of Equation 3.27 is,

$$z = \frac{n_i (\mathbf{H} \cdot \mathbf{i}_k)^2 + n_j (\mathbf{H} \cdot \mathbf{j}_k)^2}{1 - (\mathbf{H} \cdot \mathbf{N}_k)^2}$$
(3.29)

where n_i and n_j are spectral lobe shape parameters, N_k is the unit normal vector of facet k, **H** is a unit angle bisector to **L** and **V**, and each \mathbf{i}_k and \mathbf{j}_k form an orthonormal basis with each N_k which spans \mathcal{R}^3 . Recall that the vector **L** is a unit vector which points from the RSO to the Sun, and **V** is a unit vector which points from the RSO to the viewer. Finally, these all come together to give the visual magnitude of the object measured from a single

site:

$$M_v = -26.74 - 2.5 \log_{10} \sum_{k=1}^{n_f} \rho \frac{(\mathbf{L} \cdot \mathbf{N}_k) (\mathbf{V} \cdot \mathbf{N}_k)}{d^2} A_k$$
(3.30)

where A_k is the area of facet k, and d is the distance from the RSO to the observer. The sum is over the total number of assumed facets on the spacecraft. The parameters n_i and n_j of the BRDF will be assumed equal to 75 for the spacecraft body and 550 for the panel.

3.4.2 Astrometric Measurement Model

An additional source of information from a topocentric optical measurement site is the pointing direction of the observation platform, which can come from the plate-solving techniques of astrometry. The fundamental astrometric observation equation is of the slant range from each site,

$$\rho_i = \mathbf{r}_E - \mathbf{r}_i \tag{3.31}$$

where \mathbf{r}_{E} is the distance to the spacecraft LVLH frame E, and \mathbf{r}_{i} is the position vector of site *i*. Useful to the brightness model, note that the slant range vector ρ_{i} and the RSO-to-viewer direction vector \mathbf{V}_{i} are parallel and point in opposite directions, so that,

$$\mathbf{V}_i = -\frac{\rho_i}{||\rho_i||} \tag{3.32}$$

Denoting the latitude of site i by λ_i , the slant range vector is, in inertial coordinates,

$$\rho_{i} = \begin{bmatrix} x - ||\mathbf{r}_{i}||\cos\Theta\cos\lambda_{i} \\ y - ||\mathbf{r}_{i}||\sin\Theta\cos\lambda_{i} \\ z - ||\mathbf{r}_{i}||\sin\lambda_{i} \end{bmatrix}$$
(3.33)

where x, y, and z are the components of the vector,

$$\hat{\mathbf{r}}_{E} = \begin{bmatrix} x & y & z \end{bmatrix}^{T}$$
(3.34)

Now, the topocentric right ascension (RA) α_i and declination (DEC) δ_i are given by,

$$\alpha_i = \tan_2^{-1}(\rho_i^2, \rho_i^1)$$
(3.35a)

$$\delta_i = \sin^{-1} \frac{\rho_i^3}{||\rho_i||} \tag{3.35b}$$

where $\tan_2^{-1}(y, x)$ is the four-quadrant arctangent function.

3.4.3 Unified Measurement and Dynamical Model

The states to be estimated in the present thesis will be those which are most common: the attitude parameters β_0 , β_1 , β_2 , and β_3 , and the angular velocities of the body ω_1 , ω_2 , and ω_3 . The estimation state vector is thus given by,

$$\mathbf{x}^T = \begin{bmatrix} \beta^T & \omega^T \end{bmatrix}$$
(3.36)

The kinematic relationship between the attitude parameters (i.e. quaternions) and the body-frame angular velocity is given by Equation 2.10, while equations of motion for ω will come from some assumed form of attitude motion. The propellant mass displacement states will be treated as unknown or "hidden" dynamics. The measurement vectors at the North and South sites, $\tilde{\mathbf{y}}_n$ and $\tilde{\mathbf{y}}_s$ respectively, are simply the apparent magnitude expression in Equation 3.30 with \mathbf{V}_i differing for each site, and the azimuth and elevation at each site. For a geostationary object, \mathbf{V} and d will be approximately constant, while \mathbf{L} will depend on the Sun-RSO-Earth phase angle φ . By defining $\mathbf{w}(t) \in \mathcal{R}^7$, $\mathbf{v}_n(t) \in \mathcal{R}^3$, and $\mathbf{v}_s(t) \in \mathcal{R}^3$ to be zero-mean Gaussian white noise processes with respective covariances Q, R_n , and R_s :

$$\mathbf{w}(t) \sim \mathcal{N}(0, Q)$$
 and $\mathbf{v}_{n,s}(t) \sim \mathcal{N}(0, R_{n,s})$

$$\dot{\mathbf{x}} = \begin{bmatrix} \frac{1}{2}B(\beta)\omega\\ f(\omega,t) \end{bmatrix} + \mathbf{w}(t)$$
(3.37)

where,

$$f(\omega, t) = \dot{\omega} = I^{-1} \left[\left[I\omega \right] \times \omega + \mathbf{T}(t) \right]$$
(3.38)

is the kinetic equation relating the attitude motion to some arbitrary time-dependent torque and *I* is the inertia matrix of the body of interest, and the measurement model is given by,

$$\tilde{\mathbf{y}} = \mathbf{h}(\mathbf{x}, t) = \begin{bmatrix} M_n(\mathbf{x}) & \alpha_n(\mathbf{x}) & \delta_n(\mathbf{x}) & M_s(\mathbf{x}) & \alpha_s(\mathbf{x}) & \delta_s(\mathbf{x}) \end{bmatrix}^T + \begin{bmatrix} \mathbf{v}_n(t) \\ \mathbf{v}_s(t) \end{bmatrix}$$
(3.39)

The model described in this section connects the measured quantities to the dynamical states \mathbf{x} , the first step in the estimation problem. To extend the above to include polarimetric data, simply add its components to the measurement vector as so:

$$\tilde{\mathbf{y}} = \begin{bmatrix} M_n & \alpha_n & \delta_n & \mathbf{S}_n^T & M_s & \alpha_s & \delta_s & \mathbf{S}_s^T \end{bmatrix}^T + \begin{bmatrix} \mathbf{v}_n(t) \\ \mathbf{v}_s(t) \end{bmatrix}$$
(3.40)

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where the noise vectors are now $\mathbf{v}_{n,s}(t) \in \mathcal{R}^7$.

3.5. Sequential Filtering Via Unscented Kalman Filter

When estimating states, there are several available options as previously discussed, but the unscented Kalman filter (UKF) will be developed and applied to this system. A Kalman filter fundamentally operates on a predictor-corrector algorithm, relying on the user's knowledge of system dynamics and noise from unknown processes. While there exist alternative methods for nonlinear system state estimation, such as the extended
Kalman filter, the UKF provides a key benefit of avoiding computation of any Jacobians or other derivatives. For an overview of derivations and applications of many types of state estimation techniques, see Crassidis and Junkins (2012). For a detailed derivation of the standard UKF, see Julier, Uhlmann, and Durrant-Whyte (1995, 2000).

An important thing to note when implementing a UKF with quaternions, the nonlinear constraint from Equation 2.8 can cause the covariance matrix to become singular. Additionally, small quaternion rotations are applied multiplicatively, which in turn does not allow for direct implementation of the UKF structure (Linares, Jah, Crassidis, Leve, & Kelecy, 2014). The UKF presented in Crassidis and Markley (2003) overcame this issue by first transforming the quaternions to equivalent generalized Rodriguez parameters (GRPs) for the local error calculation since they are additive for small changes, then describing the global attitude with quaternions. This implementation will be reviewed then modified and applied to this system.

First, the standard discrete-time Kalman state estimate and state covariance update equations are written as,

$$\hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k^- + K_k \mathbf{e}_k^- \tag{3.41a}$$

$$P_k^+ = P_k^- - K_k P_k K_k^T$$
(3.41b)

where $\hat{\mathbf{x}}_k^-$ and $\hat{\mathbf{x}}_k^+$ are respectively the predicted and corrected values of the state at timestep k, K_k is the Kalman gain matrix, and \mathbf{e}_k^- is the measurement error or "innovations process" given by,

$$\mathbf{e}_{k}^{-} \equiv \tilde{\mathbf{y}}_{k} - \hat{\mathbf{y}}_{k}^{-} = \tilde{\mathbf{y}}_{k} - \mathbf{h}(\hat{\mathbf{x}}_{k}^{-}, k)$$
(3.42)

In this equation, $\tilde{\mathbf{y}}_k$ represents the true measured value at timestep k, while $\hat{\mathbf{y}}_k^-$ represents the predicted measurement at timestep k based on the measurement model **h**, which is

generally a function of the predicted value of the state and the current timestep. The covariance of \mathbf{e}_k^- is denoted P_k^{ee} , and is used to compute K_k as,

$$K_k = P_k^{xy} (P_k^{ee})^{-1} (3.43)$$

where P_k^{xy} is the cross-correlation matrix between $\hat{\mathbf{x}}_k^-$ and $\hat{\mathbf{y}}_k^-$.

To propagate the state, a number of "sigma points" are calculated from the 2n columns of the following matrix:

$$\Sigma = \begin{bmatrix} \sqrt{(n+\lambda)P_k^+} & -\sqrt{(n+\lambda)P_k^+} \end{bmatrix}$$
(3.44)

where the square root of a matrix M is defined to be the matrix Z such that $M = ZZ^T$, and $\lambda = \alpha^2(n + \kappa) - n$ is a scaling parameter. A popular approach for computing the matrix square root is by using a Cholesky decomposition, but this can lead to divergence issues and occasionally a non-positive semi-definite covariance matrix (Daid, Busvelle, & Aidene, 2021). For this reason, the principal matrix square root will be used since it generally yields better results.

Let σ_i be the i^{th} column of Σ , and define the corresponding sigma point χ_i as,

$$\chi_i = \hat{\mathbf{x}}_k^+ + \sigma_i \tag{3.45}$$

The divergence from the standard UKF structure necessarily begins here, since this "error distribution" is computed additively. To allow for this, define the GRP sigma point as,

$$\chi_{i}^{\delta \mathbf{p}} = \begin{bmatrix} \delta \mathbf{p}_{i} \\ \hat{\omega}_{k} + \delta \omega_{i} \end{bmatrix}$$
(3.46)

where $\delta \mathbf{p}_i$ is a small error GRP and $\delta \mathbf{p}_0 = \mathbf{0}$. Next, let the covariance matrix be interpreted as the covariance of the error GRP rather than the covariance of the quaternion estimate.

The sigma points are then calculated not for quaternions but for GRPs:

$$\chi_i^{\delta \mathbf{p}+} = \begin{bmatrix} \mathbf{0} \\ \hat{\omega}_k^+ \end{bmatrix} + \sigma_i^{\delta \mathbf{p}}$$
(3.47)

Next, transform the error GRP to error quaternions using the following relationship:

$$\delta\beta = \begin{bmatrix} \delta q_0 \\ \delta \mathbf{q} \end{bmatrix}$$
(3.48)

where,

$$\delta q_0 = \frac{-a||\delta \mathbf{p}||^2 + f\sqrt{f^2 + (1 - a^2)||\delta \mathbf{p}||^2}}{f^2 + ||\delta \mathbf{p}||^2}$$
(3.49)

and

$$\delta \mathbf{q} = f^{-1}(a+q_0)\delta \mathbf{p} \tag{3.50}$$

where $a \in [0, 1]$ and f are parameters of the GRP. When a = 0 and f = 1, this equation give the Gibbs vector, and when a = f = 1 it gives the standard vector of modified Rodrigues parameters. It is common in this type of attitude filtering to let f = 2(a + 1) so that $||\delta \mathbf{p}|| = \delta \theta$ for small attitude errors. Of note, in the simulations for this thesis, a = 1and thus f = 4. Now, the the error quaternions are used to compute a distribution of global sigma point quaternions:

$$\chi_i^{\beta+} = \begin{bmatrix} \hat{\beta}_k^+ \bigotimes \delta \beta_i \\ \hat{\omega}_k^+ + \delta \omega_i \end{bmatrix}$$
(3.51)

where \otimes denotes Hamilton multiplication, defined for quaternions as,

$$\beta_{3} = \beta_{2} \bigotimes \beta_{1} = \begin{bmatrix} a_{1}a_{2} - b_{1}b_{2} - c_{1}c_{2} - d_{1}d_{2} \\ a_{1}b_{2} + b_{1}a_{2} + c_{1}d_{2} - d_{1}c_{2} \\ a_{1}c_{2} - b_{1}d_{2} + c_{1}a_{2} + d_{1}b_{2} \\ a_{1}d_{2} + b_{1}c_{2} - c_{1}b_{2} + d_{1}a_{2} \end{bmatrix}$$
(3.52)

for $\beta_1 = [a_1 \ b_1 \ c_1 \ d_1]^T$ and $\beta_2 = [a_2 \ b_2 \ c_2 \ d_2]^T$. This operation is equivalent to rotation by β_2 of an object with an orientation described by β_1 . Equivalently, the quaternion β_3 describes the orientation obtained by rotating from the inertial frame first by β_1 and then by β_2 . The global quaternion sigma points are then propagated using the dynamical model of the system from Equations 3.37 and 3.38:

$$\chi_{i,k+1}^{\beta-} \text{ from } \dot{\mathbf{x}} = \begin{bmatrix} \frac{1}{2}B(\beta)\omega\\ f(\omega,t) \end{bmatrix} \text{ with I.C. } \chi_{i,k}^{\beta+}$$
(3.53)

From the resultant distribution of predicted quaternion sigma points $\chi_{i,k+1}^{\beta-}$, a new distribution of error quaternions is calculated as,

$$\delta\beta_{i,k+1}^{-} = \beta_{i,k+1}^{-} \bigotimes \left(\beta_{0,k+1}^{-}\right)^{-1} \tag{3.54}$$

where $(*)^{-1}$ in this case indicates the complex conjugate. Now the predicted error GRPs are computed as,

$$\delta \mathbf{p}_{i,k+1}^{-} = f \frac{\delta \mathbf{q}_{i,k+1}^{-}}{a + \delta q_{0,k+1}^{-}}$$
(3.55)

and finally, the predicted mean and covariance of the error GRP sigma points are given by,

$$\hat{\mathbf{x}}_{k+1}^{-} = \sum_{i=0}^{2n} w_i \chi_{i,k+1}^{\delta \mathbf{p}-}$$
(3.56)

and

$$P_{k+1}^{-} = Q + \sum_{i=0}^{2n} \left[\chi_{i,k+1}^{\delta \mathbf{p}-} \hat{\mathbf{x}}_{k+1}^{-} \right]$$
(3.57)

where the weights w_i are given by,

$$w_0 = \frac{\lambda}{n+\lambda} \tag{3.58a}$$

$$w_i = \frac{1}{2(n+\lambda)} \tag{3.58b}$$

and Q is the dynamical process noise. To compute the innovations covariance P_{k+1}^{ee} , the cross-correlation matrix P_{k+1}^{xy} , and subsequently the Kalman gain K_{k+1} , the same process from Equations 3.44 through 3.51 is then applied with the predicted state and covariance to find a new distribution of sigma points $\chi_{i,k+1}^{\beta-}$. These are then used to figure a measurement distribution from Equation 3.39:

$$\gamma_{i,k+1}^{-} = \mathbf{h}(\chi_{i,k+1}^{\beta-})$$
(3.59)

The predicted measurement is then given by,

$$\hat{\mathbf{y}}_{k+1} = \sum_{i=0}^{2n} w_i \gamma_{i,k+1}^- \tag{3.60}$$

and the innovations covariance is,

$$P_{k+1}^{ee} = R + \sum_{i=0}^{2n} \left[\gamma_{i,k+1}^{-} - \hat{\mathbf{y}}_{k+1} \right] \left[\gamma_{i,k+1}^{-} - \hat{\mathbf{y}}_{k+1} \right]^{T}$$
(3.61)

where R is the measurement process noise. In the dual-site case it is given by,

$$R = \operatorname{diag}\left(\begin{bmatrix} R_n & R_s \end{bmatrix} \right) \tag{3.62}$$

The cross-correlation matrix is then given by,

$$P_{k+1}^{xy} = \sum_{i=0}^{2n} \left[\chi_{i,k+1}^{\delta \mathbf{p}-} - \hat{\mathbf{x}}_{k+1}^{\delta \mathbf{p}-} \right] \left[\gamma_{i,k+1}^{-} - \hat{\mathbf{y}}_{k+1} \right]^{T}$$
(3.63)

Finally, the Kalman gain is,

$$K_{k+1} = P_{k+1}^{xy} \left(P_{k+1}^{ee} \right) \tag{3.64}$$

and the updated error GRP estimate is given by,

$$\hat{\mathbf{x}}_{k+1}^{\delta \mathbf{p}+} = \begin{bmatrix} \mathbf{0} \\ \hat{\omega}_{k+1}^{-} \end{bmatrix} + K_{k+1} (\tilde{\mathbf{y}}_{k+1} - \hat{\mathbf{y}}_{k+1})$$
(3.65)

which is then transformed back into a global quaternion representation using Equations 3.48 through 3.51, giving an updated state estimate $\hat{\mathbf{x}}_{k+1}^{\beta+}$. From here, the entire process is repeated for the duration of the measurement data set.

3.5.1 Filter Tuning

An important step in filtering is to tune the input parameters, being the weights w_i in both the propagation and update stages, as well as the process noise matrices Q and R, and the initial state estimate covariance P_0 . To start, the method for getting a reasonable estimate of the state process noise matrix will come down to estimating the matrix given by,

$$Q = \begin{bmatrix} \sigma_{grp}^2 I_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & \sigma_{\omega}^2 I_{3\times3} \end{bmatrix}$$
(3.66)

where the σ^2 values are the variance of the process noise for each state. The approach for estimating these was to first compute $\dot{\mathbf{x}}$ using Equation 3.37 given some "worst-case scenario" conditions. The resulting quaternion component was then converted to an error

GRP vector. Then, the noise covariances were set to,

$$\sigma_{grp} = \left\| \dot{\mathbf{x}}^{\delta p} \right\| \Delta t \tag{3.67}$$

$$\sigma_{\omega} = \|\dot{\mathbf{x}}^{\omega}\|\,\Delta t \tag{3.68}$$

Thus, the resulting Q is,

$$Q = \begin{bmatrix} (8 \times 10^{-10})I_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & (2.8 \times 10^{-20})I_{3\times3} \end{bmatrix}$$
(3.69)

Right out of the box, the above matrix yields a decent starting point for further tuning if necessary. The measurement noise covariance matrix R is much more straight-forward to estimate, since the variance of each of the measurement data sets can be directly estimated from detrended data. For example, the variance of visual magnitude is taken to be,

$$\sigma_{mv}^2 = \operatorname{var}\left(M_v(t) - \tilde{M}_v(t)\right) \tag{3.70}$$

where $\tilde{M}_v(t)$ is a polynomial best-fit for the data set. To account for synodic effects, a second-degree polynomial detrend was chosen for computation of $\tilde{M}_v(t)$.

3.6. Simulation

Now that the foundations of the experiment have been thoroughly laid out, this section will describe the several test cases for the simulated data sets. First, the spacecraft attitude motion is simulated using Equation 3.24 for a 3-hour window beginning at 02:12:07.455 UTC on 2020-05-20. This (oddly specific) time was chosen to correspond with a set of collected data. The initial attitude was chosen such that \hat{b}_2 points along $-\hat{n}_3$ and \hat{b}_1 points toward nadir. The rotational motion of the solar panel was chosen to be "on tracks" such that for every timestep its surface normal is given by,

$$\mathbf{N}_{panel} = \frac{\mathbf{L} - \mathbf{L} \cdot \hat{p}_1}{||\mathbf{L} - \mathbf{L} \cdot \hat{p}_1||}$$
(3.71)

where L is the unit vector pointing from the RSO to the Sun and $\hat{p}_1 = -\hat{b}_2$ is panel axis of rotation.

After the attitude was simulated, measurement data sets are simulated at 0.2 second intervals for both sites. Measurement noise was given standard deviations of 0.01 for angle data, 0.18 for brightness data, and 1×10^{-4} for each of the Stokes vector components.

Finally, the simulated measurement data was input to the UKF for estimation of the simulated states. In total, there are eight different cases for implementation in the filter, both for single-site and for dual-site. In every single-site case, the North site at ERAU was chosen. To be concise, the cases are summarized below:

- · Angles and noisy photometric data only
- · Angles and clean photometric data only
- · Angles, noisy photometric data, and stokes vector
- · Angles, clean photometric data, and stokes vector

The initial error was chosen such that roll, pitch, and yaw were 5 degrees from truth. This is consistent with the initial conditions from Linares, Jah, Crassidis, and Nebelecky (2014). The purpose in testing with clean photometric data is twofold. First, it is important to know that the filter will converge when the noise is minimal. Additionally, better performance of cleaner data can serve as a justification for investment in better (but likely more expensive) instrumentation.

3.7. Detection of Vibrational Modes

Vibrational mode detection was approached from the non-model-based approach given by the fractional Fourier transform (FrFT). The process of figuring the correct transform order was to firt compute the Wigner-Ville distribution function for a parametrically detrended signal. This is to say, the WVDF for the signal given by,

$$x_0(t_{n-1} + \delta t) = x(t_{n-1} + \delta t) - \tilde{x}(t_{n-1} + \delta t)$$
(3.72)

where $\delta t \in [0, \Delta t]$ such that $\Delta t = t_n - t_{n-1} \forall n = 1, 2, ..., f_s t_{max}/k$, with k being a window tuning parameter. $\tilde{x}(t_{n-1} + \delta t)$ is the best fit line of the data on the interval $t \in [t_{n-1}, t_n]$. Of note with this, an additional frequency of $\tilde{f} = 1/\Delta t$ may be introduced to the signal $x_0(t)$ which is not necessarily present in x(t). Next, the highest-amplitude portion was identified, and its slope in the time-frequency domain was calculated to find an appropriate rotation angle α_0 . After this, the FrFT algorithm was applied for $\alpha = \alpha_0$ to maximize the signal power density at the corresponding frequency $f = f_0$, and again for $\alpha = \alpha_0 + \pi/4$ to evenly distribute the energy across both time and frequency domains.

3.8. Experimental Setup

The electro-optical system (EOS) was an 11" Celestron Rowe-Ackerman-Schmidt Astrograph (RASA) with an equatorial mount, equipped with a ZWO ASI-1600MM/Cool monochrome CMOS detector. The right-ascension and declination of GOES-16 (NORAD 41866) was found according to its publicly available TLEs, and the integration time was set to 200 milliseconds. Because this object is geostationary, tracking was achieved by simply pointing at it and shutting off the telescope mount power.

4. Simulation Results

This section first presents the results of simulating the attitude motion from Equation 3.24 and the measurement data from Equations 3.30, 3.35, and 2.73, then shows the results of each filter case as described at the end of the previous chapter. While the spring-mass approximation to propellant slosh realistically only holds true under the small-amplitude assumption, the masses were placed at the tank edges for the sake of a "worst-case scenario" test of fluid-gas free-surface motion.

4.1. Model Validation

First to be shown are two-minute simulations of the slosh mass positions and torque from the solar panel, τ_{sp} . These are shown in Figures 4.1 and 4.2, respectively. Of note, the motion of the fuel and oxidizer masses in both directions is cleanly sinusoidal, and as such the fundamental frequency is easily verifiable. On the other hand, τ_{sp} appears chaotic in its behavior since there is not any clear periodicity.



Figure 4.1 Two minutes of simulated slosh mass position data.

It is unclear that even the expected fundamental frequency of 0.25 Hz is present in Figure 4.2, so the angular velocity history for this same simulation is shown in Figure 4.3.



Figure 4.2 Two minutes of simulated solar panel torque data.

As an additional source of confidence that the attitude motion is primarily confined to the nominal direction, the quaternion history is shown in Figure 4.4.



Figure 4.3 Two minutes of simulated angular velocity data.



Figure 4.4 Two minutes of simulated quaternion data.

Finally, to test the long-term stability of the simulation, the attitude motion was simulated for six hours. The resulting quaternion history is shown in Figure 4.5, in which it is clearly not numerically unstable and is exhibiting expected



Figure 4.5 Six hours of simulated quaternion data.

4.2. Imaging and Light Curve Simulation

With the dynamical model behaving satisfactorily, the next step was to verify that the observation geometry is correct. The way this was accomplished was to create a projected image of the RSO model onto an imaging plane centered on the observer. The result of this is shown in Figure 4.6.



Figure 4.6 Projection of GOES 16 wire frame onto image plane.



Figure 4.7 Comparison of visual magnitude prediction to collected data.

Next, a simulated set of visual magnitude data was compared to a set of collected data. This allowed for the specular and diffuse reflectance parameters to be tuned until the data trends were in agreement with one another. This is shown in Figure 4.7.

4.3. Simultaneous Light Curves and Angle Measurements

The next step in confirming that the models are behaving reasonably is to consider how the northern measurements might compare to the southern measurements. For brightness, one would expect the southern measurement of visual magnitude to take a lower value (i.e. a higher brightness), since the southern site is physically closer to GOES 16. The right-ascension should be approximately the same for both sites if they are at the same longitude, but the declination should be higher for the southern site since GOES 16 will appear higher in the sky. A clean measurement set for both of these is shown in Figure 4.8, and it is clearly in agreement with the above intuition.



Figure 4.8 Simulated clean measurements of M_v and ΔM_v (top) and RA/DEC (bottom) from North and South sites.

5. Results

This chapter is split into two sections. The first section provides an overview of the results for detection of vibrational modes using the FrFT algorithm. What is important to note here is that two of the three frequencies become local maximums for certain rotation kernels. The second section provides an overview of the filter results, showing that there is a slight improvement by the addition of a second site and that there is a great improvement by the addition of polarimetric data.

5.1. Identification of Vibrational Modes

First, the ordinary fast Fourier transform of the signal is shown in Figure 5.1. The primary frequencies expected are those due to the solar panel oscillations and due to the propellant slosh. The expected frequencies are summarized in Table 5.1. Clearly, these frequencies are all "lost in the noise" and no additional information may be discerned from the data by using FFT.



Figure 5.1 Ordinary fast Fourier transform.

Table 5.1

List of frequencies expected to have peaks in the spectrum.

Source	Frequency	Unit
Solar Panel	0.250	Hz
Fuel	8.711	mHz
Oxidizer	79.37	mHz

Next, the Wigner-Ville energy distribution function for the DOLP measured from the north site is shown in Figure 5.2. The energy peak follows a clear linear trend in the time-frequency domain, so it may be assumed that this contains a significant driver of the signal. The angle by which the WVDF should be rotated in the t - f plane can be directly calculated from two points on the peak. The resulting fractional Fourier transform from this is shown in Figure 5.3.

What is remarkable about this seemingly noisy plot is that the absolute maximum over the domain between 0 and 300 mHz occurs at 79.18 mHz. This coincides extremely closely to the expected frequency of the oxidizer, with a percent error of 2.4%. Next, the rotation by an additional $\pi/4$ radians results in the power distribution shown in Figure 5.4.



Figure 5.2 Wigner-Ville Distribution Function for the first 30 minutes of DOLP.

Again, there is an interesting occurrence in that the absolute maximum over the frequency domain occurs at 249.5 mHz, since this coincides with the fundamental frequency of the solar panel vibrations. In this case, the percent error is 2%. While it is surprising that the frequency of the fuel does not show up in either case, this could easily be due to its extremely low velocity relative to the body.



Figure 5.3 Fractional Fourier transform with kernel to maximize signal.



Figure 5.4 Fractional Fourier transform with additional rotation by $\pi/4$.

5.2. Filtering

This section presents the results from the unscented Kalman filter for each of the eight test cases described in Chapter 3. First will be the results for a filter implementation which considers only photometric and astrometric data, of which the single-site noisy case may serve as a baseline for performance. Next to be presented are the implementations which include polarimetric data. There will be four plots for each case. These are, in order, the error and covariance of the principal rotation angle Φ , the quaternion estimates compared to truth, the error quaternion, and the error of the body-frame angular velocity components.

5.2.1 Photometric and Astrometric Data

As promised, first consider Figures 5.5–5.8, which show filter performance for noisy single-site photometric and angle measurements with no polarimetric consideration. These will serve as the baseline of performance.



Figure 5.5 Error and covariance of angle with noisy single-site photometric data.



Figure 5.6 Quaternion history with noisy single-site photometric data.



Figure 5.7 Error quaternion history with noisy single-site photometric data.



Figure 5.8 Error in angular velocity with noisy single-site photometric data.

Next is the result for the dual-site case with noisy photometric data, shown in Figures 5.9–5.12. Note here that there is higher sensitivity to the measurements, and changes occur more rapidly.



Figure 5.9 Error and covariance of angle with noisy dual-site photometric data.



Figure 5.10 Quaternion history with noisy dual-site photometric data.



Figure 5.11 Error quaternion history with noisy dual-site photometric data.



Figure 5.12 Error in angular velocity with noisy dual-site photometric data.

Next, for the sake of comparison to the perfect cases with no noise to consider the benefits to better instruments, Figures 5.13–5.16 and Figures 5.17–5.20 respectively show the single and dual-site filter performance with the absence of noise.



Figure 5.13 Error and covariance of angle with clean single-site photometric data.



Figure 5.14 Quaternion history with clean single-site photometric data.



Figure 5.15 Error quaternion history with clean single-site photometric data.



Figure 5.16 Error in angular velocity with clean single-site photometric data.



Figure 5.17 Error and covariance of angle with clean dual-site photometric data.



Figure 5.18 Quaternion history with clean dual-site photometric data.



Figure 5.19 Error quaternion history with clean dual-site photometric data.



Figure 5.20 Error in angular velocity with clean dual-site photometric data.

Both single-site and dual-site measurements perform similarly in this case, though the dual-site case shows a steeper slope in the first 15 minutes of the angle error. There is no clear benefit in either case aside from the higher sensitivity of the dual-site filter. After fully processing the three hour window of data for the noisy single-site case, the principal rotation angle error was around 2° . In the dual-site case, the steady-state error is marginally improved–hovering around 1.8° –and the convergence begins sooner than for single-site.

5.2.2 Addition of Polarimetric Data

Next to be considered is the filter performance in the presence of polarimetric data. In the same fashion as in the previous section, first to be considered is that for single-site with noisy photometric data.



Figure 5.21 Error and covariance of angle with noisy single-site photometric data and including polarimetric measurements.



Figure 5.22 Quaternion history with noisy single-site photometric data and including polarimetric measurements.



Figure 5.23 Error quaternion history with noisy single-site photometric data and including polarimetric measurements.



Figure 5.24 Error in angular velocity with noisy single-site photometric data and including polarimetric measurements.

It is immediately obvious that there is a significant improvement over all cases which had no polarimetric consideration. Next, consider the dual-site case, in Figures 5.25–5.28.



Figure 5.25 Error and covariance of angle with noisy dual-site photometric data and including polarimetric measurements.



Figure 5.26 Quaternion history with noisy dual-site photometric data and including polarimetric measurements.



Figure 5.27 Error quaternion history with noisy dual-site photometric data and including polarimetric measurements.



Figure 5.28 Error in angular velocity with noisy dual-site photometric data and including polarimetric measurements.

As in the prior cases, the dual-site error converges much more quickly, but once again the overall performance appears to be only marginally better than for the single-site case. The estimates of angular velocity were still divergent from truth. Finally, the single and dual-site polarimetric cases which contain pristine photometric data are presented in Figures 5.29–5.32 and Figures 5.33–5.36, respectively.



Figure 5.29 Error and covariance of angle with single-site clean photometric data and including polarimetric measurements.



Figure 5.30 single-site error quaternion history with clean photometric data and including polarimetric measurements.



Figure 5.31 single-site error quaternion history with clean photometric data and including polarimetric measurements.



Figure 5.32 single-site error in angular velocity with clean photometric data and including polarimetric measurements.



Figure 5.33 Error and covariance of angle with clean dual-site photometric data and including polarimetric measurements.



Figure 5.34 Quaternion history with clean dual-site photometric data and including polarimetric measurements.



Figure 5.35 Error quaternion history with clean dual-site photometric data and including polarimetric measurements.



Figure 5.36 Error in angular velocity with clean dual-site photometric data and including polarimetric measurements.

Importantly, it is clear when comparing this section to the previous section that the addition of polarimetric data greatly improves the performance of the filter in both single-site and dual-site cases. The stark performance increase, even in the presence of significant photometric noise, suggests that for attitude estimation the benefit to the

addition of a polarimeter is much more clear than that of the addition of a second observation site. The dual-site polarimetric cases do in fact perform better than their single-site counterparts, but again only by a small amount overall with a significantly faster initial convergence speed.

6. Conclusion

This chapter will highlight the key results from the simulations of Chapter 4 and the vibrational mode detection and filtering results of Chapter 5. Following this, a number of suggestions for future research will be provided, concerning the measurement geometry and techniques, methods of filtering and analysis, as well as for the complex dynamical modeling.

6.1. Overall Performance

First, the BRDF model was easily made to match real collected data for GOES-16. While the dynamical simulation results of Chapter 4 are not backed by experimental data, the long-term numerical stability of the solution and the appearance of expected slosh and solar panel vibration frequencies instills confidence from an intuitive viewpoint that the derived equations of motion are accurate representations of a dynamical system under the given assumptions.

For detecting these vibrational modes, the Fractional Fourier Transform does show itself as a potentially valuable tool, since it is able to detect two of the three fundamental frequencies present in the attitude motion by using only the degree of linear polarization with a relatively large amount of noise assumed. With further noise reduction and/or longer data sets, it may be possible to detect the vibrational modes with a higher degree of certainty and subsequently associate them with their corresponding internal processes. Subsequently, inferences may be made regarding the appropriate dynamical process noise for the early stages in tuning the filter.

For the purpose of attitude filtering, it is evident that there is some benefit to the addition of a second site for observations and there is a clear benefit to the addition of a four-state polarimeter. Adding both is of course preferable, but the biggest benefit comes from the addition of the polarimetric data. Across all simulations, the dual site cases proved to be more sensitive to measurements. Additionally, the cases with polarimetric
data included performed significantly better than those without. There are a few conclusions to be made from these observations.

First is that dual-site data collection when separated by only about a degree or so of latitude is worth investigating further. If the seismology is not necessarily of interest, longer integration times and greater spacing between measurements may be desirable, since this allows for higher signal-to-noise ratio in visual magnitude and longer integration time of the dynamical model between timesteps in the filter propagation stage. This could in turn reduce the effects of floating-point error, which may or may not have played a role in the convergence times. One must approach this carefully, of course, since integration times which are too long may cause unwanted uncertainty with astrometric measurements due to streaking effects.

Of note, the only states which reliably coverged were the attitude states. The angular velocities diverged in every test case. This is not unsurprising, however, since the collected measurements depend only on the observation geometry and spacecraft attitude, which would mean the largest effect of the correction step in the UKF will be on the attitude estimate, regardless of what happens to the angular velocity estimates.

The better performance of the filter with polarimetric measurements can likely be attributed to the addition of four unique states which will have very different values from one another for any given orientation of the RSO. The performance of the filter with noisy photometric and polarimetric data was comparable to the performance of the filter with perfect photometric data and excluding polarimetric data. This is to say that the performance can be made comparable to that from perfect photometric measurements by adding a somewhat-decent polarimeter–without requiring extremely precise photometric measurements. A cost assessment would be a useful addition to this, since the polarimeter will provide another clear benefit of only requiring the weather to be clear in a single location while still yielding excellent results. However, polarimeters are notoriously complicated to design and implement, and because they are usually custom-built to meet the needs of any particular task, they can also become a significant expense.

6.2. Recommendations for Further Investigation

When there are tens or hundreds of dials to be manually tuned, as there were here, it seems inevitable that there will always be some combination of inputs which is better than another. For this reason, it is natural to believe that the performance of the filters presented may be improved upon by spending more time tuning the input parameters to the dynamical propagation stage, the covariances, the process noise, and the sigma weights at each of the stages. Alternatively, an adaptive filtering technique could potentially be applied to avoid manual tuning altogether. There may also be some merit in the use of a higher order filter, which could help with the convergence of the angular velocity estimates.

There is also merit in investigating whether a larger separation between observers produces better results, since the single degree of separation may not be sufficient for a significant benefit. The marginal-at-best improvements of the filter performance with the second site may be due in part to this. Additionally, it may be beneficial to extend the observation window, since a longer time frame may allow for the angular velocity estimates to converge once the attitude error is sufficiently reduced.

Related less so to attitude filtering and more so to attitude dynamical modeling, there should be more work into studying the effects of different models for solar panel vibration. Because the Euler-Bernoulli beam is confined to a single dimension, it fails to capture any potential lateral or twisting effects which may or may not have a noticeable effect when coupled to the greater body motion. Additionally, the assumption was made for the sake of development that the solar panel consists of a single plate. This did allow for some approximations to be made, but there should be future investigation of the effects of this assumption on the body dynamics. Furthermore, a more thorough study of the body-panel coupling effects should be performed with consideration for bending,

stretching, and twisting in all directions in order to better quantify the significance of these effects and whether some or all of them can be neglected.

For making educated inferences about the shape, materials, origin, mission, and other activities of a vehicle, non-model-based approaches are often preferred since they require minimal assumptions. The discrete Fourier transform is likely insufficient for detecting small oscillations due to stationkeeping maneuvers or other relatively impulsive sources of seismic activity, but wavelet analysis of light curves may allow for better resolution of changes in modes over time due to the nature of the time-frequency dependence of spectrograms (Dianetti & Crassidis, 2018). This could be an excellent source of information for early characterization prior to model development. Additionally, a more rigorous approach to the fractional Fourier transform may be valuable since it was only given superficial treatment in this thesis. This could include the use of more signal types, a range of signal noise characteristics, or different combinations of pre- and post-processing.

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APPENDIX A - CODE FOR COMPUTATION AND EVALUATION OF EQUATIONS OF MOTION

The code used to find the equations of attitude motion for the high-fidelity simulation is presented below. There is some required manual manipulation of the outputs from the first script, but this can be done using Notepad, or something similar.

```
1 % Attitude motion lagrangians
_2 clear
3 syms w1 w2 w3 dw1 dw2 dw3 ddw1 ddw2 ddw3 xo yo dxo dyo ddxo ddyo xf ...
      yf dxf dyf ddxf ddyf b0 b1 b2 b3
4 syms wbe wpb
\mathbf{5}
6
v w_b = [w1 w2 w3].';
s w_e = w_b - [wbe 0 0].';
9 w_p = w_b + [0 wpb 0].';
10 beta = [b0 b1 b2 b3].';
11 Ixx = 16480;
12 Iyy = 8549;
13 Izz = 12671;
_{14} Pxy = 62;
_{15} Pxz = -518;
_{16} Pyz = 315;
17
18 I = [Ixx, Pxy, Pxz;
        Pxy, Iyy, Pyz;
19
        Pxz, Pyz, Izz];
20
21
22 \, dt = 3;
_{23} tmax = 15;
_{24} xmax = .7683;
25
26 mf fill = 865.35/1627.7;
27 mo_fill = 36.35/699.7;
28 [mo, m0o, ho, h0o, ko, fo] = cylSloshParam(.105,2,1369.9,mo_fill);
29 [mf, m0f, hf, h0f, kf, ff] = cylSloshParam(.521,2,960.2863,mf_fill);
30
_{31} % hOo = dt/tmax*xmax;
32
_{33} ro = 0.7887;
_{34} rf = 1.5598;
_{35} % ho = .05*h0o;
_{36} % hf = 1;
_{37} % h0f = 0;
_{38} r_o = [ho ro 0]';
39 r_f = [hf rf 0]';
```

```
_{40} r0o = [h0o ro 0].';
_{41} rOf = [hOf rf 0].';
42
43 r_f = [xf yf 0].' + r_f;
44 r_0 = [xo yo 0].' + r_0;
_{45} vf = [dxf dyf 0].';
_{46} vo = [dxo dyo 0].';
47 v_f = vf + cross(w_b,r_f);
48 v_o = vo + cross(w_b,r_o);
49 v00 = cross(w_b, r00);
50 vOf = cross(w_b,rOf);
51
52 % mo = 0.1*mo_fill;
53 \% m00 = 0.9*mo_{fill};
54 % mf = 0.1*mf_fill;
55 % mOf = 0.9*mf_fill;
56 \% \text{ kf} = 1;
57 \% ko = 1;
58
59 r e = [2.5 0 0]';
60 r_b = [0 0 0]';
f_{1} r_{p} = [0 - 4.732 0]';
62 I_e = 0;
63 I_b = I;
64 I_p = 0;
65 \text{ me} = 1;
66 \text{ mb} = 0;
67 \text{ mp} = 1;
68
69 re = crossmatrix(r_e);
70 rb = crossmatrix(r_b);
71 rp = crossmatrix(r_p);
72
         0.5*w_e.'*(I_e + me*(re*re.'))*w_e ...
73 L =
       + 0.5*w_b.'*(I_b + mb*(rb*rb.'))*w_b ...
74
       + 0.5*w_p.'*(I_p + mp*(rp*rp.'))*w_p ...
75
       + 0.5*mo*(v_o.'*v_o) + 0.5*m0o*(v0o.'*v0o)
                                                      . . .
76
       + 0.5*mf*(v_f.'*v_f) + 0.5*mOf*(vOf.'*vOf) ...
77
       - 0.5*ko*(r_o.'*r_o) - 0.5*kf*(r_f.'*r_f);
78
79
80 % Equations of motion for omega from quasi-lagrangian approach. Missing
81 % torge vector.
82
83 [zero_omega, rhs_om] = quasi_lag(L,[w1 dw1 w2 dw2 w3 dw3]);
84 zero_omega = vpa(zero_omega);
85 rhs_om = vpa(rhs_om);
86
  [zero_else, rhs_el] = Lagrange(L,[xf dxf ddxf yf dyf ddyf xo dxo ...
87
      ddxo yo dyo ddyo w1 dw1 ddw1 w2 dw2 ddw2 w3 dw3 ddw3]);
s8 rhs_el = vpa(subs(rhs_el(1:4),[dw1 dw2 dw3],[0 0 0]));
89 zero_else = vpa(zero_else(1:4));
```

```
1 % Lagrange is a function that calculate equations of motion (Lagrange's
_2 % equations) d/dt(dL/d(dq))- dL/dq=0. It Uses the Lagrangian that ...
      is a function that summarizes the
3 % dynamics of the system. Symbolic Math Toolbox is required.
4 %
5 % Equations=Lagrange(Lag,V)
6 %
7 % Lag = Lagrange of the system (symbolic).
        = System Variables (symbolic) [q1 dq1 ddq1 q2 ddq2 ... qn dqn
8 % V
9 % ddqn].
10 % Equations = [1 X DOF] (Degrees of freedom of the system).
11
  %
12 %
13 % *****Examples*******
14 %
        *Falling mass*
15 %
16 % syms x dx ddx t m
                            %Define the symbolic variables.
17 % L=0.5*m*dx<sup>2</sup> + m*g*x; %Define the Lagragian.
18 % Equations=Lagrange(L,[x dx ddx]) %Calculate the equations
19
  %
20 % returns
               m*ddx-g*m
21 %
22 % *Pendulum on a movable support*
23 %
_{24} % syms x dx ddx theta dtheta ddtheta t m M
                                                 %Define the symbolic ...
      variables.
25 %
26 % L=0.5*(M+m)+dx^2+ m*dx*l*dtheta*cos(theta)+ ...
27 % 0.5*m*l^2*dtheta^2+m*g*l*cos(theta)
                                                 %Define the Lagragian.
28 % Equations=Lagrange(L,[theta,dtheta,ddtheta,x,dx,ddx]) %Calculate the
29 % equations
30 %
               [ m*l*(ddx*cos(theta)+l*ddtheta+g*sin(theta)),
31 % returns
                  2*ddx+m*l*ddtheta*cos(theta)-m*l*dtheta^2*sin(theta)]
32 %
33
34 function [M, rhs]=Lagrange(Lag,V)
35 syms t;
36 Var=length(V)/3;
37 Vt = V;
38 % Generic time derivatives w.r.t. each variable
39 for cont0=1:1:Var
       Vt(cont0*3-2)=str2sym( strcat('f',num2str(cont0),'(t)') );
40
       Vt(cont0*3-1)=diff(Vt((cont0*3)-2),t);
41
       Vt(cont0*3)=diff(Vt((cont0*3)-2),t,2);
42
43 end
44
45 % Equations of motion
46 for cont0=1:1:Var
47
       % Derivative w.r.t. q_dot for variable number cont0
48
       L1=simplify(diff(Lag,V(cont0*3-1)));
49
50
51
       Dposx=L1;
52
```

```
53
54
      \% Replace each variable in dL/d(q_dot) with generic function name
      % f1, f2, f3, df1, df2, df3, etc...
55
      for cont=1:1:Var*3
56
          Dposx=subs(Dposx,V(cont),Vt(cont));
57
58
      end
      \% Compute time derivatives for each function
59
      L1=diff(Dposx,t);
60
61
      % Replace newly differentiated variables with corresponding
62
      % user-defined variables
63
      for cont=Var*3:-1:1
                                  %
64
          L1=subs(L1,Vt(cont),V(cont));
65
66
      end
67
      % Derivative w.r.t. x for variable number cont0
68
      L2=simplify(diff(Lag,V(cont0*3-2)));
69
70
71
72
      L1F=L1-L2;
      L1F=simplify(expand(L1F));
73
      74
      M(cont0) = L1F;
75
76 end
77 M = M.';
rhs = -subs(M,V(3:3:end),zeros(size(V(3:3:end))));
```

```
1 function [M,rhs] = quasi_lag(Lag,V)
2 % Modified version of the Lagrange function by Ivanovitch on the MATLAB
3 % file exchange which allows for quasi-coordinates
6 syms t;
7 Var = length(V)/2;
 omega = V(1:2:end);
  omega = [0 - omega(3) omega(2);
           omega(3) 0 -omega(1);
           -omega(2) omega(1) 0];
15 % Generic time derivatives w.r.t. each variable, assigning
  for n=1:1:Var
      Vt(n*2-1)=str2sym( strcat('f',num2str(n),'(t)') );
       Vt(n*2) = diff(Vt((n*2)-1), t);
  for n=1:1:Var
       % Derivative w.r.t. w n for variable number
       dw_n(n,1)=simplify(diff(Lag,V(n*2-1)));
      Dposx=dw_n(n,1);
       \% Replace each variable in dL/d(q_dot) with generic function name
       % f1, f2, f3, df1, df2, df3, etc...
       for cont=1:1:Var*2
           Dposx=subs(Dposx,V(cont),Vt(cont));
       end
       % Compute time derivatives for each function
      L1=diff(Dposx,t);
       % Replace newly differentiated generic functions with ...
          corresponding w_n
       for cont=Var*2:-1:1
```

```
50 \, dw = V(2:2:end);
51 rhs = -subs(M,dw,zeros(size(dw)));
```

L1F=simplify(expand(L1));

L1F=collect(L1F,V(n*2));

L1=subs(L1,Vt(cont),V(cont));

 $\overline{4}$ $\mathbf{5}$

9

10

11 12

13 14

16

1718

2223

24

2526 27

 28 29

30

31

32

33 $\mathbf{34}$

35

36 37

38

39

40 41

42

43

44

end

M(n,1) = L1F;

 $_{49}$ M = M + omega*dw_n;

19 **end** 20 21

```
1 function [m, m0, h, h0, k, f] = cylSloshParam(diameter, height, ...
      density, fill, sigma)
2
3 hfl = height*fill;
4 param = 3.68*hfl/diameter;
5 r = diameter/2;
6
7 if nargin < 5
       sigma = 26.8e-3;
                            % [N/m] Propellant Surface Tension
8
9 end
10 g = sigma/density/r^2;
11
12 mT = 1/4*pi*density*diameter^2*hfl;
13
14 k = mT*g/(1.19*hfl)*tanh(param)^2;
15 m = mT*diameter/(4.4*hfl)*tanh(param);
16 \text{ mO} = \text{mT} - \text{m};
17
18 l_1 = diameter/3.68*tanh(param);
19 l_0 = mT/m0*( hfl/2 - diameter^2/(8*hfl) ) - l_1*m/m0;
20
21 I_rig = mT*diameter^2*( 1/12*(hfl/diameter)^2 + 1/16 );
^{22}
23 right = I_rig + mT*hfl^2/4 - mT*diameter^2/8*( 1.995 - ...
      diameter/hfl*( 1.07*cosh(param) - 107 )/( sinh(param) ) );
^{24}
25 IO = right - m*l_1 - m0*l_0^2;
26
_{27} h = hfl - height/2 - 1_1;
_{28} h0 = hfl - height/2 - 1_0;
29
30 f = sqrt(k/m)/2/pi;
```

```
1 function dx = crazyAttitude(t,x,Q)
2
_{3} beta = x(1:4);
4 beta = beta/norm(beta);
5 w1 = x(5);
w^{2} = x(6);
7 \text{ w3} = x(7);
s w = [w1; w2; w3];
9 xf = x(8);
10 yf = x(9);
11 xo = x(10);
_{12} yo = x(11);
13 dxf = x(12);
14 dyf = x(13);
15 \, dxo = x(14);
_{16} dyo = x(15);
17
18 Awd = [w3*(207.93117449286365976083650654198*w1 + ...
      8600.9008714718077418212242532412*w2 + 315.0*w3 - ...
      24.241017634971381511377330753021*(xf - ...
      0.078296528001835868915136984469427)*(1.0*w1*(yf + 1.5598) - ...
      w2*(xf - 0.078296528001835868915136984469427)) - ...
      0.28268134517197307520319782270235*(xo - ...
      0.92459175324661524442149129754398)*(1.0*w1*(yo + 0.7887) - ...
      w2*(xo - 0.92459175324661524442149129754398))) - 1.0*w2*(315.0*w2 ...
      -518.0*w1 + 13216.509781650478054855750767714*w3 + \ldots
      24.241017634971381511377330753021*(xf - ...
      0.078296528001835868915136984469427)*(dyf + w3*(xf - ...
      0.078296528001835868915136984469427)) + ...
      0.28268134517197307520319782270235*(xo - ...
      0.92459175324661524442149129754398)*(dyo + w3*(xo - ...
      0.92459175324661524442149129754398)) - ...
      24.241017634971381511377330753021*(dxf - 1.0*w3*(yf + ...
      1.5598))*(yf + 1.5598) - 0.28268134517197307520319782270235*(dxo ...
      - 1.0*w3*(yo + 0.7887))*(yo + 0.7887));
          w1*(315.0*w2 - 518.0*w1 + ...
19
             13216.509781650478054855750767714*w3 + ...
             24.241017634971381511377330753021*(xf - ...
             0.078296528001835868915136984469427)*(dyf + w3*(xf - ...
             0.078296528001835868915136984469427)) + ...
             0.28268134517197307520319782270235*(xo - ...
             0.92459175324661524442149129754398)*(dyo + w3*(xo - ...
             0.92459175324661524442149129754398)) - ...
             24.241017634971381511377330753021*(dxf - 1.0*w3*(yf + ...
             1.5598) (yf + 1.5598) - ...
             0.28268134517197307520319782270235*(dxo - 1.0*w3*(yo + ...
             0.7887))*(yo + 0.7887)) - ...
             w3*(16973.608910178670313034526514473*w1 + ...
             207.93117449286365976083650654198*w2 - 518.0*w3 + ...
             24.241017634971381511377330753021*(yf + ...
             1.5598 * (1.0*w1*(yf + 1.5598) - w2*(xf - ...
             0.078296528001835868915136984469427)) + ...
             0.28268134517197307520319782270235*(yo + ...
```

	0.7887)*(1.0*w1*(yo + 0.7887) - w2*(xo
	0.92459175324661524442149129754398)));
20	1.0*w2*(16973.608910178670313034526514473*w1 +
	207.93117449286365976083650654198*w2 - 518.0*w3 +
	24.241017634971381511377330753021*(yf +
	1.5598)*(1.0*w1*(vf + 1.5598) - w2*(xf
	0.078296528001835868915136984469427)) +
	$0.28268134517197307520319782270235*(v_0 + $
	(7887)*(1.0*u1*(vo + 0.7887) - v2*(vo -
	0.02450175324661524442140120754308))) =
	(0.52405175324001524442145125754350))) =
	$1.0 \times 1 \times (207.931174492605659760656505054196 \times 1^{-7}$
	8600.9008/14/180//418212242532412*W2 + 315.0*W3
	24.24101/6349/13815113//330/53021*(xf
	0.078296528001835868915136984469427)*(1.0*w1*(yf + 1.5598)
	- w2*(xf - 0.078296528001835868915136984469427))
	0.28268134517197307520319782270235*(xo
	0.92459175324661524442149129754398)*(1.0*w1*(yo + 0.7887)
	- w2*(xo - 0.92459175324661524442149129754398)))];
21	
22	Bddx = [48.482035269942763022754661506042*dyf*w3
	0.07075174974474017497438893542494*xf
	37.811139307028360881446360508562*w1*w2 +
	24.241017634971381511377330753021*w2^2*xf +
	24.241017634971381511377330753021*w3^2*xf
	1.8979875160495338824515689150396*w2^2
	1.8979875160495338824515689150396*w3^2
	24.241017634971381511377330753021*w1*w2*vf +
	0.005539616355067932900655062128276
	1 8070875160/0533882/515680150306***1***2 =
20	1.00700701004000000000000000000000000000
	40.40203209942103022734001000042*011*0000100000000000000000000
	0.07075174974474017497438895542494*y1 +
	24.241017034971301511377330753021*W1 2*y1 +
	24.24101/6349/13815113/7330/53021*W3 2*YI +
	37.811139307028360881446360508562*W1 2 +
	37.811139307028360881446360508562*₩3 2
	24.241017634971381511377330753021*w1*w2*xf
	0.11035857925184572492505186147582;
24	0.5653626903439461504063956454047*dyo*w3
	0.070557643610695650049358107480657*xo
	0.22295077693713516441276212276534*w1*w2 +
	0.28268134517197307520319782270235*w2^2*xo +
	0.28268134517197307520319782270235*w3^2*xo
	0.26136484054266620110390759496883*w2^2
	0.26136484054266620110390759496883*w3^2
	0.28268134517197307520319782270235*w1*w2*yo +
	0.065237015410962931153466730624082;
25	0.26136484054266620110390759496883*w1*w2
	0.5653626903439461504063956454047*dxo*w3
	0.070557643610695650049358107480657*vo +
	0.28268134517197307520319782270235*w1^2*vo +
	$0.28268134517197307520319782270235*w3^2*vo +$
	$0.22295077693713516441276212276534*w1^2 +$
	0,2220001100011001011210212210001,W1 Z ' ***

0.22295077693713516441276212276534*w3^2 - ...

	0.28268134517197307520319782270235*w1*w2*xo -
	0.055648813515755659193928739369994];
26	
27	a11 = 24.241017634971381511377330753021*yf^2 +
	75.622278614056721762892721017124*yf +
	0.28268134517197307520319782270235*yo^2 +
	0.44590155387427032882552424553069*yo +
	17032.76256654754346884157889308;
28	a12 = 1.8979875160495338824515689150396*yf
	0.22295077693713516441276212276534*xo
	37.811139307028360881446360508562*xf +
	0.26136484054266620110390759496883*yo
	24.241017634971381511377330753021*xf*yf
	0.28268134517197307520319782270235*xo*yo +
	211.09779387013372354349511565582;
29	a13 = -518.0;
30	a21 = 24.241017634971381511377330753021*xf^2
	3.7959750320990677649031378300792*xf +
	0.28268134517197307520319782270235*xo^2
	0.52272968108533240220781518993766*xo +
	8601.2911330806596148247267307948;
31	a22 = 10.124905471006730051432159046044*dxf^2
	20.249810942013460102864318092087*dxf*w3*yf
	31.585655107352595068447763360038*dxf*w3 +
	10.124905471006730051432159046044*dyf^2 +
	20.249810942013460102864318092087*dyf*w3*xf
	1.2651422206697530065757570216888*dyf*w3 +
	7.1610439628783865410091821774456*w1^2*xf^2
	34.059922066350554934505134640825*w1^2*xf*yf
	54.021463836210318577169311093675*w1^2*xf +
	10.124905471006730051432159046044*w1^2*yf^2 +
	33.713608082432416488234875041409*w1^2*yf +
	27.980785481967138578368225474924*w1^2 +
	51.089883099525832401757701961237*w1*w2*xf^2
	60.749432826040380308592954276262*w1*w2*xf*yf
	101.14082424729724946470462512423*w1*w2*xf +
	3.7954266620092590197272710650665*w1*w2*yf +
	6.1195278620074601960512104421364*w1*w2 +
	60.749432826040380308592954276262*w2^2*xf^2
	7.5908533240185180394545421301331*w2^2*xf +
	0.23712589363494267313397787523582*w2^2 +
	27.410854904891846643873500269533*w3^2*xf^2
	3.4250818384562290034797161242937*w3^2*xf +
	10.124905471006730051432159046044*w3^2*yf^2 +
	31.585655107352595068447763360038*w3^2*yf +
	24.740646395982614018828947966067*w3^2 +
	0.28268134517197307520319782270235*xo^2
	0.52272968108533240220781518993766*xo +
	8647.2878828420876001762304337093;
32	a23 = 315.0;
33	a31 = -518.0;
34	a32 = 315.0;
35	a33 = 24.241017634971381511377330753021*xf^2
	3.7959750320990677649031378300792*xf +

. . .

```
0.28268134517197307520319782270235*xo<sup>2</sup> - ...
       0.52272968108533240220781518993766*xo + ...
       24.241017634971381511377330753021*yf<sup>2</sup> + ...
       75.622278614056721762892721017124*yf + ...
       0.28268134517197307520319782270235*yo<sup>2</sup> + ...
       0.44590155387427032882552424553069*yo + ...
       13276.053699628203083666305623875;
_{36} A = [a11 a12 a13;
       a21 a22 a23;
37
        a31 a32 a33];
38
39
40 b11 = 24.241017634971381511377330753021;
_{41} b22 = 24.241017634971381511377330753021;
42 b33 = 0.28268134517197307520319782270235;
43 b44 = 0.28268134517197307520319782270235;
44
45 B = diag([b11 b22 b33 b44]);
46
47 \text{ bw11} = 0;
48 \text{ bw12} = 0;
49 bw13 = - 24.241017634971381511377330753021*yf - ...
       37.811139307028360881446360508562;
50 \text{ bw21} = 0;
51 \text{ bw}22 = 0;
52 bw23 = 24.241017634971381511377330753021*xf - ...
       1.8979875160495338824515689150396;
53 \text{ bw31} = 0;
_{54} bw32 = 0;
55 bw33 = - 0.28268134517197307520319782270235*yo - ...
       0.22295077693713516441276212276534;
56 \text{ bw}41 = 0;
57 \text{ bw}42 = 0;
58 bw43 = 0.28268134517197307520319782270235*xo - ...
       0.26136484054266620110390759496883;
59
60 \text{ Bw} = [bw11 bw12 bw13;
         bw21 bw22 bw23;
61
         bw31 bw32 bw33;
62
         bw41 bw42 bw43];
63
64
_{65} F = [A, zeros(3,4);
       Bw, B];
66
67
68 accel = F \setminus [Awd; Bddx];
69
_{70} wd = accel(1:3) + Q;
71 ddx = accel(4:end);
72
_{73} dx = [1/2*BmatEP(beta)*w;
         wd:
74
75
          dxf;
         dyf;
76
          dxo;
77
         dyo;
78
```

79 ddx];