Ballooning Modes in 3D Finite-β Stellarators

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The evaluation of Mercier[1] and resistive interchange[2] modes is followed up by generalizing the JMC code, which evaluates the current density from 3D equilibria as the basic quantity for the investigation of localized instabilities, in order to consider ballooning modes[3] in general 3D equilibria. (Previous work[4] was restricted to low-β approximations.) To this end, Boozer's coordinates[5] are constructed from the equilibria because they may conveniently be used to analyze the ballooning equation. A form of the Mercier criterion is used which is valid on field lines with rational twist. A typical field line of this type in a stellarator with five periods (such as W VII-AS) would have, for example, $\iota_T = 2/5$, where ι_T is the total twist, so that localized $m = 5, n = 2$ modes are considered. The analysis of such modes should not be plagued by existence problems of the equilibrium associated with ι_p (twist per period) being rational of much higher order (2/25 in the above example). The ballooning equation is cast in a form which avoids numerically dangerous cancellations and shows, explicitly, the order with respect to β of the various terms. Thus, the principal stabilizing terms (magnetic well diminished by its diamagnetic part and shear) and destabilizing terms (square of the parallel current density and unfavourable contributions from the field line curvature) are clearly borne out. Moreover, this form of the ballooning equation readily yields the asymptotic behaviour (for large arguments) which recovers the Mercier criterion in an analogous form. This feature not only facilitates testing of the code but may also be used to obtain the ballooning stability condition by asymptotic matching. Results are presented for tokamaks (as test cases) and 3D stellarators such as ATF and W VII-AS.

In Boozer's coordinates s, θ, ϕ, where s is the flux label and θ, ϕ are poloidal and toroidal angle-like variables, respectively, the local shear σ can be written in the following form

$$\begin{aligned}\sigma &= |\nabla s|^{-4}(\nabla s \times \vec{B}) \cdot \vec{\nabla} \times (\nabla s \times \vec{B}) \\ &= F_T'^2 \iota' / \sqrt{g} + \vec{B} \cdot \nabla(I g_{\theta s}/\sqrt{g}|\nabla s|^2 - J g_{\phi s}/\sqrt{g}|\nabla s|^2)\end{aligned}$$

[1] Mercier, C., in Plasma Physics and Contr. Nucl.Fusion Res.1961, Nucl.Fus.1962, Suppl.2, IAEA, Vienna (1962) 801; R. Großmann, F. Herrnegger, J. Nührenberg, Proc.Sherwood Theory Conf., Arlington 1983, 1S21

[2] J. Nührenberg, R. Großmann, Proc.Sherwood Theory Conf., Incline Village 1984, 2R6; 5th Int.Workshop on Stellarators, Schloß Ringberg 1984, Comm.EC, EUR 9618 EN, I, 339

[3] D. Correa-Restrepo, Z.Naturforsch.33a (1978) 789

[4] H.L. Berk, M.N. Rosenbluth, J.L. Shohet, Phys.Fluids **26** (1983) 2616; W.A. Cooper, T.C. Hender, Plasma Phys.and Contr.Fus. **26** (1984) 921

[5] A. Boozer, Phys.Fluids **23** (1980) 904

Here, all functions are normalized to one field period of the equilibrium; F_T, J toroidal flux and current, F_P, I poloidal flux and current, $\iota_p = F_P'/F_T'$, $' = d/ds$, $p'V' = I'F_T' + J'F_P'$, $\sqrt{g}B^2 = -F_T'I - F_P'J$. The ballooning equation can then be reduced to the following equation for F (which indicates instability if F vanishes twice)

$$\frac{d}{d\phi}\left\{a^{-1}[1+(\bar{\sigma}\phi+\tilde{\sigma})^2]\frac{dF}{d\phi}\right\}+(\bar{D}\phi+\tilde{D})F=0$$

Here, ϕ is used as the variable along the field lines and the coefficients a^{-1}, $\bar{\sigma}$, $\tilde{\sigma}$, $\bar{D}$, $\tilde{D}$ have to be used as function of this variable,

$$\begin{aligned}
a =& \sqrt{g}|\nabla s|^2 \\
\bar{\sigma} =& -F_T'\iota'|\nabla s|^2/B \\
\tilde{\sigma} =& (Ig_{\theta s} - Jg_{\phi s})/\sqrt{g}B \\
\bar{D} =& p'\iota'F_T'^{-1}(IB^{-2}{}_{,\theta} - JB^{-2}{}_{,\phi}) \\
\tilde{D} =& [\sqrt{g}p'^2/B^2 - p'\sqrt{g}_{,s} - p'B^{-2}(IF_T'' + JF_P'') + p'\sqrt{g}\vec{B}\cdot\nabla(\tilde{\beta}B^2)]/F_T'^2
\end{aligned}$$

Here, $\tilde{\beta}$ is the periodic part of the function β in the covariant representation of $\vec{B}$ ($\vec{B} = \nabla\chi + \beta\nabla s$). $\tilde{\beta}$, $\bar{D}$, and $\tilde{D}$ can be reduced as follows

$$\begin{aligned}
\tilde{\beta} &= -(F_T'g_{\phi s} + F_P'g_{\theta s})/\sqrt{g} \\
\sqrt{g}\vec{B}\cdot\nabla\tilde{\beta} &= p'(\sqrt{g} - V') \\
\bar{D} &= -F_T'^{-1}\iota'\sqrt{g}\vec{B}\cdot\nabla X, \qquad X = j_{\parallel}/B \\
-p'B^{-2}(IF_T'' + JF_P'') &= \sqrt{g}V'^{-1}[I'F_T'' + J'F_P'' + \iota'F_T'^2(I'J - J'I)/(IF_T' + JF_P')]
\end{aligned}$$

Thus, $\bar{\sigma}$ represents the stabilizing influence of the shear, $\tilde{D}$ (for stellarators with vanishing net toroidal current) mainly the stabilizing influence of an outwardly increasing B^2 diminished by the diamagnetic effect, $\bar{D}$ the terms connected with $j_{\parallel}$ and with the shear known from Mercier's criterion. In accordance with the explicit resemblance of this form of the ballooning equation with Mercier's criterion, the well-known asymptotic analysis, $F \propto \phi^{\nu} + F_1\phi^{\nu-1} + \ldots$, readily recovers that criterion in the form of the indicial equation

$$\nu = -.5 \pm [(.5 - \langle a\mathcal{D}/\bar{\sigma}^2\rangle)^2 - \langle a/\bar{\sigma}^2\rangle(\langle a\mathcal{D}^2/\bar{\sigma}^2\rangle + \langle\tilde{D}\rangle - \langle\mathcal{D}\rangle)]^{\frac{1}{2}}$$

where $\mathcal{D} = \int Dd\phi = \iota'X$. This equation may also be used to obtain a partial test of four of the five coefficients of the ballooning equation which is solved in the following way. Relatively low order rational toroidal twist numbers are considered and the contracted variable $\bar{\phi} = N^{-1}m_\iota^{-1}\phi$ is used where $\iota_T = \frac{n_\iota}{m_\iota}$, N number of field periods. The coefficients of the ballooning equation are periodic in $\bar{\phi}$, so that the above averages $\langle\ \rangle$ are well defined. Stellarator equilibria of the usual symmetry are considered, so that it is possible to consider ballooning modes symmetric with respect to $\bar{\phi}$. Thus, $F(0) = 1, (dF/d\phi)(0) = 0$ is used and a ballooning instability is found if $F = 0$ for $\bar{\phi} > 0$.

The 3D equilibrium code used so far is BETA[1]; the JMC evaluation code only uses the flux invariants and the geometry of the equilibrium obtained, reconstructs the remainder of the equilibrium information, constructs Boozer's coordinates and obtains the coefficients of the ballooning equation. By way of example, Fig.1 shows $a, \tilde{D}$ for W VII-AS for $\iota_T = \frac{2}{5}$, note the three different scales: the equilibrium period, the poloidal period, and the period of the closed field line. All results shown are obtained with finite but reasonably fine meshes – the choices being guided by the experience in extrapolating Mercier's criterion – and are being complemented by extrapolation studies; Fourier convergence problems typical of magnetic coordinates also need further study.

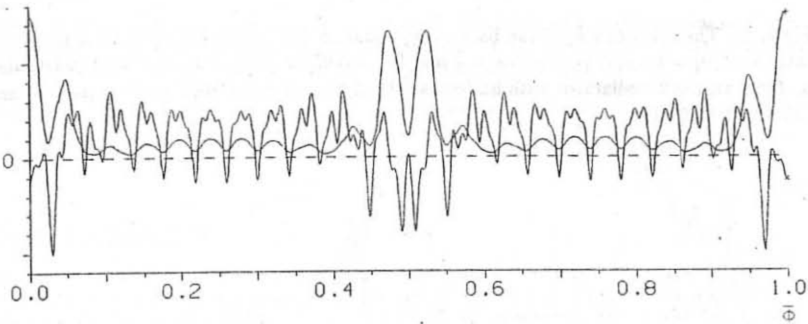

Fig.1. The coefficients a (+) and $\tilde{D}$ (×) for a W VII-AS equilibrium with $\iota_T = \frac{2}{5}$ and $\langle\beta\rangle = 0.028$.

Test results have been obtained for axisymmetric equilibria and a simple unstable $\ell = 2$ stellarator. Figure 2a shows F for $\iota_T = \frac{2}{3}$ in a standard circular tokamak at $\langle\beta\rangle = 0.02$; 0.03, below and above the ballooning stability boundary. Figure 2b shows F for $\iota_T = \frac{2}{5}$ in an $\ell = 2$ stellarator with five periods. Since for this twist value and a purely elliptic boundary no vacuum magnetic well is present, instability is seen at all β values and the ballooning character ($F = 0$ at $\bar{\phi} < n_\iota^{-1}$) prevails for $\langle\beta\rangle > 0.002$.
Results for W VII-AS and ATF are shown in Fig.3. The common feature is that the instabilities found are of the Mercier type, i.e. driven by the parallel current density ($\langle aD^2/\bar{\sigma}^2\rangle$) and occurring in the asymptotic range. Apparently, in these cases ballooning instability proper occurs at larger β-values than Mercier instability (in contrast to the tokamak situation). It will be interesting to see whether this behaviour persists in configurations in which Mercier instability occurs at higher β-values. In this connection we mention an ℓ =1, 2, 3 equilibrium encountered in the search for stable medium-β stellarators. The equilibrium shown in Fig.4 is Mercier stable at $\langle\beta\rangle = 0.05$. Ballooning stability results for this configuration are being obtained.

[1]F. Bauer, O. Betancourt, and P. Garabedian Magnetohydrodynamic Equilibrium and Stability of Stellarators (Springer, New York, 1984

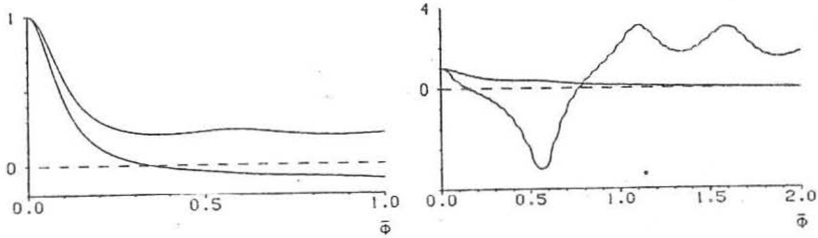

Fig.2. a. The solutions F of the ballooning equation for a circular tokamak with aspect ratio 3, $\iota_T(0) = 1, \iota_T(1) = \frac{1}{3}$ for $\iota_T = \frac{2}{3}$ and $\langle\beta\rangle = 0.02$ (stable) and $\langle\beta\rangle = 0.03$ (unstable). b. F for an $\ell = 2$ stellarator with half-axes ratio 2, aspect ratio 7.5, 5 periods, $\iota_T = \frac{2}{5}$ and $\langle\beta\rangle = 0.0012; 0.01$.

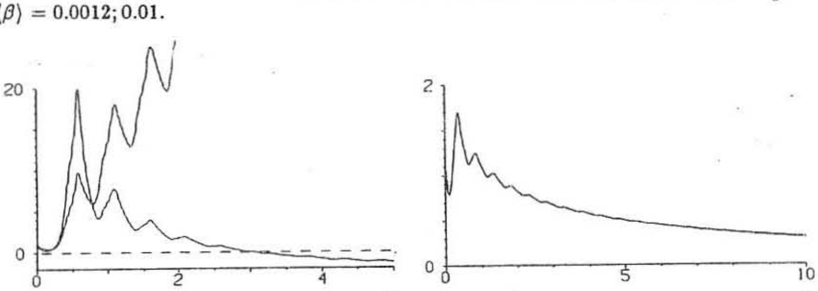

Fig.3. a. F for W VII-AS equilibria with $\langle\beta\rangle = 0.014$ (stable) and 0.028 (unstable) for $\iota_T = \frac{2}{5}$. b. F for an ATF equilibrium (standard flux-conserving case) with $\langle\beta\rangle = 0.033$ for $\iota_T = \frac{2}{3}$. This case is weakly Mercier unstable; $F = 0$ at $\bar{\phi} \approx 100$.

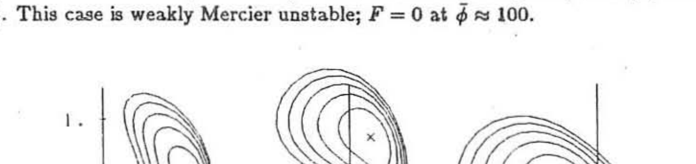

Fig.4. Flux surfaces of an $\ell = 1, 2, 3$ equilibrium which is stable with respect to Mercier and resistive interchange modes at $\langle\beta\rangle = 0.05$.