

# Clifford's Identity and Generalized Cayley-Menger Determinants

Federico Thomas and Josep M. Porta

**Abstract** Distance geometry is usually defined as the characterization and study of point sets in  $\mathbb{R}^k$ , the  $k$ -dimensional Euclidean space, based on the pairwise distances between their points. In this paper, we use Clifford's identity to extend this kind of characterization to sets of  $n$  hyperspheres embedded in  $\mathbb{S}^{n-3}$  or  $\mathbb{R}^{n-3}$  where the role of the Euclidean distance between two points is replaced by the so-called *power* between two hyperspheres. By properly choosing the value of  $n$  and the radii of these hyperspheres, Clifford's identity reduces to conditions in terms of generalized Cayley-Menger determinants which has been previously obtained on the basis of a case-by-case analysis.

**Key words:** Clifford's identity, Cayley-Menger determinants, Distance Geometry.

## 1 Introduction

In general, when using distance geometry to characterize point sets based on the pairwise distances between the points, no more constraints are enforced than the mere requirement that all distances are positive [14]. Therefore, this characterization is said to be performed in a semimetric space because the triangle inequality, the tetragonal inequality, and any other higher-order inequality, are not necessarily considered. Nevertheless, since distance geometry performs all characterizations in terms of Cayley-Menger determinants, these constraints, and those related to the orientation of simplices, can be incorporated in a simple way [18]. This generalization permits to formulate and solve such relevant problems in Kinematics as the inverse kinematics of serial robots and the forward kinematics of parallel ones (see [20] and the references therein).

---

Federico Thomas and Josep M. Porta  
Institut de Robòtica i Informàtica Industrial (CSIC-UPC),  
Llorens Artigas 4-6, 08028 Barcelona, Spain. e-mail: {fthomas, porta}@iri.upc.es

Another important generalization of distance geometry consists in considering other geometric elements than points. To this end, Cayley-Menger determinants must be extended to involve the *distance* between these other geometric elements. In this paper, we will see that this generalization is possible by adopting the concept of *power* instead of the standard Euclidean distance. As a result, the generalizations of Cayley-Menger determinants presented, for example, in [15], naturally arise as particular cases of Clifford's identity, instead of relying on a case-by-case analysis.

This paper is organized as follows. Section 2 gives the definition of the power of two circles, a generalization of Steiner's power of a point and a circle. The limit cases (those involving at least one circle with zero or infinite radius) are treated in Section 3, where the straightforward generalization to hyperspheres is also given. Some further generalizations are detailed in Section 4. Section 5 gives the most general form of Clifford's identity and explains how generalized Cayley-Menger determinants — involving points, hyperplanes and hyperspheres — can be deduced from it. Finally, Section 6 identifies some points deserving further research efforts.

## 2 The power of two circles

In 1826, Steiner defined the power of a point  $P$  with respect to a circle  $c$  of radius  $r$  as  $h = d^2 - r^2$ , where  $d$  is the distance between  $P$  and the center  $O$  of the circle [9, 21]. According to this definition, points inside the circle have negative power, points outside have positive power, and points on the circle have zero power. For external points, the power equals the square of the length of a tangent from the point to the circle. Steiner proved that for any line through  $P$  intersecting  $c$  in points  $P'$  and  $P''$ , the power of the point with respect to the circle coincides, up to sign, to the product of the lengths of the segments from  $P$  to  $P'$  and  $P$  to  $P''$ .

In 1872, Darboux extended the idea of the power of a point with respect to a circle to the power of two circles [8]. Clifford also gave the same definition in two papers that appeared in his collected works [3, 4].

Let  $c_i$  denote a circle with center at  $O_i$  and radius  $r_i$ , and let  $d_{i,j}$  denote the distance between the centers of the circles  $c_i$  and  $c_j$ . Then, the mutual power of the circles  $c_i$  and  $c_j$  is defined as:

$$(c_i c_j) = d_{i,j}^2 - r_i^2 - r_j^2. \quad (1)$$

When both circles intersect, the power thus defined reduces to

$$(c_i c_j) = -2r_i r_j \cos \theta_{i,j}, \quad (2)$$

where  $\theta_{i,j}$  is the angle of intersection of the two circles. As a consequence, when they cut at right angles, their power vanishes and vice versa.

### 3 Limit cases

In terms of Cartesian coordinates,  $c_i$  with center coordinates  $(x_i, y_i)$  can be expressed as:

$$c_i : (x - x_i)^2 + (y - y_i)^2 - r_i^2 = x^2 + y^2 + 2f_i x + 2g_i y + h_i = 0, \quad (3)$$

where  $f_i = -x_i$ ,  $g_i = -y_i$ , and  $h_i = x_i^2 + y_i^2 - r_i^2$ . Then, the power of two circles given in (1) can be rewritten as:

$$(c_i c_j) = h_i + h_j - 2f_i f_j - 2g_i g_j, \quad (4)$$

which is sometimes also referred to as the product of  $c_i$  and  $c_j$  [7]. This alternative formula is useful to analyze the limit cases in which one circle degenerates into a line possibly at infinity. Next, we analyze three relevant cases. All others follow directly.

1. *One circle degenerates into a line.* Considering a line as a circle of infinite radius, let us set  $r_i = \delta \rightarrow \infty$ . Then, using (1), we have that

$$\lim_{\delta \rightarrow \infty} (c_i c_j) = \lim_{\delta \rightarrow \infty} [(p_{i,j} + \delta)^2 - \delta^2 - r_j^2] = 2p_{i,j} \delta, \quad (5)$$

where  $p_{i,j}$  is the perpendicular distance from the center of  $c_j$  to the line. The side of the line on which  $p_{i,j}$  is positive is regarded as the outside of the infinite circle.

2. *One circle degenerates into a line at infinity.* The equation of the line at infinity can be written as

$$x^2 + y^2 + 2f_i x + 2g_i y - \delta^2 = 0, \quad (6)$$

where  $\delta \rightarrow \infty$ . Identifying (6) and (3), we have that  $h_i = -\delta^2$ . Then, the power of  $c_i$  and  $c_j$ ,  $c_i$  being the line at infinity, can be expressed as:

$$(c_i c_j) = \lim_{\delta \rightarrow \infty} (-\delta^2 + h_j - 2f_i f_j - 2g_i g_j) = -\delta^2. \quad (7)$$

Observe that the negative sign is consistent with the previous case because the line at infinity wraps all finite points.

3. *Both circles degenerate into lines.* If we set  $r_i = r_j = \delta \rightarrow \infty$  in (2), then

$$\lim_{\delta \rightarrow \infty} (c_i c_j) = -2\delta^2 \cos \theta_{i,j}. \quad (8)$$

where  $\theta_{i,j}$  is the angle between both lines. Observe that the power of two coincident lines is  $-2\delta^2$ .

The cases in which at least one of the radii is zero follow directly. Moreover, observe that the definition of power of two circles can be directly extended to the power of two hyperspheres, and that its value for the limit cases obtained above also apply to this generalization. The resulting powers for all possible combinations are summarized in Table 1. To deal with powers involving points, hyperplanes and

**Table 1** The powers between points, hyperspheres, and hyperplanes ( $\delta \rightarrow \infty$ ).

$c_i \setminus c_j$	point	hypersphere	hyperplane	hyperplane at $\infty$
point	$d_{i,j}^2$	$d_{i,j}^2 - r_j^2$	$2\delta p_{j,i}$	$-\delta^2$
hypersphere	$d_{i,j}^2 - r_i^2$	$d_{i,j}^2 - r_i^2 - r_j^2$	$2\delta p_{j,i}$	$-\delta^2$
hyperplane	$2\delta p_{i,j}$	$2\delta p_{i,j}$	$-2\delta^2 \cos \theta_{i,j}$	0
hyperplane at $\infty$	$-\delta^2$	$-\delta^2$	0	NA

hyperspheres at the same time, it is important, for dimensional consistency, to keep the factors  $\delta$  and  $\delta^2$ , contrarily to what is done, for example, in [11]. As a result, all powers have the dimension of length squared as  $\delta$  has length dimension.

#### 4 Further generalizations

Let us suppose that  $c_i$  and  $c_j$  are two circles, on a sphere of radius  $R$ , with spherical radii  $r_i$  and  $r_j$ , respectively. Let  $\omega_{i,j}$  denote the length of the arc joining their centers. Then, the power of  $c_i$  and  $c_j$  is defined as [2, §105]

$$(c_i c_j) = \cos(\kappa \omega_{i,j}) - \cos(\kappa r_i) \cos(\kappa r_j), \quad (9)$$

where  $\kappa = 1/R$  denotes the curvature of the sphere.

If  $c_i$  and  $c_j$  intersect and the angle of intersection between both circles is denoted by  $\theta_{i,j}$ , then we have that

$$(c_i c_j) = \sin(\kappa r_i) \sin(\kappa r_j) \cos \theta_{i,j}, \quad (10)$$

which is zero when the circles cut orthogonally.

If  $\kappa \rightarrow 0$  (*i.e.*, the sphere becomes a plane),  $\cos(\kappa \alpha) \rightarrow 1 - \kappa^2 \alpha^2$ . Then, it is easy to verify that

$$\lim_{\kappa \rightarrow 0} (c_i c_j) = -\kappa^2 (\omega_{i,j}^2 - r_i^2 - r_j^2), \quad (11)$$

which is consistent with the definition of power of two circles on a plane given by (1) up to the factor  $-\kappa^2$ .

Other generalization of the concept of power, that have received little attention, can be found in the literature. For example, in [12], Lachlan extended the concept of power to certain pairs of conics following the work of Casey in [1]. Laguerre also defined the power of a point with respect to any algebraic curve of degree  $n$  to be the product of the distances from the point to the intersections of a circle through the point with the curve, divided by the  $n$ th power of the diameter  $d$  [13].

Laguerre showed that this number is independent of the diameter. In the case when the algebraic curve is a circle the result does not coincide with the power of a point with respect to a circle defined by Steiner, but differs from it by a factor of  $d^2$ . The topic of the power of a point with respect to an algebraic curve has recently been revisited in [22] following the work of Neville [16, 17], which was influenced by that of Laguerre.

## 5 Clifford's identity

Clifford's identity states that, given two sets of five circles on the unit sphere  $\mathbb{S}^2$ , their mutual powers are related through the following relationship [3]:

$$\begin{vmatrix} (c_1c_6) & (c_1c_7) & (c_1c_8) & (c_1c_9) & (c_1c_{10}) \\ (c_2c_6) & (c_2c_7) & (c_2c_8) & (c_2c_9) & (c_2c_{10}) \\ (c_3c_6) & (c_3c_7) & (c_3c_8) & (c_3c_9) & (c_3c_{10}) \\ (c_4c_6) & (c_4c_7) & (c_4c_8) & (c_4c_9) & (c_4c_{10}) \\ (c_5c_6) & (c_5c_7) & (c_5c_8) & (c_5c_9) & (c_5c_{10}) \end{vmatrix} = 0, \quad (12)$$

where powers have to be computed according to equation (9).

The great mathematics historian Julian Coolidge pointed out in [5, p. 135] that this identity was first presented by Darboux in [8] and that Frobenius announced that he had long been familiar with it and proceeded to publish his results in [10]. Nevertheless, Coolidge himself later realized [6, p. 168] that Clifford presented this result for the first time in [3]. In this later reference, Clifford's identity is presented in an alternative form: that involving two sets of six spheres in  $\mathbb{R}^3$ , in which case powers have to be computed according to equation (1).

Although Clifford's identity has been presented for  $\mathbb{S}^2$  or  $\mathbb{R}^3$ , thus involving five or six hyperspheres, respectively, both versions are easily stepped up or down to as many dimensions as required. Indeed, we can present Clifford's identity in all its generality as

$$\begin{vmatrix} (c_1c_{n+1}) & (c_1c_{n+2}) & \cdots & (c_1c_{2n}) \\ (c_2c_{n+1}) & (c_2c_{n+2}) & \cdots & (c_2c_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (c_nc_{n+1}) & (c_nc_{n+2}) & \cdots & (c_nc_{2n}) \end{vmatrix} = 0, \quad (13)$$

where  $\{c_1, \dots, c_n\}$  and  $\{c_{n+1}, \dots, c_{2n}\}$  are two sets of  $n > 4$  hyperspheres (which includes points and hyperplanes as degenerate cases) embedded in  $\mathbb{S}^{n-3}$  or  $\mathbb{R}^{n-3}$ .

Many interesting relationships can be obtained as particular cases of (13). Due to space limitations, next we only analyze different cases in  $\mathbb{R}^2$  and, to better appreciate the advantages of the presented approach, this analysis follows the same sequence of cases presented in [15].

In  $\mathbb{R}^2$ , Clifford's identity involves two sets of five circles. If both sets are equal, and their radii are zero, Clifford's identity reduces to

$$\begin{vmatrix} 0 & d_{12}^2 & d_{13}^2 & d_{14}^2 & d_{15}^2 \\ d_{21}^2 & 0 & d_{23}^2 & d_{24}^2 & d_{25}^2 \\ d_{31}^2 & d_{32}^2 & 0 & d_{34}^2 & d_{35}^2 \\ d_{41}^2 & d_{42}^2 & d_{43}^2 & 0 & d_{45}^2 \\ d_{51}^2 & d_{52}^2 & d_{53}^2 & d_{54}^2 & 0 \end{vmatrix} = 0. \quad (14)$$

Let us remind that  $d_{i,j} = d_{j,i}$  is the distance between points  $c_i$  and  $c_j$ . If  $c_1 = c_6$  is a line instead of a point, after taking the common factor  $\delta$  of the first row and column out of the resulting determinant, we have that

$$\begin{vmatrix} 0 & 2p_{12} & 2p_{13} & 2p_{14} & 2p_{15} \\ 2p_{12} & 0 & d_{23}^2 & d_{24}^2 & d_{25}^2 \\ 2p_{13} & d_{32}^2 & 0 & d_{34}^2 & d_{35}^2 \\ 2p_{14} & d_{42}^2 & d_{43}^2 & 0 & d_{45}^2 \\ 2p_{15} & d_{52}^2 & d_{53}^2 & d_{54}^2 & 0 \end{vmatrix} = 0. \quad (15)$$

In this case  $p_{1,j}$  is the oriented distance between the line  $c_1$  and the point  $c_j$ . If  $c_1$  is a line at infinity, after taking the common factor  $-\delta^2$  of the first row and column out of the resulting determinant, the identity (15) becomes

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & d_{23}^2 & d_{24}^2 & d_{25}^2 \\ 1 & d_{32}^2 & 0 & d_{34}^2 & d_{35}^2 \\ 1 & d_{42}^2 & d_{43}^2 & 0 & d_{45}^2 \\ 1 & d_{52}^2 & d_{53}^2 & d_{54}^2 & 0 \end{vmatrix} = 0, \quad (16)$$

that is, the standard Cayley-Menger determinant for four points on a plane. Now, let us assume that  $c_5 = c_{10}$  is also a line instead of a point. Then, after taking the common factor  $\delta$  of the last row and column out of the resulting determinant, we have that

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & d_{23}^2 & d_{24}^2 & 2p_{25} \\ 1 & d_{32}^2 & 0 & d_{34}^2 & 2p_{35} \\ 1 & d_{42}^2 & d_{43}^2 & 0 & 2p_{45} \\ 0 & 2p_{52} & 2p_{53} & 2p_{54} & -2 \end{vmatrix} = 0, \quad (17)$$

which concurs with the result presented in [15]. If  $c_4 = c_9$  is also a line instead of a point, after taking the common factor  $\delta$  of the third row and column out of the determinant, we have that

$$\begin{vmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & d_{23}^2 & 2p_{24} & 2p_{25} \\ 1 & d_{32}^2 & 0 & 2p_{34} & 2p_{35} \\ 0 & 2p_{42} & 2p_{43} & -2 & -2 \cos \theta_{45} \\ 0 & 2p_{52} & 2p_{53} & -2 \cos \theta_{54} & -2 \end{vmatrix} = 0, \quad (18)$$

which coincides with the one derived in [15] up to a factor of  $-2$  in the last two rows, which does not alter the result. Finally, if  $c_3 = c_8$  is also a line, it is straightforward to show that

$$\begin{vmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2p_{23} & 2p_{24} & 2p_{25} \\ 0 & 2p_{32} & -2 & -2\cos\theta_{34} & -2\cos\theta_{35} \\ 0 & 2p_{42} & -2\cos\theta_{43} & -2 & -2\cos\theta_{45} \\ 0 & 2p_{52} & -2\cos\theta_{53} & -2\cos\theta_{54} & -2 \end{vmatrix} = 0, \quad (19)$$

from which we can conclude that

$$\begin{vmatrix} 1 & \cos\theta_{34} & \cos\theta_{35} \\ \cos\theta_{43} & 1 & \cos\theta_{45} \\ \cos\theta_{53} & \cos\theta_{54} & 1 \end{vmatrix} = 0. \quad (20)$$

In other words, the location of point  $c_2 = c_7$  is irrelevant and only the angles between the three finite lines,  $c_3 = c_8$ ,  $c_4 = c_9$ , and  $c_5 = c_{10}$ , are mutually constrained. This concurs with the result presented in [15] without having to analyze separately this particular case.

Finally, it is interesting to observe that the determinant in (20) can be interpreted as the Gramian of three points on the unit circle. Gramians can actually be seen as the counterpart in  $\mathbb{S}^n$  of Cayley-Menger determinants in  $\mathbb{R}^n$  [19, 23].

## 6 Conclusion

In the past, the identities in Section 5 involving generalized Cayley-Menger determinants have been individually proved. Nevertheless, we have shown how Clifford's identity, together with Table 1, is enough to derive them in a simple and unified way. The analysis presented, as an example, is valid for  $\mathbb{R}^2$ , but its generalization to higher dimensions does not offer any extra complexity than that derived from the increment in the number of cases.

The analysis of Clifford's identity in  $\mathbb{S}^n$  deserves further attention because it should be possible to obtain the results for  $\mathbb{R}^n$  from equivalent expressions in  $\mathbb{S}^n$  as a limit in which the curvature of the embedding sphere tends to zero.

Another venue for further investigation concerns the generalization of Clifford's identity to conics and, in general, to arbitrary algebraic curves following Laguerre's ideas.

**Acknowledgements** This work was partially supported by the Spanish Ministry of Economy and Competitiveness through the projects DPI2017-88282-P and MDM-2016-0656.

## References

1. Casey, J.: On bicircular quartics. *Transactions of the Irish Academy* **24**, 457–569 (1871)
2. Casey, J.: *A Treatise on Spherical Trigonometry, and its Application to Geodesy and Astronomy, with Numerous Examples*. Longmans, Green, & Co., London and New York (1889)
3. Clifford, W.K.: On the power-coordinates in general. In: R. Tucker (ed.) *Mathematical Papers*, pp. 546–555. Macmillan and Co., London (1882)
4. Clifford, W.K.: On the power of spheres. In: R. Tucker (ed.) *Mathematical Papers*, pp. 332–336. Macmillan and Co., London (1882)
5. Coolidge, J.L.: *A Treatise On The Circle And The Sphere*. Oxford University Press (1916)
6. Coolidge, J.L.: *A History of Geometrical Methods*. Oxford University Press (1940)
7. Cox, H.: On systems of circles and bicircular quartics. *Quarterly Journal of Mathematics of Pure and Applied Mathematics* **19**, 74–124 (1883)
8. Darboux, G.: Sur les relations entre les groupes de points, de cercles et de sphères dans le plan et dans l'espace. *Annales Scientifiques de l'École Normale Supérieure* **1**, 323–392 (1872)
9. Fried, M.: *Mathematics as the science of patterns - Jacob Steiner and the power of a point*. Convergence (2010)
10. Frobenius, F.G.: Anwendungen der determinantentheorie auf die geometrie des maasses. *Journal für die reine und angewandte Mathematik* (79), 185–247 (1875)
11. Johnson, W.: Some theorems relating to groups of circles and spheres. *American Journal of Mathematics* **14**(2), 97–114 (1892)
12. Lachlan, R.: On systems of circles and spheres. *Proceedings of the Royal Society of London, Philosophical Transactions of the Royal Society* **177**, 481–625 (1886)
13. Laguerre, E.N.: Sur les courbes planes algébriques. *Comptes rendus des séances de l'Académie des sciences* **60**, 18–22 (1865)
14. Menger, K.: New foundation of euclidean geometry. *American Journal of Mathematics* **53**(4), 721–745 (1931)
15. Michelucci, D., Fofou, S.: Using Cayley-Menger determinants for geometric constraint solving. In: G. Elber, N. Patrikalakis, P. Brunet (eds.) *Proceedings of the Ninth ACM Symposium on Solid modeling and Applications*, pp. 285–290. Genoa, Italy (2004)
16. Neville, E.H.: Products of lengths of tangents and of normals. *The Mathematical Gazette* **38**(325), 192–193 (1954)
17. Neville, E.H.: The power of a point for a curve. *The Mathematical Gazette* **40**(331), 11–14 (1956)
18. Porta, J.M., Rull, A., Thomas, F.: Sensor localization from distance and orientation constraints. *Sensors* **16**(7), 1–19 (2016)
19. Porta, J.M., Sarabandi, S., Thomas, F.: Angle-bound smoothing with applications in kinematics. In: *IFTToMM Asian Mechanism and Machine Science (Asian MMS 2018)*. Bengaluru, India (2018)
20. Porta, J.M., Thomas, F.: Yet another approach to the Gough-Stewart platform forward kinematics. In: *Proceedings of the IEEE International Conference in Robotics and Automation*. Brisbane, Australia (2018)
21. Steiner, J.: Einige geometrische betrachtungen. *Journal für die reine und angewandte Mathematik* (1), 161–184 (1826)
22. Suceava, B.D., Vajiac, A.I., Vajiac, M.B.: The power of a point for some real algebraic curves. *The Mathematical Gazette* **92**(523), 22–28 (2008)
23. Thomas, F., Pérez-Gracia, A.: A new insight into the coupler curves of the RCCC four-bar linkage. In: *7th IFTToMM International Workshop on Computational Kinematics*. Poitiers, France (2017)