$E_{6(6)}$ exceptional Drinfel'd algebras

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ABSTRACT: The exceptional Drinfel'd algebra (EDA) is a Leibniz algebra introduced to provide an algebraic underpinning with which to explore generalised notions of U-duality in M-theory. In essence, it provides an M-theoretic analogue of the way a Drinfel'd double encodes generalised T-dualities of strings. In this note we detail the construction of the EDA in the case where the regular U-duality group is $E_{6(6)}$. We show how the EDA can be realised geometrically as a generalised Leibniz parallelisation of the exceptional generalised tangent bundle for a six-dimensional group manifold G, endowed with a Nambu-Lie structure. When the EDA is of coboundary type, we show how a natural generalisation of the classical Yang-Baxter equation arises. The construction is illustrated with a selection of examples including some which embed Drinfel'd doubles and others that are not of this type.

Contents

1 Introduction

Dualities play an important role in our understanding of string theory. One of the best-understood dualities is T-duality, which relates string theory on backgrounds with $\mathrm{U}(1)^d$ isometries, with the backgrounds related by $\mathrm{O}(d,d)$ transformations. These T-dualities are already visible in perturbative string theory, and are enlarged into U-dualities in the non-perturbative framework of M-theory [1, 2]. Generalisations of Abelian T-dualities exist for backgrounds with non-Abelian isometries, leading to non-Abelian T-duality (NATD) [3], and for backgrounds without any isometries, called Poisson-Lie T-duality (PLTD) [4, 5]. Instead of the isometry algebra, PLTD is controlled by an underlying Drinfel'd double.

Unlike Abelian T-duality, which is an equivalence between string theories on different backgrounds to all orders in the string coupling and string length, these generalised T-duality are currently best understood at the supergravity level and their status as true dualities of the string genus expansion remains doubtful [6]. Very recent work [7–10] has provided indications that such dualities do persist under α' corrections. Both NATD and PLTD have led to fruitful results. For example, NATD has been successfully used as solution-generating mechanisms of supergravity [11], leading to the discovery of new minimally supersymmetric AdS backgrounds starting with [12, 13] (see [14] for a review and further references). Moreover, there is a close connection between PLTD and the (modified) classical Yang-Baxter equation which controls integrable deformations of σ-models [15, 16].

The non-perturbative generalisation of Poisson-Lie T-duality to a U-duality version in M-theory, or more conservatively as a solution-generating mechanism of 11-dimensional supergravity, has long been an open problem, which was recently addressed in [17, 18] and further elaborated on in [19–21]. Building on the interpretation of PLTD and Drinfel'd doubles within Double Field Theory (DFT), [17, 18, 22–24] used Exceptional Field Theory (ExFT)/Exceptional Generalised Geometry to propose a natural generalisation of the Drinfel'd double for dualities along four spacetime dimensions. This "Exceptional Drinfel'd Algebra" (EDA) was shown to lead to a new solution-generating mechanism of 11-dimensional supergravity that suggests a notion of Poisson-Lie U-duality, as well as a generalisation of the classical Yang-Baxter equation. Other recent works [25–27] have considered closely related ideas, although the detailed relation between these approaches and the EDA is not completely apparent.

In this paper, we will further develop the ideas of [17, 18] by constructing EDAs and Poisson-Lie U-duality amongst six directions. We choose six dimensions, because important new features arise when dualities are considered in six directions. This is because now the 6-form can completely wrap the six directions we are considering. As a result, the $\mathfrak{e}_{6(6)}$ algebra contains a generator, corresponding to a hexavector, which will generate new kinds of dualities and deformations which have no counterpart in PLTD, as we will see.

The outline of the rest of this paper is as follows: in section 2 we describe the EDA from a purely algebraic perspective. In section 3 we show how the EDA can be realised within exceptional generalised geometry as a Leibniz parallelisation of a particular type of group manifold G, that we will call a $(3,6)$-Nambu-Lie group. We then consider more closely the case of a coboundary EDA in section 4 whose structure is governed by a generalisation of the Yang Baxter equation. We provide a range of examples in section 5 of EDAs both coboundary and otherwise, some of which have Drinfel'd doubles as subalgebras, and other which do not have such an interpretation. The aim of these examples is not to provide here a full classification, which could form an interesting investigation in its own right, but rather to highlight the various features that can arise.

2 The $E_{6(6)}$ EDA

Before specialising to the case of $E_{6(6)}$ we begin by presenting some generalities of the Exceptional Drinfel'd Algebra. The EDA, $\mathfrak{d}_n$, is a Leibniz algebra which is a subalgebra

D	$E_{n(n)}$	H_n	R_1	R_2	R_3	R_4
7	SL(5)	$\mathrm{USp}(4)/\mathbb{Z}_2$	**10**	$\mathbf{\bar{5}}$	**5**	$\mathbf{\overline{10}}$
6	Spin(5,5)	$\mathrm{USp}(4) \times \mathrm{USp}(4)/\mathbb{Z}_2$	**16**	**10**	$\mathbf{\overline{16}}$	**45**
5	$\mathrm{E}_{6(6)}$	$\mathrm{USp}(8)/\mathbb{Z}_2$	**27**	$\mathbf{\overline{27}}$	**78**	$\mathbf{\overline{351}'}$

Table 1. The split real form of exceptional groups $E_{n(n)}$ with $D = 11 - n$, their maximal compact subgroups H_n and representations $R_1 \ldots R_4$ appearing in the tensor hierarchy of ExFT. In this work we will be mostly concerned with representations R_1 and R_2 which will be associated to the generalised tangent bundles E and N respectively.

of $E_{n(n)}$,[1] admitting a "maximally isotropic" subalgebra, as we will define shortly. In table 1 we provide details of the representations of $E_{n(n)}$ inherited from the exceptional field theory (ExFT) approach[2] to eleven-dimensional supergravity that are useful to the present construction.

We denote the generators of $\mathfrak{d}_n$ by $\{T_A\}$, with the index A inherited from the $\bar{R}_1$ representation of $E_{n(n)}$ ExFT and their product by

$$T_A \circ T_B = X_{AB}{}^C T_C \,, \tag{2.1}$$

with $X_{AB}{}^C$ structure constants which are not necessarily antisymmetric in their lower indices. The product obeys the Leibniz identity, namely

$$T_A \circ (T_B \circ T_C) = (T_A \circ T_B) \circ T_C + T_B \circ (T_A \circ T_C) \,, \tag{2.2}$$

which implies for the structure constants

$$X_{AC}{}^D X_{BD}{}^E - X_{BC}{}^D X_{AD}{}^E + X_{AB}{}^D X_{DC}{}^E = 0 \,. \tag{2.3}$$

Note that if the Leibniz algebra is a Lie algebra, i.e. the $X_{AB}{}^C$ are antisymmetric in their lower indices, then this reduces to the Jacobi identity.

We place two further (linear) requirements on the EDA. Firstly, we demand that there is a maximal *Lie* subalgebra $\mathfrak{g}$ spanned by $\{T_a\} \subset \{T_A\}$ obeying

$$\mathfrak{g} \otimes \mathfrak{g}|_{\bar{R}_2} = 0 \,, \tag{2.4}$$

in which the representation $\bar{R}_2$ is found in table 1. We call such a subalgebra $\mathfrak{g}$ maximally isotropic. We will be interested here in the case that $\dim \mathfrak{g} = n$ as this is relevant to the M-theory context.[3] Since $G = \exp \mathfrak{g}$ acts adjointly on $\mathfrak{d}_n$, it follows that G should be endowed with a trivector and hexavector. We will further require that these objects give rise to a 3- and 6-bracket on $\mathfrak{g}^*$, thereby imposing some further restrictions on the structure

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[1]In general one can allow for EDAs as subalgebras of $E_{n(n)} \times \mathbb{R}^+$, see [18]. However, we will not deal with the extra $\mathbb{R}^+$ factor here.

[2]See [28] for a contemporary review of ExFT.

[3]There is another inequivalent way to maximally solve the condition eq. (2.4) with $\dim \mathfrak{g} = n-1$ leading to a IIB scenario [29, 30].

constants $X_{AB}{}^C$.[4] These additional requirements imply that the EDA can be given a geometrical realisation in terms of certain generalised frames whose action is mediated by the generalised Lie derivative (3.4), as we will show in section 3.

Let us now discuss these restrictions in detail.

2.1 Linear constraints

We now study in detail the consequence of the requirements of the maximally isotropic subalgebra $\mathfrak{g}$ and its adjoint action. Since these constraints arise from placing requirements directly to the form of $X_{AB}{}^C$ we describe them as linear constraints; this is to be contrasted with quadratic constraints of the form $X^2 = 0$ that arise from the Leibniz identity.

Firstly, since $\mathfrak{g}$ is a Lie algebra, we immediately have

$$T_a \circ T_b = f_{ab}{}^c \, T_c \,, \tag{2.5}$$

with $f_{ab}{}^c$ antisymmetric in a, b. Secondly, the adjoint action[5] of $g \in G = \exp\mathfrak{g}$ on $\mathfrak{d}_n$ implies that

$$g \cdot T_A \cdot g^{-1} = (A_g)_A{}^B T_B \,, \tag{2.6}$$

with $(A_g)_A{}^B \in E_{6(6)}$ since $\mathfrak{g} \subset \mathfrak{d}_n \subset \mathfrak{e}_{6(6)}$. Let us denote the adjoint action of $g \in G$ on $\mathfrak{g}$ by a_g. It is now useful to branch T_A, which transform in the $\bar{R}_1 = \overline{\mathbf{27}}$ representation, and $\mathfrak{e}_{6(6)}$ according to the decomposition of $E_{6(6)}$ into GL(6):

$$\begin{aligned} \overline{\mathbf{27}} &\longrightarrow \overline{\mathbf{6}} \oplus \mathbf{15} \oplus \overline{\mathbf{6}} \,, \\ \mathbf{78} &\longrightarrow \mathbf{36} \oplus \mathbf{20} \oplus \mathbf{1} \oplus \overline{\mathbf{20}} \oplus \mathbf{1} \,. \end{aligned} \tag{2.7}$$

Now $g \cdot T_A \cdot g^{-1}$ takes the form:

$$\begin{aligned} g \cdot T_a \cdot g^{-1} &= (a_g)_a{}^b \, T_b \,, \\ g \cdot T^{a_1 a_2} \cdot g^{-1} &= -\lambda_g^{a_1 a_2 c} \, (a_g)_c{}^b \, T_b + (a_g^{-1})_{b_1}{}^{a_1} (a_g^{-1})_{b_2}{}^{a_2} \, T^{b_1 b_2} \,, \\ g \cdot T^{a_1 \dots a_5} \cdot g^{-1} &= \left(\lambda_g^{a_1 \dots a_5 c} + 5\, \lambda_g^{[a_1 a_2 a_3} \lambda_g^{a_4 a_5] c} \right) (a_g)_c{}^b \, T_b \\ &\quad - 10\, \lambda_g^{[a_1 a_2 a_3} (a_g^{-1})_{b_1}{}^{a_4} (a_g^{-1})_{b_2}{}^{a_5]} \, T^{b_1 b_2} \\ &\quad + (a_g^{-1})_{b_1}{}^{[a_1} \dots (a_g^{-1})_{b_5}{}^{a_5]} \, T^{b_1 \dots b_5} \,, \end{aligned} \tag{2.8}$$

where

$$T_A = \left\{ T_a,\, T^{a_1 a_2},\, T^{a_1 \dots a_5} \right\} , \tag{2.9}$$

are the GL(6) representations of T_A according to (2.7), whilst $(a_g)_a{}^b$ parameterize the GL(6) transformation while $\lambda_g^{a_1 a_2 a_3}$ and $\lambda_g^{a_1 \dots a_6}$ parameterise the $E_{6(6)}$ transformations generated

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[4]It is worth emphasising that these are impositions beyond simply demanding that $\mathfrak{g}$ be a maximal isotropic.

[5]To be more precise we inherit an action via the rack product:

$$g \cdot T_A \cdot g^{-1} \equiv g^{-1} \rhd T_A \equiv T_A + h \circ T_A + \frac{1}{2} h \circ (h \circ T_A) + \cdots \qquad (g^{-1} \equiv e^h) \,.$$

by the generators in the **20** and **1**, respectively.[6] Thus, G admits a totally antisymmetric trivector λ^{abc} and totally antisymmetric hexavector $\lambda^{a_1 \dots a_6}$ which control its adjoint action on the generators T^{ab} and $T^{a_1 \dots a_5}$.

Equations (2.8) imply that $(\lambda_g)^{abc}$ and $(\lambda_g)^{a_1 \dots a_6}$ vanish at the identity, i.e.

$$(\lambda_e)^{abc} = (\lambda_e)^{a_1 \dots a_6} = 0, \tag{2.10}$$

and they inherit a group composition rule

$$
\begin{aligned}
\lambda_{hg}^{a_1 a_2 a_3} &= \lambda_g^{a_1 a_2 a_3} + (a_g^{-1})_{c_1}{}^{a_1} (a_g^{-1})_{c_2}{}^{a_2} (a_g^{-1})_{c_3}{}^{a_3} \lambda_h^{c_1 c_2 c_3}, \\
\lambda_{hg}^{a_1 \dots a_6} &= \lambda_g^{a_1 \dots a_6} + (a_g^{-1})_{c_1}{}^{a_1} \dots (a_g^{-1})_{c_6}{}^{a_6} \lambda_h^{c_1 \dots c_6} \\
&\quad + 10\, \lambda_g^{[a_1 a_2 a_3} (a_g^{-1})_{c_1}{}^{a_4} (a_g^{-1})_{c_2}{}^{a_5} (a_g^{-1})_{c_3}{}^{a_6]} \lambda_h^{c_1 c_2 c_3},
\end{aligned}
\tag{2.11}
$$

for $g, h \in G$.

Finally, we come to the second condition on the EDA, i.e. the existence of a 3- and 6-bracket on $\mathfrak{g}^*$. This is equivalent to imposing the following differential conditions on λ^{abc} and $\lambda^{a_1 \dots a_6}$:

$$
\begin{aligned}
d\lambda^{a_1 a_2 a_3} &= r^b \left(f_b{}^{a_1 a_2 a_3} + 3\, f_{bc}{}^{[a_1} \lambda^{|c|a_2 a_3]} \right), \\
d\lambda^{a_1 \dots a_6} &= r^b \left(f_b{}^{a_1 \dots a_6} + 6\, f_{bc}{}^{[a_1} \lambda^{|c|a_2 \dots a_6]} + 10\, f_b{}^{[a_1 a_2 a_3} \lambda^{a_4 a_5 a_6]} \right),
\end{aligned}
\tag{2.12}
$$

where $r = r^a\, T_a$ are the right-invariant 1-forms on G obeying $dr^a = \frac{1}{2} f_{bc}{}^a r^b \wedge r^c$ and we have dropped the subscript g on $\lambda^{(3)}$ and $\lambda^{(6)}$. The $f_b{}^{a_1 \dots a_3}$ and $f_b{}^{a_1 \dots a_6}$ are structure constants for a 3- and 6-bracket and are totally antisymmetric in their upper indices. In fact, as we will see in section 2.2, the Leibniz identity implies further properties of the trivector and hexavector, in particular that they define a certain Nambu 3- and 6-bracket which are compatible with the Lie bracket on G. Therefore, it seems apt to call G a $(3, 6)$-Nambu-Lie Group.

With the above conditions, the EDA takes the following form

$$
\begin{aligned}
T_a \circ T_b &= f_{ab}{}^c\, T_c, \\
T_a \circ T^{b_1 b_2} &= f_a{}^{b_1 b_2 c}\, T_c + 2\, f_{ac}{}^{[b_1}\, T^{b_2]c}, \\
T_a \circ T^{b_1 \dots b_5} &= -f_a{}^{b_1 \dots b_5 c}\, T_c + 10\, f_a{}^{[b_1 b_2 b_3}\, T^{b_4 b_5]} - 5\, f_{ac}{}^{[b_1}\, T^{b_2 \dots b_5]c}, \\
T^{a_1 a_2} \circ T_b &= -f_b{}^{a_1 a_2 c}\, T_c + 3\, f_{[c_1 c_2}{}^{[a_1}\, \delta_b^{a_2]}\, T^{c_1 c_2}, \\
T^{a_1 a_2} \circ T^{b_1 b_2} &= -2\, f_c{}^{a_1 a_2 [b_1}\, T^{b_2]c} + f_{c_1 c_2}{}^{[a_1}\, T^{a_2] b_1 b_2 c_1 c_2}, \\
T^{a_1 a_2} \circ T^{b_1 \dots b_5} &= 5\, f_c{}^{a_1 a_2 [b_1}\, T^{b_2 \dots b_5]c}, \\
T^{a_1 \dots a_5} \circ T_b &= f_b{}^{a_1 \dots a_5 c}\, T_c - 10\, f_b{}^{[a_1 a_2 a_3}\, T^{a_4 a_5]} - 20\, f_c{}^{[a_1 a_2 a_3}\, \delta_b^{a_4}\, T^{a_5]c} \\
&\quad + 5\, f_{bc}{}^{[a_1}\, T^{a_2 \dots a_5]c} + 10\, f_{c_1 c_2}{}^{[a_1}\, \delta_b^{a_2}\, T^{a_3 a_4 a_5]c_1 c_2}, \\
T^{a_1 \dots a_5} \circ T^{b_1 b_2} &= 2\, f_c{}^{a_1 \dots a_5 [b_1}\, T^{b_2]c} - 10\, f_c{}^{[a_1 a_2 a_3}\, T^{a_4 a_5] b_1 b_2 c}, \\
T^{a_1 \dots a_5} \circ T^{b_1 \dots b_5} &= -5\, f_c{}^{a_1 \dots a_5 [b_1}\, T^{b_2 \dots b_5]c}.
\end{aligned}
\tag{2.13}
$$

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[6]The generators of $\mathfrak{e}_{6(6)}$ corresponding to the $\overline{\mathbf{20}}$ and the other **1** in (2.7) are not allowed by the assumption (2.5).

2.2 Leibniz identity constraints

We will now study the compatibility conditions between the Lie algebra $\mathfrak{g}$, the 3-bracket and 6-bracket, as well as their appropriate "closure" conditions that are required for the EDA to satisfy the Leibniz identity of eq. (2.2). This yields a number of immediate constraints. In particular, we obtain the following fundamental identities, i.e. generalisations of Jacobi for higher brackets,

$$0 = 3 f_{[a_1 a_2}{}^c f_{a_3] c}{}^b \,, \tag{2.14}$$

$$0 = f_a{}^{dc_1 c_2} f_d{}^{b_1 b_2 b_3} - 3 f_d{}^{c_1 c_2 [b_1} f_a{}^{b_2 b_3] d} + f_{d_1 d_2}{}^{[c_1} f_a{}^{c_2] b_1 b_2 b_3 d_1 d_2} \,, \tag{2.15}$$

$$0 = f_a{}^{dc_1 \ldots c_5} f_d{}^{b_1 \ldots b_6} - 6 f_d{}^{c_1 \ldots c_5 [b_1} f_a{}^{b_2 \ldots b_6] d} \,, \tag{2.16}$$

as well as compatibility conditions between the dual structure constants $f_b{}^{a_1 a_2 a_3}$, $f_b{}^{a_1 \ldots a_6}$ and the Lie algebra structure constants. These compatibility conditions take the form of cocycle conditions

$$0 = f_{a_1 a_2}{}^c f_c{}^{b_1 b_2 b_3} + 6 f_{c[a_1}{}^{[b_1|} f_{a_2]}{}^{c|b_2 b_3]} \,, \tag{2.17}$$

$$0 = f_{a_1 a_2}{}^c f_c{}^{b_1 \ldots b_6} + 12 f_{c[a_1}{}^{[b_1|} f_{a_2]}{}^{c|b_2 \ldots b_6]} - 20 f_{[a_1}{}^{[b_1 b_2 b_3} f_{a_2]}{}^{b_4 b_5 b_6]} \,, \tag{2.18}$$

as well as the additional constraint

$$f_{d_1 d_2}{}^a f_c{}^{d_1 d_2 b} = 0 \,. \tag{2.19}$$

If we only consider EDAs $\mathfrak{d}_n$ with $n \leq 6$, as we are doing here, the conditions given by the above eqs. (2.14)–(2.19) are equivalent to imposing the Leibniz identity. This is because in $n \leq 6$, the fundamental identity for the six-bracket implies that $f_b{}^{a_1 \ldots a_6} = 0$. However, since the structure we are studying here will also exist for $n > 6$, we will keep the remaining discussion as dimension-independent as possible, whilst keeping in mind that for $n > 6$, the Leibniz identity will lead to further or modified compatibility conditions between $f_{ab}{}^c$, $f_b{}^{a_1 a_2 a_3}$ and $f_b{}^{a_1 \ldots a_6}$. These additional constraints will need to be studied using EDAs based on $E_{7(7)}$ and higher.

Before interpreting these constraints, we remark that the Leibniz identity ensures, much as the structure constants of a Lie algebra $\mathfrak{g}$ are invariant under $G = \exp \mathfrak{g}$ acting adjointly, that the EDA structure constants enjoy an invariance

$$X_{AB}{}^D (A_g)_D{}^C = (A_g)_A{}^D (A_g)_B{}^E X_{DE}{}^C \,. \tag{2.20}$$

Substitution of eq. (2.8) here results in a variety of identities that we shall revisit later on.

2.2.1 Fundamental identities

Let us now introduce the 3-bracket $\{\,\}_3$ and 6-bracket $\{\,\}_6$ on $\mathfrak{g}^*$ with structure constants $f_b{}^{a_1 a_2 a_3}$ and $f_b{}^{a_1 \ldots a_6}$, respectively, i.e.

$$\{x,\, y,\, z\}_3 = f_a{}^{b_1 b_2 b_3} x_{b_1} y_{b_2} z_{b_3} \,,$$
$$\{u,\, v,\, w,\, x,\, y,\, z\}_6 = f_a{}^{b_1 \ldots b_6} u_{b_1} v_{b_2} w_{b_3} x_{b_4} y_{b_5} z_{b_6} \,. \tag{2.21}$$

The conditions (2.14) and (2.15) imply that the 3- and 6-brackets satisfy

$$
\begin{aligned}
\{x_1,\, x_2,\, \{x_3,\, x_4,\, x_5\}_3\}_3 = {} & \{\{x_1,\, x_2,\, x_3\}_3\,,\, x_4,\, x_5\}_3 + \{x_3,\, \{x_1,\, x_2,\, x_4\}_3\,,\, x_5\}_3 \\
& + \{x_3,\, x_4,\, \{x_1,\, x_2,\, x_5\}_3\}_3 \\
& - \{\Delta(x_1), x_2, x_3, x_4, x_5\}_6 + \{\Delta(x_2), x_1, x_3, x_4, x_5\}_6\,, \\
\{x_1,\, \ldots,\, x_5,\, \{y_1,\, \ldots,\, y_6\}_6\}_6 = {} & \{\{x_1,\, \ldots,\, x_5,\, y_1\}_6\,,\, y_2,\, \ldots,\, y_6\}_6 \\
& + \{y_1,\, \{x_1,\, \ldots,\, x_5,\, y_2\}_6\,,\, y_3,\, \ldots,\, y_6\}_6 \\
& + \{y_1,\, y_2,\, \{x_1,\, \ldots,\, x_5,\, y_3\}_6\,,\, y_4,\, \ldots,\, y_6\}_6 \\
& + \{y_1,\, \ldots,\, y_3,\, \{x_1,\, \ldots,\, x_5,\, y_4\}_6\,,\, y_5,\, y_6\}_6 \\
& + \{y_1,\, \ldots,\, y_4,\, \{x_1,\, \ldots,\, x_5,\, y_5\}_6\,,\, y_6\}_6 \\
& + \{y_1,\, \ldots,\, y_5,\, \{x_1,\, \ldots,\, x_5,\, y_6\}_6\}_6\,,
\end{aligned}
\tag{2.22}
$$

for all $x_1,\, \ldots,\, x_5,\, y_1,\, \ldots,\, y_6 \in \mathfrak{g}^*$, and where we used the Lie bracket on $\mathfrak{g}$ to define the ad-invariant co-product Δ on $\mathfrak{g}^*$

$$
\begin{aligned}
\Delta : \mathfrak{g}^* &\longrightarrow \mathfrak{g}^* \wedge \mathfrak{g}^*\,, \\
ad_x \Delta(y) &= \Delta(ad_x y)\,,\ \forall\, x \in \mathfrak{g},\, y \in \mathfrak{g}^*\,,
\end{aligned}
\tag{2.23}
$$

which is given, assuming a basis $\{T^a\}$ for $\mathfrak{g}^*$, by

$$
\Delta(x_a\, T^a) = \frac{1}{2} f_{bc}{}^a\, x_a\, T^b \wedge T^c\,.
\tag{2.24}
$$

We see that the 6-bracket must satisfy the fundamental identity for Nambu 6-brackets, while the 3-bracket's fundamental identity is modified by the 6-bracket and the co-product defined by the structure constants of $\mathfrak{g}$.

2.2.2 Compatibility conditions

The first set of compatibility conditions, eqs. (2.17) and (2.18), between the 3- and 6-brackets and the Lie algebra $\mathfrak{g}$ imply that $f_b{}^{a_1 a_2 a_3}$ defines a $\mathfrak{g}$-cocycle and that $f_b{}^{a_1 \ldots a_6}$ is an f_3-twisted $\mathfrak{g}$-cocycle, as follows. $f_b{}^{a_1 a_2 a_3}$ and $f_b{}^{a_1 \ldots a_6}$ define $\Lambda^3 \mathfrak{g}$- and $\Lambda^6 \mathfrak{g}$-valued 1-cochains

$$
\begin{aligned}
f_3 : \mathfrak{g} &\longrightarrow \Lambda^3 \mathfrak{g}\,, \\
f_6 : \mathfrak{g} &\longrightarrow \Lambda^6 \mathfrak{g}\,,
\end{aligned}
\tag{2.25}
$$

defined by

$$
\begin{aligned}
f_3(x) &= \frac{1}{3!} x^b\, f_b{}^{a_1 a_2 a_3}\, T_{a_1} \wedge T_{a_2} \wedge T_{a_3}\,,\ \forall\, x = x^a T_a \in \mathfrak{g}\,, \\
f_6(x) &= \frac{1}{6!} x^b\, f_b{}^{a_1 \ldots a_6}\, T_{a_1} \wedge \ldots \wedge T_{a_6}\,,\ \forall\, x = x^a T_a \in \mathfrak{g}\,.
\end{aligned}
\tag{2.26}
$$

Using the coboundary operator $d : \mathfrak{g}^* \otimes \Lambda^p \mathfrak{g} \longrightarrow \Lambda^2 \mathfrak{g}^* \otimes \Lambda^p \mathfrak{g}$, for $p = 3$ and $p = 6$ here,

$$
df_3(x, y) \equiv ad_x f_3(y) - ad_y f_3(x) - f_3([x, y])\,,
\tag{2.27}
$$

$$
df_6(x, y) \equiv ad_x f_6(y) - ad_y f_6(x) - f_6([x, y])\,,
\tag{2.28}
$$

the conditions (2.17) and (2.18) are more elegantly stated as

$$df_3(x, y) = 0, \qquad df_6(x, y) + f_3(x) \wedge f_3(y) = 0. \tag{2.29}$$

The coboundary operator is nilpotent with $d : \Lambda^p \mathfrak{g} \longrightarrow \mathfrak{g}^* \otimes \Lambda^p \mathfrak{g}$ defined as

$$d\rho_p(x) = ad_x \rho_p, \tag{2.30}$$

for all $x \in \mathfrak{g}$ and $\rho_p \in \Lambda^p \mathfrak{g}$. Therefore, the cocycle conditions (2.29) can be solved by the (twisted) coboundaries

$$f_3 = d\rho_3, \qquad f_6 = d\rho_6 + \frac{1}{2}\rho_3 \wedge d\rho_3. \tag{2.31}$$

In components, these are equivalent to

$$\begin{aligned}
f_a{}^{b_1 b_2 b_3} &= 3 f_{ac}{}^{[b_1} \rho^{|c|b_2 b_3]}, \\
f_a{}^{b_1 \dots b_6} &= 6 f_{ac}{}^{[b_1|} \rho^{c|b_2 \dots b_6]} + 30 f_{ac}{}^{[b_1} \rho^{|c|b_2 b_3} \rho^{b_4 b_5 b_6]}.
\end{aligned} \tag{2.32}$$

The coboundary case is related to a generalisation of Yang-Baxter deformations. The trivector $\rho^{a_1 a_2 a_3}$ and the hexavector $\rho^{a_1 \dots a_6}$ correspond to the M-theoretic analogue of the classical r-matrix. The equations corresponding to the classical Yang-Baxter equations for the r-matrices are implied by substituting the solutions (2.32) to the fundamental identities (2.15) and (2.16). We will discuss this further in section 4.

Finally, the additional constraint (2.19) implies that the ad-invariant co-product Δ on $\mathfrak{g}^*$ (2.23) defines a commuting subspace of the 3-bracket:

$$\{\Delta(x_1), x_2\}_3 = 0, \ \forall x_1, x_2 \in \mathfrak{g}^*. \tag{2.33}$$

3 $E_{6(6)}$ EDA from generalised frame fields

We now provide a geometric realisation of the $E_{6(6)}$ EDA by constructing a Leibniz parallelisation [31–36] of the exceptional generalised tangent bundle [37–43]

$$E \cong TM \oplus \Lambda^2 T^* M \oplus \Lambda^5 T^* M, \tag{3.1}$$

in which we identify the manifold $M = G = \exp \mathfrak{g}$. We will also be interested in a second bundle

$$N \cong T^* M \oplus \Lambda^4 T^* M \oplus (T^* M \otimes \Lambda^6 T^* M). \tag{3.2}$$

The action of sections of these bundles,

$$V = v + \nu_2 + \nu_5 \in \Gamma(E), \quad W = w + \omega_2 + \omega_5 \in \Gamma(E), \quad \mathcal{X} = \chi_1 + \chi_4 + \chi_{1,6} \in \Gamma(N), \tag{3.3}$$

is mediated by the generalised Lie derivative [39–41] defined as

$$\mathcal{L}_V W = [v, w] + (L_v \omega_2 - \iota_w d\nu_2) + (L_v \omega_5 - \iota_w d\nu_5 - \omega_2 \wedge d\nu_2), \tag{3.4}$$

$$\mathcal{L}_V \mathcal{X} = L_v \chi_1 + (L_v \chi_4 - \chi_1 \wedge d\nu_2) + (L_v \chi_{1,6} + j\chi_4 \wedge d\nu_2 + j\chi_1 \wedge d\nu_5). \tag{3.5}$$

We define [40] a symmetric bilinear map $\langle \cdot, \cdot \rangle : E \times E \to N$ as

$$\langle V, W \rangle = (\iota_v \omega_2 + \iota_w \nu_2) + (\iota_v \omega_5 - \nu_2 \wedge \omega_2 + \iota_w \nu_5) + (j\nu_2 \wedge \omega_5 + j\omega_2 \wedge \nu_5), \qquad (3.6)$$

such that the generalized Lie derivative satisfies

$$\langle \mathcal{L}_U V, W \rangle + \langle V, \mathcal{L}_U W \rangle = \mathcal{L}_U \langle V, W \rangle \quad \forall U, V, W \in \Gamma(E). \qquad (3.7)$$

The parallelisation consists of a set of sections $E_A \in \Gamma(E)$ that:

- form a globally defined basis for $\Gamma(E)$

- give rise to an $E_{6(6)}$ element,[7] $E_A{}^M$, whose matrix entries are the components of E_A

- realise the algebra of the EDA through the generalised Lie derivative

$$\mathcal{L}_{E_A} E_B = -X_{AB}{}^C E_C, \qquad (3.8)$$

where the constants $X_{AB}{}^C$ are the same as those defined through the relations in eq. (2.13) and obey the Leibniz identity.

The parallelisation can be directly constructed in terms of the right-invariant Maurer-Cartan one-forms on G, r^a, their dual vector fields e_a, and the trivector, $\lambda^{a_1 a_2 a_3}$, and hexavector, $\lambda^{a_1 \ldots a_6}$. This can thus be thought of as a special example of the more general prescription of [44], where we only make use of the aforementioned geometric data on the $(3,6)$-Nambu-Lie Group G. Following the decomposition of EDA generators we write $E_A = \{E_a, E^{a_1 a_2}, E^{a_1 \ldots a_5}\}$ with

$$E_a = e_a, \qquad E^{a_1 a_2} = -\lambda^{a_1 a_2 b} e_b + r^{a_1} \wedge r^{a_2},$$
$$E^{a_1 \ldots a_5} = \left(\lambda^{a_1 \ldots a_5 b} + 5\lambda^{[a_1 a_2 a_3} \lambda^{a_4 a_5] b}\right) e_b - 10 \lambda^{[a_1 a_2 a_3} r^{a_4} \wedge r^{a_5]} + r^{a_1} \wedge \ldots \wedge r^{a_5}. \qquad (3.9)$$

It is straightforward, but indeed quite lengthy, to verify that these furnish the EDA algebra. A first check is to see that after using the identities (2.12) to evaluate derivatives we can go to the identity of M where $\lambda^{a_1 a_2 a_3}$ and $\lambda^{a_1 \ldots a_6}$ vanish. One then has to use the adjoint invariance conditions that follow from eq. (2.20) to conclude that this holds away from the identity.

If we specialise now to the case of $f_b{}^{a_1 \ldots a_6} = 0$, which we recall is enforced for $n \leq 6$ by the fundamental identities, we find quickly an immediate consequence of eq. (2.20) is that $d\lambda^{a_1 \ldots a_6} = 0$, and since $\lambda^{a_1 \ldots a_6}$ vanishes at the identity, it must be identically zero. The remaining adjoint invariance conditions can be combined to imply that

$$f_{ab}{}^c \lambda^{abd} = 0, \quad f_{af}{}^{[b_1|} \lambda^{f|b_2 b_3} \lambda^{b_4 b_5 b_6]} = 0, \quad f_a{}^{[b_1 b_2 b_3} \lambda^{b_4 b_5 b_6]} = 0,$$
$$f_d{}^{b_1 b_2 c} \lambda^{a_1 a_2 d} - 3 f_d{}^{a_1 a_2 [c} \lambda^{b_1 b_2] d} - 3 f_{de}{}^{[b_1} \lambda^{b_2 c] d} \lambda^{a_1 a_2 e} - 3 f_{de}{}^{[a_1} \lambda^{a_2] d[b_1} \lambda^{b_2 c] e} = 0. \qquad (3.10)$$

These conditions are sufficient to ensure that frame algebra is obeyed.

---

[7] An extension of this setup is to allow $E_A{}^M$ to be elements of $E_{6(6)} \times \mathbb{R}^+$, though for simplicity in the presentation we shall demand no $\mathbb{R}^+$ weighting.

We also define the generalized frame field $\mathcal{E}_\mathcal{A}$, which is a section of N, through

$$\langle E_A, E_B \rangle = \eta_{AB}{}^\mathcal{C} \, \mathcal{E}_\mathcal{C} \,, \tag{3.11}$$

where $\eta_{AB}{}^\mathcal{C}$ is an invariant tensor of the $E_{n(n)}$. For $E_{6(6)}$ this tensor is related to the symmetric invariant (see the appendix for details) such that the explicit form of $\mathcal{E}_\mathcal{A}$ has components

$$\mathcal{E}^a = r^a \,, \qquad \mathcal{E}^{a_1 \dots a_4} = 4 \, \lambda^{[a_1 a_2 a_3} \, r^{a_4]} + r^{a_1 \dots a_4} \,,$$
$$\mathcal{E}^{a', a_1 \dots a_6} = (\lambda^{a_1 \dots a_6} \, r^{a'} - 30 \, \lambda^{a'[a_1 a_2} \lambda^{a_3 a_4 a_5} \, r^{a_6]}) - 15 \, \lambda^{a'[a_1 a_2} \, r^{a_3 \dots a_6]} + j r^{a'} r^{a_1 \dots a_6} \,. \tag{3.12}$$

Here we denote $r^{a_1 \dots a_m} = r^{a_1} \wedge \cdots \wedge r^{a_m}$ and make use of the j-wedge contraction of [37] to deal with mixed symmetry fields.[8] One can consider now the action of the frame field E_A on these $\mathcal{E}_\mathcal{A}$ and by virtue of eq. (3.11), again find that they furnish the EDA algebra, albeit in a different representation as described in the appendix.

3.1 Relation to supergravity realisations and dualisations

Whilst the main scope of the present work is to outline the algebraic considerations of the EDA, here we provide the reader a brief orientation as to how the results have utility in supergravity. The most conservative view is that the construction above provides a topdown and explicit means to uplift solutions of particular lower-dimensional gauged supergravities. However, as we will discuss, when an EDA admits multiple isotropic subalgebras, we obtain a notion of duality between solutions of 11-dimensional or maximal 10-dimensional supergravity.

In this work we have mainly considered the $E_{6(6)}$ EDA, $\mathfrak{d}_6$, though the following arguments hold more generally for $\mathfrak{d}_n$. We associate the group manifold G corresponding to the isotropic sub-algbera $\mathfrak{g}$, which for a moment we assume to be of dimension n, with the "internal" sector of a full eleven-dimensional space-time. We thus complement the local coordinates x^i ($i = 1, \dots, n$) on G with a set of "external" coordinates y^μ ($\mu = 0, \dots, 10-n$). In Exceptional Field Theory the internal components of the bosonic content of eleven-dimensional supergravity are packaged into a generalised metric $\mathcal{M}_{IJ}(x, y)$ that parameterizes the coset $E_{n(n)}/H_n$ (here H_n is the maximal compact subgroup of E_n and the indices I, J run over the representation R_1 as described in table 1). Using the generalised frame fields above we can construct a consistent truncation of 11-dimensional supergravity on G, via the *generalised Scherk Schwarz* Ansatz

$$\mathcal{M}_{IJ}(x, y) = E_I{}^A(x) \, E_J{}^B(x) \, \hat{\mathcal{M}}_{AB}(y) \,. \tag{3.13}$$

A sequence of works [31–36] have demonstrated that when such an Ansatz is inserted in the eleven-dimensional equations of motion of ExFT, the equations of motion reduce to

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[8]For a $p+1$-form α and a $(n-p)$-form β, we define

$$(j\alpha \wedge \beta)_{i, i_1 \dots i_n} = \frac{n!}{p!(n-p)!} \alpha_{i[i_1 \dots i_p} \beta_{i_{p+1} \dots i_n]} \,.$$

those of an $(11-n)$-dimensional maximal gauged supergravity, in which the EDA structure constants, $X_{AB}{}^C$, specify the gauge group. The consistency of this procedure is ensured by the frame algebra of eq. (3.8). The $\hat{\mathcal{M}}_{AB}(y)$ are a set of scalar fields, parameterising the coset space $E_{n(n)}/H_n$ in this lower-dimensional theory, see for example [45].[9]

It is fruitful to turn this logic around. Suppose a solution (including, in particular, the profiles of $\hat{\mathcal{M}}_{AB}(y)$) to the lower dimensional supergravity is known. Then the ansatz (3.13) can be used to uplift this solution to one in eleven-dimensions [35, 36]. To complete the process one has to use the well-established dictionary between $\mathcal{M}_{IJ}(x,y)$ and conventional supergravity fields. An intuition for the sorts of internal geometries that arise can be obtained by examining the case when $\hat{\mathcal{M}}_{AB}$ is constant and we refer the reader to recent works for examples of this [17, 20, 21].

An intriguing possibility is that inequivalent sets of frame fields, E_A and $\tilde{E}_A$ say, both give rise to the same frame algebra with structure constants $X_{AB}{}^C$. In this case one would be able to uplift a lower dimensional solution in two inequivalent ways:

$$\mathcal{M}_{IJ}(x,y) = E_I{}^A(x)\,E_J{}^B(x)\,\hat{\mathcal{M}}_{AB}(y)\,, \quad \tilde{\mathcal{M}}_{IJ}(\tilde{x},y) = \tilde{E}_I{}^A(\tilde{x})\,\tilde{E}_J{}^B(\tilde{x})\,\hat{\mathcal{M}}_{AB}(y)\,. \quad (3.14)$$

It is natural to think of this as a duality relation at the level of supergravity. In particular, one solution of the lower-dimensional gauged supergravity is then uplifted to two different solutions in the full eleven-dimensional supergravity and these two solutions can be viewed as being dual to each other. Indeed, when the EDA admits two different isotropic sub-algebras $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ we can systematically construct such a pair E_A and $\tilde{E}_A$ for which the pursuant notion of duality can be considered as Poisson-Lie U-duality.[10] Here the generators of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ must be related by a constant transformation[11] $\mathcal{T}_A{}^B$. The duality relation between $\mathcal{M}_{IJ}$ and $\tilde{\mathcal{M}}_{IJ}$ can then be viewed as arising from the local action of

$$\mathcal{T}_I{}^J(x,\tilde{x}) = \tilde{E}_I{}^A(\tilde{x})\mathcal{T}_A{}^B(E^{-1})^J{}_B(x)\,. \quad (3.15)$$

This view is the U-duality analogue of the considerations in [22–24, 46, 47].

A subtlety here is that the two isotropic sub-algebras $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ could be of dimension n and $n-1$ respectively. The latter (when not trivially obtained as $\mathfrak{g} = \tilde{\mathfrak{g}} + \mathfrak{u}(1)$) are naturally associated to solutions of the ten-dimensional IIB supergravity[12] and the associated duality would be relating M-theory and IIB solutions.

3.1.1 The embedding tensor

In the context of the generalised Scherk-Schwarz reduction described above the EDA structure constants, which specify the gauge group of the lower dimensional theory, are known as the embedding tensor. To facilitate contact with the literature [48] for the particular

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[9]There are also further fields arising in this reduction procedure, for instance those coming from fields with mixed internal/external components that we omit from the present discussion.

[10]It can also be the case that $\tilde{E}_A$ don't apparently follow from an EDA construction, which leads to a wider still notion of duality. This feature was seen explicitly in [21] in which an EDA uplift of certain $CSO(p,q,r)$ gaugings was given complementing the existing uplift of [36].

[11]Including potentially actions in $GL(\dim \mathbf{R_1})$ outside $E_{n(n)}$.

[12]In the jargon of Execptional Field Theory, they correspond to a IIB solution of the section condition.

case of the $n = 6$ EDA we can express this in terms of the $\overline{\mathbf{27}}$ and $\mathbf{351}$ representations of $E_{6(6)}$ as

$$X_{AB}{}^C = d_{ABD}Z^{CD} + 10d_{ADS}d_{BRT}d^{CDR}Z^{ST} - \frac{3}{2}\vartheta_{[A}\delta^C_{B]} - \frac{15}{2}d_{ABD}d^{CDE}\vartheta_E\,. \quad (3.16)$$

The components of the antisymmetric Z^{AB} are determined to be

$$
\begin{aligned}
Z^{ab} &= 0\,, \\
Z_{a_1a_2}{}^b &= -\frac{5}{2\cdot 4!\sqrt{10}}\,f_d{}^{dbc_1...c_4}\,\epsilon_{a_1a_2c_1...c_4}\,(= 0)\,, \\
Z_{a_1...a_5}{}^b &= -\frac{5}{2\sqrt{10}}\,f_d{}^{dbc}\,\epsilon_{a_1...a_5c}\,, \\
Z_{a_1a_2,b_1b_2} &= -\frac{5}{3!\sqrt{10}}\,(f_{[a_1}{}^{c_1c_2c_3}\,\epsilon_{a_2]c_1c_2c_3b_1b_2} - f_{[b_1}{}^{c_1c_2c_3}\,\epsilon_{b_2]c_1c_2c_3a_1a_2})\,, \\
Z_{a_1a_2,b_1...b_5} &= -\frac{5}{2\sqrt{10}}\,(f_{a_1a_2}{}^c\,\epsilon_{cb_1...b_5} - 10f_{[b_1b_2}{}^c\,\epsilon_{b_3b_4b_5]ca_1a_2})\,, \\
Z_{a_1...a_5,b_1...b_5} &= 0\,,
\end{aligned}
\quad (3.17)
$$

and those of ϑ_A (sometimes called the trombone gauging) to be

$$\vartheta_a = \frac{f_{ac}{}^c}{3}\,, \quad \vartheta^{a_1a_2} = -\frac{f_c{}^{ca_1a_2}}{3}\,, \quad \vartheta^{a_1...a_5} = -\frac{f_c{}^{ca_1...a_5}}{3}\,(= 0)\,. \quad (3.18)$$

4 Yang-Baxter-ology

4.1 EDA via ρ-twisting

In the context of DFT, Yang-Baxter deformations can be understood as the $\mathrm{O}(d,d)$ transformation, generated by a bivector, acting on a Drinfel'd double with vanishing dual structure constants [49–54]. The bivector that generates the transformation is then related to the classical r-matrix, the dual structure constants are coboundaries and the requirement that the $\mathrm{O}(d,d)$ transformed algebra is a Drinfel'd double is precisely the classical Yang-Baxter equation.

This suggests a natural generalisation of Yang-Baxter deformations to EDAs [17, 18]. We begin with an EDA $\widehat{\mathfrak{d}}_6$ with only the structure constants, $f_{ab}{}^c$, corresponding to a maximally isotropic Lie subalgebra $\mathfrak{g}$, non-vanishing and $f_a{}^{bcd} = f_a{}^{b_1...b_6} = 0$, i.e.

$$
\begin{aligned}
\hat{T}_a \circ \hat{T}_b &= f_{ab}{}^c\,\hat{T}_c\,, \\
\hat{T}_a \circ \hat{T}^{b_1b_2} &= 2\,f_{ac}{}^{[b_1}\,\hat{T}^{b_2]c}\,, \\
\hat{T}_a \circ \hat{T}^{b_1...b_5} &= -5\,f_{ac}{}^{[b_1}\,\hat{T}^{b_2...b_5]c}\,, \\
\hat{T}^{a_1a_2} \circ \hat{T}_b &= 3\,f_{[c_1c_2}{}^{[a_1}\,\delta^{a_2]}_{b]}\,\hat{T}^{c_1c_2}\,, \\
\hat{T}^{a_1a_2} \circ \hat{T}^{b_1b_2} &= f_{c_1c_2}{}^{[a_1}\,\hat{T}^{a_2]b_1b_2c_1c_2}\,, \\
\hat{T}^{a_1a_2} \circ \hat{T}^{b_1...b_5} &= 0\,,
\end{aligned}
$$

$$\hat{T}^{a_1 \dots a_5} \circ \hat{T}_b = 5\, f_{bc}{}^{[a_1}\, \hat{T}^{a_2 \dots a_5]c} + 10\, f_{c_1 c_2}{}^{[a_1}\, \delta_b^{a_2}\, \hat{T}^{a_3 a_4 a_5]c_1 c_2} \,,$$
$$\hat{T}^{a_1 \dots a_5} \circ \hat{T}^{b_1 b_2} = 0 \,,$$
$$\hat{T}^{a_1 \dots a_5} \circ \hat{T}^{b_1 \dots b_5} = 0 \,. \tag{4.1}$$

We denote the structure constants collectively as $\hat{X}_{AB}{}^C$.

We now perform an $E_{6(6)}$ transformation of the above EDA by a trivector, ρ^{abc}, and hexavector, $\rho^{a_1 \dots a_6}$, which will play the analogue of the classical r-matrix. The corresponding $E_{6(6)}$ group element is given by

$$C_A{}^B \equiv \left(e^{\frac{1}{6!}\, \rho^{a_1 \dots a_6}\, R_{a_1 \dots a_6}} e^{\frac{1}{3!}\, \rho^{a_1 a_2 a_3}\, R_{a_1 a_2 a_3}} \right)_A{}^B \,, \tag{4.2}$$

in which the generators $R_{a_1 a_2 a_3}$ and $R_{a_1 \dots a_6}$ are specified in the appendix. Explicitly we have that

$$(C_A{}^B) = \begin{pmatrix} \delta_a^b & 0 & 0 \\ \frac{\rho^{b a_1 a_2}}{\sqrt{2!}} & \delta_{b_1 b_2}^{a_1 a_2} & 0 \\ \frac{\tilde{\rho}^{b; a_1 \dots a_5}}{\sqrt{5!}} & \frac{20\, \delta_{b_1 b_2}^{[a_1 a_2}\, \rho^{a_3 a_4 a_5]}}{\sqrt{2!\, 5!}} & \delta_{b_1 \dots b_5}^{a_1 \dots a_5} \end{pmatrix} \,, \tag{4.3}$$

where

$$\tilde{\rho}^{b; a_1 \dots a_5} \equiv \rho^{b a_1 \dots a_5} + 5\, \rho^{b[a_1 a_2}\, \rho^{a_3 a_4 a_5]} \,. \tag{4.4}$$

Equivalently, we twist the generators by the group element (4.2) resulting in

$$T_a = \hat{T}_a \,, \qquad T^{a_1 a_2} = \hat{T}^{a_1 a_2} + \rho^{b a_1 a_2}\, \hat{T}_b \,,$$
$$T^{a_1 \dots a_5} = \hat{T}^{a_1 \dots a_5} + 10\, \rho^{[a_1 a_2 a_3}\, \hat{T}^{a_4 a_5]} + \tilde{\rho}^{b; a_1 \dots a_5}\, \hat{T}_b \,. \tag{4.5}$$

For the twisted generators, we obtain $T_A \circ T_B = X_{AB}{}^C\, T_C$ with

$$X_{AB}{}^C \equiv C_A{}^D\, C_B{}^E\, (C^{-1})_F{}^C\, \hat{X}_{DE}{}^F \,. \tag{4.6}$$

We now require that the new algebra defines an EDA $\mathfrak{d}_6$. This imposes conditions on ρ^{abc} and $\rho^{a_1 \dots a_6}$ and we will interpret these as analogues of the classical Yang-Baxter equation. From the products $T_a \circ T^{b_1 b_2}$ and $T_a \circ T^{b_1 \dots b_5}$, the dual structure constants are identified as

$$f_a{}^{b_1 b_2 b_3} = 3\, f_{ac}{}^{[b_1}\, \rho^{|c|b_2 b_3]} \,, \qquad f_a{}^{b_1 \dots b_6} = 6\, f_{ac}{}^{[b_1|}\, \tilde{\rho}^{c; |b_2 \dots b_6]} (= 0) \,, \tag{4.7}$$

which take the form of (twisted) coboundaries (2.32) and the $(= 0)$ holds for $\mathfrak{d}_6$. The remaining products impose a number of conditions, of which the following is particularly intriguing

$$3\, f_{d_1 d_2}{}^{[b_1}\, \rho^{b_2 b_3]d_1}\, \rho^{a_1 a_2 d_2} = f_{d_1 d_2}{}^{[a_1}\, \tilde{\rho}^{a_2]; b_1 b_2 b_3 d_1 d_2} \,. \tag{4.8}$$

This is a natural generalisation of the classical Yang-Baxter (YB) equation which we will elaborate more on later. In general, we get a further set of conditions which are

required to ensure that the new algebra defines an EDA $\mathfrak{d}_6$. Mostly these additional conditions appear rather cumbersome but we note the requirement that

$$\rho^{a_1 a_2 b} f_{a_1 a_2}{}^c = 0 \,. \tag{4.9}$$

With $f_a{}^{b_1 \dots b_6} = 0$, the Bianchi identity for $f_{ab}{}^c$ together with the generalised Yang-Baxter equation eq. (4.8) and compatibility condition eq. (4.9) imply the fundamental identity for $f_a{}^{b_1 \dots b_3}$. Indeed since the Leibniz identity (2.2) is $E_{6(6)}$-invariant, it is guaranteed to hold for $X_{AB}{}^C$. Therefore, we see that the generalised Yang-Baxter equation (4.8) together with the other conditions obtained by imposing that the new algebra is an EDA imply that the new dual structure constants satisfy their fundamental identities (2.15) and (2.16) and the condition (2.19).

In [25], a different approach was taken using a generalisation of the open/closed string map to propose a generalisation of the classical YB equation for a trivector deformation of 11-dimensional supergravity. The approach of [25] is not limited to group manifolds, unlike the present case, but also only considers trivector deformations. However, when specialising [25] to group manifolds and considering our deformations with $\rho^{a_1 \dots a_6} = 0$, the resulting equation of [25] is different and, in particular, weaker than the YB equation we find here (4.8) with $\rho^{a_1 \dots a_6} = 0$, or indeed the SL(5) case discussed in [17, 18]. Indeed, as shown in [26] based on explicit examples, the proposed YB-like equation of [25] is not sufficient to guarantee a solution of the equations of motion of 11-dimensional supergravity, while our deformations subject to the above conditions preserve the equations of motion of 11-dimensional supergravity by construction.

4.2 Nambu 3- and 6-brackets from ρ-twisting

The trivector $\rho^{a_1 a_2 a_3}$ and hexavector $\rho^{a_1 \dots a_6}$ define 3- and 6-brackets via (2.32). First define the maps

$$\rho_3 : \mathfrak{g}^* \wedge \mathfrak{g}^* \longrightarrow \mathfrak{g} \,, \qquad \tilde{\rho}_6 : \Lambda^5 \mathfrak{g}^* \longrightarrow \mathfrak{g} \,, \tag{4.10}$$

as

$$\begin{aligned} \rho_3(x_1, x_2) &= \rho^{abc} \, (x_1)_b \, (x_2)_c \, T_a \,, \\ \tilde{\rho}_6(x_1, \dots, x_5) &= \tilde{\rho}^{a_1; a_2 \dots a_6} \, (x_1)_{a_2} \dots (x_5)_{a_6} \, T_{a_1} \,, \ \forall \ x_1, \, \dots, \, x_5 \in \mathfrak{g}^* \,. \end{aligned} \tag{4.11}$$

This allows the generalised Yang-Baxter equation eq. (4.8) to be cast in a basis independent way

$$x_1 \left(\left(ad_{\rho_3(y_1, y_2)} \rho_3 \right) (x_2, x_3) \right) + y_1 \left(\tilde{\rho}_6(\Delta(y_2), x_1, x_2, x_3) \right) - y_2 \left(\tilde{\rho}_6(\Delta(y_1), x_1, x_2, x_3) \right) = 0 \,, \tag{4.12}$$

for all $y_1, y_2, x_1, x_2, x_3 \in \mathfrak{g}^*$. Note that the first term in (4.12) is automatically antisymmetric in (x_1, x_2, x_3) due to antisymmetry of $\rho^{a_1 a_2 a_3}$.

Then, the associated 3- and 6-brackets are defined as

$$\begin{aligned} \{x_1, x_2, x_3\}_3 &= ad_{\rho_3(x_1, x_2)} x_3 + ad_{\rho_3(x_2, x_3)} x_1 + ad_{\rho_3(x_3, x_1)} x_2 \,, \\ \{x_1, \dots, x_6\}_6 &= ad_{\tilde{\rho}_6(x_1, \dots, x_5)} x_6 + \text{cyclic permutations} \,, \end{aligned} \tag{4.13}$$

for all $x_1, \ldots, x_6 \in \mathfrak{g}^*$. Alternatively, to match more with the usual discussion of classical r-matrices in integrability, we can use the Cartan-Killing form on $\mathfrak{g}$ to define 3- and 6-brackets on $\mathfrak{g}$. For this, it is more convenient to define ρ_3' and ρ_6' as

$$\rho_3' : \mathfrak{g} \wedge \mathfrak{g} \longrightarrow \mathfrak{g}, \qquad \tilde{\rho}_6' : \Lambda^5 \mathfrak{g} \longrightarrow \mathfrak{g}, \tag{4.14}$$

with

$$\rho_3'(x_1, x_2) = \rho_3(\kappa(x_1), \kappa(x_2)), \qquad \tilde{\rho}_6'(x_1, \ldots, x_6) = \tilde{\rho}_6(\kappa(x_1), \ldots, \kappa(x_5)), \tag{4.15}$$

and where κ is the Cartan-Killing metric viewed as a map $\kappa : \mathfrak{g} \longrightarrow \mathfrak{g}^*$. Now, the 3- and 6-brackets on $\mathfrak{g}$ are defined as

$$\begin{aligned}\{x_1, x_2, x_3\}_3 &= [x_1, \rho_3'(x_2, x_3)] + [x_2, \rho_3'(x_3, x_1)] + [x_3, \rho_3'(x_1, x_2)], \\ \{x_1, \ldots, x_6\}_6 &= [x_1, \tilde{\rho}_6'(x_2, \ldots, x_6)] + \text{cyclic permutations}.\end{aligned} \tag{4.16}$$

The generalised Yang-Baxter equation (4.8) together with the other constraints required such that the new algebra is an EDA, such as (4.9), imply that the 3- and 6-brackets defined above in (4.13) and (4.16) satisfy their fundamental identities (2.15) and (2.16).

4.3 The generalised YB equation

To understand better the generalised YB equation obtained above, let us adopt a tensor product notation $\rho_{124} = \rho^{abc} T_a \otimes T_b \otimes 1 \otimes T_c \otimes 1$ etc. such that the indices denote the contracted slots in a tensor product of $\mathfrak{g}$. Assuming that (4.9) holds and that $\rho^{a_1 \cdots a_6} = 0$ we have that (4.8) becomes

$$\begin{aligned}&[\rho_{123}, \rho_{145}] + [\rho_{123}, \rho_{245}] + [\rho_{123}, \rho_{345}] \\ &+ \frac{1}{2} \big([\rho_{124} + \rho_{125}, \rho_{345}] + [\rho_{234} + \rho_{235}, \rho_{145}] + [\rho_{314} + \rho_{315}, \rho_{245}]\big) = 0.\end{aligned} \tag{4.17}$$

Introducing a (anti-)symmetrizer in the tensor product $\sigma_{[123],[45]}$, allows this equation to be concisely given as

$$\sigma_{[123],[45]}[\rho_{123} + \rho_{234}, \rho_{145}] = 0. \tag{4.18}$$

Suppose that we have a preferred $q \in \mathfrak{g}$ such that

$$\rho_{123} = r_{12} \otimes q_3 + r_{23} \otimes q_1 - r_{13} \otimes q_2, \quad r_{12} = \sum_{a,b \neq q} r^{ab} T_a \otimes T_b = -r_{21}, \tag{4.19}$$

with r_{12} neutral (i.e. $[r_{12}, q_1] = 0$) then we find eq. (4.17) becomes

$$\mathtt{YB}_{[12|4|} \otimes q_{3]} \otimes q_5 - \mathtt{YB}_{[12|5|} \otimes q_{3]} \otimes q_4 = 0, \tag{4.20}$$

in which

$$\mathtt{YB}_{123} = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}], \tag{4.21}$$

is the classical Yang-Baxter equation for r.

Recall that the classical YB equation arises from the quantum one

$$\mathbb{R}_{13}\mathbb{R}_{12}\mathbb{R}_{32} = \mathbb{R}_{32}\mathbb{R}_{12}\mathbb{R}_{13} \tag{4.22}$$

as the leading terms in the 'classical' expansion $\mathbb{R}_{12} = 1 + \hbar\, r_{12} + O(\hbar^2)$. An obvious question is if there is an equivalent 'quantum' version of eq. (4.18)? We give here one proposal (with no claim of first principle derivation or uniqueness) for such a starting point. Let us define[13] in the semi-classical limit

$$\mathbb{R}_{i;jk} = 1 + \hbar\, \rho_{ijk} + O(\hbar^2)\,, \tag{4.23}$$

$$\mathbb{R}_{ij;kl} = 1 + \frac{\hbar}{4}\left(\rho_{ijk} + \rho_{ijl} - \rho_{ikl} - \rho_{jkl}\right) + O(\hbar^2)\,. \tag{4.24}$$

Then eq. (4.17) follows from

$$\sigma_{[123],[45]}\mathbb{R}_{1;23}\mathbb{R}_{23;45}\mathbb{R}_{1;45} = \sigma_{[123],[45]}\mathbb{R}_{1;45}\mathbb{R}_{23;45}\mathbb{R}_{1;23}\,. \tag{4.25}$$

This view point is very suggestive that this may just represent a standard Yang-Baxter equation for the scattering of $\wedge^2\mathfrak{g}, \wedge^2\mathfrak{g}$ and $\mathfrak{g}$ obtained by S-matrix fusion. Here we leave an exploration of this as an open direction; further work is required to understand which quantum R-matrices give rise under S-matrix fusion to an $\mathbb{R}_{i;jk}$ and $\mathbb{R}_{ij;kl}$ with the expansion (4.23) and what are the resultant ρ_{ijk}. Conversely one might ask if there exist solutions of (4.25) compatible (4.23) but that are not obtained from fusion?

Restoring $\rho^{a_1\cdots a_6}$ we can amend this equation to

$$\sigma_{[123],[45]}[\rho_{123} + \rho_{234}, \rho_{145}] = \frac{1}{2}\left(\rho_{12345;5} + \rho_{12345;4}\right)\,, \tag{4.26}$$

where, for example,

$$\rho_{12345;5} \equiv \rho^{abcdef}\, T_a \otimes T_b \otimes T_c \otimes T_d \otimes [T_e, T_f]\,. \tag{4.27}$$

This is somewhat suggestive of a contact term in the YB relation that may lead to a quantum version of the form

$$\sigma_{[123],[45]}\mathbb{R}_{1;23}\mathbb{R}_{23;45}\mathbb{R}_{1;45} - \sigma_{[123],[45]}\mathbb{R}_{1;45}\mathbb{R}_{23;45}\mathbb{R}_{1;23} = \sigma_{[123],[45]}\mathbb{R}_{12345;5}\,. \tag{4.28}$$

Pictographically this is indicated in figure 1.

5 Examples

In this section we wish to present a range of examples of the EDA, both of coboundary type and otherwise. We will give some broad general classes that correspond to embedding the algebraic structure underlying existing T-dualities of the type II theory. In addition, in the absence of a complete classification, here we provide a selection of specific examples.

---

[13]We use lower case Roman indices to denote tensor product locations.
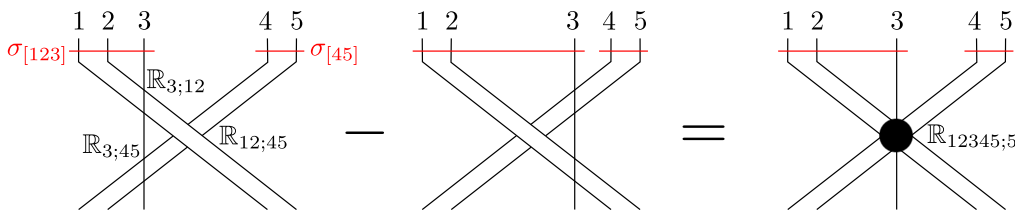
Figure 1. A proposed schematic for the generalised Yang-Baxter equation. The red lines indicate anti-symmetrisation and the black circle is a contact term that gives rise in the semi-classical limit to a contribution involving ρ_6.

5.1 Abelian

When the subalgebra $\{T_a\}$ is Abelian, arbitrary $\rho^{a_1 a_2 a_3}$ and $\rho^{a_1 \cdots a_6}$ are solutions of Yang-Baxter-like equations. However $f_a{}^{b_1 b_2 b_3} = 0$ and $f_a{}^{b_1 \cdots b_6} = 0$ and the EDA is Abelian.

5.2 Semi-Abelian EDAs and three algebras

The algebraic structure corresponding to non-Abelian T-duality is a semi-abelian Drinfel'd double i.e. a double constructed from some $n-1$ dimensional Lie-algebra (representing the non-Abelian isometry group of the target space) together with a $U(1)^{n-1}$ (or perhaps $\mathbb{R}_+^{n-1}$) factor. An analogue here would be to take $f_{ab}{}^c \neq 0$ and $f_a{}^{b_1 \cdots b_3} = 0$, this however is not especially interesting. More intriguing is to consider the analogue of the picture *after* non-Abelian T-dualisation has been performed in which the $U(1)^{n-1}$ would be viewed as the physical space. This motivates the case of semi-Abelian EDAs with $f_{ab}{}^c = 0$ but $f_a{}^{b_1 \cdots b_3} \neq 0$.

In this case the Leibniz identities reduce to the fundamental identities

$$f_a{}^{dc_1 c_2} f_d{}^{b_1 b_2 b_3} - 3 f_d{}^{c_1 c_2 [b_1} f_a{}^{b_2 b_3] d} = 0, \qquad f_a{}^{b_1 \cdots b_6} = 0. \tag{5.1}$$

Each solution for this identity gives an EDA. To identify these one can use existing classification efforts and considerations of three algebras that followed in light of their usage [55] to describe theories of interacting multiple M2 branes.

The first case to consider are the Euclidean three algebras, such that $f^{b_1 \cdots b_4} = f_a{}^{b_1 \cdots b_3} \delta^{ab_4}$ is totally antisymmetric. Here the fundamental identity is very restrictive and results in a unique possibility: the four-dimensional Euclidean three algebra [56–58], whose structure constants are just the antisymmetric symbol, complemented with two $U(1)$ directions. Relaxing the requirement of a positive definite invariant inner product allows a wider variety [59–64]. Dispensing the requirement of an invariant inner product (which thus far appears unimportant for the EDA) allows non-metric three algebras [65–67].[14]

5.3 r-matrix EDAs

We now consider coboundary EDAs given in terms of an r-matrix as in eq. (4.19) obeying the YB equation (4.21). Splitting the generators of $\mathfrak{g}$ into $T_{\bar{a}}$ with $\bar{a} = 1, \ldots, 5$ and T_6

---

[14]In addition there are three algebra structures [68, 69] in which $f_d{}^{abc}$ is not totally antisymmetric in its upper indices. These can be used to describe interacting 3d theories with lower supersymmetry. It is a unclear if they could play role in the context of EDAs.

(identified with the generator q appearing in (4.19)) we have the non-vanishing components $\rho^{\bar{a}\bar{b}6} = r^{\bar{a}\bar{b}}$. Furthermore the condition (4.9) requires that

$$r^{\bar{a}\bar{b}} f_{\bar{a}\bar{b}}{}^6 = r^{\bar{a}\bar{b}} f_{\bar{a}\bar{b}}{}^{\bar{c}} = r^{\bar{a}\bar{b}} f_{\bar{a}6}{}^{\bar{c}} = r^{\bar{a}\bar{b}} f_{\bar{a}6}{}^6 = 0 \,, \tag{5.2}$$

in which the last two equalities match the statement that r is neutral under T_6. In such a setup, the dual structure constants are specified as

$$f_{\bar{a}}{}^{\bar{b}_1\bar{b}_2 6} = 2 f_{\bar{a}\bar{c}}{}^{[\bar{b}_1} r^{|\bar{c}|\bar{b}_2]} + f_{\bar{a}6}{}^6 r^{\bar{b}_1\bar{b}_2} \,, \quad f_{\bar{a}}{}^{\bar{b}_1\bar{b}_2\bar{b}_3} = 3 f_{\bar{a}6}{}^{[\bar{b}_1} r^{\bar{b}_2\bar{b}_3]} \,, \quad f_6{}^{\bar{b}_1\bar{b}_2 6} = f_6{}^{\bar{b}_1\bar{b}_2\bar{b}_3} = 0 \,. \tag{5.3}$$

Assuming further that $\bar{\mathfrak{g}} = \mathrm{span}(T_{\bar{a}})$ is a sub-algebra of $\mathfrak{g}$ then $r^{\bar{a}\bar{b}}$ defines an r-matrix on $\bar{\mathfrak{g}}$ obeying the YB equation. Consequently $\tilde{f}^{\bar{b}_1\bar{b}_2}{}_{\bar{a}} = -2 f_{\bar{a}\bar{c}}{}^{[\bar{b}_1} r^{|\bar{c}|\bar{b}_2]}$ are the structure constants of a dual Lie algebra $\bar{\mathfrak{g}}_R$ and $\bar{\mathfrak{d}} = \bar{\mathfrak{g}} \oplus \bar{\mathfrak{g}}_R$ is a Drinfel'd double. Thus we have a family of embeddings of the Drinfel'd double into the EDA specified by $f_{\bar{a}6}{}^6$ and $f_{\bar{a}6}{}^{\bar{b}}$. When $\mathfrak{g} = \bar{\mathfrak{g}} \oplus u(1)$ is a direct sum (such that $f_{\bar{a}6}{}^6 = f_{\bar{a}6}{}^{\bar{b}} = 0$), then this is precisely an example of the non-metric three algebra of [65–67]. We emphasise though that not every (coboundary) double can be embedded in this way; one must still ensure that equation (5.2) holds.

5.4 Explicit examples

We now present a selection of explicit examples that illustrate coboundary and non-coboundary EDAs.

5.4.1 Trivial non-examples based on $\mathbf{SO}(p, q)$

To illustrate that the EDA requirements are indeed quite restrictive we can first consider the case of $\mathfrak{g} = \mathfrak{so}(p, q)$ with $p + q = 4$. A direct consideration of the Leibniz identities reveals that there is no non-zero solution for $f_a{}^{b_1...b_3}$ (in fact the cocycle conditions alone determine this). Equally the Leibniz identities admit only trivial solutions in the case of $\mathfrak{iso}(p, q) = \mathfrak{so}(p, q) \ltimes \mathbb{R}_+^{p+q}$ with $p + q = 3$.

5.4.2 An example both coboundary and non-coboundary solutions

We consider an indecomposable nilpotent Lie algebra $N_{6,22}$ of [70] specified by the structure constants[15]

$$f_{12}{}^3 = 1 \,, \quad f_{13}{}^5 = 1 \,, \quad f_{15}{}^6 = c_0 \,, \quad f_{23}{}^4 = 1 \,, \quad f_{24}{}^5 = 1 \,, \quad f_{34}{}^6 = c_0 \,. \tag{5.4}$$

We find a family of solutions

$$\begin{aligned}
f_1{}^{356} &= d_1 \,, & f_1{}^{456} &= d_2 \,, & f_2{}^{156} &= d_3 \,, & f_2{}^{346} &= -d_3 \,, \\
f_2{}^{356} &= d_4 \,, & f_2{}^{456} &= d_5 \,, & f_3{}^{456} &= -d_1 + d_3 \,,
\end{aligned} \tag{5.5}$$

which indeed satisfies the closure constraints. In particular, if we choose $c_0 = 0$, we can clearly see that this EDA contains a 10D Drinfel'd double $\{T_{\bar{a}}, T^{\bar{a}6}\}$ $(\bar{a} = 1, \ldots, 5)$ as a Lie subalgebra.

---

[15] Here we introduced a parameter c_0 for convenience, which is 1 in [70].

We can also find ρ^{abc} by considering the coboundary Ansatz. Supposing $c_0 \neq 0$, the general solution to the generalised Yang-Baxter equation and compatibility condition is

$$\rho^{146} = e_1\,, \quad \rho^{156} = e_2\,, \quad \rho^{256} = e_3\,, \quad \rho^{346} = -e_2\,, \quad \rho^{356} = e_4\,, \quad \rho^{456} = e_5\,, \tag{5.6}$$

where $e_1 = 0$ or $e_3 = 0$. The corresponding structure constants are

$$
\begin{aligned}
&f_1{}^{356} = e_3\,, &&f_1{}^{456} = e_2\,, &&f_2{}^{156} = e_1\,, &&f_2{}^{346} = -e_1\,, \\
&f_2{}^{356} = -2e_2\,, &&f_2{}^{456} = e_4\,, &&f_3{}^{456} = e_1 - e_3\,.
\end{aligned} \tag{5.7}
$$

This means that only when d_i $(i = 1, \ldots, 5)$ have the form

$$d_1 = e_3\,, \quad d_2 = e_2\,, \quad d_3 = e_1\,, \quad d_4 = -2e_2\,, \quad d_5 = e_4\,, \tag{5.8}$$

and satisfy $d_1 = 0$ or $d_3 = 0$, the cocycle becomes the coboundary.

5.4.3 An example with ρ^6

In the previous example ρ^6 is absent. By considering the Lie algebra of the form $\mathfrak{g} = \mathfrak{g}_4 \oplus u(1) \oplus u(1)$, where $\mathfrak{g}_4$ denotes a real 4D Lie algebra that is classified in [71], one can construct a number of examples[16] (based on unimodular Lie algebras) that admit ρ^6. To illustrate this let us consider the case that $\mathfrak{g}_4 = A_{4,1}$ specified by structure constants

$$f_{24}{}^1 = 1\,, \quad f_{34}{}^2 = 1\,. \tag{5.9}$$

We find the generalised Yang-Baxter and compatibility equations admit the following family of solutions:

$$
\rho^{123} = d_1\,, \quad \rho^{125} = d_2\,, \quad \rho^{126} = d_3\,, \quad \rho^{135} = \frac{d_1 d_4 d_8}{2d_0}\,, \quad \rho^{136} = \frac{d_1 d_5 d_8}{2d_0}\,, \quad \rho^{145} = d_4\,,
$$

$$
\rho^{146} = d_5\,, \quad \rho^{156} = d_6\,, \quad \rho^{256} = d_7\,, \quad \rho^{356} = d_8\,, \quad \rho^{456} = \frac{2d_0}{d_1}\,, \quad \rho^{123456} = d_0\,. \tag{5.10}
$$

The corresponding dual structure constants are

$$
\begin{aligned}
&f_2{}^{156} = \frac{2d_0}{d_1}\,, &&f_3{}^{125} = d_4\,, &&f_3{}^{126} = d_5\,, &&f_3{}^{256} = \frac{2d_0}{d_1}\,, \\
&f_4{}^{125} = -\frac{d_1 d_4 d_8}{2d_0}\,, &&f_4{}^{126} = -\frac{d_1 d_5 d_8}{2d_0}\,, &&f_4{}^{156} = -d_7\,, &&f_4{}^{256} = -d_8\,.
\end{aligned} \tag{5.11}
$$

5.4.4 An r-matrix EDA

In order to find a non-trivial example of the r-matrix EDAs, we consider a solvable Lie algebra $N_{6,29}^{\alpha\beta}$ of [72] defined by structure constants:

$$f_{13}{}^3 = 1\,, \quad f_{15}{}^5 = \alpha\,, \quad f_{16}{}^6 = 1\,, \quad f_{23}{}^3 = 1\,, \quad f_{24}{}^4 = 1\,, \quad f_{25}{}^5 = \beta\,, \quad f_{46}{}^3 = -1\,, \tag{5.12}$$

---

[16]In the notation of [71] examples with $\rho^6 \neq 0$ are found when $\mathfrak{g}_4$ is one of the following: $A_{3,1} + \mathfrak{u}(1)$, $A_{3,4}^{-1} + \mathfrak{u}(1)$, $A_{3,5}^0 + \mathfrak{u}(1)$, $A_{4,1}$, $A_{4,2}^{-2}$, $A_{4,5}^{a,b,-(a+b)}$, $A_{4,6}^{-2b,b}$.

where $\alpha^2 + \beta^2 \neq 0$. This algebra contains a subalgebra generated by $\bar{\mathfrak{g}} = \text{span}(T_{\bar{a}})$ ($\bar{a} = 1, \ldots, 5$) and is non-trivial in the sense that it satisfies $f_{\bar{a}6}{}^{\bar{b}} \neq 0$ and $f_{\bar{a}6}{}^6 \neq 0$.

Supposing $\alpha\beta \neq 0$, we find the general solution for ρ^{abc} is given by

$$\rho^{135} = c_1, \quad \rho^{235} = -c_1, \quad \rho^{356} = c_2, \quad \rho^{345} = c_3, \tag{5.13}$$

where $c_1 = 0$ when $\alpha \neq \beta$. The corresponding dual structure constants are

$$
\begin{aligned}
&f_1{}^{135} = (1+\alpha)c_1, &&f_1{}^{235} = -(1+\alpha)c_1, &&f_1{}^{345} = (1+\alpha)c_3, &&f_1{}^{356} = (2+\alpha)c_2, \\
&f_2{}^{135} = (1+\beta)c_1, &&f_2{}^{235} = -(1+\beta)c_1, &&f_2{}^{345} = (2+\beta)c_3, &&f_2{}^{356} = (1+\beta)c_2, \\
&f_4{}^{345} = -c_1, &&f_6{}^{356} = -c_1.
\end{aligned}
$$
$$\tag{5.14}$$

This solution contains an r-matrix EDA as a particular case $c_1 = c_3 = 0$.

6 Conclusion and outlook

In this work we have consolidated the exploration of exceptional Drinfel'd algebras introduced in [17, 18] extending the construction to the context of the $E_{6(6)}$ exceptional group. The algebraic construction here requires the introduction of a new feature: we have to consider not only a Lie algebra $\mathfrak{g}$ together with a three-algebra specified by $f_3 \equiv f_a{}^{b_1 \ldots b_3}$ as in [17, 18], but we have to also include a six-algebra $f_6 \equiv f_a{}^{b_1 \ldots b_6}$. The Leibniz identities that the EDA must obey enforce a set of fundamental (Jacobi-like) identities for the three- and six-algebra as well as some compatibility conditions. These compatibility conditions require that f_3 be a $\mathfrak{g}$-cocycle and f_6 be an f_3-twisted $\mathfrak{g}$-cocycle. In terms of the $\mathfrak{g}$ coboundary operator d this can be stated as

$$df_3 = 0, \quad df_6 + f_3 \wedge f_3 = 0. \tag{6.1}$$

We can solve this requirement with a coboundary Ansatz, $f_3 = d\rho_3$ and $f_6 = d\rho_6 + \frac{1}{2}\rho_3 \wedge d\rho_3$, reminiscent of the way a Drinfel'd double can be constructed through an r-matrix. Indeed, we find a generalised version for the Yang Baxter equation for ρ_3, concisely expressed as

$$\sigma_{[123],[45]}[\rho_{123} + \rho_{234}, \rho_{145}] = \frac{1}{2}\left(\rho_{12345;5} + \rho_{12345;4}\right). \tag{6.2}$$

We proposed a 'quantum' relation from which this classical equation can be obtained. This feature, and the resultant interplay between one-, three-, and six-algebras, opens up many interesting avenues for further exploration.

The construction of the EDA is closely motivated by considerations within exceptional generalised geometry. We have shown how the EDA can be realised as a generalised Leibniz parallelisation of the exceptional generalised tangent bundle of a group manifold G. The data required to construct this mean that G is equipped with a 3-bracket and a 6-bracket which invites the consideration of Nambu-Lie groups.

Now we come to solving the various constraint equations that govern the structure of the EDA. The first thing to note is that due to the dimension, the only solutions to the

fundamental identities have vanishing f_6 (and consequently a trivial 6-bracket on G). We believe however that in higher dimension this condition is less stringent and that there will solutions for which the structure described above is exhibited in full.

We then provided a range of examples that illustrate the various features here. We have examples with and without Drinfel'd double subalgebras, and examples that are both of coboundary type (specified by a ρ_3 and ρ_6) and not of coboundary type. All of the coboundary examples presented here (and indeed in all the numerous other examples we have found) can be obtained from the procedure of ρ-twisting i.e. starting with a semi-Abelian EDA and applying an $E_{6(6)}$ transformation parametrised by the ρ_3 and ρ_6. Despite the dimensionality induced restriction to $f_6 = 0$, there are examples for which $\rho_6 \neq 0$. We provide examples where ρ_3 can be parametrised in terms of a Yang-Baxter r-matrix for a lower dimensional algebra, as well as where this is not the case.

There are several exciting open directions here that we share in the hope that others may wish to develop them further:

- Extensions of the EDA to $E_{7(7)}$ and higher are likely to shed further light on the structures involved. As the space gets larger there is more scope to find interesting solutions.

- It would be interesting to develop a more general classification of EDA solutions.

- One feature of the EDA is that they may admit multiple decompositions into physical spaces, and a resultant notion of duality. Further development should go into this very interesting aspect.

- Here we make some robust requirements that result in structures compatible with maximally supersymmetric gauged supergravities. It would likely be interesting to see how the requirements of the EDA can be consistently relaxed to lower supersymmetric settings, for example using [73].

- On a mathematical note perhaps the most intriguing area of all is to develop the 'quantum' equivalent of the classical EDA proposed here.

Acknowledgments

A $E_{6(6)} \times \mathbb{R}^+$ algebra and conventions

The matrix representations of the $E_{n(n)}$ generators $(t_\alpha)_A{}^B$ in the R_1-representation are

$$K^c{}_d \equiv \begin{pmatrix} \delta_a^c \, \delta_d^b & 0 & 0 \\ 0 & -2\, \delta_{[b_1|d|}^{a_1 a_2} \, \delta_{b_2]}^c & 0 \\ 0 & 0 & -5\, \delta_{[b_1...b_4|d|}^{a_1...a_5} \, \delta_{b_5]}^c \end{pmatrix} + \frac{\delta_d^c}{9-n}\, \mathbf{1}\,,$$

$$R^{c_1 c_2 c_3} \equiv \begin{pmatrix} 0 & -\frac{3!\, \delta_{ab_1 b_2}^{c_1 c_2 c_3}}{\sqrt{2!}} & 0 \\ 0 & 0 & -\frac{5!\, \delta_{b_1...b_5}^{a_1 a_2 c_1 c_2 c_3}}{\sqrt{5!\,2!}} \\ 0 & 0 & 0 \end{pmatrix}, \qquad R^{c_1...c_6} \equiv \begin{pmatrix} 0 & 0 & -\frac{6!\, \delta_{ab_1...b_5}^{c_1...c_6}}{\sqrt{5!}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$R_{c_1 c_2 c_3} \equiv \begin{pmatrix} 0 & 0 & 0 \\ \frac{3!\, \delta_{c_1 c_2 c_3}^{b a_1 a_2}}{\sqrt{2!}} & 0 & 0 \\ 0 & \frac{5!\, \delta_{b_1 b_2 c_1 c_2 c_3}^{a_1...a_5}}{\sqrt{2!\,5!}} & 0 \end{pmatrix}, \qquad R_{c_1...c_6} \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{6!\, \delta_{c_1...c_6}^{b a_1...a_5}}{\sqrt{5!}} & 0 & 0 \end{pmatrix}, \tag{A.1}$$

and that of the $\mathbb{R}^+$ generator is simply $(t_0)_A{}^B = -\delta_A^B$.

The non-vanishing components of symmetric invariant tensor of $E_{6(6)}$, d^{ABC}, are

$$d^a{}_{b_1 b_2 c_1...c_5} = \frac{2!\, \delta_{[b_1}^a\, \epsilon_{b_2]c_1...c_5}}{\sqrt{10}}\,, \qquad d_{a_1 a_2 b_1 b_2 c_1 c_2} = \frac{\epsilon_{a_1 a_2 b_1 b_2 c_1 c_2}}{\sqrt{10}}\,, \tag{A.2}$$

and those of d_{ABC} are given by the same with indices in opposite positions. Using this we have the useful tensor given by

$$\eta_{AB}{}^{\mathcal{C}} \equiv \sqrt{10}\, d_{ABD}\, k^{D\mathcal{C}}\,, \tag{A.3}$$

in which, $k^{A\mathcal{B}}$ connects $R_1 = \mathbf{27}$ and $R_2 = \overline{\mathbf{27}}$,

$$(k^{A\mathcal{B}}) \equiv \begin{pmatrix} 0 & 0 & \frac{\delta_b^a\, \epsilon_{b_1...b_6}}{\sqrt{6!}} \\ 0 & -\frac{\epsilon_{a_1 a_2 b_1...b_4}}{\sqrt{2!\,4!}} & 0 \\ \frac{\epsilon_{a_1...a_5 b}}{\sqrt{5!}} & 0 & 0 \end{pmatrix}, \qquad (k_{A\mathcal{B}}) \equiv \begin{pmatrix} 0 & 0 & \frac{\epsilon^{b_1...b_5 a}}{\sqrt{5!}} \\ 0 & -\frac{\epsilon^{b_1 b_2 a_1...a_4}}{\sqrt{2!\,4!}} & 0 \\ \frac{\delta_b^a\, \epsilon^{a_1...a_6}}{\sqrt{6!}} & 0 & 0 \end{pmatrix}. \tag{A.4}$$

This plays a similar role to that of the $O(d,d)$ invariant inner product of generalised geometry and indeed can be used to construct the Y-tensor [41] given by $Y_{CD}^{AB} = \eta^{AB}{}_{\mathcal{E}}\, \eta_{CD}{}^{\mathcal{E}} = 10\, d^{ABE}\, d_{CDE}$ that is ubiquitous in exceptional field theory/generalised geometry.

The invariance of $\eta_{AB}{}^{\mathcal{C}}$ under $E_{n(n)} \times \mathbb{R}^+$ symmetry can be expressed as

$$(t_{\hat\alpha})_A{}^D\, \eta_{DB}{}^{\mathcal{C}} + (t_{\hat\alpha})_B{}^D\, \eta_{AD}{}^{\mathcal{C}} = \eta_{AB}{}^{\mathcal{D}}\, (t_{\hat\alpha})_{\mathcal{D}}{}^{\mathcal{C}}\,. \tag{A.5}$$

From this relation, we find the matrix representation $(t_\alpha)_{\mathcal{A}}{}^{\mathcal{B}}$ in the R_2-representation is

$$K^c{}_d \equiv \begin{pmatrix} -\delta^a_d\,\delta^c_b & 0 & 0 \\ 0 & -4\delta^{a_1...a_4}_{de_1e_2e_3}\,\delta^{ce_1e_2e_3}_{b_1...b_4} & 0 \\ 0 & 0 & -6\delta^{a_1...a_6}_{de_1...e_5}\,\delta^{ce_1...e_5}_{b_1...b_6}\,\delta^{a'}_{b'} - \delta^{a_1...a_6}_{b_1...b_6}\,\delta^{a'}_d\,\delta^c_{b'} \end{pmatrix} + \frac{2\,\delta^c_d}{9-n}\,\mathbf{1}\,,$$

$$R^{c_1c_2c_3} \equiv \begin{pmatrix} 0 & -\sqrt{4!}\delta^{ac_1c_2c_3}_{b_1...b_4} & 0 \\ 0 & 0 & -3\sqrt{30}\delta^{a_1...a_4d_1d_2}_{b_1...b_6}\,\delta^{c_1c_2c_3}_{b'd_1d_2} \\ 0 & 0 & 0 \end{pmatrix}, \quad R^{c_1...c_6} \equiv \begin{pmatrix} 0 & 0 & \sqrt{6!}\delta^{c_1...c_6}_{b_1...b_6}\,\delta^{a'}_{b'} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$R_{c_1c_2c_3} \equiv \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{4!}\delta^{a_1...a_4}_{bc_1c_2c_3} & 0 & 0 \\ 0 & 3\sqrt{30}\delta^{a_1...a_6}_{b_1...b_4d_1d_2}\,\delta^{a'd_1d_2}_{c_1c_2c_3} & 0 \end{pmatrix}, \qquad R_{c_1...c_6} \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\sqrt{6!}\delta^{a_1...a_6}_{c_1...c_6}\,\delta^{a'}_{b'} & 0 & 0 \end{pmatrix}.$$
$$\tag{A.6}$$

In terms of these generators we can express the EDA product as,

$$T_A \circ T_B = \widehat{\Theta}_A{}^{\hat{\alpha}}\,(t_{\hat{\alpha}})_B{}^C\,T_B\,,\tag{A.7}$$

where $\{t_{\hat{\alpha}}\} \equiv \{t_0,\,t_\alpha\}$, $\{t_\alpha\} \equiv \{K^a{}_b,\,R^{a_1a_2a_3},\,R^{a_1\cdots a_6},\,R_{a_1a_2a_3},\,R_{a_1\cdots a_6}\}$. The explicit form of $\widehat{\Theta}_A{}^{\hat{\alpha}}$ is as follows:

$$\begin{aligned}
\widehat{\Theta}_a{}^\alpha\,t_\alpha &\equiv f_{ab}{}^c\,K^b{}_c + \frac{1}{3!}\,f_a{}^{c_1c_2c_3}\,R_{c_1c_2c_3} + \frac{1}{6!}\,f_a{}^{c_1\cdots c_6}\,R_{c_1\cdots c_6}\,, \\
\widehat{\Theta}^{a_1a_2\alpha}\,t_\alpha &\equiv -f_{c_1c_2}{}^{[a_1}\,R^{a_2]c_1c_2} - f_c{}^{da_1a_2}\,K^c{}_d\,, \\
\widehat{\Theta}^{a_1\cdots a_5\alpha}\,t_\alpha &\equiv -\frac{5}{2}\,f_{c_1c_2}{}^{[a_1}\,R^{a_2\cdots a_5]c_1c_2} + 10\,f_c{}^{[a_1a_2a_3}\,R^{a_4a_5]c} - f_c{}^{da_1\cdots a_5}\,K^c{}_d\,, \\
\widehat{\Theta}_a{}^0 &\equiv \frac{f_{ac}{}^c}{9-n}\,, \quad \widehat{\Theta}^{a_1a_20} \equiv -\frac{f_c{}^{ca_1a_2}}{9-n}\,, \quad \widehat{\Theta}^{a_1\cdots a_50} \equiv -\frac{f_c{}^{ca_1\cdots a_5}}{9-n}\,.
\end{aligned}\tag{A.8}$$

We can now recast the algebra of frame fields (3.8) as

$$\mathcal{L}_{E_A}E_B = -\widehat{\Theta}_A{}^{\hat{\alpha}}\,(t_{\hat{\alpha}})_B{}^C\,E_C\,,\tag{A.9}$$

such that making us of (3.7), (3.11), and (A.5), we can easily find that the generalized frame field in the R_2-representation also transforms covariantly as

$$\mathcal{L}_{E_A}\mathcal{E}_{\mathcal{B}} = -\widehat{\Theta}_A{}^{\hat{\alpha}}\,(t_{\hat{\alpha}})_{\mathcal{B}}{}^{\mathcal{C}}\,\mathcal{E}_{\mathcal{C}}\,.\tag{A.10}$$

References

[1] C.M. Hull and P.K. Townsend, *Unity of superstring dualities*, *Nucl. Phys. B* **438** (1995) 109 [hep-th/9410167] [INSPIRE].

[2] E. Witten, *String theory dynamics in various dimensions*, *Nucl. Phys. B* **443** (1995) 85 [hep-th/9503124] [INSPIRE].

[3] X.C. de la Ossa and F. Quevedo, *Duality symmetries from nonAbelian isometries in string theory*, *Nucl. Phys. B* **403** (1993) 377 [`hep-th/9210021`] [InSPIRE].

[4] C. Klimčík and P. Ševera, *Poisson-Lie T duality and loop groups of Drinfeld doubles*, *Phys. Lett. B* **372** (1996) 65 [`hep-th/9512040`] [InSPIRE].

[5] C. Klimčík and P. Ševera, *Dual nonAbelian duality and the Drinfeld double*, *Phys. Lett. B* **351** (1995) 455 [`hep-th/9502122`] [InSPIRE].

[6] A. Giveon and M. Roček, *On nonAbelian duality*, *Nucl. Phys. B* **421** (1994) 173 [`hep-th/9308154`] [InSPIRE].

[7] B. Hoare, N. Levine and A.A. Tseytlin, *Integrable σ-models and 2-loop RG flow*, *JHEP* **12** (2019) 146 [`arXiv:1910.00397`] [InSPIRE].

[8] F. Hassler and T. Rochais, *α′-Corrected Poisson-Lie T-duality*, *Fortsch. Phys.* **68** (2020) 2000063 [`arXiv:2007.07897`] [InSPIRE].

[9] R. Borsato and L. Wulff, *Quantum correction to generalized T-dualities*, *Phys. Rev. Lett.* **125** (2020) 201603 [`arXiv:2007.07902`] [InSPIRE].

[10] T. Codina and D. Marqués, *Generalized Dualities and Higher Derivatives*, *JHEP* **10** (2020) 002 [`arXiv:2007.09494`] [InSPIRE].

[11] K. Sfetsos and D.C. Thompson, *On non-abelian T-dual geometries with Ramond fluxes*, *Nucl. Phys. B* **846** (2011) 21 [`arXiv:1012.1320`] [InSPIRE].

[12] Y. Lozano, E. Ó Colgáin, D. Rodríguez-Gómez and K. Sfetsos, *Supersymmetric AdS$_6$ via T Duality*, *Phys. Rev. Lett.* **110** (2013) 231601 [`arXiv:1212.1043`] [InSPIRE].

[13] G. Itsios, C. Núñez, K. Sfetsos and D.C. Thompson, *Non-Abelian T-duality and the AdS/CFT correspondence: new $N = 1$ backgrounds*, *Nucl. Phys. B* **873** (2013) 1 [`arXiv:1301.6755`] [InSPIRE].

[14] D.C. Thompson, *An Introduction to Generalised Dualities and their Applications to Holography and Integrability*, *PoS* **CORFU2018** (2019) 099 [`arXiv:1904.11561`] [InSPIRE].

[15] C. Klimčík, *Yang-Baxter σ-models and dS/AdS T duality*, *JHEP* **12** (2002) 051 [`hep-th/0210095`] [InSPIRE].

[16] C. Klimčík, *On integrability of the Yang-Baxter σ-model*, *J. Math. Phys.* **50** (2009) 043508 [`arXiv:0802.3518`] [InSPIRE].

[17] Y. Sakatani, *U-duality extension of Drinfel'd double*, *Prog. Theor. Exp. Phys.* **2020** (2020) 023B08 [`arXiv:1911.06320`] [InSPIRE].

[18] E. Malek and D.C. Thompson, *Poisson-Lie U-duality in Exceptional Field Theory*, *JHEP* **04** (2020) 058 [`arXiv:1911.07833`] [InSPIRE].

[19] Y. Sakatani and S. Uehara, *Non-Abelian U-duality for membranes*, *Prog. Theor. Exp. Phys.* **2020** (2020) 073B01 [`arXiv:2001.09983`] [InSPIRE].

[20] L. Hlavaty, *Classification of 6D Leibniz algebras*, *Prog. Theor. Exp. Phys.* **2020** (2020) 071B01 [`arXiv:2003.06164`] [InSPIRE].

[21] C.D.A. Blair, D.C. Thompson and S. Zhidkova, *Exploring Exceptional Drinfeld Geometries*, *JHEP* **09** (2020) 151 [`arXiv:2006.12452`] [InSPIRE].

[22] F. Hassler, *Poisson-Lie T-duality in Double Field Theory*, *Phys. Lett. B* **807** (2020) 135455 [`arXiv:1707.08624`] [InSPIRE].

[23] S. Demulder, F. Hassler and D.C. Thompson, *Doubled aspects of generalised dualities and integrable deformations*, *JHEP* **02** (2019) 189 [`arXiv:1810.11446`] [ɪɴSPIRE].

[24] Y. Sakatani, *Type II DFT solutions from Poisson-Lie T-duality/plurality*, *Prog. Theor. Exp. Phys.* **2019** (2019) 073B04 [`arXiv:1903.12175`] [ɪɴSPIRE].

[25] I. Bakhmatov, N.S. Deger, E.T. Musaev, E. Ó Colgáin and M.M. Sheikh-Jabbari, *Tri-vector deformations in d = 11 supergravity*, *JHEP* **08** (2019) 126 [`arXiv:1906.09052`] [ɪɴSPIRE].

[26] I. Bakhmatov, K. Gubarev and E.T. Musaev, *Non-abelian tri-vector deformations in d = 11 supergravity*, *JHEP* **05** (2020) 113 [`arXiv:2002.01915`] [ɪɴSPIRE].

[27] E.T. Musaev, *On non-abelian U-duality of 11D backgrounds*, `arXiv:2007.01213` [ɪɴSPIRE].

[28] D.S. Berman and C.D.A. Blair, *The Geometry, Branes and Applications of Exceptional Field Theory*, *Int. J. Mod. Phys. A* **35** (2020) 2030014 [`arXiv:2006.09777`] [ɪɴSPIRE].

[29] C.D.A. Blair, E. Malek and J.-H. Park, *M-theory and Type IIB from a Duality Manifest Action*, *JHEP* **01** (2014) 172 [`arXiv:1311.5109`] [ɪɴSPIRE].

[30] O. Hohm and H. Samtleben, *Exceptional Field Theory I: $E_{6(6)}$ covariant Form of M-theory and Type IIB*, *Phys. Rev. D* **89** (2014) 066016 [`arXiv:1312.0614`] [ɪɴSPIRE].

[31] M. Graña, R. Minasian, M. Petrini and D. Waldram, *T-duality, Generalized Geometry and Non-Geometric Backgrounds*, *JHEP* **04** (2009) 075 [`arXiv:0807.4527`] [ɪɴSPIRE].

[32] G. Aldazabal, W. Baron, D. Marqués and C. Núñez, *The effective action of Double Field Theory*, *JHEP* **11** (2011) 052 [*Erratum JHEP* **11** (2011) 109] [`arXiv:1109.0290`] [ɪɴSPIRE].

[33] D. Geissbühler, *Double Field Theory and N = 4 Gauged Supergravity*, *JHEP* **11** (2011) 116 [`arXiv:1109.4280`] [ɪɴSPIRE].

[34] D.S. Berman, E.T. Musaev, D.C. Thompson and D.C. Thompson, *Duality Invariant M-theory: Gauged supergravities and Scherk-Schwarz reductions*, *JHEP* **10** (2012) 174 [`arXiv:1208.0020`] [ɪɴSPIRE].

[35] K. Lee, C. Strickland-Constable and D. Waldram, *Spheres, generalised parallelisability and consistent truncations*, *Fortsch. Phys.* **65** (2017) 1700048 [`arXiv:1401.3360`] [ɪɴSPIRE].

[36] O. Hohm and H. Samtleben, *Consistent Kaluza-Klein Truncations via Exceptional Field Theory*, *JHEP* **01** (2015) 131 [`arXiv:1410.8145`] [ɪɴSPIRE].

[37] P. Pires Pacheco and D. Waldram, *M-theory, exceptional generalised geometry and superpotentials*, *JHEP* **09** (2008) 123 [`arXiv:0804.1362`] [ɪɴSPIRE].

[38] D.S. Berman and M.J. Perry, *Generalized Geometry and M-theory*, *JHEP* **06** (2011) 074 [`arXiv:1008.1763`] [ɪɴSPIRE].

[39] D.S. Berman, H. Godazgar, M. Godazgar and M.J. Perry, *The Local symmetries of M-theory and their formulation in generalised geometry*, *JHEP* **01** (2012) 012 [`arXiv:1110.3930`] [ɪɴSPIRE].

[40] A. Coimbra, C. Strickland-Constable and D. Waldram, *$E_{d(d)} \times \mathbb{R}^+$ generalised geometry, connections and M-theory*, *JHEP* **02** (2014) 054 [`arXiv:1112.3989`] [ɪɴSPIRE].

[41] D.S. Berman, M. Cederwall, A. Kleinschmidt and D.C. Thompson, *The gauge structure of generalised diffeomorphisms*, *JHEP* **01** (2013) 064 [`arXiv:1208.5884`] [ɪɴSPIRE].

[42] A. Coimbra, C. Strickland-Constable and D. Waldram, *Supergravity as Generalised Geometry II: $E_{d(d)} \times \mathbb{R}^+$ and M-theory*, *JHEP* **03** (2014) 019 [`arXiv:1212.1586`] [ɪɴSPIRE].

[43] O. Hohm and H. Samtleben, *Exceptional Form of D = 11 Supergravity*, *Phys. Rev. Lett.* **111** (2013) 231601 [arXiv:1308.1673] [INSPIRE].

[44] G. Inverso, *Generalised Scherk-Schwarz reductions from gauged supergravity*, *JHEP* **12** (2017) 124 [arXiv:1708.02589] [INSPIRE].

[45] H. Samtleben, *Lectures on Gauged Supergravity and Flux Compactifications*, *Class. Quant. Grav.* **25** (2008) 214002 [arXiv:0808.4076] [INSPIRE].

[46] A. Çatal-Özer, *Non-Abelian T-duality as a Transformation in Double Field Theory*, *JHEP* **08** (2019) 115 [arXiv:1904.00362] [INSPIRE].

[47] M. Bugden, *Non-abelian T-folds*, *JHEP* **03** (2019) 189 [arXiv:1901.03782] [INSPIRE].

[48] A. Le Diffon and H. Samtleben, *Supergravities without an Action: Gauging the Trombone*, *Nucl. Phys. B* **811** (2009) 1 [arXiv:0809.5180] [INSPIRE].

[49] T. Araujo, I. Bakhmatov, E. Ó Colgáin, J. Sakamoto, M.M. Sheikh-Jabbari and K. Yoshida, *Yang-Baxter σ-models, conformal twists, and noncommutative Yang-Mills theory*, *Phys. Rev. D* **95** (2017) 105006 [arXiv:1702.02861] [INSPIRE].

[50] J.-i. Sakamoto, Y. Sakatani and K. Yoshida, *Homogeneous Yang-Baxter deformations as generalized diffeomorphisms*, *J. Phys. A* **50** (2017) 415401 [arXiv:1705.07116] [INSPIRE].

[51] J.-I. Sakamoto and Y. Sakatani, *Local β-deformations and Yang-Baxter σ-model*, *JHEP* **06** (2018) 147 [arXiv:1803.05903] [INSPIRE].

[52] I. Bakhmatov, E. Ó Colgáin, M.M. Sheikh-Jabbari and H. Yavartanoo, *Yang-Baxter Deformations Beyond Coset Spaces (a slick way to do TsT)*, *JHEP* **06** (2018) 161 [arXiv:1803.07498] [INSPIRE].

[53] I. Bakhmatov and E.T. Musaev, *Classical Yang-Baxter equation from β-supergravity*, *JHEP* **01** (2019) 140 [arXiv:1811.09056] [INSPIRE].

[54] A. Çatal-Özer and S. Tunali, *Yang-Baxter Deformation as an O(d, d) Transformation*, *Class. Quant. Grav.* **37** (2020) 075003 [arXiv:1906.09053] [INSPIRE].

[55] J. Bagger and N. Lambert, *Modeling Multiple M2's*, *Phys. Rev. D* **75** (2007) 045020 [hep-th/0611108] [INSPIRE].

[56] P.-A. Nagy, *Prolongations of Lie algebras and applications*, *J. Lie Theory* **23** (2013) 1 [arXiv:0712.1398] [INSPIRE].

[57] G. Papadopoulos, *M2-branes, 3-Lie Algebras and Plucker relations*, *JHEP* **05** (2008) 054 [arXiv:0804.2662] [INSPIRE].

[58] J.P. Gauntlett and J.B. Gutowski, *Constraining Maximally Supersymmetric Membrane Actions*, *JHEP* **06** (2008) 053 [arXiv:0804.3078] [INSPIRE].

[59] J. Gomis, G. Milanesi and J.G. Russo, *Bagger-Lambert Theory for General Lie Algebras*, *JHEP* **06** (2008) 075 [arXiv:0805.1012] [INSPIRE].

[60] S. Benvenuti, D. Rodríguez-Gómez, E. Tonni and H. Verlinde, *N = 8 superconformal gauge theories and M2 branes*, *JHEP* **01** (2009) 078 [arXiv:0805.1087] [INSPIRE].

[61] P.-M. Ho, Y. Imamura and Y. Matsuo, *M2 to D2 revisited*, *JHEP* **07** (2008) 003 [arXiv:0805.1202] [INSPIRE].

[62] P. de Medeiros, J.M. Figueroa-O'Farrill and E. Mendez-Escobar, *Metric Lie 3-algebras in Bagger-Lambert theory*, *JHEP* **08** (2008) 045 [arXiv:0806.3242] [INSPIRE].

[63] P.-M. Ho, Y. Matsuo and S. Shiba, *Lorentzian Lie* (3-)*algebra and toroidal compactification of M/string theory*, *JHEP* **03** (2009) 045 [arXiv:0901.2003] [INSPIRE].

[64] P. de Medeiros, J. Figueroa-O'Farrill, E. Mendez-Escobar and P. Ritter, *Metric* 3-*Lie algebras for unitary Bagger-Lambert theories*, *JHEP* **04** (2009) 037 [arXiv:0902.4674] [INSPIRE].

[65] H. Awata, M. Li, D. Minic and T. Yoneya, *On the quantization of Nambu brackets*, *JHEP* **02** (2001) 013 [hep-th/9906248] [INSPIRE].

[66] A. Gustavsson, *Selfdual strings and loop space Nahm equations*, *JHEP* **04** (2008) 083 [arXiv:0802.3456] [INSPIRE].

[67] U. Gran, B.E.W. Nilsson and C. Petersson, *On relating multiple M2 and D2-branes*, *JHEP* **10** (2008) 067 [arXiv:0804.1784] [INSPIRE].

[68] J. Bagger and N. Lambert, *Three-Algebras and N = 6 Chern-Simons Gauge Theories*, *Phys. Rev. D* **79** (2009) 025002 [arXiv:0807.0163] [INSPIRE].

[69] F.-M. Chen, *Symplectic Three-Algebra Unifying N = 5, 6 Superconformal Chern-Simons-Matter Theories*, *JHEP* **08** (2010) 077 [arXiv:0908.2618] [INSPIRE].

[70] J. Patera, R.T. Sharp, P. Winternitz and H. Zassenhaus, *Invariants of Real Low Dimension Lie Algebras*, *J. Math. Phys.* **17** (1976) 986 [INSPIRE].

[71] R.O. Popovych, V.M. Boyko, M.O. Nesterenko and M.W. Lutfullin, *Realizations of real low-dimensional Lie algebras*, *J. Phys. A* **36** (2003) 7337 [math-ph/0301029] [INSPIRE].

[72] P. Turkowski, *Solvable Lie algebras of dimension six*, *J. Math. Phys.* **31** (1990) 1344.

[73] E. Malek, *Half-Maximal Supersymmetry from Exceptional Field Theory*, *Fortsch. Phys.* **65** (2017) 1700061 [arXiv:1707.00714] [INSPIRE].