

WEIGHTED LATTICE POINT SUMS IN LATTICE POLYTOPES, UNIFYING DEHN–SOMMERVILLE AND EHRHART–MACDONALD

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ABSTRACT. Let V be a real vector space of dimension n and let $M \subset V$ be a lattice. Let $P \subset V$ be an n -dimensional polytope with vertices in M , and let $\varphi: V \rightarrow \mathbb{C}$ be a homogeneous polynomial function of degree d . For $q \in \mathbb{Z}_{>0}$ and any face F of P , let $D_{\varphi,F}(q)$ be the sum of φ over the lattice points in the dilate qF . We define a generating function $G_{\varphi}(q, y) \in \mathbb{Q}[q][y]$ packaging together the various $D_{\varphi,F}(q)$, and show that it satisfies a functional equation that simultaneously generalizes Ehrhart–Macdonald reciprocity and the Dehn–Sommerville relations. When P is a simple lattice polytope (i.e., each vertex meets n edges), we show how G_{φ} can be computed using an analogue of Brion–Vergne’s Euler–Maclaurin summation formula.

1. INTRODUCTION

1.1. Let V be a real vector space of dimension n and let $M \subset V$ be a lattice. Let $P \subset V$ be an n -dimensional *polytope*, i.e., the closed convex hull of finitely many points in V . We assume further that P is a *lattice polytope*, which means the vertices of P lie in M , and that P is *simple*; this means that each vertex meets n edges. (See, e.g., [1] for terminology and background on lattice polytopes.) In this paper we simultaneously consider three important concepts for P :

- *The Dehn–Sommerville relations.* Let \mathcal{F} be the set of faces of P , let $\mathcal{F}(k)$ be the subset of faces of dimension k , and let $f_k(P) = |\mathcal{F}(k)|$. We define, as usual, the *h -polynomial* $h(P, t) = \sum_{k=0}^n h_k(P) t^k$ by

$$(1) \quad h(P, t) := f_n(P)(t-1)^n + f_{n-1}(P)(t-1)^{n-1} + \cdots + f_0(P).$$

(For instance, if P is a simplex, then $h(P, t) = t^n + t^{n-1} + \cdots + 1$.) The Dehn–Sommerville relations say that $h_k(P) = h_{n-k}(P)$ for all k .

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- *The Ehrhart polynomial and Ehrhart–Macdonald reciprocity.* For any $q \in \mathbb{Z}_{>0}$, let qP denote the q th dilate of P and let $\mathcal{E}_P(q) := |M \cap qP|$. Then Ehrhart and Macdonald [7, 11] proved that $\mathcal{E}_P(q)$ evaluates to a polynomial in q that satisfies the symmetry

$$(2) \quad \mathcal{E}_P(q) = (-1)^n \mathcal{E}_{P^\circ}(-q),$$

where P° is the interior of P . (This holds for *any* lattice polytope, not just simple ones.)

- *Euler–Maclaurin summation.* Let $\varphi: V \rightarrow \mathbb{C}$ be a polynomial function. Let $h = (h_F)_{F \in \mathcal{F}(n-1)}$ be a multiparameter indexed by the facets (faces of codimension 1) of P , and let $P(h)$ be the deformation of P obtained by independent small parallel translations of its facets according to h . The Euler–Maclaurin formula [5, 14] shows how to compute the finite sum $\sum_{m \in M \cap P} \varphi(m)$ via an explicit differential operator in the $\partial/\partial h_F$ acting on $\int_{P(h)} \varphi(x) dx$, thought of as a function of h .

We will introduce a two-variable polynomial and prove two fundamental theorems for it: one that simultaneously generalizes the Dehn–Sommerville and Ehrhart–Macdonald relations, and one that gives an Euler–Maclaurin formula.

1.2. Let us be more precise about our main results. Assume that the polynomial φ is homogeneous of degree $\deg \varphi$. For any face $F \in \mathcal{F}$, let

$$(3) \quad D_{\varphi, F}(q) := \sum_{m \in M \cap qF} \varphi(m).$$

It is known that $D_{\varphi, F}(q)$ is a polynomial in q of degree $n + \deg \varphi$ and constant term $D_{\varphi, F}(0) = \varphi(0)$ [4, Proposition 4.1]. Let

$$(4) \quad G_\varphi(q, y) := (y + 1)^{\deg \varphi} \sum_{F \in \mathcal{F}} (y + 1)^{\dim F} (-y)^{\text{codim } F} D_{\varphi, F}(q).$$

Our first main result is the following functional relation for the polynomial $G_\varphi(q, y)$.

1.3. Theorem. $G_\varphi(q, y) = (-y)^{n + \deg \varphi} G_\varphi(-q, \frac{1}{y})$.

In fact, we prove a slightly more general result than Theorem 1.3 that applies to all lattice polytopes P , simple or not (Theorem 2.6 below).

We now explicate how Theorem 1.3 implies some of the aforementioned results. First, suppose $\varphi = 1$ and $q = 0$; then each $D_{\varphi, F}$ equals 1. The generating function in Theorem 1.3 becomes

$$(5) \quad \sum_{F \in \mathcal{F}} (y + 1)^{\dim F} (-y)^{\text{codim } F} = \sum_{k=0}^n (-y)^{\dim P - k} (y + 1)^k f_k(P).$$

Expanding the right of (5) and comparing with (1), one sees that the coefficient of y^k in (5) is $(-1)^k h_{n-k}(P)$. Thus Theorem 1.3 in this case is equivalent to the Dehn–Sommerville relations $h_k(P) = h_{n-k}(P)$.

Second, when $\varphi = 1$ and $q > 0$ is a positive integer, then the constant term of $G_1(q, y)$ is $\mathcal{E}_P(q) = |M \cap qP|$. The leading term of $G_1(q, y)$ is an alternating sum over the face lattice \mathcal{F} of the lattice point enumerators $\mathcal{E}_F(q)$ and, up to sign, nothing other than the computation of $\mathcal{E}_{P^\circ}(q)$ by inclusion-exclusion. Thus the relation implied by Theorem 1.3 between the coefficients of y^n and y^0 is exactly Ehrhart reciprocity (2).

1.4. Our second main result is a formula for $G_\varphi(q, y)$ in the spirit of the *Todd operator* formulas of Khovanskii–Puklikov [14] and Brion–Vergne [5] for Euler–Maclaurin summation. To state it we require more notation. Let $\langle \cdot, \cdot \rangle$ be the pairing between V and its dual V^* . Let $N \subset V^*$ be the lattice dual to M . Any facet $F \in \mathcal{F}(n-1)$ is the intersection of P with an affine hyperplane

$$H_F = \{x \mid \langle x, u_F \rangle + \lambda_F = 0\},$$

where the normal vector u_F is taken to be a primitive vector in N . Thus

$$P = \{x \in V \mid \langle x, u_F \rangle + \lambda_F \geq 0 \text{ for all } F \in \mathcal{F}(n-1)\}.$$

As above, let $h = (h_F)_{F \in \mathcal{F}(n-1)}$ be a multiparameter indexed by the facets of P , and let $\tilde{P}_q(h)$ be the deformation by h of the $q(y+1)$ dilate of P :

$$(6) \quad \tilde{P}_q(h) := \{x \in V \mid \langle x, u_F \rangle + q(y+1)\lambda_F + h_F \geq 0 \text{ for all } F \in \mathcal{F}(n-1)\}.$$

1.5. Theorem. *There is a differential operator $\text{Td}_y(P, \partial/\partial h)$ in the derivatives $(\partial/\partial h_F)_{F \in \mathcal{F}(n-1)}$ such that*

$$G_\varphi(q, y) = \text{Td}_y(P, \partial/\partial h) \left(\int_{\tilde{P}_q(h)} \varphi(x) dx \right) \Big|_{h=0}.$$

The differential operator in Theorem 1.5 will be given explicitly, after the necessary notation is developed (see (16) below and the preceding lines).

1.6. As mentioned above, we actually prove a generalization of Theorem 1.3 that does not assume P to be simple. Since $q = 0$ and $\varphi = 1$ in Theorem 1.3 recovers the Dehn–Sommerville relations, which in turn are a manifestation of Poincaré duality for the rational cohomology $H^*(X_P; \mathbb{Q})$ of the toric variety X_P attached to P (see, e.g., [8]), it is natural to expect that the correct generalization should somehow involve the intersection cohomology of X_P , in other words, the *g-polynomials*. This is indeed the case.

It is thus natural to ask whether one can prove an analogous generalization of Theorem 1.5 for general lattice polytopes P . Work of Brion–Vergne [5] gives an analogue of the Euler–Maclaurin formula for such polytopes, and when applied to our setup gives explicit Todd operator formulas for the leading and constant terms

(in y) of $G_\varphi(q, y)$. Their technique is to consider simple deformations P' of P and then to take the limit as one collapses P' back down to P . However, this does not lead to a Todd operator formula for the other terms of $G_\varphi(q, y)$ in general. It would be interesting to generalize the results of [5] to the generating function $G_\varphi(q, y)$.

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2. THE RECIPROCITY THEOREM

2.1. The goal of this section is to modify (4) for a general lattice polytope such that Theorem 1.3 holds. We begin by recalling some notation. For more details, see, e.g., [16, §3.14].

Let P be a general polytope of dimension n (not necessarily a lattice polytope). As above, let \mathcal{F} be its set of faces, regarded as a poset under inclusion. We enlarge \mathcal{F} to \mathcal{F}^- by adjoining an extra element $\mathbf{0}$ that is defined to be smaller than any $F \in \mathcal{F}$; the element $\mathbf{0}$ should be thought of as corresponding to the “empty” face of P with dimension $\dim \mathbf{0} = -1$. We make \mathcal{F}^- into a ranked poset with rank function ρ by putting $\rho(F) = \dim F + 1$ and $\rho(\mathbf{0}) = 0$.

2.2. We define polynomials $f_P, g_P \in \mathbb{Z}[x]$ as follows:¹

- If $\rho(P) = 0$, we put $f_P(x) = g_P(x) = 1$.
- Otherwise, if $\rho(P) = n + 1 > 0$, then $f_P(x)$ is a polynomial $\sum_{l=0}^n f_l x^l$ of degree n . We recursively define

$$g_P(x) = f_0 + (f_1 - f_0)x + (f_2 - f_1)x^2 + \cdots + (f_m - f_{m-1})x^m,$$

where $m = \lfloor n/2 \rfloor$, and

$$(7) \quad f_P(x) = \sum_{\mathbf{0} \leq F \leq P} g_F(x)(x-1)^{n-\rho(F)}.$$

Note that the sum is taken over *proper* faces of P , which makes f_P well defined by induction.

With this setup, the following *master duality* theorem for the polynomial f_P holds (see, e.g., [16, Theorem 3.14.9]):

¹This (standard) definition of the f polynomial is dual to the definition of the h -polynomial (1). The f -polynomial favors simplicial polytopes, in that Dehn–Sommerville holds with no g -polynomial corrections. The h -polynomial, on the other hand, favors simple polytopes.

2.3. Theorem. *Let $n = \dim P$. Then*

$$f_P(x) = x^n f_P\left(\frac{1}{x}\right).$$

Equivalently, if $f_P = \sum_{i=0}^n a_i x^i$, then $a_i = a_{n-i}$.

We will also need the following identity of the f and g polynomials:

2.4. Lemma. *Let P be a polytope of dimension n . Then*

$$(8) \quad x^{\dim P+1} g_P\left(\frac{1}{x}\right) = \sum_{\mathbf{0} \leq F \leq P} g_F(x) (x-1)^{n-\dim F}$$

where the sum is taken over all faces of P , including P itself.

Proof. The proof is a simple computation and arises in the proof of [16, Theorem 3.14.9]. Indeed, with $f_P = \sum_{i=0}^n a_i x^i$,

$$g_P + (x-1)f_P = (a_m - a_{m+1})x^{m+1} + (a_{m+1} - a_{m+2})x^{m+2} + \dots$$

where $m = \lfloor n/2 \rfloor$. Applying Theorem 2.3,

$$g_P + (x-1)f_P = x^{n+1} g_P\left(\frac{1}{x}\right).$$

Inserting the definition (7) of f_P completes the proof. \square

2.5. Now assume that P is a lattice polytope. For any face $F \leq P$, let $\mathcal{P}_P(F)$ be the dual face of F in the polar polytope to P . For example, if P is simple, $\mathcal{P}_P(F)$ is a simplex for any proper face F . We define a polynomial $\tilde{g}_F(x)$ by

$$\tilde{g}_F(x) = g_{\mathcal{P}_P(F)}(x).$$

Note that \tilde{g}_F depends on the larger polytope P in which F is a face, although this is not part of the notation. As in the introduction, let φ be a homogeneous polynomial and define $D_{\varphi,F}(q)$ by (3). We extend the definition (4) of $G_\varphi(q, y)$ by

$$(9) \quad G_\varphi(q, y) := (y+1)^{\deg \varphi} \sum_{F \in \mathcal{F}} (y+1)^{\dim F} (-y)^{\text{codim } F} D_{\varphi,F}(q) \tilde{g}_F\left(-\frac{1}{y}\right).$$

Note that if P is simple then $\tilde{g}_F = 1$ for all faces of P , and this definition coincides with (4).²

2.6. Theorem. *For a general lattice polytope P , the function $G_\varphi(q, y)$ satisfies the relation in Theorem 1.3.*

We shall need the following lemma:

²We remark that the factor $(y+1)^{\deg \varphi}$ is not really needed for G_φ , at least as far as the results in this section are concerned. This factor appears naturally when one considers the Todd operator formula, so it is reasonable to include it here.

2.7. Lemma. *Let P be a lattice polytope of dimension n and let φ be a homogeneous polynomial function. Let $q > 0$ be an integer. Define*

$$D_{\varphi,P}(q) := \sum_{m \in M \cap qP} \varphi(m),$$

$$D_{\varphi,P}^{\circ}(q) := \sum_{m \in M \cap (qP)^{\circ}} \varphi(m),$$

where P° denotes the interior of P . Then as functions of q , both D and D° are polynomials of degree $\deg \varphi + \dim P$, and

$$(10) \quad D_{\varphi,P}(-q) = (-1)^{\deg \varphi + \dim P} D_{\varphi,P}^{\circ}(q).$$

Proof. These statements are proved by Brion–Vergne in [4, Proposition 4.1] for any simple lattice polytope. Their later paper [5] derives an Euler–Maclaurin formula for any general lattice polytope P by first passing to a simple perturbation P' and computing on P' as in [4]. This implies the result. \square

Proof of Theorem 2.6. Let n be the dimension of P and d the degree of φ . Write $G = G_{\varphi}(q, y)$ and $G' = (-y)^{n+d} G(-q, \frac{1}{y})$. We begin with the definition

$$G = (y+1)^d \sum_{F \in \mathcal{F}} (y+1)^{\dim F} (-y)^{\operatorname{codim} F} D_{\varphi,F}(q) \tilde{g}_F(-\frac{1}{y})$$

and replace each $D_{\varphi,F}$ with the sum over the faces of F of the functions D_{φ}° to obtain

$$G = (y+1)^d \sum_{F \in \mathcal{F}} (y+1)^{\dim F} (-y)^{\operatorname{codim} F} \tilde{g}_F(-\frac{1}{y}) \sum_{E \leq F} D_{\varphi,E}^{\circ}(q).$$

After interchanging the sums and swapping the labels of E and F ,

$$(11) \quad G = (y+1)^d \sum_{F \leq P} D_{\varphi,F}^{\circ}(q) \sum_{F \leq E \leq P} (y+1)^{\dim E} (-y)^{\operatorname{codim} E} \tilde{g}_E(-\frac{1}{y}).$$

Now consider G' . If we apply Lemma 2.7 then

$$(12) \quad G' = (y+1)^d \sum_{F \leq P} (y+1)^{\dim F} D_{\varphi,F}^{\circ}(q) \tilde{g}_F(-y).$$

Comparing (11) and (12), we see that we need the following identity for any face F of P :

$$(13) \quad (y+1)^{\dim F} \tilde{g}_F(-y) = \sum_{F \leq E \leq P} (y+1)^{\dim E} (-y)^{\operatorname{codim} E} \tilde{g}_E(-\frac{1}{y}).$$

We claim that this follows from Lemma 2.4. To see this, one observes that the polynomial \tilde{g}_F is the g -polynomial of the dual face $\mathcal{P}_P(F)$, and that the sum over $F \leq E \leq P$ is the same as the sum over the face poset for $\mathcal{P}_P(F)$. Applying this and putting $x = -y$ gives (8). \square

2.8. Notice that the reciprocity law (12) suggests another definition of the polynomial $G_\varphi(q, y)$ from (4)

$$G_\varphi(q, y) := (y + 1)^{\deg \varphi} \sum_{F \in \mathcal{F}} (y + 1)^{\dim F} D_{\varphi, F}^\circ(q)$$

and its extended version from (9)

$$(14) \quad G_\varphi(q, y) := (y + 1)^{\deg \varphi} \sum_{F \in \mathcal{F}} (y + 1)^{\dim F} D_{\varphi, F}^\circ(q) \tilde{g}_F(-y).$$

3. THE TODD OPERATOR FORMULA

3.1. For the rest of the paper we assume that P is simple. We begin by introducing the notation we need to define the Todd operator.

Let $f \in \mathcal{F}(n - l)$ be a face of codimension l , and let H_f be the affine subspace spanned by f . Since P is simple, there are exactly l hyperplanes in $\{H_F \mid F \in \mathcal{F}(n - 1)\}$ whose intersection is H_f . Let $\sigma_f \subset V^*$ be the convex cone generated by the corresponding normal vectors $\{u_F \mid F \in \mathcal{F}(n - 1), F \supset f\}$. The cone σ_f is called the *normal cone* to f .

3.2. The set $\Sigma = \{\sigma_f \mid f \in \mathcal{F}\}$ of all normal cones forms an acute rational polyhedral fan in V^* . This means the following:

- (1) Each $\sigma \in \Sigma$ contains no nontrivial linear subspace.
- (2) If σ' is a face of $\sigma \in \Sigma$, then $\sigma' \in \Sigma$.
- (3) If $\sigma, \sigma' \in \Sigma$, then $\sigma \cap \sigma'$ is a face of each.
- (4) Given $\sigma \in \Sigma$, there exists a finite set $S \subset N$ such that any point in σ can be written as $\sum \rho_s s$, where $s \in S$ and $\rho_s \geq 0$.

Moreover, P being simple implies that Σ is simplicial, which means that in (4) we can take $\#S = \dim \sigma$ for all σ . The fan Σ is called the *normal fan* to P .

3.3. Let $\rho \in \Sigma$ be a rational 1-dimensional cone. Then ρ contains a unique primitive point, which we call the *spanning point* of ρ . For any cone σ , we denote by $\sigma(1)$ the set of spanning points of all 1-dimensional faces of σ and write

$$\Sigma(1) := \bigcup_{\sigma \in \Sigma} \sigma(1).$$

There is bijection between $\Sigma(1)$ and $\mathcal{F}(n - 1)$: if $\rho \in \Sigma(1)$, then the spanning point of ρ is a unique normal vector u_F , which determines the corresponding facet F .

For any cone $\sigma \in \Sigma$, let $U(\sigma)$ be the sublattice of N generated by the spanning points of σ . Set

$$N(\sigma) := N \cap (U(\sigma) \otimes \mathbb{Q}) \quad \text{and} \quad \text{Ind } \sigma := [N(\sigma) : U(\sigma)].$$

If $\text{Ind } \sigma = 1$, then σ is called *unimodular*. The polytope P is called *nonsingular* if normal cones are unimodular.³ Let $G(\sigma)$ be the finite group $N(\sigma)/U(\sigma)$.

3.4. For any $\sigma \in \Sigma$, define

$$Q(\sigma) := \left\{ \sum_{s \in \sigma(1)} \rho_s s \mid 0 \leq \rho_s < 1 \right\}.$$

Note that $\text{Vol } Q(\sigma) = \text{Ind } \sigma$, and $Q(\sigma) \cap N(\sigma) = \{0\}$ if and only if σ is unimodular. Furthermore, the set $Q(\sigma) \cap N(\sigma)$ is in bijection with the finite group $G(\sigma)$ under the map $N(\sigma) \rightarrow N(\sigma)/U(\sigma)$. Put

$$\Gamma_\Sigma := \bigcup_{f \in \mathcal{F}} Q(\sigma_f) \cap N.$$

Then $\Gamma_\Sigma = \{0\}$ if and only if P is nonsingular.

3.5. As in the introduction, let y be a real variable, and let $h = (h_F)_{F \in \mathcal{F}(n-1)}$ be a real multivariable indexed by the facets of P . As before let $\tilde{P}_q(h)$ be the deformation by h of the $q(y+1)$ dilate of P defined in (6). The polytope $\tilde{P}_q(h)$ depends on y , but we suppress this from the notation.

If $q = 1$ and $y = 0$ then $\tilde{P}_1(0) = P$; furthermore if $q \neq 0$ and $y \neq -1$, then $\tilde{P}_q(h)$ is isomorphic to P for small h ; in this case the integral

$$(15) \quad I(\tilde{P}_q(h)) = I_\varphi(\tilde{P}_q(h)) := \int_{\tilde{P}_q(h)} \varphi(x) dx$$

therefore converges for small h (here we take the measure on V that gives a fundamental domain of M unit volume). We will compute the function $G_\varphi(q, y)$ by applying a differential operator to $I(\tilde{P}_q(h))$, the Todd- y operator. To define it, we need yet more notation.

3.6. For each facet $F \in \mathcal{F}(n-1)$, let $\xi_F: V^* \rightarrow \mathbb{R}$ be the unique piecewise-linear continuous function defined by

- $\xi_F(s) = 1$ if $s \in \Sigma(1)$ is the spanning point corresponding to F ,
- $\xi_F(s') = 0$ for all other $s' \in \Sigma(1)$, and
- ξ_F is linear on all the cones of Σ .

Put $a_F(x) = \exp(2\pi i \xi_F(x))$ for all $x \in V$.

Suppose $g \in \Gamma_\Sigma \cap \sigma$. Then the pair (g, σ) determines a tuple of roots of unity as follows. If s_1, \dots, s_l are the spanning points of σ , and F_1, \dots, F_l are the corresponding facets, then we can attach to (g, σ) the tuple $(a_1(g), \dots, a_l(g))$, where we have written a_i for a_{F_i} . We are now ready to define the Todd- y operator:

³This condition is the same as the toric variety X_P determined by P being nonsingular [8].

3.7. Definition. Let a be a complex number and x a real variable. We define $\text{Td}_y(a, \partial/\partial x)$ to be the differential operator given formally by the power series

$$\frac{\partial/\partial x(1 + ay \exp(-\partial/\partial x(y + 1)))}{1 - a \exp(-\partial/\partial x(y + 1))} = \frac{(y + 1)\partial/\partial x}{1 - a \exp(-\partial/\partial x(y + 1))} - y \partial/\partial x = \sum_{k=0}^{\infty} c(a, k, y) \left(\frac{\partial}{\partial x}\right)^k.$$

Table 1 gives some examples of the polynomials $c(a, k, y)$. We remark that $c(1, k, 0) =$

k	$c(a, k, y)$
1	$-a(y + 1)/a - 1$
2	$-a(y + 1)^2/(a - 1)^2$
3	$-a(a + 1)(y + 1)^3/2(a - 1)^3$
4	$-a(a^2 + 4a + 1)(y + 1)^4/6(a - 1)^4$
5	$-a(a^3 + 11a^2 + 11a + 1)(y + 1)^5/24(a - 1)^5$

TABLE 1. Sample coefficients $c(a, k, y)$.

$B_k/k!$, where B_k is the k -th Bernoulli number.⁴ If $a \neq 1$, then

$$-(k - 1)!(a - 1)^k c(a, k, y)/a(y + 1)^k$$

is the *Eulerian polynomial* for the symmetric group S_{k-1} (see, e.g., [10]).

3.8. Recall that h is a multivariable with components h_F indexed by the facets of P . For any $g \in \Gamma_\Sigma$, we define

$$\text{Td}_y(g, \partial/\partial h) := \prod_{F \in \mathcal{F}(n-1)} \text{Td}_y(a_F(g), \partial/\partial h_F)$$

and

$$(16) \quad \text{Td}_y(P, \partial/\partial h) := \sum_{g \in \Gamma_\Sigma} \text{Td}_y(g, \partial/\partial h).$$

This concludes our setup and makes the statement of Theorem 1.5 precise. We now turn to its proof.

⁴With this convention the Bernoulli numbers are $B_0 = 1$, $B_1 = \frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, \dots , and $B_{2k-1} = 0$ for $k > 1$. Note that for many authors $B_1 = -\frac{1}{2}$.

4. PROOF OF THEOREM 1.5

To prove Theorem 1.5 we adapt arguments in [4] to incorporate the parameter y . For any face $f \in \mathcal{F}$, let $C_f \subset V$ be the convex cone generated by elements $p - p'$ with $p \in P$ and $p' \in f$. The cone C_f is called the *tangent cone* to P at f . The normal cone σ_f is the dual cone to C_f . We also denote by $\mathcal{F}^f \subset \mathcal{F}(n-1)$ the subset of facets of P containing f .

Let $V_{\mathbb{C}} = V \otimes \mathbb{C}$ be the complexification of V , and let $V_{\mathbb{C}}^*$ be its dual space. We extend the pairing $\langle \cdot, \cdot \rangle$ to $V_{\mathbb{C}}$ and $V_{\mathbb{C}}^*$. Let $z \in V_{\mathbb{C}}^*$ and consider the integral

$$I(P)(z) := \int_P \exp \langle x, z \rangle dx$$

and the exponential sum

$$D(P)(z) := \sum_{m \in M \cap P} \exp \langle m, z \rangle.$$

Brion–Vergne [4] gave explicit formulas for $I(P)$ and $D(P)$ for generic z ; we recall them here. For any vertex $v \in P$, we have the normal cone σ_v with spanning points $\{u_F \mid F \in \mathcal{F}^v\}$. Let $\{m_v^F \mid F \in \mathcal{F}^v\}$ be the dual basis. The points m_v^F are rational generators for the tangent cone C_v and, in particular, lie along the edges of P through v . Let $M(v) \subset V$ be the lattice they generate. Then any $\gamma \in G(\sigma_v) = N/U(\sigma_v)$ determines a character $\chi_\gamma: M(v)/M \rightarrow \mathbb{C}^\times$ via

$$\chi_\gamma(m) = \exp(2\pi i \langle m, \tilde{\gamma} \rangle),$$

where $\tilde{\gamma} \in N$ is any representative of γ .

4.1. Proposition. *For $z \in V_{\mathbb{C}}^*$ generic,*

$$(17) \quad D(P)(z) = \sum_{v \in \mathcal{F}(0)} \frac{\exp \langle v, z \rangle}{|G(\sigma_v)|} \sum_{\gamma \in G(\sigma_v)} \prod_{F \in \mathcal{F}^v} \frac{1}{1 - \chi_\gamma(m_v^F) \exp \langle m_v^F, z \rangle}$$

and

$$(18) \quad I(P)(z) = (-1)^n \sum_{v \in \mathcal{F}(0)} \exp \langle v, z \rangle (|\det(m_v^F)|_{F \in \mathcal{F}^v}) \prod_{F \in \mathcal{F}^v} \frac{1}{\langle m_v^F, z \rangle}.$$

Proof. See [4, Propositions 3.9 and 3.10]. □

4.2. Lemma. *Let $E \in \mathcal{F}$ be a face of P , let $\mathcal{F}_E \subset \mathcal{F}$ be the subset of faces of E , and let $D(E)(z) = \sum_{m \in M \cap E} \exp \langle m, z \rangle$. Then for z generic,*

$$D(E)(z) = \sum_{v \in \mathcal{F}_E(0)} \frac{\exp \langle v, z \rangle}{|G(\sigma_v)|} \sum_{\gamma \in G(\sigma_v)} \prod_{\substack{F \in \mathcal{F}^v \\ F \supseteq E}} \frac{1}{1 - \chi_\gamma(m_v^F) \exp \langle m_v^F, z \rangle}.$$

Proof. The proof is essentially the same as that of (17). The main point is that if one considers a single vertex v in (17), then the sum over $G(\sigma_v)$ induces character sums that equal 1 on $C_v \cap M$ and 0 on $C_v \cap (M(v) \setminus M)$. These sums have the same effect on $M \cap E$ for any face $E \subset P$. Furthermore, for any vertex v of E , the points m_v^F in the dual basis lie along edges of E exactly for the facets F not containing E . \square

Now we build an exponential version of our generating function:

$$(19) \quad \tilde{G}_{\text{disc}}(z, y) := \sum_{E \in \mathcal{F}} (y+1)^{\dim E} (-y)^{\text{codim } E} D(E)(z).$$

4.3. Lemma. *For z generic,*

$$(20) \quad \tilde{G}_{\text{disc}}(z, y) = \sum_{v \in \mathcal{F}(0)} \frac{\exp \langle v, z \rangle}{|G(\sigma_v)|} \sum_{\gamma \in G(\sigma_v)} \prod_{F \in \mathcal{F}^v} \left(\frac{y+1}{1 - \chi_\gamma(m_v^F) \exp \langle m_v^F, z \rangle} - y \right).$$

Proof. This follows from Lemma 4.2 and the fact that P is simple. Indeed, consider expanding the products over the sets \mathcal{F}^v . At each vertex v one sees products over all possible subsets of the edges emanating from v . Each subset determines a unique face containing v . If we take a face E and collect the terms corresponding to these edge subsets for the vertices of E , we obtain exactly the expression in Lemma 4.2 for $D(E)(z)$. \square

Next we consider an integral version of $\tilde{G}_{\text{disc}}(z, y)$. As before, let $h = (h_F)_{F \in \mathcal{F}(n-1)}$ be a multiparameter indexed by the facets of P , and recall (cf. (6)) that $\tilde{P}_1(h)$ is the deformation by h of the $(y+1)$ -dilate of P :

$$\tilde{P}_1(h) = \{x \in V \mid \langle x, u_F \rangle + (y+1)\lambda_F + h_F \geq 0 \text{ for all } F \in \mathcal{F}(n-1)\}.$$

Given any vertex $v \in P$, the corresponding vertex in $\tilde{P}_1(h)$ is

$$v(h) = (y+1)v - \sum_{F \in \mathcal{F}^v} h_F m_v^F.$$

We define

$$\tilde{G}_{\text{cont}}(z, y) := I(\tilde{P}_1(h))(z) = \int_{\tilde{P}_1(h)} \exp \langle x, z \rangle dx.$$

4.4. Lemma. *We have*

$$(21) \quad \tilde{G}_{\text{cont}}(z, y) = (-1)^n \sum_{v \in \mathcal{F}(0)} \frac{\exp \langle (y+1)v - \sum_{F \in \mathcal{F}^v} h_F m_v^F, z \rangle}{|G(\sigma_v)|} \prod_{F \in \mathcal{F}^v} \frac{1}{\langle m_v^F, z \rangle}.$$

Proof. This follows from (18) with P replaced by $\tilde{P}_1(h)$, together with the observation that $1/|G(\sigma_v)| = |\det(m_v^F)|_{F \in \mathcal{F}^v}$. \square

We now consider the action of the operator $\mathrm{Td}_y(P, \partial/\partial h)$ on $\tilde{G}_{\mathrm{cont}}$. In particular we will compute the action on the terms for the different vertices in (21) and will ultimately compare the result with the corresponding terms in (20). Put

$$\tilde{G}_{\mathrm{cont}}(v, z, y) := \frac{\exp\langle (y+1)v - \sum_{F \in \mathcal{F}^v} h_F m_v^F, z \rangle}{|G(\sigma_v)|} \prod_{F \in \mathcal{F}^v} \frac{1}{\langle m_v^F, z \rangle}.$$

4.5. Lemma. *Let $\gamma \in \Gamma_\Sigma$ and let y be generic. Then $\mathrm{Td}_y(\gamma, \partial/\partial h)\tilde{G}_{\mathrm{cont}}(v, z, y) = 0$ unless $\gamma \in \sigma_v$. In the latter case,*

$$(22) \quad \mathrm{Td}_y(\gamma, \partial/\partial h)\tilde{G}_{\mathrm{cont}}(v, z, y)|_{h=0} = \frac{\exp\langle (y+1)v, z \rangle}{|G(\sigma_v)|} \prod_{F \in \mathcal{F}^v} \left(\frac{y+1}{1 - a_F(\gamma) \exp\langle (y+1)m_v^F, z \rangle} - y \right).$$

Proof. The first statement is proved in [4, Proof of Theorem 3.12]. The second follows from a direct computation using the identity (with $a \in \mathbb{C}$, x and u real variables)

$$\mathrm{Td}_y(a, \partial/\partial x) \exp xu|_{x=0} = \frac{u(y+1)}{1 - a \exp(-u(y+1))} - uy. \quad \square$$

4.6. Theorem. *Let z be generic. Then*

$$(23) \quad \mathrm{Td}_y(P, \partial/\partial h)\tilde{G}_{\mathrm{cont}}(z, y)|_{h=0} = \tilde{G}_{\mathrm{disc}}((y+1)z, y).$$

Proof. This follows from comparison of Lemmas 4.3 and 4.5. Indeed, by Lemma 4.5 only the γ giving elements in $G(\sigma_v)$ are relevant for computing $\mathrm{Td}_y(P, \partial/\partial h)$ on $\tilde{G}_{\mathrm{cont}}(v, z, y)$. Furthermore, if $\gamma \in G(\sigma_v)$ and $F \in \mathcal{F}(n-1)$ contains v , then a direct computation shows

$$a_F(\gamma) = \chi_\gamma(m_v^F).$$

Thus we have equality in the vertex contributions to each side of (23), after we replace z in $\tilde{G}_{\mathrm{disc}}$ with $(y+1)z$. \square

Proof of Theorem 1.5. We take the Taylor expansion on both sides of (23) with respect to z , after replacing the deformed dilate $\tilde{P}_1(h)$ with the h -deformation of the $(y+1)$ -dilate of qP , which is $\tilde{P}_q(h)$. \square

5. RELATION TO THE HIRZEBRUCH–RIEMANN–ROCH THEOREM

In recent years, a bridge between geometry has allowed one to prove beautiful results in geometry and combinatorics using tools from algebraic geometry. Many combinatorial results have their avatars in algebraic geometry and vice versa. In particular, the polynomial $G_\varphi(q, y)$ can be regarded as a generalization of the Hirzebruch χ_y -genus for a singular toric variety.

In this section, we show that Theorem 1.5 agrees with the representation of the normalized Hirzebruch class of a toric variety studied by Maxim–Schürrmann [13].

Since we treat lattice polytopes whose toric varieties are not necessarily smooth, we have to involve different approaches to the study of singular varieties such as orbifolds, motivic approach, intersection homology theory, etc.

5.1. Let $X = X_\Sigma$ be a complete toric variety of dimension n defined by the fan Σ . Denote by $\widehat{\Omega}_X^p$ the sheaf of Zariski differential p -forms on X . Recall that $\widehat{\Omega}_X^p$ is defined as $\widehat{\Omega}_X^p := i_*\Omega_U^p$, where $i: U \hookrightarrow X$ is the inclusion of the nonsingular locus U into X . Given an ample Cartier divisor D on X , let $\mathcal{O}_X(D)$ be the corresponding invertible. Let $P = P_D$ be the *support polytope* of D . For now, we suppose that the class of D is nontrivial in the Picard group of X .

The χ_y -characteristic (or *generalized Hirzebruch polynomial* of D) is defined by

$$\begin{aligned} \chi_y(X, \mathcal{O}_X(D)) &:= \sum_{p \geq 0} \chi(X, \widehat{\Omega}_X^p \otimes \mathcal{O}_X(D)) \cdot y^p \\ &= \sum_{p \geq 0} \left(\sum_{i \geq 0} (-1)^i \dim_{\mathbb{C}} H^i(X, \widehat{\Omega}_X^p \otimes \mathcal{O}_X(D)) \right) y^p. \end{aligned}$$

In particular, the χ_y -genus of a toric variety is defined as

$$\chi_y(X) := \sum_{j, p \geq 0} (-1)^{j-p} \dim_{\mathbb{C}} \mathrm{Gr}_F^p H_c^j(X; \mathbb{C}) \cdot y^p,$$

where F denotes the Hodge–Deligne filtration on $H_c^j(X; \mathbb{C})$.

The combinatorial expression for $\chi_y(X, \mathcal{O}_X(D))$ in terms of weighted sums of numbers of lattice points in faces of the polytope P_D was first obtained in [12] and also reproved in [13, Corollary 4.3].

5.2. Theorem. *Let X be a complete simplicial toric variety with ample Cartier divisor D . Then the χ_y -characteristic has the following combinatorial representation in terms of sums of lattice points over faces of the support polytope P of D :*

$$\begin{aligned} (24) \quad \chi_y(X, \mathcal{O}_X(D)) &= \sum_{F \in \mathcal{F}_P} (y+1)^{\dim F} (-y)^{\mathrm{codim} F} |F \cap M| \\ &= \sum_{F \in \mathcal{F}_P} (y+1)^{\dim F} |F^\circ \cap M|. \end{aligned}$$

5.3. In [12], the formula (24) is called the *Bott formula for toric varieties*, since it generalizes a result due to Bott, who treated $X = \mathbb{P}^n$, $\widehat{\Omega}_X^p = \Omega_{\mathbb{P}^n}^p$ and $\mathcal{O}_X(D) = \mathcal{O}_{\mathbb{P}^n}(a)$. We see that $\chi_y(X, \mathcal{O}_X(qD))$ from (24) coincides with $G_\varphi(q, y)$ in (4) and (14) when $\varphi \equiv 1$. In fact, the restriction $\varphi \equiv 1$ is not necessary. One can consider $\varphi = e^z$ as in Section 4 by working instead with the equivariant character $\sum_p \sum_i (-1)^i \mathrm{Tr}(e^z, H^i(X, \widehat{\Omega}_X^p \otimes \mathcal{O}_X(qD))) y^p$ of the torus $\mathbb{T} \subset X^5$.

⁵We thank an anonymous referee for pointing this out to us.

5.4. Here we briefly explain work of Maxim–Schürmann [13] that studies characteristic classes of singular toric varieties. First, we recall the motivic Chern and Hirzebruch classes of singular complex algebraic varieties as constructed by Brasselet–Schürmann–Yokura [2].

Let $K_0(\text{var}/X)$ be the relative Grothendieck group of complex algebraic varieties over X , as introduced by Looijenga and Bittner in relation to motivic integration, and let $G_0(X)$ be the Grothendieck group of coherent sheaves of \mathcal{O}_X -modules. Then the *motivic Chern class transformation*

$$mC_y(X) : K_0(\text{var}/X) \rightarrow G_0(X) \otimes \mathbb{Z}[y]$$

generalizes the total λ -class $\lambda^y(T^*X)$ of the cotangent bundle to the setting of singular spaces. The *un-normalized Hirzebruch class transformation* is defined by the composition

$$T_{y*} := td_* \circ mC_y : K_0(\text{var}/X) \rightarrow H_*(X) \otimes \mathbb{Q}[y]$$

as a class version of a χ_y -genus of X . Here the cohomology $H_*(X)$ denotes either the Chow groups $A_*(X)$, or the even degree Borel–Moore homology groups $H_{2*}^{BM}(X, \mathbb{Z})$, and

$$td_* : G_0(-) \rightarrow H_*(-) \otimes \mathbb{Q}$$

is the Todd transformation. The *normalized Hirzebruch class transformation* is defined via the normalization functor $\widehat{T}_{y*} := \Psi_{(1+y)} \circ T_{y*}$, where

$$\Psi_{(1+y)} : H_*(X) \otimes \mathbb{Q}[y] \rightarrow H_*(X) \otimes \mathbb{Q}[y, (1+y)^{-1}]$$

is given in degree k by multiplication by $(1+y)^{-k}$. In fact, \widehat{T}_{y*} actually takes values in $H_*(X) \otimes \mathbb{Q}[y]$ (see [2, Theorem 3.1]); this implies, for instance, that one can set the parameter y equal to -1 , and can thus generalize \widehat{T}_{-1*} to the total rational Chern class.

Now the *motivic un-normalized* and *normalized homology Hirzebruch classes* are defined respectively as

$$T_{y*}(X) := T_{y*}([\text{id}_X]), \quad \widehat{T}_{y*}(X) := \widehat{T}_{y*}([\text{id}_X]);$$

these generalize the Hirzebruch classes of X that appear in the Hirzebruch–Riemann–Roch theorem when X is smooth. Namely, assume X is smooth of dimension n , and let $\{x_j\}$ be the Chern roots of the tangent bundle T_X . Then the two formal power series

$$Q_y(x) := \frac{x(1 + ye^{-x})}{1 - e^{-x}}, \quad \widehat{Q}_y(x) := \frac{x(1 + ye^{-x(1+y)})}{1 - e^{-x(1+y)}} = 1 + \frac{1-y}{2}x + \dots$$

define two classes

$$T_y^*(T_X) = \prod_{j=1}^n Q(x_j), \quad \widehat{T}_y^*(T_X) = \prod_{j=1}^n \widehat{Q}(x_j) \in H^*(X) \otimes \mathbb{Q}[y],$$

and

$$T_{y*}(X) = T_y^*(T_X) \cap [X], \quad \widehat{T}_{y*}(X) = \widehat{T}_y^*(T_X) \cap [X].$$

We can now state Maxim–Schürmann’s result:

5.5. Theorem (Maxim–Schürmann [13]). *Let $X = X_\Sigma$ be a simplicial toric variety of dimension n with the normal fan $\Sigma = \Sigma_P$ to the polytope P . Suppose that the generators of the rational cohomology (or Chow) ring of X are the classes $[D_F]$ defined by the \mathbb{Q} -Cartier divisors corresponding to the faces of codimension 1 of P . Then the normalized Hirzebruch class of X is given by*

$$(25) \quad \widehat{T}_{y*}(X) = \left(\sum_{g \in \Gamma_\Sigma} \prod_{F \in \mathcal{F}(n-1)} \frac{[D_F](1 + y a_F(g) e^{-[D_F](y+1)})}{1 - a_F(g) e^{-[D_F](y+1)}} \right) \cap [X].$$

5.6. Now we connect Theorem 5.5 to our work. The main observation is that the Todd differential operator in Theorem 1.5

$$\mathrm{Td}_y(P, \partial/\partial h) = \left(\sum_{g \in \Gamma_\Sigma} \prod_{F \in \mathcal{F}(n-1)} \frac{\partial/\partial h_F(1 + y a_F(g) e^{-\partial/\partial h_F(1+y)})}{1 - a_F(g) e^{-\partial/\partial h_F(1+y)}} \right)$$

has the same structure as the normalized Hirzebruch class in (25). This correspondence for $y = 0$ was first established by M. Brion and M. Vergne in [3] (see also [6, Theorem 13.5.6]). The generic correspondence $[D_F] \rightarrow \partial/\partial h_F$ and the relation of $G_\varphi(q, y)$ with the polynomial $\chi_y(X, \mathcal{O}_X(D))$ can be proved by the same technique as in [3, Theorem 4.5]; this will be published elsewhere.

6. EXAMPLES

6.1. We conclude by giving some examples of our results. We begin with Theorem 2.6.

Let P be the square pyramid with vertices $(0, 0, 0)$, $(1, 1, 1)$, $(1, -1, 1)$, $(-1, 1, 1)$, $(-1, -1, 1)$ shown in Figure 1. Let $q > 0$ be an integer. We consider the generating function $G_\varphi(q, y)$ for different functions φ .

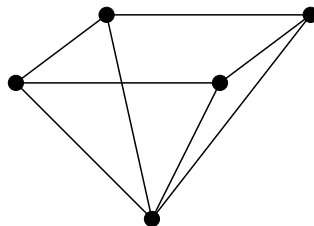


FIGURE 1. The square pyramid P .

The polytope P fails to be simple only at the bottom vertex $v = (0, 0, 0)$. The dual face $\mathcal{P}_P(v)$ is a square, and $g_{\text{square}}(x) = 1 + x$ (in general, the g -polynomial of an m -gon is $1 + (m-3)x$). Thus $\tilde{g}_v(-\frac{1}{y}) = 1 - \frac{1}{y}$, and the only effect of the non-simplicity of P is that, when we form the generating function $G_\varphi(q, y)$, the contribution of the vertices to (9) is

$$\varphi(q, q, q) + \varphi(q, -q, q) + \varphi(-q, q, q) + \varphi(-q, -q, q) + \varphi(0, 0, 0)(1 - \frac{1}{y}).$$

Suppose first $\varphi = 1$. Then

$$\begin{aligned} G_1(q, y) &= \left(\frac{4q^3}{3} - 4q^2 + \frac{11q}{3} - 1 \right) y^3 \\ &\quad + \left(4q^3 - 4q^2 - q + 2 \right) y^2 + \left(4q^3 + 4q^2 - q - 2 \right) y + \frac{4q^3}{3} + 4q^2 + \frac{11q}{3} + 1. \end{aligned}$$

One can see the Ehrhart polynomial for P in the constant term, and that for P° in the leading term. It is visible that G_1 satisfies $G_1(q, y) = (-y)^3 G_1(-q, \frac{1}{y})$, and this relation applied to the leading and constant terms is nothing other than Ehrhart reciprocity.

Denote by $\text{Vol}(P)$ the volume of polytope P of dimension n normalized so that the volume of the simplex spanned by the origin and basis vectors is equal to 1. Expand the polynomial $G_1(q, y)$:

$$G_1(q, y) = \sum_{p=1}^n L_p(q) y^p.$$

Then it is easy to see that $L_p(q)$ is the (generalized Ehrhart) polynomial in q of degree n whose leading term is $\binom{n}{p} \text{Vol}(P) q^n$. Indeed, consider the expansion from (14)

$$G_1(q, y) = \sum_{F \in \mathcal{F}} (y+1)^{\dim F} \mathcal{E}_F^\circ(q) \tilde{g}_F(-y),$$

where $\mathcal{E}_F^\circ(q) = |M \cap qF^\circ| = \text{Vol}(P) q^n + a_1 q^{n-1} + \dots$, and notice that the coefficient of the leading term of g -polynomial is 1 according to 2.2. In the example above for the square pyramid, $\text{Vol}(P) = 4/3$ and the highest order terms of $L_0(q)$ and $L_3(q)$ are $\frac{4}{3}q^3$, and of $L_1(q)$ and $L_2(q)$ are $4q^3$.

Next we take a linear polynomial $\varphi = ax_1 + bx_2 + cx_3$. Note that the symmetry of P implies that we expect that the final answer should be independent of a and b . Indeed, after summing over faces of P we find

$$\begin{aligned} G_\varphi(q, y) &= y^4 \left(cq^4 - \frac{10cq^3}{3} + \frac{7cq^2}{2} - \frac{7cq}{6} \right) + y^3 \left(4cq^4 - \frac{20cq^3}{3} + 4cq^2 - \frac{cq}{3} \right) \\ &\quad + y^2 \left(6cq^4 + cq^2 \right) + y \left(4cq^4 + \frac{20cq^3}{3} + 4cq^2 + \frac{cq}{3} \right) + cq^4 + \frac{10cq^3}{3} + \frac{7cq^2}{2} + \frac{7cq}{6}. \end{aligned}$$

This has degree 4 in y , as expected. One can also see the expected reciprocity law $G_\varphi(q, y) = (-y)^4 G_\varphi(-q, \frac{1}{y})$.

For the amusement of the reader, we finish with a larger example: $\varphi = ax_1^2 + bx_2^2 + cx_3^2$. The resulting $G_\varphi(q, y)$ equals

$$\begin{aligned}
& y^5 \left(\frac{4aq^5}{15} - \frac{4aq^4}{3} + \frac{7aq^3}{3} - \frac{5aq^2}{3} + \frac{2aq}{5} + \frac{4bq^5}{15} - \frac{4bq^4}{3} \right. \\
& \quad \left. + \frac{7bq^3}{3} - \frac{5bq^2}{3} + \frac{2bq}{5} + \frac{4cq^5}{5} - 3cq^4 + \frac{11cq^3}{3} - \frac{3cq^2}{2} + \frac{cq}{30} \right) \\
& + y^4 \left(\frac{4aq^5}{3} - 4aq^4 + 5aq^3 - 3aq^2 + \frac{2aq}{3} + \frac{4bq^5}{3} - 4bq^4 \right. \\
& \quad \left. + 5bq^3 - 3bq^2 + \frac{2bq}{3} + 4cq^5 - 9cq^4 + 9cq^3 - \frac{5cq^2}{2} - \frac{cq}{2} \right) \\
& + y^3 \left(\frac{8aq^5}{3} - \frac{8aq^4}{3} + \frac{10aq^3}{3} - \frac{4aq^2}{3} + \frac{8bq^5}{3} - \frac{8bq^4}{3} \right. \\
& \quad \left. + \frac{10bq^3}{3} - \frac{4bq^2}{3} + 8cq^5 - 6cq^4 + \frac{26cq^3}{3} - cq^2 - \frac{5cq}{3} \right) \\
& + y^2 \left(\frac{8aq^5}{3} + \frac{8aq^4}{3} + \frac{10aq^3}{3} + \frac{4aq^2}{3} + \frac{8bq^5}{3} + \frac{8bq^4}{3} \right. \\
& \quad \left. + \frac{10bq^3}{3} + \frac{4bq^2}{3} + 8cq^5 + 6cq^4 + \frac{26cq^3}{3} + cq^2 - \frac{5cq}{3} \right) \\
& + y \left(\frac{4aq^5}{3} + 4aq^4 + 5aq^3 + 3aq^2 + \frac{2aq}{3} + \frac{4bq^5}{3} + 4bq^4 \right. \\
& \quad \left. + 5bq^3 + 3bq^2 + \frac{2bq}{3} + 4cq^5 + 9cq^4 + 9cq^3 + \frac{5cq^2}{2} - \frac{cq}{2} \right) \\
& + \frac{4aq^5}{15} + \frac{4aq^4}{3} + \frac{7aq^3}{3} + \frac{5aq^2}{3} + \frac{2aq}{5} + \frac{4bq^5}{15} + \frac{4bq^4}{3} \\
& \quad + \frac{7bq^3}{3} + \frac{5bq^2}{3} + \frac{2bq}{5} + \frac{4cq^5}{5} + 3cq^4 + \frac{11cq^3}{3} + \frac{3cq^2}{2} + \frac{cq}{30}.
\end{aligned}$$

6.2. Let $P = C_n^\Delta$ be the *cross-polytope* (or *co-cube*). By definition P is the convex hull of the standard basis vectors e_1, \dots, e_n and their negatives $-e_1, \dots, -e_n$ in $M_{\mathbb{R}} \simeq \mathbb{R}^n$. For example, when $n = 3$, the polytope C_3^Δ is the octahedron. It is known that the polar dual polytope of P is the unit cube C_n , whose associated toric variety is

isomorphic to the product $\mathbb{P}^1 \times \mathbb{P}^1 \times \dots \times \mathbb{P}^1$. The g -polynomial of the cube was computed by I. Gessel [15, §2.6]:

$$(26) \quad g(C_n, x) = \sum_{k=0}^m \frac{1}{n-k+1} \binom{n}{k} \binom{2n-2k}{n} (x-1)^k, \quad m = \lfloor n/2 \rfloor.$$

To find the function $G_\varphi(q, y)$ of C_n^Δ defined in (14), we need the explicit form of the polynomial

$$K_F(y) := (1+y)^{\dim F} \tilde{g}_F(-y),$$

where F is any face of C_n^Δ . Using (26), we have

$$K_F(y) = (1+y)^{\dim F} \sum_{k=0}^{m_F} \frac{1}{\dim F - k + 1} \binom{\dim F}{k} \binom{2 \dim F - 2k}{\dim F} (-y-1)^k.$$

Putting all this together, we obtain

$$G_\varphi(q, y) = \sum_{F \in \mathcal{F}} \sum_{k=0}^{m_F} \frac{(-1)^k}{\dim F - k - 1} \binom{\dim F}{k} \binom{2 \dim F - 2k}{\dim F} (y+1)^{\dim F + k - \deg \varphi} D_{\varphi, F}^\circ(q).$$

6.3. Finally we consider an example of Theorem 1.5. Let P be the triangle with vertices at $(0,0)$, $(2,0)$, and $(0,1)$. The polygon P together with its normal fan Σ are shown in Figure 2.



FIGURE 2. The triangle P and its normal fan Σ .

In the normal fan the shaded regions represent the sets $Q(\sigma)$. One can see that the set Γ_Σ contains two lattice points $g_0 = (0,0)$ and $g_1 = (0,-1)$, shown in white. It is clear that the all the functions $\{a_F \mid F \in \mathcal{F}\}$ are identically 1 on g_0 , and that

$a_F(g_1) \neq 1$ if and only if F is one of F_2 or F_3 , and that for either of these $a_F(g_1) = -1$. Thus our Todd- y operator has the form

$$(27) \quad \text{Td}_y(P, \partial/\partial h) = \text{Td}_y(1, \partial/\partial h_1) \text{Td}_y(1, \partial/\partial h_2) \text{Td}_y(1, \partial/\partial h_3) \\ + \text{Td}_y(1, \partial/\partial h_1) \text{Td}_y(-1, \partial/\partial h_2) \text{Td}_y(-1, \partial/\partial h_3).$$

First consider putting $\varphi = 1$. The function $E_\varphi(\tilde{P}_q(h))$ is then just the volume of the deformed dilate $\tilde{P}_q(h)$, which is

$$(28) \quad \text{Vol } \tilde{P}_q(h) = \frac{(2h_1 + h_2 + h_3 + 2q(y+1))^2}{4}.$$

Applying (27) to (28) and putting $h_1 = h_2 = h_3 = 0$, we obtain

$$(q^2 - 2q + 1)y^2 + (2q^2 - 1)y + q^2 + 2q + 1.$$

It is easy to check directly that this agrees with $G_1(q, y)$.

Now suppose φ is a generic homogeneous linear function $\varphi(x_1, x_2) = ax_1 + bx_2$. Then our integral becomes

$$(29) \quad E_\varphi(\tilde{P}_q(h)) = \frac{1}{24}(2h_1 + h_2 + h_3 + 2q(y+1))^2 \\ \cdot (2a(2h_1 - 2h_2 + h_3 + 2q(y+1)) + b(-4h_1 + h_2 + h_3 + 2q(y+1))).$$

Applying (27) to (29) and setting $h_1 = h_2 = h_3 = 0$ yields

$$(y+1) \left(y^2 \left(\frac{2aq^3}{3} - \frac{3aq^2}{2} + \frac{5aq}{6} + \frac{bq^3}{3} - \frac{bq^2}{2} + \frac{bq}{6} \right) \right. \\ \left. + y \left(\frac{4aq^3}{3} - \frac{aq}{3} + \frac{2bq^3}{3} - \frac{2bq}{3} \right) + \frac{2aq^3}{3} + \frac{3aq^2}{2} + \frac{5aq}{6} + \frac{bq^3}{3} + \frac{bq^2}{2} + \frac{bq}{6} \right).$$

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