Causal canonical decomposition of hysteresis systems

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Abstract

Hysteresis is a special type of behavior found in many areas including magnetism, mechanics, biology, economics, etc. One of the characteristics of hysteresis systems is that they are approximately rate independent for slow inputs. It is possible to express this characteristic in mathematical language by decomposing hysteresis operators as the sum of a rate independent component and a nonhysteretic component which vanishes in steady state for slow inputs. This decomposition -called *canonical decomposition*- is possible for a class of hysteresis operators for which a continuous input leads to a continuous output and a continuous hysteresis loop. The canonical decomposition can be obtained using the concept of *confluence* which is an equation that continuous hysteresis operators should verify.

On the other hand, hysteresis systems are causal which means that their output depends on the current and/or previous values of the input but not on the future values of that input. Are the components of the canonical decomposition also causal? The answer is en general negative. The lack of causality of these components means that they cannot be written in the form of differential equations, integro-differential equations, partial differential equations, partial integro-differential equations and many other useful structures.

This paper proposes a new decomposition of hysteresis operators called *causal canonical decomposition* in which the rate independent component and the nonhysteretic component are both causal. The main tool to obtain the causal canonical decomposition is a new mathematical equation that we call *uniform confluence*. Using this equation we show that the causal canonical decomposition is unique. The concepts introduced in the paper are applied to the hysteretic scalar semilinear Duhem model as a case study.

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1. Introduction and problem statement

`The formal definition of causality.` Let U, X, Y be arbitrary sets. Let $\mathcal{U}$ be the set of all functions $u : \mathbb{R}_+ \to U$, $\mathcal{X}$ the set of all functions $x : \mathbb{R}_+ \to X$, and $\mathcal{Y}$ the set of all functions $y : \mathbb{R}_+ \to Y$. Then for any $(u, x_0) \in \mathcal{U} \times X$ we define $\mathcal{H}(u, x_0) = \mathcal{T} \circ [\Phi(u, x_0)]$ where Φ is a function (called operator in this work) $\Phi : \mathcal{U} \times X \to \mathcal{X}$, and $\mathcal{T} : X \to Y$ is a function, so that $\mathcal{H} : \mathcal{U} \times X \to \mathcal{Y}$.

The elements of $\mathcal{U}$ are called inputs, and the argument $t \in \mathbb{R}_+$ of the input u is called time. This means that the value of the input u at time t is $u(t)$. The function $\mathcal{H}(u, x_0) : \mathbb{R}_+ \to Y$ is called the output that corresponds to the input $u \in \mathcal{U}$ and the initial state $x_0 \in X$. The function $\Phi(u, x_0) : \mathbb{R}_+ \to Y$ is called the state that corresponds to the input $u \in \mathcal{U}$ and the initial state $x_0 \in X$. If $\mathcal{T}$ is the identity function, then the state is also the output.

Definition 1. *Let $\mathcal{U}_1 \subset \mathcal{U}$ and $X_1 \subset X$. We say that the operator Φ is $(\mathcal{U}_1, X_1)$–causal if the following holds. For all $u_1, u_2 \in \mathcal{U}_1$, all $x_0 \in X_1$ and all $t \in [0, \infty[$ we have: if $\forall \tau \in [0, t], u_1(\tau) = u_2(\tau)$, then $\forall \tau \in [0, t], \big[\Phi(u_1, x_0)\big](\tau) = \big[\Phi(u_2, x_0)\big](\tau)$ [8, p. 60].*

What Definition 1 says is that, if Φ is causal, then $\left[\Phi\left(u, x_0\right)\right](t)$ is independent of the values $u(\tau)$ for $\tau > t$. In other words, if we know x_0 and the restriction of u to the time interval $[0,t]$ then we know the restriction of $\Phi\left(u, x_0\right)$ to the time interval $[0,t]$.

An example of a causal system is the differential equation $\dot{x} = f(x,u)$ and $x(0) = x_0$ when it has a unique solution. The same can be said of any partial differential equation or partial integro-differential equation that has a unique solution for a given initial condition.

Finally, observe that if Φ is causal then $\mathcal{H}$ also is causal.

`The canonical decomposition of hysteresis systems`. The issue *'what is hysteresis'* is reviewed extensively in our paper [4, Section 2]. Hysteresis consists in that slow inputs produce a loop in the steady-state part of the graph output-versus-input.

In this paper we consider the class of hysteresis systems for which continuous inputs produce continuous outputs and continuous hysteresis loops. This class is studied in Ref. [5] using the concept of confluence introduced by the author. It is shown that any confluent operator Φ can be decomposed as

$$\Phi = \Phi^{\circ} + \Phi^{\ddagger}, \tag{1}$$

where Φ° is rate independent and $\Phi^{\ddagger}$ vanishes in steady state for slow inputs [5, Theorem 6.1]. Rate independence means that the graph output-versus-input of Φ° is invariant under a change in time scale.

Equation (1) is called the *canonical decomposition* of the confluent operator Φ. The operator Φ° is called the *rate-independent component* of Φ, and $\Phi^{\ddagger}$ the *nonhysteretic component* of Φ.

However, even when Φ is causal, the rate-independent components Φ° and $\Phi^{\ddagger}$ may not be causal.

`Is the lack of causality of the components` Φ° `and` $\Phi^{\ddagger}$ `a problem?` Whether physical systems are all causal or whether there are noncausal physical processes is an interesting question. However, this issue is not relevant to our study. What we can assert is that when Φ° and $\Phi^{\ddagger}$ are not causal, they cannot be written in the form of differential equations, integro-differential equations, partial differential equations, partial integro-differential equations and many other structures that have a unique solution for any given initial condition. In the context of hysteresis systems, causality is a customary assumption, see for example Ref. [8, p. 60].

We formulate two problems inspired from Equation (1):

(i) The direct problem that we call **'causal model construction'**: given a rate independent and confluent causal operator Φ_1, combine it to other operators in order to obtain a rate-dependent causal and confluent operator Φ such that, for slow inputs, we have $\Phi \simeq \Phi_1$ in steady state.

(ii) The inverse problem that we call **'causal canonical decomposition'**: given a causal and confluent operator Φ, find a rate independent causal and confluent operator $\Phi^{\star}$ such that, for slow inputs, we have $\Phi \simeq \Phi^{\star}$ in steady state.

An example of a solution to the causal model construction problem (i) is proposed in [5, Section 7].

`Aim of the paper`. Our purpose is to propose sufficient conditions on Φ so that the causal canonical decomposition problem (ii) has a solution. Moreover, we provide a way to construct that solution.

`Organization of the paper`. Section 2 provides the notations that are used throughout the paper. Section 3 provides those results of Ref. [5] that are relevant to the present paper. Section 4 introduces the concept of uniform confluence. The main result of this paper is Theorem 16. An example that illustrates the concepts introduced in this paper is presented in Section 5. Comments on the obtained results are given in Section 6. Conclusions are provided in Section 7.

2. Mathematical notation

The restriction of a function f to a set S is denoted $f|_S$.

An ordered pair a,b is denoted (a,b) whilst the open interval $\{t \in \mathbb{R} \mid a < t < b\}$ is denoted $]a,b[$. The set of nonnegative integers is denoted $\mathbb{N} = \{0,1,\dots\}$ and the set of nonnegative real numbers is denoted $\mathbb{R}_+ = [0,\infty[$.

We say that a subset of $\mathbb{R}$ is measurable when it is Lebesgue measurable. Consider a function $g : I \subset \mathbb{R}_+ \to \mathbb{R}^n$ where $n \in \mathbb{N}\backslash\{0\}$ and I an interval. We say that g is measurable when $\{x \in I : g(x) > a\}$ is measurable for all $a \in \mathbb{R}$. For a measurable function $g : I \to \mathbb{R}^n$, $\|g\|$ denotes the essential supremum of the function $|g|$ where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^n$.

$C^0(I,J)$ denotes the set of continuous functions defined from the interval $I \subset \mathbb{R}_+$ to the topological space J.

We consider the Sobolev space $W^{1;\infty}(\mathbb{R}_+,\mathbb{R}^n)$ of absolutely continuous functions $u:\mathbb{R}_+\to\mathbb{R}^n$. For this class of functions, the derivative $\dot{u}$ is measurable, and we have $\|u\|<\infty$, $\|\dot{u}\|<\infty$.

Let $a\in\mathbb{Z}$ and $b\in\mathbb{R}$. Define the set $[\![a,\infty[\![= \{k\in\mathbb{Z} \mid k\geq a\}$. Let $f:[\![a,\infty[\![\times]b,\infty[\to\mathbb{R}$. We say that

$$\lim_{(k,\gamma)\to\infty} f(k,\gamma)=0$$

if $\forall\epsilon>0$, $\exists k_\epsilon\in[\![a,\infty[\![$, $\exists\gamma_\epsilon\in]b,\infty[$ such that $\forall k\in[\![k_\epsilon,\infty[\![, \forall\gamma\in[\gamma_\epsilon,\infty[$ we have $|f(k,\gamma)|\leq\epsilon$.

Let $a_1,a_2\in\mathbb{Z}$. Then we define the set $[\![a_1,a_2]\!]=\{k\in\mathbb{Z}\mid a_1\leq k\leq a_2\}$.

For any $\gamma\in]0,\infty[$ define the linear change is time scale $s_\gamma:\mathbb{R}\to\mathbb{R}$ by $s_\gamma(t)=t/\gamma,\forall t\in\mathbb{R}$.

For any $a\in\mathbb{R}$ define the translation $\tau_a:\mathbb{R}\to\mathbb{R}$ by $\tau_a(t)=t+a,\forall t\in\mathbb{R}$.

A function $u\in C^0(\mathbb{R}_+,\mathbb{R}^n)$ is said to be T–periodic, where $T\in]0,\infty[$, if $u(t+T)=u(t),\forall t\in\mathbb{R}_+$.

For any $T\in]0,\infty[$ we define Ω_T as the set of all T–periodic functions $u\in C^0(\mathbb{R}_+,\mathbb{R}^n)$, and $\Omega=\cup_{T\in]0,\infty[}\Omega_T$.

3. Background results from Ref. [5]

This section presents a summary of those results of Ref. [5] that are relevant to the present work.

3.1. Time-scale change

Let $T\in]0,\infty[$ and $\gamma\in]0,\infty[$. Let $f_{T;\gamma}\in C^0(\mathbb{R}_+,\mathbb{R}_+)$ be a function that satisfies (a)–(c).

(a) $f_{T;\gamma}$ is strictly increasing on $\mathbb{R}_+$.
(b) $f_{T;\gamma}(0)=0$, $\lim_{t\to\infty} f_{T;\gamma}(t)=\infty$, and $f_{T;\gamma}(t+k\gamma T)=f_{T;\gamma}(t)+kT,\forall t\in[0,\gamma T],\forall k\in\mathbb{N}$.
(c) $f^{-1}_{T;\gamma}\in C^0(\mathbb{R}_+,\mathbb{R}_+)$, and $f^{-1}_{T;\gamma}(t+kT)=f^{-1}_{T;\gamma}(t)+k\gamma T,\forall t\in[0,T],\forall k\in\mathbb{N}$.

The set of all such functions $f_{T;\gamma}$ is denoted $\mathbb{I}_{T;\gamma}$.
For any $\gamma\in]0,\infty[$ define the set $\mathbb{I}_{T;1}\circ s_\gamma=\{f_{T;1}\circ s_\gamma, f_{T;1}\in\mathbb{I}_{T;1}\}$.

Proposition 2. $\mathbb{I}_{T;1}\circ s_\gamma=\mathbb{I}_{T;\gamma}$.

Proposition 2 provides a practical way to construct the set $\mathbb{I}_{T;\gamma}$: pick all elements of $\mathbb{I}_{T;1}$ and compose each element with s_γ to get all elements of $\mathbb{I}_{T;\gamma}$.

For any $T\in]0,\infty[$ define the set $s_{\frac{1}{T}}\circ\mathbb{I}_{1;1}\circ s_T=\{s_{\frac{1}{T}}\circ f_{1;1}\circ s_T, f_{1;1}\in\mathbb{I}_{1;1}\}$.

Proposition 3. $s_{\frac{1}{T}}\circ\mathbb{I}_{1;1}\circ s_T=\mathbb{I}_{T;1}$.

Proposition 3 provides a practical way to construct the set $\mathbb{I}_{T;1}$: pick all $f_{1;1}\in\mathbb{I}_{1;1}$ and compute $s_{\frac{1}{T}}\circ f_{1;1}\circ s_T$ to get all elements of $\mathbb{I}_{T;1}$.

Corollary 4. $\mathbb{I}_{T;\gamma}=s_{\frac{1}{T}}\circ\mathbb{I}_{1;1}\circ s_{\gamma T}$.

3.2. The frame

Let Ξ be a Banach space with norm $|\cdot|$. Let $T\in]0,\infty[$, then $C^0\big([0,T],\Xi\big)$ is a Banach space with respect to the norm $\|\cdot\|$ defined by $\|a\|=\sup_{t\in[0,T]}|a(t)|$ for any $a\in C^0\big([0,T],\Xi\big)$. For any $w\in C^0\big(\mathbb{R}_+,\Xi\big)$ define $\|w\|=\sup_{t\in\mathbb{R}_+}|w(t)|$. Define the set $\mathcal{B}(\mathbb{R}_+,\Xi)=\big\{w\in C^0\big(\mathbb{R}_+,\Xi\big)\mid \|w\|<\infty\big\}$.

Let $n,m\in\mathbb{N}\setminus\{0\}$. We consider an operator $\Phi:C^0(\mathbb{R}_+,\mathbb{R}^n)\times\Xi\to C^0(\mathbb{R}_+,\Xi)$, an operator $\mathcal{H}:C^0(\mathbb{R}_+,\mathbb{R}^n)\times\Xi\to C^0(\mathbb{R}_+,\mathbb{R}^m)$, and a continuous function $\mathcal{T}:\Xi\to\mathbb{R}^m$ such that for any $(u,x_0)\in C^0(\mathbb{R}_+,\mathbb{R}^n)\times\Xi$ we have $\mathcal{H}(u,x_0)=\mathcal{T}\circ[\Phi(u,x_0)]$.

We also set that $[\Phi(u,x_0)](0)=x_0$ for all $(u,x_0)\in C^0(\mathbb{R}_+,\mathbb{R}^n)\times\Xi$.

3.3. Confluence

Define the set $\mathrm{Id} \in \mathbb{S} \subset \mathbb{I}_{1;1}$ where $\mathrm{Id} : \mathbb{R}_+ \to \mathbb{R}_+$ is the identity function defined by $\mathrm{Id}(t) = t, \forall t \in \mathbb{R}_+$. For any $T, \gamma \in \,]0, \infty[$ define the set

$$\mathbb{S}_{T;\gamma} = \{f_{T;\gamma} = s_{\frac{1}{T}} \circ f_{1;1} \circ s_{\gamma T}, f_{1;1} \in \mathbb{S}\} = s_{\frac{1}{T}} \circ \mathbb{S} \circ s_{\gamma T}.$$

Then $s_\gamma \in \mathbb{S}_{T;\gamma} \subset \mathbb{I}_{T;\gamma}$.
Define also the set

$$\mathbb{S}_T = \bigcup_{\gamma \in]0,\infty[} \mathbb{S}_{T;\gamma}.$$

Definition 5. *Let $(T, u, x_0) \in \,]0, \infty[\times \Omega_T \times \Xi$. We say that the operator Φ is confluent with respect to $(u, x_0, \mathbb{S})$ if there exists a function $\alpha_{u;\Phi;x_0;\mathbb{S}} \in C^0\big([0, T], \Xi\big)$ such that the following holds. Define the function $\wp_\Phi : \mathbb{N}\times]0, \infty[\to \mathbb{R}_+ \cup \{\infty\}$ by*

$$\forall (k, \gamma) \in \mathbb{N}\times]0, \infty[,\quad \wp_\Phi(k, \gamma) = \sup_{f_{T;\gamma} \in \mathbb{S}_{T;\gamma}} \left\| \Big(\Phi(u \circ f_{T;\gamma}, x_0) \circ f^{-1}_{T;\gamma} \circ \tau_{kT}\Big) |_{[0,T]} - \alpha_{u;\Phi;x_0;\mathbb{S}} \right\|_{\Xi;[0,T]}. \tag{2}$$

Then,

$$\lim_{(k,\gamma)\to\infty} \wp_\Phi(k, \gamma) = 0. \tag{3}$$

Proposition 6. *The function $\alpha_{u;\Phi;x_0;\mathbb{S}}$ does not depend on the set $\mathbb{S}$.*

Owing to Proposition 6 we use the notation $\alpha_{u;\Phi;x_0}$ instead of $\alpha_{u;\Phi;x_0;\mathbb{S}}$.
When Φ is confluent with respect to $(u, x_0, \mathbb{S})$, the set

$$\mathcal{Q}_{u;\Phi;x_0} = \big\{ \big(u(t), \alpha_{u;\Phi;x_0}(t)\big), t \in [0, T] \big\} \tag{4}$$

is called the *hysteresis loop of the operator Φ with respect to (u, x_0)*.

Proposition 7. *If Φ is confluent with respect to $(u, x_0, \mathbb{S})$ then $\mathcal{H}$ is confluent with respect to $(u, x_0, \mathbb{S})$ and $\alpha_{u;\mathcal{H};x_0} = \mathcal{T} \circ \alpha_{u;\Phi;x_0}$.*

Proposition 7 says that $\mathcal{H}$ is confluent whenever Φ is confluent. This is why we focus on the study of Φ in this paper.

3.4. Some inferences from Ref. [5]

Proposition 8. *Let $T \in \,]0, \infty[$, $u \in \Omega_T$, and $x_0 \in \Xi$. If the operator Φ is confluent with respect to $(u, x_0, \mathbb{S})$ then there exists $\gamma_{u;x_0} \in \,]0, \infty[$ such that $\forall \gamma \geq \gamma_{u;x_0}$ we have the following: $\forall f_{T;\gamma} \in \mathbb{S}_{T;\gamma}, \Phi(u \circ f_{T;\gamma}, x_0) \in \mathcal{B}(\mathbb{R}_+, \Xi)$. In particular $\forall \gamma \geq \gamma_{u;x_0}, \Phi(u \circ s_\gamma, x_0) \in \mathcal{B}(\mathbb{R}_+, \Xi)$.*

Proposition 9. *Let $u_0 \in \mathbb{R}^n$ and $x_0 \in \Xi$. Define $\mathbf{u}_0 \in \Omega_{T=1}$ by $\mathbf{u}_0(t) = u_0, \forall t \in \mathbb{R}_+$ (observe that the function $\mathbf{u}_0$ can be considered 1–periodic). If the operator Φ is confluent with respect to $(\mathbf{u}_0, x_0, \mathbb{S})$ then there exists $\ell \in \Xi$ such that $\lim_{t\to\infty} \big[\Phi(\mathbf{u}_0, x_0)\big](t) = \ell$. Moreover, $\forall t \in [0, 1], \alpha_{\mathbf{u}_0;\Phi;x_0}(t) = \ell$.*

Definition 10. *Let $\mathscr{U} \subset C^0(\mathbb{R}_+, \mathbb{R}^n)$, $\mathscr{W}$ a set of homeomorphisms of $\mathbb{R}_+$, and $\mathscr{X} \subset \Xi$. We say that Φ is $(\mathscr{U}, \mathscr{W}, \mathscr{X})$*–rate independent *if $\forall u \in \mathscr{U}, \forall f \in \mathscr{W}, \forall x_0 \in \mathscr{X}$ we have $\Phi(u \circ f, x_0) = \Phi(u, x_0) \circ f$.*

Observe that if Φ is $(\mathscr{U}, \mathscr{W}, \mathscr{X})$–rate independent then $\mathcal{H}$ is also $(\mathscr{U}, \mathscr{W}, \mathscr{X})$–rate independent.

Theorem 11. *The statements* (a) *and* (b) *are equivalent.*

(a) *The operator Φ is confluent with respect to $(u, x_0, \mathbb{S})$ for all $(u, x_0) \in \Omega \times \Xi$.*

(b) *There exist unique operators $\Phi^\circ : \Omega \times \Xi \to \Omega$ and $\Phi^\ddagger : \Omega \times \Xi \to C^0(\mathbb{R}_+, \Xi)$ such that* (i)–(iii) *hold.*

(i) $\Phi = \Phi^\circ + \Phi^\ddagger$.
(ii) $\forall T \in \,]0, \infty[$, Φ° *is $(\Omega_T, \bigcup_{\gamma \in]0,\infty[} \mathbb{S}_{T;\gamma}, \Xi)$–rate independent.*
(iii) *Let $T \in \,]0, \infty[$ be arbitrary. For all $u \in \Omega_T$ and all $x_0 \in \Xi$ we have*
(iii-1) $\Phi^\circ(u, x_0)$ *is T–periodic.*
(iii-2) $\displaystyle\lim_{(k,\gamma)\to\infty} \left(\sup_{f_{T;\gamma} \in \mathbb{S}_{T;\gamma}} \big\| \Phi^\ddagger(u \circ f_{T;\gamma}, x_0)|_{[k\gamma T,\infty[} \big\| \right) = 0.$

The operator Φ° is constructed as follows:

$$\big[\Phi^\circ(u, x_0)\big](t + kT) = \alpha_{u;\Phi;x_0}(t), \forall (t, k, T, u, x_0) \in [0, T] \times \mathbb{N}\times \,]0, \infty[\, \times \Omega_T \times \Xi.$$

4. Uniform confluence

This section introduces the concept of uniform confluence which is a sufficient condition for obtaining a causal decomposition of the operator Φ. The section is organized into three subsections: Section 4.1 provides the definition of uniform confluence along with some propositions needed to get the decomposition. Section 4.2 presents the causal canonical decomposition of Φ which is the main result of the paper. Some consequences of this decomposition are analyzed in Section 4.3.

4.1. Preliminary results

Proposition 12. *Let* $T \in]0,\infty[$, $u \in \Omega_T$, $x_0 \in \Xi$, *and* $\mathrm{Id} \in \mathbb{S} \subset \mathbb{I}_{1;1}$. *Suppose that* (i)–(ii) *hold.*

(i) *The operator* Φ *is confluent with respect to* $(u, x_0, \mathbb{S})$.

(ii) $\exists \beta_{u;\Phi;x_0} \in C^0(\mathbb{R}_+, \Xi)$ *such that*

$$\lim_{\gamma\to\infty}\left(\sup_{f_{T;\gamma}\in\mathbb{S}_{T;\gamma}}\left\|\Phi(u\circ f_{T;\gamma}, x_0)\circ f_{T;\gamma}^{-1} - \beta_{u;\Phi;x_0}\right\|\right) = 0. \tag{5}$$

Then (a)–(d) *hold.*

(a) $\beta_{u;\Phi;x_0} \in \mathcal{B}(\mathbb{R}_+, \Xi)$.

(b) *For any* $k \in \mathbb{N}$ *define the function* $\beta_{u;\Phi;x_0,k} \in C^0\big([0,T], \Xi\big)$ *by* $\beta_{u;\Phi;x_0;k} = (\beta_{u;\Phi;x_0} \circ \tau_{kT})\,|_{[0,T]}$. *Then we have*

$$\lim_{k\to\infty}\|\beta_{u;\Phi;x_0,k} - \alpha_{u;\Phi;x_0}\| = 0. \tag{6}$$

(c) $\beta_{u;\Phi;x_0}(0) = x_0$.

(d) *Let* $\mathbf{u}_0$ *be the function of Proposition 9. Then* $\forall t \in \mathbb{R}_+$ *we have* $\big[\Phi(\mathbf{u}_0, x_0)\big](t) = \beta_{\mathbf{u}_0;\Phi;x_0}(t) = x_0$.

Proof. (a) From Proposition 8 there exists $\gamma_{\mathbf{u}_0;x_0} \in]0,\infty[$ such that $\forall \gamma \geq \gamma_{\mathbf{u}_0;x_0}$ we have $\forall f_{T;\gamma} \in \mathbb{S}_{T;\gamma}, \Phi(\mathbf{u}_0 \circ f_{T;\gamma}, x_0) \in \mathcal{B}(\mathbb{R}_+, \Xi)$. Then (ii) gives (a).

(b) Since Φ is confluent with respect to $(\mathbf{u}_0, x_0)$, it comes that $\forall \epsilon > 0, \exists k_\epsilon \in \mathbb{N}, \exists \gamma_\epsilon \in [\gamma_{\mathbf{u}_0;x_0}, \infty[$ such that $\forall \gamma \in [\gamma_\epsilon, \infty[, \forall k \in [\![k_\epsilon, \infty[\![$ we have

$$\sup_{f_{T;\gamma}\in\mathbb{S}_{T;\gamma}}\left\|\Big(\Phi(\mathbf{u}_0\circ f_{T;\gamma}, x_0)\circ f_{T;\gamma}^{-1}\circ\tau_{kT}\Big)\,|_{[0,T]} - \alpha_{\mathbf{u}_0;\Phi;x_0}\right\| \leq \frac{\epsilon}{2}. \tag{7}$$

On the other hand, by Equation (5) there exists $\gamma'_\epsilon \in [\gamma_{\mathbf{u}_0;x_0}, \infty[$ such that $\forall \gamma \in [\gamma'_\epsilon, \infty[$ and $\forall k \in \mathbb{N}$ we have

$$\sup_{f_{T;\gamma}\in\mathbb{S}_{T;\gamma}}\left\|\Big(\Phi(\mathbf{u}_0\circ f_{T;\gamma}, x_0)\circ f_{T;\gamma}^{-1}\circ\tau_{kT}\Big)\,|_{[0,T]} - \underbrace{(\beta_{\mathbf{u}_0;\Phi;x_0}\circ\tau_{kT})\,|_{[0,T]}}_{\beta_{\mathbf{u}_0;\Phi;x_0,k}}\right\| \leq \frac{\epsilon}{2}. \tag{8}$$

Taking $\gamma = \max(\gamma_\epsilon, \gamma'_\epsilon)$, Equations (7) and (8) show that $\forall k \in [\![k_\epsilon, \infty[\![$ we have $\|\beta_{\mathbf{u}_0;\Phi;x_0;k} - \alpha_{\mathbf{u}_0;\Phi;x_0}\| \leq \epsilon$ which proves (b).

(c) Equation (5) and $\big[\Phi(\mathbf{u}_0 \circ f_{T;\gamma}, x_0) \circ f_{T;\gamma}^{-1}\big](0) = x_0$ provide (c).

(d) In this case T can be taken to be 1 and $\mathbf{u}_0$ can be considered 1–periodic. Taking $f_{T;\gamma} = s_\gamma$ Equation (5) reduces to

$$\lim_{\gamma\to\infty}\left\|\Phi(\mathbf{u}_0, x_0)\circ s_{\frac{1}{\gamma}} - \beta_{\mathbf{u}_0;\Phi;x_0}\right\| = 0. \tag{9}$$

Thus, $\forall \epsilon > 0, \exists \gamma_\epsilon \in [\gamma_{\mathbf{u}_0;x_0}, \infty[$ such that

$$\forall \gamma \in [\gamma_\epsilon, \infty[, \forall t \in \mathbb{R}_+, \left|\Phi(\mathbf{u}_0, x_0)\circ s_{\frac{1}{\gamma}}(t) - \beta_{\mathbf{u}_0;\Phi;x_0}(t)\right| \leq \epsilon. \tag{10}$$

Taking $t = 0$ in Equation (10) gives

$$\big[\Phi(\mathbf{u}_0, x_0)\big](0) = \beta_{\mathbf{u}_0;\Phi;x_0}(0) = x_0. \tag{11}$$

Taking $t \in]0,\infty[$ in Equation (10) gives

$$\forall \sigma \in [\gamma_\epsilon t, \infty[,\ \left|\big[\Phi(\mathbf{u}_0, x_0)\big](\sigma) - \beta_{\mathbf{u}_0;\Phi;x_0}(t)\right| \leq \epsilon. \tag{12}$$

Proposition 9 along with Equation (12) show that there exists $\ell \in \Xi$ such that

$$\forall t \in \,]0,\infty[, \; \beta_{\mathbf{u}_0;\Phi;x_0}(t) = \ell. \tag{13}$$

The continuity of $\beta_{\mathbf{u}_0;\Phi;x_0}$ ensures that $\beta_{\mathbf{u}_0;\Phi;x_0}(0) = \ell$ so that $\ell = x_0$.

Now, suppose that there exists $t_0 \in \,]0,\infty[$ such that $\big[\Phi(\mathbf{u}_0, x_0)\big](t_0) \neq x_0$. Define the quantity $\epsilon_1 = |[\Phi(\mathbf{u}_0, x_0)](t_0) - x_0|/2$. There exists $\gamma_{\epsilon_1} \in [\gamma_{\mathbf{u}_0;x_0}, \infty[$ such that Inequality (10) holds with ϵ_1 instead of ϵ. Taking $t = t_0/\gamma_{\epsilon_1}$ and $\gamma = \gamma_{\epsilon_1}$ in (10) leads to a contradiction.

□

Proposition 13. *Let $T \in \,]0,\infty[$, $u \in \Omega_T$, $x_0 \in \Xi$, and* $\mathrm{Id} \in \mathbb{S} \subset \mathbb{I}_{1;1}$. *Suppose that $\exists \beta_{u;\Phi;x_0} \in C^0(\mathbb{R}_+, \Xi)$ such that Equation (5) holds. Then the assertions* (i) *and* (b) *of Proposition (12) are equivalent.*

Proof. (i) ⇒ (b) is proved is Proposition 12.
(b) ⇒ (i). Take $\epsilon \in \,]0,\infty[$, then $\exists k_\epsilon \in \mathbb{N}$ such that $\forall k \in [\![k_\epsilon, \infty[\![$ we have

$$\left\|(\beta_{u;\Phi;x_0} \circ \tau_{kT})|_{[0,T]} - \alpha_{u;\Phi;x_0}\right\| \leq \frac{\epsilon}{2}. \tag{14}$$

From Equation (5) it comes that $\exists \gamma_\epsilon \in \,]0,\infty[$ such that $\forall \gamma \in [\gamma_\epsilon, \infty[$ we have

$$\sup_{f_{T;\gamma} \in \mathbb{S}_{T;\gamma}} \left\|\left(\Phi(u \circ f_{T;\gamma}, x_0) \circ f_{T;\gamma}^{-1} \circ \tau_{kT}\right)|_{[0,T]} - (\beta_{u;\Phi;x_0} \circ \tau_{kT})|_{[0,T]}\right\| \leq \frac{\epsilon}{2}, \forall k \in \mathbb{N}. \tag{15}$$

Combining (14) and (15) we get (3). □

Definition 14. *Let* $\mathrm{Id} \in \mathbb{S} \subset \mathbb{I}_{1;1}$. *Let $\Omega' \subset \Omega$ be such that $u \circ \mathbb{S}_T \subset \Omega'$ for all $u \in \Omega' \cap \Omega_T$. Let $\Xi' \subset \Xi$. We say that Φ is uniformly confluent with respect to $(\Omega', \Xi', \mathbb{S})$ if (P_0)–(P_2) hold.*

(P_0) *Φ is $\big(\Omega', \Xi'\big)$–causal.*
(P_1) *$\forall (u, x_0) \in \Omega' \times \Xi'$, the operator Φ is confluent with respect to $(u, x_0, \mathbb{S})$.*
(P_2) *$\forall (T, u, x_0) \in \,]0,\infty[\times \Omega_T \cap \Omega' \times \Xi', \exists \beta_{u;\Phi;x_0} \in \mathcal{B}(\mathbb{R}_+, \Xi)$ such that*

$$\lim_{\gamma\to\infty} \left(\sup_{f_{T;\gamma} \in \mathbb{S}_{T;\gamma}} \left\|\Phi(u \circ f_{T;\gamma}, x_0) \circ f_{T;\gamma}^{-1} - \beta_{u;\Phi;x_0}\right\|_{\Xi;\mathbb{R}_+} \right) = 0. \tag{16}$$

Proposition 15. *Let* $\mathrm{Id} \in \mathbb{S} \subset \mathbb{I}_{1;1}$. *Let $\Omega' \subset \Omega$ be such that $u \circ \mathbb{S}_T \subset \Omega'$ for all $u \in \Omega' \cap \Omega_T$. Let $\Xi' \subset \Xi$. Suppose that Φ is uniformly confluent with respect to $(\Omega', \Xi', \mathbb{S})$. Define the operator $\mathbf{b} : \Omega' \times \Xi' \to C^0(\mathbb{R}_+, \Xi)$ by $\mathbf{b}(u, x_0) = \beta_{u;\Phi;x_0}$. Then* (a)–(c) *hold.*

(a) *$\mathbf{b}$ is $\big(\Omega', \Xi'\big)$–causal.*
(b) *$\forall T \in \,]0,\infty[$, the operator $\mathbf{b}$ is $(\Omega_T \cap \Omega', \mathbb{S}_T, \Xi')$–rate independent.*
(c) *Let $x_0 \in \Xi'$. Suppose that $\exists u_0 \in \mathbb{R}^n$ such that $\forall t \in \mathbb{R}_+$ we have $u(t) = u_0$, and $u \in \Omega'$. Then $\forall t \in \mathbb{R}_+$ we have $\big[\mathbf{b}(u, x_0)\big](t) = x_0$.*

Proof. (a) Let $x_0 \in \Xi'$, $\theta \in \mathbb{R}_+$ and $u, v \in \Omega'$. Suppose that $\forall t \in [0, \theta], u(t) = v(t)$. Then $\forall \gamma \in \,]0,\infty[, \forall t \in [0, \theta]$, $\big[\Phi(u \circ s_\gamma, x_0)\big] \circ s_{\frac{1}{\gamma}}(t) = \big[\Phi(v \circ s_\gamma, x_0)\big] \circ s_{\frac{1}{\gamma}}(t)$ since Φ is $\big(\Omega', \Xi'\big)$–causal and $u \circ s_\gamma, v \circ s_\gamma \in \Omega'$. Then, (a) follows from Equation (16).
(b) $\forall \lambda, \gamma \in \,]0,\infty[$, if $f_{T;\lambda} \in \mathbb{S}_{T;\lambda}$ then $f_{T;\lambda} \circ s_{\frac{\gamma}{\lambda}} \in \mathbb{S}_{T;\gamma}$. Let $u \in \Omega_T \cap \Omega'$ then $u \circ f_{T;\lambda} \in \Omega_{\lambda T} \cap \Omega'$. From Equation (16) we get

$$\lim_{\gamma\to\infty} \Big\|\Phi(u \circ \underbrace{f_{T;\lambda} \circ s_{\frac{\gamma}{\lambda}}}_{f_{T;\gamma}}, x_0) \circ \underbrace{s_{\frac{\lambda}{\gamma}} \circ f_{T;\lambda}^{-1}}_{f_{T;\gamma}^{-1}} - \beta_{u;\Phi;x_0}\Big\| = 0, \tag{17}$$

$$\lim_{\gamma'\to\infty} \Big\|\Phi(\underbrace{u \circ f_{T;\lambda}}_{v} \circ s_{\gamma'}, x_0) \circ s_{\frac{1}{\gamma'}} - \beta_{\underbrace{u \circ f_{T;\lambda}}_{v};\Phi;x_0}\Big\| = 0, \tag{18}$$

Taking $\gamma' = \gamma/\lambda$ and combining Equations (17) and (18) it comes that $\beta_{u \circ f_{T;\lambda};\Phi;x_0} \circ f_{T;\lambda}^{-1} = \beta_{u;\Phi;x_0}$ which gives (b).
(c) is a direct consequence of Proposition 12 (d). □

4.2. The causal canonical decomposition

Theorem 16. *Let* $\mathrm{Id} \in \mathbb{S} \subset \mathbb{I}_{1;1}$. *Let* $\Omega' \subset \Omega$ *be such that* $u \circ \mathbb{S}_T \subset \Omega'$ *for all* $u \in \Omega' \cap \Omega_T$. *Let* $\Xi' \subset \Xi$. *The statements* (i) *and* (ii) *are equivalent.*

(i) Φ *is uniformly confluent with respect to* $(\Omega', \Xi', \mathbb{S})$.

(ii) *There exist unique operators* $\Phi^\star : \Omega' \times \Xi' \to C^0(\mathbb{R}_+, \Xi)$ *and* $\Phi^\dagger : \Omega' \times \Xi' \to C^0(\mathbb{R}_+, \Xi)$ *such that* (ii-a)–(ii-d) *hold.*

(ii-a) $\Phi = \Phi^\star + \Phi^\dagger$.

(ii-b) $\Phi^\star$ *and* $\Phi^\dagger$ *are* (Ω', Ξ')*–causal.*

(ii-c) $\forall T \in]0, \infty[$, $\Phi^\star$ *is* $(\Omega_T \cap \Omega', \mathbb{S}_T, \Xi')$*–rate independent.*

(ii-d) $\forall (T, u, x_0) \in]0, \infty[\times \Omega_T \cap \Omega' \times \Xi'$ *we have*

(ii-d-1) $\displaystyle\lim_{\gamma \to \infty} \left(\sup_{f_{T;\gamma} \in \mathbb{S}_{T;\gamma}} \left\| \Phi^\dagger(u \circ f_{T;\gamma}, x_0) \right\| \right) = 0,$

(ii-d-2) *the sequence* $\left\{ \left(\Phi^\star(u, x_0) \circ \tau_{kT} \right)|_{[0,T]} \right\}_{k \in \mathbb{N}}$ *converges in* $C^0\big([0, T], \Xi\big)$.

Proof. (i) $\Rightarrow$ (ii)

Existence. Take $\Phi^\star = \mathbf{b}$ and $\Phi^\dagger = \Phi - \mathbf{b}$. Then (ii-a)–(ii-d) hold by Propositions 12 and 15.

Uniqueness. Suppose that (ii-a)–(ii-d) hold for the pair $(\Phi^\star, \Phi^\dagger)$. Then Equation (16) along with (ii-a), (ii-c) and (ii-d) lead to $\Phi^\star(u, x_0) = \beta_{u;\Phi;x_0}$ for all $(u, x_0) \in \Omega' \times \Xi'$. Thus $\Phi^\star = \mathbf{b}$ so that $\Phi^\dagger = \Phi - \mathbf{b}$.

(ii) $\Rightarrow$ (i)

Let $(T, u, x_0) \in]0, \infty[\times \Omega_T \cap \Omega' \times \Xi'$. Observe that $\left\| \Phi^\dagger(u \circ f_{T;\gamma}, x_0) \right\| = \left\| \Phi^\dagger(u \circ f_{T;\gamma}, x_0) \circ f_{T;\gamma}^{-1} \right\|$ so that (ii-d-1) gives

$$\lim_{\gamma \to \infty} \left(\sup_{f_{T;\gamma} \in \mathbb{S}_{T;\gamma}} \left\| \Phi^\dagger(u \circ f_{T;\gamma}, x_0) \circ f_{T;\gamma}^{-1} + \underbrace{\Phi^\star(u \circ f_{T;\gamma}, x_0) \circ f_{T;\gamma}^{-1}}_{= \Phi^\star(u, x_0)} - \Phi^\star(u, x_0) \right\| \right) = 0 \tag{19}$$

which gives Equation (16) by taking $\beta_{u;\Phi;x_0} = \Phi^\star(u, x_0)$. Then (P_1) is obtained from Equation (16) and (ii-d-2). Proposition 8, (P_1), and Equation (16) show that $\beta_{u;\Phi;x_0} \in \mathcal{B}(\mathbb{R}_+, \Xi)$ which gives (P_2). Finally Φ is $\big(\Omega', \Xi'\big)$–causal owing to (ii-b).

□

The operator $\Phi^\star$ is called the *causal rate-independent component* of Φ whilst the operator $\Phi^\dagger$ is called the *causal nonhysteretic component* of Φ. Equality (ii-a) is called the *causal canonical decomposition* of Φ.

Remark 17. *Let* $\mathrm{Id} \in \mathbb{S} \subset \mathbb{I}_{1;1}$. *Let* $\Omega' \subset \Omega$ *be such that* $u \circ \mathbb{S}_T \subset \Omega', \forall u \in \Omega' \cap \Omega_T$. *Let* $\Xi' \subset \Xi$. *Suppose that* Φ *is uniformly confluent with respect to* $(\Omega', \Xi', \mathbb{S})$. *Then* $\forall (u, x_0) \in \Omega' \times \Xi'$, $\big[\Phi^\star(u, x_0)\big](0) = x_0$ *by Proposition 12 (c). This means that* $\forall (u, x_0) \in \Omega' \times \Xi'$, $\big[\Phi^\dagger(u, x_0)\big](0) = 0$. *Also, let* $u_0 \in \mathbb{R}^n$ *and* $x_0 \in \Xi'$. *Define* $\mathbf{u}_0 \in \Omega$ *by* $\mathbf{u}_0(t) = u_0, \forall t \in \mathbb{R}_+$, *and suppose that* $\mathbf{u}_0 \in \Omega'$. *Then, by Proposition 12 (d) we have* $\forall t \in \mathbb{R}_+$, $\big[\Phi^\dagger(\mathbf{u}_0, x_0)\big](t) = 0$.

4.3. Some properties of uniformly confluent operators

In this section, we assume that Φ satisfies the semigroup property. That is,

Assumption 18. $\forall (u, x_0) \in C^0(\mathbb{R}_+, \mathbb{R}^n) \times \Xi$ *define* $x = \Phi(u, x_0)$. *Then* $\forall t_1, t_2 \in \mathbb{R}_+$ *such that* $t_1 \leq t_2$ *we have* $x(t_2) = \big[\Phi\big(u \circ \tau_{t_1}, x(t_1)\big)\big](t_2 - t_1)$ *[8, p. 60].*

Our aim is to analyze the consequences that derive from Assumption 18.

Proposition 19. *Suppose that* Φ *is* $\big(C^0(\mathbb{R}_+, \mathbb{R}^n), \Xi\big)$*–causal, and that* Φ *satisfies Assumption 18. Moreover, let* $\mathrm{Id} \in \mathbb{S} \subset \mathbb{I}_{1;1}$. *Let* $\Omega' \subset \Omega$ *be such that* $u \circ \mathbb{S}_T \subset \Omega', \forall u \in \Omega' \cap \Omega_T$ *and* $u \circ \tau_t \in \Omega', \forall u \in \Omega' \cap \Omega_T$. *Suppose that* Φ *is uniformly confluent with respect to* $(\Omega', \Xi, \mathbb{S})$.

Then (a)–(d) *hold.*

(a) *Let* $(\theta_1, \theta_2, u_0, u, x_0) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \times C^0(\mathbb{R}_+, \mathbb{R}^n) \times \Xi$ *be such that* $0 \leq \theta_1 < \theta_2$ *and* $u(t) = u_0, \forall t \in [\theta_1, \theta_2]$*. Then* $\forall t \in [\theta_1, \theta_2]$ *we have* $\big[\Phi(u, x_0)\big](t) = \big[\Phi(u, x_0)\big](\theta_1)$*. In other words: whenever the input* u *is constant on the interval* $[\theta_1, \theta_2]$*, the corresponding output* $\Phi(u, x_0)$ *is also constant on the same interval.*

(b) *Let* $(\theta, u_0, u, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n \times C^0(\mathbb{R}_+, \mathbb{R}^n) \times \Xi$ *be such that* $u(t) = u_0, \forall t \in [\theta, \infty[$*. Then* $\forall t \in [\theta, \infty[$ *we have* $\big[\Phi(u, x_0)\big](t) = \big[\Phi(u, x_0)\big](\theta)$*. In other words: whenever the input* u *is constant on the interval* $[\theta_1, \infty[$*, the corresponding output* $\Phi(u, x_0)$ *is also constant on the same interval.*

(c) *Let* $(T, \theta_1, \theta_2, u_0, u, x_0) \in]0, \infty[\times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \times \Omega_T \cap \Omega' \times \Xi$ *be such that* $0 \leq \theta_1 < \theta_2 \leq T$ *and* $u(t) = u_0, \forall t \in [\theta_1, \theta_2]$*. Then* $\forall t \in [\theta_1, \theta_2]$ *we have* $\big[\Phi^\star(u, x_0)\big](t) = \big[\Phi^\star(u, x_0)\big](\theta_1)$*. Also, let* $k \in \mathbb{N}$*, then* $\forall t \in [\theta_1 + kT, \theta_2 + kT]$ *we have* $\big[\Phi^\star(u, x_0)\big](t) = \big[\Phi^\star(u, x_0)\big](\theta_1 + kT)$*. In other words: whenever the input* u *is periodic and constant on the interval* $[\theta_1, \theta_2]$*, the causal rate-independent component* $\Phi^\star(u, x_0)$ *is also constant on the same interval. This means that the nonhysteretic component* $\Phi^\dagger(u, x_0)$ *is also constant on the same interval.*

Proof. (a) Define the function $v \in C^0(\mathbb{R}_+, \mathbb{R}^n)$ by $v(t) = u(t), \forall t \in [0, \theta_1]$ and $v(t) = u(\theta_1), \forall t \in [\theta_1, \infty[$. Since Φ is $\big(C^0(\mathbb{R}_+, \mathbb{R}^n), \Xi\big)$–causal we have

$$\forall t \in [0, \theta_2], \big[\Phi(u, x_0)\big](t) = \big[\Phi(v, x_0)\big](t). \tag{20}$$

On the other hand, by Assumption 18 (semigroup property) we have

$$\forall t \in [\theta_1, \infty[\,, \big[\Phi(v, x_0)\big](t) = \Big[\Phi\big(v \circ \tau_{\theta_1}, \big[\Phi(v, x_0)\big](\theta_1)\big)\Big](t - \theta_1). \tag{21}$$

Since $v \circ \tau_{\theta_1}(t') = u(\theta_1), \forall t' \in \mathbb{R}_+$ it follows from Proposition 12 (d) that

$$\forall t' \in \mathbb{R}_+, \Big[\Phi\big(v \circ \tau_{\theta_1}, \big[\Phi(v, x_0)\big](\theta_1)\big)\Big](t') = \big[\Phi(v, x_0)\big](\theta_1). \tag{22}$$

Combining Equations (22) and (20) it comes that

$$\forall t' \in \mathbb{R}_+, \Big[\Phi\big(v \circ \tau_{\theta_1}, \big[\Phi(v, x_0)\big](\theta_1)\big)\Big](t') = \big[\Phi(u, x_0)\big](\theta_1). \tag{23}$$

Taking $t \in [\theta_1, \theta_2]$ in (20)–(21) and $t' = t - \theta_1$ in (23) we get (a).
(b) is a direct consequence of (a).
(c) By (a) it comes that $\forall \gamma \in]0, \infty[, \forall t \in [\theta_1, \theta_2]$, $\big[\Phi(u \circ s_\gamma, x_0)\big] \circ s_{\frac{1}{\gamma}}(t) = \big[\Phi(u \circ s_\gamma, x_0)\big] \circ s_{\frac{1}{\gamma}}(\theta_1)$. Then, (c) follows from Equation (16).

□

5. Case study

In this section we use the scalar semilinear Duhem model as a case study to illustrate the concept of uniform confluence. The main result of this section is Theorem 21.
The scalar semilinear Duhem model is composed of a state equation (24a)–(24b) and an algebraic output equation (24c) such that [6]:

$$\begin{aligned} \dot{x}(t) &= g_1\big(\dot{u}(t)\big)\big(A_1 x(t) + B_1 u(t) + E_1\big) + g_2\big(\dot{u}(t)\big)\big(A_2 x(t) + B_2 u(t) + E_2\big) \\ &\qquad \text{for almost all } t \in \mathbb{R}_+, && \text{(24a)} \\ x(0) &= x_0, && \text{(24b)} \\ y(t) &= Cx(t) + Du(t), \forall t \in \mathbb{R}_+. && \text{(24c)} \end{aligned}$$

In Equations (24) we have $A_1, A_2 \in \mathbb{R}$ with $A_1 < 0 < A_2$; $B_1, B_2, E_1, E_2, D \in \mathbb{R}$, $0 \neq C \in \mathbb{R}$. We consider that $u \in W^{1;\infty}(\mathbb{R}_+, \mathbb{R})$ whereas the properties of $y : \mathbb{R}_+ \to \mathbb{R}$ and $x : \mathbb{R}_+ \to \mathbb{R}$ will be analyzed in

Theorem 20. The functions $g_1 : \mathbb{R} \to \mathbb{R}$ and $g_2 : \mathbb{R} \to \mathbb{R}$ are continuous and satisfy $g_1(w) = 0$ for $w \leq 0$, $g_2(w) = 0$ for $w \geq 0$. Define

$$\bar{g}_1(w) = \frac{g_1(w)}{|w|}, \forall w \neq 0, \tag{25}$$

$$\bar{g}_2(w) = \frac{g_2(w)}{|w|}, \forall w \neq 0. \tag{26}$$

As in Ref. [6] we assume that[1]

$$\lim_{w \downarrow 0} \bar{g}_1(w) = 1 \text{ and } \lim_{w \uparrow 0} \bar{g}_2(w) = -1. \tag{27}$$

A unique absolutely continuous solution of (24a) exists on $\mathbb{R}_+$ [4, Section 11.1].

Theorem 20. *[3] Consider the semilinear Duhem model given by Equations (24). Then we have* $x, y \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$.

Define the operator $\Phi : W^{1,\infty}(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R} \to W^{1,\infty}(\mathbb{R}_+, \mathbb{R})$ by the relation $\Phi(u, x_0) = x$.
Define the set

$$\mathbb{S} = \left\{ f_{1;1} \in \mathbb{I}_{1;1} \mid f_{1;1}, f_{1;1}^{-1} \in W^{1;\infty}(\mathbb{R}_+, \mathbb{R}_+) \right\}.$$

For any $d \in [1, \infty[$ and any $T, \gamma \in]0, \infty[$ define the set

$$\mathbb{S}^*_{T;\gamma;d} = \left\{ f_{T;\gamma} = f_{T;1} \circ s_\gamma \text{ where } f_{T;1} = s_{\frac{1}{T}} \circ f_{1;1} \circ s_T \mid f_{1;1} \in \mathbb{S}, \|\dot{f}_{1;1}\| \leq d \text{ and } \|\dot{\overbrace{f_{1;1}^{-1}}}\| \leq d \right\}. \tag{28}$$

Note that $\mathrm{Id} \in \mathbb{S}^*_{1;1;d}$, $\|\dot{f}_{T;1}\| \leq d$, and $\|\dot{\overbrace{f_{T;1}^{-1}}}\| \leq d$.
Define also the set

$$\mathbb{S}^*_{T;d} = \bigcup_{\gamma \in]0,\infty[} \mathbb{S}^*_{T;\gamma;d}.$$

Theorem 21. *Φ is uniformly confluent with respect to $\left(W^{1;\infty}(\mathbb{R}_+, \mathbb{R}) \cap \Omega, \mathbb{R}, \mathbb{S}^*_{1;1;d}\right)$. We have*

$$\left[\dot{\overbrace{\Phi^\star(u, x_0)}}\right](t) = \dot{u}(t)\Big(A_1 \left[\Phi^\star(u, x_0)\right](t) + B_1 u(t) + E_1\Big),$$
$$\text{for almost all } t \in \mathbb{R}_+ \text{ such that } \dot{u}(t) \geq 0, \tag{29a}$$

$$\left[\dot{\overbrace{\Phi^\star(u, x_0)}}\right](t) = \dot{u}(t)\Big(A_2 \left[\Phi^\star(u, x_0)\right](t) + B_2 u(t) + E_2\Big),$$
$$\text{for almost all } t \in \mathbb{R}_+ \text{ such that } \dot{u}(t) \leq 0, \tag{29b}$$

$$\left[\Phi^\star(u, x_0)\right](0) = x_0. \tag{29c}$$

Also, $\Phi^\circ(u, x_0)$ satisfies (29a)–(29b) substituting $\Phi^\star$ by Φ°. However, the initial condition $\left[\Phi^\circ(u, x_0)\right](0)$ may be different from x_0.

Proof. We have to prove Properties (P_0), (P_1), and (P_2) of Definition 14.
Proof of (P_0).
It is shown in [4, Section 11.2] that Φ is $\left(W^{1;\infty}(\mathbb{R}_+, \mathbb{R}), \mathbb{R}\right)$–causal.
Proof of (P_2).
Let $(T, u, x_0) \in]0, \infty[\times W^{1,\infty}(\mathbb{R}_+, \mathbb{R}) \cap \Omega_T \times \mathbb{R}$. For any $\gamma \in]0, \infty[$ let $f_{T;\gamma} \in \mathbb{S}^*_{T;\gamma;d}$. Define

$$\begin{aligned} x_\gamma &= \Phi(u \circ f_{T;\gamma}, x_0), \\ \bar{x}_\gamma &= x_\gamma \circ f_{T;\gamma}^{-1}. \end{aligned} \tag{30}$$

[1]If $\lim_{w \downarrow 0} \bar{g}_1(w) = b_1 \neq 0$ and $\lim_{w \uparrow 0} \bar{g}_2(w) = -b_2 \neq 0$, the constants b_1 and b_2 are incorporated into the matrices A_1 and A_2 respectively.

Then, by [5, Proposition Appendix A.1] we get

$$\begin{aligned}\dot{f}_{T;\gamma}\big(f_{T;\gamma}^{-1}(t)\big)\dot{\bar{x}}_\gamma(t) = {} & g_1\Big(\dot{f}_{T;\gamma}\big(f_{T;\gamma}^{-1}(t)\big)\dot{u}(t)\Big)\big(A_1\bar{x}_\gamma(t)+B_1u(t)+E_1\big)\\ & + g_2\Big(\dot{f}_{T;\gamma}\big(f_{T;\gamma}^{-1}(t)\big)\dot{u}(t)\Big)\big(A_2\bar{x}_\gamma(t)+B_2u(t)+E_2\big),\\ & \text{for almost all } t\in\mathbb{R}_+.\end{aligned}\tag{31}$$

Since $\|\dot{\overbrace{f_{T;1}^{-1}}}\| \leq d$ it comes that $\dot{f}_{T;\gamma}\big(f_{T;\gamma}^{-1}(t)\big)$ can be zero or undefined only on a set of measure zero. Thus,

$$\begin{aligned}\dot{\bar{x}}_\gamma(t) = {} & \frac{1}{\dot{f}_{T;\gamma}\big(f_{T;\gamma}^{-1}(t)\big)}\Big[g_1\Big(\dot{f}_{T;\gamma}\big(f_{T;\gamma}^{-1}(t)\big)\dot{u}(t)\Big)\big(A_1\bar{x}_\gamma(t)+B_1u(t)+E_1\big)\\ & + g_2\Big(\dot{f}_{T;\gamma}\big(f_{T;\gamma}^{-1}(t)\big)\dot{u}(t)\Big)\big(A_2\bar{x}_\gamma(t)+B_2u(t)+E_2\big)\Big],\ \text{for almost all } t\in\mathbb{R}_+,\\ \bar{x}_\gamma(0) = {} & x_0.\end{aligned}\tag{32}$$

We discuss two cases:
Case 1: $g_1(w) = \max(0,w), \forall w\in\mathbb{R}$ and $g_2(w) = \min(0,w), \forall w\in\mathbb{R}$. Then Equation (32) is equivalent to

$$\dot{\bar{x}}_\gamma(t) = \dot{u}(t)\big(A_1\bar{x}_\gamma(t)+B_1u(t)+E_1\big),\ \text{for almost all } t\in\mathbb{R}_+ \text{ such that } \dot{u}(t)\geq 0,\tag{33a}$$
$$\dot{\bar{x}}_\gamma(t) = \dot{u}(t)\big(A_2\bar{x}_\gamma(t)+B_2u(t)+E_2\big),\ \text{for almost all } t\in\mathbb{R}_+ \text{ such that } \dot{u}(t)\leq 0,\tag{33b}$$
$$\bar{x}_\gamma(0) = x_0, .\tag{33c}$$

Equations (33) are independent of γ and of $f_{T;\gamma}$ so that we can write $\bar{x}_\gamma = \zeta = \Phi(u,x_0)\in W^{1;\infty}(\mathbb{R}_+,\mathbb{R})$. Thus, Property (P_2) of of Definition 14 holds with $\beta_{u;\Phi;x_0} = \zeta$.
Case 2: One of the equalities $g_1(w) = \max(0,w), \forall w\in\mathbb{R}$ or $g_2(w) = \min(0,w), \forall w\in\mathbb{R}$ fails. Observe that

$$\dot{f}_{T;\gamma}\big(f_{T;\gamma}^{-1}(t)\big) = \frac{1}{\gamma}\dot{f}_{T;1}\big(f_{T;1}^{-1}(t)\big)$$

so that

$$g_1\Big(\dot{f}_{T;\gamma}\big(f_{T;\gamma}^{-1}(t)\big)\dot{u}(t)\Big) = g_1\left(\frac{1}{\gamma}\dot{f}_{T;1}\big(f_{T;1}^{-1}(t)\big)\dot{u}(t)\right).$$

Since $\|\dot{f}_{T;1}\| \leq d$ and $u\in W^{1,\infty}(\mathbb{R}_+,\mathbb{R})$ it comes that $\|(\dot{f}_{T;1}\circ f_{T;1}^{-1})\cdot\dot{u}\| \leq d\|\dot{u}\|$ so that by (27) it comes that $\exists\gamma_0\in\,]0,\infty[$ such that for all $t\in\mathbb{R}_+$ such that $\dot{f}_{T;1}\big(f_{T;1}^{-1}(t)\big)\dot{u}(t) > 0$ we have

$$\frac{1}{2} \leq \bar{g}_1\left(\frac{1}{\gamma}\dot{f}_{T;1}\big(f_{T;1}^{-1}(t)\big)\dot{u}(t)\right) \leq \frac{3}{2}, \forall\gamma\in[\gamma_0,\infty[,\tag{34}$$

and for all $t\in\mathbb{R}_+$ such that $\dot{f}_{T;1}\big(f_{T;1}^{-1}(t)\big)\dot{u}(t) < 0$ we have

$$\frac{1}{2} \leq \bar{g}_2\left(\frac{1}{\gamma}\dot{f}_{T;1}\big(f_{T;1}^{-1}(t)\big)\dot{u}(t)\right) \leq \frac{3}{2}, \forall\gamma\in[\gamma_0,\infty[.\tag{35}$$

Let $\gamma\in[\gamma_0,\infty[$. Define the Lyapunov function $V_\gamma:\mathbb{R}_+\to\mathbb{R}_+$ by $V_\gamma(t) = \frac{1}{2}\bar{x}_\gamma^2(t), \forall t\in\mathbb{R}_+$. Then, for almost all $t\in\mathbb{R}_+$ we have

$$\begin{aligned}\dot{V}_\gamma(t) = {} & \frac{g_1\Big(\dot{f}_{T;\gamma}\big(f_{T;\gamma}^{-1}(t)\big)\dot{u}(t)\Big)}{\dot{f}_{T;\gamma}\big(f_{T;\gamma}^{-1}(t)\big)}\Big[2A_1V_\gamma(t)+\big(B_1u(t)+E_1\big)\bar{x}_\gamma(t)\Big]\\ & + \frac{g_2\Big(\dot{f}_{T;\gamma}\big(f_{T;\gamma}^{-1}(t)\big)\dot{u}(t)\Big)}{\dot{f}_{T;\gamma}\big(f_{T;\gamma}^{-1}(t)\big)}\Big[2A_2V_\gamma(t)+\big(B_2u(t)+E_2\big)\bar{x}_\gamma(t)\Big]\end{aligned}\tag{36}$$

Taking into account (34)–(35) the following holds.
For almost all $t \in \mathbb{R}_+$ such that $\dot{u}(t) > 0$ we have

$$\dot{V}_\gamma(t) \leq \dot{u}(t) \left[A_1 V_\gamma(t) + \underbrace{\frac{3}{2}\big(|B_1| \cdot \|u\| + |E_1|\big)}_{= a_1} \sqrt{V_\gamma(t)} \right]. \tag{37}$$

For almost all $t \in \mathbb{R}_+$ such that $\dot{u}(t) < 0$ we have

$$\dot{V}_\gamma(t) \leq \dot{u}(t) \left[A_2 V_\gamma(t) - \underbrace{\frac{3}{2}\big(|B_2| \cdot \|u\| + |E_2|\big)}_{= a_2} \sqrt{V_\gamma(t)} \right]. \tag{38}$$

For almost all $t \in \mathbb{R}_+$ such that $\dot{u}(t) = 0$ we have

$$\dot{V}_\gamma(t) = 0. \tag{39}$$

Equations (37)–(39) lead to

$$\dot{V}_\gamma(t) \leq |\dot{u}(t)| \left[A_3 V_\gamma(t) + a_3 \sqrt{V_\gamma(t)} \right], \text{ for almost all } t \in \mathbb{R}_+, \forall \gamma \in [\gamma_0, \infty[\tag{40}$$

with $A_3 = \max(A_1, -A_2) < 0$ and $a_3 = \max(a_1, a_2) \geq 0$. The differential inequality (40) leads to

$$|\bar{x}_\gamma(t)| \leq a_4 = \max\left(\left| \frac{a_3}{A_3} \right|, |x_0| \right), \forall t \in \mathbb{R}_+, \forall \gamma \in [\gamma_0, \infty[. \tag{41}$$

Define $\eta_\gamma = \bar{x}_\gamma - \zeta$, then for almost all $t \in \mathbb{R}_+$ such that $\dot{u}(t) \geq 0$ we have

$$\dot{\eta}_\gamma(t) = A_1 \dot{u}(t) \eta_\gamma(t) + \varepsilon_{1;\gamma}(t), \tag{42}$$
$$\eta_\gamma(0) = 0, \tag{43}$$
$$\varepsilon_{1;\gamma}(t) = \left[\frac{g_1\Big(\dot{f}_{T;\gamma}\big(f_{T;\gamma}^{-1}(t)\big)\dot{u}(t)\Big)}{\dot{f}_{T;\gamma}\big(f_{T;\gamma}^{-1}(t)\big)} - \dot{u}(t) \right] \big(A_1 \bar{x}_\gamma(t) + B_1 u(t) + E_1\big), \tag{44}$$

and for almost all $t \in \mathbb{R}_+$ such that $\dot{u}(t) \leq 0$ we have

$$\dot{\eta}_\gamma(t) = A_2 \dot{u}(t) \eta_\gamma(t) + \varepsilon_{2;\gamma}(t), \tag{45}$$
$$\eta_\gamma(0) = 0, \tag{46}$$
$$\varepsilon_{2;\gamma}(t) = \left[\frac{g_2\Big(\dot{f}_{T;\gamma}\big(f_{T;\gamma}^{-1}(t)\big)\dot{u}(t)\Big)}{\dot{f}_{T;\gamma}\big(f_{T;\gamma}^{-1}(t)\big)} - \dot{u}(t) \right] \big(A_2 \bar{x}_\gamma(t) + B_2 u(t) + E_2\big). \tag{47}$$

Consider the Lyapunov function $W_\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ by $W_\gamma(t) = \frac{1}{2}\eta_\gamma^2(t), \forall t \in \mathbb{R}_+$. Using an argument similar to the one used to obtain (40) we get

$$\begin{gathered} \dot{W}_\gamma(t) \leq |\dot{u}(t)| \left[2A_3 W_\gamma(t) + \max\big(\|\varepsilon_{1;\gamma}\|, \|\varepsilon_{2;\gamma}\|\big) \sqrt{2W_\gamma(t)} \right], \\ \text{for almost all } t \in \mathbb{R}_+, \forall \gamma \in [\gamma_0, \infty[, \end{gathered} \tag{48}$$

so that

$$\|\eta_\gamma\| = \|\bar{x}_\gamma - \zeta\| \leq \frac{\max\big(\|\varepsilon_{1;\gamma}\|, \|\varepsilon_{2;\gamma}\|\big)}{|A_3|\sqrt{2}}, \forall \gamma \in [\gamma_0, \infty[. \tag{49}$$

Taking into account (27) and (41) we have

$$\lim_{\gamma\to\infty} \|\varepsilon_{1;\gamma}\| = \lim_{\gamma\to\infty} \|\varepsilon_{2;\gamma}\| = 0. \tag{50}$$

Combining and (50) and (49) we get Property (P_2) of Definition 14 with $\beta_{u;\Phi;x_0} = \zeta$.
Proof of (P_1).
Owing to Proposition 13, (P_1) is equivalent to Property (b) of Proposition (12). The latter property has been established in Ref. [3, Theorem 1]. □

Remark 22. *When the periodic input is composed of a strictly increasing part and a strictly decreasing part, it is possible to find the explicit expression of the initial condition $\big[\Phi^\circ(u,x_0)\big](0)$, see Ref. [4, Section 11.3].*

Remark 23. *The rate-independent hysteresis process given by Equations (29) belongs to a class of hysteresis models introduced by Duhem in 1896 [2]. One of the earliest mathematical studies of the existence and uniqueness of periodic solutions for Duhem's hysteresis model has been done by Babuška in 1959 [1].*

6. Comments

Comment 1. This paper uses the decomposition $\Phi = \Phi^\circ + \Phi^\ddagger$ introduced in Ref. [5] and the decomposition $\Phi = \Phi^\star + \Phi^\dagger$ proposed in the present work. Both operators Φ° and $\Phi^\star$ are rate independent and they both approximate Φ in steady state for slow periodic inputs. What is the main difference between Φ° and $\Phi^\star$?

Consider that the input u is T–periodic, that it $(u,x_0) \in \Omega_T \times \Xi$. Then owing to Property (iii-1) of Theorem 11 the component $\Phi^\circ(u,x_0)$ is also periodic. It has been explained in the Introduction (Section 1) using a simple example that the periodic component of the solution of the considered differential equation is not causal. This lack of causality is due to the fact that the periodic solution depends on the whole periodic input u which means that this periodic solution cannot be obtained from the restriction of u to an interval $[0,t]$ with $t < T$.

A similar phenomenon happen with the operator Φ°. The output $\Phi^\circ(u,x_0)$ depends in general on the whole function u. This means that we cannot get $\big[\Phi^\circ(u,x_0)\big](t)$ using the mere knowledge of the restriction of u to the interval $[0,t]$ when $t < T$. This fact means that the operator Φ° is not causal in general.

On the other hand, the component $\Phi^\star(u,x_0)$ is not periodic in general. Indeed, the initial condition $\big[\Phi^\star(u,x_0)\big](0)$ is equal to $x_0 = \big[\Phi(u,x_0)\big](0)$, see Remark 17.

As a conclusion, even though $\Phi^\circ \simeq \Phi^\star \simeq \Phi$ in steady state for slow periodic inputs, the transient behaviors of Φ° and $\Phi^\star$ may be quite different. This is how $\Phi^\star$ is causal whilst Φ° is not.

Comment 2. Uniform confluence is not a necessary condition to obtain the decomposition $\Phi = \Phi^\star + \Phi^\dagger$. Indeed, consider the model of Figure 1 proposed in [5, Section 7]:

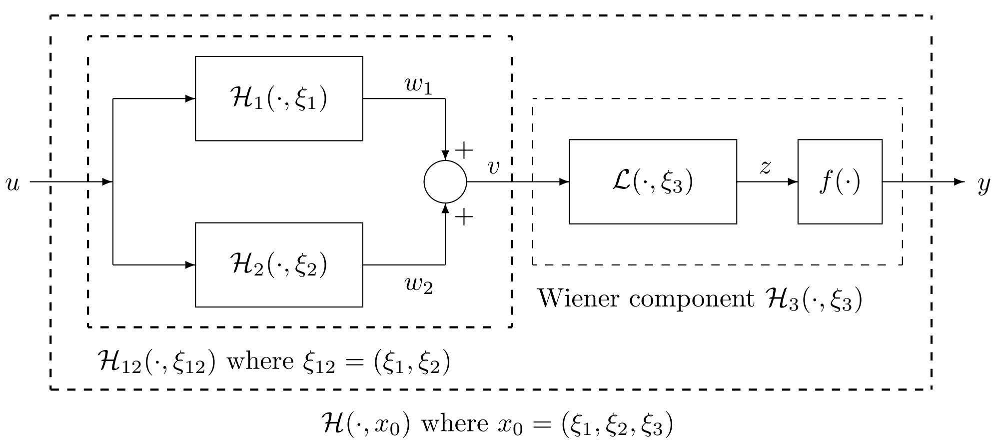

Figure 1: Flowchart of the model.

where we choose $\mathcal{H}_1$ to be a causal rate-independent operator, $\mathcal{H}_2$ a causal operator that vanishes in steady state for slow inputs, $\mathcal{L}$ a stable linear system (which is causal by construction), and f a continuous function. Then $\mathcal{H}$ is causal and confluent, and $\mathcal{H} \simeq \mathcal{H}_1$ in steady state for slow inputs [5, Section 7].

We now show that $\mathcal{H}$ is not uniformly confluent in general.

Indeed, take $\mathcal{H}_2 = 0$. Suppose furthermore that $\mathcal{H}_1$ is uniformly confluent and satisfies Assumption 18 (the semigroup property). An example of such $\mathcal{H}_1$ is the semilinear rate-independent Duhem model (29). Suppose that the input u is periodic. Then by Proposition 19 (c), if u is constant on some time interval $[\theta_1, \theta_2]$ then the output $w_1 = v$ is also constant on the same time interval. However this does not imply necessarily that y is constant on $[\theta_1, \theta_2]$ because of the presence of the dynamic linear process $\mathcal{L}$. Since $\mathcal{H}$ satisfies Assumption 18 but not Property (c) of Proposition 19 it cannot be uniformly confluent.

Figure 2 illustrates the results of this comment.

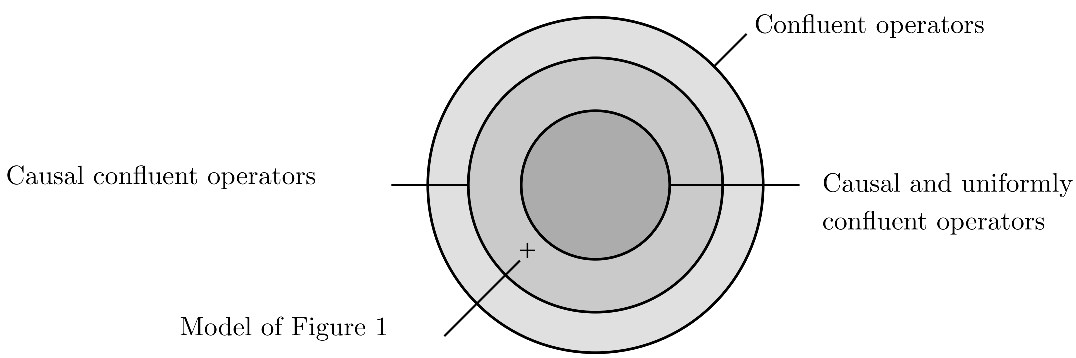

Figure 2: The model of Figure 1 is not uniformly confluent in general.

7. Conclusions

To the best of our knowledge, this is the first research work that addresses the causal decomposition of hysteresis systems into a causal rate-independent component and a causal nonhysteretic component that vanishes in steady state for slow inputs.

We presented a sufficient condition -uniform confluence- to obtain the causal decomposition of causal confluent operators. We provided a way to construct that causal rate-independent component as a uniform limit of specified functions, and we illustrated this construction using the hysteretic semilinear Duhem model. We also proved that uniform confluence is not a necessary condition to get a causal decomposition. The problem of finding *necessary and sufficient* conditions for a causal decomposition of confluent operators is an open problem.

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Compliance with Ethical Standards

Conflict of Interest: The author declares that they have no conflict of interest.

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