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ON THE EXPONENTIAL DIOPHANTINE EQUATION $P_n^x + P_{n+1}^x + \cdots + P_{n+k-1}^x = P_m$

Florian Luca* — Euloge Tchammou** — Alain Togbé***, [c]

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ABSTRACT. In this paper, we find all the solutions of the title Diophantine equation in positive integers (m, n, k, x), where P_i is the i^{th} term of the Pell sequence.

1. Introduction

Let $(P_n)_{n\geq 0}$ be the Pell sequence given by

$$P_0 = 0,\ P_1 = 1 \quad \text{and} \quad P_{n+2} = 2P_{n+1} + P_n \qquad \text{for all } n \geq 0.$$

It is well-known that

$$P_n^2 + P_{n+1}^2 = P_{2n+1} \quad \text{holds for all } n \geq 0. \tag{1.1}$$

From this identity, we see that the equation

$$P_n^x + P_{n+1}^x + \cdots + P_{n+k-1}^x = P_m \tag{1.2}$$

has the solution $m = 2n + 1$ with $(x, k) = (2, 2)$, for all $n \geq 1$. We call this a trivial solution. Another trivial solution is given by $x = k = 1$ and $m = n$. We will ignore such solutions. We prove the following theorem.

Theorem 1.1. *The Diophantine equation* (1.2) *has only trivial solutions in positive integers* (m, n, k, x).

We use Baker's method to prove our main result.

2. Some properties of the Pell sequence

Let $\alpha = 1+\sqrt{2}$ and $\beta = 1-\sqrt{2}$ be the roots of the characteristic quadratic equation $x^2-2x-1 = 0$ of the Pell sequence $(P_n)_{n\geq 0}$. The Binet formula

$$P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}} \quad \text{holds for all } n \geq 0. \tag{2.1}$$

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[c] Corresponding author.

This easily implies that the inequalities

$$\alpha^{n-2} \leq P_n \leq \alpha^{n-1} \tag{2.2}$$

hold for all integers $n \geq 1$. It is easy to prove that

$$\frac{P_n}{P_{n+1}} \leq \frac{3}{7} \tag{2.3}$$

holds for all $n \geq 2$. It is also easy to check that the inequality

$$P_1 + \cdots + P_n < P_{n+1} \tag{2.4}$$

holds for all $n \geq 1$.

3. The small cases

We find it convenient to rule out the small cases here, namely the cases when $k \in \{1, 2\}$ and $x = 1$. We will later rule out the case of $x = 2$ in Section 7. Note that for $k = 1$, the equation becomes $P_n^x = P_m$, which has only trivial solutions (see [11: Theorem 1]).

If $k = 2$, then we get only trivial solutions by the main result in [12]. Thus, $k \geq 3$, so $n+k \geq 4$. Furthermore, from inequality (2.4), we have

$$P_{n+k} < P_n + P_{n+1} + \cdots + P_{n+k} < P_{n+k+1}.$$

Therefore, there is no non-trivial solution for $x = 1$.

4. Linear forms in logarithms

The proof of our main theorem uses lower bounds for linear forms in logarithms of algebraic numbers and a version of the Baker-Davenport reduction method. So let us recall some results. For any non-zero algebraic number η of degree d over $\mathbb{Q}$, whose minimal polynomial over $\mathbb{Z}$ is $a_0 \prod_{i=1}^{d} \left(X - \eta^{(i)}\right)$ (with $a_0 > 0$), we denote by

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log \max\left(1, \left|\eta^{(i)}\right|\right)\right)$$

the usual absolute logarithmic height of η. With this notation, Matveev proved the following theorem (see [9]).

Theorem 4.1. *Let $\alpha_1, \ldots, \alpha_r$ be real algebraic numbers and let $b_1, \ldots, b_r$ be nonzero integers. Let D be the degree of the number field $\mathbb{Q}(\alpha_1, \ldots, \alpha_r)$ over $\mathbb{Q}$ and let A_j be a positive real number satisfying*

$$A_j \geq \max\left\{Dh(\alpha_j),\ |\log \alpha_j|,\ 0.16\right\} \quad \text{for } j = 1, \ldots, r.$$

Assume that

$$B \geq \max\{|b_1|, \ldots, |b_r|\}.$$

If $\Lambda = \prod_{j=1}^{r} \alpha_j^{b_j} - 1 \neq 0$, then

$$\left|\prod_{j=1}^{r} \alpha_j^{b_j} - 1\right| \geq \exp\left(-1.4 \cdot 30^{r+3} \cdot r^{4.5} \cdot D^2 (1 + \log D)(1 + \log B) A_1 \ldots A_r\right).$$

When $r = 2$ and α_1, α_2 are positive and multiplicatively independent, we can use a result of Laurent, Mignotte and Nesterenko [7]. Namely, let in this case B_1, B_2 be real numbers larger than 1 such that

$$\log B_i \geq \max\left\{h(\alpha_i), \frac{|\log \alpha_i|}{D}, \frac{1}{D}\right\}, \qquad \text{for } i = 1, 2,$$

and put

$$b' := \frac{|b_1|}{D \log B_2} + \frac{|b_2|}{D \log B_1}.$$

Put

$$\Lambda := b_1 \log \alpha_1 + b_2 \log \alpha_2. \tag{4.1}$$

We note that $\Lambda \neq 0$ because α_1 and α_2 are multiplicatively independent. The following result is due to Laurent, Mignotte and Nesterenko ([7: Corollary 2, p. 288]).

Theorem 4.2 (Laurent, Mignotte, Nesterenko). *With the above notations, assuming that α_1, α_2 are positive and multiplicatively independent, then*

$$\log |\Lambda| > -24.34 D^4 \left(\max\left\{\log b' + 0.14, \frac{21}{D}, \frac{1}{2}\right\}\right)^2 \log B_1 \log B_2. \tag{4.2}$$

Note that $\Gamma = \mathrm{e}^{\Lambda} - 1$ in case $r = 2$, which explains the connection between Theorems 4.1 and 4.2.

5. Reduction method

In 1998, Dujella and Pethő in [6: Lemma 5(a)] gave a version of the reduction method based on the Baker-Davenport lemma [1]. We next present the following lemma from [4], which is an immediate variation of the result due to Dujella and Pethő [6], and will be one of the key tools used in this paper to reduce the upper bounds on x or m of the Diophantine equation (1.2).

Lemma 5.1. *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational γ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let*

$$\varepsilon = \|\mu q\| - M \cdot \|\gamma q\|,$$

where $\|\cdot\|$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no solution of the inequality

$$0 < |m\gamma - n + \mu| < AB^{-k}$$

in positive integers m, n and k with

$$m \leq M \qquad \text{and} \qquad k \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

The above lemma cannot be applied when $\mu = 0$ (since then $\varepsilon < 0$). In this case, we use the following classical result in the theory of Diophantine approximation, which is the well-known Legendre criterion (see [10: Theorem 8.2.4]).

Lemma 5.2 (Legendre). (i) *Let τ be real number and x, y integers such that*

$$\left|\tau - \frac{x}{y}\right| < \frac{1}{2y^2}. \tag{5.1}$$

Then $x/y = p_k/q_k$ is a convergent of τ. Furthermore,

$$\left|\tau - \frac{x}{y}\right| \geq \frac{1}{(a_{k+1} + 2)y^2}. \tag{5.2}$$

(ii) *If* x, y *are integers with* $y \geq 1$ *and*
$$|y\tau - x| < |q_k\tau - p_k|,$$
then $y \geq q_{k+1}$.

6. A inequality for m in terms of n, k, x

Recall that we are working on equation (1.2) and we are now assuming that $k \geq 3,\ x \geq 3$. Observe that
$$P_n^x + P_{n+1}^x + \cdots + P_{n+k-1}^x > P_{n+k-1}^x \geq \alpha^{(n+k-3)x},$$
where we used inequality (2.2). On the other hand, we have
$$\begin{aligned} P_n^x + P_{n+1}^x + \cdots + P_{n+k-1}^x &\leq (P_n + P_{n+1} + \cdots + P_{n+k-1})^x \\ &\leq (P_0 + P_1 + P_2 + \cdots + P_{n+k-1})^x < P_{n+k}^x \leq \alpha^{(n+k-1)x}, \end{aligned} \tag{6.1}$$
where we used the fact that inequality (2.4) holds for all $n \geq 1$. Thus,
$$\alpha^{(n+k-3)x} < P_n^x + P_{n+1}^x + \cdots + P_{n+k-1}^x < \alpha^{(n+k-1)x}$$
and
$$\alpha^{m-2} \leq P_m \leq \alpha^{m-1}.$$
Comparing the two bounds above, we get:
$$(n+k-3)x < m-1 < m \qquad \text{and} \qquad m-2 < (n+k-1)x,$$
so that
$$(n+k-3)x < m \leq (n+k-1)x + 1.$$
We record this as a lemma.

Lemma 6.1. *If* (m, n, k, x) *is any nontrivial solution of* (1.2) *in positive integers, then the inequalities*
$$(n+k-3)x < m \leq (n+k-1)x + 1$$
hold.

7. The case when $x = 2$

We consider the case $x = 2,\ k \geq 3$. In this case, equation (1.2) becomes
$$P_n^2 + P_{n+1}^2 + \cdots + P_{n+k-1}^2 = P_m.$$
By Lemma 6.1, we have $2(n+k-3) < m \leq 2(n+k-1)+1$. That is, $2(n+k)-5 \leq m \leq 2(n+k)-1$. However, since $P_{n+k-2}^2 + P_{n+k-1}^2 = P_{2n+2k-3}$, it follows that
$$\begin{aligned} P_m &= P_n^2 + \cdots + P_{n+k-2}^2 + P_{n+k-1}^2 \\ &\geq P_n^2 + P_{2n+2k-3} > P_{2n+2k-3}, \end{aligned}$$
so $m \geq 2n + 2k - 2$. If k is even then
$$\begin{aligned} P_m &\leq (P_n^2 + P_{n+1}^2) + \cdots + (P_{n+k-2}^2 + P_{n+k-1}^2) \\ &= P_{2n+1} + \cdots + P_{2n+2k-3} < P_{2n+2k-2}, \end{aligned}$$

so $m < 2n + 2k - 2$, which is a contradiction. The same conclusion holds when k is odd since in that case

$$\begin{aligned} P_m &\leq (P_{n-1}^2 + P_n^2) + \cdots + (P_{n+k-2}^2 + P_{n+k-1}^2) \\ &= P_{2n-1} + \cdots + P_{2n+2k-3} < P_{2n+2k-2}. \end{aligned}$$

8. Bounds on x, m in terms of $n + k$

Recall that $k \geq 3$, $x \geq 3$ and $n + k \geq 4$. Now, we rewrite equation (1.2) as

$$\frac{\alpha^m}{2\sqrt{2}} - P_{n+k-1}^x = P_n^x + P_{n+1}^x + \cdots + P_{n+k-2}^x + \frac{\beta^m}{2\sqrt{2}}.$$

So

$$\left| \frac{\alpha^m}{2\sqrt{2}} - P_{n+k-1}^x \right| \leq P_n^x + P_{n+1}^x + \cdots + P_{n+k-2}^x + \frac{|\beta|^m}{2\sqrt{2}}$$

and using again inequality (2.4), we have that

$$P_n^x + P_{n+1}^x + \cdots + P_{n+k-3}^x \leq (P_n + P_{n+1} + \cdots + P_{n+k-3})^x < P_{n+k-2}^x,$$

so that $P_n^x + P_{n+1}^x + \cdots + P_{n+k-2}^x < 2P_{n+k-2}^x$. Then

$$\left| \frac{\alpha^m}{2\sqrt{2}} - P_{n+k-1}^x \right| < 2P_{n+k-2}^x + \frac{|\beta|^m}{2\sqrt{2}} < 3P_{n+k-2}^x,$$

since $P_{n+k-2}^x \geq 1$, while $|\beta|^m / (2\sqrt{2}) < 1$. Dividing both sides of the inequality above by P_{n+k-1}^x and using the inequality (2.3), we obtain

$$\left| \alpha^m (2\sqrt{2})^{-1} P_{n+k-1}^{-x} - 1 \right| < 3 \left(\frac{P_{n+k-2}}{P_{n+k-1}} \right)^x < \frac{3}{2.3^x}. \tag{8.1}$$

Let

$$\Lambda_1 := \alpha^m (2\sqrt{2})^{-1} P_{n+k-1}^{-x} - 1, \tag{8.2}$$

which is the expression appearing under the absolute value of the left-hand side of inequality (8.1). Let us check that $\Lambda_1 \neq 0$. If $\Lambda_1 = 0$, then $\alpha^m = 2\sqrt{2} P_{n+k-1}^x$, which implies that $\alpha^{2m} = 8P_{n+k-1}^{2x} \in \mathbb{Z}$. This is a contradiction since no power of α in nonzero integer exponent can be an integer. Thus, $\Lambda_1 \neq 0$. We will use Matveev's theorem to get a lower bound for Λ_1. Put

$$r := 3, \quad \alpha_1 := \alpha, \quad \alpha_2 := 2\sqrt{2}, \quad \alpha_3 := P_{n+k-1}, \quad b_1 := m, \quad b_2 := -1,\ b_3 := -x.$$

Note that $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{Q}(\sqrt{2})$. Thus, we take $D := 2$. Since

$$h(\alpha_1) = (\log \alpha)/2, \quad h(\alpha_2) = (\log 8)/2 \quad \text{and} \quad h(\alpha_3) = \log P_{n+k-1} \leq (n + k - 2) \log \alpha,$$

we take

$$A_1 := \log \alpha, \qquad A_2 := \log 8, \qquad A_3 := 2(n + k - 2) \log \alpha.$$

Finally, Lemma 6.1 implies that $m > (n + k - 3)x \geq x$ since $n + k - 3 \geq 1$, so we take $B := m$. Hence, Matveev's theorem implies that

$$\log |\Lambda_1| \geq -1.4 \times 30^6 \times 3^{4.5} \times 2^2 \times (1 + \log 2)(\log \alpha)(\log 8) \times 2(n + k - 2) \log \alpha (1 + \log m).$$

So

$$\log |\Lambda_1| > -3.14 \times 10^{12} (n + k - 2)(1 + \log m). \tag{8.3}$$

Thus, inequalities (8.1) and (8.3) imply that

$$x < 3.77 \times 10^{12} (n + k - 2)(1 + \log m) < 6.04 \times 10^{12} (n + k - 2) \log m, \tag{8.4}$$

where we used the fact that $1 + \log m < 1.6 \log m$, for $m \geq 6$. Using again Lemma 6.1, we have that

$$\begin{aligned} m &< (n+k-1)x + 2 < 6.04 \times 10^{12}(n+k-2)(n+k-1)\log m + 2 \\ &< 6.04 \times 10^{12}(n+k)^2 \log m. \end{aligned}$$

So

$$\frac{m}{\log m} < 6.04 \times 10^{12}(n+k)^2. \tag{8.5}$$

Using the fact that for all $A \geq 3$

$$\frac{y}{\log y} < A \qquad \text{yields} \qquad y < 2A \log A,$$

with $A := 6.04 \times 10^{12}(n+k)^2$, $y := m$, we get that

$$\begin{aligned} m &< 2 \times 6.04 \times 10^{12}(n+k)^2 \log\left(6.04 \times 10^{12}(n+k)^2\right) \\ &< 1.21 \times 10^{13}(n+k)^2 \left(\log(6.04 \times 10^{12}) + 2\log(n+k)\right) \\ &< 1.21 \times 10^{13}(n+k)^2 \left(29.43 + 2\log(n+k)\right) \\ &< 2.82 \times 10^{14}(n+k)^2 \log(n+k). \end{aligned}$$

In the above chain of inequalities, we used the fact that $n + k \geq 4$, which implies that

$$29.43 + 2\log(n+k) < 23.3 \log(n+k).$$

Going back to inequality (8.4), we get that

$$\begin{aligned} x &< 6.04 \times 10^{12}(n+k-2) \log\left(2.82 \times 10^{14}(n+k)^2 \log(n+k)\right) \\ &= 6.04 \times 10^{12}(n+k-2)\left(\log(2.82 \times 10^{14}) + 2\log(n+k) + \log\log(n+k)\right) \\ &< 6.04 \times 10^{12}(n+k-2)\left(33.28 + 3\log(n+k)\right) \\ &< 6.04 \times 10^{12}(n+k-2)\left(27.1 \log(n+k)\right) \\ &< 1.64 \times 10^{14}(n+k-2)\log(n+k). \end{aligned}$$

In the above chain of inequalities, we used the fact that $\log\log(n+k) < \log(n+k)$ together with the fact that $33.28 + 3\log(n+k) < 27.1 \log(n+k)$ for $n + k \geq 4$. We record this as a lemma.

Lemma 8.1. *If (m, n, k, x) is any nontrivial solution in positive integers of equation (1.2) with $x \geq 3$, $k \geq 3$ and $n + k \geq 4$, then both inequalities*

$$x < 1.64 \times 10^{14}(n+k-2)\log(n+k); \qquad \textit{and} \qquad m < 2.82 \times 10^{14}(n+k)^2 \log(n+k)$$

hold.

9. The case of small $n + k$

Here, we assume that $4 \leq n + k \leq 90$. Then, by Lemma 8.1, we have

$$\begin{aligned} x &< 1.64 \times 10^{14}(n+k-2)\log(n+k) \\ &< 1.64 \times 10^{14} \times 88 \log 90 < 6.5 \times 10^{16}, \end{aligned}$$

and

$$\begin{aligned} m &< 2.82 \times 10^{14}(n+k)^2 \log(n+k) \\ &< 2.82 \times 10^{14} \times 90^2 \log 90 < 1.03 \times 10^{19}. \end{aligned}$$

We consider again the expression Λ_1 given by expression (8.2). Since $x \geq 3$, we have from (8.1) that

$$|\Lambda_1| < \frac{3}{2.3^3} < \frac{1}{4}.$$

By Lemma 6.1, we have $m \leq (n+k-1)x + 1 < (n+k)x \leq 90x$. We put

$$\Gamma_1 := m \log \alpha - \log(2\sqrt{2}) - x \log P_{n+k-1}.$$

Thus, $\Lambda_1 = \mathrm{e}^{\Gamma_1} - 1$. Since the inequality

$$|\Lambda_1| < \frac{1}{4} \qquad \text{implies} \qquad |\Gamma_1| < \frac{1}{2}$$

and since $|x| < 2\,|e^x - 1|$ holds for all $x \in [-1/2, 1/2]$, we get that

$$|\Gamma_1| < 2\left|\mathrm{e}^{\Gamma_1} - 1\right| = 2\,|\Lambda_1| < \frac{6}{2.3^x}.$$

So

$$\begin{aligned} 0 &< \left| m\Big(\frac{\log \alpha}{\log P_{n+k-1}}\Big) - x - \Big(\frac{\log(2\sqrt{2}}{\log P_{n+k-1}}\Big) \right| \\ &< \frac{6}{2.3^x \log P_{n+k-1}} < \frac{4}{2.3^x} < \frac{4}{(2.3^{1/90})^m}, \end{aligned} \tag{9.1}$$

where we used the fact that $\log P_{n+k-1} \geq \log P_3 \geq \log 5 > 1.5$. For us, inequality (9.1) is

$$0 < |m\gamma - x + \mu| < AB^{-m},$$

where

$$\gamma := \frac{\log \alpha}{\log P_{n+k-1}}, \qquad \mu := -\frac{\log(2\sqrt{2})}{\log P_{n+k-1}}, \qquad A := 4, \qquad B := 1.009 < 2.3^{1/90}.$$

We can take $M := 1.03 \times 10^{19}$. For the computations, if the first convergent p/q of γ such that $q > 6M$ does not satisfy the condition $\varepsilon > 0$, then we use the next convergent until we find the one that satisfies the condition. We do this for $n + k \in \{4, 5, \ldots, 90\}$. In all cases, we obtained $m \leq 18164$.

Next, since $(n + k - 3)x \leq m$, we have

$$x \leq m/(n + k - 3) < 18164/(n + k - 3).$$

A computer search with Maple revealed that there are no solutions to the equation (1.2) in the range $n + k \in \{4, 5, \ldots, 90\}$, $m \in [6, 18164]$ and $x \in [3, 18164/(n + k - 3)]$.

10. The bound on x

From now on, we suppose that $n + k \geq 91$.

Lemma 10.1. *If (k, n, m, x) is any nontrivial solution in positive integers of equation* (1.2) *with $k \geq 3$, $x \geq 3$, then $x \leq 5$.*

P r o o f. We suppose that $x \geq 6$ in order to get a contradiction. By Lemma 8.1, we have

$$\frac{x}{\alpha^{2(n+k-1)}} < \frac{1.64 \times 10^{14}(n + k - 2)\log(n + k)}{\alpha^{2(n+k-1)}} < \frac{1}{\alpha^{n+k}},$$

where we used the fact that the inequality $1.64 \times 10^{14}(n+k-2)\log(n+k) < \alpha^{n+k-2}$ holds for $n+k \geq 45$, which is the case for us. We now write

$$P_{n+k-1}^x = \frac{\alpha^{(n+k-1)x}}{8^{x/2}}\left(1 - \frac{(-1)^{n+k-1}}{\alpha^{2(n+k-1)}}\right)^x.$$

If $n+k-1$ is odd, then

$$\begin{aligned} 1 < \left(1 - \frac{(-1)^{n+k-1}}{\alpha^{2(n+k-1)}}\right)^x &= \left(1 + \frac{1}{\alpha^{2(n+k-1)}}\right)^x = \exp\left(x \log\left(1 + \frac{1}{\alpha^{2(n+k-1)}}\right)\right) \\ &< \exp\left(\frac{x}{\alpha^{2(n+k-1)}}\right) < \exp\left(\frac{1}{\alpha^{n+k}}\right) < 1 + \frac{2}{\alpha^{n+k}}, \end{aligned}$$

because $\dfrac{1}{\alpha^{n+k}} \leq \alpha^{-91}$ is very small while if $n+k-1$ is even, then

$$\begin{aligned} 1 > \left(1 - \frac{(-1)^{n+k-1}}{\alpha^{2(n+k-1)}}\right)^x &= \exp\left(x \log\left(1 - \frac{1}{\alpha^{2(n+k-1)}}\right)\right) \\ &> \exp\left(\frac{-x}{\alpha^{2(n+k-1)}}\right) > \exp\left(\frac{-1}{\alpha^{n+k}}\right) > 1 - \frac{2}{\alpha^{n+k}}, \end{aligned}$$

again because $\dfrac{1}{\alpha^{n+k}} \leq \alpha^{-91}$ is very small. Hence, we obtain

$$P_{n+k-1}^x = \frac{\alpha^{(n+k-1)x}}{8^{x/2}}\left(1 - \frac{(-1)^{n+k-1}}{\alpha^{2(n+k-1)}}\right)^x = \frac{\alpha^{(n+k-1)x}}{8^{x/2}}(1+\zeta), \qquad |\zeta| < \frac{1}{\alpha^{n+k}}.$$

In particular, $|\zeta| < 1/2$, so that

$$\alpha^{(n+k-1)x}/8^{x/2} \in ((2/3)P_{n+k-1}^x, 2P_{n+k-1}^x).$$

Thus,

$$\frac{\alpha^m}{8^{1/2}} - \frac{\alpha^{(n+k-1)x}}{8^{x/2}}(1+\zeta) = \frac{\beta^m}{8^{1/2}} + \left(\sum_{j=n}^{n+k-2} P_j^x\right).$$

Dividing across by $\alpha^{(n+k-1)x}/8^{x/2}$, we get

$$\left|\alpha^{m-(n+k-1)x} 8^{(x-1)/2} - 1\right| \leq |\zeta| + \frac{1}{8^{1/2}\alpha^m}\left(\frac{8^{x/2}}{\alpha^{(n+k-1)x}}\right) + \left(\frac{8^{x/2}}{\alpha^{(n+k-1)x}}\right)\left(\sum_{j=n}^{n+k-2} P_j^x\right).$$

We have $|\zeta| < 1/\alpha^{n+k}$. Since $\alpha^{(n+k-1)x}/8^{x/2} \in ((2/3)P_{n+k-1}^x, 2P_{n+k-1}^x)$, we also have

$$\frac{1}{8^{1/2}\alpha^m}\left(\frac{8^{x/2}}{\alpha^{(n+k-1)x}}\right) < \frac{3}{2 \cdot 8^{1/2}\alpha^m P_{n+k-1}^x} < \frac{1}{\alpha^{n+k}}.$$

Finally, since $P_\ell/P_{\ell+1} \leq 3/7$ for all $\ell \geq 2$, we have that

$$\begin{aligned} \left(\frac{8^{x/2}}{\alpha^{(n+k-1)x}}\right)\left(\sum_{j=n}^{n+k-2} P_j^x\right) &< \frac{3}{2}\left(\left(\frac{P_{n+k-2}}{P_{n+k-1}}\right)^x + \left(\frac{P_{n+k-3}}{P_{n+k-1}}\right)^x + \cdots + \left(\frac{P_n}{P_{n+k-1}}\right)^x\right) \\ &= \frac{3}{2}\left(\frac{P_{n+k-2}}{P_{n+k-1}}\right)^x\left(1 + \left(\frac{P_{n+k-3}}{P_{n+k-2}}\right)^x + \cdots + \left(\frac{P_n}{P_{n+k-2}}\right)^x\right) \\ &< \frac{1}{2.3^x}\left(2 + \left(\frac{3}{7}\right)^2 + \left(\frac{3}{7}\right)^4 + \dots\right) < \frac{2.23}{2.3^x}. \end{aligned}$$

Thus,

$$\left|\alpha^{m-(n+k-1)x} 8^{(x-1)/2} - 1\right| < \frac{2}{\alpha^{n+k}} + \frac{2.23}{2.3^x} < \frac{5}{2.3^{\min\{x,n+k\}}}.$$

Recall that $x \geq 6$. Then the above upper bound is smaller than $1/2$, so

$$|(m-(n+k-1)x)\log\alpha - (x-1)\log(2\sqrt{2})| < \frac{10}{2.3^{\min\{x,n+k\}}}. \tag{10.1}$$

The expression on the right is smaller than $1/2$, so $|m-(n+k-1)x| < 2x$. Here, we apply Theorem 4.2 with

$$r := 2, \qquad \alpha_1 := \alpha, \qquad \alpha_2 := 2\sqrt{2}, \qquad b_1 := m-(n+k-1)x, \qquad b_2 := x-1.$$

Again $\mathbb{K} = \mathbb{Q}(\sqrt{2})$ has $D = 2$. We take $\log B_1 := 1/2$, $\log B_2 := (\log 8)/2$. Thus,

$$b' = \frac{|m-(n+k-1)x|}{2\log B_2} + \frac{x-1}{2\log B_1} = \frac{|m-(n+k-1)x|}{\log 8} + x - 1 < 2x$$

since

$$\frac{|m-(n+k-1)x|}{\log 8} < \frac{2x}{\log 8} < x.$$

We thus get that

$$\begin{aligned}\log|\Lambda| &> -24.34 \times 2^4(1/2)(\log 8)/2\max\{\log(2x)+0.14, 10.5\}^2\\ &> -203(\max\{\log(2.5x), 10.5\})^2.\end{aligned}$$

Combining the above inequality with (10.1), we get

$$\min\{x, n+k\}\log(2.3) - \log(10) < 203(\max\{\log(2.5x), 10.5\})^2.$$

If the maximum in the right above is 10.5, then $\log(2.5x) \leq 10.5$ which leads to

$$x < 14{,}527. \tag{10.2}$$

Otherwise, we get

$$\min\{x, n+k\}\log(2.3) - \log(10) < 203(\log(2.5x))^2,$$

which leads to

$$\min\{x, n+k\} < 625(\log x)^2,$$

where we used the fact that

$$\log(2.5x) = \log(2.5) + \log x < 0.92 + \log x < 1.6\log x \qquad \text{for all } x \geq 6.$$

If

$$\min\{x, n+k\} = x,$$

we get $x < 625(\log x)^2$. This implies

$$x < 80{,}000. \tag{10.3}$$

Finally, it remains to consider the possibility

$$\min\{x, n+k\} = n+k.$$

In this case, we get $n+k < 625(\log x)^2$. So, by Lemma 8.1, we get that

$$\begin{aligned}n+k &< 625\left(\log(1.64\times 10^{14}(n+k-2)\log(n+k))\right)^2\\ &< 625\left(32.74 + 2\log(n+k)\right)^2\\ &< 625\left(32.74 + 2\log(n+k)\right)^2 < 56{,}406.3\left(\log(n+k)\right)^2,\end{aligned}$$

where we used the fact that $\log\log(n+k) < \log(n+k)$ and $\log(1.64\times 10^{14}) < 32.74$ and

$$32.74 + 2\log(n+k) < 9.5\log(n+k),$$

which holds for $n+k \geq 90$. Thus,

$$n+k < 1.6 \times 10^7.$$

So, again by Lemma 8.1, we get that

$$x < 1.64 \times 10^{14} \times 1.6 \times 10^7 \log(1.6 \times 10^7),$$

which gives

$$x < 4.4 \times 10^{22}. \tag{10.4}$$

In conclusion, from (10.2), (10.3) and (10.4), we have that inequality (10.4) always holds. We now return to inequality (10.1) and divide it across by $(\log\alpha)(x-1)$ to get

$$\left|\frac{\log(2\sqrt{2})}{\log\alpha} - \frac{(n+k-1)x-m}{x-1}\right| < \frac{10}{(\log\alpha)(x-1)2.3^{\min\{x,n+k\}}}. \tag{10.5}$$

Since $x \geq 6$, we have $2.3^x > (20/\log\alpha)(x-1)$. Furthermore, since $n+k \geq 91$, we have

$$\frac{2.3^{n+k}}{(20/\log\alpha)} \geq \frac{2.3^{91}}{(20/\log\alpha)} > 3.6 \times 10^{31} > x-1.$$

To summarize, the assumption $x \geq 6$ implies

$$\frac{2.3^{\min\{x,n+k\}}}{(10/\log\alpha)} > 2(x-1),$$

and therefore inequality (10.5) implies

$$\left|\frac{\log(2\sqrt{2})}{\log\alpha} - \frac{(n+k-1)x-m}{x-1}\right| < \frac{1}{2(x-1)^2}.$$

Thus, we can apply Lemma 5.2 to conclude that $((n+k-1)x-m)/(x-1) = p_t/q_t$ for some convergent p_t/q_t of $\tau := \log(2\sqrt{2})/\log\alpha$. The continued fraction of τ starts as

$$[1, 5, 1, 1, 3, 3, 1, 1, 7, 3, 1, 1, \ldots]$$

with the 46st convergent p_{46}/q_{46} satisfying $q_{46} > 2.2 \times 10^{23} > x$. Thus, by Lemma 5.2, we have

$$\begin{aligned} |(m-(n+k-1)x)\log\alpha - (x-1)\log(2\sqrt{2})| &\geq (\log\alpha)|m-(n+k-1)x-(x-1)\tau| \\ &> (\log\alpha)|p_{45} - q_{45}\tau| > 3.96 \times 10^{-24}, \end{aligned}$$

and now inequality (10.1) shows that

$$2.3^{\min\{x,n+k\}} < \frac{10 \times 10^{24}}{3.96} < 2.53 \times 10^{24}.$$

This gives $\min\{x, n+k\} \leq 67$, so $x \leq 67$, since $n+k \geq 91$. The sequence of convergents of τ is

$$1, \quad \frac{6}{5}, \quad \frac{7}{6}, \quad \frac{13}{11}, \quad \frac{46}{39}, \quad \frac{151}{128}, \quad \ldots.$$

The only convergents of the form p_t/q_t with q_t a divisor of $(x-1)$ and $x \in [6, 67]$ are the first 5 numbers above. Thus, $t \in \{0,1,2,3,4\}$. For each one of them, we get that $q_t \mid x-1$ so $x \geq q_t+1$. Thus, $x \geq \max\{6, q_t+1\}$. Now inequality (10.5) implies that

$$\left|\frac{\log(2\sqrt{2})}{\log\alpha} - \frac{p_t}{q_t}\right| < \frac{10}{(\log\alpha)\max\{5,q_t\}2.3^{\max\{6,q_t+1\}}}.$$

We checked that this last inequality fails for all $t \in \{0,1,2,3,4\}$. Thus, the assumption $x \geq 6$ is false, therefore $x \leq 5$ which is what we wanted. □

11. Bounding k

From now, we assume that $3 \le x \le 5$ and $n + k \ge 91$. We take l to be some number in $\{n, n+1, \ldots, n+k-1\}$ such that $l \ge 45$. For example, we can take $l = n + \lfloor k/2 \rfloor$ and then certainly $l \ge (n+k)/2 \ge 45$ since $n + k \ge 91$. Further, if say $k \le 46$, we can take $l = n = (n+k) - k \ge 91 - 46 = 45$. We make these choices more precise later. Let $j \in \{l+1, \ldots, n+k-1\}$. We have

$$\frac{x}{\alpha^{2j}} \le \frac{5}{\alpha^{2l+2}} < \frac{\alpha^2}{\alpha^{2l+2}} = \frac{1}{\alpha^{2l}},$$

We now write

$$P_j^x = \frac{\alpha^{jx}}{8^{x/2}} \left(1 - \frac{(-1)^j}{\alpha^{2j}}\right)^x.$$

If j is odd, then

$$\begin{aligned} 1 < \left(1 - \frac{(-1)^j}{\alpha^{2j}}\right)^x = \left(1 + \frac{1}{\alpha^{2j}}\right)^x &= \exp\left(x \log\left(1 + \frac{1}{\alpha^{2j}}\right)\right) \\ &< \exp\left(\frac{x}{\alpha^{2j}}\right) < \exp\left(\frac{1}{\alpha^{2l}}\right) \\ &< 1 + \frac{2}{\alpha^{2l}}, \end{aligned}$$

because $\dfrac{1}{\alpha^{2l}} \le \alpha^{-90}$ is very small. If j is even, then

$$\begin{aligned} 1 > \left(1 - \frac{(-1)^j}{\alpha^{2j}}\right)^x &= \exp\left(x \log\left(1 - \frac{1}{\alpha^{2j}}\right)\right) \\ &> \exp\left(\frac{-2x}{\alpha^{2j}}\right) > \exp\left(\frac{-2}{\alpha^{2l}}\right) \\ &> 1 - \frac{2}{\alpha^{2l}}, \end{aligned}$$

again because $\dfrac{1}{\alpha^{2l}} \le \alpha^{-90}$. So, we have that

$$\left|P_j^x - \frac{\alpha^{jx}}{8^{x/2}}\right| = \frac{\alpha^{jx}}{8^{x/2}} \left|\left(1 - \frac{(-1)^j}{\alpha^{2j}}\right)^x - 1\right| < \frac{\alpha^{jx}}{8^{x/2}} \left(\frac{2}{\alpha^{2l}}\right).$$

We now return to our equation (1.2) and rewrite it as

$$\frac{\alpha^m - \beta^m}{2\sqrt{2}} = P_m = P_n^x + P_{n+1}^x + \cdots + P_l^x + P_{l+1}^x + \cdots + P_{n+k-1}^x,$$

and furthermore

$$\frac{\alpha^m - \beta^m}{2\sqrt{2}} = P_n^x + P_{n+1}^x + \cdots + P_l^x + \sum_{j=l+1}^{n+k-1} \frac{\alpha^{jx}}{8^{x/2}} + \sum_{j=l+1}^{n+k-1} \left(P_j^x - \frac{\alpha^{jx}}{8^{x/2}}\right).$$

Thus,

$$\begin{aligned} &\left|\frac{\alpha^m}{8^{1/2}} - \frac{\alpha^{(l+1)x}}{8^{x/2}} \sum_{i=0}^{n+k-l-2} \alpha^{ix}\right| \\ &= \left|\frac{\beta^m}{8^{1/2}} + \sum_{j=l+1}^{n+k-1} \left(P_j^x - \frac{\alpha^{jx}}{8^{x/2}}\right) + P_n^x + P_{n+1}^x + \cdots + P_l^x\right| \end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{\alpha^m} + \sum_{j=l+1}^{n+k-1} \left| P_j^x - \frac{\alpha^{jx}}{8^{x/2}} \right| + P_n^x + P_{n+1}^x + \cdots + P_l^x \\
&< \frac{1}{\alpha^m} + \sum_{j=l+1}^{n+k-1} \left| P_j^x - \frac{\alpha^{jx}}{8^{x/2}} \right| + P_l^x \left(1 + \left(\frac{P_{l-1}}{P_l} \right)^x + \left(\frac{P_{l-2}}{P_l} \right)^x + \dots \right) \\
&< \frac{1}{\alpha^m} + \sum_{j=l+1}^{n+k-1} \frac{\alpha^{jx}}{8^{x/2}} \left(\frac{2}{\alpha^{2l}} \right) + P_l^x \left(2 + \frac{1}{3} + \frac{1}{3^2} + \dots \right) \\
&= \frac{1}{\alpha^m} + \frac{\alpha^{(l+1)x}}{8^{x/2}} \left(\frac{2}{\alpha^{2l}} \right) \sum_{i=0}^{n+k-l-2} \alpha^{ix} + 2.5\alpha^{(l-1)x}. \qquad (11.1)
\end{aligned}
$$

In the above chain of inequalities, we used the facts that $P_i/P_{i+1} \le 3/7 < 1/2.3$ for $i \ge 2$, the fact that $2.3^x > 3$ since $x \ge 2$ and $P_l^x < \alpha^{(l-1)x}$. Multiplying both sides of the above inequality (11.1) by $\alpha^{-(n+k-1)x} 8^{x/2}$, we obtain

$$
\begin{aligned}
&\left| \alpha^{m-(n+k-1)x} 8^{(x-1)/2} - \sum_{i=0}^{n+k-l-2} \alpha^{-ix} \right| \\
&\le \frac{8^{x/2}}{\alpha^{m+(n+k-1)x}} + \frac{2}{\alpha^{2l}} \sum_{i=0}^{n+k-l-2} \alpha^{-ix} + \frac{2.5 \times 8^{x/2}}{\alpha^{(n+k-l)x}} \\
&< \frac{1}{\alpha^{(n+k-1)x}} + \frac{2}{\alpha^{2l}} \sum_{i=0}^{n+k-l-2} \alpha^{-ix} + \frac{2.5}{\alpha^{(n+k-l-1.2)x}} \qquad (11.2) \\
&< \frac{3.5}{\alpha^{(n+k-l-1.2)x}} + \frac{2}{\alpha^{2l}} \sum_{i \ge 0} \frac{1}{3^i} \\
&< \frac{3.5}{\alpha^{(n+k-l-1.2)x}} + \frac{3}{\alpha^{2l}} \\
&< \frac{6.5}{\alpha^{\min\{(n+k-l-1.2)x, 2l\}}}.
\end{aligned}
$$

In the above, we used the fact that $m > (n+k-3)x \ge 88x$ (see Lemma 6.1), so $\alpha^m > \alpha^{88x} > 8^{x/2}$, the fact that $\alpha^x \ge \alpha^3 > 3$, the fact that

$$
\sum_{i \ge 0} 3^{-i} = \frac{3}{2},
$$

as well as the fact that $\sqrt{8} < \alpha^{1.2}$. On the other hand, one has

$$
\begin{aligned}
\sum_{i=0}^{n+k-l-2} \alpha^{-ix} &= \sum_{i \ge 0} \frac{1}{\alpha^{ix}} - \sum_{i \ge n+k-l-1} \frac{1}{\alpha^{ix}} \\
&= \frac{1}{1 - 1/\alpha^x} - \frac{1}{\alpha^{(n+k-l-1)x}} \left(1 + \frac{1}{\alpha^x} + \frac{1}{\alpha^{2x}} + \dots \right) \\
&= \frac{\alpha^x}{\alpha^x - 1} + \eta,
\end{aligned}
$$

where

$$
|\eta| < \frac{1}{\alpha^{(n+k-l-1)x}} \left(1 + \frac{1}{3} + \frac{1}{3^2} + \dots \right) < \frac{1.5}{\alpha^{(n+k-l-1)x}}.
$$

Hence, we obtain

$$\begin{aligned}
&\left|\alpha^{m-(n+k-1)x}8^{(x-1)/2} - \frac{\alpha^x}{\alpha^x - 1}\right| \\
&< \frac{6.5}{\alpha^{\min\{(n+k-l-1.2)x,2l\}}} + |\eta| \\
&< \frac{6.5}{\alpha^{\min\{(n+k-l-1.2)x,2l\}}} + \frac{1.5}{\alpha^{(n+k-l-1)x}} \\
&< \frac{8}{\alpha^{\min\{(n+k-l-1.2)x,2l\}}}.
\end{aligned} \tag{11.3}$$

We want to show that $(n+k-l-1.2)x \le 6$. Suppose that $(n+k-l-1.2)x > 6$. Since $2l \ge 90$, inequality (11.3) certainly implies that

$$\begin{aligned}
\left|\alpha^{m-(n+k-1)x}8^{(x-1)/2} - 1\right| &< \frac{8}{\alpha^6} + \frac{\alpha^x}{\alpha^x - 1} - 1 \\
&= \frac{8}{\alpha^6} + \frac{1}{\alpha^x - 1} < \frac{1}{4},
\end{aligned} \tag{11.4}$$

since $x \ge 3$. So

$$|(m - (n+k-1)x)\log\alpha - (x-1)\log(2\sqrt{2})| < \frac{1}{2}.$$

Hence, we have that $|m - (n+k-1)x| < 2x$. We now take $l := n + \lfloor k/2 \rfloor$. Note that $2l \ge 90$ since $l \ge 45$. We then get

$$\left|\alpha^{m-(n+k-1)x}8^{(x-1)/2} - \frac{\alpha^x}{\alpha^x - 1}\right| < \frac{8}{\alpha^{\min\{(k-\lfloor k/2 \rfloor-1.2)x,90\}}}.$$

We checked that for $x \in [3,5]$, there is no integer $t := m - (n+k-1)x$, $t \in (-2x, 2x)$ such that

$$\left|\alpha^t 8^{(x-1)/2} - \frac{\alpha^x}{\alpha^x - 1}\right| < \frac{8}{\alpha^6}.$$

The way we checked that was to check numerically that for every x in our range and for all $t \in [-2x+1, 2x-1]$, the minimum of $\left|\alpha^t 8^{(x-1)/2} - \frac{\alpha^x}{\alpha^x-1}\right|$ is $> 0.23 > \frac{8}{\alpha^6}$, which certainly shows that such t cannot exit. This shows that $(k - \lfloor k/2 \rfloor - 1.2)x \le 6$. Since $x \ge 3$, this shows that $k - \lfloor k/2 \rfloor - 1.2 \le 2$, so $k - \lfloor k/2 \rfloor \le 3.4$, showing that $k \le 6$. We now take $l = n$. Then $l = (n+k) - k \ge 91 - 6 > 45$, so this choice of l is also valid. In this case, we get again that $n + k - l - 1.2 = k - 1.2 > 0$, so inequality (11.3) becomes

$$\left|\alpha^{m-(n+k-1)x}8^{(x-1)/2} - \frac{\alpha^x}{\alpha^x - 1}\right| < \frac{8}{\alpha^{\min\{(k-1.2)x,90\}}}.$$

The preceding argument shows that $(k - 1.2)x \le 6$ and since $x \ge 3$, we get $k \le 3$. Then $k = 3$, since $k \ge 3$. Let us record what we have proved.

Lemma 11.1. *If (k, n, m, x) is any nontrivial solution of equation (1.2) in positive integers with $k \ge 3$, $x \ge 3$, then $k = 3$.*

12. The final contradiction

To finish, we take $k = 3$, $x \in [3,5]$. Lemma 6.1 shows that

$$t := m - (n+k-1)x \in (-2x, 1],$$

so $t \in [-9,1]$. Put $X := \alpha^n$. Then $\beta^n = \varepsilon X^{-1}$, where $\varepsilon = (-1)^n \in \{\pm 1\}$ according to whether n is even or odd. Now, by the Binet formulas, equation (1.2) becomes

$$\frac{\alpha^{2x+t}X^x - \beta^{2x+t}\varepsilon^x X^{-x}}{2\sqrt{2}} = P_m = \sum_{j=0}^{2} P_{n+j}^x = \sum_{j=0}^{2}\left(\frac{\alpha^j X - \beta^j \varepsilon X^{-1}}{2\sqrt{2}}\right)^x$$

or

$$\frac{\alpha^{2x+t}X^{2x} - \beta^{2x+t}\varepsilon^x}{2\sqrt{2}} = \sum_{j=0}^{2}\left(\frac{\alpha^j X^2 - \beta^j \varepsilon}{2\sqrt{2}}\right)^x. \tag{12.1}$$

For fixed $x \in [3,5]$, $t \in [-9,1]$ and $\varepsilon \in \{\pm 1\}$, this reduces to a polynomial equation in X of degree at most $2x$. We could compute all these equations and all their roots positive real roots X and check that these roots are not of the form α^n for some positive integer n; that is, that $\log X/\log\alpha$ is not a positive integer n. Instead of doing that, we will prove directly that the roots X of equation (12.1) cannot be as large as $X = \alpha^n$ for some $n = (n+k) - k \geq 91 - 3 = 88$.

Well, assume that this is so for a contradiction. The general term of the sum the right hand side is

$$\left(\frac{\alpha^j X^2 - \beta^j\varepsilon}{2\sqrt{2}}\right)^x = \frac{\alpha^{jx}X^{2x}}{8^{x/2}}\left(1 - \frac{\varepsilon\beta^j}{\alpha^j X^2}\right)^x = \frac{\alpha^{jx}X^{2x}}{8^{x/2}}(1+\varepsilon_j)^x, \qquad \varepsilon_j := -\frac{\varepsilon\beta^j}{\alpha^j X^2}.$$

Note that

$$|\varepsilon_j| = \frac{1}{\alpha^{2j}X^2} < 10^{-60}$$

is very small since $X = \alpha^n \geq \alpha^{88} > 10^{33}$ is very large. By an argument used before, since $x \leq 5$ is small, we have

$$(1+\varepsilon_j)^x = 1 + \varepsilon_{x,j}, \qquad \text{where} \qquad |\varepsilon_{x,j}| < 2x|\varepsilon_j| = \frac{2x}{\alpha^{2j}X^2}.$$

Thus,

$$\left(\frac{\alpha^{2x+t}}{2\sqrt{2}} - \sum_{j=0}^{2}\frac{\alpha^{jx}}{8^{x/2}}\right)X^{2x} - \frac{\beta^{2x+t}\varepsilon^x}{2\sqrt{2}} = \sum_{j=0}^{2}\varepsilon_{j,x}\frac{\alpha^{jx}}{8^{x/2}}X^{2x}.$$

We move the second term in the left on the right hand side and take absolute values to get that

$$\begin{aligned}
\left|\frac{\alpha^{2x+t}}{2\sqrt{2}} - \sum_{j=0}^{2}\frac{\alpha^{jx}}{8^{x/2}}\right|X^{2x} &= \left|\frac{\beta^{2x+t}\varepsilon^x}{2\sqrt{2}} + \sum_{j=0}^{2}\varepsilon_{j,x}\frac{\alpha^{jx}}{8^{x/2}}X^{2x}\right| \\
&\leq \frac{\alpha^3}{8^{1/2}} + \sum_{j=0}^{2}|\varepsilon_{j,x}|\frac{\alpha^{jx}}{8^{x/2}}X^{2x} \\
&< \alpha^2 + \frac{2x}{8^{x/2}}\sum_{j=0}^{2}\alpha^{j(x-2)}X^{2x-2} \\
&< \alpha^2 + \frac{6x}{8^{x/2}}\alpha^{2(x-2)}X^{2x-2} \\
&\leq \alpha^2 + \frac{6\times 5}{8^{3/2}}\alpha^{2\times 3}X^{2x-2} \\
&< \alpha^2 + 2\alpha^6 X^{2x-2} \\
&< 3\alpha^6 X^{2x-2}.
\end{aligned}$$

In the above chain of inequalities we used the fact that $t \geq -9$, therefore

$$2x + t \geq 2\times 3 - 9 = -3,$$

and at the end we used that $X^{2x-2} > X > 10^{33} > \alpha^2$. The minimum of the expression

$$\left|\frac{\alpha^{2x+t}}{2\sqrt{2}} - \sum_{j=0}^{2} \frac{\alpha^{jx}}{8^{x/2}}\right|$$

for $x \in [3,5]$, $t \in [-9,1]$ is > 0.06. Thus, we get

$$0.06X^{2x} < 3\alpha^6 X^{2x-2},$$

which gives

$$X^2 < \frac{3\alpha^6}{0.06} = 50\alpha^6 < \alpha^{5+6} = \alpha^{11},$$

so $X < \alpha^{5.5}$, a contradiction. Thus, there are no solutions with $n + k \geq 91$, and this completes the proof of Theorem 1.1.

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* *School of Mathematics*
University of the Witwatersrand
Private Bag X3
Wits 2050
SOUTH AFRICA

Research Group in Algebraic Structures and Applications
King Abdulaziz University
Jeddah
SAUDI ARABIA

Max Planck Institute for Mathematics
Bonn
GERMANY

Centro de Ciencias Matemáticas UNAM
Morelia
MEXICO

E-mail: florian.luca@wits.ac.za

** *Institut de Mathématiques et de Sciences Physiques*
Dangbo
BÉNIN

E-mail: tchammoue@yahoo.fr

*** *Department of Mathematics*
Statistics and Computer Science
Purdue University Northwest
1401 S, U.S. 421, Westville
IN 46391
USA

E-mail: atogbe@pnw.edu