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Moduli spaces of symmetric cubic fourfolds and locally symmetric varieties

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We realize the moduli spaces of cubic fourfolds with specified group actions as arithmetic quotients of complex hyperbolic balls or type IV symmetric domains, and study their compactifications. We prove the geometric (GIT) compactifications are naturally isomorphic to the Hodge theoretic (Looijenga, in many cases Baily–Borel) compactifications. The key ingredients of the proof are the global Torelli theorem by Voisin, the characterization of the image of the period map given by Looijenga and Laza independently, and the functoriality of Looijenga compactifications proved in the Appendix.

A list of symbols can be found on page 2680.

1. Introduction

Cubic fourfolds are intensively studied objects in algebraic geometry. There are many interesting relations and analogues between cubic fourfolds and $K3$ surfaces. The Hodge structure on the primitive middle cohomology $H_0^4(X)$ of a smooth cubic fourfold X is of $K3$ type. On the other hand, the Fano scheme of lines on a smooth cubic fourfold is a hyper-Kähler fourfold of $K3^{[2]}$ type; see [Beauville and Donagi 1985]. Similar to $K3$ surfaces, people have a good understanding of the period map for cubic fourfolds. The period map $\mathscr{P}$ gives an algebraic map from the moduli of smooth cubic fourfolds to an arithmetic quotient of a 20-dimensional type IV domain. This period map is an open embedding due to the global Torelli theorem by Voisin [1986]. The image of the period map is the complement of certain hypersurface arrangement. This was proved by Looijenga [2009] and Laza [2010] independently.

Zarhin [1983] classified the Mumford–Tate groups of $K3$-type Hodge structures. The corresponding Mumford–Tate domains are either complex hyperbolic balls or type IV domains. Examples of those Mumford–Tate groups can arise when the Hodge structures admit extra symmetries. This leads us to study moduli spaces of cubic fourfolds with specified group actions. For cubic fourfolds, any automorphisms are induced from linear automorphisms of $\mathbb{P}^5$. This is a general fact for almost all hypersurfaces in projective spaces with degree at least 3; see [Matsumura and Monsky 1963]. Moreover, in [Zheng 2019], the second author checked that any automorphism of the polarized Hodge structure on the middle cohomology of a smooth cubic fourfold is induced by a unique automorphism of the cubic fourfold. Therefore, the symmetries of polarized Hodge structures for cubic fourfolds can be detected geometrically by linear

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symmetries. These facts give rise to identifications between moduli spaces constructed by GIT and arithmetic quotients of complex hyperbolic balls or type IV domains. We next review two such examples.

One example regarding the complex hyperbolic ball is given by Looijenga and Swierstra [2007] and Allcock, Carlson and Toledo [2011] independently on the moduli space of cubic threefolds. They attach to cubic threefolds the Hodge structures of cubic fourfolds with specified automorphism with order 3. Explicitly, suppose the cubic threefold is given by a polynomial $F(x_1, \dots, x_5)$, then the corresponding symmetric cubic fourfold we looking at is $x_0^3 + F(x_1, \dots, x_5) = 0$. Via this construction, the moduli space of cubic threefolds with at worst ADE singularities is identified with the complement of an irreducible totally geodesic hypersurface in an arithmetic quotient of a complex hyperbolic ball of dimension 10. The phenomena that the image of a period map is the complement of some totally geodesic hypersurfaces in a locally symmetric variety appear in many examples besides cubic fourfolds and cubic threefolds. In fact, type IV domains and complex hyperbolic balls are the only irreducible Hermitian symmetric domains admitting totally geodesic hypersurfaces. Coming back to cubic threefolds, on the geometric side, we have the natural GIT compactification of the moduli space of cubic threefolds. On the Hodge theoretic side, there is a natural compactification of the complement of the hypersurface arrangements, building upon Baily–Borel compactification. The construction of this Hodge-theoretic compactification was carried out by Looijenga [2003a], inspired by work of Shah, consisting of two steps. The first step is a partial blowup of the boundary components of Baily–Borel compactification, sitting between toroidal compactification and Baily–Borel compactification. The second step is a successive blowup of the intersection strata of hyperplane arrangements and blowdown in the opposite direction; see the Appendix for a discussion of Looijenga compactification. The fascinating result proved in [Looijenga and Swierstra 2007; Allcock et al. 2011] is the existence of a natural isomorphism between the GIT compactification and Looijenga compactification, which are from totally different origins.

Another example regarding type IV domain was given by Laza, Pearlstein and Zhang [Laza et al. 2018] recently. They considered the moduli space of pairs consisting of a cubic threefold $F(x_1, \dots, x_5) = 0$ and a hyperplane section $H(x_1, \dots, x_5) = 0$, or equivalently the moduli of cubic fourfolds

$$x_0^2 H(x_1, \dots, x_5) + F(x_1, \dots, x_5) = 0$$

which have natural involutions $x_0 \mapsto -x_0$. The period map gives rise to an identification between the moduli space of the pairs and an arrangement complement in an arithmetic quotient of a type IV domain of dimension 14. Moreover, Laza, Pearlstein and Zhang showed that with a careful choice of linearization (which is indeed natural as we will discuss in Proposition 6.8) in the GIT construction, the pairs which give rise to symmetric cubic fourfolds with at worst ADE singularities are stable, and their moduli can be identified with the whole arithmetic quotient. Finally, they showed that the GIT compactification is isomorphic to the Baily–Borel compactification of the arithmetic quotient.

The first key observation of this work is that the phenomena in the above two examples should also appear in a much more general situation, namely, for cubic fourfolds with any given symmetry. Along this direction, we are able to unify many examples studied before (including the two above), and produce many new identifications between GIT compactifications and Hodge-theoretic compactifications. Before giving the main theorems, we introduce some notation.

For a smooth cubic fourfold X, we have the Hodge decomposition

$$H^4(X,\mathbb{C}) = H^{3,1}(X) \oplus H^{2,2}(X) \oplus H^{1,3}(X),$$

where $\dim(H^{3,1}) = \dim(H^{1,3}) = 1$, and $\dim H^{2,2} = 21$. We denote by $\varphi_X : H^4(X,\mathbb{C}) \times H^4(X,\mathbb{C}) \to \mathbb{C}$ the topological intersection pairing, whose restriction to $H^4(X,\mathbb{Z}) \times H^4(X,\mathbb{Z})$ is an integral unimodular bilinear form of signature $(21, 2)$.

Let X be a smooth cubic fourfold with an action of a finite group A. All deformations of the pair (X, A) form a quasiprojective variety $\mathcal{F}$, which is called the moduli space of smooth cubic fourfolds with this given group action. See Section 2 for a GIT construction of $\mathcal{F}$. There is a natural morphism (via forgetting the action of A) from $\mathcal{F}$ to the moduli space $\mathcal{M}$ of smooth cubic fourfolds, which is a finite morphism (see Proposition 2.8). Moreover, when the action of A realizes all the automorphisms of X, the morphism $j : \mathcal{F} \to \mathcal{M}$ is a normalization of its image.

On the other hand, we look at the induced action of A on the Hodge structure of the cubic fourfold X. This induces a character $\zeta : A \to \mathrm{GL}(H^{3,1}(X)) \cong \mathbb{C}^\times$ of A. Denote by $H^4(X)_\zeta$ the ζ-eigenspace of the action of A. There is a Hermitian form $h : H^4(X)_\zeta \times H^4(X)_\zeta \to \mathbb{C}$ defined by $h(x, y) = \varphi_X(x, \bar{y})$ for any $x, y \in H^4(X)_\zeta$. If $\zeta = \bar{\zeta}$, then h has signature $(n', 2)$ and there is an type IV domain $\mathbb{D}$ associated with $(H^4(X)_\zeta, h)$. If $\zeta \neq \bar{\zeta}$, the form h has signature $(n', 1)$ and there is an associated complex hyperbolic ball, which we still denote by $\mathbb{D}$ for the moment, with $(H^4(X)_\zeta, h)$; see Proposition 4.1. The discussion above applies to any cubic fourfolds X' in $\mathcal{F}$ and the Hodge structures on $H^4(X')_\zeta$ give rise to a period map from $\mathcal{F}$ to an arithmetic quotient $\Gamma\backslash\mathbb{D}$. Here Γ is an arithmetic group acting properly discontinuously on $\mathbb{D}$ (see the beginning of Section 4C for the definition).

Notice that n' is the dimension of the Hermitian symmetric domain $\mathbb{D}$. We denote by n the dimension of $\mathcal{F}$. The first main theorem of the paper is the following:

Theorem 1.1 (Main Theorem 1). (i) *We have the equality $n' = n$.*

(ii) *The period map $\mathscr{P} : \mathcal{F} \cong \Gamma\backslash(\mathbb{D} - \mathcal{H}_s)$ is an algebraic isomorphism. Here $\mathcal{H}_s$ is a Γ-invariant hyperplane arrangement in $\mathbb{D}$.*

(iii) *The period map $\mathscr{P}$ extends naturally to an algebraic isomorphism $\mathcal{F}_1 \cong \Gamma\backslash(\mathbb{D} - \mathcal{H}_*)$, where $\mathcal{F}_1$ is a natural partial completion of $\mathcal{F}$, adding cubic fourfolds with at worst* ADE*-singularities, and $\mathcal{H}_*$ is a Γ-invariant hyperplane arrangement contained in $\mathcal{H}_s$.*

Denote by $\overline{\mathcal{F}}$ the GIT compactification of $\mathcal{F}$; see Section 2B. For a Γ-invariant hyperplane arrangement $\mathcal{H}$ in $\mathbb{D}$, we denote by $\overline{\Gamma\backslash\mathbb{D}}^{\mathcal{H}}$ the Looijenga compactification of $\Gamma\backslash(\mathbb{D} - \mathcal{H})$; see Section A5. We characterize $\overline{\mathcal{F}}$ via:

Theorem 1.2 (Main Theorem 2). (i) *The period map $\mathscr{P}$ extends to an algebraic isomorphism $\overline{\mathcal{F}} \cong \overline{\Gamma\backslash\mathbb{D}}^{\mathcal{H}_*}$ between the two projective varieties.*

(ii) *There are two pairs* (G_1, λ_1) *and* (G_2, λ_2), *each consists of a subgroup of* $\mathrm{SL}(6, \mathbb{C})$ *and a character of the subgroup* (*see Definition 5.6*), *such that the hyperplane arrangement* $\mathcal{H}_*$ *is empty if and only if for* $i = 1$ *or* 2, *there exists* $h \in \mathrm{GL}(6, \mathbb{C})$ *with* $h^{-1}Ah \subset G_i$ *and* $\lambda(a) = \lambda_i(h^{-1}ah)$ *for any* $a \in A$. *In this case*, *the Looijenga compactification* $\overline{\Gamma\backslash\mathbb{D}}^{\mathcal{H}_*}$ *is the Baily–Borel compactification* $\overline{\Gamma\backslash\mathbb{D}}^{bb}$. *See Theorem 5.7 for a complete statement.*

In the previous works for cubic fourfolds [Looijenga 2009; Laza 2010], cubic threefolds [Allcock et al. 2011; Looijenga and Swierstra 2007] and pairs consisting of a cubic threefold and a hyperplane section [Laza et al. 2018], the extended isomorphisms between the GIT compactifications and Looijenga compactifications rely on the machinery developed in [Looijenga 2003a; 2003b]. The key observation is that the period map also identifies the GIT polarization and the automorphic bundle on the period domain. If the period map can be extended to a Zariski-open subset U such that its complement in the GIT compactification has codimension at least 2, then the coordinate ring of the GIT compactification consists of sections (of the GIT polarization) over U. On the other hand, if each nonempty intersection of members in $\mathcal{H}$ has dimension at least 2, then the Γ-invariant automorphic sections with poles along $\mathcal{H}$ form the coordinate ring of the Looijenga compactification. Therefore, the two compactifications are identified if the two conditions hold.

For each case, the hard work on GIT side is to extend the defining domain of the period map to moduli space of varieties with at worst simple singularities and obtain codimension estimate for the indeterminacy locus. On the period domain side one need to obtain dimension estimate for all possible intersections of members in $\mathcal{H}$. Usually this is achieved by careful lattice analysis. In some cases people also need the correct choice of polarization on the GIT side in order to have such an extension. In our setting for Theorems 1.1 and 1.2, the dimension estimate fails in some cases, for example when A is a cyclic group of order 7; see Remark 6.7. So the previous approach does not work for all symmetry types.

We developed a new approach by considering the functorial properties on both GIT and Hodge theory side. We explain the proof of Theorems 1.1 and 1.2 with the following diagram:

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\cong"] \arrow[d, hook] & \Gamma\backslash(\mathbb{D} - \mathcal{H}_s) \arrow[d, hook] \\
\overline{\mathcal{F}} \arrow[r, dashed] \arrow[d, "j"] & \overline{\Gamma\backslash\mathbb{D}}^{\mathcal{H}_*} \arrow[d, "\pi"] \\
\overline{\mathcal{M}} \arrow[r, "\mathscr{P}"] & \overline{\widehat{\Gamma}\backslash\widehat{\mathbb{D}}}^{\mathcal{H}_\infty}
\end{tikzcd}$$

Here $\overline{\mathcal{M}}$ is the GIT compactification of the moduli space of cubic fourfolds, $\widehat{\Gamma}\backslash\widehat{\mathbb{D}}$ is the period domain for cubic fourfolds and $\mathcal{H}_\infty$ is a ($\widehat{\Gamma}$-invariant) hyperplane arrangement. These are explained in detail in Section 3. The bottom isomorphism in the above diagram is the main result in [Looijenga 2009; Laza 2010]. The top isomorphism is Proposition 4.10 proved in Section 4, which relies essentially on the global Torelli for cubic fourfolds. The left vertical morphism j is finite due to a classical result by Luna [1975] (with a modified version for projective GIT quotients; see [Ressayre 2010]). This is included in Proposition 2.7. The right vertical morphism π is also finite, which is proved in the Appendix; see Theorem A.13. After establishing the two finiteness results (of j and π) and the horizontal bimeromorphism between $\overline{\mathcal{F}}$ and $\overline{\Gamma\backslash\mathbb{D}}^{\mathcal{H}_*}$, we show that the period map $\mathscr{P}:\mathcal{F}\cong\Gamma\backslash(\mathbb{D}-\mathcal{H}_s)$ extends to an isomorphism $\overline{\mathcal{F}}\cong\overline{\Gamma\backslash\mathbb{D}}^{\mathcal{H}_*}$ by Lemma 5.4.

Our formalism of the proof does not need the codimension and dimension estimates, and hence avoids complicated GIT and lattice analysis. Finally, we reduce the complication to the proof of the functorial property of Looijenga compactifications. This allows us to deal with the theory uniformly and systematically for all symmetry types of cubic fourfolds. To the best of our knowledge, this formalism is new and may have further applications.

In many cases, the hyperplane arrangement $\mathcal{H}_*$ is empty; hence the Looijenga compactification is simply Baily–Borel compactification (for example, [Laza et al. 2018]). We discuss in Section 5 a criterion (Theorem 5.7) based on the symmetry type for the emptiness of $\mathcal{H}_*$. In particular, we apply the criterion to determine the emptiness of $\mathcal{H}_*$ for all symmetry type with A a prime-order cyclic group.

We end the introduction with a discussion on future works. A closely related question is to classify automorphism groups of cubic fourfolds. There are 13 conjugacy classes of prime-order automorphisms of smooth cubic fourfolds (see [González-Aguilera and Liendo 2011]). For two of them, our main theorems recover some of the main results in [Allcock et al. 2011; Looijenga and Swierstra 2007; Laza et al. 2018]. We will discuss these examples in more detail in Sections 6A and 6B.

Cubic fourfolds have very close relation with hyper-Kähler manifolds; see [Beauville and Donagi 1985; Hassett 2000]. For a smooth cubic fourfold X, its Fano scheme of lines is a polarized hyper-Kähler fourfold of $K3^{[2]}$ type. The automorphism group of a smooth cubic fourfold X is naturally identified with the automorphism group of the associated polarized hyper-Kähler manifold; see [Fu 2016]. The classification of automorphism groups of hyper-Kähler manifolds has appealed to a lot of interests recently. There is a systematic study by Mongardi in his thesis [2012; 2013; 2016]. Höhn and Mason [2019] classified all maximal finite symplectic automorphism groups of hyper-Kähler fourfolds of $K3^{[2]}$ type. Those groups are all subgroups of the Conway group. Recently, Laza and the second author classified all finite symplectic automorphism groups of smooth cubic fourfolds; see [Laza and Zheng 2019] . While related to Höhn and Mason's classification [2019], the main difference in [Laza and Zheng 2019] is that the authors are dealing with "polarized" hyper-Kähler fourfolds. Moreover, in many cases the explicit normal forms for the cubic fourfolds with a specified symplectic automorphism group are given. This classification offers a bunch of examples for Theorems 1.1 and 1.2 with $\mathbb{D}$ being type IV domains. Another closely related problem is to characterize the moduli spaces of symmetric or lattice-polarized hyper-Kähler manifolds. There are works along this direction; see [Dolgachev and Kondō 2007, Section 11; Artebani et al. 2011, Section 9; Joumaah 2016; Camere 2016, Section 3; Boissière et al. 2016, Section 5; Boissière et al. 2019]

The symmetries of the Hodge structures can also arise from degenerations of cubic fourfolds or $K3$ surfaces. For example, we consider a one-parameter degeneration of smooth cubic fourfolds to a singular cubic fourfold with only one node. The monodromy of the family gives a reflection on the primitive middle cohomology. The Hodge structures fixed by this reflection form a hyperplane in the period domain $\widehat{\mathbb{D}}$, which is a 19-dimensional type IV domain. On the geometric side, such singular cubic fourfolds naturally give rise to $K3$ surfaces of degree 6. So the proof above can also be applied to obtain comparison between GIT compactification of $K3$ surfaces of degree 6 and Baily–Borel compactification of period domain. Following this perspective, we are able to realize moduli of singular sextic curves (regarding as singular $K3$ surfaces of degree 2) as arithmetic quotients of type IV domains and again identify GIT compactification and Looijenga compactification; see [Yu and Zheng 2018].

Structure of the paper. Section 2 is devoted to the GIT construction of symmetric hypersurfaces in general. In Section 3 we review concepts about cubic fourfolds, and introduce the global Torelli theorem. Sections 4 and 5 are the main part of the paper, where we formulate and prove our main theorems. As we have mentioned, one of the key ingredients in the proof is the functorial property of Looijenga compactifications. This is treated in the Appendix. Moduli of cubic fourfolds with specified action by cyclic group is discussed in Section 6.

2. General setup: symmetric hypersurfaces

2A. ***Space of symmetric polynomials.*** Let V be a complex vector space of dimension $k+2$. Denote by $\mathrm{Sym}^d(V^*)$ the space of degree d polynomials on V. We have the natural action of $\mathrm{SL}(V)$ on $\mathrm{Sym}^d(V^*)$, namely, $g(F) = F \circ g^{-1}$ for $g \in \mathrm{SL}(V)$ and $F \in \mathrm{Sym}^d(V^*)$. The center of $\mathrm{SL}(V)$ is the group μ_{k+2} consisting of $(k{+}2)$-th roots of unity. Let A be a finite subgroup of $\mathrm{SL}(V)$ containing μ_{k+2} and denote by $\bar{A} = A/\mu_{k+2}$ the image of A in $\mathrm{PSL}(V)$. Then $\mathrm{Sym}^d(V^*)$ is a representation of A.

For any $\xi \in \mu_{k+2}$ and $F \in \mathrm{Sym}^d(V^*)$, we have $\xi(F) = \xi^{-d}F$. Let $\lambda : A \to \mathbb{C}^\times$ be a character of A such that $\lambda|_{\mu_{k+2}}$ sends $\xi \in \mu_{k+2}$ to ξ^{-d}. Let $\mathcal{V}_\lambda$ be the λ-eigenspace of $\mathrm{Sym}^d(V^*)$. We write $\mathcal{V} = \mathcal{V}_\lambda$ for short. Geometrically, an element in $\mathcal{V}$ determines a degree d hypersurface (not necessarily smooth) in $\mathbb{P}V$, whose automorphism group contains $\bar{A}$.

Two pairs (A_1, λ_1) and (A_2, λ_2) are called equivalent if and only if there exists $g \in \mathrm{SL}(V)$ such that $gA_1g^{-1} = A_2$ and $\lambda_1(a_1) = \lambda_2(ga_1g^{-1})$ for any $a_1 \in A_1$. We call an equivalence class a symmetry type, denoted by T. There is a poset structure on the space of symmetry types, namely, $T_2 \le T_1$ if T_1, T_2 are represented by (A_1, λ_1), (A_2, λ_2) respectively, such that $A_1 \subset A_2$ and $\lambda_1 = \lambda_2|_{A_1}$. Notice that the space $\mathcal{V}$ depends on the representative (A, λ) of T.

For $F \in \mathcal{V}$, we denote by $Z(F)$ the hypersurface defined by F in $\mathbb{P}V$. For $X = Z(F)$, we denote by $\mathrm{Aut}(X)$ the group of elements in $\mathrm{PSL}(V)$ preserving X, and by $\mathrm{Aut}(F)$ the preimage of $\mathrm{Aut}(X)$ in $\mathrm{SL}(V)$. From [Matsumura and Monsky 1963, Theorems 1 and 2] we have:

Theorem 2.1 (Matsumura–Monsky). *When X is smooth, $d \ge 3$, $k \ge 2$,*

(i) *the group* $\mathrm{Aut}(X)$ *is finite*,

(ii) *if $(d, k) \neq (4, 2)$, the group* $\mathrm{Aut}(X)$ *contains all biregular automorphisms of X.*

For any $X = Z(F)$, the group $\bar{A}$ is naturally a subgroup of $\mathrm{Aut}(X)$. We propose the following conditions on the symmetry type T:

Condition 2.2. The linear space $\mathcal{V}$ contains a point F defining a smooth hypersurface.

Condition 2.3. The linear space $\mathcal{V}$ contains a point F with the hypersurface $X = Z(F)$ smooth and $\bar{A} = \mathrm{Aut}(X)$.

Remark 2.4. Condition 2.3 is indeed stronger than Condition 2.2. For example, a smooth cubic fourfold with an automorphism of order 7 can be defined by a polynomial

$$F(x_0, \ldots, x_6) = x_0^2x_4 + x_1^2x_2 + x_0x_2^2 + x_3^2x_5 + x_3x_4^2 + x_1x_5^2 + ax_0x_1x_3 + bx_2x_4x_5$$

with $a, b \in \mathbb{C}$ (see Proposition 6.1). The order 7 automorphism ρ is given by $x_i \mapsto \omega^{i+1}x_i$ for ω a primitive 7-root of unity. On the other hand, such a polynomial always admits an order 3 automorphism given by $(x_0, x_1, x_2, x_3, x_4, x_5) \mapsto (x_1, x_3, x_5, x_0, x_2, x_4)$. If we take $\bar{A} = \langle\rho\rangle$ and take λ trivial, then (A, λ) is a symmetry type satisfying Condition 2.2. However, for a generic member $F \in \mathcal{V}$, the automorphism group $\mathrm{Aut}(Z(F))$ is strictly larger than $\bar{A}$. Thus the symmetry type does not satisfy Condition 2.3. See [Laza and Zheng 2019, Theorem 1.2] for more such examples.

For T satisfying Condition 2.2, a generic point in $\mathcal{V}$ defines a smooth hypersurface. We have a similar result about Condition 2.3.

Proposition 2.5. *If $T = [(A, \lambda)]$ satisfies Condition 2.3, then a generic element in $\mathcal{V}$ defines a smooth hypersurface X with $\bar{A} = \mathrm{Aut}(X)$.*

Proof. Suppose $F \in \mathcal{V}$ with $X = Z(F)$ smooth, and $A = \mathrm{Aut}(X)$. Then any small deformation F_1 of F in $\mathcal{V}$ defines a smooth hypersurface $Z(F_1)$. By Proposition 2.1 in [Zheng 2019], when F_1 is sufficiently close to F, there exists $g \in \mathrm{PSL}(V)$ such that $g\mathrm{Aut}(Z(F_1))g^{-1} \subset \mathrm{Aut}(X) = \bar{A}$. Since $F_1 \in \mathcal{V}$, we have $\bar{A} \subset \mathrm{Aut}(Z(F_1))$; hence $\bar{A} = \mathrm{Aut}(Z(F_1))$. □

2B. ***Geometric invariant theory for symmetric hypersurfaces.*** Now we assume that $d \geq 3$, $k \geq 2$. Given a symmetry type $T = [(A, \lambda)]$ satisfying Condition 2.2, let $C = \{g \in \mathrm{SL}(V) \mid gag^{-1} = a \text{ for all } a \in A\}$ and $N = \{g \in \mathrm{SL}(V) \mid gAg^{-1} = A, \lambda(gag^{-1}) = \lambda(a) \text{ for all } a \in A\}$ be two reductive subgroups of $\mathrm{SL}(V)$. For reductivity, see [Luna and Richardson 1979, Lemma 1.1].

Lemma 2.6. *There is a natural action of N on $\mathcal{V}$, under which the points in $\mathcal{V}$ defining smooth hypersurfaces are stable.*

Proof. For any $g \in N$ and $F \in \mathcal{V}$, we need to show $g(F) \in \mathcal{V}$. For any $a \in A$, we have

$$a(g(F)) = g(g^{-1}ag(F)) = g(\lambda(g^{-1}ag)F) = g\lambda(a)F = \lambda(a)g(F),$$

which implies $g(F) \in \mathcal{V}$ by definition of $\mathcal{V}$. Therefore, there is a natural action of N on $\mathcal{V}$.

Now take $F \in \mathcal{V}$ with $X = Z(F)$ smooth. Then $\mathrm{Aut}(X)$ is finite by Theorem 2.1. Since the stabilizer group of F under the action of N is a subgroup of $\mathrm{Aut}(F)$, it is also finite. Moreover, NF is closed in $\mathrm{SL}(V)F$, and the latter is closed in $\mathrm{Sym}^d(V^*)$ since $Z(F)$ is smooth. Thus NF is closed in $\mathrm{Sym}^d(V^*)$; hence also closed in $\mathcal{V}$. We conclude that F is stable under the action of N. □

Denote $\mathcal{V}^{sm} = \{F \in \mathcal{V} \mid Z(F) \text{ smooth}\}$, by $\mathcal{V}^{ss}$ the set of semistable elements in $\mathcal{V}$ under the action of N, and by $\mathbb{P}\mathcal{V}^{sm}$, $\mathbb{P}\mathcal{V}^{ss}$ their projectivizations. By Lemma 2.6, we can take $\mathcal{F} = N\backslash\!\backslash\mathbb{P}\mathcal{V}^{sm}$ to be the GIT quotient, with the GIT compactification $\overline{\mathcal{F}} = N\backslash\!\backslash\mathbb{P}\mathcal{V}^{ss}$. Different representatives of the symmetry type induce canonically isomorphic GIT-quotients. Define $\mathcal{M} = \mathrm{SL}(V)\backslash\!\backslash\mathbb{P}\mathrm{Sym}^d(V^*)^{sm}$ to be the moduli space of smooth degree d hypersurfaces in $\mathbb{P}(V)$, with the GIT compactification $\overline{\mathcal{M}} = \mathrm{SL}(V)\backslash\!\backslash\mathbb{P}\mathrm{Sym}^d(V^*)^{ss}$. We have the following proposition:

Proposition 2.7. *There is a natural morphism $j : \overline{\mathcal{F}} \to \overline{\mathcal{M}}$ sending $[F] \in \mathcal{F}$ to $[F] \in \mathcal{M}$ for any $F \in \mathcal{V}^{sm}$. This morphism is finite. When T satisfies Condition 2.3, the morphism j is a normalization of its image.*

Proof. Here we use a projective version of the main theorem in [Luna 1975]. See the argument of Proposition 8 in [Ressayre 2010]. Since A is a finite group, there exists certain symmetric power $\mathrm{Sym}^l(\mathcal{V})$ on which the A-action is trivial. Consider the $\mathrm{SL}(V)$-action on the coordinate ring $\bigoplus_m \mathrm{Sym}^{lm}(\mathrm{Sym}^d(V^*)^*)$ of $(\mathbb{P}(\mathrm{Sym}^d(V^*)), \mathcal{O}(l))$. Notice that N is of finite index in the normalizer of A in $\mathrm{SL}(V)$. By the main theorem in [Luna 1975], we have a finite morphism

$$\tilde{j} : \mathrm{Spec}\Big(\Big(\bigoplus_m \mathrm{Sym}^{lm}(\mathcal{V}^*)\Big)^N\Big) \to \mathrm{Spec}\Big(\Big(\bigoplus_m \mathrm{Sym}^{lm}(\mathrm{Sym}^d(V^*)^*)\Big)^{\mathrm{SL}(V)}\Big)$$

sending semistable points to semistable points, and preserving the cone structures. Thus $\tilde{j}$ does not contract any line; hence descends to a finite morphism $j : \overline{\mathcal{F}} \to \overline{\mathcal{M}}$. The morphism j sends $[F] \in \mathcal{F}$ to $[F] \in \mathcal{M}$ for any $F \in \mathcal{V}^{sm}$.

We claim that when T satisfies Condition 2.3, the morphism j is generically injective. Take generically $F_1, F_2 \in \mathcal{V}$ and assume $[F_1] = [F_2]$ in $\mathcal{M}$. Then there exists $g \in \mathrm{SL}(V)$ with $g(F_1) = F_2$. By the calculation

$$g^{-1}ag(F_1) = g^{-1}a(F_2) = g^{-1}\lambda(a)F_2 = \lambda(a)F_1, \tag{1}$$

we have that $g^{-1}ag \in \mathrm{SL}(V)$ is an automorphism of $Z(F_1)$. By the genericity of F_1, we have $A \cong \mathrm{Aut}(F_1)$, which implies that $g^{-1}ag \in A$. Then by equation (1) and $F_1 \in \mathcal{V}$, we have $\lambda(g^{-1}ag) = \lambda(a)$. This implies that $g \in N$, hence $[F_1] = [F_2]$ in $\mathcal{F}$. Thus j is generically injective.

Moreover, since $\overline{\mathcal{F}}$ is normal and projective, j is a normalization of its image. □

Let $T = [(A, \lambda)]$ be a symmetry type satisfying Condition 2.2. Consider the automorphism groups $\mathrm{Aut}(F)$ for all $F \in \mathcal{V}^{sm}$. There exists $F' \in \mathcal{V}^{sm}$ such that $\#\mathrm{Aut}(F')$ is minimal. Let $A' = \mathrm{Aut}(F')$. For any $a \in A'$, there exists $\lambda'(a) \in \mathbb{C}$ with $a(F') = \lambda'(a)F'$. Then we have a symmetry type $T' = [(A', \lambda')]$. It is straightforward that $T \geq T'$, and T' satisfies Condition 2.3. Similar as T, we have for T' correspondingly N', $\mathcal{V}'$ and $\overline{\mathcal{F}'}$. We have the following proposition:

Proposition 2.8. *There exists a natural finite morphism $\overline{\mathcal{F}} \to \overline{\mathcal{F}'}$.*

Proof. By Proposition 2.7, we have two finite morphisms $j : \overline{\mathcal{F}} \to \overline{\mathcal{M}}$ and $j' : \overline{\mathcal{F}'} \to \overline{\mathcal{M}}$, and the latter one is a normalization of its image. We show that j and j' have the same image. We have $j'(\overline{\mathcal{F}'}) \subset j(\overline{\mathcal{F}})$ since $\mathcal{V}' \subset \mathcal{V}$. By Proposition 2.1 in [Zheng 2019], when $F'' \in \mathcal{V}$ is sufficiently close to F', there exists $g \in \mathrm{SL}(V)$, such that $g\mathrm{Aut}(F'')g^{-1} \subset \mathrm{Aut}(F') = A'$. By minimality of $\#A'$, we have $g\mathrm{Aut}(F'')g^{-1} = A'$.

This implies that $\mathrm{Aut}(g(F'')) = A'$; hence $g(F'') \in \mathcal{V}'$. We then have $\dim(j(\overline{\mathcal{F}})) \leq \dim(j'(\overline{\mathcal{F}'}))$. By irreducibilities of the two images, they are the same.

By universal property of normalization, the morphism j factors through j'. Therefore, we have naturally a finite morphism $\overline{\mathcal{F}} \to \overline{\mathcal{F}'}$. □

Remark 2.9. The fiber of the finite morphism $\overline{\mathcal{F}} \to \overline{\mathcal{F}'}$ over $[F']$ is naturally bijective to the orbit of (A, λ) in the set of subdata of (A', λ') under the action of N'.

2C. ***Universal deformation.*** We fix a type $T = [(A, \lambda)]$ satisfying Condition 2.2, and assume $d \geq 3$ and $k \geq 2$. Next we use Luna's étale slice theorem to describe the local structure of $\mathcal{F}$, and construct the universal family of smooth degree d k-folds of type T. We essentially follow the argument in [Zheng 2019, Section 2]. For Luna's étale slice theorem and its proof, see [Luna 1973] or [Vinberg and Popov 1994].

Denote by G the centralizer of $\bar{A}$ in $\mathrm{PSL}(V)$. Recall that $\mathbb{P}\mathcal{V}^{sm}$ is the space of smooth degree d k-folds of symmetry type (A, λ). As a closed subvariety of the affine variety $\mathbb{P}\mathrm{Sym}^d(V^*)^{sm}$, the variety $\mathbb{P}\mathcal{V}^{sm}$ is also affine. There is a natural action of G on $\mathbb{P}\mathcal{V}^{sm}$. For any $x \in \mathbb{P}\mathcal{V}^{sm}$, we denote by Gx the orbit of x and by G_x the stabilizer of x. By Lemma 2.6, Gx is closed in the affine variety $\mathbb{P}\mathcal{V}^{sm}$ and G_x is finite. For a G_x-invariant subvariety S of X containing x, there is an action of G_x on $G \times S$ given by $g(h, y) = (hg^{-1}, gy)$ for any $g \in G_x$, $h \in G$, $y \in S$. We denote by $G \times^{G_x} S$ the quotient of $G \times S$ by this action. By Luna's étale slice theorem, there exists a smooth, locally closed, G_x-invariant subvariety S containing x, such that

(i) the image of $\kappa : G \times^{G_x} S \to \mathbb{P}\mathcal{V}^{sm}$, denoted by U, is Zariski-open and G-invariant,

(ii) the morphism $\kappa : G \times^{G_x} S \to U$ is étale,

(iii) the morphism $G \backslash\backslash \kappa : G_x \backslash\backslash S \to G \backslash\backslash U$ is étale,

(iv) the above two morphisms induce an isomorphism
$$G \times^{G_x} S \cong U \underset{G\backslash\backslash U}{\times} G_x \backslash\backslash S. \tag{2}$$

We can shrink S in the analytic category such that

(v) S is G_x-invariant, contractible and contains x, with $U = \kappa(G \times^{G_x} S)$ a G-invariant open subset of $\mathbb{P}\mathcal{V}^{sm}$,

(vi) the morphism between analytic spaces: $G_x \backslash\backslash S \to G \backslash\backslash U$ is an isomorphism.

From (2), we have an isomorphism between analytic spaces,
$$G \times^{G_x} S \cong U,$$
by which we have a principal G_x-bundle $G \times S \to U$. In particular, $G \times S \to U$ is a covering map.

Definition 2.10. For any symmetry type T, we define a category $\mathcal{C}^T_{d,k}$ as follows. The objects are families of degree d k-folds of type T with a specified central fiber. The morphisms are holomorphic maps between families, sending central fiber to central fiber and compatible with the action of $\bar{A}$.

Proposition 2.11. *The family $\mathscr{X}_S$ of degree d k-folds of type T over S has the following universal property. For any subfamily $\mathscr{X}_{S'} \to S' \subset U$ of degree d k-folds of type T containing a central fiber X' with an isomorphism $f : X' \cong X$ compatible with the actions of $\overline{A}$, we have a unique morphism in the category $\mathcal{C}^T_{d,k}$:*

$$\begin{tikzcd} \mathscr{X}_{S'} \arrow[r, "\tilde{f}"] \arrow[d] & \mathscr{X}_S \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

such that the restriction of $\tilde{f}$ to X' is f. Moreover, for any two fibers X_1, X_2 of $\mathscr{X}_S$ with an isomorphism $g : X_1 \to X_2$ compatible with the actions of $\overline{A}$, we can extend g uniquely to a morphism $\widetilde{g} : \mathscr{X}_S \to \mathscr{X}_S$ in $\mathcal{C}^T_{d,k}$.

Proof. The base S' lies in U and is covered by $G \times S$. Thus we have a unique lifting $S' \hookrightarrow G \times S$, sending x' to (f^{-1}, x). In other words, we have uniquely a morphism $\tilde{f} : \mathscr{X}_{S'} \to \mathscr{X}_S$, which restricts to f on X'.

Now suppose X_1, X_2 are two fibers of $\mathscr{X}_S$ with an isomorphism $g : X_1 \cong X_2$. Denote by x_1, x_2 the corresponding base points in S. Then (g, x_1), $(\mathrm{id}, x_2) \in G \times S$ have the same image in U. Since $G \times S \to U$ is a principal G_x-bundle, the two pairs (g, x_1) and (id, x_2) are G_x-equivalent; hence $g \in G_x$. The proposition follows. □

We have the following lemma, which is used in the proof of Proposition 4.8. Since it holds for general degree d k-folds, we state and prove it here.

Lemma 2.12. *Let*

$$\begin{tikzcd} \mathscr{X} \arrow[r, hook] \arrow[d] & S \times \mathbb{P}V \arrow[dl] \\ S & \end{tikzcd}$$

be a family of smooth degree d k-folds, with the base S contractible. Suppose there is a group $\widetilde{A}$, such that for all $s \in S$, the fiber $\mathscr{X}_s$ admits a biregular action of $\widetilde{A}$, with induced actions on $H^n(\mathscr{X}_s, \mathbb{Z})$ compatible with respect to the local trivialization. Then there exists an action of $\widetilde{A}$ on the whole family $\mathscr{X} \to S$ inducing on each fiber the existing action.

To prove this, we need another lemma from [Javanpeykar and Loughran 2017, Proposition 2.12; Matsumura and Monsky 1963]:

Lemma 2.13. *For $d \geq 3$, $k \geq 2$, and a smooth degree d k-fold X, the induced action of* $\mathrm{Aut}(X)$ *on $H^k(X, \mathbb{Z})$ is faithful.*

Proof of Lemma 2.12. Without loss of generality, we can assume the action of $\widetilde{A}$ on each $H^n(\mathscr{X}_s, \mathbb{Z})$ is faithful. Take any $s \in S$. By Proposition 2.1 in [Zheng 2019], there is a universal hypersurface family $\mathscr{X}'$ of $\mathscr{X}_s$, such that any isomorphism between two fibers (may coincide) of $\mathscr{X}'$ comes from an automorphism of the central fiber $\mathscr{X}_s$. There exists an open neighborhood U of s in S, with a unique morphism $\mathscr{X}|_U \to \mathscr{X}'$. Then for any $s' \in U$, the action of $\widetilde{A}$ on $\mathscr{X}_{s'}$ is induced by a subgroup $\widetilde{A}'$ of $\mathrm{Aut}(\mathscr{X}_s)$. By Lemma 2.13, and the compatibility of induced actions of $\widetilde{A}$ on $\mathscr{X}_s$ and $\mathscr{X}_{s'}$, we have $\widetilde{A} = \widetilde{A}'$ as subgroups of $\mathrm{Aut}(\mathscr{X}_s)$. Therefore, the actions of $\widetilde{A}$ on fibers of $\mathscr{X} \to S$ glue to an action of $\widetilde{A}$ on the whole family. □

3. Period map for smooth cubic fourfolds

In this section we recall some fundamental facts on the period map for cubic fourfolds, the main references are [Voisin 1986; Hassett 2000; Looijenga 2009; Laza 2009; 2010].

Take $(d, k) = (3, 4)$. Then we have $\mathcal{M}$ the moduli of smooth cubic fourfolds, as a Zariski-open subset of its GIT compactification $\overline{\mathcal{M}}$. Let X be a smooth cubic fourfold. We denote by φ_X the intersection pairing on $H^4(X, \mathbb{Z})$. Then $(H^4(X, \mathbb{Z}), \varphi_X)$ is an odd unimodular lattice of signature $(21, 2)$. Denote by η_X the square of the hyperplane class of X. Then $H_0^4(X, \mathbb{Z}) := \eta_X^{\perp}$ is an even lattice of discriminant 3. Now we define $(\Lambda, \Lambda_0, \eta)$ to be an abstract data isomorphic to $(H^4(X, \mathbb{Z}), H_0^4(X, \mathbb{Z}), \eta_X)$. This does not depend on the choice of the cubic fourfold X.

Definition 3.1. A marking of the cubic fourfold X is an isomorphism $\Phi : H^4(X, \mathbb{Z}) \cong \Lambda$ of lattices sending η_X to η.

Two marked cubic fourfolds (X_1, Φ_1) and (X_2, Φ_2) are called equivalent if there exists a linear isomorphism $g : X_1 \to X_2$ such that $\Phi_1 = g^*\Phi_2$. Let $\mathcal{M}^m$ be the set of equivalence classes of marked cubic fourfolds. From [Zheng 2019, Section 3], we have:

Proposition 3.2. *The set $\mathcal{M}^m$ is a complex manifold in a natural way.*

Next we define the period domain and period map for cubic fourfolds. Let

$$\widetilde{\mathbb{D}} := \mathbb{P}\{x \in (\Lambda_0)_{\mathbb{C}} \mid \varphi(x, x) = 0, \varphi(x, \bar{x}) < 0\}.$$

This is an analytically open subset of a quadric hypersurface in $\mathbb{P}(\Lambda_0)_{\mathbb{C}}$, and has two connected components. We have naturally a holomorphic map

$$\widetilde{\mathscr{P}} : \mathcal{M}^m \to \widetilde{\mathbb{D}}$$

sending $(X, \Phi) \in \mathcal{M}^m$ to $\Phi(H^{3,1}(X))$. It is called the local period map for cubic fourfolds.

Let $\widehat{\mathbb{D}}$ be one connected component of $\widetilde{\mathbb{D}}$ and $\widehat{\Gamma}$ be the index 2 subgroup of $\mathrm{Aut}(\Lambda, \varphi, \eta)$ which leaves the component $\widehat{\mathbb{D}}$ stable. Then $\widehat{\Gamma}$ is an arithmetic group acting on $\widehat{\mathbb{D}}$, and $\widetilde{\mathscr{P}}$ descends to

$$\mathscr{P} : \mathcal{M} \to \widehat{\Gamma} \backslash \widehat{\mathbb{D}},$$

which is called the (global) period map for cubic fourfolds.

Remark 3.3. The subgroup $\widehat{\Gamma}$ consists of elements in Γ with spinor norm 1. Since there exist vectors in Λ_0 with self intersection -2, the group $\widehat{\Gamma}$ is of index 2 in $\mathrm{Aut}(\Lambda, \varphi, \eta)$.

The global Torelli theorem was originally proved by Voisin [1986], with an erratum based on some work by Laza [2009]:

Theorem 3.4 (Voisin). *The period map $\mathscr{P}$ is an open embedding.*

Remark 3.5. In fact, the period map $\mathscr{P}$ is algebraic; see the discussion in [Hassett 2000, Proposition 2.2.3].

We give a lemma which is constantly used; see [Zheng 2019, Proposition 1.3].

Lemma 3.6. *Take X a smooth cubic fourfold. Then* $\mathrm{Aut}(X) \cong \mathrm{Aut}(H^4(X, \mathbb{Z}), \varphi_X, \eta_X, H^{3,1}(X))$.

We have a refined version of Theorem 3.4:

Proposition 3.7 (Voisin, Hassett, Looijenga, Laza). *The local period map $\widetilde{\mathscr{P}} : \mathcal{M}^m \to \widetilde{\mathbb{D}}$ is an open embedding, with image being the complement of a hyperplane arrangement invariant under the action of* $\mathrm{Aut}(\Lambda, \eta)$ *on $\widetilde{D}$.*

Proof. Combining Theorem 3.4 and Lemma 3.6 we have injectivity. The characterization of the image of $\widetilde{\mathscr{P}}$ is due to Looijenga [2009] and Laza [2010, Theorem 1.1], a more precise version is discussed in Proposition 4.7. □

4. Period maps for symmetric cubic fourfolds

4A. ***Local period map for symmetric cubic fourfolds.*** In this section we are going to discuss the local and global period maps for symmetric cubic fourfolds. Let $(d, k) = (3, 4)$, and fix a symmetry type $T = [(A, \lambda)]$ satisfying Condition 2.2. We first introduce the local period domains with actions of arithmetic groups. Take $X = Z(F)$ for a generic point $F \in \mathcal{V}$. Recall that the action of A on X induces an action of A on $H^{3,1}(X)$. This action is a character $\zeta : A \to \mathbb{C}^\times$ with trivial restriction on μ_{k+2}. We denote

$$H^4(X)_\zeta = \{x \in H^4(X) \mid ax = \zeta(a)x \text{ for all } a \in A\}.$$

Define a Hermitian form $h : H^4(X)_\zeta \times H^4(X)_\zeta \to \mathbb{C}$ by $h(x, y) = \varphi(x, \bar{y})$. Denote by σ_X the action of A on $H^4(X, \mathbb{Z})$. Let σ be an action of A on Λ, making (Λ, η, σ) isomorphic to $(H^4(X, \mathbb{Z}), \eta_X, \sigma_X)$. Denote by $\Lambda_\zeta \subset \Lambda_0 \otimes \mathbb{C}$ the ζ-eigenspace of the action of A on $(\Lambda_0)_{\mathbb{C}}$.

Proposition 4.1. *The Hermitian form h has signature $(n', 2)$ if $\zeta = \bar{\zeta}$ (this is also equivalent to $\zeta(A) \subset \mu_2$); it has signature $(n', 1)$ otherwise. Here n' is a nonnegative integer independent of the choice of X.*

Proof. Notice that the lattice $H^4(X, \mathbb{Z})$ has signature $(21, 2)$, with negative part $H^{3,1}(X) \oplus H^{1,3}(X)$. If $\zeta(A)$ is not contained in μ_2, we have $\zeta \neq \bar{\zeta}$. Since $H^{1,3}$ lies in $\bar{\zeta}$-eigenspace, the signature of h is $(n', 1)$.

For the case $\zeta(A) \subset \mu_2$, both $H^{3,1}(X)$ and $H^{1,3}(X)$ are contained in H_ζ; thus h has signature $(n', 2)$. □

An isomorphism $\Phi : (H^4(X, \mathbb{Z}), \eta_X, \sigma_X) \cong (\Lambda, \eta, \sigma)$ is called a T-marking of X. We consider pairs consisting of a smooth cubic fourfold and its T-marking. Two such pairs (X_1, Φ_1) and (X_2, Φ_2) are equivalent if there exists $g \in G$ such that $\Phi_1 = g^*\Phi_2$. Letting $\mathcal{F}^m$ be the set of equivalence classes of such pairs, we have:

Proposition 4.2. *The set $\mathcal{F}^m$ is naturally a complex manifold.*

Proof. First we describe the local charts on $\mathcal{F}^m$. Take a point $(X, \Phi) \in \mathcal{F}^m$, and take a universal deformation $\mathscr{X}_S \to S$ of X as in Proposition 2.11. Since S is contractible, the local system $R^4\pi_*(\mathbb{Z})$ is trivializable over S and the T-marking Φ of X naturally extends to a T-marking for every fiber of $\mathscr{X}_S \to S$. Thus we have a map

$$\alpha : S \to \mathcal{F}^m.$$

We first show that α is injective. Suppose X_1, X_2 are two fibers of $\mathscr{X}_S$, with Φ_1, Φ_2 the induced T-markings by Φ, such that (X_1, Φ_1) and (X_2, Φ_2) represent the same point in $\mathcal{F}^m$. Then there exists $g : X_1 \cong X_2$ with $\Phi_2 = \Phi_1 \circ g^*$. By Proposition 2.11 we have $g \in G_x$ and $\Phi = \Phi \circ g^*$; hence $g^* = \mathrm{id}$. By Lemma 3.6 we have $g = \mathrm{id}$. Thus α is injective.

By definition, $\mathcal{F}^m$ is covered by countably many such $\alpha(S)$, and they form a basis of a topology. To show $\mathcal{F}^m$ is a complex manifold, we need to prove that the topology is Hausdorff. Suppose not, then we have two nonseparated points $(X, \Phi), (X', \Phi') \in \mathcal{F}^m$. Then X and X' are isomorphic (because $\mathcal{F}$ is separated). Without loss of generality, we assume $X' = X$. Take $\mathscr{X}_S \to S$ the universal family as in Proposition 2.11, and

$$\alpha, \alpha' : S \to \mathcal{F}^m$$

induced by Φ and Φ'. Now since (X, Φ) and (X', Φ') are nonseparated, we have $\alpha(S) \cap \alpha'(S) \neq \varnothing$. Thus there exists $x_1 \in S$ with corresponding cubic fourfold X_1, such that the two pairs (X_1, Φ) and (X_1, Φ') represent the same point in $\mathcal{F}^m$. Then there is an automorphism g of X_1, such that $\Phi' = \Phi \circ g^*$. Proposition 2.11 implies that g is also an automorphism of X and satisfies the above relation. Thus $(X, \Phi) = (X, \Phi')$ in $\mathcal{F}^m$, a contradiction. We showed the Hausdorff property. We conclude that $\mathcal{F}^m$ is naturally a complex manifold. □

Remark 4.3. Proposition 4.2 can be generalized to degree d k-folds ($d \geq 3$, $k \geq 2$) with specified automorphism group. The argument is the same.

When h has signature $(n', 1)$, we define $\mathbb{D}_T = \mathbb{P}\{x \in \Lambda_\zeta \mid \varphi(x, \bar{x}) < 0\}$, which is a hyperbolic complex ball of dimension n'; when h has signature $(n', 2)$, define $\mathbb{D}_T$ to be a component of

$$\mathbb{P}\{x \in (\Lambda_0)_\zeta \mid \varphi(x, x) = 0, \varphi(x, \bar{x}) < 0\},$$

which is a type IV symmetric domain of dimension n'.

We define the local period map for symmetric cubic fourfolds of type T as the map from $\mathcal{F}^m$ to $\mathbb{D}_T \sqcup \overline{\mathbb{D}_T}$, sending (X, Φ) to $\Phi(H^{3,1}(X))$, still denoted by $\widetilde{\mathscr{P}}$. Suppose $\mathbb{D}_T$ is a type IV domain and $\mathcal{F}^m$ is connected, then we make the choice of $\mathbb{D}_T$ such that $\widetilde{\mathscr{P}}$ has image in $\mathbb{D}_T$. Actually, the two situations, $\mathcal{F}^m$ being connected or not, both happen. See Proposition 4.9 for a precise argument.

4B. ***Properties of local period maps for symmetric cubic fourfolds.*** We need to review basic works by Laza [2009; 2010]. In [Laza 2009] stable and semistable cubic fourfolds are classified. One of the main theorems is:

Theorem 4.4 [Laza 2009]. *A cubic fourfold with at worst* ADE-*singularities is stable.*

Independently, Looijenga [2009] and Laza [2010] proved that the period map $\mathscr{P} : \mathcal{M} \to \widehat{\Gamma}\backslash\widehat{\mathbb{D}}$ extends to the moduli space $\mathcal{M}_1$ of cubic fourfolds with at worst ADE singularities, and characterized its image. The results are gathered in the following theorem:

Theorem 4.5 [Laza 2010]. *The period map $\mathscr{P}:\mathcal{M}\to\widehat{\Gamma}\backslash\widehat{\mathbb{D}}$ has image $\widehat{\Gamma}\backslash(\widehat{\mathbb{D}}-\mathcal{H}_\infty-\mathcal{H}_\Delta)$, and extends holomorphically to*

$$\mathscr{P}:\mathcal{M}_1\to\widehat{\Gamma}\backslash\widehat{\mathbb{D}}$$

with image $\widehat{\Gamma}\backslash(\widehat{\mathbb{D}}-\mathcal{H}_\infty)$. Here $\mathcal{H}_\infty,\mathcal{H}_\Delta$ are two $\widehat{\Gamma}$-invariant hyperplane arrangements in $\widehat{\mathbb{D}}$, with the quotients $\widehat{\Gamma}\backslash\mathcal{H}_\infty$ and $\widehat{\Gamma}\backslash\mathcal{H}_\Delta$ irreducible.

Remark 4.6. This characterization of the image $\mathscr{P}(\mathcal{M})$ was conjectured by Hassett [2000]. Hassett defined the special cubic fourfolds, some of which correspond to polarized $K3$ surfaces. The hyperplane arrangements $\mathcal{H}_\Delta$ and $\mathcal{H}_\infty$ are two particular ones, parametrizing nodal cubic fourfolds and secant lines of the determinantal cubic fourfold, and corresponding to $K3$ surfaces of degree 6 and 2 respectively; see [Hassett 2000, Sections 4.2 and 4.4].

We have also the following marked version of Theorem 4.5:

Proposition 4.7. *The local period map $\widetilde{\mathscr{P}}:\mathcal{M}^m\to\widetilde{\mathbb{D}}$ has image $\widetilde{\mathbb{D}}-\mathcal{H}_\infty-\mathcal{H}_\Delta-\overline{\mathcal{H}_\infty}-\overline{\mathcal{H}_\Delta}$.*

Proof. By Theorem 4.5, the image of $\widetilde{\mathscr{P}}$ lies in $\widetilde{\mathbb{D}}-\mathcal{H}_\infty-\mathcal{H}_\Delta-\overline{\mathcal{H}_\infty}-\overline{\mathcal{H}_\Delta}$. Take any point x in $\widetilde{\mathbb{D}}-\mathcal{H}_\infty-\mathcal{H}_\Delta-\overline{\mathcal{H}_\infty}-\overline{\mathcal{H}_\Delta}$. By Theorem 4.5 the point $[x]\in\widehat{\Gamma}\backslash(\widehat{\mathbb{D}}-\mathcal{H}_\infty-\mathcal{H}_\Delta)$ lies in the image of $\mathscr{P}:\mathcal{M}\to\widehat{\Gamma}\backslash\widehat{\mathbb{D}}$. Thus the orbit $\mathrm{Aut}(\Lambda,\eta)x$ intersects with $\widetilde{\mathscr{P}}(\mathcal{M}^m)$. Notice that the set $\widetilde{\mathscr{P}}(\mathcal{M}^m)$ is $\mathrm{Aut}(\Lambda,\eta)$-invariant; hence contains the orbit $\mathrm{Aut}(\Lambda,\eta)x$. We showed the surjectivity. ☐

For a specified type T, we write $\mathbb{D}=\mathbb{D}_T$ for short. We have a natural embedding $\mathbb{D}\sqcup\overline{\mathbb{D}}\hookrightarrow\widetilde{\mathbb{D}}$. Denote $\mathcal{H}_s=\mathbb{D}\cap(\mathcal{H}_\Delta\cup\mathcal{H}_\infty\cup\overline{\mathcal{H}_\Delta}\cup\overline{\mathcal{H}_\infty})$ and $\mathcal{H}_*=\mathbb{D}\cap(\mathcal{H}_\infty\cup\overline{\mathcal{H}_\infty})$. The local period map $\widetilde{\mathscr{P}}:\mathcal{F}^m\to\mathbb{D}\sqcup\overline{\mathbb{D}}$ has image contained in $\mathbb{D}\sqcup\overline{\mathbb{D}}-\mathcal{H}_s-\overline{\mathcal{H}_s}$.

Proposition 4.8. *The local period map $\widetilde{\mathscr{P}}:\mathcal{F}^m\to\mathbb{D}\sqcup\overline{\mathbb{D}}$ is an open embedding, with image either $\mathbb{D}-\mathcal{H}_s$ or $\mathbb{D}\sqcup\overline{\mathbb{D}}-\mathcal{H}_s-\overline{\mathcal{H}_s}$. In particular, $n'=n$.*

Proof. We have a closed embedding $\pi:\mathbb{D}\sqcup\overline{\mathbb{D}}\hookrightarrow\widetilde{\mathbb{D}}$. There is a natural map $j:\mathcal{F}^m\to\mathcal{M}^m$. Suppose (X_1,Φ_1), (X_2,Φ_2) represent the same point in $\mathcal{M}^m$, then there exists a linear isomorphism $g:X_1\cong X_2$ such that

$$g^*=\Phi_1^{-1}\circ\Phi_2:H^4(X_2,\mathbb{Z})\to H^4(X_1,\mathbb{Z})$$

Since Φ_1,Φ_2 are compatible with the actions of A on $H^4(X_1,\mathbb{Z})$, $H^4(X_2,\mathbb{Z})$, so is g^*. Lemma 3.6 implies that g is compatible with the actions of A on X_1,X_2. Thus (X_1,Φ_1), (X_2,Φ_2) represent the same point in $\mathcal{F}^m$. We showed the injectivity of j.

Combining this with the commutative diagram

$$\begin{tikzcd}
\mathcal{F}^m \arrow[r, "\widetilde{\mathscr{P}}"] \arrow[d, hook, "j"] & \mathbb{D}\sqcup\overline{\mathbb{D}} \arrow[d, hook, "\pi"] \\
\mathcal{M}^m \arrow[r, hook, "\widetilde{\mathscr{P}}"] & \widetilde{\mathbb{D}}
\end{tikzcd}$$

we obtain the injectivity of $\widetilde{\mathscr{P}}:\mathcal{F}^m\to\mathbb{D}\sqcup\overline{\mathbb{D}}$. In particular, $n\le n'$.

Since the differential of $\widetilde{\mathscr{P}} : \mathcal{M}^m \to \widetilde{\mathbb{D}}$ is injective everywhere, so is the differential of $\widetilde{\mathscr{P}} : \mathcal{F}^m \to \mathbb{D} \sqcup \overline{\mathbb{D}}$.

Take $(X, \Phi) \in \mathcal{F}^m$. Let $x = \Phi(H^{3,1}(X)) \in \mathbb{D} \sqcup \overline{\mathbb{D}}$ and y be any point in the component of $\mathbb{D} \sqcup \overline{\mathbb{D}}$ containing x. Since both $\mathbb{D} - \mathcal{H}_s$ and $\overline{\mathbb{D}} - \overline{\mathcal{H}_s}$ are connected, there exists a path

$$\gamma : [0, 1] \to \mathbb{D} \sqcup \overline{\mathbb{D}} - \mathcal{H}_s - \overline{\mathcal{H}_s}$$

with $\gamma(0) = x$ and $\gamma(1) = y$. The path γ has a unique lifting in $\mathcal{M}^m$. By Proposition 3.7, we can choose a family $\mathscr{X} \to [0, 1]$ of cubic fourfolds, with marking Φ of every fiber, such that $(\mathscr{X}_0, \Phi) = (X, \Phi)$ and $\Phi(H^{3,1}(\mathscr{X}_s)) = \gamma(s)$, for all $s \in [0, 1]$. Since $\gamma(s) \in \mathbb{D} \sqcup \overline{\mathbb{D}}$, the Hodge structure on $H^4(X_s, \mathbb{Z})$ has an action of A induced by Φ. By Lemma 3.6, there exist actions of A on $\mathscr{X}_s$ for any $s \in [0, 1]$, inducing compatible actions on $H^4(\mathscr{X}_s, \mathbb{Z})$. By Lemma 2.12, actions of A are of the same type T. Thus we obtain a lifting of γ in $\mathcal{F}^m$, hence $y \in \widetilde{\mathscr{P}}(\mathcal{F}^m)$.

If $\widetilde{\mathscr{P}}(\mathcal{F}^m) \subset \mathbb{D}$, then $\widetilde{\mathscr{P}}(\mathcal{F}^m) = \mathbb{D} - \mathcal{H}_s$; otherwise $\widetilde{\mathscr{P}}(\mathcal{F}^m)$ intersects with both $\mathbb{D}$ and $\overline{\mathbb{D}}$, which implies that $\widetilde{\mathscr{P}}(\mathcal{F}^m) = \mathbb{D} \sqcup \overline{\mathbb{D}} - \mathcal{H}_s - \overline{\mathcal{H}}_s$. □

We introduce an involution on $\mathcal{M}^m$. Take any smooth cubic fourfold $X = Z(F)$, and a marking $\Phi : H^4(X, \mathbb{Z}) \to \Lambda$. Let $X' = Z(\bar{F})$. There exists a homeomorphism τ from X to X' given by the complex conjugation. Let ι be the involution on $\mathcal{M}^m$ sending (X, Φ) to $(X', \Phi \circ \tau^*)$. Consider a smooth cubic fourfold $X = Z(F)$ such that F has real coefficients. Then τ is a diffeomorphism of X, and τ^* sends $H^{3,1}(X)$ to $H^{1,3}(X)$. Therefore, choosing any marking Φ of X, the points $[(X, \Phi)]$ and $[(X, \Phi \circ \tau^*)]$ lie in different components of $\mathcal{M}^m$. This implies that the involution ι exchanges the two components of $\mathcal{M}^m$.

Next we give criteria on the number of connected components of $\mathcal{F}^m$. For a symmetry type $T = [(A, \lambda)]$, we define the complex conjugate $\overline{T}$ of T to be $[(\widetilde{A}, \widetilde{\lambda})]$, where $\widetilde{A}$ is the complex conjugate of A, and $\widetilde{\lambda}(a) = \lambda(\bar{a})$ for all $a \in \widetilde{A}$. From the definition, the involution ι exchanges the two spaces $\mathcal{F}^m_T$ and $\mathcal{F}^m_{\overline{T}}$.

Proposition 4.9. *Given a symmetry type $T = [(A, \lambda)]$:*

(i) *If ζ is not real, then $\mathcal{F}^m$ is connected.*

(ii) *If $T = \overline{T}$, then $\mathcal{F}^m$ has two components.*

(iii) *If T satisfies Condition 2.3, and $T \neq \overline{T}$, then $\mathcal{F}^m$ is connected.*

Proof. Suppose ζ is not real, then $\widetilde{\mathscr{P}}(\mathcal{F}^m)$ lies in the ball attached to (Λ_ζ, h). Thus $\mathcal{F}^m$ is connected.

Suppose $T = \overline{T}$, then $\mathcal{F}^m$ is preserved by ι. Thus $\mathcal{F}^m$ has two components.

Suppose $\mathcal{F}^m$ has two components, then $\widetilde{\mathscr{P}}(\mathcal{F}^m) = \mathbb{D} \sqcup \overline{\mathbb{D}} - \mathcal{H}_s - \overline{\mathcal{H}_s}$. Thus $\mathcal{F}^m$ is preserved by ι. Thus $\mathcal{F}^m_{\overline{T}} = \mathcal{F}^m_T$. This can not happen if T satisfies Condition 2.3 and $T \neq \overline{T}$. The third part follows. □

4C. *Global period map.* In this section we are going to define the global period domain for symmetric cubic fourfolds of type T as an arithmetic quotient of $\mathbb{D}$, and study the global period map.

Let $(d, k) = (3, 4)$ and fix a symmetry type $T = [(A, \lambda)]$ satisfying Condition 2.2. Let $\Gamma = \{\rho \in \widehat{\Gamma} \mid \rho \bar{A} \rho^{-1} = \bar{A}\}$ be the normalizer of $\bar{A}$ in $\widehat{\Gamma}$. Take $\rho \in \widehat{\Gamma}$ and a point $x \in \Lambda_\zeta$. We claim that $\rho x \in \Lambda_\zeta$. In fact, taking any $a \in A$, we have

$$a\rho x = \rho\rho^{-1} a\rho x = \rho\zeta(\rho^{-1} a\rho)x = \zeta(\rho^{-1} a\rho)\rho x.$$

Since $\rho \in \widehat{\Gamma}$, we have $\rho[x] \in \widehat{\mathbb{D}}$. The two characters ζ and $\rho^{-1}\zeta\rho$ both give nondefinite eigensubspaces of $\Lambda_{\mathbb{C}}$. We conclude that $\zeta = \rho^{-1}\zeta\rho$; hence $\rho x \in \Lambda_\zeta$. This gives a natural action of Γ on $\mathbb{D}$.

Let N_A be the normalizer of A in $\mathrm{Aut}((\Lambda_0)_{\mathbb{Q}}, \varphi)$, which is a reductive algebraic subgroup. The group Γ is an arithmetic subgroup of N_A; see the Appendix. The arithmetic quotient $\Gamma\backslash\mathbb{D}$ is a quasiprojective variety thanks to the Baily–Borel compactification (see Section A3 in the Appendix). From our assumption that the local period map $\widetilde{\mathscr{P}}$ for $\mathcal{F}^m$ takes values in $\mathbb{D}$, we can take $(\mathcal{F}^m)^1$ to be the connected component of $\mathcal{F}^m$ such that $\widetilde{\mathscr{P}}((\mathcal{F}^m)^1) = \mathbb{D} - \mathcal{H}_s$. Notice that when $\mathcal{F}^m$ is connected, we have $(\mathcal{F}^m)^1 = \mathcal{F}^m$.

Proposition 4.10. *The local period map $\widetilde{\mathscr{P}} : (\mathcal{F}^m)^1 \to \mathbb{D} - \mathcal{H}_s$ descends to an algebraic isomorphism $\mathscr{P} : \mathcal{F} \cong \Gamma\backslash(\mathbb{D} - \mathcal{H}_s)$.*

Proof. There are natural analytic morphisms from $\mathcal{F}^m$ to $\mathcal{F}$, and $\mathbb{D} - \mathcal{H}_s$ to $\Gamma\backslash(\mathbb{D} - \mathcal{H}_s)$. We define the global period map $\mathscr{P} : \mathcal{F} \to \Gamma\backslash(\mathbb{D} - \mathcal{H}_s)$ as follows. Take $F \in \mathcal{V}^{sm}$. We choose a T-marking Φ of $X = Z(F)$, such that $\Phi(H^{3,1}(X)) \in \mathbb{D}$ (this also means that $(F, \Phi) \in (\mathcal{F}^m)^1$). We define

$$\mathscr{P}([F]) = [\widetilde{\mathscr{P}}(X, \Phi)].$$

We show this map is well-defined. Take $F_1, F_2 \in \mathcal{V}^{sm}$ with T-markings Φ_1, Φ_2 respectively. Suppose there exists $g \in N$, such that $g(F_1) = F_2$. We have an induced map

$$g^* : H^4(Z(F_2), \mathbb{Z}) \to H^4(Z(F_1), \mathbb{Z}).$$

Next we show $\rho = \Phi_1 g^* \Phi_2^{-1} \in \Gamma$. Denote $a' = gag^{-1}$. Since $g \in N$, we have $a' \in A$. We have the following commutative diagram:

$$\begin{array}{ccccccc}
\Lambda & \xrightarrow{\Phi_2^{-1}} & H^4(Z(F_2), \mathbb{Z}) & \xrightarrow{g^*} & H^4(Z(F_1), \mathbb{Z}) & \xrightarrow{\Phi_1} & \Lambda \\
\downarrow{\scriptstyle a'} & & \downarrow{\scriptstyle a'^*} & & \downarrow{\scriptstyle a^*} & & \downarrow{\scriptstyle a} \\
\Lambda & \xrightarrow{\Phi_2^{-1}} & H^4(Z(F_2), \mathbb{Z}) & \xrightarrow{g^*} & H^4(Z(F_1), \mathbb{Z}) & \xrightarrow{\Phi_1} & \Lambda
\end{array}$$

This implies that, as automorphisms of Λ, $a' = \rho^{-1} a \rho$. Thus $\rho \in \Gamma$. We then have a well-defined analytic morphism $\mathscr{P} : \mathcal{F} \to \Gamma\backslash(\mathbb{D} - \mathcal{H}_s)$.

By definition we have the following commutative diagram:

$$\begin{array}{ccc}
(\mathcal{F}^m)^1 & \xrightarrow{\widetilde{\mathscr{P}}} & \mathbb{D} - \mathcal{H}_s \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \pi} \\
\mathcal{F} & \xrightarrow{\mathscr{P}} & \Gamma\backslash(\mathbb{D} - \mathcal{H}_s)
\end{array} \tag{3}$$

We next show that $\mathscr{P} : \mathcal{F} \to \Gamma\backslash(\mathbb{D} - \mathcal{H}_s)$ is an isomorphism.

We first show the injectivity. Suppose that $(F_1, \Phi_1), (F_2, \Phi_2) \in \mathcal{F}^m$, with $\Phi_1(H^{3,1}(Z(F_1)))$ and $\Phi_2(H^{3,1}(Z(F_2)))$ representing the same point in $\Gamma\backslash(\mathbb{D} - \mathcal{H}_s)$. Then there exists $\rho \in \Gamma$, such that $\rho\Phi_1(H^{3,1}(Z(F_1))) = \Phi_2(H^{3,1}(Z(F_2)))$. The map

$$\Phi_2^{-1}\rho\Phi_1 : H^4(Z(F_1), \mathbb{Z}) \to H^4(Z(F_2), \mathbb{Z})$$

preserves the polarized Hodge structures. By Lemma 3.6, we have $g \in \mathrm{SL}(V)$, with gF_2 equals to F_1 after rescaling of F_2, and $g^* = \Phi_2^{-1}\rho\Phi_1$. For any $a \in A$, we have $a^* : H^4(Z(F_1), \mathbb{Z}) \to H^4(Z(F_1), \mathbb{Z})$. The $g^{-1}ag$ acts on $Z(F_2)$, and this induces

$$(g^{-1}ag)^* = g^*a^*g^{*-1} = (\Phi_2^{-1}\rho\Phi_1)(\Phi_1^{-1}a\Phi_1)(\Phi_1^{-1}\rho^{-1}\Phi_2) = \Phi_2^{-1}\rho a\rho^{-1}\Phi_2.$$

Since $\rho \in \Gamma$, we have $\rho a \rho^{-1} \in A$. Again by Lemma 3.6, we have $g^{-1}ag \in A$. Since

$$g^{-1}agF_2 = g^{-1}aF_1 = \lambda(a)g^{-1}F_1 = \lambda(a)F_2,$$

we have $\lambda(g^{-1}ag) = \lambda(a)$. We conclude $g \in N$. Thus $\mathscr{P}$ is injective.

By Proposition 4.8, the composition of

$$(\mathcal{F}^m)^1 \to \mathbb{D} - \mathcal{H}_s \to \Gamma\backslash(\mathbb{D} - \mathcal{H}_s)$$

is surjective. By commutativity of diagram (3), the composition of

$$(\mathcal{F}^m)^1 \to \mathcal{F} \to \Gamma\backslash(\mathbb{D} - \mathcal{H}_s)$$

is also surjective; hence $\mathscr{P} : \mathcal{F} \to \Gamma\backslash(\mathbb{D} - \mathcal{H}_s)$ is surjective.

The algebraicity of $\mathscr{P}$ can be deduced from its extension to certain compactifications on both sides; see Theorem 5.3. An alternative argument follows the proof of Proposition 2.2.3 in [Hassett 2000] using Baily–Borel compactification and the Borel extension theorem. □

5. Compactifications

In this section we are going to study the compactifications of both two sides of $\mathscr{P} : \mathcal{F} \to \Gamma\backslash(\mathbb{D} - \mathcal{H}_s)$. The essential ingredient is the identification between the GIT compactification of the moduli space of cubic fourfolds and the Looijenga compactification of the global period domain, proved by Looijenga [2009] and Laza [2010] independently. Depending on this, we will prove Theorem 1.2(i), and then deduce Theorem 1.1(iii). In Theorem 5.7 (=Theorem 1.2(ii)), we give a criterion when the Looijenga compactification is actually Baily–Borel compactification.

Let $(d, k) = (3, 4)$. Recall that from Theorem 4.5 we have the isomorphism $\mathscr{P} : \mathcal{M}_1 \cong \widehat{\Gamma}\backslash(\widehat{\mathbb{D}} - \mathcal{H}_\infty)$. From [Looijenga 2009; Laza 2010] we have:

Theorem 5.1 (Looijenga, Laza). *The period map $\mathscr{P}$ extends to an isomorphism $\mathscr{P} : \overline{\mathcal{M}} \to \overline{\widehat{\Gamma}\backslash\widehat{\mathbb{D}}}^{\mathcal{H}_\infty}$.*

Recall that $\mathcal{H}_* = \mathbb{D} \cap (\mathcal{H}_\infty \cup \overline{\mathcal{H}}_\infty)$, which is a Γ-invariant hyperplane arrangement in $\mathbb{D}$. We have a morphism between locally symmetric varieties

$$\Gamma\backslash\mathbb{D} \to \mathrm{Aut}(\Lambda, \eta)\backslash\widetilde{\mathbb{D}} \cong \widehat{\Gamma}\backslash\widehat{\mathbb{D}}.$$

We can construct the Looijenga compactification $\overline{\Gamma\backslash\mathbb{D}}^{\mathcal{H}_*}$ of $\Gamma\backslash(\mathbb{D} - \mathcal{H}_*)$ (see the Appendix). From Theorem A.13, we have:

Proposition 5.2. *There exists a finite morphism* $\pi : \overline{\Gamma\backslash\mathbb{D}}^{\mathcal{H}_*} \to \overline{\widehat{\Gamma}\backslash\widehat{\mathbb{D}}}^{\mathcal{H}_\infty}$. *If* T *satisfies Condition 2.3, then this morphism is a normalization of its image.*

We now state our main theorem:

Theorem 5.3. *The global period map* $\mathscr{P} : \mathcal{F} \cong \Gamma\backslash(\mathbb{D} - \mathcal{H}_s)$ *extends to an algebraic isomorphism* $\mathscr{P} : \overline{\mathcal{F}} \cong \overline{\Gamma\backslash\mathbb{D}}^{\mathcal{H}_*}$.

We need the following fact in algebraic geometry. We give the proof for the reader's convenience.

Lemma 5.4. *Let* $f_1 : Z_1 \to Y$ *and* $f_2 : Z_2 \to Y$ *be finite morphisms between irreducible algebraic varieties. Suppose* Z_1, Z_2 *are normal. Moreover, suppose that there exists Zariski-open subset* U_i *of* Z_i, $i = 1$ *or* 2, *with a biholomorphic map* $g : U_1 \to U_2$, *such that* $f_1 = f_2 \circ g$. *Then* g *extends to an algebraic isomorphism* $Z_1 \to Z_2$.

Proof. Without loss of generality, we assume that Y is affine. Let $\mathbb{C}(Z)$ be the field of rational functions on an irreducible algebraic variety Z, and $M(Z)$ the field of meromorphic functions. We claim $g^*\mathbb{C}(Z_2) = \mathbb{C}(Z_1)$. Let $x \in \mathbb{C}(U_2) = \mathbb{C}(Z_2)$. Since $\mathbb{C}(U_2)$ is a finite extension of $\mathbb{C}(Y)$, g^*x is finite over $\mathbb{C}(U_1)$. We can find a Zariski-open subset U_1° of U_1, with a Galois covering $\widetilde{U} \to U_1^\circ$, such that $g^*x \in \mathbb{C}(\widetilde{U})$. Since $g^*x \in M(U_1^\circ)$, it is invariant under the action of Deck transformations. Thus $g^*x \in \mathbb{C}(U_1^\circ) = \mathbb{C}(Z_1)$. The claim follows.

The coordinate ring $\mathbb{C}[Z_i]$ is the integral closure of $\mathbb{C}[Y]$ in $\mathbb{C}(Z_i)$. So $g^*\mathbb{C}[Z_2] = \mathbb{C}[Z_1]$. Thus g extends to an algebraic isomorphism $Z_1 \cong Z_2$. □

Proof of Theorem 5.3. We have the commutative diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\cong"] \arrow[d, hook] & \Gamma\backslash(\mathbb{D} - \mathcal{H}_s) \arrow[d, hook] \\
\overline{\mathcal{F}} \arrow[r, dashed] \arrow[d, "j"] & \overline{\Gamma\backslash\mathbb{D}}^{\mathcal{H}_*} \arrow[d, "\pi"] \\
\overline{\mathcal{M}} \arrow[r, "\mathscr{P}"] & \overline{\widehat{\Gamma}\backslash\widehat{\mathbb{D}}}^{\mathcal{H}_\infty}
\end{tikzcd} \tag{4}$$

with both j, π finite morphisms. The commutativity is straightforward from the definitions of the maps. Since $\mathcal{F}$ is Zariski-open in $\overline{\mathcal{F}}$, the image $j(\mathcal{F})$ contains a Zariski-open subset of $j(\overline{\mathcal{F}})$. Thus $j(\overline{\mathcal{F}})$ is the closure of $j(\mathcal{F})$ in $\overline{\mathcal{M}}$. The same argument shows that $\pi(\overline{\Gamma\backslash\mathbb{D}}^{\mathcal{H}_*})$ is the closure of $\pi(\Gamma\backslash(\mathbb{D} - \mathcal{H}_s))$ in $\overline{\widehat{\Gamma}\backslash\widehat{\mathbb{D}}}^{\mathcal{H}_\infty}$. By commutativity of diagram (4), the two images $j(\mathcal{F})$ and $\pi(\Gamma\backslash(\mathbb{D} - \mathcal{H}_s))$ are identified via $\mathscr{P}$, so are $j(\overline{\mathcal{F}})$ and $\pi(\overline{\Gamma\backslash\mathbb{D}}^{\mathcal{H}_*})$. By Propositions 2.7, 5.2 and Lemma 5.4, we have an identification between $\overline{\mathcal{F}}$ and $\overline{\Gamma\backslash\mathbb{D}}^{\mathcal{H}_*}$ which extends $\mathscr{P} : \mathcal{F} \cong \Gamma\backslash(\mathbb{D} - \mathcal{H}_s)$. This identification is the extended global period map $\mathscr{P} : \overline{\mathcal{F}} \cong \overline{\Gamma\backslash\mathbb{D}}^{\mathcal{H}_*}$. □

The proof of the above theorem does not use algebraicity of $\mathscr{P}$. Actually, we can deduce algebraicity of $\mathscr{P}$ from Theorem 5.3. At this point, we have already finished the proof of Theorem 1.1(i), (ii) and Theorem 1.2(i). In the rest of this section, we prove Theorem 1.1(iii) and Theorem 1.2(ii).

Let $\mathcal{V}_1$ be the subset of $\mathcal{V}$ consisting of cubic forms of type T defining cubic fourfolds with at worst ADE-singularities. The points in $\mathcal{V}_1$ are stable with respect to the action of $\mathrm{SL}(V)$ on $\mathrm{Sym}^3(V^*)$; hence also stable with respect to the action of N on $\mathcal{V}$. Define $\mathcal{F}_1 = N\backslash\!\backslash\mathbb{P}\mathcal{V}_1$ to be the moduli space of cubic fourfolds of type T with at worst ADE-singularities. We have:

Proposition 5.5. *The period map $\mathscr{P}:\mathcal{F}\to\Gamma\backslash(\mathbb{D}-\mathcal{H}_s)$ extends to an algebraic isomorphism $\mathscr{P}:\mathcal{F}_1\cong\Gamma\backslash(\mathbb{D}-\mathcal{H}_*)$.*

Proof. From the definition we have $j(\mathcal{F}_1)=j(\overline{\mathcal{F}})\cap\mathcal{M}_1$ and $j^{-1}(j(\mathcal{F}_1))=\mathcal{F}_1$. From Proposition 2.7, the morphism $j:\mathcal{F}_1\to\mathcal{M}_1$ is finite. On the other hand, we have

$$\pi(\Gamma\backslash(\mathbb{D}-\mathcal{H}_*))=\pi(\overline{\Gamma\backslash\mathbb{D}}^{\mathcal{H}_*})\cap\widehat{\Gamma}\backslash(\widehat{\mathbb{D}}-\mathcal{H}_\infty)$$

and

$$\pi^{-1}(\pi(\Gamma\backslash(\mathbb{D}-\mathcal{H}_*)))=\Gamma\backslash(\mathbb{D}-\mathcal{H}_*).$$

From Proposition 5.2, the morphism $\pi:\Gamma\backslash(\mathbb{D}-\mathcal{H}_*)\to\widehat{\Gamma}\backslash(\widehat{\mathbb{D}}-\mathcal{H}_\infty)$ is finite. By Theorems 4.5 and 5.3, the two images $j(\mathcal{F}_1)$ and $\pi(\Gamma\backslash(\mathbb{D}-\mathcal{H}_*))$ are identified via $\mathscr{P}$. By Lemma 5.4, we have the algebraic isomorphism $\mathscr{P}:\mathcal{F}_1\cong\Gamma\backslash(\mathbb{D}-\mathcal{H}_*)$. □

If the hyperplane arrangement $\mathcal{H}_*$ is empty, then the Looijenga compactification of $\Gamma\backslash\mathbb{D}$ is actually the Baily–Borel compactification. Next we give a criterion of emptiness of $\mathcal{H}_*$ from the perspective of GIT. Following Section 6 of [Laza 2009], there is a rational curve χ parametrizing certain semistable cubic fourfolds, given by

$$F_{a,b}(x_0,\dots,x_5)=\begin{vmatrix} x_0 & x_1 & x_2+2ax_5\\ x_1 & x_2-ax_5 & x_3\\ x_2+2ax_5 & x_3 & x_4\end{vmatrix}+bx_5^3$$

where $(a:b)\in\mathrm{WP}(1:3)$ with $\mathrm{WP}(1:3)$ the weighted projective space of weight $(1:3)$. We denote by $X_{(a:b)}$ the cubic fourfold defined by $F_{a,b}$. Denote

$$F_0(x_0,\dots,x_4)=\begin{vmatrix} x_0 & x_1 & x_2\\ x_1 & x_2 & x_3\\ x_2 & x_3 & x_4\end{vmatrix},$$

then $F_{a,b}(x_0,\dots,x_5)=F_0(x_0,\dots,x_4)+ax_5(4x_1x_3-3x_2^2-x_0x_4)+(b-4a^3)x_5^3$.

We next define two pairs (G_1,λ_1) and (G_2,λ_2) which will be used in Theorem 5.7. Here for $i=1$ or 2, G_i is a subgroup of $\mathrm{SL}(V)$ and $\lambda_i:G_i\to\mathbb{C}^\times$ is a character of G_i. As we will discuss below, the pairs (G_1,λ_1) and (G_2,λ_2) are essentially symmetries for $F_{1,0}$ and $F_{0,1}$ respectively.

The cubic fourfold $X_{(1:0)}$ is called the determinantal cubic fourfold. The singular locus of $X_{(1:0)}$ is a rational surface. Explicitly, take V_3 to be a complex vector space of dimension 3 and denote by

$[y_0 : y_1 : y_2]$ a homogeneous coordinate for $\mathbb{P}V_3$. Consider an embedding $\mathbb{P}V_3 \hookrightarrow \mathbb{P}V$ defined by $[y_0 : y_1 : y_2] \mapsto [x_0 : \cdots : x_5]$ with $x_0 = y_0^2,\ x_1 = y_0y_1,\ x_2 - x_5 = y_1^2,\ x_0 + 2x_5 = y_0y_2,\ x_3 = y_1y_2,\ x_4 = y_2^2$. This induces a natural morphism from $\mathrm{GL}(V_3)$ to $\mathrm{GL}(V)$. The image of $\mathbb{P}V_3$ in $\mathbb{P}V$ is called the Veronese surface, and it is the singular locus of $X_{(1:0)}$. Actually the singular cubic fourfold $X_{(1:0)}$ is the secant variety of the Veronese surface in $\mathbb{P}^5$, and the linear automorphism group of $X_{(1:0)}$ can be identified with $\mathrm{PSL}(V_3)$. For each $g \in \mathrm{GL}(V_3)$ there is a complex number $\lambda_1(g)$ such that $gF_{1,0} = \lambda_1(g)F_{1,0}$. We hence obtain a character λ_1 of $\mathrm{GL}(V_3)$. By standard theory on general linear group, there exists an integer k such that $\lambda_1(g) = \det(g)^k$ for any $g \in \mathrm{GL}(V_3)$. To know k, we only need to compute $\lambda_1(g)$ for a special g. Take $g : (y_0, y_1, y_2) \mapsto (ty_0, y_1, y_2)$. Then

$$g : (x_0, x_1, x_2 + 2x_5, x_2 - x_5, x_3, x_4) \mapsto (t^2x_0, tx_1, t(x_2 + 2x_5), x_2 - x_5, x_3, x_4).$$

Thus $gF_{1,0} = t^2F_{1,0}$. This implies that $\lambda_1(g) = t^2 = \det(g)^2$ and we have $k = 2$. In conclusion, for any $g \in \mathrm{GL}(V_3)$ we have $\lambda_1(g) = \det(g)^2$.

For $b \neq 0$, the singular locus of the cubic fourfold $X_{(a:b)}$ is a rational curve. Explicitly, take V_2 to be a complex vector space of dimension 2 and denote by $[y_0 : y_1]$ a homogeneous coordinate for $\mathbb{P}V_2$. Let V_5 be the subspace of V defined by $x_5 = 0$. Consider an embedding $\mathbb{P}V_2 \hookrightarrow \mathbb{P}V_5$ defined by $[y_0 : y_1] \mapsto [x_0 : \cdots : x_4]$ with $x_i = y_0^{4-i}y_i$ for $i = 0, 1, 2, 3, 4$. This also induces a natural morphism from $\mathrm{GL}(V_2)$ to $\mathrm{GL}(V)$. Then the singular locus of $X_{(a:b)}$ is the image of $\mathbb{P}V_2 \hookrightarrow \mathbb{P}V_5 \hookrightarrow \mathbb{P}V$. By [Laza 2009, Proposition 6.6 and its proof] the linear automorphism group of $X_{(a:b)}$ for a generic choice $(a : b) \in \mathrm{WP}(1 : 3)$ is $\mathrm{PSL}(V_2)$. For any $g \in \mathrm{GL}(V_2)$, there exists a complex number $\lambda_2(g)$ such that $gF_0 = \lambda_0(g)F_0$. A similar calculation as before gives $\lambda_0(g) = \det(g)^6$.

When $(a, b) = (0, 1)$, we have extra automorphisms of $X_{(a:b)}$ given by taking scalars on x_5. Suppose $(g, u) \in \mathrm{GL}(V_5) \times \mathbb{C}^\times$ is an automorphism of $F_{0,1} = F_0 + x_5^3$. Since $gF_0 = \det(g)^6F_0$ and $u(x_0^3) = u^3x_0^3$, we must have $\det(g)^6 = u^3$. Thus $\det(g)^2/u$ is a third root of unity. The following definition is then natural:

Definition 5.6. (i) Let G_1 be the intersection of $\mathrm{SL}(V)$ with the image of $\mathrm{GL}(V_3) \to \mathrm{GL}(V)$.

(ii) Let $\widetilde{G_2}$ be the subgroup of $\mathrm{GL}(V_2) \times \mathbb{C}^*$ consisting of elements (g, u) such that $(\det g)^2/u$ is a third root of unity. Let G_2 be the intersection of $\mathrm{SL}(V)$ with the image of the natural map $\widetilde{G_2} \to \mathrm{GL}(V)$.

Both G_1 and G_2 contain the center of $\mathrm{SL}(V)$. The restriction of λ_1 to G_1 is still denoted by λ_1. For G_2, we have a character $\lambda_2 : (g, u) \mapsto \lambda_0(g) = \det(g)^6 = u^3$. The next theorem gives a criterion on emptiness of $\mathcal{H}_*$. We will apply this criterion to prime-order groups (Proposition 6.5).

Theorem 5.7. *For a symmetry type (A, λ) satisfying Condition 2.2, the following three statements are equivalent*:

(i) *The hyperplane arrangement $\mathcal{H}_*$ is nonempty.*

(ii) *The space $\mathbb{P}\mathcal{V}_\lambda$ intersects with the orbit $\mathrm{PSL}(V)\chi$ of the rational curve χ in $\mathbb{P}\mathrm{Sym}^3(V^*)$.*

(iii) *For $i = 1$ or 2, there exists $h \in \mathrm{SL}(V)$ such that $h^{-1}Ah \subset G_i$ and for any $a \in A$ we have $\lambda(a) = \lambda_i(h^{-1}ah)$. If this is satisfied, we say that (A, λ) factors through (G_i, λ_i).*

Proof. We first show the equivalence of (i) and (ii). If (ii) holds, the intersection points survive after taking GIT quotients since the $\mathrm{PSL}(V)$ orbits of points in χ are closed. Conversely, suppose $j(\overline{\mathcal{F}})$ intersects with the image of χ at $[F]$ in $\overline{\mathcal{M}}$. We can always take the representative F in $\mathcal{V}_\lambda$ has closed N-orbit. According to the main theorem in [Luna 1975], the $\mathrm{PSL}(V)$-orbit of $[F] \in \mathbb{P}\mathrm{Sym}^3(V^*)$ is also closed. So F represents an element in $\mathrm{PSL}(V)\chi$.

Secondly we recall that the blow-up and blow-down construction in Looijenga compactification $\overline{\widehat{\Gamma}\backslash\widehat{\mathbb{D}}}^{\mathcal{H}_\infty}$ gives a stratum corresponding to χ. We claim that $\mathcal{H}_*$ is nonempty if and only if the image of $\overline{\Gamma\backslash\mathbb{D}}^{\mathcal{H}_*}$ in $\overline{\widehat{\Gamma}\backslash\widehat{\mathbb{D}}}^{\mathcal{H}_\infty}$ intersects with the stratum. From the proof of functoriality of semitoric compactification in Section A4, we know that $\mathbb{D}^\Sigma$ intersects with $\overline{\mathcal{H}}_\infty$ if and only if $\mathbb{D}$ intersects with $\mathcal{H}_\infty$. So the image of $\overline{\Gamma\backslash\mathbb{D}}^{\mathcal{H}_*}$ intersects with the stratum if and only if $\mathbb{D}$ intersects with $\mathcal{H}_\infty$. By diagram (4), the intersection of $j(\overline{\mathcal{F}})$ with the image of χ in $\overline{\mathcal{M}}$ is equivalent to the intersection of the image of $\pi(\overline{\Gamma\backslash\mathbb{D}}^{\mathcal{H}_*})$ with the stratum corresponding to χ. The equivalence of (i) and (ii) follows.

Next we show the equivalence of (ii) and (iii). Suppose (iii) is satisfied, then for $i = 1$ or 2, there exists $h \in \mathrm{GL}(V)$ such that $h^{-1}Ah \subset G_i$ and $\lambda(a) = \lambda_i(h^{-1}ah)$ for any $a \in A$. Then $hF_{1,0}$ or $hF_{0,1}$ lies in $\mathcal{V}_\lambda$. This implies (ii).

Suppose (ii) holds. Then there is a member $F_{a,b}$ in χ, and an element $h \in \mathrm{GL}(V)$, such that $hF_{a,b} \in \mathcal{V}_\lambda$. If $b = 0$ or $a = 0$, then (A, λ) factors through (G_1, λ_1) or (G_2, λ_2). Otherwise, we claim that the linear automorphism group of $X_{(a:b)}$ is indeed $\mathrm{PSL}(V_2)$; hence (A, λ) factors through both (G_1, λ_1) and (G_2, λ_2). Let $g \in \mathrm{GL}(V)$ be an automorphism of $F_{a,b}$. Since the singular locus of $F_{a,b}$ is the image $\mathbb{P}V_2 \hookrightarrow \mathbb{P}V$, the automorphism g fixes $\mathbb{P}V_2$ which is the smallest subspace of $\mathbb{P}V$ containing the singular locus. Moreover, g is induced by an element of $\mathrm{GL}(V_2)$. Since $F_{a,b}(x_0, \dots, x_5)$ contains no monomial with the degree of x_5 equal to 2, the action of g on the coordinate x_5 is by a scalar. By $a \neq 0$ we conclude that g fixes x_5. Therefore, the linear automorphism group of $X_{(a:b)}$ is $\mathrm{PSL}(V_2)$. The discussion above shows that (ii) and (iii) are equivalent. □

6. Examples and related constructions

In this section we apply our theorems to specific examples. We will first review the classification of prime-order automorphisms of smooth cubic fourfolds [González-Aguilera and Liendo 2011, Theorem 3.8] in Section 6A, then in Section 6B we will show how our results recover a main theorem in [Laza et al. 2018].

6A. ***Prime-order automorphisms of smooth cubic fourfolds.*** The classification of prime-order automorphisms of smooth cubic fourfolds was given in [González-Aguilera and Liendo 2011, Theorem 3.8]. For the reader's convenience we present the result in this section. (There was a small mistake in [González-Aguilera and Liendo 2011, Theorem 3.8]. The second example with $p = 5$ contains only singular cubic fourfolds. This is pointed out in [Boissière et al. 2016, Remark 6.3]).

Proposition 6.1 [González-Aguilera and Liendo 2011]. *Let ω be a prime p-th root of unity and $\rho = (m_0, \dots, m_5)$ be the automorphism of $V \cong \mathbb{C}^6$ given by $(x_0, \dots, x_5) \mapsto (\omega^{m_0}x_0, \dots, \omega^{m_5}x_5)$. The list of* *smooth cubic polynomials F preserved by the action under ρ is as follows*:

$T_2^1: \rho=(0,0,0,0,0,1)$, $\quad n=14$, $\quad F=L_3(x_0,\dots,x_4)+x_5^2L_1(x_0,\dots,x_4)$.

$T_2^2: \rho=(0,0,0,0,1,1)$, $\quad n=12$,
$$F=L_3(x_0,\dots,x_3)+x_4^2L_1(x_0,\dots,x_3)+x_4x_5M_1(x_0,\dots,x_3)+x_5^2N_1(x_0,\dots,x_3).$$

$T_2^3: \rho=(0,0,0,1,1,1)$, $\quad n=10$,
$$F=L_3(x_0,x_1,x_2)+x_0L_2(x_3,x_4,x_5)+x_1M_2(x_3,x_4,x_5)+x_2N_2(x_3,x_4,x_5).$$

$T_3^1: \rho=(0,0,0,0,0,1)$, $\quad n=10$, $\quad F=L_3(x_0,\dots,x_4)+x_5^3$.

$T_3^2: \rho=(0,0,0,0,1,1)$, $\quad n=4$, $\quad F=L_3(x_0,\dots,x_3)+M_3(x_4,x_5)$.

$T_3^3: \rho=(0,0,0,0,1,2)$, $\quad n=8$, $\quad F=L_3(x_0,\dots,x_3)+x_4^3+x_5^3+x_4x_5M_1(x_0,\dots,x_3)$.

$T_3^4: \rho=(0,0,0,1,1,1)$, $\quad n=2$, $\quad F=L_3(x_0,x_1,x_2)+M_3(x_3,x_4,x_5)$.

$T_3^5: \rho=(0,0,0,1,1,2)$, $\quad n=7$,
$$F=L_3(x_0,x_1,x_2)+M_3(x_3,x_4)+x_5^3+x_3x_5L_1(x_0,x_1,x_2)+x_4x_5M_1(x_0,x_1,x_2).$$

$T_3^6: \rho=(0,0,1,1,2,2)$, $\quad n=8$,
$$F=L_3(x_0,x_1)+M_3(x_2,x_3)+N_3(x_4,x_5)+\sum_{i=1,2;j=3,4;k=5,6}a_{ijk}x_ix_jx_k.$$

$T_3^7: \rho=(0,0,1,1,2,2)$, $\quad n=6$,
$$F=x_2L_2(x_0,x_1)+x_3M_2(x_0,x_1)+x_4^2L_1(x_0,x_1)$$
$$+x_4x_5M_1(x_0,x_1)+x_5^2N_1(x_0,x_1)+x_4N_2(x_2,x_3)+x_5O_2(x_2,x_3).$$

$T_5^1: \rho=(0,0,1,2,3,4)$, $\quad n=4$,
$$F=L_3(x_0,x_1)+x_2x_5L_1(x_0,x_1)+x_3x_4M_1(x_0,x_1)+x_2^2x_4+x_2x_3^2+x_3x_5^2+x_4^2x_5.$$

$T_7^1: \rho=(1,2,3,4,5,6)$, $\quad n=2$, $\quad F=x_0^2x_4+x_1^2x_2+x_0x_2^2+x_3^2x_5+x_3x_4^2+x_1x_5^2+ax_0x_1x_3+bx_2x_4x_5$

$T_{11}^1: \rho=(0,1,3,4,5,9)$, $\quad n=0$, $\quad F=x_0^3+x_1^2x_5+x_2^2x_4+x_2x_3^2+x_1x_4^2+x_3x_5^2$.

Here the lower index is the prime p, the polynomials L_i, M_i, N_i are of degree i, and n is the dimension of the corresponding GIT-*quotient.*

Remark 6.2. This classification offers 13 symmetry types with $\#\bar{A}$ a prime number 2, 3, 5, 7 or 11. Those symmetry types may not satisfy Condition 2.3. See previous discussion in Remark 2.4.

By Griffiths residue calculus [1969a; 1969b], for a smooth cubic fourfold $X=Z(F)$, the complex line $H^{3,1}(X)$ is generated by $\mathrm{Res}_X(\Omega/F^2)$. Here $\Omega=\Sigma_{i=0}^5(-1)^ix_idx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_5$. By direct calculation, we have:

Proposition 6.3. (i) *For type $T=T_2^2, T_3^3, T_3^4, T_3^6, T_5^1, T_7^1, T_{11}^1$, we have $\zeta=1$.*

(ii) *For type $T=T_2^1, T_2^3$, we have $\zeta=-1$*

(iii) *For type $T=T_3^1, T_3^2, T_3^5, T_3^7$, we have that $\zeta(\rho)$ is equal to ω or $\overline{\omega}$.*

We already proved that $\mathscr{P}(\mathcal{F}^m)$ is either $\mathbb{D}-\mathcal{H}_s$ or $\mathbb{D}\sqcup\overline{\mathbb{D}}-\mathcal{H}_s-\overline{\mathcal{H}_s}$. From Proposition 4.9, we have:

Proposition 6.4. (i) *If $T = T_3^1, T_3^2, T_3^5, T_3^7$, then $\mathbb{D}$ is a complex hyperbolic ball and $\widetilde{\mathscr{P}}(\mathcal{F}^m) = \mathbb{D} - \mathcal{H}_s$.*
(ii) *If $T = T_2^1, T_2^2, T_2^3, T_3^3, T_3^4, T_3^6$ or T_7^1, then $\mathbb{D}$ is a type* IV *domain and $\widetilde{\mathscr{P}}(\mathcal{F}^m) = \mathbb{D} \sqcup \overline{\mathbb{D}} - \mathcal{H}_s - \overline{\mathcal{H}_s}$.*

Now we apply Theorem 5.7 for prime-order cases.

Proposition 6.5. *For $T = T_2^1, T_2^3, T_3^2, T_3^3, T_3^4, T_3^7, T_{11}^1$, we obtain isomorphisms between* GIT *compactifications $\overline{\mathcal{F}}$ with Baily–Borel compactifications $\overline{\Gamma\backslash\mathbb{D}}^{bb}$. For $T = T_2^2, T_3^1, T_3^5, T_3^6, T_5^1, T_7^1$, the corresponding Looijenga compactifications are not Baily–Borel compactifications.*

Proof. We do the calculation for $p = 2$ and 3; the other cases are similar. Suppose $p = 2$. If (A, λ) factors through (G_1, λ_1), a generator of $\bar{A}$ corresponds (up to conjugate) to $g = \mathrm{diag}(1, 1, -1) \in \mathrm{GL}(V_3)$ with order 2. The image of g in $\mathrm{GL}(V)$ is $\mathrm{diag}(1, 1, 1, 1, -1, -1)$. If (A, λ) factors through (G_2, λ_2), then we take $(g, u) \in \mathrm{GL}(V_2) \times \mathbb{C}^*$ such that $g = \mathrm{diag}(1, -1)$ and u is a third root of unity. The image of (g, u) in $\mathrm{GL}(V)$ is $\mathrm{diag}(1, 1, 1, 1, -1, -1)$. In both two cases, we obtain $\mathrm{diag}(1, 1, 1, 1, -1, -1) \in \mathrm{SL}(V)$ and the values of both λ_1 and λ_2 are equal to 1. By Theorem 5.7, the symmetry type T_2^2 does not give Baily–Borel compactification and T_2^1, T_2^3 give Baily–Borel compactifications.

Suppose $p = 3$. If (A, λ) factors through (G_1, λ_1), then a generator of $\bar{A}$ corresponds to $g = \mathrm{diag}(1, 1, \omega) \in \mathrm{GL}(V_3)$ with order 3. The image of g in $\mathrm{GL}(V)$ is $\mathrm{diag}(1, 1, 1, \omega, \omega, \omega^2)$, with value of λ_1 equal to $\det(g)^2 = \omega^2$. If (A, λ) factors through (G_2, λ_2), then we take $(g, u) \in \mathrm{GL}(V_2) \times \mathbb{C}^\times$ with $g = \mathrm{diag}(1, \omega)$ or $\mathrm{diag}(1, 1)$, and u a third root of unity. The image of (g, u) in $\mathrm{GL}(V)$ is $\mathrm{diag}(1, 1, \omega, \omega, \omega^2, u)$ or $\mathrm{diag}(1, 1, 1, 1, 1, u)$. For these elements, the values of λ_2 equal to $u^3 = 1$. We conclude that for T_3^1, T_3^5, T_3^6 we do not obtain Baily–Borel compactification, and for other T_3^i we do. □

Remark 6.6. Notice that Proposition 6.5 is compatible with results in previous literature. For $T = T_2^1$, we have $\mathcal{H}_*$ is empty and we obtain Baily–Borel compactification. This is proved in [Laza et al. 2018] via a lattice-theoretic argument. For $T = T_3^1$, the arrangement $\mathcal{H}_*$ is not empty and we do not obtain Baily–Borel compactification. This coincides with the work in [Looijenga and Swierstra 2007; Allcock et al. 2011].

Remark 6.7. Notice that for the symmetry type T_7^1, the hyperplane arrangement $\mathcal{H}_*$ is nonempty and the dimension of each member is 1. This is one of the examples in which the approach adopted in previous works does not apply; see the discussion right after Theorem 1.2.

6B. *Examples revisit.* Take $T = T_3^1$. Then $T = [(\bar{A} = \mu_3, \lambda = 1)]$ satisfies Condition 2.3. The space $\mathcal{F}$ can be identified with the moduli space of smooth cubic threefolds. The local period domain $\mathbb{D}$ is a complex hyperbolic ball of dimension 10 with an action of an arithmetic group Γ. Then Theorems 1.1 and 1.2 recover the main results in [Looijenga and Swierstra 2007; Allcock et al. 2011]. By Proposition 6.5, the hyperplane arrangement $\mathcal{H}_*$ is nonempty. Actually, from [Looijenga and Swierstra 2007; Allcock et al. 2011], the quotients $\Gamma\backslash\mathcal{H}_s$ has two irreducible components, and $\Gamma\backslash\mathcal{H}_*$ is irreducible.

Take $T = T_2^1$. Then $T = [(\bar{A} = \mu_2, \lambda = 1)]$ satisfies Condition 2.3. In this case, the moduli space $\mathcal{F}$ turns out to be the moduli space of pairs consisting of a cubic threefold and a hyperplane section. This was recently studied in [Laza et al. 2018]. Denote by $\mathcal{W}_1 = H^0(\mathbb{P}^4, \mathcal{O}(3))$ the space of cubic forms in $x_0, \ldots, x_4$ and by $\mathcal{W}_2 = H^0(\mathbb{P}^4, \mathcal{O}(1))$ the space of linear forms in $x_0, \ldots, x_4$. We have an identification $\mathcal{W}_1 \oplus \mathcal{W}_2 \cong \mathcal{V}$ sending (L_3, L_1) to $L_3 + x_5^2 L_1$. In [Laza et al. 2018], the authors defined $\mathcal{F}$ to be a GIT-quotient of $(\mathbb{P}\mathcal{W}_1 \times \mathbb{P}\mathcal{W}_2, \mathcal{O}(3) \boxtimes \mathcal{O}(1))$ by $\mathrm{SL}(5, \mathbb{C})$. Direct calculation shows that $N = C = \mathrm{SL}(5, \mathbb{C}) \times Z \subset \mathrm{SL}(V)$, where $Z = \{\mathrm{diag}(u, u, u, u, u, u^{-5}) \mid u \in \mathbb{C}^\times\}$ is the center. The following proposition gives the relation of our constructions with that in [Laza et al. 2018]:

Proposition 6.8. *We have identification between polarized projective varieties*:

$$Z \backslash\!\backslash (\mathbb{P}\mathcal{V}, \mathcal{O}(1)) \cong (\mathbb{P}\mathcal{W}_1 \times \mathbb{P}\mathcal{W}_2, \mathcal{O}(3) \boxtimes \mathcal{O}(1)).$$

Proof. It is equivalent to show

$$\bigoplus_k (H^0(\mathbb{P}\mathcal{V}, \mathcal{O}(k)))^Z \cong \bigoplus_k H^0(\mathbb{P}\mathcal{W}_1 \times \mathbb{P}\mathcal{W}_2, \mathcal{O}(3k) \boxtimes \mathcal{O}(k))$$

as graded algebras. The action of Z on $\mathcal{W}_1$ has weight 3, and on $\mathcal{W}_2$ weight -9.

We have the direct sum decomposition

$$\mathrm{Sym}^m(\mathcal{V}^*) = \bigoplus_{k+l=m} \mathrm{Sym}^k \mathcal{W}_1^* \otimes \mathrm{Sym}^l \mathcal{W}_2^*$$

with Z-action of weight $-3k+9l$. The weight zero part has $k = 3l$ and $m = 4l$. So we obtain identification between the two polarized varieties. □

Moreover, by Proposition 6.5, the hyperplane arrangement $\mathcal{H}_*$ is empty in this case, and we obtain identification between $\overline{\mathcal{F}}$ and Baily–Borel compactification $\overline{\Gamma \backslash \mathbb{D}}^{bb}$. It is straightforward to see that the arithmetic group Γ is exactly the one used in [Laza et al. 2018]. Therefore, we recover the main result in [Laza et al. 2018].

Appendix: Locally symmetric varieties and Looijenga compactifications

It is well-known that the normalization of each stratum in the orbifold loci of a locally Hermitian symmetric variety is still a locally Hermitian symmetric variety. For the reader's convenience, we include a discussion of this fact in Section A1. In the rest of the appendix, we prove that a similar result (Theorem A.13) holds for Looijenga compactifications. This is first observed by Looijenga [2016, p. 72]. We provide the complete formalism and the details of the proof.

We will recall the construction of Looijenga compactifications [2003a; 2003b] of arithmetic quotients $\mathbb{X}$ of complex hyperbolic balls or type IV domains. There are two steps. The first is constructing the semitoric blowup $\overline{\mathbb{X}}^\Sigma$, which is an intermediate compactification of arithmetic quotient $\mathbb{X}$ sitting between Baily–Borel and toroidal compactifications. We will recall the geometric construction of Baily–Borel compactifications of complex hyperbolic balls and type IV domains in Section A3, and recall the semitoric blow-up construction in Section A4. The second step is taking successive blow-up constructions along the hyperplane arrangement in $\overline{\mathbb{X}}^\Sigma$ and blow-down constructions of certain induced strata (we will sketch this in Section A5).

The idea of the proof of Theorem A.13 is that natural morphisms between arithmetic quotients (of balls or type IV domains) can be extended to morphisms between Baily–Borel compactifications, semitoric compactifications and Looijenga compactifications. Moreover, the extensions are finite morphisms. We call this the functorial property. We will prove the functorial property for Baily–Borel compactification in Section A3, for semitoric compactifications in Section A4, and for Looijenga compactification in Section A5. The existence of the extension in the Baily–Borel case is done in [Kiernan and Kobayashi 1972]. Harris [1989] proved the functorial properties for toroidal compactifications of locally symmetric varieties. The other part of our results in Sections A3, A4, A5, up to our knowledge, are new. Our proof follows the same idea in [Harris 1989]. We need to verify that the combinatorial data associated with the ambient hyperplane arrangements induces the same type of combinatorial data for subspaces, and match each stratum accordingly.

A1. ***Orbifold loci of locally symmetric varieties.*** In this section we show the normalization of an orbifold stratum of a locally Hermitian symmetric variety is again a locally Hermitian symmetric variety.

Let G be a real reductive algebraic group with compact center. Let K be a maximal compact subgroup of G. Let $\mathbb{D} = G/K$ be the corresponding symmetric space. Assume $\mathbb{D}$ is Hermitian symmetric and G has a $\mathbb{Q}$-structure. Let $\Gamma \subset G(\mathbb{Q})$ be an arithmetic subgroup. For simplicity, we assume the action of Γ on $\mathbb{D}$ is faithful. Denote by $\mathbb{X} = \Gamma\backslash\mathbb{D}$ the arithmetic quotient. This is naturally a quasiprojective variety due to Baily–Borel compactification [1966]. There is a natural orbifold structure on $\mathbb{X}$. We consider the orbifold locus indexed by certain finite subgroup $A \subset \Gamma$. More precisely, we take $A \subset \Gamma$ fixing some point $x \in \mathbb{D}$. Without loss of generality, we assume K to be the stabilizer of $x \in \mathbb{D}$ under the action of G. Then $A \subset K$. Denote by G_A, K_A and Γ_A the corresponding normalizers of A in G, K and Γ, respectively. Then G_A is again a real reductive algebraic group with compact center and K_A is a maximal compact subgroup (see [Looijenga 2016, pp. 37–38]). There is a natural holomorphic embedding

$$G_A/K_A \hookrightarrow \mathbb{D} = G/K.$$

Define $\mathbb{D}_A := G_A/K_A$. This is a Hermitian symmetric subspace of $\mathbb{D}$. We have the following proposition:

Proposition A.1. *The group Γ_A is an arithmetic subgroup in $G_A(\mathbb{Q})$ and the map $\pi : \Gamma_A\backslash\mathbb{D}_A \to \Gamma\backslash\mathbb{D}$ is finite. Furthermore, if A is the stabilizer of x under the action of Γ, then this map gives a normalization of its image.*

Proof. Due to the extension theorem of Baily–Borel compactifications (see Theorem 2 in [Kiernan and Kobayashi 1972]), the map π is algebraic and proper. We show π is finite. It suffices to show π is quasifinite, namely, having finite fibers. Take any $y \in \mathbb{D}_A$. Suppose we have a point $y' = \rho y$ for $\rho \in \Gamma$. Then $\rho^{-1}A\rho$ is contained in the stabilizer group of y. Actually, the Γ_A-orbits of such points y' are one-to-one corresponding to subgroups with form $\rho^{-1}A\rho$ in the stabilizer group of y, hence finitely many.

If A is the stabilizer group of x, a generic point in $\mathbb{X}_A := \Gamma_A\backslash\mathbb{D}_A$ also has A as stabilizer group. We first show that π is generically injective in this case. Take generically $x_1, x_2 \in \mathbb{D}_A$, and assume they $[x_1] = [x_2]$ in $\Gamma\backslash\mathbb{D}$. Then there exists $\rho \in \Gamma$ such that $\rho x_1 = x_2$. Since both x_1, x_2 have stabilizer group A, we have $\rho A \rho^{-1} = A$; hence $\rho \in \Gamma_A$. This implies that $[x_1] = [x_2]$ in $\Gamma_A \backslash \mathbb{D}_A$. We have π a finite and birational morphism from a normal variety to its image, hence a normalization of its image. $\square$

Remark A.2. The same construction also works for any finite volume locally Hermitian symmetric varieties. The difference from the arithmetic case is that Γ_A is not automatically a lattice. We need to use the compactification in finite volume case (see Theorem 1 in [Mok and Zhong 1989]) to show that the orbifold locus also admits a compactification, which implies the finiteness of the volume by Yau's Schwarz lemma [1978].

A2. ***Orbifold loci of ball and type* IV *quotients.*** We now focus on arithmetic quotients of balls and type IV domains.

We fix some notation that will be used in the rest of the appendix. Let $(V_{\mathbb{Q}}, \varphi)$ be a vector space over $\mathbb{Q}$ with nondegenerate rational bilinear form φ of signature $(2, N)$. Let $V = V_{\mathbb{Q}} \otimes \mathbb{C}$. Notice that here $V_{\mathbb{Q}}$ is not necessarily the middle cohomology of cubic fourfold. Similar to Section 3, the type IV domain $\widehat{\mathbb{D}}$ attached to $(V_{\mathbb{Q}}, \varphi)$ is a component of

$$\widehat{\mathbb{D}} \sqcup \overline{\widehat{\mathbb{D}}} = \mathbb{P}\{x \in V \mid \varphi(x, x) = 0, \varphi(x, \bar{x}) > 0\}.$$

Denote by $\widehat{G}$ the subgroup of $\mathrm{Aut}(\varphi)(\mathbb{R})$ (of index 2) respecting the component $\widehat{\mathbb{D}}$. Let $\widehat{\Gamma} \subset \widehat{G}$ be an arithmetic subgroup. The corresponding locally Hermitian symmetric variety is $\widehat{\mathbb{X}} = \widehat{\Gamma} \backslash \widehat{\mathbb{D}}$. Let A be a finite subgroup of $\widehat{\Gamma}$. Let ζ be a character of A, such that there exists $x \in V$ with $\varphi(x, x) = 0$ and $\varphi(x, \bar{x}) > 0$, and $a(x) = \zeta(a)x$ for all $a \in A$. Denote by V_ζ the ζ-subspace of V. Then there is a natural Hermitian form h on V_ζ defined by $h(x, y) = \varphi(x, \bar{y})$. If $\zeta = \bar{\zeta}$, this Hermitian form has signature $(2, n)$ and we obtain a type IV subdomain $\mathbb{D}$ of $\widehat{\mathbb{D}}$. Otherwise the signature is $(1, n)$ and we obtain a complex hyperbolic ball $\mathbb{B}$ inside $\widehat{\mathbb{D}}$. Indeed, let

$$G := \{g \in \widehat{G} \mid gAg^{-1} = A\}$$

be an algebraic subgroup over $\mathbb{Q}$. The fixed locus of A in $\mathbb{D}$ is $G(\mathbb{R})/K$, where K is maximal compact subgroup of $G(\mathbb{R})$. Denote $\Gamma = \{\rho \in \widehat{\Gamma} \mid \rho^{-1} A \rho = A\}$. As in Section 4, we have Γ an arithmetic subgroup of $G(\mathbb{Q})$ acting on $\mathbb{B}$ or $\mathbb{D}$. Then we have a natural map $\Gamma \backslash \mathbb{D} \to \widehat{\Gamma} \backslash \widehat{\mathbb{D}}$ or $\Gamma \backslash \mathbb{B} \to \widehat{\Gamma} \backslash \widehat{\mathbb{D}}$. We consider the following condition:

Condition A.3. The group A is the stabilizer of a generic point of $\mathbb{D}$ or $\mathbb{B}$.

If A satisfies this condition, Proposition A.1 implies that the morphism $\pi : \Gamma \backslash \mathbb{B} \to \widehat{\Gamma} \backslash \widehat{\mathbb{D}}$ or $\pi : \Gamma \backslash \mathbb{D} \to \widehat{\Gamma} \backslash \widehat{\mathbb{D}}$ is the normalization of its image.

We will consider a larger set of type IV subdomains. Taking $W_{\mathbb{Q}}$ to be a $\mathbb{Q}$-subspace of $V_{\mathbb{Q}}$ with signature $(2, n)$, we have the associated type IV subdomain $\mathbb{D}$ inside $\widehat{\mathbb{D}}$ with the action of an arithmetic group $\Gamma_W = \{\rho \in \widehat{\Gamma} \mid \rho(W) = W\}$. Take $V_{\mathbb{Z}}$ to be an integral structure on $V_{\mathbb{Q}}$ such that $\Gamma \subset \mathrm{Aut}(V_{\mathbb{Z}})$ has finite index. Denote $W_{\mathbb{Z}} := W_{\mathbb{Q}} \cap V_{\mathbb{Z}}$. For $x \in \mathbb{D}$, define $\mathrm{Pic}(x) := V_x^{1,1} \cap V_{\mathbb{Z}}$ to be the Picard lattice of x where x is viewed as a weight two Hodge structure on $V_{\mathbb{Z}}$. Then for generic $x \in \mathbb{D}$, we have $\mathrm{Pic}(x) = W_{\mathbb{Z}}^{\perp}$.

We have the following lemma:

Lemma A.4. *For A satisfying Condition A.3 and $W = V_\zeta$, we have $\Gamma_A = \Gamma_W$.*

Proof. It is straightforward that $\Gamma_A \subset \Gamma_W$, and they both act on $\mathbb{D}$. Take any $\rho \in \Gamma_W$ and a generic point x in $\mathbb{D}$. Then A is contained in the stabilizer group of ρx. Thus both A and $\rho^{-1}A\rho$ are contained in the stabilizer group of x. Since x is generic, we have $\rho^{-1}A\rho = A$ by Condition A.3. So $\rho \in \Gamma_A$. We showed that $\Gamma_W \subset \Gamma_A$. □

With this lemma, we will simply denote by Γ the arithmetic group acting on $\mathbb{D}$. We have:

Proposition A.5. *For any $\mathbb{Q}$-subspace $W_\mathbb{Q}$ (of $V_\mathbb{Q}$) with signature $(2, n)$, we have a morphism $\pi : \Gamma\backslash\mathbb{D} \to \widehat{\Gamma}\backslash\widehat{\mathbb{D}}$, which is the normalization of its image.*

Proof. Properness is by [Kiernan and Kobayashi 1972]. Take a generic point x in $\mathbb{D}$. Suppose $\rho \in \widehat{\Gamma}$ sends x to $\rho x \in \mathbb{D}$. The Picard lattice $\mathrm{Pic}(\rho x)$ of ρx contains $W_\mathbb{Z}^\perp$; hence $\rho^{-1}(W_\mathbb{Z}^\perp) \subset \mathrm{Pic}(x)$. Since x is generic, we have $\mathrm{Pic}(x) = W_\mathbb{Z}^\perp$. This implies that $\rho(W_\mathbb{Z}^\perp) = W_\mathbb{Z}^\perp$; hence $\rho(W) = W$. Thus $\rho \in \Gamma_W$.

Finally, we show finiteness. Take a point $x \in \mathbb{D}$. For any $\rho \in \widehat{\Gamma}$, we have $\rho^{-1}(W_\mathbb{Z}^\perp)$ contained in the Picard lattice $\mathrm{Pic}(x)$. The set $\widehat{\Gamma}x$ is a disjoint union of some Γ-orbits, each of which corresponds to the image of certain primitive embedding of $W_\mathbb{Z}^\perp$ into $\mathrm{Pic}(x)$. The orthogonal complement of $W_\mathbb{Z}^\perp$ in $\mathrm{Pic}(x)$ is positive definite with discriminant at most $\det(W_\mathbb{Z}^\perp)\det(\mathrm{Pic}(x))$. By reduction theory of lattice, there are finitely many such primitive embeddings. □

A3. *Functoriality of Baily–Borel compactification.* In this section we recall Baily–Borel compactifications of arithmetic quotients of complex hyperbolic balls or type IV domains; see [Baily and Borel 1966; Looijenga 2003a; 2003b].

We deal with type IV domain $\widehat{\mathbb{D}}$ first. The boundary components of Baily–Borel compactifications corresponds to $\mathbb{Q}$-isotropic planes J or $\mathbb{Q}$-isotropic lines I. Let

$$\pi_{J^\perp} : \mathbb{P}(V) - \mathbb{P}(J^\perp) \to \mathbb{P}(V/J^\perp) \quad \text{and} \quad \pi_{I^\perp} : \mathbb{P}(V) - \mathbb{P}(I^\perp) \to \mathbb{P}(V/I^\perp)$$

be the natural projections. The image $\pi_{J^\perp}\widehat{\mathbb{D}}$ is isomorphic to upper half plane. The image $\pi_{I^\perp}\widehat{\mathbb{D}}$ is a point. Adding rational boundary components, we have

$$\widehat{\mathbb{D}}^{bb} := \widehat{\mathbb{D}} \sqcup \coprod_J \pi_{J^\perp}\widehat{\mathbb{D}} \sqcup \coprod_I \pi_{I^\perp}\widehat{\mathbb{D}}$$

with suitable topology and ringed space structure. The Baily–Borel compactification is the quotient $\Gamma\backslash\widehat{\mathbb{D}}^{bb}$ as a projective variety.

Given $W_\mathbb{Q} \subset V_\mathbb{Q}$ with signature $(2, n)$. Let $\mathbb{D}$ be the corresponding type IV domain. We have a natural map from $\mathbb{D}$ to $\widehat{\mathbb{D}}$, inducing $\Gamma\backslash\mathbb{D} \to \widehat{\Gamma}\backslash\widehat{\mathbb{D}}$. According to Theorem 2 in [Kiernan and Kobayashi 1972], this holomorphic map can be extended to Baily–Borel compactifications, sending boundary components into boundary components.

Proposition A.6 (type IV to type IV). *There is a natural finite extension of $\pi : \Gamma\backslash\mathbb{D} \to \widehat{\Gamma}\backslash\widehat{\mathbb{D}}$ to Baily–Borel compactifications*

$$\pi : \overline{\Gamma\backslash\mathbb{D}}^{bb} \to \overline{\widehat{\Gamma}\backslash\widehat{\mathbb{D}}}^{bb}.$$

If A satisfies Condition A.3, the map is a normalization of its image.

Proof. Let $W := V_\zeta$ in this proof. The boundary components of $\mathbb{D}^{bb}$ correspond to rational isotropic planes J and rational isotropic lines I in W. From the natural embedding $W \hookrightarrow V$, they also have associated boundary components in $\widehat{\mathbb{D}}^{bb}$. Under the natural commutative diagram

$$\begin{array}{ccc} \mathbb{P}(W) - \mathbb{P}(J^\perp) & \xrightarrow{\pi_{J^\perp}} & \mathbb{P}(W/J^\perp) \\ \downarrow & & \downarrow \\ \mathbb{P}(V) - \mathbb{P}(J^\perp) & \xrightarrow{\pi_{J^\perp}} & \mathbb{P}(V/J^\perp) \end{array}$$

we have isomorphisms $\pi_{J^\perp}\mathbb{D} \to \pi_{J^\perp}\widehat{\mathbb{D}}$, and similar maps $\pi_{I^\perp}\mathbb{D} \to \pi_{I^\perp}\widehat{\mathbb{D}}$, which induce an extension $\mathbb{D}^{bb} \to \widehat{\mathbb{D}}^{bb}$ equivariant under the action of $\Gamma \to \widehat{\Gamma}$. After taking quotients, we have an extension map $\mathbb{X}^{bb} \to \widehat{\mathbb{X}}^{bb}$. By Proposition A.1, this map is generically injective and it is finite over $\widehat{\Gamma}\backslash\widehat{\mathbb{D}}$. Let Γ_J be the stabilizer of J under the action of Γ. The projection of Γ in $\mathrm{GL}(J)$ (or equivalently $\mathrm{GL}(V/J^\perp)$) is arithmetic. The boundary component corresponding to J is the quotient of $\pi_{J^\perp}\widehat{\mathbb{D}}$ by Γ_J, hence a modular curve. The restriction to the boundary component corresponding to each J is a nonconstant map between modular curves, hence finite. The restriction to boundary components corresponding to each I is automatically finite. So we have an algebraic finite morphism between normal varieties $\mathbb{X}^{bb} \to \widehat{\mathbb{X}}^{bb}$. If A satisfies Condition A.3, then this morphism is generically injective by Proposition A.1, hence a normalization of its image. $\square$

We recall the Baily–Borel compactification of Ball quotient. Let K be a CM field and W_K a finite-dimensional vector space over K with

$$h_K : W_K \times W_K \to K$$

a K-valued Hermitian form. For each embedding $\iota : K \hookrightarrow \mathbb{C}$, we define $W_\iota := W_K \otimes_\iota \mathbb{C}$, then we have a Hermitian form $h_\iota : W_\iota \times W_\iota \to \mathbb{C}$. Assume the form h_ι has signature $(1, n)$ under embedding $\iota = \iota_1$ or $\bar{\iota}_1$, and is definite otherwise. The complex ball $\mathbb{B}$ is defined to be the set of positive lines in W_{ι_1}. The boundary components of Baily–Borel compactification correspond to K-isotropic lines I in W_K and we denote $\mathbb{B}^{bb} := \mathbb{B} \sqcup \coprod_I \pi_{I^\perp}\mathbb{B}$. When the totally real part of K is not $\mathbb{Q}$, there exists complex embedding ι such that (W_ι, h_ι) is definite, which implies that any isotropic vector in W_K must be zero. Thus in this case the boundary set is empty.

Now consider the action of A on V with $\zeta \neq \bar{\zeta}$. Let K be the cyclotomic field generated by $\zeta(A)$. Take W_K to be the ζ-eigenspace of $V_K := V_\mathbb{Q} \otimes K$ under the action of A.

Lemma A.7. *The K-vector space W_K is isotropic under φ.*

Proof. Taking any $x, y \in W_K$, we need to show $\varphi(x, y) = 0$. Take $a \in A$ such that $\zeta(a)$ is not real. Then $\zeta(a)^2 \neq 1$. By

$$\varphi(x, y) = \varphi(ax, ay) = \varphi(\zeta(a)x, \zeta(a)y) = \zeta(a)^2\varphi(x, y),$$

we have $\varphi(x, y) = 0$. $\square$

There is a natural Hermitian form h of signature $(1, n)$ on W_K, given by $h(x, y) = \varphi(x, \bar{y})$ for all $x, y \in W_K$. The Galois conjugates of K define eigenspaces of V under the action of A. The sum of all those eigenspaces is a subspace of V defined over $\mathbb{Q}$. Then we have the ball $\mathbb{B}$ consisting of positive lines in W and we denote $\overline{(\Gamma\backslash\mathbb{B})}^{bb} := \Gamma\backslash\mathbb{B}^{bb}$ the Baily–Borel compactification of $\mathbb{X} = \Gamma\backslash\mathbb{B}$.

Proposition A.8 (ball to type IV). *There is a natural finite extension of* $\pi : \Gamma\backslash\mathbb{B} \to \widehat{\Gamma}\backslash\widehat{\mathbb{D}}$ *to Baily–Borel compactifications*

$$\pi : \overline{\Gamma\backslash\mathbb{B}}^{bb} \to \overline{\widehat{\Gamma}\backslash\widehat{\mathbb{D}}}^{bb}.$$

If A satisfies Condition A.3, the map is a normalization of its image.

Proof. Similar as the proof for type IV case, we need to identify the boundary components on both sides. The ball and its boundaries are defined as above by W_K. If K is not a quadratic extension of $\mathbb{Q}$, then the boundary set is empty; hence $\Gamma\backslash\mathbb{B}$ is already compact. If K is, then each K-isotropic line I together with its complex conjugate $\bar{I}$ defines a rational isotropic plane in $V_{\mathbb{Q}}$. So there is a natural extension map $\mathbb{B}^{bb} \to \widehat{\mathbb{D}}^{bb}$ which is equivariant under the action of $\Gamma \to \widehat{\Gamma}$. After taking quotient on both sides, we have a finite algebraic map $\pi : \mathbb{X}^{bb} \to \widehat{\mathbb{X}}^{bb}$. If A satisfies Condition A.3, then this morphism is generically injective by Proposition A.1, hence a normalization of its image. $\square$

Remark A.9. Similar constructions of ball quotients appears in the arithmetic examples of Deligne–Mostow theory; see [Deligne and Mostow 1986; Looijenga 2007]. In both constructions, if the cyclotomic field generated by the corresponding characters is not $\mathbb{Q}(\sqrt{-1})$ or $\mathbb{Q}(\sqrt{-3})$, then the Baily–Borel compactification is compact.

A4. ***Functoriality of semitoric compactifications.*** We first briefly sketch the semitoric blow-up constructions of complex hyperbolic balls and type IV domains with respect to certain hyperplane arrangements; see [Looijenga 2003a; 2003b]. Semitoric compactification with respect to a hyperplane arrangement is the minimal blowup of certain boundary components in Baily–Borel compactification, such that the closure of every hypersurface is Cartier at the boundary.

Let $\widehat{\mathcal{H}}$ be a hyperplane arrangement on $\widehat{\mathbb{D}}$ defined by a set of negative vectors $v \in V_{\mathbb{Q}}$, which form finitely many orbits under the action of $\widehat{\Gamma}$. We recall some definitions and notation in [Looijenga 2003b]. Each rational isotropic line I in $V_{\mathbb{Q}}$ realizes $\widehat{\mathbb{D}}$ as a tube domain, with real cone denoted by

$$C_I \subset (I^{\perp}/I \otimes I)(\mathbb{R}).$$

Each rational isotropic plane J determines a half line on the boundaries of the C_I for any $I \subset J$. The union of these cones is called the conical locus of $\widehat{\mathbb{D}}$. Let $C_{I,+}$ be the convex hull of $\overline{C}_I \cap (I^{\perp}/I \otimes I)(\mathbb{Q})$, which is the union of C_I with rational isotropic half lines corresponding to J containing I. The hyperplane arrangement $\widehat{\mathcal{H}}$ determines an admissible decomposition $\Sigma(\widehat{\mathcal{H}})$ of the conical locus. More precisely, it is a Γ-invariant choice of locally rational cone decomposition of $C_{I,+}$ such that the support for isotropic half line corresponding to J is independent of those $I \subset J$; see Section 6 of [Looijenga 2003b] for details.

For each member σ of $\Sigma(\widehat{\mathcal{H}})$ contained in $C_{I,+}$, we define a corresponding vector subspace V_σ of V as follows. When σ is the half line corresponding to an isotropic plane J, then

$$V_\sigma := \Big(\bigcap_{J\subset H} H\Big) \cap J^\perp.$$

Otherwise V_σ is the span of σ in $I^\perp$, which is also the intersection among $I^\perp$ and those $H \in \widehat{\mathcal{H}}$ containing I. Here we identify $H \subset V$ with $H \in \widehat{\mathcal{H}}$. We have a projection $\pi_\sigma : \mathbb{D} \to \mathbb{P}(V/V_\sigma)$. The semitoric compactification is denoted by $\overline{\mathbb{X}}^\Sigma = \Gamma\backslash\mathbb{D}^\Sigma$. Here $\mathbb{D}^\Sigma := \coprod_{\sigma\in\Sigma} \pi_\sigma\widehat{\mathbb{D}}$. The space $\overline{\mathbb{X}}^\Sigma$ has a natural map to $\widehat{\mathbb{X}}^{bb}$ respecting the stratifications. We have two different types of boundary components. One is finite quotient of an abelian torsor over the modular curve $\widehat{\Gamma}_J\backslash\pi_{J^\perp}\widehat{D}$. The abelian torsor is modeled over vector group $J^\perp/V_\sigma$ quotient by a lattice. The other is an algebraic torus torsor over a point $\pi_{I^\perp}\widehat{D}$, which is the boundary stratum in the quotient of an infinite-type toric variety induced by the cone decomposition of $C_{I,+}$. In particular, each cone of codimension k corresponds to algebraic torus torsor of dimension k.

Given $W_{\mathbb{Q}} \subset V_{\mathbb{Q}}$ a sublattice of signature $(2, n)$, with $\mathbb{D}$ the associated type IV domain, we have the intersection $\mathcal{H} := \mathbb{D} \cap \widehat{\mathcal{H}}$ a Γ-invariant hyperplane arrangement in $\mathbb{D}$. We also have the semitoric blowup of $\mathbb{D}$ with respect to $\mathcal{H}$.

Theorem A.10 (type IV to type IV). *There is a natural finite extension of $\pi : \Gamma\backslash\mathbb{D} \to \widehat{\Gamma}\backslash\widehat{\mathbb{D}}$ to semitoric compactifications*

$$\pi : \overline{\Gamma\backslash\mathbb{D}}^{\Sigma(\mathcal{H})} \to \overline{\widehat{\Gamma}\backslash\widehat{\mathbb{D}}}^{\Sigma(\widehat{\mathcal{H}})}.$$

If A satisfies Condition A.3, the map is a normalization of its image.

Proof. We first show the existence of $\pi : \overline{\Gamma\backslash\mathbb{D}}^{\Sigma(\mathcal{H})} \to \overline{\widehat{\Gamma}\backslash\widehat{\mathbb{D}}}^{\Sigma(\widehat{\mathcal{H}})}$ as a morphism between two projective varieties, then prove finiteness.

The subdomain is induced by (W, φ). The isotropic lines and planes in W are naturally viewed as boundary data of both $\mathbb{D}$ and $\widehat{\mathbb{D}}$. The conical locus of $\mathbb{D}$ is naturally embedded into that of $\widehat{\mathbb{D}}$.

Suppose $\sigma \in \Sigma(\mathcal{H})$ does not correspond to a rational isotropic plane of W. Then we have a rational isotropic line I, such that σ is contained in $C_{I,W,+}$ and intersects with $C_{I,W}$. For each H containing I, the intersection $H\cap C_{I,W}$ being not empty is equivalent to $H\cap\mathbb{D}$ being not empty. Then there exists $\tau \in \Sigma(\widehat{\mathcal{H}})$ such that $\sigma = \tau \cap W$. We denote by $\widehat{\sigma}$ the minimal element among all such τ. Thus $\sigma = C_{I,W} \cap \widehat{\sigma}$, which implies $W_\sigma = V_{\widehat{\sigma}} \cap W$.

Let $\sigma \in \Sigma(\mathcal{H})$ correspond to an isotropic plane J contained in both W and a hyperplane H. Suppose v is a normal vector of H and $v = w + w^\perp$ the decomposition in $V = W \oplus W^\perp$. We have $\varphi(v, v) < 0$. The hyperplane H intersects with $\mathbb{D}$ if and only if $\varphi(w, w) < 0$. Since the orthogonal complement of w in $W_{\mathbb{Q}}$ contains the isotropic plane J, we have either $\varphi(w, w) < 0$ or $\varphi(w, w) = 0$. Suppose the latter case happens, then $w \in J$ since otherwise $\langle J, w\rangle$ is an isotropic subspace of rank 3 contained in $W_{\mathbb{Q}}$, which is impossible. Thus in this case $H \supset J^\perp \cap W$. The above argument holds for any $H \in \mathcal{H}$ containing σ; hence $W_\sigma = V_\sigma \cap W$. In this case we also denote $\widehat{\sigma} = \sigma$.

For $\sigma = \{0\} \in \Sigma(\mathcal{H})$, just take $\widehat{\sigma} = \{0\} \in \Sigma(\widehat{\mathcal{H}})$. Then for every $\sigma \in \Sigma(\mathcal{H})$, we have a natural holomorphic map $\pi_\sigma \mathbb{D} \to \pi_{\widehat{\sigma}} \widehat{\mathbb{D}}$ which is apparently injective. Taking union among σ, we have

$$\coprod_{\sigma\in\Sigma(\mathcal{H})} \pi_\sigma \mathbb{D} \to \coprod_{\sigma\in\Sigma(\mathcal{H})} \pi_{\widehat{\sigma}} \widehat{\mathbb{D}} \hookrightarrow \coprod_{\tau\in\Sigma(\widehat{\mathcal{H}})} \pi_\tau \widehat{\mathbb{D}}$$

with the composition continuous. After taking quotients by the equivariant actions on both sides, we obtain a finite map between the boundary components. Actually, for those rational isotropic planes J, we obtain finite morphisms between Abelian torsors; for those rational isotropic lines I, we obtain finite morphisms between algebraic torus torsors. If A satisfies Condition A.3, then π is generically injective by Proposition A.1, hence a normalization of its image. $\square$

Remark A.11. The injectivity of $\coprod_{\sigma\in\Sigma(\mathcal{H})} \pi_\sigma \mathbb{D} \to \coprod_{\tau\in\Sigma(\widehat{\mathcal{H}})} \pi_\tau \widehat{\mathbb{D}}$ is already known by [Looijenga 2003b, paragraph after Lemma 7.1].

For $\zeta \neq \bar{\zeta}$, we have ball $\mathbb{B}$ attached to $W = V_\zeta$. We next describe the semitoric compactification of $\mathbb{B}$ with respect to $\mathcal{H}$. Here we identify elements in $\mathcal{H}$ with hypersurfaces in W. The cusp points correspond to isotropic lines I in W_K. Let

$$j(I) = \Big(\bigcap_{H\in\mathcal{H}, H\supset I} H\Big) \cap I_W^\perp$$

and $\pi_I : \mathbb{P}(W) - \mathbb{P}(j(I)) \to \mathbb{P}(W/j(I))$. Define

$$\overline{\mathbb{X}}^j = \Gamma\backslash\Big(\mathbb{B} \sqcup \coprod_I \pi_{j(I)}\mathbb{B}\Big).$$

It naturally maps to the Baily–Borel compactification. The boundary component over each cusp point is an abelian torsor modeled over the vector space $I_W^\perp/j(I)$ quotient by a lattice.

Theorem A.12 (ball to type IV). *There is a natural finite extension of $\pi : \Gamma\backslash\mathbb{B} \to \widehat{\Gamma}\backslash\widehat{\mathbb{D}}$ to semitoric compactifications*

$$\pi : \overline{\Gamma\backslash\mathbb{B}}^j \to \overline{\widehat{\Gamma}\backslash\widehat{\mathbb{D}}}^{\Sigma(\widehat{\mathcal{H}})}.$$

If A satisfies Condition A.3, the map is a normalization of its image.

Proof. If K is not a quadratic extension of $\mathbb{Q}$, then $\mathbb{X}$ is compact and the theorem holds. Now assume that K is a quadratic extension of $\mathbb{Q}$. Namely, $K = \mathbb{Q}(\sqrt{-D})$ for certain positive integer D. Take any isotropic line I in W_K. Suppose a nonzero generator of I is $e + \sqrt{-D}f$, where $e, f \in V_\mathbb{Q}$. Then $\varphi(e+\sqrt{-D}f, e-\sqrt{-D}f) = 0$. From Lemma A.7 we have $\varphi(e+\sqrt{-D}f, e+\sqrt{-D}f) = 0$. This implies that $J = \langle e, f\rangle$ is an isotropic plane in $V_\mathbb{Q}$.

We claim that $j(I) = W \cap V_J$. Take $H \in \widehat{\mathcal{H}}$ with orthogonal vector $v \in V_\mathbb{Q}$. Under the orthogonal decomposition $V_K = W_K \oplus \overline{W_K} \oplus V'$, we can decompose v as $v = v_W + \overline{v_W} + v'$. Then $\varphi(\mathrm{Re}(v_W), J) = 0$. From Lemma A.7 we have $\varphi(v_W, I) = 0$. Therefore, $\varphi(\mathrm{Im}(v_W), I) = 0$ and hence $\varphi(\mathrm{Im}(v_W), J) = 0$.

Since $(V_{\mathbb{Q}}, \varphi)$ has signature $(2, N)$, the orthogonal complement of J in $V_{\mathbb{Q}}$ is negative semidefinite. Thus $\varphi(\mathrm{Re}(v_W), \mathrm{Re}(v_W)) \le 0$ and $\varphi(\mathrm{Im}(v_W), \mathrm{Im}(v_W)) \le 0$. We then have

$$\varphi(v_W, \overline{v_W}) = \varphi(\mathrm{Re}(v_W), \mathrm{Re}(v_W)) + \varphi(\mathrm{Im}(v_W), \mathrm{Im}(v_W)) \le 0.$$

Suppose $\varphi(v_W, \overline{v_W}) < 0$, then $H \cap \mathbb{B} \neq \varnothing$. Suppose $\varphi(v_W, \overline{v_W}) = 0$, then v_W is an isotropic line in W_K. The vectors $\mathrm{Re}(v_W)$ and $\mathrm{Im}(v_W)$ in $V_{\mathbb{Q}}$ are then isotropic. These two vectors are orthogonal to J; hence they belong to J. We deduce that $H \supset I_W^{\perp}$. By the definition of $j(I)$ and V_J, we conclude the claim.

We now have naturally an injective map $\pi_{j(I)}\mathbb{B} \to \pi_J\widehat{\mathbb{D}}$. Taking the union among those isotropic lines I, we have an injective map

$$\mathbb{B} \sqcup \coprod_I \pi_{j(I)}\mathbb{B} \hookrightarrow \coprod_{\sigma \in \Sigma(\widehat{\mathcal{H}})} \pi_\sigma \widehat{\mathbb{D}}.$$

After taking quotients by the equivariant actions on both sides, we obtain a morphism $\pi : \overline{\Gamma\backslash\mathbb{B}}^{j} \to \overline{\Gamma\backslash\widehat{\mathbb{D}}}^{\Sigma(\widehat{\mathcal{H}})}$. Actually, the restriction of this π to the boundary component corresponding to I is a finite morphism between Abelian torsors. We conclude that there is natural extension

$$\pi : \overline{(\Gamma\backslash\mathbb{B})}^{j} \to \overline{(\widehat{\Gamma}\backslash\widehat{\mathbb{D}})}^{\Sigma(\widehat{\mathcal{H}})}$$

which is a finite morphism between projective varieties. If A satisfies Condition A.3, this π is generically injective, hence a normalization of its image. □

A5. ***Main theorem.*** In this section, we first describe the construction of Looijenga compactification $\overline{\mathbb{X}}^{\mathcal{H}}$ of $\mathbb{X}^\circ := \mathbb{X} - \Gamma\backslash\mathcal{H}$. We need to successively blow up nonempty intersections of components of $\Gamma\backslash\mathcal{H}$, and then contract the strict transformations of $\Gamma\backslash\mathcal{H}$ via a natural associated semiample line bundle on the blowup. We then prove existence and finiteness of morphism between Looijenga compactifications on both sides of $\mathbb{X} \to \widehat{\mathbb{X}}$.

The blow-up and blow-down constructions with respect to hyperplane arrangement in any normal analytic variety with a properly given line bundle are discussed in the first three sections in [Looijenga 2003a]. Looijenga applied this general theory to $(\overline{\mathbb{X}}^{\Sigma(\mathcal{H})}, \Gamma\backslash\mathcal{H}, \mathcal{L})$, where $\mathbb{X}$ is either arithmetic quotient of type IV domain $\mathbb{D}$ or ball $\mathbb{B}$, and $\mathcal{L}$ is the natural automorphic line bundle; see [Looijenga 2003a, Theorem 5.7; 2003b, Theorem 7.4].

We now describe the blow-up and blow-down constructions before quotient by the arithmetic groups. The Looijenga compactifications are obtained by the modified spaces quotient by the arithmetic groups. Denote by $\mathrm{PO}(\mathcal{H})$ the set of nonempty intersections of elements in $\mathcal{H}$ as hyperplanes in $\mathbb{D}$ (or $\mathbb{B}$). Let $L \in \mathrm{PO}(\mathcal{H})$ also denote its closure in $\mathbb{D}^\Sigma$ (or $\mathbb{B}^j$). Denote $c(L) := \mathrm{codim}(L) - 1$.

We first look at the semitoric compactification $\mathbb{D}^\Sigma$ of $\mathbb{D}$. Denote by $(\mathbb{D}^\Sigma)^\circ$ the arrangement complement of $\mathcal{H}$ in $\mathbb{D}^\Sigma$. Choose $L \in \mathrm{PO}(\mathcal{H})$ a minimal member. Blowing up along L replaces L by the projectivization of its normal bundle, which is isomorphic to $L \times \mathbb{P}^{c(L)}$. The modified space, denoted by $\mathrm{Bl}_L(\mathbb{D}^\Sigma)$, has natural topology, arrangement (the strict transform of the previous one) and automorphic line bundle. The strict transforms of those hypersurfaces passing through L form a hyperplane arrangement in $\mathbb{P}^{c(L)}$, and we denote the complement by $(\mathbb{P}^{c(L)})^\circ$. The complement of the new arrangement in $\mathrm{Bl}_L(\mathbb{D}^\Sigma)$ is the disjoint union $(\mathbb{D}^\Sigma)^\circ \sqcup L \times (\mathbb{P}^{c(L)})^\circ$. After blowing up successively until hypersurfaces disjoint, we obtain the final blowup $\widetilde{\mathbb{D}}$. This is a disjoint union of $(\mathbb{D}^\Sigma)^\circ$ with $L \times (\mathbb{P}^{c(L)})^\circ$ for all such minimal L appearing in each step.

Now we can contract $L \times (\mathbb{P}^{c(L)})^\circ$ along the direction of L for all such L, and obtain $\mathbb{D}^*$ with natural quotient topology. Set-theoretically, $L \times (\mathbb{P}^{c(L)})^\circ$ is contracted to $(\mathbb{P}^{c(L)})^\circ$. We have the following description (see [Looijenga 2003b]):

$$\mathbb{D}^* = \coprod_{L \in \mathrm{PO}(\mathcal{H})} \pi_L \mathbb{D}^\circ \sqcup \coprod_{\sigma \in \Sigma(\mathcal{H})} \pi_\sigma \mathbb{D}^\circ.$$

Notice that for σ being the vertex, π_σ is identity and $\pi_\sigma \mathbb{D}^\circ = \mathbb{D}^\circ$.

The spaces $\mathbb{D}^\Sigma$, $\widetilde{\mathbb{D}}$, $\mathbb{D}^*$ constructed above all have natural ringed space structure. Namely, we have the structure sheaves consisting of continuous functions with analytic restriction to each stratum. The group Γ naturally acts on those ringed spaces respecting the stratification. The topological quotient space $\overline{\mathbb{X}}^{\mathcal{H}} := \Gamma \backslash \mathbb{D}^*$ has normal analytic structure respecting the stratification; see [Looijenga 2003b, Theorem 7.4]. According to the Riemann extension theorem, the quotient ringed space structure and the analytic structure on $\overline{\mathbb{X}}^{\mathcal{H}}$ coincide.

For the case of ball, parallel argument gives $\widetilde{\mathbb{B}}$ and $\mathbb{B}^*$. We have

$$\mathbb{B}^* = \mathbb{B}^\circ \sqcup \coprod_{L \in \mathrm{PO}(\mathcal{H})} \pi_L \mathbb{B}^\circ \sqcup \coprod_{I} \pi_{j(I)} \mathbb{B}^\circ.$$

This also has natural ringed structure, and $\overline{\mathbb{X}}^{\mathcal{H}} \cong \Gamma \backslash \mathbb{B}^*$ as analytic spaces.

Theorem A.13 (Main Theorem). *There is a natural extension of $\pi : \Gamma \backslash (\mathbb{D} - \mathcal{H}) \to \widehat{\Gamma} \backslash (\widehat{\mathbb{D}} - \widehat{\mathcal{H}})$ to Looijenga compactifications*

$$\pi : \overline{\Gamma \backslash \mathbb{D}}^{\mathcal{H}} \to \overline{\widehat{\Gamma} \backslash \widehat{\mathbb{D}}}^{\widehat{\mathcal{H}}}$$

which is a finite morphism. If A satisfies Condition A.3, the map is a normalization of its image. The same result holds for ball quotients.

Proof. From Theorem A.10, we have natural morphisms from $\mathbb{D}^\Sigma$ to $\widehat{\mathbb{D}}^\Sigma$. Near each boundary component, there is a contraction map from a neighborhood to the boundary itself. The arrangement in total space is the pullback of smooth arrangement on the boundary. According to the map defined near the boundary components, we know that any $H \in \widehat{\mathcal{H}}$ not intersecting with $\mathbb{D}$ is still away from $\mathbb{D}^\Sigma$ after taking its closure. From Corollary 7.15 in Chapter II in [Hartshorne 1977], we have injective map $\widetilde{\mathbb{D}} \to \widetilde{\widehat{\mathbb{D}}}$ respecting the ringed space structures. Notice that the automorphic line bundle on $\mathbb{D}^\Sigma$ is the pull back of that on $\widehat{\mathbb{D}}^\Sigma$; hence we have an injective map on the stratum $L \times (\mathbb{P}^{c(L)})^\circ$ to $\widehat{L} \times (\mathbb{P}^{c(\widehat{L})})^\circ$ which is linear on the second component. Here $\widehat{L}$ is a minimal member used in certain step of the successive blow-up construction of $\widehat{\mathbb{D}}$, and L is the induced member on the smaller subspace by intersecting with $\widehat{L}$. After blowing down, we have a natural injective map $\mathbb{D}^* \to \widehat{\mathbb{D}}^*$ respecting the ringed space structures.

This morphism descends to a morphism $\pi : \Gamma\backslash\mathbb{D}^* \to \widehat{\Gamma}\backslash\widehat{\mathbb{D}}^*$, still in the category of ringed spaces. We then have an analytic morphism

$$\pi : \overline{\mathbb{X}}^{\mathcal{H}} \to \overline{\widehat{\mathbb{X}}}^{\widehat{\mathcal{H}}}.$$

This analytic morphism extends $\pi : \mathbb{X}^\circ \to \widehat{\mathbb{X}}^\circ$, and sends boundary strata into boundary strata. Combining with Theorem A.10, the extended morphism π here is finite. If A satisfies Condition A.3, it is generically injective and hence a normalization of its image.

The same argument also holds for ball quotients. $\square$

List of symbols

(d, k)	dimension and degree of a hypersurface
V	complex vector space of dimension $k+2$
F	degree d polynomial in $k+2$ variables
X	degree d k-fold; cubic fourfold when $(d, k) = (3, 4)$
A	a finite subgroup of $\mathrm{SL}(V)$, containing the center μ_{k+2} of $\mathrm{SL}(V)$
$\bar{A}$	image of A in $\mathrm{PSL}(V)$
λ	character of A with specified restriction to μ_{k+2}
T	equivalence class of (A, λ), called symmetry type of degree d k-fold
$\mathcal{V}$	eigenspace of $\mathrm{Sym}^d(V^*)$ corresponding to (A, λ)
C	centralizer of A in $\mathrm{SL}(V)$
N	a reductive group acting on $\mathcal{V}$
$\mathcal{V}^{sm}/\mathcal{V}^{ss}$	space of smooth/semistable points in $\mathcal{V}$
$\mathcal{F}$	GIT quotient of $\mathcal{V}^{sm}$ by N
$\mathcal{F}^m$	moduli space of cubic fourfolds with T-markings
$\mathcal{F}_1$	moduli space of cubic fourfolds of type T, which admits at worst ADE singularities
$\overline{\mathcal{F}}$	GIT quotient of $\mathcal{V}^{ss}$ by N, which is a compactification of $\mathcal{F}$
$\mathcal{M}$	moduli space of smooth cubic fourfolds
$\overline{\mathcal{M}}$	GIT compactification of $\mathcal{M}$
$(\Lambda_0)\Lambda$	(primitive) middle cohomology lattice of a smooth cubic fourfold
φ	topological intersection pairing on Λ
η	square of the hyperplane class
$\widehat{\mathbb{D}}$	local period domain for cubic fourfolds
$\widehat{\Gamma}$	monodromy group of the universal family of smooth cubic fourfolds
$\mathcal{H}_\Delta/\mathcal{H}_\infty$	$\widehat{\Gamma}$-invariant hyperplane arrangement in $\widehat{\mathbb{D}}$
ζ	character of A, induced by the action of A on $H^{3,1}(X)$
Λ_ζ	eigenspace of $(\Lambda_0)_{\mathbb{C}}$ corresponding to the character ζ

σ_X/σ	representation of A on $H^4(X,\mathbb{Z})/\Lambda$
h_X/h	Hermitian form on $H^4(X)_\zeta/\Lambda_\zeta$
$\mathbb{D}$	local period domain for cubic fourfolds of symmetry type T
Γ	normalizer of $\bar{A}$ in $\widehat{\Gamma}$
$\mathcal{H}_s/\mathcal{H}_*$	Γ-invariant hyperplane arrangements in $\mathbb{D}$
$\overline{\Gamma\backslash\mathbb{D}}^{\mathcal{H}_*}$	Looijenga compactification of $\Gamma\backslash(\mathbb{D}-\mathcal{H}_*)$
$\widetilde{\mathscr{P}}$	local period map
$\mathscr{P}$	global period map

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References

[Allcock et al. 2011] D. Allcock, J. A. Carlson, and D. Toledo, *The moduli space of cubic threefolds as a ball quotient*, Mem. Amer. Math. Soc. **985**, Amer. Math. Soc., Providence, RI, 2011. MR Zbl

[Artebani et al. 2011] M. Artebani, A. Sarti, and S. Taki, "$K3$ surfaces with non-symplectic automorphisms of prime order", *Math. Z.* **268**:1-2 (2011), 507–533. MR Zbl

[Baily and Borel 1966] W. L. Baily, Jr. and A. Borel, "Compactification of arithmetic quotients of bounded symmetric domains", *Ann. of Math.* (2) **84** (1966), 442–528. MR Zbl

[Beauville and Donagi 1985] A. Beauville and R. Donagi, "La variété des droites d'une hypersurface cubique de dimension 4", *C. R. Acad. Sci. Paris Sér. I Math.* **301**:14 (1985), 703–706. MR Zbl

[Boissière et al. 2016] S. Boissière, C. Camere, and A. Sarti, "Classification of automorphisms on a deformation family of hyper-Kähler four-folds by p-elementary lattices", *Kyoto J. Math.* **56**:3 (2016), 465–499. MR Zbl

[Boissière et al. 2019] S. Boissière, C. Camere, and A. Sarti, "Complex ball quotients from manifolds of $K3^{[n]}$-type", *J. Pure Appl. Algebra* **223**:3 (2019), 1123–1138. MR Zbl

[Camere 2016] C. Camere, "Lattice polarized irreducible holomorphic symplectic manifolds", *Ann. Inst. Fourier* (*Grenoble*) **66**:2 (2016), 687–709. MR Zbl

[Deligne and Mostow 1986] P. Deligne and G. D. Mostow, "Monodromy of hypergeometric functions and nonlattice integral monodromy", *Inst. Hautes Études Sci. Publ. Math.* **63** (1986), 5–89. MR Zbl

[Dolgachev and Kondō 2007] I. V. Dolgachev and S. Kondō, "Moduli of $K3$ surfaces and complex ball quotients", pp. 43–100 in *Arithmetic and geometry around hypergeometric functions* (Istanbul, 2005), edited by R.-P. Holzapfel et al., Progr. Math. **260**, Birkhäuser, Basel, 2007. MR Zbl

[Fu 2016] L. Fu, "Classification of polarized symplectic automorphisms of Fano varieties of cubic fourfolds", *Glasg. Math. J.* **58**:1 (2016), 17–37. MR Zbl

[González-Aguilera and Liendo 2011] V. González-Aguilera and A. Liendo, "Automorphisms of prime order of smooth cubic n-folds", *Arch. Math.* (*Basel*) **97**:1 (2011), 25–37. MR Zbl

[Griffiths 1969a] P. A. Griffiths, "On the periods of certain rational integrals, I", *Ann. of Math.* (2) **90** (1969), 460–495. MR Zbl

[Griffiths 1969b] P. A. Griffiths, "On the periods of certain rational integrals, II", *Ann. of Math.* (2) **90** (1969), 496–541. MR Zbl

[Harris 1989] M. Harris, "Functorial properties of toroidal compactifications of locally symmetric varieties", *Proc. Lond. Math. Soc.* (3) **59**:1 (1989), 1–22. MR Zbl

[Hartshorne 1977] R. Hartshorne, *Algebraic geometry*, Grad. Texts in Math. **52**, Springer, 1977. MR Zbl

[Hassett 2000] B. Hassett, "Special cubic fourfolds", *Compositio Math.* **120**:1 (2000), 1–23. MR Zbl

[Höhn and Mason 2019] G. Höhn and G. Mason, "Finite groups of symplectic automorphisms of hyper-Kähler manifolds of type $K3^{[2]}$", *Bull. Inst. Math. Acad. Sin.* (*N.S.*) **14**:2 (2019), 189–264. MR Zbl

[Javanpeykar and Loughran 2017] A. Javanpeykar and D. Loughran, "Complete intersections: moduli, Torelli, and good reduction", *Math. Ann.* **368**:3-4 (2017), 1191–1225. MR Zbl

[Joumaah 2016] M. Joumaah, "Non-symplectic involutions of irreducible symplectic manifolds of $K3^{[n]}$-type", *Math. Z.* **283**:3-4 (2016), 761–790. MR Zbl

[Kiernan and Kobayashi 1972] P. Kiernan and S. Kobayashi, "Satake compactification and extension of holomorphic mappings", *Invent. Math.* **16** (1972), 237–248. MR Zbl

[Laza 2009] R. Laza, "The moduli space of cubic fourfolds", *J. Algebraic Geom.* **18**:3 (2009), 511–545. MR Zbl

[Laza 2010] R. Laza, "The moduli space of cubic fourfolds via the period map", *Ann. of Math.* (2) **172**:1 (2010), 673–711. MR Zbl

[Laza and Zheng 2019] R. Laza and Z. Zheng, "Automorphisms and periods of cubic fourfolds", preprint, 2019. arXiv

[Laza et al. 2018] R. Laza, G. Pearlstein, and Z. Zhang, "On the moduli space of pairs consisting of a cubic threefold and a hyperplane", *Adv. Math.* **340** (2018), 684–722. MR Zbl

[Looijenga 2003a] E. Looijenga, "Compactifications defined by arrangements, I: The ball quotient case", *Duke Math. J.* **118**:1 (2003), 151–187. MR Zbl

[Looijenga 2003b] E. Looijenga, "Compactifications defined by arrangements, II: Locally symmetric varieties of type IV", *Duke Math. J.* **119**:3 (2003), 527–588. MR Zbl

[Looijenga 2007] E. Looijenga, "Uniformization by Lauricella functions: an overview of the theory of Deligne–Mostow", pp. 207–244 in *Arithmetic and geometry around hypergeometric functions* (Istanbul, 2005), edited by R.-P. Holzapfel et al., Progr. Math. **260**, Birkhäuser, Basel, 2007. MR Zbl

[Looijenga 2009] E. Looijenga, "The period map for cubic fourfolds", *Invent. Math.* **177**:1 (2009), 213–233. MR Zbl

[Looijenga 2016] E. Looijenga, "Moduli spaces and locally symmetric varieties", pp. 33–75 in *Development of moduli theory* (Kyoto, 2013), edited by O. Fujino et al., Adv. Stud. Pure Math. **69**, Math. Soc. Japan, Tokyo, 2016. MR Zbl

[Looijenga and Swierstra 2007] E. Looijenga and R. Swierstra, "The period map for cubic threefolds", *Compos. Math.* **143**:4 (2007), 1037–1049. MR Zbl

[Luna 1973] D. Luna, "Slices étales", pp. 81–105 in *Sur les groupes algébriques*, Mém. Soc. Math. France **33**, Soc. Math. France, Paris, 1973. MR Zbl

[Luna 1975] D. Luna, "Adhérences d'orbite et invariants", *Invent. Math.* **29**:3 (1975), 231–238. MR Zbl

[Luna and Richardson 1979] D. Luna and R. W. Richardson, "A generalization of the Chevalley restriction theorem", *Duke Math. J.* **46**:3 (1979), 487–496. MR Zbl

[Matsumura and Monsky 1963] H. Matsumura and P. Monsky, "On the automorphisms of hypersurfaces", *J. Math. Kyoto Univ.* **3** (1963), 347–361. MR Zbl

[Mok and Zhong 1989] N. Mok and J. Q. Zhong, "Compactifying complete Kähler–Einstein manifolds of finite topological type and bounded curvature", *Ann. of Math.* (2) **129**:3 (1989), 427–470. MR Zbl

[Mongardi 2012] G. Mongardi, "Symplectic involutions on deformations of $K3^{[2]}$", *Cent. Eur. J. Math.* **10**:4 (2012), 1472–1485. MR Zbl

[Mongardi 2013] G. Mongardi, “On symplectic automorphisms of hyper-Kähler fourfolds of $K3^{[2]}$ type”, *Michigan Math. J.* **62**:3 (2013), 537–550. MR Zbl

[Mongardi 2016] G. Mongardi, “Towards a classification of symplectic automorphisms on manifolds of $K3^{[n]}$ type”, *Math. Z.* **282**:3-4 (2016), 651–662. MR Zbl

[Ressayre 2010] N. Ressayre, “Geometric invariant theory and the generalized eigenvalue problem”, *Invent. Math.* **180**:2 (2010), 389–441. MR Zbl

[Vinberg and Popov 1994] E. B. Vinberg and V. L. Popov, “Invariant theory”, pp. 123–278 in *Algebraic geometry, IV*, edited by A. N. Parshin and I. R. Shafarevich, Encyc. Math. Sci. **55**, Springer, 1994.

[Voisin 1986] C. Voisin, “Théorème de Torelli pour les cubiques de $\mathbb{P}^5$”, *Invent. Math.* **86**:3 (1986), 577–601. Correction in **172**:2 (2008), 455–458. MR Zbl

[Yau 1978] S. T. Yau, “A general Schwarz lemma for Kähler manifolds”, *Amer. J. Math.* **100**:1 (1978), 197–203. MR Zbl

[Yu and Zheng 2018] C. Yu and Z. Zheng, “Moduli of singular sextic curves via periods of $K3$ surfaces”, preprint, 2018. arXiv

[Zarhin 1983] Y. G. Zarhin, “Hodge groups of $K3$ surfaces”, *J. Reine Angew. Math.* **341** (1983), 193–220. MR Zbl

[Zheng 2019] Z. Zheng, “Orbifold aspects of certain occult period maps”, *Nagoya Math. J.* (online publication November 2019).

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