Received: 14 August 2020 | Accepted: 3 November 2020

DOI: 10.1002/pamm.202000337

The Loewner framework for nonlinear identification and reduction of Hammerstein cascaded dynamical systems

Dimitrios S. Karachalios[1,*], **Ion Victor Gosea**[1,**], and **Athanasios C. Antoulas**[1,2,3,***]

[1] Max Planck Institute for Dynamics of Complex Technical Systems, Data-Driven System Reduction and Identification (DRI) group, Sandtorstraße 1, Magdeburg 39106

[2] Rice University Houston, Electrical and Computer Engineering Department, 6100 Main St., Houston, TX 77005

[3] Baylor College of Medicine, 1 Baylor Plaza, Houston, TX 77030

We present an algorithm for data-driven identification and reduction of nonlinear cascaded systems with Hammerstein structure. The proposed algorithm relies on the Loewner framework (LF) which constitutes a non-intrusive algorithm for identification and reduction of dynamical systems based on interpolation. We address the following problem: the actuator (control input) enters a static nonlinear block. Then, this processed signal is used as an input for a linear time-invariant system (LTI). Additionally, it is considered that the orders of the linear transfer function and of the static nonlinearity are not a priori known.

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1 Introduction

In some engineering applications that deal with the study of dynamical control systems, the control input enters the differential equations in a nonlinear fashion [5]. It is of interest to identify the hidden nonlinearity while at the same time reduction is needed for robust simulations and control design [1]. The LF [2–4] constitutes a non-intrusive method that uses only input-output data. The matrix pencil composed of two Loewner matrices reveals the minimality (in terms of McMillan degree) of the LTI system. By means of a singular value decomposition (SVD), one can find left and right projection matrices that are used to construct a low order model.

The Hammerstein system is characterized by two blocks connected in series, where the static nonlinear (memoryless) block is followed by a linear time-invariant system (LTI) as in Fig. 1. The scalar control input-u(t) is used as an argument to the static nonlinearity-$\mathcal{F}$ and then the signal $\mathcal{F}(u(t))$ passes through a linear time-invariant (LTI) system. The static polynomial map approximates other non-polynomial maps (Taylor series expansion) s.a. $\tanh(\cdot)$, $\exp(\cdot)$, etc. The aim is to identify the cascaded system by estimating the coefficients of the polynomial map $k_i,\ i=1,2,\ldots,n$ and the hidden LTI system by using only input-output data $(u(t),y(t))\,,\ t\geq 0$.

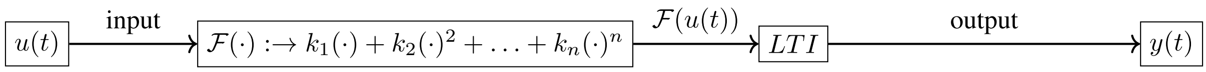

Fig. 1: The input-output scheme of a cascaded system with a static nonlinear (polynomial) map of n^{th} order followed by an LTI. The connection describes a Hammerstein nonlinear model.

The steady state output solution can be computed explicitly with the convolution integral[1], the impulse response $h(t),\ t\geq 0$ and the linear transfer function $H(j\omega),\ j\omega\in\mathbb{C}$ of the LTI as:

$$\begin{aligned} y(t) &= \left((k_1u(t)+k_2u^2(t)+\ldots+k_nu^n(t))\star h\right)(t) = k_1(u\star h)(t)+k_2(u^2\star h)(t)+\ldots+k_n(u^n\star h)(t) \\ &= k_1\int_{-\infty}^{\infty}h(\tau)u(t-\tau)d\tau+\ldots+k_n\int_{-\infty}^{\infty}h(\tau)u^n(t-\tau)d\tau = \sum_{i=1}^{n}k_i\int_{-\infty}^{\infty}h(\tau)u^i(t-\tau)d\tau. \end{aligned} \tag{1}$$

Let the singleton real input be defined as $u(t)=A\cos(\omega t)=\alpha e^{j\omega t}+\alpha e^{-j\omega t}$ with the amplitude $\alpha=A/2$, the imaginary unit j, the driving frequency $\omega>0$ and time $t\geq 0$. By substituting the above input in Eq. (1) and by making use of the binomial theorem, we conclude that:

$$y(t)=\sum_{i=1}^{n}k_i\int_{-\infty}^{\infty}h(\tau)\left(\alpha e^{j\omega(t-\tau)}+\alpha e^{-j\omega(t-\tau)}\right)^i d\tau=\sum_{i=1}^{n}\sum_{m=0}^{i}k_i\alpha^i\frac{i!}{(i-m)!m!}H(j\omega(2m-i))e^{j\omega(2m-i)} \tag{2}$$

* Corresponding author: email karachalios@mpi-magdeburg.mpg.de

** email gosea@mpi-magdeburg.mpg.de

*** email aca@rice.edu

[1] $(f\star g)(t)=\int_{-\infty}^{\infty}f(\tau)g(t-\tau)d\tau$

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At frequency ω the ℓ^{th} harmonic is computed by applying the single-sided Fourier transform in Eq. (2) as:

$$Y_{\omega,\ell}(j\ell\omega) = H(j\ell\omega)\delta(j\ell\omega)\sum_{\ell\leq i\neq 0}^{n} k_i\phi_{i,\ell},\ \ell = 0,\dots,n,$$
$$\phi_{i,\ell} = \begin{cases} 2\alpha^i \cdot {}^iC_{(i+\ell)/2},\ (i \geq \ell) \text{ and } (i+\ell)(even) \\ 0,\ (else) \end{cases},\ \text{where } {}^nC_m = \frac{n!}{(n-m)!m!} \tag{3}$$

2 The Loewner-Hammerstein identification method

As we have computed the total output of the Hammerstein cascaded system, we proceed with the method of determining the unknowns from input-output data. The symmetry in Eq. (3) allows the cancellation of the unknown contribution of the transfer function. Thus, we first determine the unknown coefficients k_i, and afterwards, we fit the LTI system by means of the LF. For this purpose, it is important to define the following invariant frequency quantities $\lambda_{p,q}$.

Definition 2.1 (Frequency invariant quantities) The $Y_{p,q}$ denotes the q^{th} harmonic at p frequency.

$$\lambda_{p,q} = \frac{Y_{p,q}}{Y_{q,p}} = \frac{\sum_{i=p}^{n} k_i\phi_{i,p}}{\sum_{i=q}^{n} k_i\phi_{i,q}}, p \neq q. \tag{4}$$

The entries $\lambda_{p,q}$ are independent of ω.

$input \setminus harmonic$	1^{st}	2^{nd}	3^{rd}	4^{th}	$\cdots$	n^{th}
$1\omega \to \boxed{\mathcal{N}}$	$Y_{1\omega,1}$	$Y_{1\omega,2}$	$Y_{1\omega,3}$	$Y_{1\omega,4}$	$\cdots$	$Y_{1\omega,n}$
$2\omega \to \boxed{\mathcal{N}}$	$Y_{2\omega,1}$	$Y_{2\omega,2}$	$Y_{2\omega,3}$	$Y_{2\omega,4}$	$\cdots$	$Y_{2\omega,n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$n\omega \to \boxed{\mathcal{N}}$	$Y_{n\omega,1}$	$Y_{n\omega,2}$	$Y_{n\omega,3}$	$Y_{n\omega,4}$	$\cdots$	$Y_{n\omega,n}$

The above harmonic map allows the construction of the following linear system. Due to the mixing linearities (i.e. $k_1u(t)$ and $(u \star h)(t)$), we can fix k_1 to an arbitrary value. For $p = 1$ and $q = 2,\dots,n$ results:

$$\begin{bmatrix} \phi_{21}-\lambda_{12}\phi_{22} & \phi_{31}-\lambda_{12}\phi_{32} & \cdots & \phi_{n1}-\lambda_{12}\phi_{n2} \\ \phi_{21} & \phi_{31}-\lambda_{13}\phi_{32} & \cdots & \phi_{n1}-\lambda_{13}\phi_{n3} \\ \phi_{21} & \phi_{31} & \cdots & \phi_{n1}-\lambda_{14}\phi_{n4} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{21} & \phi_{31} & \cdots & \phi_{n1}-\lambda_{1n}\phi_{nn} \end{bmatrix} \begin{bmatrix} k_2 \\ k_3 \\ k_4 \\ \vdots \\ k_n \end{bmatrix} = -k_1\phi_{11} \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix},\ \forall k_1 \in \mathbb{R}\setminus\{0\}. \tag{5}$$

Finally, as we have identify the scaled (k_1 arbitrary) coefficient vector $\mathbf{k} = (k_1, k_2, \dots, k_n)$, we can transform the above harmonic map into a measurement map for the linear transfer function as $H(j\ell\omega) = Y_{\omega,\ell}/\sum_{\ell\leq i\neq 0}^{n} k_i\phi_{i,\ell}$. The identification and reduction of the LTI system is done by applying the LF [2–4].

Algorithm 1: Hammerstein identification with the Loewner framework

Input: Apply signals $u(t) = \alpha\cos(\omega_i t)$ with driving frequencies $\omega_i,\ i = 1,\dots,n$ where n is the maximum nonzero harmonic index.
Output: An identified Hammerstein system.

1. Apply FT and measure $U(j\omega_i), Y_{1^{\text{st}}}(j\omega_i), Y_{2^{\text{nd}}}(2j\omega_i),\dots,Y_{n^{\text{th}}}(nj\omega_i)$ from the power spectrum.
2. Fix k_1 to an arbitrary value and determine the scaled coefficient vector $\mathbf{k} = (k_1, k_2, \dots, k_n)$ by solving the system in Eq. (5).
3. Estimate the measurements of the linear transfer function from $H(j\ell\omega) = Y_{\omega,\ell}/\sum_{\ell\leq i\neq 0}^{n} k_i\phi_{i,\ell}$.
4. Apply the linear Loewner framework for identification and reduction of the LTI.

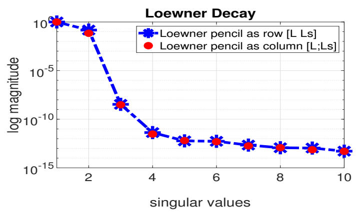

Fig. 2: The singular value decay of the Loewner matrices. $\sigma_3/\sigma_1 \sim 1e-10$.

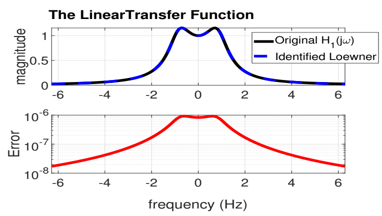

Fig. 3: The identified linear transfer function with $\|H - H_r\|_\infty \sim 1e-7$.

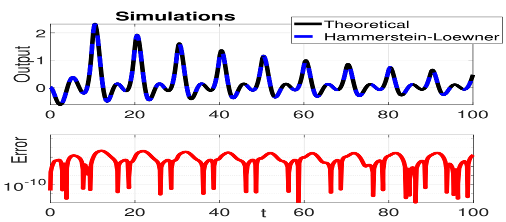

Fig. 4: The simulated identified Hammerstein system in comparison with the original one. $\|y - y_r\|_\infty \sim 1e-7$.

3 Numerical example

To illustrate the proposed method, we choose the following static nonlinearity as $\mathcal{F}(\cdot) = e^{(\cdot)} - 1$ along with the transfer function $H(j\omega) = 1/((j\omega)^2 + j\omega + 1)$. By exciting the system with $u(t) = 2\cos(\omega_i t),\ \omega_i = 2\pi[1, 2, \dots, 10]$ and with collecting the steady state snapshots, we perform Fourier transform for each signal and we solve the linear system in Eq. (5) for $n = 10$ ($Y_{10^{\text{th}}} \sim 1e-10$). The solution is $\hat{\mathbf{k}} = (1, 0.5, 0.167, 0.0417, 0.0083, 0.0014, 1.9676e-4, 2.4001e-5, 3.1005e-6, 3.7401e-7)$ which constitutes a very good approximation of the Taylor expansion coefficients of the $\mathcal{F}$. Next, we estimate the linear transfer function with the LF [2–4]. The singular value decay in Fig. 2 allows the assignment of the order $r = 2$ ($dt = 1e-3$). In Fig. 3 for the biased $k_1 = 1$ solution, we show that the identification of the transfer function is accurate. The time

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domain simulation in Fig. 4 is independent of the choice of k_1. The large input as $u(t) = 2\text{sawtooth}(0.1 \cdot 2\pi t)e^{-0.01t}\cos(0.1 \cdot 2\pi t)$ certifies that the method is able to perform well under large inputs for nonlinear Hammerstein systems.

Acknowledgements Open access funding enabled and organized by Projekt DEAL.

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