

Good reductions of Shimura varieties of Hodge type in arbitrary unramified mixed characteristic. Part I

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ABSTRACT. We prove the existence of good smooth integral models of Shimura varieties of Hodge type in arbitrary unramified mixed characteristic $(0, p)$. As a first application we provide a smooth solution (answer) to a conjecture (question) of Langlands for Shimura varieties of Hodge type. As a second application we prove the existence in arbitrary unramified mixed characteristic $(0, p)$ of integral canonical models of projective Shimura varieties of Hodge type with respect to h -hyperspecial subgroups as pro-étale covers of Néron models; this forms progress towards the proof of conjectures of Milne and Reimann. Though the second application was known before in some cases, its proof is new and more of a principle.

KEY WORDS: abelian scheme, affine group scheme, Barsotti–Tate group, deformation theory, F -crystal, Hodge cycle, integral model, and Shimura variety

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1 Introduction

Let $p \in \mathbb{N}$ be a prime. Let $\mathbb{Z}_{(p)}$ be the localization of \mathbb{Z} at its prime ideal (p) . Let $r \in \mathbb{N}^*$. Let $N \geq 3$ be a natural number relatively prime to p . Let $\mathcal{A}_{r,1,N}$

be the *Mumford moduli scheme* over $\mathbb{Z}_{(p)}$ that parameterizes isomorphism classes of *principally polarized abelian schemes* over $\mathbb{Z}_{(p)}$ -schemes of relative dimension r and endowed with a symplectic similitude level- N structure (cf. [43, Thms. 7.9 and 7.10] applied to symplectic similitude level structures instead of simply level structures).

1.1 Basic properties

The $\mathbb{Z}_{(p)}$ -schemes $\mathcal{A}_{r,1,N}$ have the following four properties:

- (i) *They are smooth and quasi-projective.*
- (ii) *If $N_1 \in \mathbb{N}\mathbb{N} \setminus p\mathbb{N}$, then the natural level-reduction morphism $\mathcal{A}_{r,1,N_1} \rightarrow \mathcal{A}_{r,1,N}$ is finite, étale, and surjective. Thus the projective limit*

$$\mathcal{M}_r := \operatorname{projlim}_{N \geq 3, (N,p)=1} \mathcal{A}_{r,1,N}$$

exists and is a regular, quasi-compact, formally smooth $\mathbb{Z}_{(p)}$ -scheme.

- (iii) *If Z is a regular, formally smooth scheme over $\mathbb{Z}_{(p)}$, then each morphism $Z_{\mathbb{Q}} \rightarrow \mathcal{M}_{r,\mathbb{Q}}$ extends uniquely to a morphism $Z \rightarrow \mathcal{M}_r$.*

- (iv) *Their geometric special fibers have level m stratifications ($m \in \mathbb{N}^*$) enjoying many properties: strata are regular, quasi-affine, equidimensional of dimensions given by explicit formulas, etc.*

Property (i) is checked in loc. cit., cf. also Serre's Lemma of [42, Ch. IV, Sect. 21, Thm. 5]. Property (ii) is well-known. Property (iii) is implied by the fact that each abelian scheme over $Z_{\mathbb{Q}}$ that has level- N structure for all $N \in \mathbb{N} \setminus (p\mathbb{N} \cup \{1, 2\})$, extends to an abelian scheme over Z (cf. the Néron–Ogg–Shafarevich criterion of good reduction and the purity result [67, Cor. 5]); such an extension is unique up to a unique isomorphism (cf. [49, Ch. IX, Cor. 1.4]). Property (iv) is an application of the deformation theories for abelian varieties and for Barsotti–Tate groups (i.e., p -divisible groups): see [45] for the case $m = 1$ and see [59, Ex. 4.5] for all $m \in \mathbb{N}^*$.

From Yoneda Lemma we get that the regular, formally smooth $\mathbb{Z}_{(p)}$ -scheme \mathcal{M}_r is uniquely determined by its generic fibre $\mathcal{M}_{r,\mathbb{Q}}$ and by the *universal property* expressed by the property (iii). Thus one can view $\mathcal{A}_{r,1,N}$ as the *best* smooth integral model of $\mathcal{A}_{r,1,N,\mathbb{Q}}$ over $\mathbb{Z}_{(p)}$. The main goal of this paper is to generalize properties (i) to (iv) to the context of Shimura varieties of Hodge type. Thus in this paper we prove the existence and the uniqueness

of *good* smooth integral models of Shimura varieties of Hodge type in unramified mixed characteristic $(0, p)$ and we list several main properties of them, including identifying cases when the theoretical non-smooth loci are actually empty. We emphasize from the very beginning that this paper brings no new contribution to either the study of non-smooth loci (when non-empty) or to ramified mixed characteristic $(0, p)$ situations. We will begin with notation and with a review of Shimura varieties.

1.2 Notation

Let $\mathbb{S} := \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m, \mathbb{C}}$ be the two dimensional torus over \mathbb{R} such that we have identifications $\mathbb{S}(\mathbb{R}) = \mathbb{G}_{m, \mathbb{C}}(\mathbb{C})$ and $\mathbb{S}(\mathbb{C}) = \mathbb{G}_{m, \mathbb{C}}(\mathbb{C}) \times \mathbb{G}_{m, \mathbb{C}}(\mathbb{C})$ with the property that the monomorphism $\mathbb{R} \hookrightarrow \mathbb{C}$ induces the map $z \rightarrow (z, \bar{z})$; here $z \in \mathbb{S}(\mathbb{R}) = \mathbb{G}_{m, \mathbb{C}}(\mathbb{C})$ is a non-zero complex number.

Let R be a commutative \mathbb{Z} -algebra. We recall that a group scheme F over R is called *reductive* if it is smooth and affine and its fibres are connected and have trivial unipotent radicals. Let $\text{Lie}(\mathfrak{h})$ be the *Lie algebra* over R of a smooth, closed subgroup scheme \mathfrak{h} of F . The group schemes $\mathbb{G}_{m, R}$ and $\mathbb{G}_{a, R}$ are over R . For a free module M of finite rank over R , let $M^\vee := \text{Hom}(M, R)$ be its dual, and let \mathbf{GL}_M be the reductive group scheme over R of linear automorphisms of M . A bilinear form $\psi : M \times M \rightarrow R$ on M is called perfect if it defines naturally an R -linear isomorphism $M \rightarrow M^\vee$. If ψ is a perfect, alternating bilinear form on M (thus the rank of M is even), then $\mathbf{Sp}(M, \psi)$ and $\mathbf{GSp}(M, \psi)$ are viewed as reductive group schemes over R .

Let k be a perfect field of characteristic p . Let $W(k)$ be the ring of p -typical Witt vectors with coefficients in k . Always $n \in \mathbb{N}^*$. Let $\mathbb{A}_f := \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ be the ring of finite adèles of \mathbb{Q} . Let $\mathbb{A}_f^{(p)}$ be the ring of finite adèles of \mathbb{Q} with the p -component omitted; we have $\mathbb{A}_f = \mathbb{Q}_p \times \mathbb{A}_f^{(p)}$. If $R \in \{\mathbb{A}_f, \mathbb{A}_f^{(p)}, \mathbb{Q}_p\}$, then the group $F(R)$ is endowed with the coarsest topology that makes all maps $R = \mathbb{G}_{a, R}(R) \rightarrow F(R)$ associated to morphisms $\mathbb{G}_{a, R} \rightarrow F$ of R -schemes to be continuous; thus $F(R)$ is a totally discontinuous locally compact group. Each continuous action of a totally discontinuous locally compact group on a scheme will be in the sense of [13, Subsubsection. 2.7.1] and it will be a right action.

1.3 Shimura varieties

A *Shimura pair* (G, \mathcal{X}) consists of a reductive group G over \mathbb{Q} and a $G(\mathbb{R})$ -conjugacy class \mathcal{X} of homomorphisms $\mathbb{S} \rightarrow G_{\mathbb{R}}$ that satisfy the three axioms [13, (2.1.1.1) to (2.1.1.3)]: the Hodge \mathbb{Q} -structure on $\mathrm{Lie}(G)$ defined by each $h \in \mathcal{X}$ is of type $\{(-1, 1), (0, 0), (1, -1)\}$, no simple factor of the adjoint group G^{ad} of G becomes compact over \mathbb{R} , and $\mathrm{Ad}(\mathbb{R})(h(i))$ is a Cartan involution of $\mathrm{Lie}(G_{\mathbb{R}}^{\mathrm{ad}})$. Here $\mathrm{Ad} : G_{\mathbb{R}} \rightarrow \mathbf{GL}_{\mathrm{Lie}(G_{\mathbb{R}}^{\mathrm{ad}})}$ is the adjoint representation. These axioms imply that \mathcal{X} has a natural structure of a hermitian symmetric domain, cf. [13, Cor. 1.1.17]. For $h \in \mathcal{X}$ we consider the Hodge cocharacter

$$\mu_h : \mathbb{G}_{m, \mathbb{C}} \rightarrow G_{\mathbb{C}}$$

which maps $z \in \mathbb{G}_{m, \mathbb{C}}(\mathbb{C})$ to $\mu_h(\mathbb{C})(z) = h_{\mathbb{C}}(\mathbb{C})(z, 1) \in G_{\mathbb{C}}(\mathbb{C})$.

The most studied Shimura pairs are constructed as follows. Let W be a vector space over \mathbb{Q} of even dimension $2r$. Let ψ be a non-degenerate, alternating bilinear form on W . Let \mathcal{Y} be the set of all monomorphisms $\mathbb{S} \hookrightarrow \mathbf{GSp}(W \otimes_{\mathbb{Q}} \mathbb{R}, \psi)$ that define Hodge \mathbb{Q} -structures on W of type $\{(-1, 0), (0, -1)\}$ and that have either $2\pi i\psi$ or $-2\pi i\psi$ as polarizations. The pair $(\mathbf{GSp}(W, \psi), \mathcal{Y})$ is a Shimura pair that defines a *Siegel modular variety*. Let L be a \mathbb{Z} -lattice of W such that ψ induces a perfect bilinear form $\psi : L \times L \rightarrow \mathbb{Z}$. Let

$$K(N) := \{g \in \mathbf{GSp}(L, \psi)(\widehat{\mathbb{Z}}) \mid g \bmod N\widehat{\mathbb{Z}} \text{ is identity}\} \text{ and } K_p := \mathbf{GSp}(L, \psi)(\mathbb{Z}_p).$$

Let $E(G, \mathcal{X}) \hookrightarrow \mathbb{C}$ be the number subfield of \mathbb{C} that is the field of definition of the $G(\mathbb{C})$ -conjugacy class of the cocharacters μ_h 's of $G_{\mathbb{C}}$, cf. [37, p. 163]. We recall that $E(G, \mathcal{X})$ is called the *reflex field* of (G, \mathcal{X}) . The *Shimura variety* $\mathrm{Sh}(G, \mathcal{X})$ is identified with the canonical model over $E(G, \mathcal{X})$ of the complex Shimura variety

$$\mathrm{Sh}(G, \mathcal{X})_{\mathbb{C}} := \mathrm{proj.lim.}_{K \in \Sigma(G)} G(\mathbb{Q}) \backslash [\mathcal{X} \times (G(\mathbb{A}_f)/K)],$$

where $\Sigma(G)$ is the set of compact, open subgroups of $G(\mathbb{A}_f)$ endowed with the inclusion relation (see [12], [13], [36], [37], [38], and [39]). Thus $\mathrm{Sh}(G, \mathcal{X})$ is an $E(G, \mathcal{X})$ -scheme together with a continuous $G(\mathbb{A}_f)$ -action. For C a compact subgroup of $G(\mathbb{A}_f)$, let

$$\mathrm{Sh}_C(G, \mathcal{X}) := \mathrm{Sh}(G, \mathcal{X})/C.$$

Let $K \in \Sigma(G)$. We recall that a classical result of Baily and Borel allows us to view

$$\mathrm{Sh}_K(G, \mathcal{X})(\mathbb{C}) := G(\mathbb{Q}) \backslash [\mathcal{X} \times (G(\mathbb{A}_f)/K)]$$

as a finite, disjoint union of normal, quasi-projective varieties over \mathbb{C} and not only of normal complex analytic spaces (see [1, Thm. 10.11]) and this makes $\mathrm{Sh}_K(G, \mathcal{X})$ to be a normal, quasi-projective $E(G, \mathcal{X})$ -scheme. If K is small enough, then $\mathrm{Sh}_K(G, \mathcal{X})$ is in fact a smooth, quasi-projective $E(G, \mathcal{X})$ -scheme. We also recall that $\mathrm{Sh}_K(G, \mathcal{X})$ is a projective $E(G, \mathcal{X})$ -scheme if and only if the \mathbb{Q} -rank of G^{ad} is 0 (i.e., the Shimura pair (G, \mathcal{X}) is compact), cf. [6, Thm. 12.3 and Cor. 12.4].

Let H be a compact, open subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$.

We recall that the group $G_{\mathbb{Q}_p}$ is called *unramified* if and only if it has a Borel subgroup and splits over an unramified, finite field extension of \mathbb{Q}_p .

See [52] for *hyperspecial subgroups* of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$. In what follows we will only use the following three properties of them:

- the group $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$ has hyperspecial subgroups if and only if $G_{\mathbb{Q}_p}$ is unramified,
- a subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$ is hyperspecial if and only if it is the group of \mathbb{Z}_p -valued points of a reductive group scheme over \mathbb{Z}_p whose generic fibre is $G_{\mathbb{Q}_p}$, and
- each hyperspecial subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$ is a maximal compact, open subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$.

Let v be a prime of $E(G, \mathcal{X})$ that divides p . Let $k(v)$ be the residue field of v . Let $e(v) \in \mathbb{N}^*$ be the absolute ramification index of v . Let $O_{(v)}$ be the localization of the ring of integers of $E(G, \mathcal{X})$ with respect to v .

Definitions 1. (a) *By an integral model of $\mathrm{Sh}_K(G, \mathcal{X})$ over $O_{(v)}$ we mean a faithfully flat $O_{(v)}$ -scheme whose generic fibre is $\mathrm{Sh}_K(G, \mathcal{X})$.*

(b) *By an integral model of $\mathrm{Sh}_H(G, \mathcal{X})$ over $O_{(v)}$ we mean a faithfully flat $O_{(v)}$ -scheme equipped with a continuous $G(\mathbb{A}_f^{(p)})$ -action whose generic fibre is the $E(G, \mathcal{X})$ -scheme $\mathrm{Sh}_H(G, \mathcal{X})$ equipped with its natural continuous $G(\mathbb{A}_f^{(p)})$ -action.*

In this paper we study integral models of $\mathrm{Sh}_K(G, \mathcal{X})$ and $\mathrm{Sh}_H(G, \mathcal{X})$ over $O_{(v)}$. The subject has a long history, the first main result being the existence of the moduli schemes $\mathcal{A}_{r,1,N}$ and \mathcal{M}_r . This is so as we have natural identifications

$$\mathcal{A}_{r,1,N,\mathbb{Q}} = \mathrm{Sh}_{K(N)}(\mathbf{GSp}(W, \psi), \mathcal{Y}) \quad \text{and} \quad \mathcal{M}_{r,\mathbb{Q}} = \mathrm{Sh}_{K_p}(\mathbf{GSp}(W, \psi), \mathcal{Y})$$

(see [12], [37], [54], etc.). In particular, see [54, Ex. 3.2.9 and Subsect. 4.1] and [12, Thm. 4.21] for the natural continuous action of $\mathbf{GSp}(W, \psi)(\mathbb{A}_f^{(p)})$ on \mathcal{M}_r .

In 1976 Langlands conjectured the existence of a good integral model of $\mathrm{Sh}_H(G, \mathcal{X})$ over $O_{(v)}$, provided H is a hyperspecial subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$ (see [31, p. 411]); unfortunately, Langlands did not explain what good is supposed to stand for. Only in 1992, an idea of Milne made it significantly clearer how to characterize and identify the good integral models. Milne's philosophy can be roughly summarized as follows (cf. [37]): under certain conditions, the good regular, formally smooth integral models should be uniquely determined by (Néron type) universal properties that are similar to the property (iii) of Subsection 1.1.

Definitions 2. (a) *We assume that $e(v) = 1$. A flat, affine group scheme $G_{\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$ that extends G (i.e., whose generic fibre is G) is called a quasi-reductive group scheme for (G, \mathcal{X}, v) if there exists a reductive, normal, closed subgroup scheme $G_{\mathbb{Z}_p}^r$ of $G_{\mathbb{Z}_p}$ equipped with a cocharacter $\mu_v : \mathbb{G}_{m, W(k(v))} \rightarrow G_{\mathbb{Z}_p}^r \times_{\mathrm{Spec}(\mathbb{Z}_p)} \mathrm{Spec}(W(k(v)))$ such that the extension of μ_v to \mathbb{C} via an (any) $O_{(v)}$ -monomorphism $W(k(v)) \hookrightarrow \mathbb{C}$ defines a cocharacter of $G_{\mathbb{C}}$ that is $G(\mathbb{C})$ -conjugate to the cocharacters μ_h of $G_{\mathbb{C}}$ introduced above ($h \in \mathcal{X}$).*

(b) *We say that a smooth $O_{(v)}$ -scheme Y of finite type is a Néron model of its generic fibre $Y_{E(G, \mathcal{X})}$ over $O_{(v)}$, if for each smooth $O_{(v)}$ -scheme Z , every morphism $Z_{E(G, \mathcal{X})} \rightarrow Y_{E(G, \mathcal{X})}$ of $E(G, \mathcal{X})$ -schemes extends uniquely to a morphism $Z \rightarrow Y$ of $O_{(v)}$ -schemes.*

Definition 2 (a) is a variation of [51, Def. 1.5]; more precisely, the group $G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$ is an h -hyperspecial subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$ in the sense of loc. cit. Definition 2 (b) is well-known, cf. [7, Ch. 1, Sect. 1.2, Def. 1].

The notion $G_{\mathbb{Z}_{(p)}}$ is a quasi-reductive group scheme for (G, \mathcal{X}, v) is far more general than the notion $G_{\mathbb{Z}_{(p)}}$ is a reductive group scheme over $\mathbb{Z}_{(p)}$ that extends G . For instance, if $G_{\mathbb{Z}_{(p)}}$ is a reductive group scheme over $\mathbb{Z}_{(p)}$, then $G_{\mathbb{Q}_p}$ splits over an unramified finite field extension of \mathbb{Q}_p but if $G_{\mathbb{Z}_{(p)}}$ is a quasi-reductive group scheme for (G, \mathcal{X}, v) , then nothing one can say in general about a finite field extension \sharp of \mathbb{Q}_p over which the group $G_{\mathbb{Q}_p}$ splits (and there exist plenty of examples in which \sharp must contain an arbitrary a priori given finite field extension of \mathbb{Q}_p). To exemplify this, we will assume for the remaining part of this paragraph that G is a \mathbb{Q} -simple adjoint group. It is a Weil restriction $\mathrm{Res}_{\mathbb{E}/\mathbb{Q}} J$, where \mathbb{E} is a totally real number field and

where J is an absolutely simple adjoint group over \mathbb{E} , cf. [13, Subsect. 2.3.4 (a)]. If there exists a reductive group scheme $G_{\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$ that extends G (i.e., if $G_{\mathbb{Q}_p}$ is unramified), then \mathbb{E} is unramified over p . We assume now that the set of primes of \mathbb{E} above p is the disjoint union of two non-empty sets S_1 and S_2 with the following two properties:

(i) each $w \in S_1$ is unramified over p and $J_{\mathbb{E}_w}$ is unramified (here \mathbb{E}_w is the completion of \mathbb{E} at w);

(ii) each $w \in S_2$ corresponds naturally to compact simple factors of $G_{\mathbb{R}} = \prod_{\tau: \mathbb{E} \rightarrow \mathbb{R}} J_{\tau, \mathbb{R}}$ (via the identification of embeddings of \mathbb{E} in \mathbb{R} with embeddings of \mathbb{E} in an algebraically closed field that contains both \mathbb{Q}_p and \mathbb{R}).

Then there exist finite, flat group schemes $G_{\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$ which are quasi-reductive group scheme for (G, \mathcal{X}, v) . For instance, we can choose $G_{\mathbb{Z}_{(p)}}$ (see [54, Cl. 3.1.3.1]) such that its extension $G_{\mathbb{Z}_p}$ to \mathbb{Z}_p is a product of the form $\mathcal{G}_1 \times_{\text{Spec}(\mathbb{Z}_p)} \mathcal{G}_2$ with

$$\mathcal{G}_1 = \prod_{w \in S_1} \text{Res}_{O_w/\mathbb{Z}_p} J_{O_w},$$

where O_w is the ring of integers of \mathbb{E}_w and where J_{O_w} is a reductive group scheme over O_w that extends $J_{\mathbb{E}_w}$. There exist no additional requirements from either S_2 or \mathcal{G}_2 and in particular \mathbb{E} can be arbitrarily ramified at a prime $w \in S_2$.

1.4 Constructing integral models

Until the end we will assume that the Shimura pair (G, \mathcal{X}) is of *Hodge type*, i.e., there exists an injective map

$$f : (G, \mathcal{X}) \hookrightarrow (\mathbf{GSp}(W, \psi), \mathcal{Y})$$

for some symplectic space (W, ψ) over \mathbb{Q} ; thus $f : G \hookrightarrow \mathbf{GSp}(W, \psi)$ is a monomorphism such that we have $f_{\mathbb{R}} \circ h \in \mathcal{Y}$ for all elements $h \in \mathcal{X}$.

We recall that we identify $\mathcal{M}_{r, \mathbb{Q}} = \text{Sh}_{K_p}(\mathbf{GSp}(W, \psi), \mathcal{Y})$. Let $L_{(p)} := L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. The schematic closure $G_{\mathbb{Z}_{(p)}}$ of G in $\mathbf{GL}_{L_{(p)}}$ is a flat, affine group scheme over $\mathbb{Z}_{(p)}$. Until the end we will also assume that we have an identity $H = K_p \cap G_{\mathbb{Q}_p}(\mathbb{Q}_p)$; thus $H = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$.

The functorial morphism $f_0 : \text{Sh}(G, \mathcal{X}) \rightarrow \text{Sh}(\mathbf{GSp}(W, \psi), \mathcal{Y})_{E(G, \mathcal{X})}$ defined by f (see [12, Cor. 5.4]) is a closed embedding as it is so over \mathbb{C}

(cf. [12, Prop. 1.15]). The morphism f_0 induces naturally a morphism of $E(G, \mathcal{X})$ -schemes

$$f_p : \mathrm{Sh}_H(G, \mathcal{X}) \rightarrow \mathrm{Sh}_{K_p}(\mathbf{GSp}(W, \psi), \mathcal{Y})_{E(G, \mathcal{X})}$$

which is a closed embedding (cf. Fact 1). Thus we can speak about the normalization \mathcal{N} of the schematic closure of $\mathrm{Sh}_H(G, \mathcal{X})$ in $\mathcal{M}_{r, O_{(v)}}$. As $G(\mathbb{A}_f^{(p)})$ acts continuously on $\mathrm{Sh}_H(G, \mathcal{X})$ and \mathcal{M}_r , it is easy to see that we get naturally an induced continuous action of $G(\mathbb{A}_f^{(p)})$ on \mathcal{N} (to be compared with [54, Prop. 3.4]). Let \mathcal{N}^s be the formally smooth locus of \mathcal{N} over $O_{(v)}$; it is a $G(\mathbb{A}_f^{(p)})$ -invariant, open subscheme of \mathcal{N} such that we have identities $\mathcal{N}_{E(G, \mathcal{X})}^s = \mathcal{N}_{E(G, \mathcal{X})} = \mathrm{Sh}_H(G, \mathcal{X})$ (cf. Lemma 1). Let

$$(\mathcal{A}, \lambda_{\mathcal{A}})$$

be the principally polarized abelian scheme over \mathcal{N} which is the natural pullback of the universal principally polarized abelian scheme over \mathcal{M}_r .

If $p > 2$ and $e(v) = 1$, let $\mathcal{N}^m := \mathcal{N}^s$. If $p = 2$ and $e(v) = 1$, let \mathcal{N}^m be the $G(\mathbb{A}_f^{(p)})$ -invariant, open subscheme of \mathcal{N}^s defined in Subsection 3.5. In this paper we study when $e(v) = 1$ the following sequence

$$\mathcal{N}^m \hookrightarrow \mathcal{N}^s \hookrightarrow \mathcal{N} \rightarrow \mathcal{M}_{r, O_{(v)}}$$

of morphisms of $O_{(v)}$ -schemes in order to prove the following four basic results that pertain to \mathcal{N}^s .

Theorem 1 (Basic Theorem). *We assume that $e(v) = 1$ (i.e., v is unramified over p) and that the $k(v)$ -scheme $\mathcal{N}_{k(v)}^s$ is non-empty. Then the following three properties hold:*

(a) *The $O_{(v)}$ -scheme \mathcal{N}^s is the unique regular, formally smooth integral model of $\mathrm{Sh}_H(G, \mathcal{X})$ over $O_{(v)}$ that satisfies the following smooth extension property: if Z is a regular, formally smooth scheme over a discrete valuation ring O which is of absolute ramification index 1 and an $O_{(v)}$ -algebra, then each morphism $Z_{E(G, \mathcal{X})} \rightarrow \mathrm{Sh}_H(G, \mathcal{X})$ of $E(G, \mathcal{X})$ -schemes extends uniquely to a morphism $Z \rightarrow \mathcal{N}^s$ of $O_{(v)}$ -schemes.*

(b) *For each algebraically closed field k of characteristic p , the natural morphism $\mathcal{N}_{W(k)}^s \rightarrow \mathcal{M}_{r, W(k)}$ induces $W(k)$ -epimorphisms at the level of complete, local rings of residue field k (i.e., it is a formally closed embedding at all k -valued point of $\mathcal{N}_{W(k)}^s$).*

(c) We also assume that the \mathbb{Q} -rank of the adjoint group G^{ad} is 0. Let $H^{(p)}$ be a compact, open subgroup of $G(\mathbb{A}_f^{(p)})$ such that \mathcal{N} is a pro-finite pro-étale cover of $\mathcal{N}/H^{(p)}$. Then $\mathcal{N}^s/H^{(p)}$ is a Néron model of its generic fibre $\text{Sh}_{H \times H^{(p)}}(G, \mathcal{X})$ over $O_{(v)}$.

Corollary 1. *We assume that $e(v) = 1$, that $G_{\mathbb{Z}_{(p)}}$ is smooth, that the $k(v)$ -scheme $\mathcal{N}_{k(v)}^m$ is non-empty, and that k is algebraically closed. Let $H^{(p)}$ be a compact, open subgroup of $G(\mathbb{A}_f^{(p)})$ such that \mathcal{N} is a pro-finite pro-étale cover of $\mathcal{N}/H^{(p)}$ and there exists a natural number $N \geq 3$ relatively prime to p such that $H \times H^{(p)}$ is a subgroup of $K(N)$ (thus we have a natural morphism $\mathcal{N}_k^s/H^{(p)} \rightarrow \mathcal{A}_{r,1,N,k} = \mathcal{M}_{r,k}/K(N)^{(p)}$, where $K(N)^{(p)}$ is the compact, open subgroup of $\mathbf{GSp}(L, \psi)(\mathbb{A}_f^{(p)})$ such that we have $K(N) = K_p \times K(N)^{(p)}$). Then each connected component of $\mathcal{N}_k^m/H^{(p)}$ is a quasi Shimura p -variety of Hodge type in the sense of [59, Def. 4.2.1] (and therefore for all $m \in \mathbb{N}^*$ it has a level m stratification that enjoys all the nice properties listed in [59, Cor. 4.3]).*

Remark 1. (a) We assume that $e(v) = 1$. From many points of view (such as zeta functions) one would like to have a very good understanding of \mathcal{N} itself. However, as of today, its formally smooth locus \mathcal{N}^s is the only open subscheme of \mathcal{N} which is uniquely determined by a universal property and for which the connected components of its geometric special fibers have under natural assumptions (see Corollary 1) level m stratifications for each $m \in \mathbb{N}^*$ that generalize the classical Ekedahl–Oort stratifications for $m = 1$ and that have all the desired good properties (strata are regular, quasi-affine, equidimensional of dimensions given by concrete formulas, etc.).

(b) Under an additional condition satisfied for instance if $G_{\mathbb{Z}_{(p)}}$ is also a quasi-reductive group scheme for (G, \mathcal{X}, v) and with the notation of Corollary 1, \mathcal{N}_k^m itself is a quasi Shimura p -variety of Hodge type in the sense of [59, Def. 4.2.1] (see Subsubsection 3.5.2 for details).

Proposition 1. *We assume that $e(v) = 1$ and that $G_{\mathbb{Z}_{(p)}}$ is a quasi-reductive group scheme for (G, \mathcal{X}, v) . Then all ordinary points of $\mathcal{N}_{k(v)}$ (i.e., all points $y : \text{Spec}(k) \rightarrow \mathcal{N}_{k(v)}$ with values in perfect fields such that the abelian variety $y^*(\mathcal{A})$ over k is ordinary) belong to $\mathcal{N}_{k(v)}^m$.*

Theorem 2 (Main Theorem). *We assume that $e(v) = 1$ and that $G_{\mathbb{Z}_{(p)}}$ is a quasi-reductive group scheme for (G, \mathcal{X}, v) .*

- (a) Then $\mathcal{N}_{k(v)}^m$ is a non-empty, open closed subscheme of $\mathcal{N}_{k(v)}$.
- (b) If the ordinary locus of $\mathcal{N}_{k(v)}$ is Zariski dense in $\mathcal{N}_{k(v)}$, then we have $\mathcal{N}^m = \mathcal{N}^s = \mathcal{N}$.
- (c) If the \mathbb{Q} -rank of the adjoint group G^{ad} is 0, then the following two properties hold:

(c.i) We have $\mathcal{N}^m = \mathcal{N}^s = \mathcal{N}$ and moreover \mathcal{N} is the integral canonical model of $\text{Sh}_H(G, \mathcal{X})$ over $O_{(v)}$ as defined in [54, Def. 3.2.3 6)].

(c.ii) Let $H^{(p)}$ be a compact, open subgroup of $G(\mathbb{A}_f^{(p)})$ such that $H \times H^{(p)}$ is contained in $K(N)$ for some $N \in \mathbb{N} \setminus (p\mathbb{N} \cup \{1, 2\})$; thus we have a natural finite morphism

$$f(N) : \text{Sh}_{H \times H^{(p)}}(G, \mathcal{X}) \rightarrow \mathcal{A}_{r,1,N,E(G,\mathcal{X})} = \text{Sh}_{K(N)}(\mathbf{GSp}(W, \psi), \mathcal{Y})_{E(G,\mathcal{X})}.$$

Then the normalization \mathcal{Q} of $\mathcal{A}_{r,1,N,O_{(v)}}$ in the ring of fractions of $\text{Sh}_{H \times H^{(p)}}(G, \mathcal{X})$ is a smooth, projective $O_{(v)}$ -scheme that can be identified with $\mathcal{N}/H^{(p)}$ and that is the Néron model of $\text{Sh}_{H \times H^{(p)}}(G, \mathcal{X})$ over $O_{(v)}$.

1.5 On contents and proofs

We detail on the contents of this Part I. Section 2 lists conventions, notation, and few basic properties that pertain to the injective map $f : (G, \mathcal{X}) \hookrightarrow (\mathbf{GSp}(W, \psi), \mathcal{Y})$ and to Hodge cycles on abelian schemes over \mathbb{Q} -schemes. In connection to Sections 3 to 5 we assume that $e(v) = 1$.

Section 3 includes crystalline applications. Until Subsection 3.3 we introduce basic notation and review three relatively recent results that pertain to Barsotti–Tate groups and that play a central role in Subsections 3.2 to 3.6, Sections 4 and 5, and Appendix B. The results are:

- (i) *de Jong extension theorem* (see [11] and Theorem 3),
- (ii) a variant of *Faltings deformation theory* (see Subsection 3.2), and
- (iii) a refinement of *a motivic conjecture of Milne* proved in [64, Thm. 1.2].

Our first main new idea is to use (ii) in order to show directly that each $W(k)$ -valued point of \mathcal{N} factors through \mathcal{N}^s . Based on this and [67, Cor. 5], in Subsection 3.3 we prove the Basic Theorem 1. Subsections 3.3 and 3.4 gather extra crystalline properties required in Sections 4 and 5 and required to prove Corollary 1 and a variant of it in Subsubsections 3.5.1 and 3.5.2; these two subsections can be viewed as an enlarged version with details of

[59, Ex. 4.6]. Proposition 1 is proved in Subsection 3.6 based on [44] and on (iii).

See Lemma 8 (a) for a simple criterion on when the $k(v)$ -scheme $\mathcal{N}_{k(v)}^m$ is non-empty. In Subsection 4.1 we apply Theorem 1 (a) and Lemma 8 (a) to prove the existence and the uniqueness of good regular, formally smooth integral models of $\mathrm{Sh}_{\tilde{H}}(G, \mathcal{X})$ over $O_{(v)}$ for a large class of compact, open subgroups \tilde{H} of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$ (the class includes all *parahoric* subgroups of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$) provided $G_{\mathbb{Q}_p}$ splits over an unramified extension of \mathbb{Q}_p (see Theorem 7). In particular, Corollary 3 can be viewed as a smooth solution (answer) to the conjecture (question) of Langlands (mentioned in the paragraph before Definitions 2) for Shimura varieties of Hodge type.

In Section 5 we use (i), Lemma 2 (i.e., [57, Cor. 4.3]), [66], and Subsection 3.3 to prove the Main Theorem 2 (see Subsections 5.1 to 5.6). Our second main new idea is to use (i) and purity results for reductive groups as in [66] in order to get that the open subscheme $\mathcal{N}_{k(v)}^m$ of $\mathcal{N}_{k(v)}$ is as well stable under specializations.

Appendices A and B review basic properties of affine group schemes and of Barsotti–Tate groups. Their subsections are numbered as A1, A2, and B1 to B6. The reader ought to refer to these subsections only when they are quoted in the main text. Modulo few parts of the notation of Subsection 2.1, Appendices A and B are entirely independent of the main text.

1.6 On literature

Referring to Theorem 1 (a), all ordinary points of $\mathcal{N}_{k(v)}$ belong to $\mathcal{N}_{k(v)}^s$ (cf. [44, Cor. 3.8]). Thus the only new part of Proposition 1 is the case $p = 2$. If the \mathbb{Q} -rank of the adjoint group G^{ad} is 0 and $\mathcal{N}^s \neq \mathcal{N}$, then Theorem 1 (c) provides Néron models over $O_{(v)}$ which are not projective and thus which are not among the Néron models obtained in either [57, Prop. 4.4.1] or [67, Thm. 31]. Besides their applications to the conjecture of Langlands, Theorems 1 and 2 are also key steps in proving the deep conjectures [50, Conjs. B 3.7 and B 3.12] and [51, Conj. 1.6].

The uniqueness of an integral canonical model of $\mathrm{Sh}_H(G, \mathcal{X})$ over $O_{(v)}$ for $e(v) < p - 1$ was proved in [54, Subsubsect. 3.2.17] (cf. also [54, Fact of Subsubsect. 3.2.12 or Rm. 3.2.4 stated for $e(v) = 1 < p - 1$] and [55, Prop. 4.1], the last reference being a correction to the last part of [54, Step B of Subsubsect. 3.2.17]). The uniqueness of an integral canonical model of

$\mathrm{Sh}_H(G, \mathcal{X})$ over $O_{(v)}$ for $e(v) \leq p - 1$ is also a particular case of [67, Cor. 30]. Moreover, a second proof of [67, Cor. 30] can be obtained based on [20, Thm. 1], which also corrects [40, Subsect. 3.6.1].

If $p \geq 5$ and $G_{\mathbb{Z}_{(p)}}$ is a reductive group scheme, then Theorem 2 (c.i) was first obtained in [54, Rm. 3.2.12, Thms. 5.1 and 6.4.1], [58, App.], and [67, Thm. 31]. If the Shimura pair (G, \mathcal{X}) is unitary (i.e., $G_{\mathbb{C}}^{\mathrm{ad}}$ is a non-trivial product of **PGL** groups) and $G_{\mathbb{Z}_{(p)}}$ is a reductive group scheme, then Theorem 2 (c.i) follows also from [58, Thm. 5.1], [54, Subsubsect. 3.2.12], and [67, Thm. 31]. If $p \geq 3$ or if $p = 2$ and the 2-rank of each geometric fibre of the abelian scheme $\mathcal{A}_{k(v)}$ over $\mathcal{N}_{k(v)}$ is 0 and if moreover $G_{\mathbb{Z}_{(p)}}$ is a reductive group scheme, then Theorem 2 (c.i) has been also claimed in [27] which relies on [40]. Similarly, if $p = 2$ and $G_{\mathbb{Z}_{(p)}}$ is a reductive group scheme, then Theorem 2 (c.i) has been also claimed in [26].

Theorem 2 (c.i) represents progress towards the proof of a conjecture of Milne (see [37, Conj. 2.7] and [54, Conj. 3.2.5]) that pertains to the existence and the uniqueness of integral canonical models of arbitrary Shimura varieties.

The published works [43], [16], [41], [68], [32], [29], [54], [55], [57], [58], [62], [63], [64], [65], [27], and [26] are the most relevant ones for the existence of good smooth integral models of Shimura varieties of Hodge type. The construction of all integral models of Shimura varieties of Hodge type (such as \mathcal{N} in Subsection 1.4) via normalizations of schematic closures in integral models of Siegel moduli varieties (i.e., in Mumford moduli schemes) used in all these references follows entirely [53] and [54], and thus are based on an original idea of Faltings shared with us in 1993. See also [23, Sect. 5] for a translation of part of [16] in terms of the existence of good smooth integral models in arbitrary ramified mixed characteristic $(0, p)$ of very simple unitary Shimura varieties.

Part II will complete the proof of the conjecture of Milne on integral canonical models for the case of Shimura varieties of abelian type (see <http://arxiv.org/abs/0712.1572>). Part of Part II is also claimed in [27] and [26].

Part I brings completely new ideas in order to:

- shorten and simplify [54];
- extend many parts of [54] that were worked out only for $p \geq 5$ to the case of small primes $p \in \{2, 3\}$;
- achieve progress towards the proofs of conjectures of Langlands, Milne, and

Reimann;

- work with large classes of subgroups of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$ which in the case when $G_{\mathbb{Q}_p}$ splits over an unramified extension of \mathbb{Q}_p include as a very particular case the class of parahoric subgroups of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$ (and therefore also the class of hyperspecial subgroups of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$).

Theorem 2 (c.ii) for $p \geq 5$ corrects an error in the proof of [54, Prop. 3.2.3.2 ii)] that invalidated [54, Rm. 6.4.1.1 2) and most of Subsubsect. 6.4.11]. This correction was acknowledged and started in [57, Rm. 4.6 (b)] and [58, Thm. 5.1 (c) and App. E.8]. We recall that [58, App.] is the published erratum to [54].

The theory of local models aims to construct a projective scheme $\mathcal{N}_{\text{local}}$ over the completion O_v of $O_{(v)}$ which among other things is expected to model the singularities of the complement $\mathcal{N} \setminus \mathcal{N}^s$ (in the pro-étale topology); for instance, see [48] and [46]. To our best knowledge, so far this theory has not been able to prove either the existence or the uniqueness of integral canonical models of Shimura varieties of Hodge type which are not of PEL type. But it has been able to say a lot about the nature of the singularities of the complement $\mathcal{N} \setminus \mathcal{N}^s$ in many cases. The most advanced work in this direction is the recent paper [28] which works in the case when H is a parahoric subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$ and $G_{\mathbb{Q}_p}$ splits over a tamely ramified extension of \mathbb{Q}_p .

If $G_{\mathbb{Q}_p}$ splits over an unramified extension of \mathbb{Q}_p (thus $e(v) = 1$), then the class of subgroups H of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$ for which our results work is a lot more general than the class of parahoric subgroups of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$ considered in [28] (see Theorem 7). But even when $e(v) = 1$ neither this paper nor its Part II says anything about the complement $\mathcal{N} \setminus \mathcal{N}^s$ in the case when it is non-empty.

2 Preliminaries

In Subsection 2.1 we include some conventions and notation to be used throughout the paper. In Subsection 2.2 we study the injective map $f : (G, \mathcal{X}) \hookrightarrow (\mathbf{GSp}(W, \psi), \mathcal{Y})$. In Subsection 2.3 we consider \mathbb{C} -valued points of $\text{Sh}(G, \mathcal{X})$ and different realizations of Hodge cycles on abelian schemes over reduced \mathbb{Q} -schemes.

2.1 Conventions and notation

We recall that p is a prime and that k is a perfect field of characteristic p . Let $\sigma := \sigma_k$ be the Frobenius automorphism of k , $W(k)$, and of the field of fractions $B(k) := W(k)[\frac{1}{p}]$ of $W(k)$. For a Barsotti–Tate group D over $W(k)$, let $H^1(D)$ be the dual of the Tate-module of $D_{B(k)}$.

Let R , M , and F be as in the beginning of Section 1. If $*$ or $*_R$ is either a morphism or an object of the category of R -schemes and if S is a commutative R -algebra, let $*_S$ be the pullback of $*$ or $*_R$ to the category of S -schemes. Let $Z(F)$, F^{ad} , and F^{der} denote the center, the adjoint group scheme, and the derived group scheme (respectively) of F . We have $F^{\text{ad}} = F/Z(F)$. The group schemes $\mathbf{SL}_{n,R}$, etc., are over R . If $F_1 \hookrightarrow F$ is a closed embedding monomorphism of group schemes over R , then we identify F_1 with its image in F and we consider intersections of subgroups of $F_1(R)$ with subgroups of $F(R)$. By the *essential tensor algebra* of $M \oplus M^\vee$ we mean the R -module

$$\mathcal{T}(M) := \bigoplus_{s,t \in \mathbb{N}} M^{\otimes s} \otimes_R M^{\vee \otimes t}.$$

Let $F^1(M)$ be a direct summand of M . Let $F^0(M) := M$ and $F^2(M) := 0$. Let $F^1(M^\vee) := 0$, $F^0(M^\vee) := \{y \in M^\vee \mid y(F^1(M)) = 0\}$, and $F^{-1}(M^\vee) := M^\vee$. Let $(F^i(\mathcal{T}(M)))_{i \in \mathbb{Z}}$ be the tensor product filtration of $\mathcal{T}(M)$ defined by the resulting exhaustive, separated filtrations $(F^i(M))_{i \in \mathbb{Z}}$ and $(F^i(M^\vee))_{i \in \mathbb{Z}}$ of M and M^\vee (respectively). We refer to $(F^i(\mathcal{T}(M)))_{i \in \mathbb{Z}}$ as the filtration of $\mathcal{T}(M)$ defined by $F^1(M)$.

We identify naturally $\text{End}(M) = M \otimes_R M^\vee$ and $\text{End}(\text{End}(M)) = M^{\otimes 2} \otimes_R M^{\vee \otimes 2}$. Let $x \in R$ be a non-divisor of 0. A family of tensors of $\mathcal{T}(M[\frac{1}{x}]) = \mathcal{T}(M)[\frac{1}{x}]$ is denoted $(u_\alpha)_{\alpha \in \mathcal{J}}$, with \mathcal{J} as the set of indexes. Let M_1 be another free R -module of finite rank. Let $(u_{1,\alpha})_{\alpha \in \mathcal{J}}$ be a family of tensors of $\mathcal{T}(M_1[\frac{1}{x}])$ indexed by the same set \mathcal{J} . By an isomorphism $(M, (u_\alpha)_{\alpha \in \mathcal{J}}) \rightarrow (M_1, (u_{1,\alpha})_{\alpha \in \mathcal{J}})$ we mean an R -linear isomorphism $M \rightarrow M_1$ that extends naturally to an $R[\frac{1}{x}]$ -linear isomorphism $\mathcal{T}(M[\frac{1}{x}]) \rightarrow \mathcal{T}(M_1[\frac{1}{x}])$ which takes u_α to $u_{1,\alpha}$ for all $\alpha \in \mathcal{J}$. We denote two tensors or bilinear forms in the same way, provided they are obtained one from another via either a reduction modulo some ideal or a scalar extension.

The notation r , N , $\mathcal{A}_{r,1,N}$, \mathcal{M}_r , $\mu_h : \mathbb{G}_{m,\mathbb{C}} \rightarrow G_{\mathbb{C}}$, $(\mathbf{GSp}(W, \psi), \mathcal{Y})$, L , $K(N)$, K_p , $E(G, \mathcal{X}) \hookrightarrow \mathbb{C}$, $\text{Sh}(G, \mathcal{X})$, $\text{Sh}_C(G, \mathcal{X}) = \text{Sh}(G, \mathcal{X})/C$, v , $k(v)$, $e(v)$, $O_{(v)}$, $f : (G, \mathcal{X}) \hookrightarrow (\mathbf{GSp}(W, \psi), \mathcal{Y})$, $L_{(p)} := L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$, $G_{\mathbb{Z}_{(p)}}$, $H = K_p \cap G_{\mathbb{Q}_p}(\mathbb{Q}_p) = G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$, $f_0 : \text{Sh}(G, \mathcal{X}) \rightarrow \text{Sh}(\mathbf{GSp}(W, \psi), \mathcal{Y})_{E(G, \mathcal{X})}$, $f_p : \text{Sh}_H(G, \mathcal{X}) \rightarrow \text{Sh}_{K_p}(\mathbf{GSp}(W, \psi), \mathcal{Y})_{E(G, \mathcal{X})}$, \mathcal{N} , \mathcal{N}^s , and $(\mathcal{A}, \lambda_{\mathcal{A}})$ will be as in

Subsections 1.1, 1.3, and 1.4. Let $d := \dim_{\mathbb{C}}(\mathcal{X}) \in \mathbb{N}$ and $l := \dim(G) \in \mathbb{N}$. Let $G^0 := G \cap \mathbf{Sp}(W, \psi)$. As $G_{\mathbb{C}}$ is the semidirect product of $G_{\mathbb{C}}^0$ and the image of any cocharacter $\mu_h : \mathbb{G}_{m, \mathbb{C}} \rightarrow G_{\mathbb{C}}$ with $h \in \mathcal{X}$, G^0 is a connected, reductive, normal subgroup of G .

2.2 On the injective map f

Let $H^{(p)}$ be an arbitrary compact, open subgroup of $G(\mathbb{A}_f^{(p)})$ such that $H \times H^{(p)} \leq K(N)$. As the morphism $f_0 : \mathrm{Sh}(G, \mathcal{X}) \rightarrow \mathrm{Sh}(\mathbf{GSp}(W, \psi), \mathcal{Y})_{E(G, \mathcal{X})}$ is a closed embedding, the induced morphisms

$$f_p : \mathrm{Sh}_H(G, \mathcal{X}) \rightarrow \mathrm{Sh}_{K_p}(\mathbf{GSp}(W, \psi), \mathcal{Y})_{E(G, \mathcal{X})}$$

and

$$f_{H^{(p)}} : \mathrm{Sh}_{H \times H^{(p)}}(G, \mathcal{X}) \rightarrow \mathrm{Sh}_{K(N)}(\mathbf{GSp}(W, \psi), \mathcal{Y})_{E(G, \mathcal{X})}$$

are pro-finite and finite (respectively). Thus we can speak about the normalization \mathcal{Q} of $\mathcal{A}_{r,1,N,O_{(v)}}$ (equivalently, of the schematic closure in $\mathcal{A}_{r,1,N,O_{(v)}}$ of the image of $f_{H^{(p)}}$) in the ring of fractions of $\mathrm{Sh}_{H \times H^{(p)}}(G, \mathcal{X})$. We recall that every $O_{(v)}$ -scheme of finite type is excellent (for instance, cf. [34, (34.A) and (34.B)]). The $O_{(v)}$ -scheme $\mathcal{A}_{r,1,N,O_{(v)}}$ is quasi-projective (cf. property (i) of Subsection 1.1) and thus it is also excellent. Therefore the $O_{(v)}$ -scheme \mathcal{Q} is normal, quasi-projective, flat, has a relative dimension equal to $\dim(\mathrm{Sh}_{H \times H^{(p)}}(G, \mathcal{X})) = \dim_{\mathbb{C}}(\mathcal{X}) = d$, and is finite over the $O_{(v)}$ -scheme $\mathcal{A}_{r,1,N,O_{(v)}}$.

Let \mathcal{Q}^s be the smooth locus of \mathcal{Q} over $O_{(v)}$; it is an open subscheme of \mathcal{Q} . As $\mathrm{Sh}(\mathbf{GSp}(W, \psi), \mathcal{Y})$ is a pro-finite pro-étale cover of $\mathcal{A}_{r,1,N,\mathbb{Q}} = \mathrm{Sh}_{K(N)}(\mathbf{GSp}(W, \psi), \mathcal{Y})$, the group $K(N)$ acts freely on $\mathrm{Sh}(\mathbf{GSp}(W, \psi), \mathcal{Y})$. Thus the subgroup $H \times H^{(p)}$ of $K(N)$ acts freely on $\mathrm{Sh}(\mathbf{GSp}(W, \psi), \mathcal{Y})$ and therefore also on $\mathrm{Sh}(G, \mathcal{X})$. Thus $\mathcal{Q}_{E(G, \mathcal{X})} = \mathrm{Sh}_{H \times H^{(p)}}(G, \mathcal{X})$ is a smooth $E(G, \mathcal{X})$ -scheme and therefore it is the open subscheme $\mathcal{Q}_{E(G, \mathcal{X})}^s$ of \mathcal{Q}^s .

Fact 1. *The finite morphism $f_p : \mathrm{Sh}_H(G, \mathcal{X}) \rightarrow \mathrm{Sh}_{K_p}(\mathbf{GSp}(W, \psi), \mathcal{Y})_{E(G, \mathcal{X})}$ is in fact a closed embedding.*

Proof: As f_0 is a closed embedding, it suffices to show that the map

$$f_p(\mathbb{C}) : \mathrm{Sh}_H(G, \mathcal{X})(\mathbb{C}) \rightarrow \mathrm{Sh}_{K_p}(\mathbf{GSp}(W, \psi), \mathcal{Y})_{E(G, \mathcal{X})}(\mathbb{C})$$

is injective. But we have canonical identifications

$$\mathrm{Sh}_H(G, \mathcal{X})(\mathbb{C}) = G(\mathbb{Q}) \backslash [\mathcal{X} \times G(\mathbb{A}_f)] / H$$

and

$$\mathrm{Sh}_{K_p}(\mathbf{GSp}(W, \psi), \mathcal{Y})_{E(G, \mathcal{X})}(\mathbb{C}) = \mathbf{GSp}(W, \psi)(\mathbb{Q}) \backslash [\mathcal{Y} \times \mathbf{GSp}(W, \psi)(\mathbb{A}_f)] / K_p$$

(cf. [13, Cor. 2.1.11]) and based on these and the fact that the intersections $G(\mathbb{A}_f^{(p)}) \cap H$ and $\mathbf{GSp}(W, \psi)(\mathbb{A}_f^{(p)}) \cap K_p$ are the trivial subgroups of $G(\mathbb{A}_f)$ and $\mathbf{GSp}(W, \psi)(\mathbb{A}_f)$ (respectively), one easily gets that $f_p(\mathbb{C})$ is an injective map. \square

Proposition 2. *The following three properties hold:*

(a) *The $O_{(v)}$ -scheme \mathcal{N} is a pro-finite pro-étale cover of \mathcal{Q} and \mathcal{Q} is the quotient of \mathcal{N} by $H^{(p)}$.*

(b) *The morphism $\mathcal{N} \rightarrow \mathcal{M}_{r, O_{(v)}}$ is finite.*

(c) *We assume that $e(v) \leq p - 1$. If Z is a regular, formally smooth scheme over a discrete valuation ring O which is of absolute ramification index at most $p - 1$ and an $O_{(v)}$ -algebra, then each morphism $Z_{E(G, \mathcal{X})} \rightarrow \mathcal{N}_{E(G, \mathcal{X})}$ extends uniquely to a morphism $Z \rightarrow \mathcal{N}$ of $O_{(v)}$ -schemes.*

Proof: Let $N_1 \in \mathbb{N}\mathbb{N} \setminus p\mathbb{N}$. Let $N_2 := N$. For $i \in \{1, 2\}$ we write $K(N_i) = K_p \times K(N_i)^{(p)}$, where the group $K(N_i)^{(p)}$ is a compact, open subgroup of $\mathbf{GSp}(W, \psi)(\mathbb{A}_f^{(p)})$. The scheme \mathcal{M}_r is a pro-finite pro-étale cover of $\mathcal{M}_r / K(N_i)^{(p)} = \mathcal{A}_{r, 1, N_i}$. Let H_i be a compact, open subgroup of $G(\mathbb{A}_f^{(p)}) \cap K(N_i)^{(p)}$; thus $\mathrm{Sh}(G, \mathcal{X})$ is a pro-finite pro-étale cover of $\mathrm{Sh}_{H \times H_i}(G, \mathcal{X})$. The morphism $\mathrm{Sh}_{H \times H_i}(G, \mathcal{X})_{\mathbb{C}} \rightarrow \mathcal{A}_{r, 1, N_i, \mathbb{C}}$ is of finite type and a formally closed embedding at each \mathbb{C} -valued point of $\mathrm{Sh}_{H \times H_i}(G, \mathcal{X})_{\mathbb{C}}$. Let \mathcal{Q}_i be the normalization of $\mathcal{A}_{r, 1, N_i, O_{(v)}}$ in the ring of fractions of $\mathrm{Sh}_{H \times H_i}(G, \mathcal{X})$; it is a finite $\mathcal{A}_{r, 1, N_i, O_{(v)}}$ -scheme and a normal, quasi-projective, flat $O_{(v)}$ -scheme of relative dimension d .

As $N_1 \in N_2\mathbb{N}^*$, we have $K(N_1)^{(p)} \leq K(N_2)^{(p)}$. We assume that H_1 is a normal subgroup of H_2 . The natural morphism $q_{12} : \mathcal{Q}_1 \rightarrow \mathcal{Q}_2 \times_{\mathcal{A}_{r, 1, N_2, O_{(v)}}} \mathcal{A}_{r, 1, N_1, O_{(v)}}$ of normal schemes is finite. We check that $q_{12, E(G, \mathcal{X})}$ is an open closed embedding. As $q_{12, E(G, \mathcal{X})}$ is a finite, étale morphism between normal

$E(G, \mathcal{X})$ -schemes of finite type, it is enough to check that the map $q_{12}(\mathbb{C}) : \mathcal{Q}_1(\mathbb{C}) \rightarrow \mathcal{Q}_2(\mathbb{C}) \times_{\mathcal{A}_{r,1,N_2,O_{(v)}}(\mathbb{C})} \mathcal{A}_{r,1,N_1,O_{(v)}}(\mathbb{C})$ is injective. We have

$$\mathrm{Sh}_{K_p \times H_i}(\mathbf{GSp}(W, \psi), \mathcal{Y})(\mathbb{C}) = \mathbf{GSp}(L, \psi)(\mathbb{Z}_{(p)}) \backslash [\mathcal{Y} \times (\mathbf{GSp}(W, \psi)(\mathbb{A}_f^{(p)})/H_i)]$$

(for instance, cf. [38, Prop. 4.11]). Also we have a natural disjoint union decomposition

$$(1) \quad \mathrm{Sh}_{H \times H_i}(G, \mathcal{X})(\mathbb{C}) = \sqcup_{[g_j] \in G(\mathbb{Q}) \backslash G(\mathbb{Q}_p)/H} C_j \backslash [\mathcal{X} \times (G(\mathbb{A}_f^{(p)})/H_i)],$$

where $g_j \in G(\mathbb{Q}_p)$ is a representative of the class $[g_j] \in G(\mathbb{Q}) \backslash G(\mathbb{Q}_p)/H$ and where the group $C_j := G(\mathbb{Q}) \cap g_j H g_j^{-1}$ does not depend on $i \in \{1, 2\}$. As we have an identity $\mathbf{GSp}(W, \psi)(\mathbb{Q}_p) = \mathbf{GSp}(W, \psi)(\mathbb{Q})K_p$ (cf. [38, Lem. 4.9]), we can write $g_j = a_j h_j$, where $a_j \in \mathbf{GSp}(W, \psi)(\mathbb{Q})$ and $h_j \in K_p$. Thus

$$C_j \leq \mathbf{GSp}(W, \psi)(\mathbb{Q}) \cap g_j K_p g_j^{-1} = \mathbf{GSp}(W, \psi)(\mathbb{Q}) \cap a_j K_p a_j^{-1} = a_j \mathbf{GSp}(L, \psi)(\mathbb{Z}_{(p)}) a_j^{-1} =: C_j^{\mathrm{bigg}}.$$

We have $C_j = G(\mathbb{Q}) \cap C_j^{\mathrm{bigg}}$. This is so as $g_j H g_j^{-1}$ is the group of \mathbb{Z}_p -valued points of the schematic closure of G in $a_j \mathbf{GSp}(L, \psi)_{\mathbb{Z}_{(p)}} a_j^{-1}$.

To show that the map $q_{12}(\mathbb{C})$ is injective, it suffices to show that each one of the following commutative diagrams indexed by j

$$\begin{array}{ccc} C_j \backslash [\mathcal{X} \times (G(\mathbb{A}_f^{(p)})/H_1)] & \xrightarrow{s_1} & \mathbf{GSp}(L, \psi)(\mathbb{Z}_{(p)}) \backslash [\mathcal{Y} \times (\mathbf{GSp}(W, \psi)(\mathbb{A}_f^{(p)})/H_1)] \\ \downarrow \pi_{12} & & \downarrow \pi_{12}^{\mathrm{bigg}} \\ C_j \backslash [\mathcal{X} \times (G(\mathbb{A}_f^{(p)})/H_2)] & \xrightarrow{s_2} & \mathbf{GSp}(L, \psi)(\mathbb{Z}_{(p)}) \backslash [\mathcal{Y} \times (\mathbf{GSp}(W, \psi)(\mathbb{A}_f^{(p)})/H_2)] \end{array}$$

is such that the maps π_{12} and s_1 define an injective map of $C_j \backslash [\mathcal{X} \times (G(\mathbb{A}_f^{(p)})/H_1)]$ into the fibre product of s_2 and π_{12}^{bigg} . Here the maps π_{12} and π_{12}^{bigg} are the natural projections. The maps s_1 and s_2 are defined by the rule: the equivalence class $[h, g]$, where $h \in \mathcal{X}$ and $g \in G(\mathbb{A}_f^{(p)})$, is mapped to the equivalence class $[a_j^{-1} h, a_j^{-1} g]$. Thus the fact that π_{12} and s_1 define an injective map of $C_j \backslash [\mathcal{X} \times (G(\mathbb{A}_f^{(p)})/H_1)]$ into the fibre product of s_2 and π_{12}^{bigg} is a direct consequence of the identity $C_j = G(\mathbb{Q}) \cap C_j^{\mathrm{bigg}}$. Thus $q_{12}(\mathbb{C})$ is injective.

Therefore $q_{12,E(G,\mathcal{X})}$ is an open closed embedding. As q_{12} is also a finite morphism of normal, flat $O_{(v)}$ -schemes of finite type, q_{12} itself is an open closed embedding. Thus \mathcal{Q}_1 is a finite étale cover of \mathcal{Q}_2 that in characteristic 0

is a finite étale cover which (as $H_1 \triangleleft H_2$) induces finite Galois covers between connected components. Therefore \mathcal{Q}_1 is a finite étale cover of \mathcal{Q}_2 which induces finite Galois covers between connected components. This implies that \mathcal{Q}_2 is the quotient of \mathcal{Q}_1 under the natural action of H_2/H_1 on it.

By allowing H_1 to vary among the normal, open subgroups of H_2 and by a natural passage to limits, we get that \mathcal{N} is a pro-finite pro-étale cover of \mathcal{Q}_2 and that $\mathcal{Q}_2 = \mathcal{N}/H_2$. Thus by taking $H_2 = H^{(p)}$, we get that $\mathcal{Q} = \mathcal{Q}_2$ and that part (a) holds.

As each morphism $q_{12} : \mathcal{Q}_1 \rightarrow \mathcal{Q}_2 \times_{\mathcal{A}_{r,1,N_2,O(v)}} \mathcal{A}_{r,1,N_1,O(v)}$ is an open closed embedding, by allowing H_1 to vary through all normal, open subgroups of H_2 we get that \mathcal{N} is an open closed subscheme of $\mathcal{Q}_2 \times_{\mathcal{A}_{r,1,N_2,O(v)}} \mathcal{M}_{r,O(v)}$ and thus part (b) holds.

To prove part (c), we recall that Z is a healthy regular scheme in the sense of either [54, Def. 3.2.1 2)] or [55] (cf. [67, Cor. 5]). Thus part (c) is implied by [54, Ex. 3.2.9 and Prop. 3.4.1], cf. definitions [54, Def. 3.2.3 2), 3), and 6)] (to be compared with the argument for the property (iii) of Subsection 1.1). \square

Remark 2. Similar arguments to the ones that checked that \mathcal{N} is a pro-finite pro-étale cover of \mathcal{N}/H_2 can be used to check that the right action of $G(\mathbb{A}_f^{(p)})$ on \mathcal{N} is indeed a continuous action in the sense of [13, Subsubsect. 2.7.1] and in what follows we will use this property without any extra comment.

Lemma 1. *The scheme \mathcal{N}^s is an open subscheme of \mathcal{N} and $\mathcal{N}_{E(G,\mathcal{X})}^s = \mathcal{N}_{E(G,\mathcal{X})}$. Moreover, if $\mathcal{N}_{k(v)}^s$ is a non-empty scheme, then \mathcal{N}^s together with the resulting action of $G(\mathbb{A}_f^{(p)})$ on it is a regular, formally smooth integral model of $\text{Sh}_H(G, \mathcal{X})$ over $O(v)$.*

Proof: As \mathcal{N} is a pro-finite pro-étale cover of the excellent, quasi-projective $O(v)$ -scheme \mathcal{Q} (see Proposition 2 (a)), $\mathcal{N}^s = \mathcal{N} \times_{\mathcal{Q}} \mathcal{Q}^s$ is an open subscheme of \mathcal{N} . As $\mathcal{Q}_{E(G,\mathcal{X})} = \mathcal{Q}_{E(G,\mathcal{X})}^s$, we have $\mathcal{N}_{E(G,\mathcal{X})}^s = \mathcal{N}_{E(G,\mathcal{X})}$. The open subscheme \mathcal{N}^s of \mathcal{N} is $G(\mathbb{A}_f^{(p)})$ -invariant. As $G(\mathbb{A}_f^{(p)})$ acts continuously on \mathcal{N} , it also acts continuously on \mathcal{N}^s . Thus, if the scheme $\mathcal{N}_{k(v)}^s$ is non-empty, then \mathcal{N}^s together with the resulting continuous action of $G(\mathbb{A}_f^{(p)})$ on it is a regular, formally smooth integral model of $\text{Sh}_H(G, \mathcal{X})$ over $O(v)$. \square

Fact 2. *We assume that there exists a simple factor G_1 of $G_{\mathbb{Q}}^{ad}$ which is an SO_{2n+1} group for some $n \in \mathbb{N}^*$. Let G_2 be the semisimple, normal subgroup*

of $G_{\overline{\mathbb{Q}}}$ whose adjoint is naturally identified with G_1 . Then G_2 is a **Spin** $_{2n+1}$ group.

Proof: The $\mathrm{Lie}(G_2)$ -module $W \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is non-trivial and its irreducible $\mathrm{Lie}(G_2)$ -submodules are associated to the weight ϖ_n of the B_n Lie type, cf. [38, p. 456]. Thus G_2 is a **Spin** $_{2n+1}$ group. \square

Lemma 2. *If the \mathbb{Q} -rank of the adjoint group G^{ad} is 0, then \mathcal{Q} is a projective $O_{(v)}$ -scheme.*

Proof: Let G' be the smallest subgroup of G such that all elements $h \in \mathcal{X}$ factor through $G'_{\mathbb{R}}$. It is a normal, reductive subgroup of G that contains G^{der} ; thus $G'^{\mathrm{ad}} = G^{\mathrm{ad}}$. Let $h' \in \mathcal{X}$ be an element such that G' is the smallest subgroup of \mathbf{GL}_W with the property that h' factors through $G'_{\mathbb{R}}$. We can assume that the \mathbb{C} -valued point $[h', 1_W] \in \mathrm{Sh}_{H \times H^{(p)}}(G, \mathcal{X})$ is definable over a number field (here 1_W is the identity element of $G(\mathbb{A}_f)$ modulo $H^{(p)}$) and that $2\pi i\psi$ is a principal polarization of the Hodge \mathbb{Z} -structure on L defined by h' . Thus G' is the Mumford–Tate group of the principally polarized Hodge \mathbb{Z} -structure on L defined by h' and ψ and this principally polarized Hodge \mathbb{Z} -structure is associated naturally to a principally polarized abelian scheme over a number field.

Let \mathcal{X}' be the $G'(\mathbb{R})$ -conjugacy class of h' . The pair (G', \mathcal{X}') is a Shimura pair whose reflex field and dimension are $E(G, \mathcal{X})$ and d (respectively). Let $H' := H \cap G'(\mathbb{Q}_p)$ and $H'^{(p)} := H^{(p)} \cap G'(\mathbb{A}_f^{(p)})$. As the \mathbb{Q} -rank of $G'^{\mathrm{ad}} = G^{\mathrm{ad}}$ is 0, as in [57, Prop. 2.7] we argue that the normalization \mathcal{Q}' of $\mathcal{A}_{r,1,N,O_{(v)}}$ in $\mathrm{Sh}_{H' \times H'^{(p)}}(G', \mathcal{X}')$ is a projective $O_{(v)}$ -scheme provided the Morita conjecture holds for all abelian varieties over number fields. We recall from [47], [57], and [33], that the Morita conjecture predicts that each abelian variety over a number field with the property that a pullback of it over \mathbb{C} has a Mumford–Tate group whose adjoint has \mathbb{Q} -rank 0, has potentially good reduction everywhere. As the Morita conjecture holds (see [33]), we get that \mathcal{Q}' is a projective $O_{(v)}$ -scheme.

The Shimura variety $\mathrm{Sh}(G', \mathcal{X}')$ is a closed subscheme of $\mathrm{Sh}(G, \mathcal{X})$ of the same dimension d and therefore it is an open closed subscheme of $\mathrm{Sh}(G, \mathcal{X})$. Thus each connected component of the normalization of $\mathcal{A}_{r,1,N,O_{(v)}}$ (equivalently of \mathcal{Q}) in the ring of fractions of $\mathrm{Sh}(G, \mathcal{X})$ is a $G(\mathbb{A}_f)$ -translation of a connected component of the normalization of $\mathcal{A}_{r,1,N,O_{(v)}}$ (equivalently of \mathcal{Q}') in the ring of fractions of $\mathrm{Sh}(G', \mathcal{X}')$. Thus, as \mathcal{Q}' is a projective $O_{(v)}$ -scheme, we get directly that \mathcal{Q} is a projective $O_{(v)}$ -scheme. \square

2.3 Tensors

The image of each $h \in \mathcal{X}$ contains $Z(\mathbf{GL}_{W \otimes_{\mathbb{Q}} \mathbb{R}})$. Thus $Z(\mathbf{GL}_W) \leq G$ and therefore each tensor of $\mathcal{T}(W^\vee)$ fixed by G belongs to the direct summand $\bigoplus_{u \in \mathbb{N}} W^{\vee \otimes u} \otimes_{\mathbb{Q}} W^{\otimes u}$ of $\mathcal{T}(W^\vee)$. We consider a family of tensors $(v_\alpha)_{\alpha \in \mathcal{J}}$ in $\bigsqcup_{u=0}^{\infty} W^{\vee \otimes u} \otimes_{\mathbb{Q}} W^{\otimes u} \subset \mathcal{T}(W^\vee)$ such that G is the subgroup of \mathbf{GL}_W that fixes v_α for all $\alpha \in \mathcal{J}$, cf. [14, Prop. 3.1 c)].

Let $\mathfrak{T} : \text{End}(W \otimes_{\mathbb{Q}} \mathbb{Q}_p) \times \text{End}(W \otimes_{\mathbb{Q}} \mathbb{Q}_p) \rightarrow \mathbb{Q}_p$ be the trace bilinear form on $\text{End}(W \otimes_{\mathbb{Q}} \mathbb{Q}_p)$. If \mathfrak{b} is a reductive subgroup of $\mathbf{GL}_{W \otimes_{\mathbb{Q}} \mathbb{Q}_p}$, then the restriction of \mathfrak{T} to $\text{Lie}(\mathfrak{b})$ is non-degenerate (cf. Lemma 11 (b)). Let $\pi_{\mathfrak{b}}$ be the projector of $\text{End}(W \otimes_{\mathbb{Q}} \mathbb{Q}_p)$ on $\text{Lie}(\mathfrak{b})$ along the perpendicular on $\text{Lie}(\mathfrak{b})$ with respect to \mathfrak{T} . If $G_{\mathbb{Q}_p}$ normalizes \mathfrak{b} , then $G_{\mathbb{Q}_p}$ fixes $\pi_{\mathfrak{b}}$.

2.3.1 Complex manifolds

For a smooth \mathbb{C} -scheme Y , let Y^{an} be the complex manifold associated naturally to Y . It is well-known that for each $u \in \mathbb{N}^*$ and for every abelian scheme $\Pi : C \rightarrow Y$ and its associated morphism $\Pi^{\text{an}} : C^{\text{an}} \rightarrow Y^{\text{an}}$ of complex manifolds, we have a natural isomorphism

$$(2) \quad R^u \Pi_*^{\text{an}}(\mathbb{C}) \rightarrow R^u \Pi_*^{\text{an}}(\Omega_{C^{\text{an}}/Y^{\text{an}}}^*)^{\nabla_C^{\text{an}}}$$

of complex sheaves on Y^{an} , where ∇_C^{an} is the connection on $R^u \Pi_*^{\text{an}}(\Omega_{C^{\text{an}}/Y^{\text{an}}}^*)$ induced by the Gauss–Manin connection on $R^u \Pi_*(\Omega_{C/Y}^*)$.

2.3.2 Hodge cycles

We will use the terminology of [14] on Hodge cycles on an abelian scheme B_X over a reduced \mathbb{Q} -scheme X . Thus we write each Hodge cycle v on B_X as a pair $(v_{\text{dR}}, v_{\text{ét}})$, where v_{dR} and $v_{\text{ét}}$ are the de Rham and the étale component of v (respectively). The étale component $v_{\text{ét}}$ at its turn has an l -component $v_{\text{ét}}^l$, for each rational prime l .

In what follows we will be interested only in Hodge cycles on B_X that involve no Tate twists and that are tensors of different essential tensor algebras. Accordingly, if X is the spectrum of a field E , then in applications $v_{\text{ét}}^p$ will be a suitable $\text{Gal}(\overline{E}/E)$ -invariant tensor of $\mathcal{T}(H_{\text{ét}}^1(B_{\overline{X}}, \mathbb{Q}_p))$, where $\overline{X} := \text{Spec}(\overline{E})$. If moreover \overline{E} is a subfield of \mathbb{C} , then we will also use the Betti realization v_B of v : it is a tensor of $\mathcal{T}(H^1((B_X \times_X \text{Spec}(\mathbb{C}))^{\text{an}}, \mathbb{Q}))$ that corresponds to v_{dR} (resp. to $v_{\text{ét}}^l$) via the canonical isomorphism that

relates the Betti cohomology of $(B_X \times_X \text{Spec}(\mathbb{C}))^{\text{an}}$ with \mathbb{Q} -coefficients with the de Rham (resp. the \mathbb{Q}_l étale) cohomology of $B_{\overline{X}}$ (see [14, Sect. 2]). We recall that v_B is also a tensor of the F^0 -filtration of the Hodge filtration of $\mathcal{T}(H^1((B_X \times_X \text{Spec}(\mathbb{C}))^{\text{an}}, \mathbb{C}))$.

2.3.3 On $\mathcal{A}_{E(G, \mathcal{X})}$

The choice of the \mathbb{Z} -lattice L of W and of the family of tensors $(v_\alpha)_{\alpha \in \mathcal{J}}$ allows a moduli interpretation of $\text{Sh}(G, \mathcal{X})$ (see [12], [13], [38], and [54, Subsect. 4.1, Lem. 4.1.3]). For instance, $\text{Sh}(G, \mathcal{X})(\mathbb{C}) = G(\mathbb{Q}) \backslash (\mathcal{X} \times G(\mathbb{A}_f))$ is the set of isomorphism classes of principally polarized abelian varieties over \mathbb{C} of dimension r , that carry a family of Hodge cycles indexed by \mathcal{J} , that have compatible level- N symplectic similitude structures for each $N \in \mathbb{N}^*$, and that satisfy few axioms. Thus the abelian scheme $\mathcal{A}_{E(G, \mathcal{X})}$ over $\mathcal{N}_{E(G, \mathcal{X})}$ is endowed with a family $(w_\alpha^A)_{\alpha \in \mathcal{J}}$ of Hodge cycles; all realizations of pullbacks of w_α^A via \mathbb{C} -valued points of $\mathcal{N}_{E(G, \mathcal{X})}^s$ correspond naturally to v_α .

Lemma 3. *Let $w \in \text{Sh}(G, \mathcal{X})(\mathbb{C})$. We denote also by w the \mathbb{C} -valued point of \mathcal{N} defined by w ; thus we can define $(A_w, \lambda_{A_w}) := w^*((\mathcal{A}, \lambda_{\mathcal{A}}))$. Let u_α^w (resp. t_α^w) be the p -component of the étale component (resp. be the de Rham component) of the Hodge cycle $w^*(w_\alpha^A)$ on A_w . We have:*

(a) *There exist isomorphisms $(H_{\acute{e}t}^1(A_w, \mathbb{Z}_p), (u_\alpha^w)_{\alpha \in \mathcal{J}}) \rightarrow (L_{(p)}^\vee \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p, (v_\alpha)_{\alpha \in \mathcal{J}})$ that take the perfect bilinear form on $H_{\acute{e}t}^1(A_w, \mathbb{Z}_p)$ defined by λ_{A_w} to a $\mathbb{G}_{m, \mathbb{Z}_p}(\mathbb{Z}_p)$ -multiple of the perfect bilinear form ψ^\vee on $L_{(p)}^\vee \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p$ defined by ψ .*

(b) *There exist isomorphisms $(H_{dR}^1(A_w, \mathbb{C}), (t_\alpha^w)_{\alpha \in \mathcal{J}}, \psi_{H_{dR}^1(A_w, \mathbb{C})}) \rightarrow (W^\vee \otimes_{\mathbb{Q}} \mathbb{C}, (v_\alpha)_{\alpha \in \mathcal{J}}, \psi^\vee)$, where $\psi_{H_{dR}^1(A_w, \mathbb{C})}$ is the perfect bilinear form on $H_{dR}^1(A_w, \mathbb{C})$ defined by λ_{A_w} .*

Proof: We write $w = [h_w, g_w] \in \text{Sh}(G, \mathcal{X})(\mathbb{C}) = G(\mathbb{Q}) \backslash (\mathcal{X} \times G(\mathbb{A}_f))$, where $h_w \in \mathcal{X}$ and $g_w \in G(\mathbb{A}_f)$. From the standard moduli interpretation of $\text{Sh}(G, \mathcal{X})(\mathbb{C})$ applied to $w \in \text{Sh}(G, \mathcal{X})(\mathbb{C})$ we get (see [12], [37], [38], and [54, p. 454]) that the complex manifold A_w^{an} associated to A_w is $L_w \backslash W \otimes_{\mathbb{Q}} \mathbb{C} / F_w^{0, -1}$, where:

(i) L_w is the \mathbb{Z} -lattice of W defined uniquely by the identity $L_w \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} = g_w(L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})$;

(ii) $W \otimes_{\mathbb{Q}} \mathbb{C} = F_w^{0, -1} \oplus F_w^{-1, 0}$ is the usual Hodge decomposition of the Hodge \mathbb{Q} -structure on W defined by $h_w \in \mathcal{X}$;

(iii) the principal polarization λ_{A_w} of A_w is defined naturally by a uniquely determined (non-zero) rational multiple of ψ ;

(iv) under the canonical identifications $H_{\text{dR}}^1(A_w/\mathbb{C}) = H_{\text{dR}}^1(A_w^{\text{an}}/\mathbb{C}) = W^\vee \otimes_{\mathbb{Q}} \mathbb{C} = L_w^\vee \otimes_{\mathbb{Z}} \mathbb{C}$, each tensor t_α^w gets identified with v_α for all $\alpha \in \mathcal{J}$ and $\psi_{H_{\text{dR}}^1(A_w, \mathbb{C})}$ gets identified with a (non-zero) complex multiple of ψ^\vee .

Thus $(H_{\text{ét}}^1(A_w, \mathbb{Z}_p), (u_\alpha^w)_{\alpha \in \mathcal{J}})$ is identified naturally with $(L_w^\vee \otimes_{\mathbb{Z}} \mathbb{Z}_p, (v_\alpha)_{\alpha \in \mathcal{J}})$ (cf. (iv)) and therefore also with a $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$ -conjugate of $(L_{(p)}^\vee \otimes_{\mathbb{Z}(p)} \mathbb{Z}_p, (v_\alpha)_{\alpha \in \mathcal{J}})$ (cf. (i)). Part (a) follows from this and from the existence of the rational multiple of ψ mentioned in the property (iii).

For each non-zero complex number ϵ , the automorphism of $(W^\vee \otimes_{\mathbb{Q}} \mathbb{C}, (v_\alpha)_{\alpha \in \mathcal{J}})$ defined by $\mu_{h_w}(\mathbb{C})(\epsilon^{-1})$ acts on the \mathbb{C} -span of ψ^\vee as the multiplication by ϵ . From this and the property (iv) we get that part (b) holds. \square

Lemma 4. *Let $m \in \mathbb{N}$. Let $\mathcal{R}_1 := \mathbb{C}[[x_1, \dots, x_m]]$, where x_1, \dots, x_m are independent variables. Let $\mathcal{I}_1 := (x_1, \dots, x_m)$ be the maximal ideal of \mathcal{R}_1 . Let $s \in \mathbb{N}^*$. Let $A_{w,s}$ be an abelian scheme over $\mathcal{R}_1/\mathcal{I}_1^s$ that is a deformation of A_w (i.e., we have $A_w = A_{w,s} \times_{\text{Spec}(\mathcal{R}_1/\mathcal{I}_1^s)} \text{Spec}(\mathcal{R}_1/\mathcal{I}_1)$) and which has a principal polarization. Then there exists a unique isomorphism*

$$I_{w,s} : H_{\text{dR}}^1(A_{w,s}/(\mathcal{R}_1/\mathcal{I}_1^s)) \rightarrow H_{\text{dR}}^1(A_w/\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{R}_1/\mathcal{I}_1^s$$

that has the following two properties:

- (i) it lifts (i.e., modulo $\mathcal{I}_1/\mathcal{I}_1^s$ is) the identity automorphism of $H_{\text{dR}}^1(A_w/\mathbb{C})$;
- (ii) under it, the Gauss–Manin connection on $H_{\text{dR}}^1(A_{w,s}/(\mathcal{R}_1/\mathcal{I}_1^s))$ becomes isomorphic to the flat connection δ on the $\mathcal{R}_1/\mathcal{I}_1^s$ -module $H_{\text{dR}}^1(A_w/\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{R}_1/\mathcal{I}_1^s$ that annihilates $H_{\text{dR}}^1(A_w/\mathbb{C}) \otimes 1$.

Proof: The uniqueness of $I_{w,s}$ is implied by the fact that $H_{\text{dR}}^1(A_w/\mathbb{C}) \otimes 1$ is the set of all elements of $H_{\text{dR}}^1(A_w/\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{R}_1/\mathcal{I}_1^s$ that are annihilated by δ . We consider an abelian scheme $\Pi : A_Y \rightarrow Y$ over a smooth \mathbb{C} -scheme Y of dimension m which is a global deformation of $A_{w,s} \rightarrow \text{Spec}(\mathcal{R}_1/\mathcal{I}_1^s)$ and which has a principal polarization. Let Z^{an} be a simply connected open complex submanifold of Y^{an} that contains the \mathbb{C} -valued point of Y defined naturally by A_w . We identify naturally $\text{Spec}(\mathcal{R}_1/\mathcal{I}_1^s)$ with a complex analytic subspace of Y^{an} and thus also of Z^{an} . We apply Formula (2) with $u = 1$ and with $\Pi : C \rightarrow Y$ replaced by $\Pi : A_Y \rightarrow Y$. The pullback of $R^1 \Pi_*^{\text{an}}(\mathbb{C})$ to Z^{an} is a constant sheaf on Z^{an} . Thus by pulling back Formula (2) to the complex analytic subspace $\text{Spec}(\mathcal{R}_1/\mathcal{I}_1^s)$ of Z^{an} , we get directly the existence of $I_{w,s}$ for which properties (i) and (ii) hold. \square

Corollary 2. *Let m , \mathcal{R}_1 , and \mathcal{I}_1 be as in Lemma 4. Let $A_{w,\infty}$ be an abelian scheme over \mathcal{R}_1 that is a deformation of A_w and that has a principal polarization. Then there exists a unique isomorphism*

$$I_{w,\infty} : H_{dR}^1(A_{w,\infty}/\mathcal{R}_1) \rightarrow H_{dR}^1(A_w/\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{R}_1$$

that has the following two properties:

- (i) *it lifts (i.e., modulo \mathcal{I}_1 is) the identity automorphism of $H_{dR}^1(A_w/\mathbb{C})$;*
- (ii) *under it, the \mathcal{I}_1 -completion of the Gauss–Manin connection on $H_{dR}^1(A_{w,\infty}/\mathcal{R}_1)$ becomes isomorphic to the \mathcal{I}_1 -completion of the flat connection δ on the \mathcal{R}_1 -module $H_{dR}^1(A_w/\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{R}_1$ that annihilates $H_{dR}^1(A_w/\mathbb{C}) \otimes 1$.*

If $w_\alpha^{\mathcal{R}_1}$ (resp. $\lambda_{A_{w,\infty}}$) is a Hodge cycle on (resp. a principal polarization of) $A_{w,\infty}$ that lifts the Hodge cycle $w^(w_\alpha^A)$ on A_w (resp. lifts the principal polarization λ_{A_w} of A_w), then the isomorphism*

$$I_{w,\infty} : \mathcal{T}(H_{dR}^1(A_{w,\infty}/\mathcal{R}_1)) \rightarrow \mathcal{T}(H_{dR}^1(A_w/\mathbb{C})) \otimes_{\mathbb{C}} \mathcal{R}_1$$

induced naturally by $I_{w,\infty}$ (and denoted in the same way) takes the de Rham realization of $w_\alpha^{\mathcal{R}_1}$ (resp. of $\lambda_{A_{w,\infty}}$) to t_α^w (resp. to the de Rham realization of λ_{A_w}).

Proof: The existence and the uniqueness of $I_{w,\infty}$ follow from Lemma 4 by taking $s \rightarrow \infty$. It is well-known that each de Rham component of a Hodge cycle on $A_{w,\infty}$ is annihilated by the Gauss–Manin connection on $\mathcal{T}(H_{dR}^1(A_{w,\infty}/\mathcal{R}_1))$. For instance, this follows from [14, Prop. 2.5] via a natural algebraization process. Thus $I_{w,\infty}(w_\alpha^{\mathcal{R}_1})$ and t_α^w are tensors of $\mathcal{T}(H_{dR}^1(A_w/\mathbb{C})) \otimes_{\mathbb{C}} \mathcal{R}_1$ which are annihilated by the \mathcal{I}_1 -completion of the flat connection on $\mathcal{T}(H_{dR}^1(A_w/\mathbb{C})) \otimes_{\mathbb{C}} \mathcal{R}_1$ induced by δ and which modulo \mathcal{I}_1 coincide. Therefore the two tensors coincide, i.e., we have $I_{w,\infty}(w_\alpha^{\mathcal{R}_1}) = t_\alpha^w$. A similar argument shows that $I_{w,\infty}$ takes $\lambda_{A_{w,\infty}}$ to the de Rham realization of λ_{A_w} . \square

3 Crystalline applications

Theorem 3 recalls a variant of the main result of [11]. In Subsection 3.1 we first introduce notation required to prove Theorems 1 and 2 and then we apply the main result of [64] in the form recalled in Theorem 9. In Subsection 3.2 we apply the deformation theory of [17, Sect. 7]. Subsection 3.3 proves the

Basic Theorem 1. Subsection 3.4 introduces de Rham realizations of certain Hodge cycles. Subsection 3.5 defines the open subscheme \mathcal{N}^m of \mathcal{N}^s , proves Corollary 1 and a variant of it, and lists few simple crystalline properties that are required in Sections 4 and 5. Subsection 3.6 proves Proposition 1 based also on Lemma 7. Throughout this section we assume that $e(v) = 1$.

For (crystalline or de Rham) Fontaine comparison theory we refer to [19], [17, Sect. 5], and [64]; see also Subsections B2, B3 and B6. We recall that k is a perfect field of characteristic p . As the Verschiebung maps of Barsotti–Tate groups are not mentioned at all in what follows, we use the terminology F -crystals (resp. filtered F -crystals) associated to Barsotti–Tate groups over k , $k[[x]]$, or $k((x))$ (resp. over $W(k)$ or $W(k)[[x]]$) instead of the terminology Dieudonné F -crystals (resp. filtered Dieudonné F -crystals) of [4, Ch. 3] and [3, Chs. 2 and 3].

Let x be an independent variable. The simplest form of [11, Thm. 1.1] says:

Theorem 3 (de Jong). *The natural functor from the category of F -crystals over $k[[x]]$ to the category of F -crystals over $k((x))$ is fully faithful.*

3.1 Basic setting

From now on until the end, the field k will be assumed to be algebraically closed and we will use the notation of Subsection 2.1. Let $z \in \mathcal{N}(W(k))$. Let

$$(A, (w_\alpha)_{\alpha \in \mathcal{J}}, \lambda_A) := z^*(\mathcal{A}, (w_\alpha^{\mathcal{A}})_{\alpha \in \mathcal{J}}, \lambda_{\mathcal{A}}).$$

Let

$$(M, F^1, \phi, \psi_M)$$

be the principally quasi-polarized filtered F -crystal over k of the principally quasi-polarized Barsotti–Tate group (D, λ_D) of (A, λ_A) . Thus $\psi_M : M \times M \rightarrow W(k)$ is a perfect, alternating bilinear form on the free $W(k)$ -module M of rank $2r$, F^1 is a maximal isotropic submodule of M with respect to ψ_M , and $\phi : M \rightarrow M$ is a σ -linear endomorphism such that we have an inclusion $pM \subset \phi(M)$ as well as identities $\psi_M(\phi(a), \phi(b)) = p\sigma(\psi_M(a, b))$ for all $a, b \in M$. The σ -linear automorphism ϕ of $M[\frac{1}{p}]$ acts on $M^\vee[\frac{1}{p}]$ by mapping $e \in M^\vee[\frac{1}{p}]$ to $\sigma \circ e \circ \phi^{-1} \in M^\vee[\frac{1}{p}]$ and it acts on $\mathcal{T}(M)[\frac{1}{p}]$ in the natural tensor product way. Let $\psi_{H^1(D)}$ and $\psi_{H_{et}^1}$ be the perfect, alternating bilinear forms on $H^1(D) = H_{et}^1(A_{\overline{B(k)}}, \mathbb{Z}_p)$ and $H_{et}^1(A_{\overline{B(k)}}, \mathbb{Z}_p)$ (respectively)

defined by λ_D . We have a canonical identification of $\text{Gal}(B(k))$ -modules (cf. property (ii) of Subsection B5):

$$(3) \quad (H^1(D), \psi_{H^1(D)}) = (H_{\text{ét}}^1(A_{\overline{B(k)}}, \mathbb{Z}_p), \psi_{H_{\text{ét}}^1}).$$

Let t_α and u_α be the de Rham component of w_α and the p -component of the étale component of w_α (respectively). We have $u_\alpha \in \mathcal{T}(H^1(D))[\frac{1}{p}]$ for all $\alpha \in \mathcal{J}$. If $(F^i(\mathcal{T}(M)))_{i \in \mathbb{Z}}$ is the filtration of $\mathcal{T}(M)$ defined by F^1 , then we have $t_\alpha \in F^0(\mathcal{T}(M))[\frac{1}{p}]$ for all $\alpha \in \mathcal{J}$. Let \mathcal{G} be the schematic closure in \mathbf{GL}_M of the subgroup of $\mathbf{GL}_{M[\frac{1}{p}]}$ that fixes t_α for all $\alpha \in \mathcal{J}$; it is a flat, affine group scheme over $W(k)$. It is known that w_α is a de Rham cycle, i.e., t_α and u_α correspond to each other via de Rham and thus also crystalline Fontaine comparison theory (see [61, Thm. 5.1.6 and Cor 5.1.7]). Thus $\phi(t_\alpha) = t_\alpha$ for all $\alpha \in \mathcal{J}$.

Let $\mu : \mathbb{G}_{m, W(k)} \rightarrow \mathbf{GL}_M$ be the inverse of the canonical split cocharacter of (M, F^1, ϕ) defined in [69, p. 512]. The cocharacter μ acts on F^1 via the weight -1 and fixes a direct supplement F^0 of F^1 in M ; therefore we have $M = F^1 \oplus F^0$. Moreover, μ fixes each tensor t_α (cf. the functorial aspects of [69, p. 513]). Thus μ factors through \mathcal{G} . Let

$$\mu : \mathbb{G}_{m, W(k)} \rightarrow \mathcal{G}$$

be the resulting factorization. We emphasize that in connection to different Kodaira–Spencer maps, in what follows we will identify naturally $\text{Hom}(F^1, F^0)$ with the direct summand $\{e \in \text{End}(M) \mid e(F^0) = 0, e(F^1) \subset F^0\} \simeq \text{Hom}(M/F^0, F^0)$ of $\text{End}(M)$.

Let $\mathcal{G}^0 = \mathcal{G} \cap \mathbf{Sp}(M, \psi_M)$. As \mathcal{G} is the semidirect product of \mathcal{G}^0 and of the image of $\mu : \mathbb{G}_{m, W(k)} \rightarrow \mathcal{G}$, we get that \mathcal{G}^0 is a flat, closed subgroup scheme of $\mathbf{GSp}(M, \psi_M)$ which is smooth or reductive if and only if \mathcal{G} is so. We consider the following family of principally quasi-polarized Dieudonné modules with a group over k associated to z :

$$\mathfrak{F} = \{(M, g\phi, \psi_M, \mathcal{G}^0) \mid g \in \mathcal{G}^0(W(k))\}.$$

Lemma 5. *The direct summand $\text{Lie}(\mathcal{G}_{B(k)}) \cap \text{Hom}(F^1, F^0)$ of $\text{End}(M)$ has rank d . Moreover $\mathcal{G}_{B(k)}$ is a form of $G_{B(k)}$ and thus a reductive group.*

Proof: To prove the lemma we can assume that k has countable transcendental degree; thus there exists an $O_{(v)}$ -monomorphism $W(k) \hookrightarrow \mathbb{C}$. Let $\mathcal{F}_{B(k)}$ be

the normalizer of $F^1[\frac{1}{p}]$ in $\mathcal{G}_{B(k)}$. Its Lie algebra is equal to $\text{Lie}(\mathcal{G}_{B(k)}) \cap \{e \in \text{End}(M)[\frac{1}{p}] | e(F^1[\frac{1}{p}]) \subset F^1[\frac{1}{p}]\}$. As μ factors through \mathcal{G} , we have a direct sum decomposition $\text{Lie}(\mathcal{G}_{B(k)}) = \text{Lie}(\mathcal{F}_{B(k)}) \oplus (\text{Lie}(\mathcal{G}_{B(k)}) \cap \text{Hom}(F^1[\frac{1}{p}], F^0[\frac{1}{p}]))$ of $B(k)$ -vector spaces. Thus the rank of $\text{Lie}(\mathcal{G}_{B(k)}) \cap \text{Hom}(F^1, F^0)$ is $\dim_{B(k)}(\text{Lie}(\mathcal{G}_{B(k)})) - \dim_{B(k)}(\text{Lie}(\mathcal{F}_{B(k)}))$ and therefore it is also equal to $\dim(\mathcal{G}_{B(k)}/\mathcal{F}_{B(k)})$.

We will use the notation of the proof of Lemma 3 for a point $w \in \text{Sh}(G, \mathcal{X})(\mathbb{C})$ that lifts the \mathbb{C} -valued point of $\mathcal{N}_{E(G, \mathcal{X})}$ defined naturally by the generic fibre of z and the $O_{(w)}$ -monomorphism $W(k) \hookrightarrow \mathbb{C}$. Let $W^\vee \otimes_{\mathbb{Q}} \mathbb{C} = F_w^{1,0} \oplus F_w^{0,1}$ be the Hodge decomposition defined by $h_w \in \mathcal{X}$ (it is the dual of the Hodge decomposition of the property (ii) of the proof of Lemma 3). We have a natural isomorphism $(M \otimes_{W(k)} \mathbb{C}, (t_\alpha)_{\alpha \in \mathcal{J}}) \rightarrow (W^\vee \otimes_{\mathbb{Q}} \mathbb{C}, (v_\alpha)_{\alpha \in \mathcal{J}})$ that takes $F^1 \otimes_{W(k)} \mathbb{C}$ to $F_w^{1,0}$, cf. Subsection B6 and Lemma 3 (b). Thus $\mathcal{G}_{B(k)}$ is a form of $G_{B(k)}$, $\mathcal{F}_{B(k)}$ is a parabolic subgroup of $\mathcal{G}_{B(k)}$, and we have $\dim(\mathcal{G}_{B(k)}/\mathcal{F}_{B(k)}) = \dim(G_{\mathbb{C}}/P_w)$, where P_w is the parabolic subgroup of $G_{\mathbb{C}}$ which is the normalizer of $F_w^{1,0}$ in $G_{\mathbb{C}}$. But $G_{\mathbb{C}}/P_w$ is the compact dual of any connected component of \mathcal{X} . Thus $\dim(G_{\mathbb{C}}/P_w) = d$ and therefore $\text{Lie}(\mathcal{G}_{B(k)}) \cap \text{Hom}(F^1, F^0)$ has rank d . \square

Theorem 4 (Key Theorem). *If $p = 2$, then we assume that D is a direct sum of connected and étale Barsotti–Tate groups (e.g., this holds if $G_{\mathbb{Z}(p)}$ is a torus). We have:*

(a) *There exist isomorphisms*

$$(M, (t_\alpha)_{\alpha \in \mathcal{J}}, \psi_M) \rightarrow (H_{\text{ét}}^1(A_{B(k)}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k), (u_\alpha)_{\alpha \in \mathcal{J}}, \psi_{H_{\text{ét}}^1}) \rightarrow (L_{(p)}^\vee \otimes_{\mathbb{Z}(p)} W(k), (v_\alpha)_{\alpha \in \mathcal{J}}, \psi^\vee).$$

(b) *The group scheme \mathcal{G} is isomorphic to $G_{W(k)} = G_{\mathbb{Z}(p)} \times_{\text{Spec}(\mathbb{Z}(p))} \text{Spec}(W(k))$.*

Proof: From Theorem 9 applied to the triple $(D, \lambda_D, (t_\alpha)_{\alpha \in \mathcal{J}})$ and from Formula (3) we get the existence of an isomorphism

$$(M, (t_\alpha)_{\alpha \in \mathcal{J}}, \psi_M) \rightarrow (H_{\text{ét}}^1(A_{B(k)}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k), (u_\alpha)_{\alpha \in \mathcal{J}}, \psi_{H_{\text{ét}}^1}).$$

Thus it suffices to prove part (a) under the extra assumption that k has a countable transcendental degree, i.e., there exists an $E(G, \mathcal{X})$ -monomorphism $B(k) \hookrightarrow \mathbb{C}$. Let $w \in \mathcal{N}_{E(G, \mathcal{X})}(\mathbb{C})$ be the composite of the resulting morphism $\text{Spec}(\mathbb{C}) \rightarrow \text{Spec}(B(k))$ with the generic fibre of z . There exists a unit ϵ of $W(k)$ such that we have isomorphisms $(H_{\text{ét}}^1(A_{B(k)}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k), (u_\alpha)_{\alpha \in \mathcal{J}}, \psi_{H_{\text{ét}}^1}) \rightarrow (L_{(p)}^\vee \otimes_{\mathbb{Z}(p)} W(k), (v_\alpha)_{\alpha \in \mathcal{J}}, \epsilon\psi^\vee)$ (cf. Lemma 3 (a)). Thus part (a) follows

once we remark that $\mu(W(k))(\epsilon)$ defines an isomorphism $(M, (t_\alpha)_{\alpha \in \mathcal{J}}, \psi_M) \rightarrow (M, (t_\alpha)_{\alpha \in \mathcal{J}}, \epsilon^{-1}\psi_M)$.

Part (b) follows from part (a). \square

Lemma 6. *Let $G_{\mathbb{Q}_p}^v$ be a normal, reductive subgroup of $G_{\mathbb{Q}_p}$ such that there exists a cocharacter $\mathbb{G}_{m, B(k(v))} \rightarrow G_{B(k(v))}^v$ whose extension to \mathbb{C} via an $O_{(v)}$ -monomorphism $B(k(v)) \hookrightarrow \mathbb{C}$ is $G(\mathbb{C})$ -conjugate to the cocharacters μ_h of $G_{\mathbb{C}}$ introduced in Subsection 1.3 ($h \in \mathcal{X}$). Let $\mathcal{G}_{B(k)}^v$ be the normal, reductive subgroup of $\mathcal{G}_{B(k)}$ which corresponds to $G_{\mathbb{Q}_p}^v$ via Fontaine comparison theory, cf. Lemmas 15 (a) and 3 (a). Then μ factors through the schematic closure \mathcal{G}^v of $\mathcal{G}_{B(k)}^v$ in \mathcal{G} .*

Proof: To prove this we can assume there exists a $W(k(v))$ -monomorphism $W(k) \hookrightarrow \mathbb{C}$. We have canonical isomorphisms $(M \otimes_{W(k)} \mathbb{C}, (t_\alpha)_{\alpha \in \mathcal{J}}) \rightarrow (W^\vee \otimes_{\mathbb{Q}} \mathbb{C}, (v_\alpha)_{\alpha \in \mathcal{J}})$ such that $F^1 \otimes_{W(k)} \mathbb{C}$ is mapped to the Hodge filtration of $W^\vee \otimes_{\mathbb{Q}} \mathbb{C}$ defined by a cocharacter $\mu_h : \mathbb{G}_{m, \mathbb{C}} \rightarrow G_{\mathbb{C}}$ with $h \in \mathcal{X}$ (see Subsection B6 and Lemma 3 (b)). We know that $\mu_{\mathbb{C}}$ is $\mathcal{G}(\mathbb{C})$ -conjugate to some (any) μ_h , cf. Lemma 18. From this and the very definition of $G_{\mathbb{Q}_p}^v$ we get that μ factors through \mathcal{G}^v . \square

3.2 Local deformation theory

Let \mathcal{G}' be the universal smoothing of \mathcal{G} , cf. Subsection A1. As $\mathcal{G}_{B(k)} = \mathcal{G}'_{B(k)}$ is a form of $G_{B(k)}$, it is a reductive group over $B(k)$ of dimension l . Thus the relative dimension of \mathcal{G}' over $\mathrm{Spec}(W(k))$ is also l . Let R be the completion of the local ring of \mathcal{G}' at the identity element of \mathcal{G}'_k . Let $g_{\mathrm{univ}} \in \mathcal{G}'(R)$ be the natural (universal) element. Let U be the connected, unipotent, smooth, closed subgroup scheme of either \mathcal{G} or \mathcal{G}' whose Lie algebra is $\mathrm{Lie}(\mathcal{G}_{B(k)}) \cap \mathrm{Hom}(F^1, F^0)$ (cf. Subsection B2 and Subsubsection B4.1). As the rank of $\mathrm{Lie}(\mathcal{G}_{B(k)}) \cap \mathrm{Hom}(F^1, F^0)$ is d (cf. Lemma 5), U is isomorphic to $\mathbb{G}_{a, W(k)}^d$.

We fix an identification $R = W(k)[[x_1, \dots, x_l]]$ such that the identity section of \mathcal{G}' is defined by the ideal $\mathfrak{I} := (x_1, \dots, x_l)$ of R . Let Φ_R be the Frobenius lift of R that is compatible with σ and we have $\Phi_R(x_i) = x_i^p$ for all $i \in \{1, \dots, l\}$. The \mathfrak{I} -adic completion $\hat{\Omega}_{R/W(k)}$ of $\Omega_{R/W(k)}$ is a free R -module that has $\{dx_1, \dots, dx_l\}$ as an R -basis. Let $d\Phi_R : \hat{\Omega}_{R/W(k)} \rightarrow \hat{\Omega}_{R/W(k)}$ be the differential map of Φ_R . Let $M_R := M \otimes_{W(k)} R$ and $F_R^1 := F^1 \otimes_{W(k)} R$. We consider the Φ_R -linear endomorphism

$$\Phi := g_{\mathrm{univ}}(\phi \otimes \Phi_R) : M_R \rightarrow M_R.$$

Let $\nabla : M_R \rightarrow M_R \otimes_R \hat{\Omega}_{R/W(k)}$ be the unique connection on M_R such that we have $\nabla \circ \Phi = (\Phi \otimes d\Phi_R) \circ \nabla$; it is integrable and nilpotent modulo p (see Subsection B6). See properties (i) of (iii) of Subsubsection B4.1 for three main properties of ∇ and for the fact that there exists a unique $\text{Ker}(\mathbb{G}_{m,W(k)}(R) \rightarrow \mathbb{G}_{m,W(k)}(R/\mathfrak{J}))$ -multiple ψ_{M_R} of the perfect, alternating bilinear form ψ_M on M_R such that we have an identity $\psi_{M_R}(\Phi(a), \Phi(b)) = p\Phi_R(\psi_{M_R}(a, b))$ for all $a, b \in M_R$.

There exists a unique principally quasi-polarized Barsotti–Tate group (D_R, λ_{D_R}) over R which modulo \mathfrak{J} is (D, λ_D) and whose principally quasi-polarized filtered F -crystal over R/pR is the quintuple $(M_R, F_R^1, \Phi, \nabla, \psi_{M_R})$, cf. Lemmas 16 and 17.

Let (B_R, λ_{B_R}) be the principally polarized abelian scheme over R which modulo \mathfrak{J} is (A, λ_A) and whose principally quasi-polarized Barsotti–Tate group is (D_R, λ_{D_R}) , cf. Serre–Tate deformation theory and Grothendieck algebraization theorem. Let

$$\tau_R : \text{Spec}(R) \rightarrow \mathcal{M}_r$$

be the natural morphism that corresponds to (B_R, λ_{B_R}) and its level- N symplectic similitude structures which lift those of (A, λ_A) (here $N \in \mathbb{N} \setminus (p\mathbb{N} \cup \{1, 2\})$). We have a canonical identification $H_{\text{dR}}^1(B_R/R) = M_R = M \otimes_{W(k)} R$, cf. [2, Ch. V, Subsect. 2.3] and [4, Prop. 2.5.8]. Under this identification, the following two properties hold:

- (i) the perfect form on M_R defined by the principal polarization λ_{B_R} of B_R is identified with ψ_{M_R} ;
- (ii) for all $s \in \mathbb{N}^*$, the connection on $H_{\text{dR}}^1(B_R/R)/\mathfrak{J}^s H_{\text{dR}}^1(B_R/R) = M_R/\mathfrak{J}^s M_R$ induced by ∇ is the Gauss–Manin connection of $B_R \times_{\text{Spec}(R)} \text{Spec}(R/\mathfrak{J}^s)$ (cf. [2, Ch. V, Prop. 3.6.4] and the fact that R/\mathfrak{J}^s is p -adically complete).

Theorem 5 (Faltings). *For each $\alpha \in \mathcal{J}$, the tensor $t_\alpha \in \mathcal{T}(M) \otimes_{W(k)} R[\frac{1}{p}] = \mathcal{T}(M_R)[\frac{1}{p}]$ is the de Rham component of a Hodge cycle on $B_{R[\frac{1}{p}]}$.*

Proof: We recall that B_R is a deformation of A over R . As $t_\alpha \in \mathcal{T}(M)[\frac{1}{p}]$ is the de Rham component of the Hodge cycle w_α on $A_{B(k)}$ and due to the property (i) of Subsubsection B4.1, the theorem is a result of Faltings whose essence is outlined in [54, Rm. 4.1.5] and whose proof is presented here.

As $\mathcal{A}_{r,1,N}$ is a quasi-projective $\mathbb{Z}_{(p)}$ -scheme and as the set \mathcal{J} is countable, it suffices to prove the theorem in the case when there exists a morphism

$e_k : \text{Spec}(\mathbb{C}) \rightarrow \text{Spec}(W(k))$ of $\text{Spec}(W(k(v)))$ -schemes. We will view \mathbb{C} as a $W(k)$ -algebra via e_k . Let $\mathcal{R} := \mathbb{C}[[x_1, \dots, x_l]]$ and $\mathcal{S} := \mathbb{C}[[x_1, \dots, x_d]]$. Let $\mathcal{I} := \mathfrak{I}\mathcal{R} = (x_1, \dots, x_l)\mathcal{R}$ and \mathcal{I}_0 be the maximal ideals of \mathcal{R} and \mathcal{S} (respectively).

Let $(B_{\mathcal{R}}, (t_{\alpha})_{\alpha \in \mathcal{J}}, \lambda_{B_{\mathcal{R}}})$ be the pullback of $(B_R, (t_{\alpha})_{\alpha \in \mathcal{J}}, \lambda_{B_R})$ via the morphism of schemes defined by the natural $W(k)$ -monomorphism

$$R = W(k)[[x_1, \dots, x_l]] \hookrightarrow \mathbb{C}[[x_1, \dots, x_l]] = \mathcal{R}.$$

To prove the theorem it suffices to show that the tensor $t_{\alpha} \in \mathcal{T}(M) \otimes_{W(k)} \mathcal{R} = \mathcal{T}(M_R \otimes_R \mathcal{R}) = \mathcal{T}(H_{\text{dR}}^1(B_{\mathcal{R}}/\mathcal{R}))$ is the de Rham component of a Hodge cycle on $B_{\mathcal{R}}$.

Let $(C_{\mathcal{S}}, (w_{\alpha}^{\mathcal{S}})_{\alpha \in \mathcal{J}}, \lambda_{C_{\mathcal{S}}})$ be the pullback of $(\mathcal{A}, (w_{\alpha}^{\mathcal{A}})_{\alpha \in \mathcal{J}}, \lambda_{\mathcal{A}})$ via a formally étale morphism $\text{Spec}(\mathcal{S}) \rightarrow \mathcal{N}_{\mathbb{C}}^{\mathfrak{s}}$ whose composite with the closed embedding $\text{Spec}(\mathbb{C}) \hookrightarrow \text{Spec}(\mathcal{S})$ is the point $z \circ e_k \in \mathcal{N}_{\mathbb{C}}(\mathbb{C}) = \mathcal{N}^{\mathfrak{s}}(\mathbb{C})$. Let $\mathcal{W} := H_{\text{dR}}^1(C_{\mathcal{S}}/\mathcal{S})$. Let $\psi_{\mathcal{W}}$ be the perfect, alternating bilinear form on \mathcal{W} defined by $\lambda_{C_{\mathcal{S}}}$. Let $t_{\alpha}^{\mathcal{S}} \in \mathcal{T}(\mathcal{W})$ be the de Rham component of $w_{\alpha}^{\mathcal{S}}$. Let Δ be the Gauss–Manin connection on \mathcal{W} defined by $C_{\mathcal{S}}$. We recall that ψ^{\vee} is the alternating bilinear form on W^{\vee} (or on $L_{(p)}^{\vee}$) defined naturally by ψ .

From Corollary 2 and (the proof of) Lemma 3 (b) we get that there exists $\epsilon \in \mathbb{Q} \setminus \{0\}$ for which there exist an isomorphism

$$I : (\mathcal{W}, (t_{\alpha}^{\mathcal{S}})_{\alpha \in \mathcal{J}}, \psi_{\mathcal{W}}) \rightarrow (W^{\vee} \otimes_{\mathbb{Q}} \mathcal{S}, (v_{\alpha})_{\alpha \in \mathcal{J}}, \epsilon\psi^{\vee})$$

under which the \mathfrak{I}_0 -completion of Δ becomes the \mathfrak{I}_0 -completion of the flat connection on $W^{\vee} \otimes_{\mathbb{Q}} \mathcal{S}$ that annihilates $W^{\vee} \otimes 1$. As there exist isomorphisms of $(W^{\vee} \otimes_{\mathbb{Q}} \mathbb{C}, (v_{\alpha})_{\alpha \in \mathcal{J}})$ that take ψ^{\vee} to $\epsilon\psi^{\vee}$, we can assume that $\epsilon = 1$. We fix an isomorphism I with $\epsilon = 1$ and we view it as an identification. For each $\beta \in \mathbb{G}_{m, \mathbb{C}}(\mathcal{R})$, there exist isomorphisms of $(W^{\vee} \otimes_{\mathbb{Q}} \mathcal{R}, (v_{\alpha})_{\alpha \in \mathcal{J}})$ that take ψ^{\vee} to $\beta\psi^{\vee}$. Thus, based on the construction of M_R and on either Lemma 3 (b) or the proof of Lemma 5, we get that there exist isomorphisms

$$I_A : (M_R \otimes_R \mathcal{R}, (t_{\alpha})_{\alpha \in \mathcal{J}}, \psi_{M_R}) \rightarrow (W^{\vee} \otimes_{\mathbb{Q}} \mathcal{R}, (v_{\alpha})_{\alpha \in \mathcal{J}}, \psi^{\vee}).$$

By induction on $s \in \mathbb{N}^*$ we show that there exists a unique morphism of \mathbb{C} -schemes

$$J_s : \text{Spec}(\mathcal{R}/\mathcal{I}^s) \rightarrow \text{Spec}(\mathcal{S})$$

that has the following property:

(i) *There exists an isomorphism ξ_s between the reduction of $(B_{\mathcal{R}}, (t_\alpha)_{\alpha \in \mathcal{J}}, \lambda_{B_{\mathcal{R}}})$ modulo \mathcal{I}^s and the pullback $J_s^*((C_{\mathcal{S}}, (t_\alpha^{\mathcal{S}})_{\alpha \in \mathcal{J}}, \lambda_{C_{\mathcal{S}}}))$ which modulo $\mathcal{I}/\mathcal{I}^s$ is defined by $1_{A_{\mathbb{C}}} = 1_{C_{\mathcal{S}} \times_{\text{Spec}(\mathcal{S})} \text{Spec}(\mathbb{C})} = 1_{B_{\mathcal{R}} \times_{\text{Spec}(\mathcal{R})} \text{Spec}(\mathbb{C})}$.*

As $\mathcal{N}_{E(G, \mathcal{X})}^s$ is a closed subscheme of $\mathcal{M}_{T, E(G, \mathcal{X})}$ (cf. Fact 1) and as $\text{Spec}(\mathcal{S}) \rightarrow \mathcal{N}_{\mathbb{C}}^s$ is formally étale, the deformation $(C_{\mathcal{S}}, \lambda_{C_{\mathcal{S}}})$ of the principally polarized abelian variety $(A, \lambda_A)_{\mathbb{C}}$ is versal, i.e., the Kodaira–Spencer map \mathfrak{K} of Δ is injective and its image is a free \mathcal{S} -module of rank d which is a direct summand of its codomain. This implies the uniqueness of J_s .

The existence of J_1 is obvious. For $s \geq 2$ the passage from the existence of J_{s-1} to the existence of J_s goes as follows. Let $J'_s : \text{Spec}(\mathcal{R}/\mathcal{I}^s) \rightarrow \text{Spec}(\mathcal{S})$ be an arbitrary morphism of \mathbb{C} -schemes that lifts J_{s-1} . Let Δ_s be the connection on $\mathcal{W} \otimes_{\mathcal{S}} \mathcal{R}/\mathcal{I}^s = W^\vee \otimes_{\mathbb{Q}} \mathcal{R}/\mathcal{I}^s$ which is the extension of the connection Δ on \mathcal{W} via J'_s (the last identification is defined naturally by I). Let ∇_s be the Gauss–Manin connection on $H_{\text{dR}}^1(B_R/R) \otimes_R \mathcal{R}/\mathcal{I}^s = M_R \otimes_R \mathcal{R}/\mathcal{I}^s$ defined by $B_R \times_{\text{Spec}(R)} \text{Spec}(\mathcal{R}/\mathcal{I}^s)$; it is the extension of the connection ∇ on M_R (cf. property (ii) of Subsection 3.2) and thus it annihilates each tensor $t_\alpha \in \mathcal{T}(M_R) \otimes_R \mathcal{R}/\mathcal{I}^s$ (cf. property (i) of Subsubsection B4.1). From Lemma 4 we get:

(ii) *There exists a unique isomorphism $I_{A,s} : M_R \otimes_R \mathcal{R}/\mathcal{I}^s \rightarrow W^\vee \otimes_{\mathbb{Q}} \mathcal{R}/\mathcal{I}^s$ which lifts a fixed isomorphism between $(M_R \otimes_R \mathcal{R} \otimes_{\mathcal{R}} \mathcal{R}/\mathcal{I}, (t_\alpha)_{\alpha \in \mathcal{J}}) = (H_{\text{dR}}^1(A_{\mathbb{C}}/\mathbb{C}), (t_\alpha)_{\alpha \in \mathcal{J}})$ and $(W^\vee \otimes_{\mathbb{Q}} \mathbb{C}, (v_\alpha)_{\alpha \in \mathcal{J}})$ obtained as in Lemma 3 (b) and such that under it ∇_s becomes the flat connection δ_s on $W^\vee \otimes_{\mathbb{C}} \mathcal{R}/\mathcal{I}^s$ that annihilates $W^\vee \otimes 1$.*

We denote also by $I_{A,s}$ the isomorphism $\mathcal{T}(M_R \otimes_R \mathcal{R}/\mathcal{I}^s) \rightarrow \mathcal{T}(W^\vee \otimes_{\mathbb{Q}} \mathcal{R}/\mathcal{I}^s)$ induced by $I_{A,s}$. As $I_{A,s}(t_\alpha)$ and v_α are two tensors of $W^\vee \otimes_{\mathbb{C}} \mathcal{R}/\mathcal{I}^s$ that are annihilated by δ_s and that coincide modulo $\mathcal{I}/\mathcal{I}^s$, we get that we have $I_{A,s}(t_\alpha) = v_\alpha$ for all $\alpha \in \mathcal{J}$. An argument similar to the one above involving $\epsilon \in \mathbb{Q} \setminus \{0\}$ shows that we can assume that $I_{A,s}$ takes ψ_{M_R} to ψ^\vee . Thus we can choose I_A such that it lifts $I_{A,s}$. We will view the reduction $I_{A,s}$ of I_A modulo \mathcal{I}^s as an identification. Therefore we will also identify $\nabla_s = \delta_s$.

From the existence of I and the fact that $I_{A,s}$ is the reduction of I_A modulo \mathcal{I}^s , we get that there exists an isomorphism

$$\begin{aligned} \zeta_s : J_s^*((\mathcal{W}, (t_\alpha^{\mathcal{S}})_{\alpha \in \mathcal{J}}, \psi_{\mathcal{W}})) &= (W^\vee \otimes_{\mathbb{Q}} \mathcal{R}/\mathcal{I}^s, (v_\alpha)_{\alpha \in \mathcal{J}}, \psi^\vee) \rightarrow \\ \rightarrow (M_R \otimes_R \mathcal{R}/\mathcal{I}^s, (t_\alpha)_{\alpha \in \mathcal{J}}, \psi_{M_R}) &= (W^\vee \otimes_{\mathbb{Q}} \mathcal{R}/\mathcal{I}^s, (v_\alpha)_{\alpha \in \mathcal{J}}, \psi^\vee) \end{aligned}$$

with the properties that it lifts the identity automorphism of $W^\vee \otimes_{\mathbb{Q}} \mathbb{C}$ and that:

(iii) *It respects the Gauss–Manin connections, i.e., it takes Δ_s to $\nabla_s = \delta_s$.*

From the uniqueness part of the property (ii) we also get that:

(iv) *The reduction of ζ_s modulo \mathcal{I}^{s-1} is the isomorphism defined by ξ_{s-1} .*

Let $F_{A,s}^1$ and $F_{C,s}^1$ be the Hodge filtrations of $W^\vee \otimes_{\mathbb{Q}} \mathcal{R}/\mathcal{I}^s$ defined naturally by $B_{\mathcal{R}}$ and $J_s^*(C_S)$ (respectively) via the above identifications. The direct summands $F_{A,s}^1$ and $\zeta_s(F_{C,s}^1)$ of $W^\vee \otimes_{\mathbb{Q}} \mathcal{R}/\mathcal{I}^s$ coincide modulo $\mathcal{I}^{s-1}/\mathcal{I}^s$, cf. property (iv). Moreover, there exist direct sum decompositions

$$W^\vee \otimes_{\mathbb{Q}} \mathcal{R}/\mathcal{I}^s = F_{A,s}^1 \oplus F_{A,s}^0 = F_{C,s}^1 \oplus F_{C,s}^0$$

defined by cocharacters $\mu_{A,s}$ and $\mu_{C,s}$ of the reductive subgroup scheme $G_{\mathcal{R}/\mathcal{I}^s}$ of $\mathbf{GL}_{W^\vee \otimes_{\mathbb{Q}} \mathcal{R}/\mathcal{I}^s}$ (here $\mathbb{G}_{m, \mathcal{R}/\mathcal{I}^s}$ through $\mu_{*,s}$ fixes $F_{*,s}^0$ and acts via the weight -1 on $F_{*,s}^1$). The existence of $\mu_{A,s}$ is a direct consequence of the existence of the cocharacter $\mu : \mathbb{G}_{m, W(k)} \rightarrow \mathcal{G}$ (see paragraph before Lemma 5) and of the definition of F_R^1 (see Subsection 3.2) while the existence of $\mu_{C,s}$ is well-known.

As $F_{A,s}^1$ and $\zeta_s(F_{C,s}^1)$ coincide modulo $\mathcal{I}^{s-1}/\mathcal{I}^s$, we can choose $\mu_{A,s}$ and $\mu_{C,s}$ such that $\zeta_s^{-1} \mu_{A,s} \zeta_s$ and $\mu_{C,s}$ commute modulo $\mathcal{I}^{s-1}/\mathcal{I}^s$ and thus coincide modulo $\mathcal{I}^{s-1}/\mathcal{I}^s$. Thus based on [15, Vol. II, Exp. IX, Thm. 3.6], there exists an element $g_s \in \text{Ker}(G(\mathcal{R}/\mathcal{I}^s) \rightarrow G(\mathcal{R}/\mathcal{I}^{s-1}))$ such that we have $\zeta_s^{-1} \mu_{A,s} \zeta_s = g_s \mu_{C,s} g_s^{-1}$. We have $\zeta_s(g_s(F_{C,s}^1)) = F_{A,s}^1$. We would like to mention that the original approach of Faltings used the strictness of filtrations of morphisms between Hodge \mathbb{R} -structures in order to get the existence of the element g_s .

The image of \mathfrak{K} is a free \mathcal{S} -module that has rank d and that is equal to the image of $\text{Lie}(G_S)$ into the codomain of \mathfrak{K} . Thus we can replace J'_s by another morphism $J_s : \text{Spec}(\mathcal{R}/\mathcal{I}^s) \rightarrow \text{Spec}(\mathcal{S})$ that lifts J_{s-1} and such that under it and $I_{A,s}$ the Hodge filtration $F_{C,s}^1$ gets replaced by $g_s(F_{C,s}^1)$. Thus ζ_s becomes the de Rham realization of an isomorphism ξ_s between the reduction of $(B_{\mathcal{R}}, (t_\alpha)_{\alpha \in \mathcal{J}}, \lambda_{B_{\mathcal{R}}})$ modulo \mathcal{I}^s and $J_s^*((C_S, (t_\alpha^S)_{\alpha \in \mathcal{J}}, \lambda_{C_S}))$ which lifts ξ_{s-1} , cf. deformation theory of abelian varieties. Thus the morphism J_s has the desired properties. This ends the induction.

Let $J_\infty : \text{Spec}(\mathcal{R}) \rightarrow \text{Spec}(\mathcal{S})$ be the morphism defined by J_s 's ($s \in \mathbb{N}^*$). The isomorphism ξ_s is uniquely determined by the property (i) and this implies that ξ_{s+1} lifts ξ_s . From this and Grothendieck algebraization theorem

we get the existence of an isomorphism

$$\xi_\infty : (B_{\mathcal{R}}, (t_\alpha)_{\alpha \in \mathcal{J}}, \lambda_{B_{\mathcal{R}}}) \rightarrow J_\infty^*((C_S, (t_\alpha^S)_{\alpha \in \mathcal{J}}, \lambda_{C_S}))$$

which modulo \mathcal{I} is defined by $1_{A_{\mathcal{C}}}$ and which lifts each ξ_s with $s \in \mathbb{N}$. Thus for each $\alpha \in \mathcal{J}$, the tensor $t_\alpha \in \mathcal{T}(M) \otimes_{W(k)} \mathcal{R}$ is the de Rham component of the Hodge cycle on $B_{\mathcal{R}}$ which is the pullback of the Hodge cycle $J_\infty^*(w_\alpha^S)$ on $J_\infty^*(C_S)$ via the isomorphism $B_{\mathcal{R}} \rightarrow J_\infty^*(C_S)$ that defines ξ_∞ . \square

3.3 Proof of Theorem 1

In this subsection we prove the Basic Theorem 1. Let O be an $O_{(v)}$ -algebra which is a discrete valuation ring of absolute ramification index 1. We choose the field k such that we have a $O_{(v)}$ -monomorphism $O \hookrightarrow W(k)$. Let Z be a regular, formally smooth O -scheme equipped with a morphism $\chi : Z_{E(G, \mathcal{X})} \rightarrow \text{Sh}_H(G, \mathcal{X}) = \mathcal{N}_{E(G, \mathcal{X})}^s$. Thus χ extends uniquely to a morphism $\chi_Z : Z \rightarrow \mathcal{N}$, cf. Proposition 2 (c). To prove Theorem 1 (a) we have to show that χ_Z factors through \mathcal{N}^s . It suffices to check this under the extra assumptions that $O = W(k)$ and $Z = \text{Spec}(O)$. We will use the notation of Subsection 3.1 for the point $z := \chi_Z \in \mathcal{N}(W(k))$.

Let $y : \text{Spec}(k) \hookrightarrow \mathcal{N}_{W(k)}$ be the closed embedding defined by the special fibre of $z \in \mathcal{N}(W(k))$. Let O_y^{bigg} (resp. O_y) be the completion of the local ring of y in $\mathcal{M}_{r, W(k)}$ (resp. in $\mathcal{N}_{W(k)}$). As \mathcal{Q} is a normal, flat $O_{(v)}$ -scheme of relative dimension d and as \mathcal{N} is a pro-finite pro-étale cover of \mathcal{Q} (cf. Proposition 2 (a)), the local ring O_y is normal and noetherian of dimension $1 + d$. The natural homomorphism $n_y : O_y^{\text{bigg}} \rightarrow O_y$ (associated at y to the morphism $\mathcal{N}_{W(k)} \rightarrow \mathcal{M}_{r, W(k)}$) is finite, cf. Proposition 2 (b). Let $h_z : O_y^{\text{bigg}} \rightarrow R$ be the $W(k)$ -homomorphism that defines $\tau_R : \text{Spec}(R) \rightarrow \mathcal{M}_r$.

Let $S := W(k)[[x_1, \dots, x_d]]$ and let $\mathfrak{I}_0 := (x_1, \dots, x_d)$ be its ideal. We consider a closed embedding $c_R : \text{Spec}(S) \hookrightarrow \text{Spec}(R)$ such that the following two properties hold (cf. Subsubsection B4.2 and Lemma 5):

- (i) it is defined by a $W(k)$ -epimorphism $e_z : R \twoheadrightarrow S$ satisfying $e_z(\mathfrak{I}) \subset \mathfrak{I}_0 \subset S$;
- (ii) the pullback of $(M_R, F_R^1, \Phi, \nabla, \psi_{M_R})$ via the closed embedding $\text{Spec}(S/pS) \hookrightarrow \text{Spec}(R/pR)$, is a principally quasi-polarized filtered F -crystal over S/pS whose Kodaira–Spencer map is injective and has an image equal to the direct summand $\text{Lie}(U) \otimes_{W(k)} S$ of $\text{Hom}(F^1, F^0) \otimes_{W(k)} S \simeq \text{Hom}(F^1, M/F^1) \otimes_{W(k)} S$.

From the property (ii) we get that the composite morphism $\tau_S := \tau_R \circ c_R : \text{Spec}(S) \rightarrow \mathcal{M}_r$ is defined naturally by a $W(k)$ -epimorphism $s_z^{\text{bigg}} := e_z \circ h_z : O_y^{\text{bigg}} \twoheadrightarrow S$.

The existence of the isomorphism ξ_∞ (see the end of the proof of Theorem 5) implies that the morphism $\tau_R : \text{Spec}(R) \rightarrow \mathcal{M}$ factors through \mathcal{N} in such a way that modulo the ideal \mathfrak{J} of R it defines the point $z \in \mathcal{N}(W(k))$. Thus there exists a $W(k)$ -epimorphism $s_z : O_y \twoheadrightarrow S$ such that we have $s_z^{\text{bigg}} = s_z \circ n_y$, i.e., the following diagram is commutative

$$\begin{array}{ccc} O_y^{\text{bigg}} & \xrightarrow{n_y} & O_y \\ \downarrow h_z & & \downarrow s_z \\ R & \xrightarrow{e_z} & S. \end{array}$$

By reasons of dimensions of local, noetherian, normal rings, the $W(k)$ -epimorphism $s_z : O_y \twoheadrightarrow S$ is an isomorphism. Thus $\mathcal{N}_{W(k)}$ is formally smooth at z and therefore z factors through \mathcal{N}^s . Therefore Theorem 1 (a) holds and y is a k -valued point of $\mathcal{N}_{W(k)}^s$.

As s_z is an isomorphism, the $W(k)$ -homomorphism $n_y : O_y^{\text{bigg}} \rightarrow O_y$ is onto. Therefore the natural $W(k)$ -morphism $\mathcal{N}_{W(k)}^s \rightarrow \mathcal{M}_{r,W(k)}$ is a formally closed embedding at $y \in \mathcal{N}_{W(k)}^s(k)$. The role of $z \in \mathcal{N}(W(k))$ is that of an arbitrary $W(k)$ -valued of \mathcal{N} (and thus due to Theorem 1 (a)) of \mathcal{N}^s . Thus the $W(k)$ -morphism $\mathcal{N}_{W(k)}^s \rightarrow \mathcal{M}_{r,W(k)}$ is a formally closed embedding at every k -valued point of $\mathcal{N}_{W(k)}^s$. Thus Theorem 1 (b) also holds.

We check that the Theorem 1 (c) holds. Let Z be a smooth $O_{(v)}$ -scheme such that we have a morphism $\chi : Z_{E(G,\mathcal{X})} \rightarrow \text{Sh}_{H \times H^{(p)}}(G, \mathcal{X})$. From Proposition 2 (b) and Lemma 2 we get that $\mathcal{N}/H^{(p)}$ has a finite étale cover which is projective; thus $\mathcal{N}/H^{(p)}$ is a proper $O_{(v)}$ -scheme. From this and the valuative criterion of properness, we get that there exists an open subscheme U_Z of Z such that it contains $Z_{E(G,\mathcal{X})}$, the complement of U_Z in Z has codimension in Z at least 2, and the morphism χ extends uniquely to a morphism $\chi_{U_Z} : U_Z \rightarrow \mathcal{N}/H^{(p)}$. From the classical purity theorem of Zariski, Nagata and Grothendieck (see [21, Exp. X, Thm. 3.4 (i)]) we get that the pro-finite pro-étale cover $U_Z \times_{\mathcal{N}/H^{(p)}} \mathcal{N} \rightarrow U_Z$ extends uniquely to a pro-finite pro-étale cover $Z_\infty \rightarrow Z$. From this and Theorem 1 (a) we get that the natural morphism $U_Z \times_{\mathcal{N}/H^{(p)}} \mathcal{N} \rightarrow \mathcal{N}$ extends uniquely to a morphism $Z_\infty \rightarrow \mathcal{N}$. This implies that the morphism χ extends uniquely to a morphism $\chi_Z : Z \rightarrow \mathcal{N}/H^{(p)}$. Thus $\mathcal{N}/H^{(p)}$ is a Néron model of its generic fibre

$\mathrm{Sh}_{H \times H^{(p)}}(G, \mathcal{X})$ over $O_{(v)}$, i.e., Theorem 1 (c) holds. This ends the proof of the Basic Theorem 1. \square

3.4 de Rham realizations of Hodge cycles

We denote also by τ_R the factorization of $\tau_R : \mathrm{Spec}(R) \rightarrow \mathcal{M}_r$ through either \mathcal{N} or (cf. Theorem 1 (a)) \mathcal{N}^s which modulo \mathfrak{J} is the $W(k)$ -valued point $z \in \mathcal{N}(W(k)) = \mathcal{N}^s(W(k))$. As $s_z : O_y \rightarrow S$ is a $W(k)$ -isomorphism and as we have a $W(k)$ -epimorphism $e_z : R \twoheadrightarrow S$, the morphism $\tau_R : \mathrm{Spec}(R) \rightarrow \mathcal{N}^s$ is formally smooth. Under the canonical identification $H_{\mathrm{dR}}^1(B_R/R) = M_R = M \otimes_{W(k)} R$, the pullback of w_α^A via the morphism $\mathrm{Spec}(R[\frac{1}{p}]) \rightarrow \mathcal{N}_{E(G, \mathcal{X})} = \mathrm{Sh}_H(G, \mathcal{X})$ defined by τ_R , is a Hodge cycle on $B_{R[\frac{1}{p}]}$ whose de Rham component $t'_\alpha \in \mathcal{T}(M) \otimes_{W(k)} R[\frac{1}{p}]$ modulo $\mathfrak{J}[\frac{1}{p}]$ is t_α modulo $\mathfrak{J}[\frac{1}{p}]$. In fact we have $t'_\alpha = t_\alpha$ for all $\alpha \in \mathcal{J}$. This follows either from the existence of ξ_∞ (see end of the proof of Theorem 5) or (in Faltings' approach) from the fact that there exists no non-trivial tensor of $\mathcal{T}(M) \otimes_{W(k)} \mathfrak{J}[\frac{1}{p}]$ fixed by Φ . Similarly, the de Rham realization of the pullback of λ_A via the morphism $\mathrm{Spec}(R[\frac{1}{p}]) \rightarrow \mathcal{N}_{E(G, \mathcal{X})} = \mathrm{Sh}_H(G, \mathcal{X})$ defined by τ_R , is ψ_{M_R} .

3.5 The open subscheme \mathcal{N}^m of \mathcal{N}^s

For $p > 2$ let $\mathcal{N}^m := \mathcal{N}^s$. If $p = 2$, then let \mathcal{N}^m be the maximal open subscheme of \mathcal{N}^s with the property that for each algebraically closed field k of characteristic p and for every $z \in \mathcal{N}^m(W(k))$, Theorem 4 (a) (and thus also Theorem 4 (b)) holds. Thus regardless of the parity of p , for each such field k and for every $z \in \mathcal{N}^m(W(k))$, Theorem 4 (a) holds.

Proposition 3. *The following two properties hold:*

- (a) *Always \mathcal{N}^m is a $G(\mathbb{A}_f^{(p)})$ -invariant, open subscheme of \mathcal{N}^s .*
- (b) *If Theorem 4 (a) holds for $z \in \mathcal{N}^s(W(k))$, then $z \in \mathcal{N}^m(W(k))$.*

Proof: The right translations of z by elements of $G(\mathbb{A}_f^{(p)})$ correspond to passages to isogenies prime to p of the abelian scheme \mathbb{A} . Therefore the triple $(M, \phi, (t_\alpha)_{\alpha \in \mathcal{J}})$ depends only on the $G(\mathbb{A}_f^{(p)})$ -orbit of z . Thus, if Theorem 4 (a) holds for z , then Theorem 4 (a) also holds for every point in the $G(\mathbb{A}_f^{(p)})$ -orbit of z . This implies that part (a) holds.

To check part (b), let $H^{(p)}$, \mathcal{Q} , and \mathcal{Q}^s be as in Subsection 2.2. By enlarging N we can assume that the triple $(\mathcal{A}, (w_\alpha^A)_{\alpha \in \mathcal{J}}, \lambda_{\mathcal{A}})$ is the pullback of an analogue triple \mathcal{T} over \mathcal{Q} . Let $\text{Spec}(Q)$ be an affine, open subscheme of \mathcal{Q}^s such that the composite $z_{H^{(p)}}$ of $z : \text{Spec}(W(k)) \rightarrow \mathcal{N}^s$ with $\mathcal{N}^s \rightarrow \mathcal{Q}^s$ factors through $\text{Spec}(Q)$. Let $(M_Q, (t_\alpha^Q)_{\alpha \in \mathcal{J}}, \psi_{M_Q})$ be the de Rham realization of the pullback triple \mathcal{T}_Q (of \mathcal{T} to $\text{Spec}(Q)$). Let F_Q^1 be the direct summand of M_Q which is the Hodge filtration associated to \mathcal{T}_Q . By shrinking $\text{Spec}(Q)$, we can assume that M_Q and F_Q^1 are free Q -module of ranks $2r$ and r (respectively). The existence of the formally smooth morphism $\tau_R : \text{Spec}(R) \rightarrow \mathcal{N}^s$ implies that we have isomorphisms (cf. Subsection 3.4)

$$(4) \quad \begin{aligned} & (M_Q \otimes_Q R, F_Q^1 \otimes_Q R, (t_\alpha^Q)_{\alpha \in \mathcal{J}}, \psi_{M_Q}) \rightarrow (M_R, F_R^1, (t_\alpha)_{\alpha \in \mathcal{J}}, \psi_{M_R}) \\ & = (M \otimes_{W(k)} R, F^1 \otimes_{W(k)} R, (t_\alpha)_{\alpha \in \mathcal{J}}, \epsilon_M \psi_M), \end{aligned}$$

where $\epsilon_M \in \text{Ker}(\mathbb{G}_{m, W(k)}(R) \rightarrow \mathbb{G}_{m, W(k)}(R/\mathfrak{J}))$. We note that $\mu(R)(\epsilon_M)$ defines an isomorphism $(M \otimes_{W(k)} R, F^1 \otimes_{W(k)} R, (t_\alpha)_{\alpha \in \mathcal{J}}, \psi_M) \rightarrow (M \otimes_{W(k)} R, F^1 \otimes_{W(k)} R, (t_\alpha)_{\alpha \in \mathcal{J}}, \epsilon_M \psi_M)$. As Theorem 4 (a) holds for $z \in \mathcal{N}^s(W(k))$, we also have isomorphisms $(M \otimes_{W(k)} R, (t_\alpha)_{\alpha \in \mathcal{J}}, \psi_M) \rightarrow (L_{(p)}^\vee \otimes_{\mathbb{Z}_{(p)}} R, (v_\alpha)_{\alpha \in \mathcal{J}}, \psi^\vee)$. From the last three sentences and Artin approximation theorem ([7, Ch. 3, Sect. 3.6, Thm. 16]) we get that there exists a smooth, affine morphism $\text{Spec}(Q') \rightarrow \text{Spec}(Q)$ through which $z_{H^{(p)}} : \text{Spec}(W(k)) \rightarrow \text{Spec}(Q)$ and the natural factorization $\text{Spec}(R) \rightarrow \text{Spec}(Q)$ of τ_R factor naturally producing morphisms $z'_{H^{(p)}} : \text{Spec}(W(k)) \rightarrow \text{Spec}(Q')$ and $\text{Spec}(R) \rightarrow \text{Spec}(Q')$ and such that we have an isomorphism

$$(M_Q \otimes_Q Q', (t_\alpha^Q)_{\alpha \in \mathcal{J}}, \psi_{M_Q}) \rightarrow (L_{(p)}^\vee \otimes_{\mathbb{Z}_{(p)}} Q', (v_\alpha)_{\alpha \in \mathcal{J}}, \psi^\vee)$$

whose extension to R (via $\text{Spec}(R) \rightarrow \text{Spec}(Q')$) defines (4). The image $\text{Im}(\text{Spec}(Q') \rightarrow \text{Spec}(Q))$ is an open subscheme of \mathcal{Q}^s whose pullback to \mathcal{N}^s is (due to the last isomorphism) an open subscheme of \mathcal{N}^m that contains the point $z \in \mathcal{N}^s(W(k))$. Thus part (b) holds. \square

3.5.1 Proof of Corollary 1

We assume that $G_{\mathbb{Z}_{(p)}}$ is smooth over $\mathbb{Z}_{(p)}$ and that the $k(v)$ -scheme $\mathcal{N}_{k(v)}^m$ is non-empty. We fix a connected component \mathcal{C}^m of $\mathcal{N}_k^m/H^{(p)}$ and consider a point $y_1 \in \mathcal{C}^m(k)$. Not to introduce extra notation, we will assume that

$z \in \mathcal{N}^m(W(k))$ is such that its image $[\mathcal{N}^m/H^{(p)}](W(k))$ lifts the point $y_1 \in \mathcal{C}^m(k)$. As we will need to vary y_1 , we will denote $Q_1 = Q$ and $Q'_1 = Q'$. We have the following obvious property:

(i) *We can assume that $\text{Spec}(Q'_1) \rightarrow \text{Spec}(Q_1)$ is an étale morphism. If $\Phi_{Q'_1}$ is a Frobenius lift of the p -adic completion Q'^{\wedge}_1 of Q'_1 compatible with σ , then the Frobenius of $M_{Q_1} \otimes_{Q_1} Q'^{\wedge}_1$ can be identified via an isomorphism*

$$(5) \quad \iota_{y_1} : (M_{Q_1} \otimes_{Q_1} Q'^{\wedge}_1, F_{Q_1}^1 \otimes_{Q_1} Q'^{\wedge}_1, (t_\alpha^{Q_1})_{\alpha \in \mathcal{J}}, \psi_{M_{Q_1}}) \rightarrow (M \otimes_{W(k)} Q'^{\wedge}_1, F^1 \otimes_{W(k)} Q'^{\wedge}_1, (t_\alpha)_{\alpha \in \mathcal{J}}, \epsilon_1 \psi_M)$$

with the Frobenius endomorphism $g_{Q'_1}(\phi \otimes \Phi_{Q'_1})$ of $M \otimes_{W(k)} Q'^{\wedge}_1$ for a suitable element $g_{Q'_1} \in \mathbf{GL}_M(Q'^{\wedge}_1)$ which modulo the p -adic completion of the ideal of Q'_1 that defines $z'_{H^{(p)}}$ is the identity. Here ϵ_1 is a unit of Q'^{\wedge}_1 . From the property (v) of Subsubsection B4.1 applied in the context of the quintuple $(M_R, F_R^1, \Phi, \nabla, (t_\alpha)_{\alpha \in \mathcal{J}})$ of Subsection 3.2, we get that the Frobenius of $M_{Q_1} \otimes_{Q_1} Q'^{\wedge}_1$ fixes each $t_\alpha^{Q_1}$ with $\alpha \in \mathcal{J}$. Thus $g_{Q'_1}(\phi \otimes \Phi_{Q'_1})$ fixes each t_α with $\alpha \in \mathcal{J}$ and therefore we have $g_{Q'_1} \in \mathcal{G}(Q'^{\wedge}_1)$.

As Theorem 4 (a) holds for $z \in \mathcal{N}^m(W(k))$ and as $G_{\mathbb{Z}_{(p)}}$ is smooth over $\mathbb{Z}_{(p)}$, we get that \mathcal{G} is a smooth, closed subgroup scheme of $\mathbf{GSp}(M, \psi_M)$. Thus \mathcal{G}^0 is also smooth over $W(k)$.

As $\mathcal{G}_{B(k)}^0$ is the subgroup of $\mathbf{Sp}(M[\frac{1}{p}], \psi_M)$ that fixes t_α for each $\alpha \in \mathcal{J}$ and as ϕ fixes each t_α , we get that under the natural action of ϕ on $\text{End}(M[\frac{1}{p}])$, we have $\phi(\text{Lie}(\mathcal{G}_{B(k)}^0)) = \mathcal{G}_{B(k)}^0$. From this and the existence of the direct sum decomposition $M = F^1 \oplus F^0$ defined by μ (see Subsection 3.1) we get that the two axioms of [59, Subsect. 4.1] hold for the triple (M, ϕ, \mathcal{G}^0) .

Let \mathcal{F} be the normalizer of F^1 in \mathcal{G} and let $\mathcal{F}^0 = \mathcal{G}^0 \cap \mathcal{F}$. From [10, Lem. 2.1.5 and Prop. 2.1.8 (3)] we get that \mathcal{F} is smooth over $W(k)$ and the product morphism $U \times_{\text{Spec}(W(k))} \mathcal{F} \rightarrow \mathcal{G}$ is an open embedding. As \mathcal{F} is the semiproduct of \mathcal{F}^0 and the image of μ , \mathcal{F}^0 is smooth over $W(k)$ and the product morphism $U \times_{\text{Spec}(W(k))} \mathcal{F}^0 \rightarrow \mathcal{G}^0$ is an open embedding.

To prove the Corollary 1 it suffices to show that \mathcal{C}^m equipped with the morphism $\mathcal{C}^m \rightarrow \mathcal{A}_{r,1,N,k}$ is a quasi Shimura p -variety of Hodge type relative to \mathfrak{F} in the sense of [59, Def. 4.2.1], i.e., the axioms (i) of (iii) of loc. cit. hold for \mathcal{C}^m (more precisely, for the morphism $\mathcal{C}^m \rightarrow \mathcal{A}_{r,1,N,k}$).

Axiom (i) of [59, Def. 4.2.1] holds for \mathcal{C}^m as \mathcal{C}^m is smooth over k of dimension d (the role of e_- of loc. cit. is played here by the rank d of the $W(k)$ -module $\text{Lie}(\mathcal{G}_{B(k)}) \cap \text{Hom}(F^1, F^0) = \text{Lie}(\mathcal{G}_{B(k)}^0) \cap \text{Hom}(F^1, F^0) = \text{Lie}(U)$). Axiom (ii) of [59, Def. 4.2.1] holds as it is just the modulo p variant

of Theorem 1 (b) for \mathcal{C}^m .

Let $\rho_{y_1} : \text{Spec}(Q'_1/pQ'_1) \rightarrow \mathcal{C}^m$ be the étale morphism induced naturally by the étale morphism $\text{Spec}(Q'_1) \rightarrow \text{Spec}(Q_1)$. Let $W(Q'_1/pQ'_1)$ be the ring of p -typical Witt vectors with coefficients in Q'_1/pQ'_1 . By shrinking Q'_1 we can assume that we have a homomorphism $Q'_1{}^\wedge \rightarrow W(Q'_1/pQ'_1)$ which lifts the identity $Q'_1{}^\wedge/pQ'_1{}^\wedge = Q'_1/pQ'_1$ and such that the following property holds (cf. the definition of g_{univ} in Subsection 3.2, the fact that the product morphism $U \times_{W(k)} \mathcal{F} \rightarrow \mathcal{G}$ is an open embedding, and the property (ii) of Subsection 3.3):

(ii) *The composite morphism $\text{Spec}(Q'_1{}^\wedge) \rightarrow \mathcal{G}^0/\mathcal{F}^0 = \mathcal{G}/\mathcal{F}$ induced by $g_{Q'_1}$ is formally étale.*

We consider a finite number of points $y_1, \dots, y_t \in \mathcal{C}^m(k)$ such that we have $\cup_{i=1}^t \text{Im}(\rho_{y_i}) = \mathcal{C}^m$; here $\rho_{y_i} : \text{Spec}(Q'_1/pQ'_1) \rightarrow \mathcal{C}^m$ for $i \in \{2, \dots, t\}$ is constructed similarly to ρ_{y_1} .

To end the proof of Corollary 1 it suffices to show that the axioms (iii.a) to (iii.d) of [59, Def. 4.2.1] hold in the context of the family of morphisms $(\rho_{y_i})_{i \in \{1, \dots, t\}}$. Axiom (iii.a) of [59, Def. 4.2.1] holds for \mathcal{C}^m as it just says that $\cup_{i=1}^t \text{Im}(\rho_{y_i}) = \mathcal{C}^m$ and that the domain of each ρ_{y_i} is affine.

As \mathcal{G} is the semidirect product of \mathcal{G}^0 and the image of μ , by replacing the isomorphism of (5) with an automorphism of $(M \otimes_{W(k)} Q'_1{}^\wedge, F^1 \otimes_{W(k)} Q'_1{}^\wedge, (t_\alpha)_{\alpha \in \mathcal{J}}, \epsilon_1 \psi_M)$ defined by a $Q'_1{}^\wedge$ -valued point of the image of μ , we can assume that $\epsilon_1 = 1$. This implies that in fact we have $g_{Q'_1} \in \mathcal{G}^0(Q'_1{}^\wedge)$. Now the fact that the axiom (iii.b) of [59, Def. 4.2.1] holds for \mathcal{C}^m follows from the formally étale part of the property (ii) via a natural extension through the homomorphism $Q'_1{}^\wedge \rightarrow W(Q'_1/pQ'_1)$.

Let $i, j \in \{1, \dots, t\}$. Let $\nu_i : \text{Spec}(Q'_i) \rightarrow \mathcal{N}_{W(k)}^m/H^{(p)}$ be an étale morphism analogues to the morphism $\nu_1 : \text{Spec}(Q'_1) \rightarrow \mathcal{N}_{W(k)}^m/H^{(p)}$ induced naturally by the composite morphism $\text{Spec}(Q'_1) \rightarrow \text{Spec}(Q_1) \rightarrow \mathcal{Q}_{W(k)}^s \rightarrow \mathcal{N}_{W(k)}^s/H^{(p)}$ and let $\text{Spec}(Q'_{i,j})$ define the cartesian product of ν_i and ν_j . As we have $\epsilon_i = \epsilon_j = 1$, the extensions of the isomorphisms ι_{y_i} and ι_{y_j} to $W(Q'_{ij}/pQ'_{ij})$ via composite homomorphisms $Q'_i{}^\wedge \rightarrow Q'_{i,j} \rightarrow W(Q'_{ij}/pQ'_{ij})$ and $Q'_j{}^\wedge \rightarrow Q'_{i,j} \rightarrow W(Q'_{ij}/pQ'_{ij})$ that lift the homomorphisms $Q'_i/pQ'_i \rightarrow Q'_{ij}/pQ'_{ij}$ and $Q'_j/pQ'_j \rightarrow Q'_{ij}/pQ'_{ij}$ (respectively), when viewed without filtrations differ by an automorphism of $(M \otimes_{W(k)} W(Q'_{ij}/pQ'_{ij}), (t_\alpha)_{\alpha \in \mathcal{J}}, \psi_M)$ and thus by an element $h_{ij} \in \mathcal{G}^0(W(Q'_{ij}/pQ'_{ij}))$. Thus the axiom (iii.c) of [59, Def. 4.2.1] holds for \mathcal{C}^m .

Axiom (iii.d) of [59, Def. 4.2.1] holds for \mathcal{C}^m as it just says that regardless of what choices are made in Subsubsection B4.2 to define the closed embedding $\mathrm{Spec}(S) \rightarrow \mathrm{Spec}(R)$ used in Subsection 3.3, the morphism $O_y/pO_y \rightarrow S/pS$ of Subsection 3.3 induced by the isomorphism $s_Z : O_y \rightarrow S$ does exist and it is an isomorphism and moreover the variant of Theorem 10 described in Subsubsection B4.2 holds. \square

3.5.2 Variant of Corollary 1

The isomorphism class of the quintuple $(M, (t_\alpha)_{\alpha \in \mathcal{J}}, \psi_M, \mathcal{G}^0, \mathcal{G})$ does not depend on $z \in \mathcal{N}^m(W(k))$ (as Theorem 4 (a) holds). But the isomorphism class of the family \mathfrak{F} does depend in general on the connected component \mathcal{C}_∞^m of \mathcal{N}_k^m which contains the k -valued point $y \in \mathcal{N}^m(k)$ defined by z . This is so as in general the $\mathcal{G}^0(W(k))$ -conjugacy class $[\mu]$ of the cocharacter $\mu : \mathbb{G}_{m, W(k)} \rightarrow \mathcal{G}$ does depend on the connected component \mathcal{C}_∞^m , despite the fact that it is easy to check that the $\mathcal{G}^0(B(k))$ -conjugacy class of $\mu_{B(k)} : \mathbb{G}_{m, B(k)} \rightarrow \mathcal{G}_{B(k)}$ does not depend on $z \in \mathcal{N}^m(W(k))$.

If the $\mathcal{G}^0(W(k))$ -conjugacy class $[\mu]$ does not depend on \mathcal{C}_∞^m and if the hypotheses of Corollary 1 hold, then (the isomorphism class of) the family \mathfrak{F} does not depend on $z \in \mathcal{N}^m(W(k))$ and from this and Corollary 1 we get that $\mathcal{N}_k^m/H^{(p)}$ itself is a quasi Shimura p -variety of Hodge type relative to \mathfrak{F} .

In this paragraph we check that if the hypotheses of Corollary 1 hold and if moreover $G_{\mathbb{Z}(p)}$ is a quasi-reductive group scheme for (G, \mathcal{X}, v) , then $\mathcal{N}_k^m/H^{(p)}$ itself is a quasi Shimura p -variety of Hodge type relative to \mathfrak{F} . To check this we can assume that k has a countable transcendental degree and thus that we have an $E(G, \mathcal{X})$ -monomorphism $B(k) \hookrightarrow \mathbb{C}$. Let \mathcal{G}^r be the schematic closure in \mathcal{G} of the normal, reductive subgroup $\mathcal{G}_{B(k)}^r$ of $\mathcal{G}_{B(k)}$ which corresponds to $G_{\mathbb{Q}_p}^r$ via Fontaine comparison theory, cf. Lemma 6 applied with $\mathcal{G}_{B(k)}^v = \mathcal{G}_{B(k)}^r$ and Definition 2 (a). As Theorem 4 (a) holds for $z \in \mathcal{N}^m(W(k))$, \mathcal{G}^r is isomorphic to $G_{\mathbb{Z}_p}^r \times_{\mathrm{Spec}(\mathbb{Z}_p)} \mathrm{Spec}(W(k))$ and thus it is a reductive group scheme. From Lemma 6 we get that μ factors through the normal, reductive subgroup scheme of \mathcal{G}^r of \mathcal{G} . Thus \mathcal{G}^r is the semiproduct of $\mathcal{G}^{r,0} = \mathcal{G}^r \cap \mathbf{Sp}(M, \psi_M)$ and of the image of μ and therefore $\mathcal{G}^{r,0}$ is a normal, reductive subgroup scheme of \mathcal{G}^r (or of \mathcal{G} or \mathcal{G}^0). The $\mathcal{G}(\mathbb{C})$ -conjugacy class of $\mu_{\mathbb{C}}$ does not depend on $z \in \mathcal{N}^m(W(k))$ as via isomorphisms $(M \otimes_{W(k)} \mathbb{C}, (t_\alpha)_{\alpha \in \mathcal{J}}) \rightarrow (W^\vee \otimes_{\mathbb{Q}} \mathbb{C}, (v_\alpha)_{\alpha \in \mathcal{J}})$ it corresponds to the $G(\mathbb{C})$ -conjugacy class of the cocharacters $\mu_h : \mathbb{G}_{m, \mathbb{C}} \rightarrow G_{\mathbb{C}}$ with $h \in \mathcal{X}$, cf. proof of Lemma 6. It is well-known that the last two sentences imply that $[\mu]$ equals to the

$\mathcal{G}^{r,0}(W(k))$ -conjugacy class of μ and it does not depend on $z \in \mathcal{N}^m(W(k))$. Thus $\mathcal{N}_k^m/H^{(p)}$ itself is a quasi Shimura p -variety of Hodge type relative to \mathfrak{F} , cf. previous paragraph.

The following lemma will be used in the proof of Lemma 7 and in Subsections 5.4 and 5.6.

Lemma 7. *Let $\mu : \mathbb{G}_{m,W(k)} \rightarrow \mathcal{G}$ and $M = F^1 \oplus F^0$ be as in Subsection 3.1. Let $y \in \mathcal{N}^s(k)$ be defined by $z \in \mathcal{N}^s(W(k)) = \mathcal{N}(W(k))$. Let $\mu_1 : \mathbb{G}_{m,W(k)} \rightarrow \mathcal{G}$ be a cocharacter such that we have a direct sum decomposition $M = F_1^1 \oplus F_1^0$ with the properties that $\mathbb{G}_{m,W(k)}$ acts through μ_1 on each F_1^i via the weight $-i$ and we have $F_1^1/pF_1^1 = F^1/pF^1$. Then the following three properties hold:*

(a) *There exists a point $z_1 \in \mathcal{N}^s(W(k))$ that lifts $y \in \mathcal{N}^s(k)$ and such that the principally quasi-polarized filtered F -crystal over k of $z_1^*(\mathcal{A}, \lambda_{\mathcal{A}})$ is (M, F_1^1, ϕ, ψ_M) .*

(b) *If $p > 2$, then a point z_1 as in part (a) is unique.*

(c) *We assume that $p = 2$ and that y factors through \mathcal{N}^m . We also assume that one of the following two conditions holds:*

(c.i) *the abelian variety A_k is ordinary and \mathcal{G} is smooth;*

(c.ii) *$G_{\mathbb{Z}_{(p)}}$ is a quasi-reductive group scheme for (G, \mathcal{X}, v) .*

Then there exist exactly 2^a such z_1 's, where a is the multiplicity of the Newton polygon slope -1 for $(\text{Lie}(\mathcal{G}), \phi)$. Moreover, if (c.i) holds, then we can choose μ_1 and z_1 such that the abelian scheme $z_1^(\mathcal{A})$ is the canonical lift of A_k .*

Proof: For $n \in \mathbb{N}^*$ let $W_n(k) := W(k)/p^n W(k)$. By induction on $n \in \mathbb{N}^*$ we show that there exists a point $z(n) \in \mathcal{N}^s(W(k)) = \mathcal{N}(W(k))$ that has the following three properties:

(i) *it lifts $y \in \mathcal{N}^s(k)$;*

(ii) *for $n \geq 2$ the $W_{n-1}(k)$ -valued points of \mathcal{N}^s defined by $z(n-1)$ and $z(n)$ coincide;*

(iii) *the principally quasi-polarized filtered F -crystal over k of $z(n)^*(\mathcal{A}, \lambda_{\mathcal{A}})$ is a quadruple of the form $(M, F_1^1(n), \phi, \psi_M)$, where $F_1^1(n)$ is congruent to F_1^1 modulo p^n .*

Let $z(1) := z$; obviously the base of the induction for $n = 1$ holds. The passage from n to $n + 1$ goes as follows. Not to introduce extra notation by

replacing z with $z(n)$, we can assume that $z(n) = z$; thus we have $F^1/p^n F^1 = F_1^1/p^n F_1^1$. Let $v \in p\text{Lie}(U)$ be such that for $u := 1_M + v$ we have $u(F^1) = F_1^1$, cf. Lemma 12. As $F^1/p^n F^1 = F_1^1/p^n F_1^1$, we have $v \in p^n \text{Lie}(U)$.

As the image of the Kodaira–Spencer map of ∇ is $\text{Lie}(U) \otimes_{W(k)} R$ (cf. property (iii) of Subsubsection B4.1) and as the morphism $\tau_R : \text{Spec}(R) \rightarrow \mathcal{N}^s$ is formally smooth, from the relation $v \in p^n \text{Lie}(U)$ we get that there exists a lift $z(n+1) \in \mathcal{N}^s(W(k))$ of $z(n)$ modulo p^n such that the principally quasi-polarized filtered F -crystal over k of $z(n+1)^*(\mathcal{A}, \lambda_{\mathcal{A}})$ is $(M, F_1^1(n+1), \phi, \psi_M)$, where $F_1^1(n+1)$ is congruent to $u(F^1) = F_1^1$ modulo p^{n+1} (this holds even if $p = 2$; for instance, the proof of [63, Prop. 6.4.6 (b)] applies entirely in the slightly more general context of our present lemma). This ends the induction.

From the property (ii) we get that there exists a point $z_1 \in \mathcal{N}^s(W(k))$ that lifts $z(n)$ modulo p^n for all $n \in \mathbb{N}^*$. Thus z_1 also lifts y , cf. property (i). The principally quasi-polarized filtered F -crystal over k of $z_1^*(\mathcal{A}, \lambda_{\mathcal{A}})$ is (M, F_1^1, ϕ, ψ_M) , cf. property (iii). Thus part (a) holds. Part (b) follows from Theorem 1 (b) and the Grothendieck–Messing deformation theory.

If the condition (c.i) holds, then part (c) follows from Theorem 1 (b) and Theorem 11 (c) (more precisely, the constructions of Subsection 3.2 needed to prove Theorem 1 (b) in Subsection 3.3 are exactly the constructions of Subsubsection B4.1 and Theorem 10).

We are left to check that part (c) holds if the condition (c.ii) holds. Let $G_{\mathbb{Z}_p}^r = G_{\mathbb{Z}_2}^r$ be as in Definition 2 (a). As y factors through \mathcal{N}^m , the normal, closed subgroup scheme \mathcal{G}^r of \mathcal{G} obtained as in Lemma 6 but for $G_{\mathbb{Q}_p}^v = G_{\mathbb{Q}_2}^r$, is a reductive group scheme over $W(k)$ through which μ and thus through which μ_1 's factor as well (note that U is a closed subgroup scheme of \mathcal{G}^r). Thus this case of part (c) also follows from Theorem 1 (b) and Theorem 11 (c) and (d) applied to (M, ϕ, \mathcal{G}^r) instead of (M, ϕ, \mathcal{G}) , with the cocharacter μ_1 chosen such that we have $\phi(F_1^1) = 2F_1^1$ (see the second paragraph of the proof of Theorem 11 for the existence of such a μ_1). \square

3.6 Proof of Proposition 1

Let $y \in \mathcal{N}(k)$ be such that $A_k := y^*(\mathcal{A})$ is an ordinary abelian variety. From [44, Cor. 3.8] we get that y factors through \mathcal{N}^s . Thus to prove the Proposition 1 we can assume that $p = 2$ and we have to show that $y \in \mathcal{N}^m(k)$. We will use the previous notation for a lift $z \in \mathcal{N}^s(W(k)) = \mathcal{N}(W(k))$ of y . We have a direct sum decomposition $M = F_1^1 \oplus F_1^0$ such that $\phi(F_1^0) = F_1^0$ and $\phi(F_1^1) = 2F_1^1$; obviously, $F_1^1/2F_1^1 = F^1/2F^1$. The cocharacter μ_1 :

$\mathbb{G}_{m,W(k)} \rightarrow \mathbf{GL}_M$ associated to it factors through \mathcal{G} , cf. second paragraph of the proof of Theorem 11. Based on Lemma 7 (a) we can assume that $F^1 = F_1^1$ and $\mu = \mu_1$; thus F^1 is the Hodge filtration of M which defines the canonical lift A_{can} of A_k . The $\text{Gal}(B(k))$ -module $H^1(D_{\text{can}})$ is canonically identified with a $\text{Gal}(B(k))$ -submodule of $\frac{1}{2}H^1(D)$ which contains $2H^1(D)$ (see [64, Subsubsects. 2.2.1 and 2.2.3] applied as in the second paragraph of the proof of [64, Lem. 2.2.5]). Let $H^1(D_{\text{can}}) = H^1(D_{\text{can}})_1 \oplus H^1(D_{\text{can}})_0$ be the direct sum decomposition that corresponds naturally to the direct sum decomposition $(M, F^1, \phi) = (F^1, F^1, \phi) \oplus (F^0, 0, \phi)$. As we have a short exact sequence $0 \rightarrow H^1(D_{\text{can}})_0 \rightarrow H^1(D) \rightarrow H^1(D_{\text{can}})_1 \rightarrow 0$, there exists $c \in \frac{1}{2}\text{Hom}(H^1(D_{\text{can}})_1, H^1(D_{\text{can}})_0)$ such that we have $(1_{H^1(D_{\text{can}})} + c)(H^1(D_{\text{can}})) = H^1(D)$. Let $\mu^{\acute{e}t} : \mathbb{G}_{m,\mathbb{Z}_2} \rightarrow \mathbf{GL}_{H^1(D_{\text{can}})}$ be the cocharacter that fixes $H^1(D_{\text{can}})_0$ and that acts on $H^1(D_{\text{can}})_1$ via the weight -1 . We consider an isomorphism $(H^1(D), (u_\alpha)_{\alpha \in \mathcal{J}}) \rightarrow (L_{(2)}^\vee \otimes_{\mathbb{Z}_{(2)}} \mathbb{Z}_2, (v_\alpha)_{\alpha \in \mathcal{J}})$ (cf. Lemma 3 (a)) to be viewed as an identification.

Let $G_{\mathbb{Z}_2}^r = G_{\mathbb{Z}_p}^r$ be as in Definition 2 (a); it is a reductive, normal, closed subgroup scheme of $G_{\mathbb{Z}_2}$ and thus of $\mathbf{GL}_{H^1(D)}$, cf. last identification. We know that $\mu_{\mathbb{Q}_2}^{\acute{e}t}$ is the étale counterpart of the cocharacter $\mu_{B(k)}$ of $\mathcal{G}_{B(k)}$, i.e., they correspond to each other via Fontaine comparison theory (the functorial isomorphism i_D of the property (v) of Subsection B2 preserves the direct sum decompositions of the previous paragraph). Thus from Lemma 6 we get that $\mu_{\mathbb{Q}_2}^{\acute{e}t}$ factors through $G_{\mathbb{Q}_2}^r$. Let $U_{\text{bigg}}^{\acute{e}t}$ (resp. $U^{\acute{e}t}$) be the unipotent radical of the parabolic subgroup scheme of $\mathbf{GL}_{H^1(D)}$ (resp. of the parabolic subgroup scheme $P_{\mathbb{Z}_2}^r$ of $G_{\mathbb{Z}_2}^r$) that normalizes $H^1(D)_0$ (cf. the existence of $\mu_{\mathbb{Q}_2}^{\acute{e}t}$ and the fact that the \mathbb{Z}_2 -schemes of parabolic subgroup schemes of reductive group schemes over \mathbb{Z}_2 are projective, see [15, Vol. III, Exp. XXVI, Cor. 3.5]). As a $P_{\mathbb{Z}_2}^r(\mathbb{Q}_2)$ -conjugate of $\mu_{\mathbb{Q}_2}^{\acute{e}t}$ extends to a cocharacter of $G_{\mathbb{Z}_2}^r$, it is easy to see that we have $G_{\mathbb{Z}_2}^r \cap U_{\text{bigg}}^{\acute{e}t} = U^{\acute{e}t}$ (this is a particular case of [10, Lem. 2.1.5 and Prop. 2.1.8 (3)]).

We claim that there exists an element $g \in G_{\mathbb{Z}_2}^r(\mathbb{Q}_2)$ such that we have $g(H^1(D_{\text{can}})) = H^1(D) = L_{(2)}^\vee \otimes_{\mathbb{Z}_{(2)}} \mathbb{Z}_2$. It suffices to show that the reduction \bar{e} of $e := 2c \in \text{Lie}(U_{\text{bigg}}^{\acute{e}t})$ modulo 2 belongs to $\text{Lie}(U_{\mathbb{F}_2}^{\acute{e}t})$. Let $G_{1,\mathbb{Z}_2}^r := (1_{H^1(D_{\text{can}})} - c)G_{\mathbb{Z}_2}^r(1_{H^1(D_{\text{can}})} + c)$; it is a reductive, closed subgroup scheme $\mathbf{GL}_{H^1(D_{\text{can}})}$. For $t \in W(k)$, $\mu^{\acute{e}t}((1 + 2t)^{-1})$ normalizes $H^1(D) \otimes_{\mathbb{Z}_2} W(k) = (1_{H^1(D_{\text{can}})} + c)(H^1(D_{\text{can}})) \otimes_{\mathbb{Z}_2} W(k)$ and thus its conjugate under $1_{H^1(D_{\text{can}})} - c$ belongs to $G_{1,\mathbb{Z}_2}^r(W(k))$. Therefore $1_{H^1(D_{\text{can}})/2H^1(D_{\text{can}})} - \bar{t}\bar{e}$ belongs to $G_{1,\mathbb{Z}_2}^r(k)$ for all $\bar{t} \in k$ and thus $\bar{e} \in \text{Lie}(G_{1,\mathbb{F}_2}^r)$. Conjugating via $1_{H^1(D_{\text{can}})} + c$ we get that

$\bar{e} \in \text{Lie}(G_{\mathbb{F}_2}^r) \cap \text{Lie}(U_{\text{bigg}, \mathbb{F}_2}^{\text{ét}}) = \text{Lie}(U_{\mathbb{F}_2}^{\text{ét}})$. Thus the claim holds.

Let $z_{\text{can}} : \text{Spec}(W(k)) \rightarrow \mathcal{M}_{r, O(v)}$ be the morphism which is the canonical lift of the composite morphism $\text{Spec}(k) \rightarrow \mathcal{N} \rightarrow \mathcal{M}_{r, O(v)}$ which is defined naturally by y and which factors through the ordinary locus of $\mathcal{M}_{r, k(v)}$. From the above claim we get that the generic fibres of z_{can} and z define $B(k)$ -valued points of $\mathcal{M}_{r, E(G, \mathcal{X})}$ which are images of complex points of $\text{Sh}(G, \mathcal{X})$ that differ by the right translation through the element $g \in G(\mathbb{Q}_2) \leq G(\mathbb{A}_f)$, cf. proof of Lemma 3. Therefore we have a unique factorization $z_{\text{can}} : \text{Spec}(W(k)) \rightarrow \mathcal{N}$ with the property that each t_α is the crystalline realization of the Hodge cycle $z_{\text{can}, B(\mathbb{F})}^*(\mathcal{A}_{E(G, \mathcal{X})})$ on $A_{\text{can}, B(\mathbb{F})}$. We know that $z_{\text{can}} : \text{Spec}(W(k)) \rightarrow \mathcal{N}$ factors through \mathcal{N}^s (cf. Theorem 1 (a)) and even through \mathcal{N}^m (cf. Theorem 4 (a)). From this and the existence of $g \in G_{\mathbb{Z}_2}^r(\mathbb{Q}_2)$ such that $g(H^1(D_{\text{can}})) = H^1(D)$ we get that Theorem 4 (a) holds for $z \in \mathcal{N}^s(W(k))$ and therefore we have $z \in \mathcal{N}^m(W(k))$ and $y \in \mathcal{N}^m(k)$. Thus Proposition 1 holds. \square

4 Applications to integral models

In this section we take k to be an algebraic closure of $k(v)$. This implies that there exist $O(v)$ -monomorphisms $W(k) \hookrightarrow \mathbb{C}$.

Lemma 8 presents a simple criterion on when the $k(v)$ -scheme $\mathcal{N}_{k(v)}^m$ is non-empty or when the $W(k)$ -valued points of $\mathcal{N}_{W(k)}^m$ are Zariski dense. In Subsection 4.1 we apply Theorem 1 (a) and Lemma 8 (a) to prove the existence of good smooth integral models of $\text{Sh}_{\tilde{H}}(G, \mathcal{X})$ over $O(v)$ for a large class of maximal compact, open subgroups \tilde{H} of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$. Corollary 3 can be viewed as a smooth solution (answer) to the conjecture (question) of Langlands of [31, p. 411] for Shimura varieties of Hodge type. Theorem 7 shows that, in the case when $G_{\mathbb{Q}_p}$ splits over an unramified extension of \mathbb{Q}_p , Subsection 4.1 extends naturally to the case of parahoric subgroups of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$.

Lemma 8. *We assume that one of the following two conditions holds:*

- (i) *there exists a smooth, affine group scheme $G_{\mathbb{Z}(p)}^v$ over $\mathbb{Z}(p)$ that extends G (i.e., it has G as its generic fibre), that has a special fibre $G_{\mathbb{F}_p}^v$ of the same rank as G , and that has the property that there exists a homomorphism $G_{\mathbb{Z}(p)}^v \rightarrow G_{\mathbb{Z}(p)}$ which extends the identity automorphism of G ;*
- (ii) *we have $e(v) = 1$ and $G_{\mathbb{Z}(p)}$ is a quasi-reductive group scheme for (G, \mathcal{X}, v) in the sense of Definition 2 (a).*

(a) Then $e(v) = 1$ and the $k(v)$ -scheme $\mathcal{N}_{k(v)}^m$ (and thus also $\mathcal{N}_{k(v)}^s$) is non-empty.

(b) Then the $W(k)$ -valued points of $\mathcal{N}_{W(k)}^m$ are Zariski dense in $\mathcal{N}_{W(k)}^m$.

Proof: We prove part (a). We first assume that (i) holds. Each torus of $G_{\mathbb{F}_p}^v$ lifts to a torus of $G_{\mathbb{Z}_p}^v$, cf. [15, Vol. II, Exp. XII, Cor. 1.10]. Thus $G_{\mathbb{Z}_p}^v$ has tori of rank equal to the rank of G . Let $T_{\mathbb{Z}(p)}^v$ be a torus of $G_{\mathbb{Z}(p)}^v$ of the same rank as G and such that there exists $h^v \in \mathcal{X}$ which factors through $T_{\mathbb{R}}^v$. Its existence is implied by [22, Lem. 5.5.3]. The pair $(T_{\mathbb{Q}}^v, \{h^v\})$ is a Shimura subpair of (G, \mathcal{X}) and therefore we have an inclusion $E(G, \mathcal{X}) \subset E(T_{\mathbb{Q}}^v, \{h^v\})$ of reflex fields. Each prime of $E(T_{\mathbb{Q}}^v, \{h^v\})$ that divides v is unramified over p (cf. [38, Prop. 4.6 and Cor. 4.7]) and thus we have $e(v) = 1$. The intersection $H^v := H \cap T_{\mathbb{Z}(p)}^v(\mathbb{Q}_p)$ is the unique hyperspecial subgroup $T_{\mathbb{Z}(p)}^v(\mathbb{Z}_p)$ of $T_{\mathbb{Z}(p)}^v(\mathbb{Q}_p)$. Therefore there exists an integral model \mathcal{Z}^v of $\mathrm{Sh}_{H^v}(T_{\mathbb{Q}}^v, \{h^v\})$ over the spectrum of the normalization of $O_{(v)}$ in $E(T_{\mathbb{Q}}^v, \{h^v\})$ which is a pro-finite pro-étale cover of $\mathrm{Spec}(O_{(v)})$, cf. either [37, Rm. 2.16] or [54, Ex. 3.2.8]. In particular, \mathcal{Z}^v is a regular, formally étale, faithfully flat $O_{(v)}$ -scheme. The functorial morphism $\mathrm{Sh}_{H^v}(T_{\mathbb{Q}}^v, \{h^v\}) \rightarrow \mathrm{Sh}_H(G, \mathcal{X})$ of $E(G, \mathcal{X})$ -schemes extends uniquely to a morphism $\mathcal{Z}^v \rightarrow \mathcal{N}^s$ of $O_{(v)}$ -schemes, cf. Theorem 1 (a). There exist points $z \in \mathcal{Z}^v(W(k))$. Let $(v_\alpha)_{\alpha \in \mathcal{J}^v}$ be a family of tensors of $\mathcal{T}(W^v)$ such that $T_{\mathbb{Q}}^v$ is the subgroup of \mathbf{GL}_{W^v} that fixes v_α for all $\alpha \in \mathcal{J}^v$. We can assume that $\mathcal{J} \subset \mathcal{J}^v$ and that for each $\alpha \in \mathcal{J}$, the tensor v_α is the tensor introduced in Subsection 2.3. We will use the notation of Subsection 3.1 for $z \in \mathcal{Z}^v(W(k))$. From Theorem 4 (a) applied to the point $z \in \mathcal{Z}^v(W(k))$ we get that there exists an isomorphism $(M, (t_\alpha)_{\alpha \in \mathcal{J}^v}) \rightarrow (L_{(p)}^v \otimes_{\mathbb{Z}(p)} W(k), (v_\alpha)_{\alpha \in \mathcal{J}^v})$ (each t_α with $\alpha \in \mathcal{J}^v$, is the de Rham realization of the Hodge cycle on $A_{B(k)}$ that corresponds naturally to v_α). Thus as $\mathcal{J} \subset \mathcal{J}^v$, Theorem 4 (a) holds for the $W(k)$ -valued point of \mathcal{N}^s defined by z . From this and Proposition 3 (b) we get that this last point factors through \mathcal{N}^m . Therefore the $k(v)$ -scheme $\mathcal{N}_{k(v)}^m$ is non-empty.

We now assume that (ii) holds; thus $e(v) = 1$. Let $G_{\mathbb{Z}_p}^r$ and μ_v be as in Definition 2 (a). Let $T_{\mathbb{F}_p}^r$ be a maximal torus of $G_{\mathbb{F}_p}^r$. Due to the existence of μ_v , $T_{\mathbb{F}_p}^r$ has positive rank. The torus $T_{\mathbb{F}_p}^r$ lifts to a torus $T_{\mathbb{Z}_p}^r$ of $G_{\mathbb{Z}_p}^r$, cf. [15, Vol. II, Exp. XII, Cor. 1.10]. Let T_{0, \mathbb{Q}_p}^v be a maximal torus of $G_{\mathbb{Q}_p}$ which has $T_{\mathbb{Q}_p}^r$ as a subtorus. Let T^v be a maximal torus of G such that there exists an element $h^v \in \mathcal{X}$ which factors through $T_{\mathbb{R}}^v$ and moreover $T_{\mathbb{Q}_p}^v$ is H -conjugate to T_{0, \mathbb{Q}_p}^v . Again, the existence of T^v is implied by [22, Lem. 5.5.3]. Thus (up

to H -conjugation) we can assume that we have $T_{0, \mathbb{Q}_p}^v = T_{\mathbb{Q}_p}^v$.

The intersection $H^v := H \cap T^v(\mathbb{Q}_p)$ is not necessarily the maximal compact, open subgroup of $T^v(\mathbb{Q}_p)$ and the subgroup $T^v(\mathbb{Q})H^v$ of $T^v(\mathbb{Q}_p)$ is not necessarily $T^v(\mathbb{Q}_p)$. But the intersection $T_{\mathbb{Z}_p}^r(\mathbb{Q}_p) \cap H$ is the unique hyperspecial subgroup $T_{\mathbb{Z}_p}^r(\mathbb{Z}_p)$ of $T_{\mathbb{Z}_p}^r(\mathbb{Q}_p)$. We fix an $O_{(v)}$ -monomorphism $W(k(v)) \hookrightarrow \mathbb{C}$ as in Definition 2 (a). As μ_{h^v} and $\mu_{v, \mathbb{C}}$ are $G(\mathbb{C})$ -conjugate and as $G_{\mathbb{C}}^r$ is a normal subgroup of $G_{\mathbb{C}}$, μ_{h^v} factors through the intersection $T_{\mathbb{C}}^v \cap G_{\mathbb{C}}^r$ and therefore through $T_{\mathbb{C}}^r = T_{\mathbb{Z}_p}^r \times_{\text{Spec}(\mathbb{Z}_p)} \text{Spec}(\mathbb{C})$. Thus as $T_{\mathbb{Z}_p}^r$ splits over a finite, unramified extension of \mathbb{Z}_p , we get that the field of definition $E(T_{\mathbb{Q}}^v, \{h^v\})$ of μ_{h^v} is a number subfield of \mathbb{C} that contains $E(G, \mathcal{X})$ and that is unramified over v . From the class field theory (see [30, Th. 4 of p. 220]) and the reciprocity map of [37, pp. 163–164] we easily get that each connected component of $\text{Sh}_{H^v}(T_{\mathbb{Q}}^v, \{h^v\})_{\mathbb{C}}$ is defined over the spectrum of an abelian extension of $E(T_{\mathbb{Q}}^v, \{h^v\})$ unramified over all primes of $E(T_{\mathbb{Q}}^v, \{h^v\})$ that divide v . Thus there exists an integral model \mathcal{Z}^v of $\text{Sh}_{H^v}(T_{\mathbb{Q}}^v, \{h^v\})$ over the normalization of $O_{(v)}$ in $E(T_{\mathbb{Q}}^v, \{h^v\})$ which has the same properties as above. Let $z \in \mathcal{Z}^v(W(k))$.

Let $(v_{\alpha})_{\alpha \in \mathcal{J}^r}$ be a family of tensors of $\mathcal{T}(W^v \otimes_{\mathbb{Q}} \mathbb{Q}_p)$ such that $T_{\mathbb{Q}_p}^r$ is the subgroup of $\mathbf{GL}_{W^v \otimes_{\mathbb{Q}} \mathbb{Q}_p}$ that fixes v_{α} for all $\alpha \in \mathcal{J}^r$. We can assume that $\mathcal{J} \subset \mathcal{J}^r$ and that for each $\alpha \in \mathcal{J}$, the tensor v_{α} is the tensor introduced in Subsection 2.3.

We will use the notation of Subsection 3.1 for $z \in \mathcal{Z}^v(W(k))$ and for k of countable transcendental degree. Let $\rho_D : \text{Gal}(B(k)) \rightarrow \mathbf{GL}_{H_{\text{ét}}^1(A_{B(k)}, \mathbb{Q}_p)}(\mathbb{Q}_p) \rightarrow \mathbf{GL}_{L_{(p)}^v \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}_p}(\mathbb{Q}_p)$ be the p -adic Galois representation associated to the Barsotti–Tate group D of A . Let $\mathcal{D}_{\mathbb{Q}_p}^{\text{ét}}$ be the schematic closure of $\text{Im}(\rho_D)$ in $\mathbf{GL}_{L_{(p)}^v \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}_p}$; it is a connected group (cf. Subsection B1) which is a subgroup of $T_{\mathbb{Q}_p}^v$. As the groups $T_{\mathbb{Q}_p}^v$, $T_{\mathbb{Q}_p}^r$, and $G_{\mathbb{Q}_p}^r$ are normalized by $\mathcal{D}_{\mathbb{Q}_p}^{\text{ét}}$, we can speak about the subgroups $\mathcal{T}_{B(k)}^r$, $\mathcal{T}_{B(k)}^v$, and $\mathcal{G}_{B(k)}^r$ of $\mathcal{G}_{B(k)}$ that correspond to $T_{\mathbb{Q}_p}^v$, $T_{\mathbb{Q}_p}^r$, and $G_{\mathbb{Q}_p}^r$ (respectively) via Fontaine comparison theory for D (cf. Lemma 15 (a)). The generic fibre of μ factors through $\mathcal{T}_{B(k)}^v$ (cf. Subsection 3.1 applied in the context of $z \in \mathcal{Z}^v(W(k))$) and through $\mathcal{G}_{B(k)}^r$ (cf. Lemma 6 applied with $G_{\mathbb{Q}_p}^v = G_{\mathbb{Q}_p}^r$ to the image of $z \in \mathcal{Z}^v(W(k))$ in $\mathcal{N}^s(W(k))$) and thus it factors through $\mathcal{T}_{B(k)}^r = \mathcal{T}_{B(k)}^v \cap \mathcal{G}_{B(k)}^r$. From this and Lemma 15 (b) we get that $\mathcal{D}_{\mathbb{Q}_p}^{\text{ét}}$ is a subgroup of $T_{\mathbb{Q}_p}^r$. This implies that each v_{α} with $\alpha \in \mathcal{J}^r$ defines naturally an étale Tate-cycle u_{α} on $D_{B(k)}$.

As $T_{\mathbb{Z}_p}^r$ is a torus, from Theorem 9 applied to the pair $(D, (u_{\alpha})_{\alpha \in \mathcal{J}^r})$, from

Formula (3), and from Lemma 3 (b) applied to $\mathrm{Sh}(T_{\mathbb{Q}}^{\vee}, \{h^{\vee}\})$ we get that there exist isomorphisms

$$(M, (t_{\alpha})_{\alpha \in \mathcal{J}^r}, \psi_M) \rightarrow (H_{\acute{e}t}^1(A_{B(k)}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(k), (u_{\alpha})_{\alpha \in \mathcal{J}^r}, \psi_{H_{\acute{e}t}^1}) \rightarrow (L_{(p)}^{\vee} \otimes_{\mathbb{Z}_{(p)}} W(k), (v_{\alpha})_{\alpha \in \mathcal{J}^r}, \psi^{\vee})$$

(each $t_{\alpha} \in \mathcal{T}(M[\frac{1}{p}])$ with $\alpha \in \mathcal{J}^r$ corresponds to u_{α} via Fontaine comparison theory for D). As $\mathcal{J} \subset \mathcal{J}^r$, from this and Proposition 3 (b) we get that the image of $z \in \mathcal{Z}^{\vee}(W(k))$ in $\mathcal{N}^s(W(k))$ belongs to $\mathcal{N}^m(W(k))$. Thus the $k(v)$ -scheme $\mathcal{N}_{k(v)}^m$ is non-empty, i.e., part (a) holds.

We prove part (b). If (i) holds, let $T_{\mathbb{Q}_p}^r := T_{\mathbb{Q}_p}^{\vee}$. Thus $T_{\mathbb{Q}_p}^r$ is well-defined regardless of which one of the conditions (i) and (ii) holds. Due to Formula (1) and the fact that \mathcal{N}^m is $G(\mathbb{A}_f^{(p)})$ -invariant, to prove that the $W(k)$ -valued points of $\mathcal{N}_{W(k)}^m$ are Zariski dense, it suffices to show that for each open subset \mathcal{K} of \mathcal{X} and for every element of $G(\mathbb{Q}) \backslash G(\mathbb{Q}_p) / H$, we can choose a representative $g_j \in G(\mathbb{Q}_p) \leq G(\mathbb{A}_f)$ of this element and we can choose $(T_{\mathbb{Q}}^{\vee}, \{h^{\vee}\})$ such that $h^{\vee} \in \mathcal{K}$ and the elements of $T_{\mathbb{Q}_p}^r(\mathbb{Q}_p) \cap H$ act via left translation trivially on the image of g_j in $G(\mathbb{Q}_p) / H$ (this is so as from the class field theory and the reciprocity map of [37, pp. 163–164] we easily get that the complex point $[h^{\vee}, g_j]$ of $\mathrm{Sh}_{H^{\vee}}(T_{\mathbb{Q}}^{\vee}, \{h^{\vee}\})$ is defined over the spectrum of an abelian extension of $E(T_{\mathbb{Q}}^{\vee}, \{h^{\vee}\})$ unramified over all primes of $E(T_{\mathbb{Q}}^{\vee}, \{h^{\vee}\})$ that divide v).

If (i) holds, then the existence of $T_{\mathbb{Z}_{(p)}}^{\vee}$ implies that $G_{\mathbb{Q}_p}$ splits over a finite unramified extension of \mathbb{Q}_p and therefore we have $G(\mathbb{Q}_p) = G(\mathbb{Q})H$ (cf. [38, Lem. 4.10]). This implies that we can take g_j to be the identity element and based on [22, Lem. 5.5.3] we can assume that $h^{\vee} \in \mathcal{K}$.

If (ii) holds, then g_j can be any representative and we choose $T_{\mathbb{Z}_p}^r$ so that it is also a maximal torus of $g_j G_{\mathbb{Z}_p}^r g_j^{-1}$ (this is argued similarly to [62, Subsect. 5.2] based on [52, pp. 43–44, Subsect. 3.4, and Subsubsect. 3.8.1]); based on [22, Lem. 5.5.3] we can assume that $h^{\vee} \in \mathcal{K}$ and that $T_{\mathbb{Q}_p}^{\vee}$ is $H \cap g_j H g_j^{-1}$ -conjugate to $T_{0, \mathbb{Q}_p}^{\vee}$ and thus that the elements of $T_{\mathbb{Q}_p}^r(\mathbb{Q}_p) \cap H$ act via left translation trivially on the image of g_j in $G(\mathbb{Q}_p) / H$. We conclude that part (b) holds. \square

4.1 Integral models for maximal compact, open subgroups

Let \tilde{H} be a maximal compact, open subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$. Let $\tilde{G}_{\mathbb{Z}_p}$ be a smooth, affine group scheme over \mathbb{Z}_p that extends $G_{\mathbb{Q}_p}$ and such that we

have $\tilde{H} = \tilde{G}_{\mathbb{Z}_p}(\mathbb{Z}_p)$, cf. [52, p. 52]. Let $\tilde{G}_{\mathbb{Z}_p}$ be the smooth, affine group scheme over \mathbb{Z}_p that extends G and whose extension to \mathbb{Z}_p is $\tilde{G}_{\mathbb{Z}_p}$, cf. [54, Cl. 3.1.3.1]. Let $\tilde{L}_{(p)}$ be a \mathbb{Z}_p -lattice of W such that the monomorphism $G \hookrightarrow \mathbf{GL}_W$ extends to a homomorphism $\tilde{G}_{\mathbb{Z}_p} \rightarrow \mathbf{GL}_{\tilde{L}_{(p)}}$, cf. [25, Part I, 10.9].

Lemma 9. *We can modify the \mathbb{Z} -lattice L of W and the injective map $f : (G, \mathcal{X}) \hookrightarrow (\mathbf{GSp}(W, \psi), \mathcal{Y})$, such that we have an identity $H = \tilde{H}$ and $L_{(p)}$ is a $\tilde{G}_{\mathbb{Z}_p}$ -module (but we emphasize that the resulting homomorphism $\tilde{G}_{\mathbb{Z}_p} \rightarrow \mathbf{GL}_{L_{(p)}}$ of smooth group schemes over \mathbb{Z}_p is not necessarily a closed embedding).*

Proof: Let \tilde{L} be the \mathbb{Z} -lattice of W such that we have $\tilde{L}[\frac{1}{p}] = L[\frac{1}{p}]$ and $\tilde{L} \otimes_{\mathbb{Z}} \mathbb{Z}_p = \tilde{L}_{(p)}$. If ψ induces a perfect form on \tilde{L} , then by replacing L with \tilde{L} we get that $H = \tilde{H}$. This is so as the fact that \tilde{H} is a maximal compact, open subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$ implies that the monomorphism $\tilde{H} \hookrightarrow G_{\mathbb{Q}_p}(\mathbb{Q}_p) \cap \mathbf{GL}_{\tilde{L} \otimes_{\mathbb{Z}} \mathbb{Z}_p}(\mathbb{Z}_p)$ is an isomorphism. If ψ does not induce a perfect form on \tilde{L} , then we will have to modify f as follows.

Let $L'_1 := \tilde{L} \oplus \tilde{L}^\vee$. Let $W_1 := L'_1 \otimes_{\mathbb{Z}} \mathbb{Q}$ and $L'_{1,(p)} := L'_1 \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Let ψ'_1 be a perfect, alternating bilinear form on L'_1 such that the group scheme $\mathbf{SL}_{\tilde{L}}$, when viewed naturally as a closed subgroup scheme of $\mathbf{SL}_{L'_1}$, is in fact a subgroup scheme of $\mathbf{Sp}(L'_1, \psi'_1)$. We can assume that \tilde{L} and \tilde{L}^\vee are both maximal isotropic \mathbb{Z} -lattices of W_1 with respect to ψ'_1 (this automatically holds if $r > 1$). Let $\tilde{G}_{\mathbb{Z}_p}^0$ be the schematic closure in $\tilde{G}_{\mathbb{Z}_p}$ of G^0 ; it is a flat, closed subgroup scheme of $\mathbf{SL}_{\tilde{L} \otimes_{\mathbb{Z}} \mathbb{Z}_p}$ and thus also of $\mathbf{GSp}(L'_{1,(p)}, \psi'_1)$. The subgroup scheme of $\mathbf{GSp}(L'_{1,(p)}, \psi'_1)$ generated by $Z(\mathbf{GL}_{L'_{1,(p)}})$ and $\tilde{G}_{\mathbb{Z}_p}^0$ is a group scheme which is naturally identified with $\tilde{G}_{\mathbb{Z}_p}$ itself.

Let \mathfrak{A} be the free \mathbb{Z}_p -module of alternating bilinear forms on $L'_1 \otimes_{\mathbb{Z}} \mathbb{Z}_p$ fixed by $\tilde{G}_{\mathbb{Z}_p}^0$. There exist elements of $\mathfrak{A} \otimes_{\mathbb{Z}_p} \mathbb{R}$ that define polarizations of the Hodge \mathbb{Q} -structure on W_1 defined by a fixed element $h \in \mathcal{X}$, cf. [13, Cor. 2.3.3]. Thus the real vector space $\mathfrak{A} \otimes_{\mathbb{Z}_p} \mathbb{R}$ has a non-empty, open subset of such polarizations, cf. [13, Subsubsection. 1.1.18 (a)]. A standard application to \mathfrak{A} of the approximation theory for independent valuations, shows the existence of an alternating bilinear form ψ_1 on $L'_1 \otimes_{\mathbb{Z}} \mathbb{Z}_p$ that is fixed by $\tilde{G}_{\mathbb{Z}_p}^0$, that is congruent to ψ'_1 modulo p , and that defines a polarization

of the mentioned Hodge \mathbb{Q} -structure. Thus there exists an injective map $f_1 : (G, \mathcal{X}) \hookrightarrow (\mathbf{GSp}(W_1, \psi_1), \mathcal{Y}_1)$ of Shimura pairs.

As ψ_1 is congruent to ψ'_1 modulo p , it is a perfect, alternating bilinear form on $L'_1 \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. Let L_1 be a \mathbb{Z} -lattice of W_1 such that ψ_1 induces a perfect, alternating bilinear form on L_1 and we have $L_{1,(p)} := L_1 \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} = L'_{1,(p)}$; thus $L_{1,(p)}$ is a $\tilde{G}_{\mathbb{Z}_{(p)}}$ -module. As above we argue that $\tilde{H} = G_{\mathbb{Q}_p}(\mathbb{Q}_p) \cap \mathbf{GL}_{L_1 \otimes_{\mathbb{Z}} \mathbb{Z}_p}(\mathbb{Z}_p)$. Therefore the lemma holds. \square

Theorem 6. *Let \tilde{H} be a maximal compact, open subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$. Let $\tilde{G}_{\mathbb{Z}_{(p)}}$ be a smooth, affine group scheme over $\mathbb{Z}_{(p)}$ that has G as its generic fibre and such that $\tilde{H} = \tilde{G}_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p)$ (see beginning of Subsection 4.1). We assume that one of the following two conditions holds:*

- (i) *the special fibre $\tilde{G}_{\mathbb{F}_p}$ of $\tilde{G}_{\mathbb{Z}_p}$ has a torus of the same rank as G (e.g., this holds if $G_{\mathbb{Q}_p}$ splits over an unramified extension of \mathbb{Q}_p , cf. [52, Sects. 1.10 and 3.4]);*
- (ii) *we have $e(v) = 1$ and $\tilde{G}_{\mathbb{Z}_{(p)}}$ is a quasi-reductive group scheme for (G, \mathcal{X}, v) .*

Then there exists a unique regular, formally smooth integral model \tilde{N}^s of $Sh_{\tilde{H}}(G, \mathcal{X})$ over $O_{(v)}$ that satisfies the following smooth extension property: if Z is a regular, formally smooth scheme over a discrete valuation ring O which is of absolute ramification index 1 and is an $O_{(v)}$ -algebra, then each morphism $Z_{E(G, \mathcal{X})} \rightarrow \tilde{N}^s_{E(G, \mathcal{X})}$ of $E(G, \mathcal{X})$ -schemes extends uniquely to a morphism $Z \rightarrow \tilde{N}^s$ of $O_{(v)}$ -schemes.

Proof: We can assume that the injective map $f : (G, \mathcal{X}) \rightarrow (\mathbf{GSp}(W, \psi), \mathcal{Y})$ of Shimura pairs is such that $\tilde{H} = H$ and $L_{(p)}$ is a $\tilde{G}_{\mathbb{Z}_{(p)}}$ -module, cf. Lemma 9. If (i) holds, then the condition (i) of Lemma 8 holds. If (ii) holds, let $\tilde{G}_{\mathbb{Z}_p}^r$ be a reductive, normal, closed subgroup scheme of $\tilde{G}_{\mathbb{Z}_p}$ such that there exists a cocharacter $\mu_v : \mathbb{G}_{m, W(k(v))} \rightarrow \tilde{G}_{W(k(v))}^r$ with the property that the extension of μ_v to \mathbb{C} via an (any) $O_{(v)}$ -monomorphism $W(k(v)) \hookrightarrow \mathbb{C}$ defines a cocharacter of $G_{\mathbb{C}}$ that is $G(\mathbb{C})$ -conjugate to the cocharacters μ_h ($h \in \mathcal{X}$) introduced in the beginning of Subsection 1.3. The group $G_{\mathbb{C}}^{\text{der}}$ has no simple factors that are \mathbf{SO}_{2n+1} groups for some $n \in \mathbb{N}^*$, cf. Fact 2. Therefore the natural homomorphism $\tilde{G}_{\mathbb{Z}_p}^r \rightarrow \mathbf{GL}_{L_{(p)} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p}$ is a closed embedding, cf. [56, Thm. 1.1 (d)]. Thus $\tilde{G}_{\mathbb{Z}_p}^r$ is naturally a closed subgroup scheme of $G_{\mathbb{Z}_p}$. This implies that $G_{\mathbb{Z}_{(p)}}$ is also a quasi-reductive group scheme for (G, \mathcal{X}, v) . Thus, if (ii) holds, then the condition (ii) of Lemma 8 holds.

As one of the two conditions (i) and (ii) of Lemma 8 holds, the $k(v)$ -scheme $\mathcal{N}_{k(v)}^s$ is non-empty (cf. Lemma 8 (a)). Based on Theorem 1 (a) and the fact that $\tilde{H} = H$, we get that as $\tilde{\mathcal{N}}^s$ we can take \mathcal{N}^s itself. \square

Corollary 3. *Let (G, \mathcal{X}) be a Shimura pair of Hodge type. Let v a prime of the reflex field $E(G, \mathcal{X})$ that divides a prime p with the property that the group $G_{\mathbb{Q}_p}$ is unramified. Then for each hyperspecial subgroup \tilde{H} of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$, there exists a unique regular, formally smooth integral model $\tilde{\mathcal{N}}^s$ of $Sh_{\tilde{H}}(G, \mathcal{X})$ over $O_{(v)}$ that satisfies the following smooth extension property: if Z is a regular, formally smooth scheme over a discrete valuation ring O which is of absolute ramification index 1 and is an $O_{(v)}$ -algebra, then each morphism $Z_{E(G, \mathcal{X})} \rightarrow \tilde{\mathcal{N}}_{E(G, \mathcal{X})}^s$ extends uniquely to a morphism $Z \rightarrow \tilde{\mathcal{N}}^s$ between $O_{(v)}$ -schemes.*

Proof: As \tilde{H} is a hyperspecial subgroup, we can assume that the group scheme $\tilde{G}_{\mathbb{Z}_p}$ is reductive. Thus $\tilde{G}_{\mathbb{Z}_{(p)}}$ is a reductive group scheme. Therefore the condition (i) (in fact even the condition (ii)) of Theorem 9 holds and hence the corollary follows from Theorem 6. \square

Theorem 7. *We assume that $G_{\mathbb{Q}_p}$ splits over an unramified extension of \mathbb{Q}_p (thus $e(v) = 1$). Then for each parahoric subgroup \tilde{H} of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$ in the sense of [9, Def. 5.2.6], there exists a unique regular, formally smooth integral model $\tilde{\mathcal{N}}^s$ of $Sh_{\tilde{H}}(G, \mathcal{X})$ over $O_{(v)}$ that satisfies the following smooth extension property: if Z is a regular, formally smooth scheme over a discrete valuation ring O which is of absolute ramification index 1 and is an $O_{(v)}$ -algebra, then each morphism $Z_{E(G, \mathcal{X})} \rightarrow \tilde{\mathcal{N}}_{E(G, \mathcal{X})}^s$ extends uniquely to a morphism $Z \rightarrow \tilde{\mathcal{N}}^s$ between $O_{(v)}$ -schemes.*

Proof: As \tilde{H} is a parahoric subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$, it is the subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$ that fixes all vertices v_1, \dots, v_s of a facet $\tilde{\mathcal{F}}$ of an apartment $\tilde{\mathcal{A}}$ of the building of $G_{\mathbb{Q}_p}$ over \mathbb{Q}_p . For $i \in \{1, \dots, s\}$, let \tilde{H}_i be the maximal, compact subgroup of $G_{\mathbb{Q}_p}(\mathbb{Q}_p)$ that fixes v_i . We have:

(i) $\tilde{H} = \bigcap_{i=1}^s \tilde{H}_i$.

Let $\tilde{G}_{i, \mathbb{Z}_p}$ be the smooth, affine group scheme over \mathbb{Z}_p that extends $G_{\mathbb{Q}_p}$, that satisfies the identity $\tilde{H}_i = \tilde{G}_{i, \mathbb{Z}_p}(\mathbb{Z}_p)$, and that is constructed as in [52, p. 52]. Let $\tilde{G}_{\mathbb{Z}_{(p)}}$ be the flat group scheme over $\mathbb{Z}_{(p)}$ that extends G and

such that its extension to \mathbb{Z}_p is $\tilde{G}_{i,\mathbb{Z}_p}$, cf. [54, Cl. 3.1.3.1]. Let $\tilde{G}_{\mathbb{Z}(p)}$ be the schematic closure of G embedded diagonally into the generic fibre of the product $\prod_{i=1}^s \tilde{G}_{i,\mathbb{Z}(p)}$; it is a flat, affine group scheme over $\mathbb{Z}(p)$ such that (due to property (i)) we have $\tilde{G}_{\mathbb{Z}(p)}(\mathbb{Z}_p) = \tilde{H}$.

As $G_{\mathbb{Q}_p}$ splits over an unramified extension of \mathbb{Q}_p , there exists a maximal torus \tilde{T} of $G_{\mathbb{Q}_p}$ which splits over a finite unramified Galois extension \mathbb{E}_p of \mathbb{Q}_p and which contains the maximal, split torus of $G_{\mathbb{Q}_p}$ that is related to the apartment $\tilde{\mathcal{A}}$ in such a way that the apartment of the building of $G_{\mathbb{E}_p}$ related to $\tilde{T}_{\mathbb{E}_p}$ contains $\tilde{\mathcal{A}}$ (see [52, Subsects. 1.10 and 2.6]). Let $\tilde{T}_{\mathbb{Z}_p}$ be the torus over \mathbb{Z}_p whose generic fibre is \tilde{T} ; it is a maximal torus of each $\tilde{G}_{i,\mathbb{Z}_p}$ and therefore also of the pullback of $\tilde{G}_{\mathbb{Z}(p)}$ to $\text{Spec}(\mathbb{Z}_p)$.

Based on the last two paragraphs, it is easy to check that (cf. also [9, Subsect. 5.2]):

(ii) $\tilde{G}_{\mathbb{Z}(p)}$ is a smooth group scheme over $\text{Spec}(\mathbb{Z}(p))$ whose special fibre has the same rank as G .

Based on Lemma 9, for each $i \in \{1, \dots, s\}$ there exists an injective map $f_i : (G, \mathcal{X}) \hookrightarrow (\mathbf{GSp}(W_i, \psi_i), \mathcal{Y}_i)$ and a \mathbb{Z} -lattice L_i of W_i such that ψ_i induces a perfect, alternating bilinear form $\psi_i : L_i \times L_i \rightarrow \mathbb{Z}$, $L_i \otimes_{\mathbb{Z}} \mathbb{Z}(p)$ is a $\tilde{G}_{i,\mathbb{Z}(p)}$ -module, and we have $\tilde{H}_i = G(\mathbb{Q}_p) \cap \mathbf{GSp}(L_i, \psi_i)(\mathbb{Z}_p)$. We fix an element $x \in \mathcal{X}$. By replacing each ψ_i by either itself or $-\psi_i$ we can assume that $2\pi i \psi_i$ is a polarization of the \mathbb{Q} -structure on W_i defined naturally by x for all $i \in \{1, \dots, s\}$. Let

$$(W, \psi, L) = (\oplus_{i=1}^s W_i, \oplus_{i=1}^s \psi_i, \oplus_{i=1}^s L_i).$$

We have a natural diagonal embedding $(G, \mathcal{X}) \hookrightarrow (\mathbf{GSp}(W, \psi), \mathcal{Y})$, where \mathcal{Y} is the $\mathbf{GSp}(W, \psi)(\mathbb{R})$ -conjugacy class of homomorphisms $\text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \rightarrow \mathbf{GSp}(W, \psi)_{\mathbb{R}}$ that contains all those homomorphisms that are defined naturally by elements of \mathcal{X} . The group $H = G(\mathbb{Q}_p) \cap \mathbf{GSp}(L, \psi)(\mathbb{Z}_p)$ is $\cap_{i=1}^s \tilde{H}_i = \tilde{H}$, cf. property (i). From the construction of $\tilde{G}_{\mathbb{Z}(p)}$ (as a schematic closure) we get that $L_{(p)}$ is a $\tilde{G}_{\mathbb{Z}(p)}$ -module. From this and from the rank part of the property (ii), we get that the condition (i) of Lemma 8 holds. Thus the $k(v)$ -scheme $\mathcal{N}_{k(v)}^s$ is non-empty (cf. Lemma 8 (a)). Based on Theorem 1 (a) and the fact that $\tilde{H} = H$, we get that as $\tilde{\mathcal{N}}^s$ we can take \mathcal{N}^s itself. \square

5 Proof of the Main Theorem 2

In this section we take k to be a field extension of $k(v)$ that is algebraically closed and has a countable transcendental degree. Let $(v_\alpha)_{\alpha \in \mathcal{J}}$ and $(w_\alpha^A)_{\alpha \in \mathcal{J}}$ be as in Subsection 2.3. For a point $z \in \mathcal{N}^s(W(k)) = \mathcal{N}(W(k))$, the notation $(A, (w_\alpha)_{\alpha \in \mathcal{J}}, \lambda_A)$, $(M, F^1, \phi, (t_\alpha)_{\alpha \in \mathcal{J}}, \psi_M)$, $M = F^1 \oplus F^0$, and $\mu : \mathbb{G}_{m, W(k)} \rightarrow \mathcal{G}$ is as in Subsection 3.1. Subsections 5.1 to 5.6 prove the Main Theorem 2.

Let $R_0 := W(k)[[x]]$, where x is an independent variable. Let Φ_{R_0} be the Frobenius lift of R_0 that is compatible with σ and that takes x to x^p .

5.1 Basic notation and facts

We begin the proof of the Main Theorem 2 by introducing notation and some basic facts. We have $e(v) = 1$ and $G_{\mathbb{Z}(p)}$ is a quasi-reductive group scheme for (G, \mathcal{X}, v) . We recall that \mathcal{N}^m is an open subscheme of \mathcal{N}^s (cf. Subsection 3.5) and therefore also of \mathcal{N} (cf. Lemma 1). Thus $\mathcal{N}_{k(v)}^m$ is also an open subscheme of $\mathcal{N}_{k(v)}$. Moreover, the open embedding $\mathcal{N}^m \hookrightarrow \mathcal{N}$ is a pro-finite pro-étale cover of an open embedding between quasi-projective $O(v)$ -schemes (cf. Propositions 2 (a) and 3 (a)) and the $k(v)$ -scheme $\mathcal{N}_{k(v)}^m$ is non-empty (cf. Lemma 8 (a)). Thus to show that $\mathcal{N}_{k(v)}^m$ is a non-empty, open closed subscheme of $\mathcal{N}_{k(v)}$, we only have to show that for each commutative diagram of the following type

$$\begin{array}{ccccc} \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(k[[x]]) & \longleftarrow & \mathrm{Spec}(k((x))) \\ \downarrow y & & \downarrow \tau & & \downarrow \tau_{k((x))} \\ \mathcal{N} & \longleftarrow & \mathcal{N}_{k(v)} & \longleftarrow & \mathcal{N}_{k(v)}^m, \end{array}$$

the morphism $y : \mathrm{Spec}(k) \rightarrow \mathcal{N}$ factors through the open subscheme \mathcal{N}^m of \mathcal{N} . All the horizontal arrows of this diagram are natural (closed or open) embeddings. Until Subsection 5.4 inclusive we study properties of this diagram that are needed to prove Theorem 2 in Subsections 5.4 to 5.6.

We consider the principally quasi-polarized F -crystal

$$(M_0, \Phi_0, \nabla_0, \psi_{M_0})$$

over $k[[x]]$ of $\tau^*((\mathcal{A}, \lambda_{\mathcal{A}}) \times_{\mathcal{N}} \mathcal{N}_{k(v)})$. Thus M_0 is a free R_0 -module of rank $2r$, Φ_0 is a Φ_{R_0} -linear endomorphism of M_0 , ∇_0 is an integrable and nilpotent modulo p connection on M_0 such that we have $\nabla_0 \circ \Phi_0 = (\Phi_0 \otimes d\Phi_{R_0}) \circ \nabla_0$,

and ψ_{M_0} is a perfect, alternating bilinear form on M_0 that defines a principal quasi-polarization of (M_0, Φ_0, ∇_0) .

Let \mathcal{O} be the unique local ring of R_0 that is a discrete valuation ring of mixed characteristic $(0, p)$. Let $\hat{\mathcal{O}}$ be the completion of \mathcal{O} . Let $\Phi_{\hat{\mathcal{O}}}$ be the Frobenius lift of $\hat{\mathcal{O}}$ defined by Φ_{R_0} via a natural localization and completion. Let $k_1 := \overline{k((x))}$. Let $\text{Spec}(W(k_1)) \rightarrow \text{Spec}(R_0)$ be the lift that is compatible with the Frobenius lifts σ_{k_1} and Φ_{R_0} ; under it $W(k_1)$ gets naturally the structure of a $*$ -algebra, where $*$ $\in \{R_0, \mathcal{O}, \hat{\mathcal{O}}\}$.

As the $\mathcal{O}_{(v)}$ -scheme \mathcal{N}^m is formally smooth, there exists a lift $\tilde{z}_1 : \text{Spec}(\hat{\mathcal{O}}) \rightarrow \mathcal{N}^m$ of the morphism $\tau_{k((x))} : \text{Spec}(k((x))) \rightarrow \mathcal{N}^m$ defined naturally by $\tau_{k((x))}$ and denoted in the same way. Let $z_1 : \text{Spec}(W(k_1)) \rightarrow \mathcal{N}^m$ be the composite of $\text{Spec}(W(k_1)) \rightarrow \text{Spec}(\hat{\mathcal{O}})$ with \tilde{z}_1 ; we also view \tilde{z}_1 and z_1 as valued points of either \mathcal{N}^s or \mathcal{N} . Let

$$(\tilde{A}_1, (w_{1,\alpha})_{\alpha \in \mathcal{J}}, \lambda_{\tilde{A}_1}) := \tilde{z}_1^*(\mathcal{A}, (w_\alpha^A)_{\alpha \in \mathcal{J}}, \lambda_{\mathcal{A}}) \text{ and } (A_1, \lambda_{A_1}) := z_1^*(\mathcal{A}, \lambda_{\mathcal{A}}) = (\tilde{A}_1, \lambda_{\tilde{A}_1})_{W(k_1)}.$$

For $\alpha \in \mathcal{J}$ let $n(\alpha) \in \mathbb{N}$ be such that we have $v_\alpha \in W^{\vee \otimes n(\alpha)} \otimes_{\mathbb{Q}} W^{\otimes n(\alpha)} \subset \mathcal{T}(W^\vee)$, cf. definition of v_α in Subsection 2.3. Let $t_{1,\alpha}$ be the de Rham realization of $w_{1,\alpha}$. We identify canonically $M_0 \otimes_{R_0} \hat{\mathcal{O}} = H_{\text{dR}}^1(\tilde{A}_1/\hat{\mathcal{O}})$ (cf. [2, Ch. V, Subsect. 2.3]) and thus we view each $t_{1,\alpha}$ as a tensor of $(M_0^{\otimes n(\alpha)} \otimes_{R_0} M_0^{\vee \otimes n(\alpha)}) \otimes_{R_0} \hat{\mathcal{O}}[\frac{1}{p}] \subset \mathcal{T}(M_0 \otimes_{R_0} \hat{\mathcal{O}})[\frac{1}{p}]$. Let $n_\alpha \in \mathbb{N}$ be the smallest number such that we have $p^{n_\alpha} t_{1,\alpha} \in (M_0^{\otimes n(\alpha)} \otimes_{R_0} M_0^{\vee \otimes n(\alpha)}) \otimes_{R_0} \hat{\mathcal{O}} \subset \mathcal{T}(M_0 \otimes_{R_0} \hat{\mathcal{O}})$.

Proposition 4. *For all $\alpha \in \mathcal{J}$ we have $p^{n_\alpha} t_{1,\alpha} \in M_0^{\otimes n(\alpha)} \otimes_{R_0} M_0^{\vee \otimes n(\alpha)} \subset \mathcal{T}(M_0)$.*

Proof: The tensor $p^{n_\alpha} t_{1,\alpha}$ is fixed by the σ_{k_1} -linear automorphism of $\mathcal{T}(M_0 \otimes_{R_0} B(k_1))$ defined by Φ_0 , cf. Subsection 3.1. Thus (as $\text{Spec}(W(k_1)) \rightarrow \text{Spec}(R_0)$ is a Teichmüller lift) $p^{n_\alpha} t_{1,\alpha}$ is also fixed by the $\Phi_{\hat{\mathcal{O}}}$ -linear endomorphism of $\mathcal{T}(M_0 \otimes_{R_0} \hat{\mathcal{O}})[\frac{1}{p}]$ defined by Φ_0 .

The field $k((x))$ has $\{x\}$ as a p -basis, i.e., $\{1, x, \dots, x^{p-1}\}$ is a basis of $k((x))$ over $k((x))^p = k((x^p))$. Thus the p -adic completion of the \mathcal{O} -module $\Omega_{\mathcal{O}/W(k)}$ of relative differentials is naturally isomorphic to $\mathcal{O}dx$, cf. [3, Prop. 1.3.1]. Let $\nabla_0 : M_0 \otimes_{R_0} \hat{\mathcal{O}} \rightarrow M_0 \otimes_{R_0} \hat{\mathcal{O}}dx$ be the connection which is the natural extension of the connection ∇_0 on M_0 .

The de Rham component of w_α^A is annihilated by the Gauss–Manin connection of \mathcal{A} (this is a property of Hodge cycles; for instance, it follows from [14, Prop. 2.5] applied in the context of a quotient of $\text{Sh}_H(G, \mathcal{X})$ by a small

compact, open subgroup of $G(\mathbb{A}_f^{(p)})$). Thus the tensor $p^{n_\alpha} t_{1,\alpha}$ is annihilated by the Gauss–Manin connection on $\mathcal{T}(H_{\text{dR}}^1(\tilde{A}_1/\mathcal{O})) = \mathcal{T}(M_0 \otimes_{R_0} \mathcal{O})$ of \tilde{A}_1 and thus also by the p -adic completion of this connection. Therefore $p^{n_\alpha} t_{1,\alpha}$ is annihilated by the connection $\nabla_0 : M_0 \otimes_{R_0} \mathcal{O} \rightarrow M_0 \otimes_{R_0} \mathcal{O} dx$, cf. [2, Ch. V, Prop. 3.6.4].

As the field $k((x))$ has a p -basis, each F -crystal over $k((x))$ is uniquely determined by its evaluation at the thickening naturally associated to the closed embedding $\text{Spec}(k((x))) \hookrightarrow \text{Spec}(\mathcal{O})$ (cf. [3, Prop. 1.3.3]). Thus the natural identification

$$(M_0^{\otimes n(\alpha)} \otimes_{R_0} M_0^{\vee \otimes n_\alpha}) \otimes_{R_0} \mathcal{O} = \text{End}(M_0^{\otimes n(\alpha)} \otimes_{R_0} \mathcal{O})$$

allows us to view $p^{n_\alpha} t_{1,\alpha}$ as an endomorphism of the F -crystal over $k((x))$ defined by the tensor product of $n(\alpha)$ -copies of $(M_0 \otimes_{R_0} \mathcal{O}, \Phi_0 \otimes \Phi_{\mathcal{O}}, \nabla_0)$. From this and Theorem 3 we get that $p^{n_\alpha} t_{1,\alpha}$ can be viewed as an endomorphism of the F -crystal over $k[[x]]$ defined by the tensor product of $n(\alpha)$ -copies of (M_0, Φ_0, ∇_0) and therefore in fact we have $p^{n_\alpha} t_{1,\alpha} \in M_0^{\otimes n(\alpha)} \otimes_{R_0} M_0^{\vee \otimes n(\alpha)} \subset \mathcal{T}(M_0)$. \square

5.1.1 Group schemes

Next we introduce notation that pertains to group schemes. Let $G_{\mathbb{Z}_p}^r$ be a reductive, normal, closed subgroup scheme of $G_{\mathbb{Z}_p}$ as in Definition 2 (a); we emphasize that in general $G_{\mathbb{Z}_p}^r$ is not the pullback to $\text{Spec}(\mathbb{Z}_p)$ of a closed subgroup scheme of $G_{\mathbb{Z}(p)}$. Let $\pi_r \in \text{End}(M_0 \otimes_{R_0} B(k_1))$ be the projector that corresponds to the projector $\pi_{G_{\mathbb{Q}_p}^r}$ of Subsection 2.3 via Fontaine comparison theory for (the Barsotti–Tate group of) $\tilde{A}_{1,W(k_1)}$, cf. Subsection B3. As $\pi_{G_{\mathbb{Q}_p}^r}$ is fixed by $G_{\mathbb{Q}_p}$, by enlarging the family $(v_\alpha)_{\alpha \in \mathcal{J}}$, we can assume that $\pi_{G_{\mathbb{Q}_p}^r}$ is a \mathbb{Q}_p -linear combinations of the v_α 's with $\alpha \in \mathcal{J}$. Thus π_r is a \mathbb{Q}_p -linear combination of the $t_{1,\alpha}$'s with $\alpha \in \mathcal{J}$. From this and Proposition 4 we get that in fact we have $\pi_r \in \text{End}(M_0[\frac{1}{p}])$. Thus there exists $n_r \in \mathbb{N}$ such that $p^{n_r} \pi_r \in \text{End}(M_0)$.

Let η be the field of fractions of R_0 (or of \mathcal{O}). Let $\mathcal{G}_{0,\eta}$ be the subgroup of $(\mathbf{GL}_{M_0})_\eta$ that fixes $p^{n_\alpha} t_{1,\alpha}$ for all $\alpha \in \mathcal{J}$ (this definition makes sense due to Proposition 4). The group $\mathcal{G}_{0,B(k_1)}$ corresponds to $G_{\mathbb{Q}_p}$ via Fontaine comparison theory for (the Barsotti–Tate group of) $\tilde{A}_{1,B(k_1)}$. This implies that $\mathcal{G}_{0,\eta}$ is a reductive group.

Lemma 10. *There exists a unique reductive subgroup $\mathcal{G}_{0,\eta}^r$ of $\mathcal{G}_{0,\eta}$ whose Lie algebra is $\text{Im}(\pi_r) \otimes_{R_0[\frac{1}{p}]} \eta$. The subgroup $\mathcal{G}_{0,\eta}^r$ of $\mathcal{G}_{0,\eta}$ is normal. Moreover each geometric pullback of $\mathcal{G}_{0,\eta}^{r,\text{der}}$ has no normal subgroup which is an \mathbf{SO}_{2n+1} group for some $n \in \mathbb{N}^*$.*

Proof: From Fontaine comparison theory for (the Barsotti–Tate group of) $\tilde{A}_{1,W(k_1)}$ we get that there exists a unique reductive subgroup $\mathcal{G}_{0,B(k_1)}^r$ of $\mathbf{GL}_{M_0 \otimes_{R_0} B(k_1)}$ whose Lie algebra is $\text{Im}(\pi_r) \otimes_{R_0[\frac{1}{p}]} B(k_1)$, cf. Lemma 15 (a). From Lemma 11 (a) applied with $(\mathcal{W}, \mathcal{L}, \eta, \eta_1) = (M_0 \otimes_{R_0} \eta, \text{Im}(\pi_r) \otimes_{R_0[\frac{1}{p}]} \eta, \eta, B(k_1))$, we get that there exists a unique reductive subgroup $\mathcal{G}_{0,\eta}^r$ of $\mathbf{GL}_{M_0 \otimes_{R_0} \eta}$ whose Lie algebra is $\text{Im}(\pi_r) \otimes_{R_0[\frac{1}{p}]} \eta$. The group $\mathcal{G}_{0,\eta}^r$ is a subgroup of $\mathcal{G}_{0,\eta}$, as this holds after extension to $B(k_1)$. Thus the first part of the lemma holds.

But π_r is fixed by $\mathcal{G}_{0,\eta}$ (as this holds after tensorization with $B(k_1)$, cf. Subsection B3) and thus $\text{Im}(\pi_r) \otimes_{R_0[\frac{1}{p}]} \eta$ is a $\mathcal{G}_{0,\eta}$ -submodule of $\text{Lie}(\mathcal{G}_{0,\eta})$. From this and the uniqueness part of the lemma, we get that $\mathcal{G}_{0,\eta}^r$ is a subgroup of $\mathcal{G}_{0,\eta}$ normalized by $\mathcal{G}_{0,\eta}(\eta)$ and thus also by $\mathcal{G}_{0,\eta}$. As $\mathcal{G}_{0,B(k_1)}^r$ corresponds to the normal subgroup $G_{\mathbb{Q}_p}^r$ of $G_{\mathbb{Q}_p}$ via Fontaine comparison theory for (the Barsotti–Tate group of) $\tilde{A}_{1,W(k_1)}$, from Fact 2 we get that each geometric pullback of $\mathcal{G}_{0,\eta}^{r,\text{der}}$ has no normal subgroup which is an \mathbf{SO}_{2n+1} group for some $n \in \mathbb{N}^*$. \square

Theorem 8. *The schematic closure \mathcal{G}_0^r of $\mathcal{G}_{0,\eta}^r$ in \mathbf{GL}_{M_0} is a reductive subgroup scheme over $\text{Spec}(R_0)$.*

Proof: We check that if \mathcal{V} is a local ring of R_0 which is a discrete valuation ring, then $\mathcal{G}_{0,\mathcal{V}}^r$ is a reductive group scheme over \mathcal{V} .

We first assume that $\mathcal{V} = O$. As we know that $\tilde{z}_1 \in \mathcal{N}^{\text{m}}(\mathcal{O})$, there exist isomorphisms $(M_0 \otimes_{R_0} W(k_1), (t_{1,\alpha})_{\alpha \in \mathcal{J}}) \rightarrow (L_{(p)}^\vee \otimes_{\mathbb{Z}(p)} W(k_1), (v_\alpha)_{\alpha \in \mathcal{J}})$. Therefore the schematic closure of $\mathcal{G}_{0,B(k_1)}^r$ in $\mathbf{GL}_{M_0 \otimes_{R_0} W(k_1)}$ is isomorphic to $G_{W(k_1)}^r$ and thus it is a reductive group scheme over $W(k_1)$. As the natural morphism $\text{Spec}(W(k_1)) \rightarrow \text{Spec}(\mathcal{V}) = \text{Spec}(O)$ is faithfully flat, this schematic closure is $\mathcal{G}_{0,\mathcal{V}}^r \times_{\text{Spec}(\mathcal{V})} \text{Spec}(W(k_1))$. Thus $\mathcal{G}_{0,\mathcal{V}}^r$ is a reductive group scheme over \mathcal{V} .

We now assume that $\mathcal{V} \neq O$, i.e., \mathcal{V} is of equal characteristic 0. Thus $\mathcal{G}_{0,\mathcal{V}}^r$ is a smooth, closed subgroup scheme of $\mathbf{GL}_{M_0 \otimes_{R_0} \mathcal{V}}$, cf. Cartier theorem. Its Lie algebra $\mathfrak{g}_{\mathcal{V}}$ is $(\text{Im}(p^{nr} \pi_r) \otimes_{R_0} \eta) \cap \text{End}(M_0 \otimes_{R_0} \mathcal{V}) = \text{Im}(\pi_r) \otimes_{R_0[\frac{1}{p}]} \mathcal{V}$

and thus the restriction of the trace bilinear form on $\text{End}(M_0 \otimes_{R_0} \mathcal{V})$ to $\mathfrak{g}_{\mathcal{V}}$ is perfect. From this and Lemma 11 (b) we get that the identity component of the special fibre of $\mathcal{G}_{0,\mathcal{V}}^r$ is a reductive group. Let $\mathcal{G}_{0,\mathcal{V}}^{r,\text{id}}$ be the open subgroup scheme of $\mathcal{G}_{0,\mathcal{V}}^r$ whose special fibre is the identity component of the special fibre of $\mathcal{G}_{0,\mathcal{V}}^r$. As $\mathcal{G}_{0,\mathcal{V}}^{r,\text{id}}$ is the complement in $\mathcal{G}_{0,\mathcal{V}}^r$ of a divisor of $\mathcal{G}_{0,\mathcal{V}}^r$, it is an affine $\mathcal{G}_{0,\mathcal{V}}^r$ -scheme and thus it is an affine scheme. Therefore $\mathcal{G}_{0,\mathcal{V}}^{r,\text{id}}$ is a reductive group scheme. Based on this and the second part of Lemma 10, from [56, Thm. 1.1 (d)] we get that the homomorphism $\mathcal{G}_{0,\mathcal{V}}^{r,\text{id}} \rightarrow \mathbf{GL}_{M_0 \otimes_{R_0} \mathcal{V}}$ is a closed embedding. Thus $\mathcal{G}_{0,\mathcal{V}}^{r,\text{id}} \rightarrow \mathcal{G}_{0,\mathcal{V}}^r$ is a closed embedding. Being also an open embedding, we conclude that $\mathcal{G}_{0,\mathcal{V}}^{r,\text{id}} = \mathcal{G}_{0,\mathcal{V}}^r$ is a reductive, closed subgroup scheme of $\mathbf{GL}_{M_0 \otimes_{R_0} \mathcal{V}}$.

Let $\mathcal{U} := \text{Spec}(R_0) \setminus \text{Spec}(k)$. As $\mathcal{G}_{0,\mathcal{U}}^r$ is a reductive, closed group scheme of $\mathbf{GL}_{M_0,\mathcal{U}}$ (cf. last two paragraphs), it extends uniquely to a reductive group scheme $\tilde{\mathcal{G}}_0^r$ over R_0 (cf. [66, Thm. 1.4 (b)]). The closed embedding homomorphism $\mathcal{G}_{0,\mathcal{U}}^r \rightarrow \mathbf{GL}_{M_0,\mathcal{U}}$ extends to a closed embedding homomorphism $\tilde{\mathcal{G}}_0^r \rightarrow \mathbf{GL}_{M_0}$, cf. [66, Prop. 5.1 (c)] and for $p = 2$ cf. also the last property of Lemma 10. Thus $\mathcal{G}_0^r = \tilde{\mathcal{G}}_0^r$ is a reductive, closed subgroup scheme of \mathbf{GL}_{M_0} . \square

5.2 Applying Theorem 8

Let $(M_1, F_1^1, \phi_1, \psi_{M_1})$ be the principally quasi-polarized filtered F -crystal over k_1 of (A_1, λ_{A_1}) . We have $M_1 = M_0 \otimes_{R_0} W(k_1)$, $\phi_1 = \Phi_0 \otimes \sigma_{k_1}$, and each $t_{1,\alpha} \in \mathcal{T}(M_1)_{[p]}^1$ with $\alpha \in \mathcal{J}$ is the de Rham realization of the Hodge cycle $z_1^*(w_\alpha^A)$ on $A_{1,B(k)}$. Let $\mu_1 : \mathbb{G}_{m,W(k_1)} \rightarrow \mathcal{G}_1 = \mathcal{G}_{0,W(k_1)}$ be the analogue of $\mu : \mathbb{G}_{m,W(k)} \rightarrow \mathcal{G}$ but obtained working with $z_1 \in \mathcal{N}(W(k_1))$ instead of some $z \in \mathcal{N}(W(k))$. We know that μ_1 factors through $\mathcal{G}_{0,W(k_1)}^r$, cf. Lemma 6 applied to $z_1 \in \mathcal{N}(W(k_1))$ with $G_{\mathbb{Q}_p}^v = G_{\mathbb{Q}_p}^r$.

Let \bar{F}_0^1 be the kernel of Φ_0 modulo p ; it is a free module over $k[[x]] = R_0/pR_0$ of rank r . As the cocharacter μ_1 factors through $\mathcal{G}_{0,W(k_1)}^r$, the normalizer of $\bar{F}_0^1 \otimes_{k[[x]]} k_1$ in \mathcal{G}_{0,k_1}^r is a parabolic subgroup of the reductive group \mathcal{G}_{0,k_1}^r which (as $\bar{F}_0^1 \otimes_{k[[x]]} k_1$ is defined over $k((x))$) is the pullback of a parabolic subgroup $\mathcal{F}_{0,k((x))}^r$ of $\mathcal{G}_{0,k((x))}^r$. The $k[[x]]$ -scheme of parabolic subgroup schemes of the reductive group scheme $\mathcal{G}_{0,k[[x]]}^r$ is projective, cf. [15, Vol. III, Exp. XXVI, Cor. 3.5]. Thus the schematic closure $\mathcal{F}_{0,k[[x]]}^r$ of $\mathcal{F}_{0,k((x))}^r$ in $\mathcal{G}_{0,k[[x]]}^r$ is a parabolic subgroup scheme of $\mathcal{G}_{0,k[[x]]}^r$.

As \mathcal{G}_0^r is a split reductive group scheme and μ_{1,k_1} factors through \mathcal{G}_{0,k_1}^r , there exists a cocharacter $\mu_{0,k[[x]]} : \mathbb{G}_{m,k[[x]]} \rightarrow \mathcal{G}_{0,k[[x]]}^r$ that factors through $\mathcal{F}_{0,k[[x]]}^r$ and that produces a direct sum decomposition $M_0/pM_0 = \bar{F}_0^1 \oplus \bar{F}_0^0$ such that for each $i \in \{0, 1\}$, every $\beta \in \mathbb{G}_{m,k[[x]]}(k[[x]])$ acts via $\mu_{0,k[[x]]}$ on \bar{F}_0^i as the multiplication by β^{-i} . We consider a cocharacter

$$\mu_0 : \mathbb{G}_{m,R_0} \rightarrow \mathcal{G}_0^r$$

that lifts $\mu_{0,k[[x]]}$, cf. [15, Vol. II, Exp. IX, Thms. 3.6 and 7.1]. Let $M_0 = F_0^1 \oplus F_0^0$ be the direct sum decomposition such that for each $i \in \{0, 1\}$, every element $\beta \in \mathbb{G}_{m,R_0}(R_0)$ acts via μ_0 on F_0^i as the multiplication by β^{-i} ; the notation matches, i.e., we have $F_0^i/pF_0^i = \bar{F}_0^i$.

We consider the $W(k)$ -epimorphism $R_0 \twoheadrightarrow W(k)$ whose kernel is the ideal (x) . Let

$$(M, F^1, \phi, \mathcal{G}, \mathcal{G}^r, (t_\alpha)_{\alpha \in \mathcal{J}}, \psi_M) := (M_0, F_0^1, \Phi_0, \mathcal{G}_0, \mathcal{G}_0^r, (t_{1,\alpha})_{\alpha \in \mathcal{J}}, \psi_{M_0}) \otimes_{R_0} W(k).$$

5.3 Extra crystalline applications

Let (D, λ_D) be an arbitrary principally quasi-polarized Barsotti–Tate group over $W(k)$ whose principally quasi-polarized filtered F -crystal over k is (M, F^1, ϕ) and for which we have an isomorphism

$$(6) \quad (M, (t_\alpha)_{\alpha \in \mathcal{J}}, \psi_M) \rightarrow (H^1(D) \otimes_{\mathbb{Z}_p} W(k), (u_\alpha)_{\alpha \in \mathcal{J}}, \psi_{H^1(D)}),$$

where $\psi_{H^1(D)}$ is the perfect, alternating bilinear form on $H^1(D)$ which is the étale realization of λ_D and where $u_\alpha \in \mathcal{T}(H^1(D))[\frac{1}{p}]$ corresponds to t_α via Fontaine comparison theory for D . If $p = 2$, then the existence of (D, λ_D) is implied by Theorem 11 (b) applied to $(M, \phi, \mathcal{G}^r, \psi_M)$ instead of $(M, \phi, \mathcal{G}, \psi_M)$. If $p > 2$ or if $p = 2$ and (M, ϕ) has no integral slopes, then there exists a unique Barsotti–Tate group D over $W(k)$ whose filtered F -crystal over k is (M, F^1, ϕ) (cf. [64, Prop. 2.2.6] for $p = 2$); due to the uniqueness part, ψ_M is the crystalline realization of a (unique) principal quasi-polarization λ_D of D . The fact that (6) holds in this case follows from Theorem 9.

Let $(D_{R_0}, \lambda_{D_{R_0}})$ be the principally quasi-polarized Barsotti–Tate group over R_0 which modulo the ideal (x) is (D, λ_D) and whose principally quasi-polarized F -crystal over R_0/pR_0 is the quintuple $(M_0, F_0^1, \Phi_0, \nabla_0, \psi_{M_0})$, cf. Lemmas 16 and 17. Let

$$\tau_{R_0} : \text{Spec}(R_0) \rightarrow \mathcal{M}_r$$

be the morphism that has the following two properties:

- (i) it lifts the composite of $y : \text{Spec}(k) \rightarrow \mathcal{N}$ with the morphism $\mathcal{N} \rightarrow \mathcal{M}_r$, and
- (ii) the principally quasi-polarized Barsotti–Tate group of the pullback via τ_{R_0} of the universal principally polarized abelian scheme over \mathcal{M}_r , is $(D_{R_0}, \lambda_{D_{R_0}})$.

Let

$$z_2 : \text{Spec}(W(k_1)) \rightarrow \mathcal{M}_r$$

be the composite of the morphism $\text{Spec}(W(k_1)) \rightarrow \text{Spec}(R_0)$ of Subsection 5.1 with τ_{R_0} . Let (A_2, λ_{A_2}) be the principally polarized abelian scheme over $W(k_1)$ that is the pullback through z_2 of the universal principally polarized abelian scheme over \mathcal{M}_r and let (D_2, λ_{D_2}) be its principally quasi-polarized Barsotti–Tate group. The principally quasi-polarized filtered F -crystal of (D_2, λ_{D_2}) is canonically identified with $(M_1, F_2^1, \phi_1, \psi_{M_1})$, where F_2^1 is a direct summand of M_1 of rank r . Let $(F_2^i(\mathcal{T}(M_1)))_{i \in \mathbb{Z}}$ be the filtration of $\mathcal{T}(M_1)$ defined by F_2^1 and let $(F_0^i(\mathcal{T}(M_0)))_{i \in \mathbb{Z}}$ be the filtration of $\mathcal{T}(M_0)$ defined by F_0^1 . For each $\alpha \in \mathcal{J}$, the tensor $t_{1,\alpha} \in \mathcal{T}(M_0)[\frac{1}{p}]$ is annihilated by ∇_0 , is fixed by Φ_0 , and belongs to $F_0^0(\mathcal{T}(M_0))[\frac{1}{p}]$. This implies that we have $t_{1,\alpha} \in F_2^0(\mathcal{T}(M_1))[\frac{1}{p}]$ for all $\alpha \in \mathcal{J}$. Thus as before Lemma 5 we argue that the inverse of the canonical split cocharacter of (M_1, F_2^1, ϕ_1) defined in [69, p. 512] factors through the closed subgroup scheme $\mathcal{G}_1 = \mathcal{G}_{0,W(k_1)}$ of \mathbf{GL}_{M_1} ; let $\mu_2 : \mathbb{G}_{m,W(k_1)} \rightarrow \mathcal{G}_1$ be the resulting factorization.

Due to Lemma 7 (a) applied to $z_1 \in \mathcal{N}^m(W(k_1)) \subset \mathcal{N}^s(W(k_1))$ and to $\mu_2 : \mathbb{G}_{m,W(k_1)} \rightarrow \mathcal{G}_1$, there exists a point $z_3 \in \mathcal{N}^m(W(k_1)) \subset \mathcal{N}^s(W(k_1)) = \mathcal{N}(W(k_1))$ that lifts the k_1 -valued point y_1 of \mathcal{N}^m defined naturally by z_1 (or z_2) and such that the filtered F -crystal of $(A_3, \lambda_{A_3}) := z_3^*(\mathcal{A}, \lambda_{\mathcal{A}})$ is precisely $(M_1, F_2^1, \phi_1, \psi_{M_1})$. Let (D_3, λ_{D_3}) be the principally quasi-polarized Barsotti–Tate group of (A_3, λ_{A_3}) .

5.4 Proof of Theorem 2 (a), part I

In this subsection we assume that either $p > 2$ or $p = 2$ and the 2-rank of $y_1^*(\mathcal{A}) = A_{1,k_1} = \tilde{A}_{1,k_1}$ is 0. Due to this assumption, the Barsotti–Tate groups D_2 and D_3 are the same lift of the Barsotti–Tate group of $y_1^*(\mathcal{A})$ (cf. [64, Prop. 2.2.6] for $p = 2$). Therefore the $W(k_1)$ -valued points of \mathcal{M}_r defined by z_2 and z_3 coincide. Thus z_2 factors through \mathcal{N}^s . From this and Theorem 1 (b) we get that τ_{R_0} factors through \mathcal{N}^s . Let

$z : \text{Spec}(W(k)) \rightarrow \mathcal{N}^s$ be the composite of the factorization $\text{Spec}(R_0) \rightarrow \mathcal{N}^s$ of τ_{R_0} with the closed embedding $\text{Spec}(W(k)) \hookrightarrow \text{Spec}(R_0)$ defined by the $W(k)$ -epimorphism $R_0 \twoheadrightarrow R_0/(x) = W(k)$; it lifts y . The notation matches with the one of Subsection 3.1: (D, λ_D) is the principally quasi-polarized Barsotti–Tate group of $(A, \lambda_A) := z^*(\mathcal{A}, \lambda_{\mathcal{A}})$, the principally quasi-polarized filtered F -crystal of (D, λ_D) is (M, F^1, ϕ, ψ_M) , and u_α corresponds to t_α via Fontaine comparison theory for D .

There exists an isomorphism $(M \otimes_{W(k)} W(k_1), (t_\alpha)_{\alpha \in \mathcal{J}}, \psi_M) \rightarrow (M_1, (t_{1,\alpha})_{\alpha \in \mathcal{J}}, \psi_{M_1})$, cf. proof of Proposition 3 (b). Thus, as Theorem 4 (a) holds for $z_1 \in \mathcal{N}^m(W(k_1))$, there exist isomorphisms $(M \otimes_{W(k)} W(k_1), (t_\alpha)_{\alpha \in \mathcal{J}}, \psi_M) \rightarrow (L_{(p)}^\vee \otimes_{\mathbb{Z}(p)} W(k_1), (v_\alpha)_{\alpha \in \mathcal{J}}, \psi^\vee)$ and therefore (using an argument similar to the one used to prove Lemma 14 we get that) there exist isomorphisms $(M, (t_\alpha)_{\alpha \in \mathcal{J}}, \psi_M) \rightarrow (L_{(p)}^\vee \otimes_{\mathbb{Z}(p)} W(k), (v_\alpha)_{\alpha \in \mathcal{J}}, \psi^\vee)$. From this and Lemma 3 (a) we get that Theorem 4 (a) holds for $z \in \mathcal{N}^s(W(k))$. Thus we have $z \in \mathcal{N}^m(W(k))$ (cf. Proposition 3 (b)) and therefore the morphism $y : \text{Spec}(k) \rightarrow \mathcal{N}$ factors through \mathcal{N}^m . This ends the proof of Theorem 2 (a) provided either $p > 2$ or $p = 2$ and the 2-rank of A_{1,k_1} is 0. \square

5.5 Proof of Theorem 2 (a), part II

We prove Theorem 2 (a) for $p = 2$ in the general case. Let $a \in \mathbb{N}$ be the multiplicity of the Newton polygon slope -1 for $(\text{Lie}(\mathcal{G}_1)[\frac{1}{2}], \phi_1)$. Let $u_{2,\alpha} \in \mathcal{T}(H^1(D_2))[\frac{1}{2}] = \mathcal{T}(H^1(D_3))[\frac{1}{2}]$ correspond to $t_{1,\alpha}$ and let $\psi_{H^1(D_2)}$ correspond to ψ_{M_1} via Fontaine comparison theory for D_2 .

Let $Z_0 = \text{Spec}(R_0)$. From Theorem 10 we get that the sextuple

$$(M_0, F_0^1, \Phi_0, \nabla_0, (t_{1,\alpha})_{\alpha \in \mathcal{J}}, \psi_{M_0})$$

is the pullback of the sextuple $(M_R, F_R^1, \Phi, \nabla, (t_\alpha)_{\alpha \in \mathcal{J}}, \psi_{M_R})$ of Susubsubsection B4.1 via a morphism $i_{Z_0} : Z_0 \rightarrow \text{Spec}(R)$ of $\text{Spec}(W(k))$ -schemes which lifts the identity $R/(x_1, \dots, x_l) = W(k) = R_0/(x)$. This implies that there exists an isomorphism

$$(7) \quad (M \otimes_{W(k)} W(k_1), (t_\alpha)_{\alpha \in \mathcal{J}}, \psi_M) = (M_1, (t_{1,\alpha})_{\alpha \in \mathcal{J}}, \psi_{M_1}).$$

We have a canonical identification

$$(8) \quad (H^1(D_2), (u_{2,\alpha})_{\alpha \in \mathcal{J}}) = (H^1(D), (u_\alpha)_{\alpha \in \mathcal{J}}),$$

cf. [64, Subsubsect. 3.4.2] (to be compared also with either the fifth paragraph of the proof of Theorem 11 or [64, Lem. 3.4.3]). From (6), (7), and (8) we get that there exists an isomorphism

$$(9) \quad (M_1, (t_{1,\alpha})_{\alpha \in \mathcal{J}}, \psi_M) \rightarrow (H^1(D_2) \otimes_{\mathbb{Z}_2} W(k_1), (u_{2,\alpha})_{\alpha \in \mathcal{J}}, \epsilon_2 \psi_{H^1(D_2)}),$$

with ϵ_2 a unit of $W(k_1)$. As in the proof of Theorem 4 we argue that we can assume that $\epsilon_2 = 1$.

We have exactly 2^a possibilities for a lift $z_3 \in \mathcal{N}^m(W(k_1))$ of y_1 as in the end of Subsection 5.3 (cf. Lemma 7 (c)) for which Theorem 4 (a) holds and each such z_3 is uniquely determined by (D_3, λ_{D_3}) (cf. Theorem 1 (b)). Theorem 11 (c) proves that there are exactly 2^a possibilities for (D_2, λ_{D_2}) such as (9) holds with $\epsilon_2 = 1$ and its principally quasi-polarized filtered F -crystal over k is $(M_1, F_2^1, \phi_1, \psi_{M_1})$. From the last two sentences we get that we can choose z_3 such that we have $(D_3, \lambda_{D_3}) = (D_2, \lambda_{D_2})$ as lifts of the principally quasi-polarized 2-divisible group of $y_1^*(\mathcal{A}, \lambda_{\mathcal{A}})$. Therefore the $W(k_1)$ -valued points of \mathcal{M}_r defined by z_2 and z_3 coincide and thus, as in Subsection 5.4 we argue that τ_{R_0} factors through \mathcal{N}^s and that $y : \text{Spec}(k) \rightarrow \mathcal{N}$ factors through \mathcal{N}^m . Thus Theorem 2 (a) holds. \square

5.6 Proof of Theorem 2 (b) and (c)

Theorem 2 (b) follows from Theorem 2 (a) and Proposition 1. To prove Theorem 2 (c), let \mathcal{Q} and \mathcal{Q}^s be as in Subsection 2.2. As the \mathbb{Q} -rank of the adjoint group G^{ad} is 0, \mathcal{Q} is a projective $O_{(v)}$ -scheme (cf. Lemma 2). From Proposition 3 (a) we get that \mathcal{N}^m is the pullback of a smooth, open subscheme \mathcal{Q}^m of \mathcal{Q} . To prove Theorem 2 (c.i), it suffices to show that $\mathcal{Q}^m = \mathcal{Q}$, i.e., to show that if \mathcal{C} is a connected component of $\mathcal{Q}_{W(k)}$, then we have $\mathcal{C} \subset \mathcal{Q}^m$. As $\mathcal{C}_{B(k)} \subset \mathcal{C} \cap \mathcal{Q}_{W(k)}^m$, from Lemma 8 (b) we get that the intersection $\mathcal{C}_k \cap \mathcal{Q}_k^m$ is non-empty and thus (as \mathcal{Q}^m is smooth) there exist $W(k)$ -valued points of \mathcal{C} . Thus the ring of global functions of the connected, flat, normal, projective $W(k)$ -scheme \mathcal{C} is $W(k)$. From this and [24, Ch. III, Cor. 11.3] we get that the special fibre \mathcal{C}_k of \mathcal{C} is connected. But the non-empty scheme $\mathcal{C}_k \cap \mathcal{Q}_k^m$ is an open, closed subscheme of \mathcal{C}_k , cf. Theorem 2 (a). From the last two sentences we get that $\mathcal{Q}_k^m \cap \mathcal{C}_k = \mathcal{C}_k$. Thus $\mathcal{Q}_{W(k)}^m \cap \mathcal{C} = \mathcal{C}$. Therefore Theorem 2 (c.i) holds.

We know that $\mathcal{Q} = \mathcal{N}/H^{(p)} = \mathcal{N}^s/H^{(p)}$ is a normal, projective $O_{(v)}$ -scheme and that the quotient morphism $\mathcal{N}^s = \mathcal{N} \rightarrow \mathcal{Q}$ is a pro-étale cover,

cf. beginning of Subsection 2.2 and Proposition 2 (a). Thus the $O_{(v)}$ -scheme \mathcal{Q} is smooth, i.e., we have $\mathcal{Q} = \mathcal{Q}^s$. As \mathcal{Q} is a Néron model of its generic fibre $\mathrm{Sh}_{H \times H^{(v)}}(G, \mathcal{X})$ over $O_{(v)}$ (cf. Theorem 1 (c)), Theorem 2 (c.ii) holds. \square

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APPENDIX

A On affine group schemes

Let $p \in \mathbb{N}$ be a prime. Let k be an algebraically field of characteristic p . Let $W(k)$ be the ring of p -typical Witt vectors with coefficients in k and let $B(k) := W(k)_{[p]}^{\frac{1}{p}}$ be its field of fractions.

A.1 Universal smoothenings

Let \mathcal{G} be a flat, affine group scheme over $W(k)$. For $a \in \mathcal{G}(W(k))$, the Néron measure of the defect of smoothness $\delta(a) \in \mathbb{N}$ of \mathcal{G} at a is the length of the torsion part of the finitely generated $W(k)$ -module $a^*(\Omega_{\mathcal{G}/\mathrm{Spec}(W(k))})$. As \mathcal{G} is a group scheme over $W(k)$, the value of $\delta(a)$ does not depend on a and thus we denote it by $\delta(\mathcal{G})$. We have $\delta(\mathcal{G}) \in \mathbb{N}^*$ if and only if \mathcal{G} is not smooth, cf. [7, Ch. 3, Sect. 3.3, Lem. 1]. Let \mathcal{F}_k be the schematic closure in \mathcal{G}_k of all special fibres of $W(k)$ -valued points of \mathcal{G} ; it is a reduced subgroup of \mathcal{G}_k . We write $\mathcal{F}_k = \mathrm{Spec}(R_{\mathcal{G}}/J_{\mathcal{G}})$, where $\mathcal{G} = \mathrm{Spec}(R_{\mathcal{G}})$ and where $J_{\mathcal{G}}$ is the ideal of $R_{\mathcal{G}}$ that defines \mathcal{F}_k and contains p . By the *canonical dilatation* of \mathcal{G} we mean the affine \mathcal{G} -scheme $\mathcal{G}_1 = \mathrm{Spec}(R_{\mathcal{G}_1})$, where $R_{\mathcal{G}_1}$ is the $R_{\mathcal{G}}$ -subalgebra of $R_{\mathcal{G}}_{[p]}^{\frac{1}{p}}$ generated by $\frac{x}{p}$ with $x \in J_{\mathcal{G}}$.

The $W(k)$ -scheme \mathcal{G}_1 has a canonical group scheme structure and the morphism $\mathcal{G}_1 \rightarrow \mathcal{G}$ is a homomorphism of group schemes over $W(k)$, cf. [7, Ch. 3, Sect. 3.2, Prop. 2 (d)]. Moreover the $W(k)$ -morphism $\mathcal{G}_1 \rightarrow \mathcal{G}$ has the following universal property: each $W(k)$ -morphism $Z \rightarrow \mathcal{G}$ of flat $W(k)$ -schemes whose special fibre $Z_k \rightarrow \mathcal{G}_k$ factors through the closed embedding $\mathcal{F}_k \hookrightarrow \mathcal{G}_k$, factors uniquely through $\mathcal{G}_1 \rightarrow \mathcal{G}$ (cf. [7, Ch. 3, Sect. 3.2, Prop. 1 (b)]). If \mathcal{G} is smooth, then $\mathcal{F}_k = \mathcal{G}_k$ and therefore $\mathcal{G}_1 = \mathcal{G}$.

Either \mathcal{G}_1 is smooth or we have $0 < \delta(\mathcal{G}_1) < \delta(\mathcal{G})$, cf. [7, Ch. 3, Sect. 3.3, Prop. 5]. Thus by using a sequence of at most $\delta(\mathcal{G})$ canonical dilatations (the first one of \mathcal{G} , the second one of \mathcal{G}_1 , etc.), we get the existence of a unique smooth, affine group scheme \mathcal{G}' over $W(k)$ endowed with a homomorphism $\mathcal{G}' \rightarrow \mathcal{G}$ whose generic fibre over $B(k)$ is an isomorphism and which has the following universal property: each $W(k)$ -morphism $Z \rightarrow \mathcal{G}$, with Z a smooth $W(k)$ -scheme, factors uniquely through $\mathcal{G}' \rightarrow \mathcal{G}$. One calls \mathcal{G}' the *universal smoothening* of \mathcal{G} .

A.2 On Lie algebras

Lemma 11. *Let \mathcal{W} be a finite dimensional vector space over a field η of characteristic 0. Let \mathcal{L} be a Lie subalgebra of $\text{End}(\mathcal{W})$. We assume that there exists a field extension η_1 of η such that $\mathcal{L} \otimes_{\eta} \eta_1$ is the Lie algebra of a connected (resp. reductive) subgroup \mathcal{F}_{η_1} of $\mathbf{GL}_{\mathcal{W} \otimes_{\eta} \eta_1}$.*

(a) *Then there exists a unique connected (resp. reductive) subgroup \mathcal{F} of $\mathbf{GL}_{\mathcal{W}}$ whose Lie algebra is \mathcal{L} (the notation matches, i.e., the extension of \mathcal{F} to η_1 is \mathcal{F}_{η_1}).*

(b) *The restriction $\mathfrak{t} : \mathcal{L} \times \mathcal{L} \rightarrow \eta$ of the trace bilinear form on $\text{End}(\mathcal{W})$ to \mathcal{L} is non-degenerate if and only if \mathcal{F} is a reductive subgroup of $\mathbf{GL}_{\mathcal{W}}$.*

Proof: We prove part (a). The uniqueness part is implied by [5, Ch. I, Sect. 7.1]. Loc. cit. also implies that if \mathcal{F} exists, then its extension to η_1 is indeed \mathcal{F}_{η_1} . It suffices to prove part (a) for the case when \mathcal{F}_{η_1} is connected. We consider commutative η -algebras κ such that there exists a closed subgroup scheme \mathcal{F}_{κ} of $\mathbf{GL}_{\mathcal{W} \otimes_{\eta} \kappa}$ whose Lie algebra is $\mathcal{L} \otimes_{\eta} \kappa$. Our hypotheses imply that as κ we can take η_1 . Thus as κ we can also take a finitely generated η -subalgebra of η_1 . By considering the reduction modulo a maximal ideal of this last η -algebra, we can assume that κ is a finite field extension of η which is separable as η has characteristic 0. Thus we can assume that κ is

a finite Galois extension of η . By replacing \mathcal{F}_κ with its identity component, we can assume that \mathcal{F}_κ is connected. Due to the mentioned uniqueness part, the Galois group $\text{Gal}(\kappa/\eta)$ acts naturally on the connected subgroup \mathcal{F}_κ of $\mathbf{GL}_{\mathcal{W} \otimes_\eta \kappa}$. As \mathcal{F}_κ is an affine scheme, the resulting Galois descent datum on \mathcal{F}_κ with respect to $\text{Gal}(\kappa/\eta)$ is effective (cf. [7, Ch. 6, Sect. 6.1, Thm. 5]). This implies the existence of a subgroup \mathcal{F} of $\mathbf{GL}_{\mathcal{W}}$ whose extension to κ is \mathcal{F}_κ . As $\text{Lie}(\mathcal{F}) \otimes_\eta \kappa = \text{Lie}(\mathcal{F}_\kappa) = \mathcal{L} \otimes_\eta \kappa$, we have $\text{Lie}(\mathcal{F}) = \mathcal{L}$. The group \mathcal{F} is connected as \mathcal{F}_κ is so. Thus \mathcal{F} exists, i.e., part (a) holds.

Part (b) follows from [8, Ch. I, Sect. 6, Prop. 5 and Thm. 4]. For the sake of completeness we include here a short proof of part (b). We can assume that η is algebraically closed. We first prove the if part. Using isogenies, we are reduced to the case when \mathcal{F} is either $\mathbb{G}_{m,\eta}$ or a semisimple group whose adjoint is simple. If \mathcal{F} is $\mathbb{G}_{m,\eta}$, then the \mathcal{F} -module \mathcal{W} is a direct sum of one dimensional \mathcal{F} -modules. We easily get that there exists an element $x \in \mathcal{L} \setminus \{0\}$ which is a semisimple element of $\text{End}(\mathcal{W})$ whose eigenvalues are integers. The trace of x^2 is a sum of squares of natural numbers not all zero and thus it is non-zero. If \mathcal{F} is a semisimple group whose adjoint is simple, then \mathcal{L} is a simple Lie algebra over η . From Cartan solvability criterion we get that \mathfrak{t} is non-zero, i.e., the ideal $\text{Ker}(\mathfrak{t}) = \{x \in \mathcal{L} \mid \mathfrak{t}(x, y) = 0 \forall y \in \mathcal{L}\}$ of \mathcal{L} is not \mathcal{L} . As \mathcal{L} is a simple Lie algebra over η , we get that $\text{Ker}(\mathfrak{t}) = 0$, i.e., \mathfrak{t} is non-degenerate.

To prove the only if part, we consider the unipotent radical \mathcal{U} of \mathcal{F} . Let $0 = \mathcal{W}_0 \subsetneq \mathcal{W}_1 \subsetneq \dots \subsetneq \mathcal{W}_s = \mathcal{W}$ be a strictly increasing filtration of \mathcal{W} by \mathcal{F} -modules such that \mathcal{U} acts trivially on $\mathcal{W}_i/\mathcal{W}_{i-1}$ for all $i \in \{1, \dots, s\}$. Based on the existence of this filtration, it is easy to see that $\text{Lie}(\mathcal{U}) \subset \text{Ker}(\mathfrak{t}) = 0$ and thus $\text{Lie}(\mathcal{U}) = 0$. Therefore \mathcal{U} is the trivial subgroup, i.e., \mathcal{F} is reductive. Thus part (b) holds. \square

See [60, Prop. 3.2] for a different approach to prove Lemma 11 (a).

B Complements on Barsotti–Tate groups

Let p , k , $W(k)$, and $B(k)$ be as in Appendix A. Let $\sigma := \sigma_k$ be the Frobenius automorphism of k , $W(k)$, and $B(k)$. We fix an algebraic closure $\overline{B(k)}$ of $B(k)$. Let $\text{Gal}(B(k)) := \text{Gal}(\overline{B(k)}/B(k))$. Let D be a Barsotti–Tate group over $W(k)$. Let $\mathfrak{b}^\mathfrak{t}$ be the Cartier dual of a Barsotti–Tate group \mathfrak{b} over a $W(k)$ -algebra. Let (M, ϕ) be the F -crystal of D_k (i.e., the contravariant Dieudonné module of D_k with the Verschiebung map suppressed). Thus M

is a free $W(k)$ -module of rank equal to the height of D and $\phi : M \rightarrow M$ is a σ -linear endomorphism such that we have $pM \subset \phi(M)$. Let F^1 be the direct summand of M that is the Hodge filtration defined by D . We have $\phi(M + \frac{1}{p}F^1) = M$. The rank of F^1 is the dimension of D_k . Let $M^\vee := \text{Hom}(M, W(k))$. Let $\mathcal{T}(M)$ and its filtration $(F^i(\mathcal{T}(M)))_{i \in \mathbb{Z}}$ defined by F^1 , be as in Subsection 2.1. For $f \in M^\vee[\frac{1}{p}]$ let $\phi(f) := \sigma \circ f \circ \phi^{-1} \in M^\vee[\frac{1}{p}]$. Thus ϕ acts in the usual tensor product way on $\mathcal{T}(M[\frac{1}{p}])$.

If D has a principal quasi-polarization λ_D , let $\psi_M : M \times M \rightarrow W(k)$ be the perfect, alternating form defined by λ_D . For all $a, b \in M$ we have $\psi_M(\phi(a), \phi(b)) = p\sigma(\psi_M(a, b))$. Moreover, we have $\psi_M(F^1, F^1) = 0$.

B.1 Galois modules

Let $H^1(D) := T_p(D_{B(k)}^t)(-1)$ be the dual of the Tate-module $T_p(D_{B(k)})$ of $D_{B(k)}$. Thus $H^1(D)$ is a free \mathbb{Z}_p -module of the same rank as M on which $\text{Gal}(B(k))$ acts. If D has a principal quasi-polarization λ_D , let $\psi_{H^1(D)} : H^1(D) \times H^1(D) \rightarrow \mathbb{Z}_p$ be the perfect, alternating form defined by λ_D . Let $F^0(H^1(D)) := H^1(D)$ and $F^1(H^1(D)) := 0$. Let

$$\rho_D : \text{Gal}(B(k)) \rightarrow \mathbf{GL}_{H^1(D)}(\mathbb{Z}_p)$$

be the natural Galois representation associated to $D_{B(k)}$. Let $\mathcal{D}^{\acute{e}t}$ be the schematic closure in $\mathbf{GL}_{H^1(D)}$ of $\text{Im}(\rho_D)$; it is a flat, affine group scheme over \mathbb{Z}_p . From [69, Prop. 4.2.3] one gets that the generic fibre $\mathcal{D}_{\mathbb{Q}_p}^{\acute{e}t}$ is connected. See Subsection 2.1 for $\mathcal{T}(H^1(D))$; it is naturally a $\text{Gal}(B(k))$ -module. By an *étale Tate-cycle* on $D_{B(k)}$ we mean a tensor of $\mathcal{T}(H^1(D[\frac{1}{p}])) = \mathcal{T}(H^1(D))[\frac{1}{p}]$ that is fixed by $\text{Gal}(B(k))$ (equivalently, by $\mathcal{D}_{\mathbb{Q}_p}^{\acute{e}t}$). In what follows we will fix a family $(v_\alpha)_{\alpha \in \mathcal{J}}$ of étale Tate-cycles on $D_{B(k)}$. Let $\mathcal{G}^{\acute{e}t}$ be the schematic closure in $\mathbf{GL}_{H^1(D)}$ of the subgroup of $\mathbf{GL}_{H^1(D)[\frac{1}{p}]}$ that fixes v_α for all $\alpha \in \mathcal{J}$. The flat, affine group scheme $\mathcal{D}^{\acute{e}t}$ is a closed subgroup scheme of $\mathcal{G}^{\acute{e}t}$.

B.2 Fontaine comparison theory

We refer to [19], [17], and [64] for the following review of *Fontaine comparison theory*. This theory provides us with three rings $B_{\text{crys}}^+(W(k))$, $B_{\text{crys}}(W(k))$, and $B_{\text{dR}}(W(k))$ that are $W(k)$ -algebras for which the following six properties hold:

- (i) The three rings are integral domains equipped with exhaustive and decreasing filtrations and with a Galois action. Moreover $B_{dR}(W(k))$ is a field.
- (ii) We have $W(k)$ -monomorphisms $B_{crys}^+(W(k)) \hookrightarrow B_{crys}(W(k)) \hookrightarrow B_{dR}(W(k))$.
- (iii) The ring $B_{crys}^+(W(k))$ is faithfully flat over $W(k)$ and has a natural Frobenius lift that is compatible with σ and that also extends to an endomorphism of $B_{crys}(W(k))$.
- (iv) There exists a functorial $B_{crys}^+(W(k))$ -linear monomorphism

$$i_D^+ : M \otimes_{W(k)} B_{crys}^+(W(k)) \hookrightarrow H^1(D) \otimes_{\mathbb{Z}_p} B_{crys}^+(W(k))$$

that respects the tensor product filtrations, the Galois actions, and the tensor product Frobenius endomorphisms, with the Frobenius endomorphism of $H^1(D)$ being $1_{H^1(D)}$.

- (v) The functorial $B_{dR}(W(k))$ -linear map $i_D := i_D^+ \otimes 1_{B_{dR}(W(k))}$ is a bijection that induces naturally a $B_{dR}(W(k))$ -linear isomorphism denoted in the same way

$$i_D : \mathcal{T}(M) \otimes_{W(k)} B_{dR}(W(k)) \rightarrow \mathcal{T}(H^1(D)) \otimes_{\mathbb{Z}_p} B_{dR}(W(k)).$$

- (vi) Each étale Tate-cycle v_α on $D_{B(k)}$ defines a tensor $t_\alpha := i_D^{-1}(v_\alpha) \in \mathcal{T}(M) \otimes_{W(k)} B_{dR}(W(k))$ which in fact belongs to $F^0(\mathcal{T}(M))[\frac{1}{p}] \subset \mathcal{T}(M)[\frac{1}{p}]$ and is fixed by ϕ .

Let $\mathcal{G}_{B(k)}$ be the subgroup of $\mathbf{GL}_{M[\frac{1}{p}]}$ that fixes t_α for all $\alpha \in \mathcal{J}$. As ϕ fixes each t_α we have $\phi(\text{Lie}(\mathcal{G}_{B(k)})) = \text{Lie}(\mathcal{G}_{B(k)})$. As we also have $\mathcal{G}_{\mathbb{Q}_p}^{\acute{e}t} \times_{\text{Spec}(\mathbb{Q}_p)} \text{Spec}(B_{dR}(W(k))) = i_D(\mathcal{G}_{B(k)} \times_{\text{Spec}(B(k))} \text{Spec}(B_{dR}(W(k))))i_D^{-1}$, the groups $\mathcal{G}_{\mathbb{Q}_p}^{\acute{e}t} \times_{\text{Spec}(\mathbb{Q}_p)} \text{Spec}(B(k))$ and $\mathcal{G}_{B(k)}$ are forms of each other.

Let \mathcal{G} be the schematic closure of $\mathcal{G}_{B(k)}$ in \mathbf{GL}_M . It is a flat, closed subgroup scheme of \mathbf{GL}_M . Let $\mu : \mathbb{G}_{m,W(k)} \rightarrow \mathcal{G}$ be a cocharacter that produces a direct sum decomposition $M = F^1 \oplus F^0$ such that for each $i \in \{0, 1\}$, every $\beta \in \mathbb{G}_{m,W(k)}(W(k))$ acts through μ on F^i as the multiplication with β^{-i} . For instance, we can take μ to be the factorization through \mathcal{G} of the inverse of the canonical split cocharacter $\mu_{\text{can}} : \mathbb{G}_{m,W(k)} \rightarrow \mathbf{GL}_M$ of (M, F^1, ϕ) defined in [69, p. 512]; this is so as from the functorial properties in [69, p. 513] we get that μ_{can} fixes each t_α .

We identify $\text{Hom}(F^1, F^0)$ with the direct summand $\{e \in \text{End}(M) | e(F^0) = 0, e(F^1) \subset F^0\}$ of $\text{End}(M)$. Let U_{bigg} and U be the smooth, unipotent,

closed subgroup schemes of \mathbf{GL}_M and \mathcal{G} (respectively) defined by the rule: if \diamond is an arbitrary commutative $W(k)$ -algebra, then $U_{\text{bigg}}(\diamond) := 1_{M \otimes_{W(k)} \diamond} + \text{Hom}(F^1, F^0) \otimes_{W(k)} \diamond$ and

$$U(\diamond) := 1_{M \otimes_{W(k)} \diamond} + (\text{Lie}(\mathcal{G}_{B(k)}) \cap \text{Hom}(F^1, F^0)) \otimes_{W(k)} \diamond.$$

We have $\text{Lie}(U_{\text{bigg}}) = \text{Hom}(F^1, F^0)$ and $\text{Lie}(U) = (\text{Lie}(\mathcal{G}_{B(k)}) \cap \text{Hom}(F^1, F^0))$.

Let \mathcal{H} be the centralizer of μ in \mathcal{G} and let \mathcal{P}_0 be the semidirect product of \mathcal{H} and U . If \mathcal{G} is smooth, then \mathcal{H} is a smooth closed subgroup scheme of \mathcal{G} (cf. [10, Lem. 2.1.5 and Prop. 2.1.8 (3)]) and thus also \mathcal{P}_0 is a smooth closed subgroup scheme of \mathcal{G} .

Lemma 12. *Let $\mu_1 : \mathbb{G}_{m, W(k)} \rightarrow \mathcal{G}$ be a cocharacter such that we have a direct sum decomposition $M = F_1^1 \oplus F_1^0$ with the properties that $\mathbb{G}_{m, W(k)}$ acts through μ_1 on each F_1^i via the weight $-i$ and we have $F_1^1/pF_1^1 = F^1/pF^1$. Then there exists a unique element $v \in p\text{Lie}(U)$ such that for $u := 1_M + v \in \text{Ker}(U(W(k)) \rightarrow U(k))$ we have $u(F^1) = F_1^1$.*

Proof: There exists a unique element $u \in \text{Ker}(U_{\text{bigg}}(W(k)) \rightarrow U_{\text{bigg}}(k))$ such that we have an identity $u(F^1) = F_1^1$. We write $u = 1_M + v$, where $v \in p\text{Hom}(F^1, F^0) = p\text{Lie}(U_{\text{bigg}})$. Let $\mathcal{T}(M) = \bigoplus_{i \in \mathbb{Z}} \tilde{F}^i(\mathcal{T}(M))$ be the direct sum decomposition such that $\mathbb{G}_{m, W(k)}$ acts on each $\tilde{F}^i(\mathcal{T}(M))$ through μ via the weight $-i$. The filtration $(F^i(\mathcal{T}(M)))_{i \in \mathbb{Z}}$ of $\mathcal{T}(M)$ defined by F^1 satisfies for all $i \in \mathbb{Z}$ the following identity $F^i(\mathcal{T}(M)) = \bigoplus_{j \geq i} \tilde{F}^j(\mathcal{T}(M))$. As μ and μ_1 are two cocharacters of \mathcal{G} , they fix each t_α . In particular, we have $t_\alpha \in \tilde{F}^0(\mathcal{T}(M))[\frac{1}{p}]$ and the tensor $u^{-1}(t_\alpha) = (1_M - v)(t_\alpha)$ belongs to $F^0(\mathcal{T}(M))[\frac{1}{p}]$. As $v \in \text{Hom}(F^1, F^0) \subset \tilde{F}^{-1}(\mathcal{T}(M))$, the component of $(1_M - v)(t_\alpha)$ in $\tilde{F}^{-1}(\mathcal{T}(M))[\frac{1}{p}]$ is $-v(t_\alpha)$ as well as 0. Thus v annihilates t_α for all $\alpha \in \mathcal{J}$ and therefore $v \in p\text{Hom}(F^1, F^0) \cap \text{Lie}(\mathcal{G}_{B(k)}) = p\text{Lie}(U)$. \square

Lemma 13. *We assume that the group scheme \mathcal{G} is smooth. Then $[1_M + \frac{1}{p}\text{Lie}(U)]/U(W(k))$ is the intersection of $[1_M + \frac{1}{p}\text{Lie}(U_{\text{bigg}})]/U_{\text{bigg}}(W(k))$ and $\mathcal{P}_0(B(k))/\mathcal{P}_0(W(k))$ taken inside $\mathbf{GL}_M(B(k))/\mathbf{GL}_M(W(k))$. Thus, if \mathcal{G} is a reductive group scheme over $W(k)$, then $[1_M + \frac{1}{p}\text{Lie}(U)]/U(W(k))$ is the intersection of $[1_M + \frac{1}{p}\text{Lie}(U_{\text{bigg}})]/U_{\text{bigg}}(W(k))$ and $\mathcal{G}(B(k))/\mathcal{G}(W(k))$ taken inside $\mathbf{GL}_M(B(k))/\mathbf{GL}_M(W(k))$.*

Proof: We check that if \mathcal{G} is a reductive group scheme over $W(k)$, then the natural injective map

$$\mathcal{P}_0(B(k))/\mathcal{P}_0(W(k)) \rightarrow \mathcal{G}(B(k))/\mathcal{G}(W(k))$$

is a bijection. This is equivalent to the equality $\mathcal{G}(B(k)) = \mathcal{P}_0(B(k))\mathcal{G}(W(k))$. If \mathcal{G} is a reductive group scheme over $W(k)$, \mathcal{P}_0 is a parabolic subgroup scheme (see [10, Lem. 2.1.5 and Prop. 2.1.8 (3)]) and thus the last equality follows from the fact that projective $W(k)$ -scheme $\mathcal{G}/\mathcal{P}_0$ has the same sets of $B(k)$ - and $W(k)$ -valued points.

We are left to show that if $c \in \frac{1}{p}\mathrm{Lie}(U_{\mathrm{bigg}})$ and $g \in \mathcal{P}_0(B(k))$ are such that $g(M) = (1_M + c)(M)$, then the reduction $\bar{X} \in \mathrm{Lie}(U_{\mathrm{big},k})$ of $X := pc$ modulo p is in fact an element of $\mathrm{Lie}(U_k)$.

We consider the smooth, closed subgroup schemes $\mathcal{G}_1 := g\mathcal{G}g^{-1}$ of $\mathbf{GL}_{g(M)} = \mathbf{GL}_{(1_M+c)(M)}$ and $\tilde{\mathcal{G}} := (1_M - c)\mathcal{G}_1(1_M + c)$ of \mathbf{GL}_M . Both U_{bigg} and U are closed subgroup schemes of $\mathbf{GL}_{(1_M+c)(M)}$ and moreover $U \leq \mathcal{G}_1$.

As \mathcal{G} is smooth through which μ factors, we have $U_{\mathrm{bigg}} \cap \mathcal{G} = U$ (cf. [10, Lem. 2.1.5 and Prop. 2.1.8 (3)]). Similarly, as \mathcal{G}_1 is smooth through which $g\mu g^{-1}$ factors, we have $U_{\mathrm{bigg}} \cap \mathcal{G}_1 = U$.

All 2×2 block matrices with coefficients in $W(k)$ or k will be with respect to the direct sum decomposition $M = F^1 \oplus F^0$ or its reduction modulo p . For each $t \in W(k)$, the element $\begin{pmatrix} 1+pt & 0 \\ 0 & 1 \end{pmatrix} = \mu(W(k))((1+pt)^{-1})$ belongs to $\mathcal{G}_1(W(k))$. Thus $\begin{pmatrix} 1 & 0 \\ -c & 1 \end{pmatrix} \begin{pmatrix} 1+pt & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} = \begin{pmatrix} 1+pt & 0 \\ -tX & 1 \end{pmatrix}$ belongs to $\tilde{\mathcal{G}}(W(k))$. Therefore $\begin{pmatrix} 1 & 0 \\ \bar{t}\bar{X} & 1 \end{pmatrix}$ belongs to $\tilde{\mathcal{G}}(k)$ for all $\bar{t} \in k$. Conjugating with $1_M + c$ we get that $\begin{pmatrix} 1 & 0 \\ \bar{t}\bar{X} & 1 \end{pmatrix}$ belongs to $(U_{\mathrm{bigg}} \cap \mathcal{G}_1)(k) = U(k)$ for all $\bar{t} \in k$. Thus $\bar{X} \in \mathrm{Lie}(U_k)$. \square

Theorem 9. ([64, Thm. 1.2 and Ex. 4.4.1]) *If $p = 2$, then we assume that D is a direct sum of connected and étale Barsotti–Tate groups (e.g., this holds if $\mathcal{G}^{\acute{e}t}$ is a torus). Then there exist isomorphisms*

$$\varrho_D : (M, (t_\alpha)_{\alpha \in \mathcal{J}}) \rightarrow (H^1(D) \otimes_{\mathbb{Z}_p} W(k), (v_\alpha)_{\alpha \in \mathcal{J}})$$

(in the sense of Subsection 2.1). If moreover D has a principal quasi-polarization, then there exist isomorphisms $\varrho_D : (M, (t_\alpha)_{\alpha \in \mathcal{J}}, \psi_M) \rightarrow (H^1(D) \otimes_{\mathbb{Z}_p} W(k), (v_\alpha)_{\alpha \in \mathcal{J}}, \psi_{H^1(D)})$.

If $\mathcal{G}^{\acute{e}t}$ is a reductive group scheme over \mathbb{Z}_p and if for $p = 2$ the 2-divisible group D is connected, then Theorem 9 is also proved in [27, Cor. 1.4.3].

Lemma 14. *Let k_1 be an algebraically closed field that contains k . We assume that there exists an isomorphism $(M \otimes_{W(k)} W(k_1), (t_\alpha)_{\alpha \in \mathcal{J}}) \rightarrow (H^1(D) \otimes_{\mathbb{Z}_p} W(k_1), (v_\alpha)_{\alpha \in \mathcal{J}})$. Then there exists an isomorphism $\varrho_D : (M, (t_\alpha)_{\alpha \in \mathcal{J}}) \rightarrow (H^1(D) \otimes_{\mathbb{Z}_p} W(k), (v_\alpha)_{\alpha \in \mathcal{J}})$.*

Proof: To check the existence of ϱ_D we can assume that we have $t_\alpha \in \mathcal{T}(M)$ and $v_\alpha \in H^1(D)$ for all $\alpha \in \mathcal{J}$. Thus we can speak about the affine $W(k)$ -scheme \mathfrak{P} of finite type that parameterizes isomorphisms between $(M, (t_\alpha)_{\alpha \in \mathcal{J}})$ and $(H^1(D) \otimes_{\mathbb{Z}_p} W(k), (v_\alpha)_{\alpha \in \mathcal{J}})$. We know that \mathfrak{P} has a $W(k_1)$ -valued point. As the monomorphism $W(k) \hookrightarrow W(k_1)$ is of ramification index one, from [7, Ch. 3, Sect. 3.6, Prop. 4] we get that there exists a morphism $\mathfrak{P}' \rightarrow \mathfrak{P}$ of $W(k)$ -schemes such that \mathfrak{P}' is smooth over $W(k)$ and has a $W(k_1)$ -valued point. Thus the special fibre \mathfrak{P}'_k is non-empty. As \mathfrak{P}' is smooth over $W(k)$ and has a non-empty special fibre, it has $W(k)$ -valued points. Therefore \mathfrak{P} also has $W(k)$ -valued points and thus the isomorphism ϱ_D exists. \square

B.3 Group correspondences

Let $\mathcal{F}_{\mathbb{Q}_p}^{\acute{e}t}$ be a reductive, closed subgroup of $\mathcal{G}_{\mathbb{Q}_p}^{\acute{e}t}$. The restriction to $\text{Lie}(\mathcal{F}_{\mathbb{Q}_p}^{\acute{e}t})$ of the trace bilinear form on $\text{End}(H^1(D)[\frac{1}{p}])$ is non-degenerate, cf. Lemma 11 (b). Let $\text{Lie}(\mathcal{F}_{\mathbb{Q}_p}^{\acute{e}t})^\perp$ be the perpendicular on $\text{Lie}(\mathcal{F}_{\mathbb{Q}_p}^{\acute{e}t})$ with respect to the trace bilinear form on $\text{End}(H^1(D)[\frac{1}{p}])$; we have a direct sum decomposition of \mathbb{Q}_p -vector spaces

$$\text{End}(H^1(D)[\frac{1}{p}]) = \text{Lie}(\mathcal{F}_{\mathbb{Q}_p}^{\acute{e}t}) \oplus \text{Lie}(\mathcal{F}_{\mathbb{Q}_p}^{\acute{e}t})^\perp.$$

Let $\pi^{\acute{e}t}$ be the projector of $\text{End}(H^1(D)[\frac{1}{p}])$ on $\text{Lie}(\mathcal{F}_{\mathbb{Q}_p}^{\acute{e}t})$ along $\text{Lie}(\mathcal{F}_{\mathbb{Q}_p}^{\acute{e}t})^\perp$; it is an idempotent of $\text{End}(H^1(D)[\frac{1}{p}])$ fixed by each subgroup of $\mathbf{GL}_{H^1(D)[\frac{1}{p}]}$ that normalizes $\mathcal{F}_{\mathbb{Q}_p}^{\acute{e}t}$.

If $\mathcal{D}_{\mathbb{Q}_p}^{\acute{e}t}$ normalizes $\mathcal{F}_{\mathbb{Q}_p}^{\acute{e}t}$ (e.g., this holds if $\mathcal{F}_{\mathbb{Q}_p}^{\acute{e}t}$ is a normal subgroup of $\mathcal{G}_{\mathbb{Q}_p}^{\acute{e}t}$), then $\pi^{\acute{e}t}$ is fixed by $\mathcal{D}_{\mathbb{Q}_p}^{\acute{e}t}$ and thus also by $\text{Im}(\rho_D)$ and therefore we can speak about the projector π^{crys} of $\text{End}(M[\frac{1}{p}])$ that corresponds to $\pi^{\acute{e}t}$ via Fontaine comparison theory.

Lemma 15. *We assume that $\mathcal{D}_{\mathbb{Q}_p}^{\acute{e}t}$ normalizes $\mathcal{F}_{\mathbb{Q}_p}^{\acute{e}t}$. Then the following two properties hold:*

(a) *There exists a unique reductive subgroup $\mathcal{F}_{B(k)}$ of $\mathcal{G}_{B(k)}$ whose Lie algebra is $\text{Im}(\pi^{\text{crys}})$.*

(b) *If the generic fibre of μ_{can} factors through $\mathcal{F}_{B(k)}$, then $\mathcal{D}_{\mathbb{Q}_p}^{\acute{e}t}$ is a subgroup of $\mathcal{F}_{\mathbb{Q}_p}^{\acute{e}t}$.*

Proof: We check part (a). As i_D^{-1} is a $B_{\text{dR}}(W(k))$ -linear isomorphism that takes $\pi^{\acute{e}t}$ to π^{crys} , the group $i_D^{-1}(\mathcal{F}_{\mathbb{Q}_p}^{\acute{e}t} \times_{\text{Spec}(\mathbb{Q}_p)} \text{Spec}(B_{\text{dR}}(W(k))))i_D$ is a subgroup of

$$i_D^{-1}(\mathcal{G}_{\mathbb{Q}_p}^{\acute{e}t} \times_{\text{Spec}(\mathbb{Q}_p)} \text{Spec}(B_{\text{dR}}(W(k))))i_D = \mathcal{G}_{B(k)} \times_{\text{Spec}(B(k))} \text{Spec}(B_{\text{dR}}(W(k)))$$

whose Lie algebra is $\text{Im}(\pi^{\text{crys}}) \otimes_{B(k)} B_{\text{dR}}(W(k))$. Thus as $B_{\text{dR}}(W(k))$ is a field, from Lemma 11 (a) applied with $(\mathcal{W}, \mathcal{L}, \eta, \eta_1) = (M[\frac{1}{p}], \text{Im}(\pi^{\text{crys}}), B(k), B_{\text{dR}}(W(k)))$, we get that there exists a unique reductive subgroup $\mathcal{F}_{B(k)}$ of $\mathbf{GL}_{M[\frac{1}{p}]}$ whose Lie algebra is $\text{Im}(\pi^{\text{crys}})$. As $\mathcal{F}_{B(k)} \times_{\text{Spec}(B(k))} \text{Spec}(B_{\text{dR}}(W(k)))$ is a subgroup of $\mathcal{G}_{B(k)} \times_{\text{Spec}(B(k))} \text{Spec}(B_{\text{dR}}(W(k)))$, the group $\mathcal{F}_{B(k)}$ is in fact a subgroup of $\mathcal{G}_{B(k)}$. Thus part (a) holds.

We check part (b). Let l_{can} be the Lie algebra of the image of the generic fibre of μ_{can} . As π^{crys} is fixed by ϕ , the Lie algebra $\text{Lie}(\mathcal{F}_{B(k)}) = \text{Im}(\pi^{\text{crys}})$ is normalized by ϕ . Let $\mathcal{D}_{B(k)}$ be the smallest connected subgroup of $\mathcal{F}_{B(k)}$ with the property that its Lie algebra $\text{Lie}(\mathcal{D}_{B(k)})$ contains $\phi^m(l_{\text{can}})$ for all $m \in \mathbb{Z}$. From [5, Ch. I, Sect. 7.1] we get that all conjugates of the generic fibre of μ_{can} through integral powers of ϕ factor through $\mathcal{D}_{B(k)}$ and $\mathcal{D}_{B(k)}$ is the smallest subgroup of $\mathcal{F}_{B(k)}$ that has this property. This implies that $\mathcal{D}_{B(k)}$ corresponds to $\mathcal{D}_{\mathbb{Q}_p}^{\acute{e}t}$ via Fontaine comparison theory (cf. [69, Prop. 4.2.3]), i.e., we have an identity of subgroups

$$\mathcal{D}_{\mathbb{Q}_p}^{\acute{e}t} \times_{\text{Spec}(\mathbb{Q}_p)} \text{Spec}(B_{\text{dR}}(W(k))) = i_D(\mathcal{D}_{B(k)} \times_{\text{Spec}(B(k))} \text{Spec}(B_{\text{dR}}(W(k))))i_D^{-1}$$

of $\mathbf{GL}_{H^1(D) \otimes_{\mathbb{Z}_p} B_{\text{dR}}(W(k))}$. Thus, as $\mathcal{D}_{B(k)}$ is a subgroup of $\mathcal{F}_{B(k)}$, $\mathcal{D}_{\mathbb{Q}_p}^{\acute{e}t} \times_{\text{Spec}(\mathbb{Q}_p)} \text{Spec}(B_{\text{dR}}(W(k)))$ is a subgroup of $\mathcal{F}_{\mathbb{Q}_p}^{\acute{e}t} \times_{\text{Spec}(\mathbb{Q}_p)} \text{Spec}(B_{\text{dR}}(W(k))) = i_D(\mathcal{F}_{B(k)} \times_{\text{Spec}(B(k))} \text{Spec}(B_{\text{dR}}(W(k))))i_D^{-1}$. From this part (b) follows. \square

B.4 Faltings deformation theory

Let $l \in \mathbb{N}$. Let $R = W(k)[[x_1, \dots, x_l]]$ be the ring of formal power series in l variables with coefficients in $W(k)$. Let Φ_R be the Frobenius lift of R that is compatible with σ and that takes x_i to x_i^p for all $i \in \{1, \dots, l\}$. We consider the ideal $\mathfrak{I} := (x_1, \dots, x_l)$ of R . Let $\hat{\Omega}_{R/W(k)} = \bigoplus_{i=1}^l R dx_i$ be the \mathfrak{I} -adic completion of the R -module of relative differentials $\Omega_{R/W(k)}$. Let $d\Phi_R : \hat{\Omega}_{R/W(k)} \rightarrow \hat{\Omega}_{R/W(k)}$ be the (\mathfrak{I} -adic completion of the) differential map of Φ_R .

Let (M_R, F_R^1, Φ) be a triple such that the following four axioms hold:

- (i) M_R is a free R -module of rank equal to the height of D ;
- (ii) F_R^1 is a direct summand of M_R of rank equal to the rank of F^1 ;
- (iii) $\Phi : M_R \rightarrow M_R$ is a Φ_R -linear endomorphism that induces an R -linear isomorphism $(M_R + \frac{1}{p}F_R^1) \otimes_{R, \Phi_R} R \rightarrow M_R$;
- (iv) the reduction of (M_R, F_R^1, Φ) modulo \mathfrak{J} is canonically identified with (M, F^1, ϕ) .

Let Φ act in the natural tensor way on $\mathcal{T}(M_R)[\frac{1}{p}]$. For instance, if $e \in M_R^\vee := \text{Hom}(M_R, R)$, then $\Phi(e) \in M_R^\vee[\frac{1}{p}]$ is the unique element such that we have $\Phi(e)(\Phi(a)) = \Phi_R(e(a)) \in R$ for all $a \in M_R$.

It is known that there exists a unique connection $\nabla : M_R \rightarrow M_R \otimes_R \hat{\Omega}_{R/W(k)}$ such that we have an identity $\nabla \circ \Phi = (\Phi \otimes d\Phi_R) \circ \nabla$ and moreover such a connection is automatically integrable and nilpotent modulo p , cf. either [17, Thm. 10] or [64, Thm. 3.2 and Cor. 3.3.2]. By viewing $\mathcal{T}(M_R)[\frac{1}{p}]$ as a module over the Lie algebra (associated to) $\text{End}(M_R)$, we can view also ∇ as a connection on the R -module $\mathcal{T}(M_R)[\frac{1}{p}]$ and thus it makes sense to say that it annihilates some specific tensor of $\mathcal{T}(M_R)[\frac{1}{p}]$.

Lemma 16. *There exists a unique Barsotti–Tate group D_R over R which modulo the ideal \mathfrak{J} is D and such that its filtered F -crystal over R/pR is $(M_R, F_R^1, \Phi, \nabla)$.*

Proof: Let J be an ideal of R such that R is complete in the J -adic topology (e.g., (p) , \mathfrak{J} , or $p\mathfrak{J}$). Let $\text{Spf}(R)$ be the formal scheme which is the formal completion of $\text{Spec}(R)$ along $\text{Spec}(R/J)$. The categories of Barsotti–Tate groups over $\text{Spec}(R)$ and respectively over $\text{Spf}(R)$ are canonically isomorphic, cf. [35, Ch. II, Lem. 4.16]; below we will use this fact without any extra comment.

The existence of D_R is implied by [17, Thm. 10]. The uniqueness of the fibre $D_{R/pR}$ of D_R over $\text{Spec}(R/pR)$ is implied by [3, Thm. 4.1.1]. As the ideal $p(\mathfrak{J}/\mathfrak{J}^m)$ of R/\mathfrak{J}^m has a natural nilpotent divided power structure for all $m \in \mathbb{N}^*$, from the Grothendieck–Messing deformation theory we get that D_R is the unique Barsotti–Tate group over R that lifts both D and $D_{R/pR}$ and whose filtered F -crystal is $(M_R, F_R^1, \Phi, \nabla)$. \square

Lemma 17. *We assume that D has a principal quasi-polarization λ_D . We also assume that there exists a perfect, alternating bilinear form ψ_{M_R} on M_R that lifts ψ_M (i.e., which modulo \mathfrak{J} is ψ_M), that satisfies $\psi_{M_R}(F_R^1, F_R^1) = 0$*

(i.e., F_R^1 is anisotropic with respect to ψ_{M_R}), and such that for all $a, b \in M_R$ we have $\psi_{M_R}(\Phi(a), \Phi(b)) = p\Phi_R(\psi_{M_R}(a, b))$. Then there exists a unique principal quasi-polarization λ_{D_R} of D_R which modulo the ideal \mathfrak{I} is λ_D and whose crystalline realization is ψ_{M_R} .

Proof: Let $(M_R^t, F_R^{1t}, \Phi^t, \nabla^t)$ be the filtered F -crystal over R/pR of the Cartier dual D_R^t of D_R . The form ψ_{M_R} defines naturally an isomorphism $\theta_0 : (M_R^t, F_R^{1t}, \Phi^t) \rightarrow (M_R, F_R^1, \Phi)$. As the connections ∇ and ∇^t are uniquely determined by (M_R, F_R^1, Φ) and $(M_R^t, F_R^{1t}, \Phi^t)$ (respectively), θ_0 extends to an isomorphism $\theta : (M_R^t, F_R^{1t}, \Phi^t, \nabla^t) \rightarrow (M_R, F_R^1, \Phi, \nabla)$ of filtered F -crystals over R/pR .

As the ring R/pR has a finite p -basis $\{x_1, \dots, x_l\}$ in the sense of [3, Def. 1.1.1], from the fully faithfulness part of [3, Thm. 4.1.1] we get that there exists a unique principal quasi-polarization $\lambda_{D_{R/pR}} : D_{R/pR} \rightarrow D_{R/pR}^t$ whose crystalline realization is θ ; it lifts the special fibre of λ_D . As the ideal $p(\mathfrak{I}/\mathfrak{I}^m)$ of R/\mathfrak{I}^m has a natural nilpotent divided power structure for all $m \in \mathbb{N}^*$, from the Grothendieck–Messing deformation theory we get that there exists a unique principal quasi-polarization λ_{D_R} of D_R that lifts both $\lambda_{D_{R/pR}}$ and λ_D and whose crystalline realization is ψ_{M_R} . \square

B.4.1 Explicit filtered F -crystal with tensors

Let $M = F^1 \oplus F^0$, U^{bigg} , U , \mathcal{H} , and \mathcal{P}_0 be as before Lemma 12. Let \mathcal{G}' be the universal smoothening of \mathcal{G} , cf. Subsection A1.

Until Subsection B5 we will assume that D has a principal quasi-polarization λ_D , that \mathcal{G} is a closed subgroup scheme of $\mathbf{GSp}(M, \psi_M)$, and that $R = W(k)[[x_1, \dots, x_l]]$ is the completion of the local ring of \mathcal{G}' at the identity element of \mathcal{G}'_k . Thus the relative dimension of \mathcal{G} over $W(k)$ is l . The closed embedding $U \hookrightarrow \mathcal{G}$ factors through \mathcal{G}' (cf. Subsection A1); thus U is a closed subgroup scheme of \mathcal{G}' .

Let $g_{\text{univ}} \in \mathcal{G}'(R)$ be the universal element. We define $(M_R, F_R^1) := (M, F^1) \otimes_{W(k)} R$ and $\Phi := g_{\text{univ}}(\phi \otimes \Phi_R)$. Let

$$\mathfrak{C}_{\text{univ}} := (M_R, F_R^1, \Phi, \nabla, (t_\alpha)_{\alpha \in \mathcal{J}}).$$

The $W(k)$ -algebra R is complete in the \mathfrak{I} -topology and we have $\Phi_R(\mathfrak{I}) \subset \mathfrak{I}^p$. This implies that each element of $\text{Ker}(\mathbb{G}_{m, W(k)}(R) \rightarrow \mathbb{G}_{m, W(k)}(R/\mathfrak{I}))$ is of the form $\beta\Phi_R(\beta^{-1})$ for some element $\beta \in \text{Ker}(\mathbb{G}_{m, W(k)}(R) \rightarrow \mathbb{G}_{m, W(k)}(R/\mathfrak{I}))$. As g_{univ} takes ψ_M to a $\text{Ker}(\mathbb{G}_{m, W(k)}(R) \rightarrow \mathbb{G}_{m, W(k)}(R/\mathfrak{I}))$ -multiple of ψ_M , we

get that there exists a $\text{Ker}(\mathbb{G}_{m,W(k)}(R) \rightarrow \mathbb{G}_{m,W(k)}(R/\mathfrak{J}))$ -multiple ψ_{M_R} of the perfect, alternating bilinear form ψ_M on M_R such that we have an identity

$$\psi_{M_R}(\Phi(a), \Phi(b)) = p\Phi_R(\psi_{M_R}(a, b))$$

for all $a, b \in M_R$. As 1 is the only element of $\text{Ker}(\mathbb{G}_{m,W(k)}(R) \rightarrow \mathbb{G}_{m,W(k)}(R/\mathfrak{J}))$ fixed by Φ_R , this $\text{Ker}(\mathbb{G}_{m,W(k)}(R) \rightarrow \mathbb{G}_{m,W(k)}(R/\mathfrak{J}))$ -multiple ψ_{M_R} of ψ_M is uniquely determined.

We have the following three properties:

- (i) *The connection on $\mathcal{T}(M_R) = \mathcal{T}(M) \otimes_{W(k)} R$ induced naturally by ∇ (and denoted in the same way) annihilates the tensor $t_\alpha \in \mathcal{T}(M) \otimes_{W(k)} R[\frac{1}{p}]$ for all $\alpha \in \mathcal{J}$.*
- (ii) *The connection ∇ is of the form $\delta + \gamma$, where δ is the flat connection on $M_R = M \otimes_{W(k)} R$ that annihilates $M \otimes 1$ and where $\gamma \in (\text{Lie}(\mathcal{G}_{B(k)}) \cap \text{End}(M)) \otimes_{W(k)} \hat{\Omega}_{R/W(k)}$.*
- (iii) *The Kodaira–Spencer map of the connection ∇ has an image Θ which is the direct summand $\text{Lie}(U) \otimes_{W(k)} R$ of $\text{Lie}(U_{\text{bigg}}) \otimes_{W(k)} R \rightarrow \text{Hom}(F^1, M/F^1) \otimes_{W(k)} R$.*

As ϕ fixes t_α and $\nabla \circ \Phi = (\Phi \otimes d\Phi_R) \circ \nabla$, we have $\nabla(t_\alpha) = (\Phi \otimes d\Phi_R)(\nabla(t_\alpha))$. As $d\Phi_R(x_i) = px_i^{p-1}dx_i$, by induction on $n \in \mathbb{N}^*$ we get that $\nabla(t_\alpha) \in M \otimes_{W(k)} \mathfrak{J}^n(\oplus_{i=1}^l Rdx_i)[\frac{1}{p}]$. This implies that (i) holds. Property (ii) follows from the property (i) and the fact that $\text{Lie}(\mathcal{G}_{B(k)}) \cap \text{End}(M)$ is the Lie subalgebra of $\text{End}(M)$ which annihilates t_α for all $\alpha \in \mathcal{J}$.

To check (iii), we first remark that the property (ii) implies that Θ is contained in the image of $(\text{Lie}(\mathcal{G}_{B(k)}) \cap \text{End}(M)) \otimes_{W(k)} R$ in $\text{Lie}(U_{\text{bigg}}) \otimes_{W(k)} R \rightarrow \text{Hom}(F^1, M/F^1) \otimes_{W(k)} R$ and thus it is contained in $\text{Lie}(U) \otimes_{W(k)} R$. It is easy to see that γ modulo (p, \mathfrak{J}^{p-1}) is $g_{\text{univ}}^{-1}dg_{\text{univ}}$ modulo (p, \mathfrak{J}^{p-1}) (for instance, this follows from [64, Equations (11) and (12)]). Thus, as U is a closed subgroup scheme of \mathcal{G}' and as $g_{\text{univ}} \in \mathcal{G}'(R)$ is the universal element, we get that Θ surjects onto $\text{Lie}(U) \otimes_{W(k)} R/(p, \mathfrak{J})$. From this and the inclusion $\Theta \subset \text{Lie}(U) \otimes_{W(k)} R$ we get that the property (iii) holds.

Let $m \in \mathbb{N}$, $R_1 := W(k)[[x_1, \dots, x_m]]$, and $Z := \text{Spec}(R_1)$. Let Φ_{R_1} be the Frobenius lift of R_1 that is compatible with σ and that takes x_i to x_i^p for all $i \in \{1, \dots, m\}$. We consider the ideal $\mathfrak{J}_1 := (x_1, \dots, x_m)$ of R_1 .

Let $(M_1, F_1^1, \Phi_1, \nabla_1)$ be a filtered F -crystal over R_1/pR_1 . Thus:

- (iv) Φ_1 induces an R_1 -linear isomorphism $(M_1 + \frac{1}{p}F_1^1) \otimes_{R_1, \Phi_{R_1}} R_1 \rightarrow M_1$.

Let $\mathfrak{C}_1 := (M_1, F_1^1, \Phi_1, \nabla_1, (t_{1,\alpha})_{\alpha \in \mathcal{J}})$, where $(t_{1,\alpha})_{\alpha \in \mathcal{J}}$ is a family of tensors $(t_{1,\alpha})_{\alpha \in \mathcal{J}}$ of $\mathcal{T}(M_1)[\frac{1}{p}]$ such that the following two axioms hold (here $\mathcal{T}(M_1)$ is as in Subsection 2.1):

(v) *Each tensor $t_{1,\alpha}$ is fixed by Φ_1 , is annihilated by ∇_1 , and belongs to $F^0(\mathcal{T}(M_1))[\frac{1}{p}]$ (here $(F^i(\mathcal{T}(M_1)))_{i \in \mathbb{Z}}$ is the filtration of $\mathcal{T}(M_1)$ defined by F_1^1 , cf. Subsection 2.1).*

(vi) *Its reduction modulo the ideal \mathfrak{I}_1 is $(M, F^1, \phi, (t_\alpha)_{\alpha \in \mathcal{J}})$.*

The R_1 -module M_1 is free of rank equal to the rank of M , cf. property (vi). We consider the closed embedding $z_Z : \text{Spec}(W(k)) \hookrightarrow Z$ defined by the ideal \mathfrak{I}_1 of R_1 .

Theorem 10. *The following two properties hold:*

(a) *There exists a morphism $i_Z : Z \rightarrow \text{Spec}(R)$ of $W(k)$ -schemes such that $g_{\text{univ}} \circ i_Z \circ z_Z$ is the identity section of \mathcal{G}' and \mathfrak{C}_1 is isomorphic to $i_Z^*(\mathfrak{C}_{\text{univ}})$ under an isomorphism which modulo the ideal \mathfrak{I}_1 becomes the identity automorphism 1_M of M .*

(b) *If there exists a perfect, alternating bilinear form ψ_{M_1} on M_1 which modulo \mathfrak{I}_1 is ψ_M and which is a principal quasi-polarization of the filtered F -crystal $(M_1, F_1^1, \Phi_1, \nabla_1)$ over R_1/pR_1 , then (\mathfrak{C}_1, ψ_1) is isomorphic to $i_Z^*(\mathfrak{C}_{\text{univ}}, \psi_{M_R})$ under an isomorphism which modulo the ideal \mathfrak{I}_1 becomes the identity automorphism 1_M of M .*

Proof: If \mathcal{G} is smooth, then part (a) is a particular case of [17, Thm. 10 and Rm. iii) after it]. To prove part (a) in the general case, we follow the proof of [64, Thm. 5.3]. Let D_{R_1} be the unique Barsotti–Tate group over R_1 which modulo the ideal \mathfrak{I}_1 is D and whose filtered F -crystal over R_1/pR_1 is $(M_1, F_1^1, \Phi_1, \nabla_1)$, cf. Lemma 17.

By induction on $s \in \mathbb{N}^*$ we show that there exists a morphism $i_{Z,s} : \text{Spec}(R_1/\mathfrak{I}_1^s) \rightarrow \text{Spec}(R)$ of $W(k)$ -schemes which at the level of rings maps \mathfrak{I} to $\mathfrak{I}_1/\mathfrak{I}_1^s$ and such that $i_{Z,s}^*(D_R)$ is isomorphic to D_{R_1} modulo \mathfrak{I}_1^s under a unique isomorphism \mathcal{I}_s that has the following two properties:

(i) *it lifts the identity automorphism of D ;*

(ii) *its crystalline Dieudonné realization defines an isomorphism \mathcal{E}_s between \mathfrak{C}_1 modulo \mathfrak{I}_1^s and $i_{Z,s}^*(\mathfrak{C}_{\text{univ}})$ which modulo $\mathfrak{I}_1/\mathfrak{I}_1^s$ is the identity automorphism 1_M of M .*

As $\Phi_{R_1}(\mathfrak{J}_1) \subset \mathfrak{J}_1^p$ and the ideal $\mathfrak{J}_1/\mathfrak{J}_1^s$ is complete, such an isomorphism \mathcal{E}_s is unique. We take $i_{Z,1}$ to be defined by the $W(k)$ -epimorphism $R \rightarrow R/\mathfrak{J} = W(k) = R_1/\mathfrak{J}_1$ and we take \mathcal{I}_1 and \mathcal{E}_1 to be defined by the identity automorphism of D and by 1_M (respectively). Thus the existence and the uniqueness of $i_{Z,1}$ and \mathcal{I}_1 are obvious.

For $s \geq 2$ the passage from $s-1$ to s goes as follows. We endow the ideal $\mathfrak{J}_s := \mathfrak{J}_1^{s-1}/\mathfrak{J}_1^s$ of R_1/\mathfrak{J}_1^s with the trivial divided power structure; thus $\mathfrak{J}_s^{[2]} = 0$. The uniqueness of \mathcal{I}_s is implied by the uniqueness of \mathcal{I}_{s-1} and \mathcal{E}_s , cf. Grothendieck–Messing deformation theory. To end the induction, we check that we can choose $i_{Z,s}$ such that \mathcal{I}_s and \mathcal{E}_s exist.

Let $\tilde{i}_{Z,s} : \text{Spec}(R_1/\mathfrak{J}_1^s) \rightarrow \text{Spec}(R)$ be an arbitrary morphism of $W(k)$ -schemes through which $i_{Z,s-1}$ factors naturally. We write

$$\tilde{i}_{Z,s}^*(\mathfrak{C}_{\text{univ}}) = (M \otimes_{W(k)} R_1/\mathfrak{J}_1^s, F^1 \otimes_{W(k)} R_1/\mathfrak{J}_1^s, {}_s\Phi, {}_s\nabla, (t_\alpha)_{\alpha \in \mathcal{J}}).$$

Due to the existence of \mathcal{I}_{s-1} , there exists (cf. Grothendieck–Messing deformation theory) a direct summand ${}_sF^1$ of $M \otimes_{W(k)} R_1/\mathfrak{J}_1^s$ that lifts $F^1 \otimes_{W(k)} R_1/\mathfrak{J}_1^{s-1}$ and such that the quintuple $(M_1, F_1, \Phi_1, \nabla_1)$ modulo \mathfrak{J}_1^s is isomorphic to the quintuple $(M \otimes_{W(k)} R_1/\mathfrak{J}_1^s, {}_sF^1, {}_s\Phi, {}_s\nabla)$ under an isomorphism $\tilde{\mathcal{E}}_s$ that lifts the one defined by \mathcal{E}_{s-1} . Let $t_{1,\alpha,s} \in \mathcal{T}(M \otimes_{W(k)} R_1/\mathfrak{J}_1^s)$ be the image under $\tilde{\mathcal{E}}_s$ of $t_{1,\alpha}$. As $t_{1,\alpha}$ is fixed by Φ_1 , $t_{1,\alpha,s}$ is fixed by ${}_s\Phi$. As $\tilde{\mathcal{E}}_s$ lifts \mathcal{E}_{s-1} , the reductions modulo \mathfrak{J}_s of t_α and $t_{1,\alpha,s}$ coincide. As ${}_s\Phi(\mathcal{T}(M) \otimes_{W(k)} \mathfrak{J}_s) = 0$, inside $\mathcal{T}(M) \otimes_{W(k)} R_1/\mathfrak{J}_1^s$ we have

$$t_{1,\alpha,s} - t_\alpha = {}_s\Phi(t_{1,\alpha,s} - t_\alpha) \in {}_s\Phi(\mathcal{T}(M) \otimes_{W(k)} \mathfrak{J}_s) = 0.$$

Thus we have $t_{1,\alpha,s} = t_\alpha \in \mathcal{T}(M) \otimes_{W(k)} R_1/\mathfrak{J}_1^s$ for all $\alpha \in \mathcal{J}$.

Let $v_s \in \text{Lie}(U_{\text{bigg}}) \otimes_{W(k)} \mathfrak{J}_s$ be the unique element such that we have

$$(1_{M \otimes_{W(k)} R_1/\mathfrak{J}_1^s} + v_s)(F^1 \otimes_{W(k)} R_1/\mathfrak{J}_1^s) = {}_sF^1.$$

As each $t_{1,\alpha,s} = t_\alpha$ belongs to the F^0 -filtrations defined by either ${}_sF^1$ or $F^1 \otimes_{W(k)} R_1/\mathfrak{J}_1^s$ and as the $W(k)$ -module \mathfrak{J}_s is torsionless, as in [64, proof of Thm. 5.3, bottom of p. 241 and top of p. 242] we argue that $v_s \in \text{Lie}(U) \otimes_{W(k)} \mathfrak{J}_s$. Based on this and the property (iii), as in [64, proof of Thm. 5.3, p. 242] we argue that we can replace $\tilde{i}_{Z,s}$ by another morphism $i_{Z,s} : \text{Spec}(R_1/\mathfrak{J}_1^s) \rightarrow \text{Spec}(R)$ through which $i_{Z,s-1}$ factors and for which ${}_sF^1$ gets replaced by (i.e., becomes) $F^1 \otimes_{W(k)} R_1/\mathfrak{J}_1^s$. From Grothendieck–Messing deformation theory we get that $i_{Z,s}^*(D_R)$ is isomorphic to D_{R_1} modulo \mathfrak{J}_1^s

under an isomorphism \mathcal{I}_s which lifts \mathcal{I}_{s-1} and which defines an isomorphism \mathcal{E}_s between \mathfrak{C}_1 modulo \mathfrak{I}_1^s and $i_{Z,s}^*(\mathfrak{C}_{\text{univ}})$. As \mathcal{I}_s lifts \mathcal{I}_{s-1} , the uniqueness of \mathcal{E}_{s-1} implies that \mathcal{E}_s lifts \mathcal{E}_{s-1} . This ends the induction.

We take $i_Z : Z \rightarrow \text{Spec}(R)$ such that it lifts $i_{Z,s}$ for all $s \in \mathbb{N}^*$. From the very definition of $i_{Z,1}$ we get that $g_{\text{univ}} \circ i_Z \circ z_Z$ is the identity section of \mathcal{G}' . Moreover, $i_Z^*(\mathfrak{C}_{\text{univ}})$ is isomorphic to \mathfrak{C}_1 under an isomorphism that lifts \mathcal{E}_s for all $s \in \mathbb{N}^*$. Thus part (a) holds.

Part (b) follows from part (a) and the fact that ψ_{M_1} is the unique principal quasi-polarization of $(M_1, F^1, \Phi, \nabla_1)$ which modulo \mathfrak{I}_1 is ψ_M . \square

B.4.2 Variant of Subsubsection B4.1 and Theorem 10

Let $d \in \mathbb{N}$ be the rank of $\text{Lie}(U) = \text{Lie}(\mathcal{G}_{B(k)}) \cap \text{Hom}(F^1, F^0)$. Let $S := W(k)[[x_1, \dots, x_d]]$ and $\mathfrak{I}_0 := (x_1, \dots, x_d)$ be its ideal. We consider an arbitrary closed embedding $\text{Spec}(S) \hookrightarrow \text{Spec}(R)$ such that the following two properties hold:

- (i) at the level of $W(k)$ -algebras, the ideal \mathfrak{I} of R maps to the ideal \mathfrak{I}_0 of S ;
- (ii) the pullback $\mathfrak{D}_{\text{univ}}$ of $\mathfrak{C}_{\text{univ}}$ via the closed embedding $\text{Spec}(S) \hookrightarrow \text{Spec}(R)$, has a Kodaira–Spencer map which is injective and whose image equals to the direct summand $\text{Lie}(U) \otimes_{W(k)} S$ of $\text{Lie}(U_{\text{bigg}}) \otimes_{W(k)} S \simeq \text{Hom}(F^1, M/F^1) \otimes_{W(k)} S$.

The proof of Theorem 10 applies to give us that there exists a morphism $j_Z : Z \rightarrow \text{Spec}(S)$ of $W(k)$ -schemes such that \mathfrak{C}_1 is isomorphic to $j_Z^*(\mathfrak{D}_{\text{univ}})$ under an isomorphism which modulo \mathfrak{I}_1 becomes the identity automorphism 1_M of M . As the Kodaira–Spencer map of $\mathfrak{D}_{\text{univ}}$ is injective, the morphism j_Z is unique. In simpler words, we can choose $i_Z : Z \rightarrow \text{Spec}(R)$ to factor through the closed embedding $\text{Spec}(S) \hookrightarrow \text{Spec}(R)$ and the resulting factorization is our unique morphism $j_Z : Z \rightarrow \text{Spec}(S)$.

In this paragraph we assume that \mathcal{G} is smooth over $W(k)$. This assumption implies that the normalizer $\mathcal{P} = \mathcal{P}_1$ of F^1 in \mathcal{G} is smooth over $W(k)$ and the product morphism $U \times_{\text{Spec}(W(k))} \mathcal{P} \rightarrow \mathcal{G}$ is an open embedding, cf. [10, Lem. 2.1.5 and Prop. 2.1.8 (3)]. Thus we can view $g_{\text{univ}} \in \mathcal{G}(R)$ as an R -valued point of $U \times_{\text{Spec}(W(k))} \mathcal{P} \rightarrow \mathcal{G}$ as well as of the quotient $[U \times_{\text{Spec}(W(k))} \mathcal{P}]/\mathcal{P} = U$. So the closed embedding $\text{Spec}(S) \hookrightarrow \text{Spec}(R)$ can be any closed embedding with the property that the morphism $\text{Spec}(S) \rightarrow [U \times_{\text{Spec}(W(k))} \mathcal{P}]/\mathcal{P} = U$ induced by g_{univ} is formally étale.

B.4.3 On the $p = 2$ case

The following result complements Theorem 9 for $p = 2$.

Theorem 11. *We assume that $p = 2$ and that one of the following two conditions holds:*

- (i) *the group scheme \mathcal{G} is reductive;*
- (ii) *the 2-divisible group D_k is ordinary.*

(a) *Then there exists a 2-divisible group D' over $W(k)$ which lifts D_k , whose filtered F -crystal over k is as well the triple (M, F^1, ϕ) , and for which there exists an isomorphism $\varrho_{D'} : (M, (t_\alpha)_{\alpha \in \mathcal{J}}) \rightarrow (H^1(D') \otimes_{\mathbb{Z}_2} W(k), (v_\alpha)_{\alpha \in \mathcal{J}})$. Here $v_\alpha \in \mathcal{T}(H^1(D'))[\frac{1}{2}] = \mathcal{T}(H^1(D))[\frac{1}{2}]$ is the tensor that corresponds to t_α via Fontaine comparison theory for either D' or D (cf. the canonical identification $H^1(D')[\frac{1}{2}] = H^1(D)[\frac{1}{2}]$ induced by the $B_{\text{dR}}(W(k))$ -linear isomorphism $i_{D'} \circ i_D^{-1}$).*

(b) *Let λ_{D_k} be the principal quasi-polarization of D_k which is the pullback of the principal quasi-polarization λ_D of D . Then we can assume that D' and $\varrho_{D'}$ are such that there exists a principal quasi-polarization $\lambda_{D'}$ of D' which lifts λ_{D_k} and whose étale realization is a perfect, alternating bilinear form $\psi_{H^1(D')}$ on $H^1(D')$ such that $\varrho_{D'}$ is in fact an isomorphism $\varrho_{D'} : (M, (t_\alpha)_{\alpha \in \mathcal{J}}, \psi_M) \rightarrow (H^1(D') \otimes_{\mathbb{Z}_2} W(k), (v_\alpha)_{\alpha \in \mathcal{J}}, \psi_{H^1(D')})$.*

(c) *If (ii) holds, then we moreover assume that \mathcal{G} is smooth. Then the number of D' 's (resp. of $(D', \lambda_{D'})$'s) for which part (a) (resp. (b)) holds is 2^a , where a is the multiplicity of the Newton polygon slope -1 for $(\text{Lie}(\mathcal{G})[\frac{1}{2}], \phi)$. Moreover, if we can take $D' = D$, then each other such D' is the pullback of the 2-divisible group D_R of Lemma 16 via a uniquely determined morphism $\text{Spec}(W(k)) \rightarrow \text{Spec}(R)$ that factors through the closed embedding $\text{Spec}(S) \hookrightarrow \text{Spec}(R)$ introduced in Subsubsection B4.2.*

(d) *We assume that (ii) holds and that $\phi(F^1) = 2F^1$. Then referring to part (a), as D' we can take the canonical lift of D_k .*

Proof: We prove part (a). We consider the direct sum decomposition

$$(M, \phi) = (M_0, \phi) \oplus (M_{>0}, \phi)$$

such that $\phi(M_0) = M_0$ and $\phi : M_{>0} \rightarrow M_{>0}$ is topologically nilpotent.

In this paragraph we check that there exists a cocharacter $\tilde{\mu} : \mathbb{G}_{m, W(k)} \rightarrow \mathcal{G}$ which normalizes the descending Newton polygon slope filtration of (M, ϕ)

(in particular, it normalizes $M_{>0}$) and which produces naturally a direct sum decomposition $M = \tilde{F}^1 \oplus \tilde{F}^0$ such that $\tilde{F}^1/2\tilde{F}^1 = F^1/2F^1$ (for each $i \in \{0, 1\}$), every $\beta \in \mathbb{G}_{m, W(k)}(W(k))$ acts through $\tilde{\mu}$ on \tilde{F}^i as the multiplication by β^{-i} ; this implies that we have $\tilde{F}^1 \subset M_{>0}$. If \mathcal{G} is a reductive group scheme over $W(k)$, then the existence of $\tilde{\mu}$ is a particular case of [61, Thm. 1.3.1 or Cor. 1.3.2 (a)]. If D_k is ordinary, then we have $\phi(M_{>0}) = 2M_{>0}$ and we can take $\tilde{F}^1 = M_{>0}$ and $\tilde{F}^0 = M_0$; the resulting cocharacter $\tilde{\mu} : \mathbb{G}_{m, W(k)} \rightarrow \mathbf{GL}_M$ fixes each t_α (as $\tilde{\mu}$ is the inverse of the Newton cocharacter of (M, ϕ) and as we have $\phi(t_\alpha) = t_\alpha$ for all $\alpha \in \mathcal{J}$), and therefore it factors through \mathcal{G} as desired.

Let $\tilde{D} = \tilde{D}_0 \oplus \tilde{D}_{>0}$ be the unique 2-divisible group over $W(k)$ such that the filtered F -crystals of \tilde{D}_0 and $\tilde{D}_{>0}$ are $(M_0, 0, \phi)$ and $(M_{>0}, \tilde{F}^1, \phi)$ (respectively), cf. [64, Prop. 2.2.6] for the uniqueness of $\tilde{D}_{>0}$. If D_k is ordinary, then \tilde{D} is the canonical lift of D_k . From Theorem 9 we get the existence of an isomorphism $\varrho_{\tilde{D}} : (M, (t_\alpha)_{\alpha \in \mathcal{J}}) \rightarrow (H^1(\tilde{D}) \otimes_{\mathbb{Z}_2} W(k), (\tilde{v}_\alpha)_{\alpha \in \mathcal{J}})$, where $\tilde{v}_\alpha \in \mathcal{T}(H^1(\tilde{D}))[\frac{1}{2}]$ corresponds to t_α via Fontaine comparison theory for \tilde{D} . Thus, if $\tilde{F}^1 = F^1$, then we can take $D' = \tilde{D}$.

In the general case (thus \tilde{F}^1 could now be different from F^1), we will use the deformation theory of Subsection B6 for \tilde{D} in order to prove that D' exists. If \mathcal{G} is a reductive group scheme, then we have $\mathcal{G}' = \mathcal{G}$. Let $R, \mathfrak{J}, M_R, \Phi, \nabla$ be as in Subsubsection B4.1. Let $\tilde{F}_R^1 := \tilde{F}^1 \otimes_{W(k)} R$. There exists a unique 2-divisible group \tilde{D}_R over R which modulo the ideal \mathfrak{J} is \tilde{D} and whose filtered F -crystal over $R/2R$ is $(M_R, \tilde{F}_R^1, \Phi, \nabla)$, cf. Lemma 16 applied to $(\tilde{D}, \tilde{F}_R^1)$ instead of (D, F_R^1) . Let $\tilde{\mathcal{C}}_{\text{univ}} := (M_R, \tilde{F}_R^1, \Phi, \nabla, (t_\alpha)_{\alpha \in \mathcal{J}})$ be the last filtered F -crystal endowed with the family $(t_\alpha)_{\alpha \in \mathcal{J}}$ of crystalline tensors. Let $z : \text{Spec}(W(k)) \rightarrow \text{Spec}(R)$ be the closed embedding defined by the ideal \mathfrak{J} of R . We have $z^*(\tilde{D}_R) = \tilde{D}$. We emphasize that the pullbacks of \tilde{D}_R and D_R to $\text{Spec}(R/2R)$ coincide, cf. [3, Thm. 4.1.1]. Thus a closed embedding $\text{Spec}(S) \hookrightarrow \text{Spec}(R)$ chosen as in Subsubsection B4.2 working with D_R works as well for \tilde{D}_R .

Let K be the field of fractions of R . From [64, Subsubsect. 3.4.2 and Lem. 3.4.3] we get that for each $\alpha \in \mathcal{J}$ there exists an étale Tate-cycle $\tilde{V}_\alpha \in \mathcal{T}(H^1(\tilde{D}_K))[\frac{1}{2}]$ on \tilde{D}_K which corresponds to t_α via Fontaine comparison theory for \tilde{D}_R . If $z_1 : \text{Spec}(W(k)) \rightarrow \text{Spec}(R)$ is a closed embedding, then the filtered F -crystal of $D_1 := z_1^*(\tilde{D}_R)$ is of the form (M, F_1^1, ϕ) for a suitable direct summand F_1^1 of M which lifts $F^1/2F^1$ and moreover to each t_α corresponds an étale Tate-cycle $v_{1,\alpha} \in \mathcal{T}(H^1(D_1))[\frac{1}{2}]$ on $D_{1, B(k)}$ in such a way that we have a canonical identification $(H^1(\tilde{D}_K), (\tilde{V}_\alpha)_{\alpha \in \mathcal{J}}) = (H^1(D_1), (v_{1,\alpha})_{\alpha \in \mathcal{J}})$

(see proof of [64, Lem. 3.4.3]).

Thus we have a canonical identification $(H^1(\tilde{D}), (\tilde{v}_\alpha)_{\alpha \in \mathcal{J}}) = (H^1(D_1), (v_{1,\alpha})_{\alpha \in \mathcal{J}})$. Therefore the existence of $\varrho_{\tilde{D}}$ implies the existence of an isomorphism

$$\varrho_{D_1} : (M, (t_\alpha)_{\alpha \in \mathcal{J}}) \rightarrow (H^1(D_1) \otimes_{\mathbb{Z}_2} W(k), (v_{1,\alpha})_{\alpha \in \mathcal{J}}).$$

Thus to end the proof of part (a) it suffices to show that we can choose z_1 such that we have $F_1^1 = F^1$ (and then we can take $D' = D_1$). Let $v \in 2\text{Lie}(U)$ be the unique element such that for $u := 1_M + v \in \text{Ker}(U(W(k)) \rightarrow U(k))$ we have $u(\tilde{F}^1) = F^1$, cf. Lemma 12. By denoting $z_{1,0} : \text{Spec}(k) \hookrightarrow \text{Spec}(R)$ the closed point of $\text{Spec}(R)$, by induction on $s \in \mathbb{N}^*$ we check that there exists a morphism $z_{1,s} : \text{Spec}(W_s(k)) \rightarrow \text{Spec}(R)$ which lifts $z_{1,s-1}$ and such that the Hodge filtration of $z_{1,s}^*(\tilde{D}_R)$ is the direct summand $F^1/2^s F^1$ of $M/2^s M$. We can take $z_{1,1} := z_{1,0}$. For $s \geq 2$, assuming that $z_{1,s-1}$ exists, the existence of the lift $z_{1,s}$ of $z_{1,s-1}$ is implied by the property (iii) of Subsubsection B4.1 and the relation $v \in 2\text{Lie}(U)$ (the arguments for these are the same as the ones of the proof of [63, Prop. 6.4.6 (b)] and rely on the fact that our field k is algebraically closed).

To prove part (b), we consider a direct sum decomposition

$$(M_{>0}, \phi) = (M_{(0,1)}, \phi) \oplus (M_1, \phi)$$

such that $\phi(M_1) = 2M_1$ and all Newton polygon slopes of $(M_{(0,1)}, \phi)$ belong to $(0, 1) \cap \mathbb{Q}$. As $\tilde{\mu}$ normalizes the descending Newton polygon slope filtration of (M, ϕ) , we have $M_1 \subset \tilde{F}^1$. Thus

$$(M_{>0}, \tilde{F}^1, \phi) = (M_{(0,1)}, M_{(0,1)} \cap \tilde{F}^1, \phi) \oplus (M_1, M_1, \phi)$$

and therefore we have a uniquely determined direct sum decomposition $\tilde{D}_{>0} = \tilde{D}_{(0,1)} \oplus \tilde{D}_1$: the filtered F -crystals over k of $\tilde{D}_{(0,1)}$ and \tilde{D}_1 are $(M_{(0,1)}, M_{(0,1)} \cap \tilde{F}^1, \phi)$ and (M_1, M_1, ϕ) (respectively).

As $\tilde{\mu}$ factors through \mathcal{G} and as \mathcal{G} normalizes the $W(k)$ -span of ψ_M , \tilde{F}^1 is a maximal isotropic direct summand of M with respect to ψ_M . Due to this and the uniqueness properties of $\tilde{D} = \tilde{D}_0 \oplus \tilde{D}_{(0,1)} \oplus \tilde{D}_1$, there exists a unique principal quasi-polarization $\lambda_{\tilde{D}}$ of \tilde{D} which lifts λ_{D_k} . The étale realization of $\lambda_{\tilde{D}}$ is a perfect, alternating bilinear form $\psi_{H^1(\tilde{D})}$ on $H^1(\tilde{D})$. We choose $\varrho_{\tilde{D}}$ such that we have an isomorphism $\varrho_{\tilde{D}} : (M, (t_\alpha)_{\alpha \in \mathcal{J}}, \psi_M) \rightarrow (H^1(\tilde{D}) \otimes_{\mathbb{Z}_2} W(k), (\tilde{v}_\alpha)_{\alpha \in \mathcal{J}}, \psi_{H^1(\tilde{D})})$, cf. Theorem 9. Let $\lambda_{\tilde{D}_R}$ be the unique principal quasi-polarization of \tilde{D}_R whose reduction modulo the ideal \mathfrak{J} is $\lambda_{\tilde{D}}$

and whose crystalline realization is the perfect, alternating bilinear form ψ_{M_R} on M_R , cf. Lemma 17 applied to $(\tilde{D}, \lambda_{\tilde{D}}, \tilde{F}_R^1)$ instead of (D, λ_D, F_R^1) .

The remaining part of the proof of part (b) is the same as of part (a). Briefly, it goes as follows. If $\tilde{F}^1 = F^1$, then we take $(D', \lambda_{D'}) = (\tilde{D}, \lambda_{\tilde{D}})$. If $\tilde{F}^1 \neq F^1$, then we have to consider the filtered principally quasi-polarized F -crystal (M, F_1^1, ϕ, ψ_M) of $(D_1, \lambda_{D_1}) := z_1^*(\tilde{D}_R, \lambda_{\tilde{D}_R})$ and the étale realizations $\psi_{H^1(\tilde{D}_K)}$ and $\psi_{H^1(D_1)}$ of $(\lambda_{\tilde{D}_R})_K$ and λ_{D_1} (respectively); as above one gets a canonical identification $(H^1(\tilde{D}), (\tilde{v}_\alpha)_{\alpha \in \mathcal{J}}, \psi_{H^1(\tilde{D})}) = (H^1(D_1), (v_{1,\alpha})_{\alpha \in \mathcal{J}}, \psi_{H^1(D_1)})$. If $z_1 : \text{Spec}(W(k)) \rightarrow \text{Spec}(R)$ is such that $F_1^1 = F^1$, then by taking $(D', \lambda_{D'}) = (D_1, \lambda_{D_1})$ we get that part (b) holds.

To prove part (c), based on the proof of part (b) it suffices to consider only the non-principally quasi-polarized case. To ease notation we can assume that D is one of the D' 's, cf. part (a). Thus there exists an isomorphism $\varrho_D : (M, (t_\alpha)_{\alpha \in \mathcal{J}}) \rightarrow (H^1(D) \otimes_{\mathbb{Z}_2} W(k), (v_\alpha)_{\alpha \in \mathcal{J}})$. We will consider two cases, the first one being only a particular case of the second (general) one.

Case 1. We assume that $F^1 = \tilde{F}^1$ and $D = \tilde{D}$ and thus also $D_R = \tilde{D}_R$. To the direct sum decomposition $D = \tilde{D} = \tilde{D}_0 \oplus \tilde{D}_{(0,1)} \oplus \tilde{D}_1$, corresponds a direct sum decomposition $H^1(D) = H^1(D)_0 \oplus H^1(D)_{(0,1)} \oplus H^1(D)_1$. If D' is a 2-divisible group for which part (a) holds, then we have short exact sequences $0 \rightarrow \tilde{D}_1 \rightarrow D' \rightarrow \tilde{D}_{(0,1)} \oplus \tilde{D}_0 \rightarrow 0$ and $0 \rightarrow \tilde{D}_1 \oplus \tilde{D}_{(0,1)} \rightarrow D' \rightarrow \tilde{D}_0 \rightarrow 0$ and $H^1(D')$ is a \mathbb{Z}_2 -submodule of $\frac{1}{2}H^1(D)$ that contains $2H^1(D)$ (as one can easily check based on [64, Prop. 2.2.6] and the proof of [64, Lem. 2.2.5]). We get the existence of an element $c \in \frac{1}{2}\text{Hom}(H^1(D)_1, H^1(D)_0)$ such that $H^1(D') = (1_M + c)(H^1(D))$; it is uniquely determined modulo $\text{Hom}(H^1(D)_1, H^1(D)_0)$. But as there exists an isomorphism $\varrho_{D'} : (M, (t_\alpha)_{\alpha \in \mathcal{J}}) \rightarrow (H^1(D') \otimes_{\mathbb{Z}_2} W(k), (v_\alpha)_{\alpha \in \mathcal{J}})$, there exists $g \in \mathcal{G}^{\text{ét}}(B(k))$ such that $H^1(D') \otimes_{\mathbb{Z}_2} W(k) = g(H^1(D) \otimes_{\mathbb{Z}_2} W(k))$.

We claim that we can assume that we have

$$c \in \frac{1}{2}[\text{Hom}(H^1(D)_1, H^1(D)_0) \cap \text{Lie}(\mathcal{G}_{\mathbb{Q}_2}^{\text{ét}})],$$

i.e., the image of c in the quotient group $\frac{1}{2}\text{Hom}(H^1(D)_1, H^1(D)_0)/\text{Hom}(H^1(D)_1, H^1(D)_0)$ belongs to the following subgroup

$$\frac{1}{2}[\text{Hom}(H^1(D)_1, H^1(D)_0) \cap \text{Lie}(\mathcal{G}_{\mathbb{Q}_2}^{\text{ét}})]/[\text{Hom}(H^1(D)_1, H^1(D)_0) \cap \text{Lie}(\mathcal{G}_{\mathbb{Q}_2}^{\text{ét}})].$$

This is only a variant of the Lemma 13 over \mathbb{Z}_2 instead of $W(k)$ which gets reduced to the Lemma 13 as follows. We can assume that ϱ_D maps $M_0, M_{(0,1)}$,

and M_1 onto $H^1(D)_0 \otimes_{\mathbb{Z}_2} W(k)$, $H^1(D)_{(0,1)} \otimes_{\mathbb{Z}_2} W(k)$, and $H^1(D)_1 \otimes_{\mathbb{Z}_2} W(k)$ (respectively), cf. Theorem 9. Thus we can assume that the element

$$\varrho_D g \varrho_D^{-1} \mathbf{GL}_M(W(k)) = \varrho_D (1_M + c) \varrho_D^{-1} \mathbf{GL}_M(W(k)) \in \mathbf{GL}_M(B(k)) / \mathbf{GL}_M(W(k))$$

belongs to the intersection of

$$[1_M + \frac{1}{2} \text{Lie}(U_{\text{bigg}})] / U_{\text{bigg}}(W(k)) \cap \mathcal{P}_0(B(k)) / \mathcal{P}_0(W(k))$$

taken inside $\mathbf{GL}_M(B(k)) / \mathbf{GL}_M(W(k))$ and thus it is an element of $[1_M + \frac{1}{2} \text{Lie}(U)] / U(W(k))$ (cf. Lemma 13). We note that if (ii) holds, then based on Theorem 9 we can assume that $\varrho_D g \varrho_D^{-1}$ fixes $W(k)$ -bases of M_0 and M/M_0 formed by elements fixed by ϕ and $p^{-1}\phi$ (respectively) and thus in fact in the above intersection we can replace $\mathcal{P}_0(B(k)) / \mathcal{P}_0(W(k))$ by $U(B(k)) / U(W(k))$.

This implies that there exists an element $c_{\text{crys}} \in \frac{1}{2}[\text{Hom}(M_1, M_0) \cap \text{Lie}(\mathcal{G})]$ such that $\varrho_D c \varrho_D^{-1} - c_{\text{crys}} \in \text{Hom}(M^1, M^0)$. Thus

$$c - \varrho_D^{-1} c_{\text{crys}} \varrho_D \in \text{Hom}(H^1(D)_1, H^1(D)_0) \otimes_{\mathbb{Z}_2} W(k)$$

and moreover

$$\varrho_D^{-1} c_{\text{crys}} \varrho_D \in \frac{1}{2}[\text{Hom}(H^1(D)_1, H^1(D)_0) \cap \text{Lie}(G_{\mathbb{Q}_2}^{\text{ét}})] \otimes_{\mathbb{Z}_2} W(k).$$

Therefore by replacing c with $\varrho_D^{-1} c_{\text{crys}} \varrho_D$, we get that the claim follows.

The group $\frac{1}{2}[\text{Hom}(H^1(D)_1, H^1(D)_0) \cap \text{Lie}(\mathcal{G}_{\mathbb{Q}_2}^{\text{ét}})] / [\text{Hom}(H^1(D)_1, H^1(D)_0) \cap \text{Lie}(\mathcal{G}_{\mathbb{Q}_2}^{\text{ét}})]$ has order 2^a . We conclude that the number of \mathbb{Z}_2 -lattices $H^1(D')$ of $H^1(D)_{[\frac{1}{2}]}$ such that the above properties hold (equivalently, the number of D' 's as in part (a)), is precisely 2^a . The fact that all of them are pullbacks of D_R via $W(k)$ -valued points of $\text{Spec}(R)$ follows from the fact that there exists a closed embedding $\text{Spec}(R_1) \hookrightarrow \text{Spec}(R)$ defined by an ideal of R contained in the ideal (x_1, \dots, x_l) and with $R_1 = W(k)[[x_1, \dots, x_a]]$, such that the restriction \tilde{D}_{R_1} of $\tilde{D}_R = D_R$ to $\text{Spec}(R_1)$ is a direct sum $\tilde{D}_{R_1} = \tilde{D}_{(0,1),R_1} \oplus \tilde{D}_{0,1,R_1}$, where $\tilde{D}_{0,1,R_1}$ sits in a short exact sequence $0 \rightarrow \tilde{D}_{1,R_1} \rightarrow \tilde{D}_{0,1,R_1} \rightarrow \tilde{D}_{0,R_1} \rightarrow 0$ which is a versal deformation of $\tilde{D}_1 \oplus \tilde{D}_0$ and which endows $\text{Spf}(R_1)$ with the structure of a formal subtorus of dimension a of the formal torus over $\text{Spf}(W(k))$ of deformations of the ordinary 2-divisible group $\tilde{D}_{1,k} \oplus \tilde{D}_{0,k}$ over k (here versal is used in the sense that the Kodaira–Spencer map is injective and has an image which is a direct summand of its codomain).

More precisely, if U_1 is the smooth, connected, closed subgroup scheme of U whose Lie algebra is $\text{Hom}(M_1, M_0) \cap \text{Lie}(U) = \text{Hom}(M_1, M_0) \cap \text{Lie}(\mathcal{G})$ (and thus has rank a), then the filtered F -crystal of \tilde{D}_{R_1} endowed with tensors is $\tilde{\mathfrak{D}}_1 := (M \otimes_{W(k)} R_1, \tilde{F}^1 \otimes_{W(k)} R_1, \Phi_1, \nabla, (t_\alpha)_{\alpha \in \mathcal{J}})$, where $\Phi_1 = u_1(\phi \otimes \Phi_{R_1})$ with Φ_{R_1} as in Subsubsection B4.1 for $m := a$ and with $u_1 \in U_1(R_1)$ a universal element which identifies R_1 with the completion of the local ring of U_1 at the identity element of $U_1(k)$. This is so as $\tilde{\mathfrak{D}}_1$ is the pullback of $\tilde{\mathfrak{C}}_{\text{univ}}$ via a $W(k)$ -morphism $\text{Spec}(R_1) \rightarrow \text{Spec}(R)$ which is a closed embedding and which at the level of rings maps the ideal (x_1, \dots, x_l) of R to the ideal (x_1, \dots, x_a) of R_1 , cf. Theorem 10 (a) and the fact that $\tilde{\mathfrak{D}}_1$ is versal. Each $\text{Spf}(W(k))$ -valued point of the formal torus $\text{Spf}(R_1)$ which is of order 1 or 2 corresponds uniquely to a D' as in part (a) and therefore indeed we have precisely 2^a such D' 's as in part (a) and all of them are pullbacks of D_R via $W(k)$ -valued points of $\text{Spec}(R)$. Thus part (c) holds if $D = \tilde{D}$. From the uniqueness part of Subsubsection B4.2 we get that we can assume that $\text{Spec}(R_1)$ is as well a closed subscheme of the closed subscheme $\text{Spec}(S)$ of $\text{Spec}(R)$ chosen in Subsubsection B4.2; therefore all D' 's as in part (a) are pullbacks of D_R via uniquely determined $W(k)$ -valued points of $\text{Spec}(R)$ that factor through $\text{Spec}(S)$.

Case 2. We check that part (c) holds in the general case (i.e., without assuming that $F^1 = \tilde{F}^1$ and $D = \tilde{D}$). Let \tilde{D}_S be the pullback of \tilde{D}_R constructed above via the closed embedding $\text{Spec}(S) \rightarrow \text{Spec}(R)$ of the Subsubsection B4.2. Let $\tilde{\mathfrak{D}}_{\text{univ}}$ be the pullback to $S/2S$ of $\tilde{\mathfrak{C}}_{\text{univ}}$. As in the proof of part (a) we argue that there exists a morphism $\tilde{z}' : \text{Spec}(W(k)) \rightarrow \text{Spec}(R)$ such that the Hodge filtration of M defined by $\tilde{D}' := (\tilde{z}')^*(\tilde{D}_R)$ is \tilde{F}^1 and there exists an isomorphism $\varrho_{\tilde{D}'} : (M, (t_\alpha)_{\alpha \in \mathcal{J}}) \rightarrow (H^1(\tilde{D}') \otimes_{\mathbb{Z}_2} W(k), (\tilde{v}_\alpha)_{\alpha \in \mathcal{J}})$. Let $\tilde{\mathfrak{J}}'$ be the ideal of R that defines \tilde{z}' . Let y_1, \dots, y_l be regular parameters of R such that we have an identity $\tilde{\mathfrak{J}}' = (y_1, \dots, y_l)$ between ideals of R . Let $\tilde{\Phi}_{R,1}$ be the Frobenius lift of $\text{Spec}(R)$ which is compatible with σ and takes each y_i to y_i^2 . Based on Case 1, we can assume that the morphism $\tilde{z}' : \text{Spec}(W(k)) \rightarrow \text{Spec}(R)$ factors through the closed embedding $\text{Spec}(S) \hookrightarrow \text{Spec}(R)$. Let $\tilde{z}'_S : \text{Spec}(W(k)) \rightarrow \text{Spec}(S)$ be the resulting factorization.

From Theorem 10 (a) we get that D_R and $\mathfrak{C}_{\text{univ}}$ are the pullbacks of \tilde{D}_S and $\tilde{\mathfrak{D}}_{\text{univ}}$ (respectively) via a morphism $h : \text{Spec}(R) \rightarrow \text{Spec}(S)$ that satisfies the identity $h \circ \tilde{z}' = \tilde{z}'_S$ (for this part we have to consider new Frobenius lifts of R and S ; like for R we would have to replace Φ_R by $\tilde{\Phi}_{R,1}$). Due to the

uniqueness part of Subsubsection B4.2 and the identity $h \circ \tilde{z}' = \tilde{z}'_S$, the closed embedding $\mathrm{Spec}(S) \hookrightarrow \mathrm{Spec}(R)$ is a section of $h : \mathrm{Spec}(R) \rightarrow \mathrm{Spec}(S)$.

Due to the existence of h , to prove part (c) in the general case it suffices to show that there exist exactly 2^a morphisms $z_1 : \mathrm{Spec}(W(k)) \rightarrow \mathrm{Spec}(S)$ such that the Hodge filtration of M defined by $z_1^*(\tilde{D}_S)$ is F^1 . Fixing such a morphism $z_{1,0}$ (it exists, cf. proof of part (a)), any other such morphism z_1 , induces a unique isomorphism $h_1 : \mathrm{Spec}(S) \rightarrow \mathrm{Spec}(S)$ with the properties that $\tilde{D}_S = h_1^*(\tilde{D}_S)$ and we have $h_1 \circ z_1 = z_{1,0}$. But the number of isomorphisms $h_2 : \mathrm{Spec}(S) \rightarrow \mathrm{Spec}(S)$ with the property that $\tilde{D}_S = h_2^*(\tilde{D}_S)$ is uniquely determined by the property that under it the ideal $\tilde{\mathfrak{I}}_0$ of S that defines \tilde{D} is mapped to one of the 2^a ideals of S under which one gets a 2-divisible group over $W(k)$ whose Hodge filtration is \tilde{F}^1 (cf. Case 1 applied to \tilde{D}). Thus we have 2^a such h_2 's and z_1 's and therefore part (c) holds in the general case.

Part (d) follows from Theorem 9. □

B.5 On abelian schemes

We assume that D is the Barsotti–Tate group of an abelian scheme A over $W(k)$. It is known that we have two canonical and functorial identifications:

- (i) $H_{\mathrm{dR}}^1(A/W(k)) = M$ of $W(k)$ -modules (see [2, Ch. V, Subsect. 2.3] and [4, Prop. 2.5.8]);
- (ii) $H^1(D) = H_{\acute{e}t}^1(A_{\overline{B(k)}}, \mathbb{Z}_p)$ of $\mathrm{Gal}(B(k))$ -modules.

The crystalline conjecture (see [19]) provides a $B_{\mathrm{crys}}(W(k))$ -linear isomorphism

$$i_A : H_{\mathrm{dR}}^1(A/W(k)) \otimes_{W(k)} B_{\mathrm{crys}}(W(k)) \rightarrow H_{\acute{e}t}^1(A_{\overline{B(k)}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\mathrm{crys}}(W(k))$$

that is compatible with the tensor product filtrations, with the $\mathrm{Gal}(B(k))$ -actions, and with the Frobenius endomorphisms. See [54, Subsubsection. 5.2.15] for a proof of the following property (strictly speaking, the paragraphs before loc. cit. work with a prime $p \geq 3$ but the arguments of loc. cit. work for all primes):

- (iii) under the identifications of (i) and (ii), we have $i_A = i_D^+ \otimes 1_{B_{\mathrm{crys}}(W(k))}$.

B.6 On Hodge cocharacters

In this subsection we assume that we have a monomorphism $W(k) \hookrightarrow \mathbb{C}$ and that D is the Barsotti–Tate group of an abelian scheme A over $W(k)$.

We recall that we have canonical identifications

$$(6) \quad M \otimes_{W(k)} \mathbb{C} = H_{\text{dR}}^1(A/W(k)) \otimes_{W(k)} \mathbb{C} = H_{\text{dR}}^1(A_{\mathbb{C}}/\mathbb{C}) = F^{1,0} \oplus F^{0,1},$$

where the last identity is the usual Hodge decomposition. Under (6) we can identify

$$F^1 \otimes_{W(k)} \mathbb{C} = F^{1,0}.$$

Let $A_{\mathbb{C}}^{\text{an}}$ be the complex manifold associated to $A_{\mathbb{C}}$. Let $W := H_1(A_{\mathbb{C}}^{\text{an}}, \mathbb{Q})$ be the first Betti homology group of $A_{\mathbb{C}}^{\text{an}}$ with rational coefficients. Let $W^{\vee} := \text{Hom}(W, \mathbb{Q})$. We identify naturally $W^{\vee} \otimes_{\mathbb{Q}} \mathbb{C}$ with the first Betti cohomology group $H^1(A_{\mathbb{C}}^{\text{an}}, \mathbb{C})$ and thus also with $H_{\text{dR}}^1(A_{\mathbb{C}}/\mathbb{C}) = M \otimes_{W(k)} \mathbb{C}$. Let $\mu_A : \mathbb{G}_{m, \mathbb{C}} \rightarrow \mathbf{GL}_{W^{\vee} \otimes_{\mathbb{Q}} \mathbb{C}}$ be the Hodge cocharacter that fixes $F^{0,1}$ and that acts on $F^{1,0}$ via the weight -1 .

Lemma 18. *Let the cocharacter $\mu : \mathbb{G}_{m, W(k)} \rightarrow \mathcal{G}$ be as in Subsection B2. We assume that for every $\alpha \in \mathcal{J}$ the tensor $t_{\alpha} \in \mathcal{T}(M)[\frac{1}{p}] = \mathcal{T}(H_{\text{dR}}^1(A/W(k)))[\frac{1}{p}]$ is the de Rham component of a Hodge cycle on $A_{B(k)}$. We also assume that $\mathcal{G}_{B(k)}$ is a reductive group. Then the cocharacter $\mu_A : \mathbb{G}_{m, \mathbb{C}} \rightarrow \mathbf{GL}_{M \otimes_{W(k)} \mathbb{C}}$ factors through $\mathcal{G}_{\mathbb{C}}$ and this factorization $\mu_A : \mathbb{G}_{m, \mathbb{C}} \rightarrow \mathcal{G}_{\mathbb{C}}$ is $\mathcal{G}(\mathbb{C})$ -conjugate to $\mu_{\mathbb{C}}$. Thus, if $\mathcal{G}_{B(k)}$ is a torus, then we have $\mu_A = \mu_{\mathbb{C}}$.*

Proof: Let $v_{\alpha}^B \in \mathcal{T}(W^{\vee})$ be the Betti realization of t_{α} ; it is fixed by μ_A . The identity $W^{\vee} \otimes_{\mathbb{Q}} \mathbb{C} = M \otimes_{W(k)} \mathbb{C}$ produces an identity $\mathcal{T}(W^{\vee} \otimes_{\mathbb{Q}} \mathbb{C}) = \mathcal{T}(M \otimes_{W(k)} \mathbb{C})$ under which the tensors t_{α} and v_{α}^B are as well identified. Thus the cocharacter $\mu_A : \mathbb{G}_{m, \mathbb{C}} \rightarrow \mathbf{GL}_{W^{\vee} \otimes_{\mathbb{Q}} \mathbb{C}}$ fixes t_{α} for all $\alpha \in \mathcal{J}$ and therefore it factors through $\mathcal{G}_{\mathbb{C}}$. Let $\mathcal{P}_{\mathbb{C}}$ be the parabolic subgroup of $\mathcal{G}_{\mathbb{C}}$ that normalizes $F^1 \otimes_{W(k)} \mathbb{C} = F^{1,0}$. Both the cocharacters $\mu_A : \mathbb{G}_{m, \mathbb{C}} \rightarrow \mathbf{GL}_{M \otimes_{W(k)} \mathbb{C}}$ and $\mu_{\mathbb{C}}$ factor through $\mathcal{P}_{\mathbb{C}}$ and thus a $\mathcal{P}_{\mathbb{C}}(\mathbb{C})$ -conjugate $\mu'_{\mathbb{C}}$ of $\mu_{\mathbb{C}}$ commutes with μ_A . As the commuting cocharacters $\mu'_{\mathbb{C}}$ and μ_A of $\mathcal{P}_{\mathbb{C}}$ act on $F^1 \otimes_{W(k)} \mathbb{C} = F^{1,0}$ and on $M \otimes_{W(k)} \mathbb{C} / (F^1 \otimes_{W(k)} \mathbb{C}) = H_{\text{dR}}^1(A_{\mathbb{C}}/\mathbb{C}) / F^{1,0}$ in the same way, we have $\mu'_{\mathbb{C}} = \mu_A$. Thus the cocharacters $\mu_{\mathbb{C}}$ and μ_A are $\mathcal{P}_{\mathbb{C}}(\mathbb{C})$ -conjugate and therefore they are also $\mathcal{G}(\mathbb{C})$ -conjugate. \square