# Position Analysis of a Class of $n$-RRR Planar Parallel Robots

Tommaso Marchi, Giovanni Mottola, Josep M. Porta, Federico Thomas and Marco Carricato

**Abstract** Parallel robots with a configurable platform are a class of parallel robots in which the end-effector is a closed-loop kinematic chain. In $n$-RRR planar robots the end-effector is a $n$-bar chain controlled by $n$ actuated RRR chains connected to the base. We solve the direct kinematic problem for 4, 5 and 6-RRR mechanisms by using bilateration, a method that easily lends itself to generalization. Finally, we present the results from experimental tests that have been performed on a 5-RRR prototype.

## 1 Introduction

This paper studies a class of planar parallel mechanisms having a *configurable* end-effector (EE), namely an EE whose shape can be reconfigured as the robot moves. Parallel robots with configurable platform (PRCP) are especially useful when the interaction with the environment requires additional Degrees-of-Freedom (DoFs) beyond those required for positioning and orienting the EE. For example, haptic interfaces have been developed to provide the operator with multiple points of contact for multi-finger gripping, which results in a more natural interaction with the virtual or remote environment [2]. The advantage of configurable platforms is in that they allow the number of DoFs to be increased while retaining all motors on the robot base, unlike for example adding extra actuators on the EE.

Some early works on this topic considered an EE with a parallelogram shape [18], or proposed a generalization for planar and spatial PRCPs, as well as a framework to

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compute the mobility of these mechanisms [8]. Pierrot et al. proposed a number of 4-DoF spatial PRCPs based on the Delta architecture [3,9,11]. A similar idea is found in the design of PentaG [5], a 5-DoFs spatial robot with two EEs on the configurable platform: the distance between the EEs is controlled to generate a grasping motion. A literature review on PRCPs can be found in [4].

The Direct Kinematics Problem (DKP) of parallel robots is difficult in general and more so when the EE has many DoFs, as in our case [8]. Here, we will focus on *analytical* methods, which, while slower than numerical alternatives, have the advantage of providing deeper insights on the problem than numerical methods. A common approach is to write loop-closure equations, which are then reduced to a polynomial system of equations by using the tangent half-angle substitution. The equations are then combined through algebraic manipulation into a single univariate *characteristic polynomial*. This for instance is the approach used in [10] to analyze a 3-RRR, 3-DoF planar manipulator. For this case, the authors proved that the DKP admits in general six real, distinct solutions. This result was later generalized in [7] to any 3-DoF planar robot having three independent actuated kinematic chains. Both [7, 10] consider only robots with rigid EE.

An interesting approach for the DKP of planar robots can be the *bilateration* method. This methods relies on the results in [1, 6], where the authors defined a way to express Euclidean geometry in terms of distances between points. The methods of bilateration (and trilateration for points in 3D) and their applications to robotics were reviewed in [12, 15, 17]; later, the same authors applied bilateration to solve the DKP of planar mechanisms having only revolute joints [14, 16].

This paper focuses on planar PRCPs, specifically those having $n$ $\underline{\text{R}}$RR chains connecting the base to the EE, which is a closed $n$-R chain. Here, R denotes a revolute joint and $\underline{\text{R}}$ an actuated one. The paper is organized as follows. After providing a brief introduction to bilateration in Sec. 2, we apply it to solve the DKP of the PRCPs at hand in Sec. 3. Numerical results and experimental tests on a prototype are presented in Sec. 4. Finally, Sec. 5 presents conclusions and directions for future work.

## 2 Bilateration

Bilateration is a method to locate a point $P_k$, given its distances from two other points $P_i$ and $P_j$, whose positions $\mathbf{p}_i = P_i - O$, $\mathbf{p}_j = P_j - O$ with respect to the global coordinate frame $Oxy$ are known. If $s_{i,j} = \|\mathbf{p}_i - \mathbf{p}_j\|^2$ is the squared distance between $P_i$ and $P_j$, which is independent from the choice of the coordinate frame, the *Cayley-Menger bi-determinant* is defined as:

$$D(i_1, \dots, i_n; j_1, \dots, j_n) = 2\left(-\frac{1}{2}\right)^n \begin{vmatrix} 0 & 1 & \dots & 1 \\ 1 & s_{i_1,j_1} & \dots & s_{i_1,j_n} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & s_{i_n,j_1} & \dots & s_{i_n,j_n} \end{vmatrix} \tag{1}$$

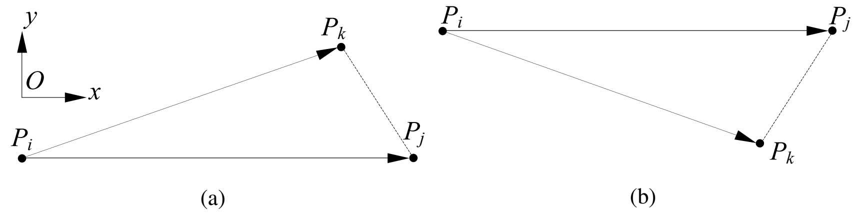



Fig. 1: Two possible solutions for bilateration.

For conciseness, we abbreviate $D(i_1, \ldots, i_n; i_1, \ldots, i_n)$ as $D(i_1, \ldots, i_n)$. It is now possible to define the transformation matrix:

$$\mathbf{Z}_{i,j,k} = \frac{1}{D(i,j)} \begin{bmatrix} D(i,j;i,k) & \mp\sqrt{D(i,j,k)} \\ \pm\sqrt{D(i,j,k)} & D(i,j;i,k) \end{bmatrix} \tag{2}$$

and then, as proved in [1],

$$P_k = P_i + \mathbf{Z}_{i,j,k}(P_j - P_i) \tag{3}$$

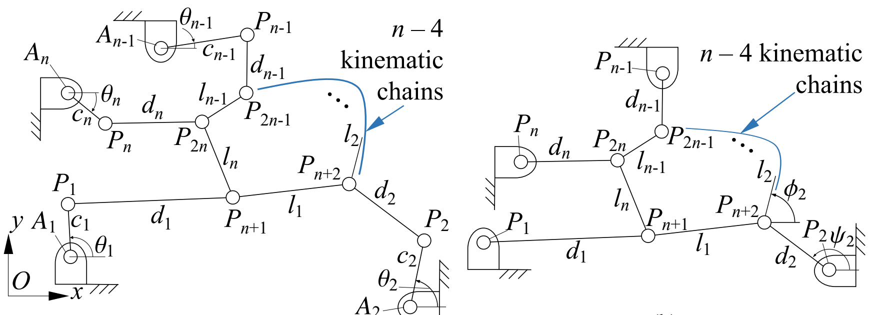
In matrix $\mathbf{Z}_{i,j,k}$, if points $P_i$, $P_j$ and $P_k$ are ordered in counterclockwise sense (see Fig. 1a), the element in the bottom-left corner is positive, while the one in the upper-right corner is negative (and viceversa when points are ordered in clockwise sense, see Fig. 1b). Every bilateration thus provides two solutions. In the DKP, we will take into account all solutions for each bilateration step, and retain only those which lead to a feasible solution for the complete mechanism.

## 3 Inverse and Direct Kinematic Problems

The schematic of a general $n$-RRR robot is shown in Fig. 2a, where points $A_i$'s ($i \in \{1, \ldots, n\}$), having position vectors $\mathbf{a}_i$, are the centers of the (actuated) R joints on the fixed base, whose joint variables are denoted by $\theta_i$; points $P_i$'s ($i \in \{1, \ldots, 2n\}$) are the centers of the mobile joints, of coordinates $\mathbf{p}_i = \left[x_{P_i}, y_{P_i}\right]^T$. Link lengths are defined as $c_i = \|\mathbf{p}_i - \mathbf{a}_i\|$, $d_i = \|\mathbf{p}_{n+i} - \mathbf{p}_i\|$ ($i \in \{1, \ldots, n\}$), $l_i = \|\mathbf{p}_{n+1+i} - \mathbf{p}_{n+i}\|$ ($i \in \{1, \ldots, n-1\}$), and $l_n = \|\mathbf{p}_{n+1} - \mathbf{p}_{2n}\|$.

The Inverse Kinematic Problem (IKP) is usually straightforward for parallel manipulators. We define the pose of the EE by the array $\boldsymbol{\pi} = \left[x_{P_{n+1}}, y_{P_{n+1}}, \phi_1, \ldots, \phi_{n-2}\right]$; here, $\phi_i$ is the angle formed by link $\overline{P_{n+1+i}P_{n+i}}$ with axis $x$. Pose $\boldsymbol{\pi}$ has $n$ components, as the input array $\boldsymbol{\theta} = [\theta_1, \ldots, \theta_n]$ of joint coordinates. The robot has $n$ DoFs and is fully actuated. The IKP requires $\boldsymbol{\theta}$ to be determined from $\boldsymbol{\pi}$. From the position of $P_{n+1}$, length $l_1$ and angle $\phi_1$ one can find point $P_{n+2}$ as

$$\mathbf{p}_{n+2} = \mathbf{p}_{n+1} + l_1 \left[\cos\phi_1, \, \sin\phi_1\right]^T \tag{4}$$




(a) (b)

Fig. 2: (a) $n$-RRR robot in a general pose; (b) $n$-RR structure with fixed actuators. For simplicity, angles $\psi_i$, $\phi_i$ are represented only for the second chain.

All other points $P_{n+3}, \ldots, P_{2n-1}$ on the EE can be found by a similar procedure. Point $P_{2n}$ can be found from $P_{n+1}, P_{2n-1}, l_{n-1}$ and $l_n$, by using bilateration (Eq. (3))[1]. Then, from the closure equation

$$\overrightarrow{OP_{n+i}} = \overrightarrow{OA_i} + \overrightarrow{A_i P_i} + \overrightarrow{P_i P_{n+i}} \tag{5}$$

of the $i$-th RRR chain, one finds

$$\mathbf{p}_{n+i} = \mathbf{a}_i + c_i \left[\cos\theta_i,\ \sin\theta_i\right]^T + d_i \left[\cos\left(\theta_i + \psi_i\right),\ \sin\left(\theta_i + \psi_i\right)\right]^T \tag{6}$$

By using the tangent-half-angle substitution and some manipulations, one obtains

$$\theta_i = 2\tan^{-1}\left(\frac{-e_{1i} \pm \sqrt{e_{1i}^2 + e_{2i}^2 - e_{3i}^2}}{e_{3i} - e_{2i}}\right) \tag{7}$$

where

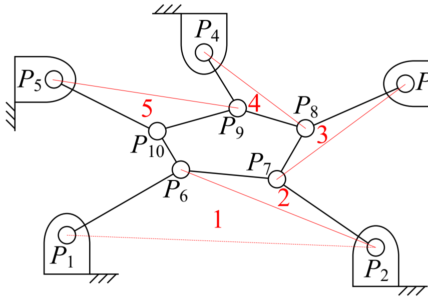
$$e_{1i} = 2c_i(y_{A_i} - y_{P_{n+i}}), \quad e_{2i} = 2c_i(x_{A_i} - x_{P_{n+i}}) \tag{8}$$

$$e_{3i} = x_{P_{n+i}}^2 + x_{A_i}^2 + y_{P_{n+i}}^2 + y_{A_i}^2 + c_i^2 - d_i^2 - 2(x_{P_{n+i}} x_{A_i} + y_{P_{n+i}} y_{A_i}) \tag{9}$$

From Eq. (7), one can see that there can be up to two real, distinct solutions for each chain; therefore, the IKP for an $n$-RRR robot can have up to $2^n$ solutions.

Solving the DKP on the other hand is more complex. First, we note that, once $\boldsymbol{\theta}$ is known, the mechanism can be simplified into an equivalent rigid structure (see Fig. 2b), by eliminating the first link: the position of $P_i$ ($i \in \{1, \ldots, n\}$) is found as

[1] This leads to two possible positions for $P_{2n}$; we assume that the EE configuration is known and thus one position can be discarded.



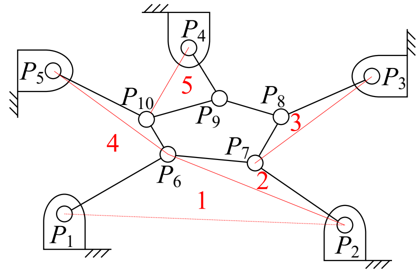




Fig. 3: (a) first bilateration approach for the 5-RRR; (b) second approach. Each bilateration step is denoted in red.

$$\mathbf{p}_i = \mathbf{a}_i + \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i) \\ \sin(\theta_i) & \cos(\theta_i) \end{bmatrix} \begin{bmatrix} c_i \\ 0 \end{bmatrix} \tag{10}$$

Therefore, the DKP reduces to the problem of finding points $P_i$ ($i \in \{n+1, \ldots, 2n\}$), knowing distances $d_i$ and $l_i$; here, we will apply the technique of bilateration. As an example, we show an application to the 5-RRR robot.

First, we define an *unknown variable* to be found: the characteristic polynomial obtained at the end will be univariate in this unknown. Any distance $s_{i,j}$ (except those that are already known from link lengths) may be chosen; for convenience, we pick $s_{2,6}$. Then, we define a *bilateration sequence*, that is, a series of successive bilateration steps. Here, we introduce the shorthand notation $(P_i, P_j) \Rightarrow P_k$ to denote a step, meaning that point $P_k$ can be found from points $P_i$ and $P_j$ and from known fixed distances, by using Eq. (3). A possible bilateration sequence is thus defined as $[(P_1, P_2) \Rightarrow P_6, (P_2, P_6) \Rightarrow P_7, (P_3, P_7) \Rightarrow P_8, (P_4, P_8) \Rightarrow P_9, (P_5, P_9) \Rightarrow P_{10}]$ (see Fig. 3a). In other words, from the coordinates of $P_1$ and $P_2$ we write the coordinates of $P_6$ as functions of $s_{2,6}$, then the coordinates of $P_7$ and so on. Finally, we find the *closure condition*, namely

$$s_{6,10} = \|\mathbf{p}_6 - \mathbf{p}_{10}\|^2 = l_5^2 \tag{11}$$

The final expression for $s_{6,10}$ will be an algebraic function in the unknown $s_{2,6}$, containing a number of nested radicals. These can be removed through algebraic manipulation; since this process quickly becomes cumbersome as the number of RRR chains increases, we resorted to automatic techniques by using MATLAB's Symbolic Math Toolbox. We thus developed a script to remove all square roots by an iterative algorithm, which leads to a univariate polynomial in $s_{2,6}$ [13]. Each root of this polynomial may correspond to a different configuration of the mechanism.

Our observations from the analysis of the $n$-RRR robots ($n \leq 6$) are the following.

- The method can be easily generalized to architectures of increasing complexity: the main difference is the time required by the symbolic analysis package to

| | $A_i$ coordinates [mm] | | $\theta_i$ input angles | |
|---|---|---|---|---|
| | $x$ | $y$ | | |
| $A_1$ | 0 | 0 | $\theta_1$ | 64.8° |
| $A_2$ | 330 | 0 | $\theta_2$ | 115.2° |
| $A_3$ | 432 | 314 | $\theta_3$ | 201.67° |
| $A_4$ | 165 | 508 | $\theta_4$ | 237.6° |
| $A_5$ | -102 | 314 | $\theta_5$ | 320.4° |

Table 1: Coordinates of the fixed points $A_i$ and input angles $\theta_i$; the links' lengths are $c_i = 160$, $d_i = 120$ and $l_i = 80$ (for $i = 1, \ldots, 5$; all lengths in millimeters).

simplify all radicals in the final closure equation, as this time rapidly increases with the number of RRR chains.

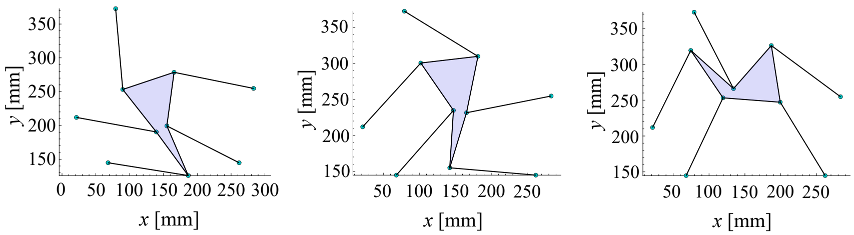
- The computational cost can be significantly reduced by a suitable choice of the bilateration sequence. For instance, it was found that using sequence $[(P_1, P_2) \Rightarrow P_6, (P_2, P_6) \Rightarrow P_7, (P_3, P_7) \Rightarrow P_8, (P_5, P_6) \Rightarrow P_{10}, (P_4, P_{10}) \Rightarrow P_9]$ (see Fig. 3b) instead of the one previously reported leads to a dramatic decrease in computational cost: with our setup (Matlab R2019a running on a processor Intel Core i7-8700 CPU @ 3.20GHz) the time required for the algebraic manipulation decreased from two days to ten minutes, as this approach reduces the accumulation of nested radicals. We empirically found that, in order to minimize computational time, the optimal sequence has approximately the same number of bilateration steps in both clockwise and counterclockwise sense.
- We conjecture that the characteristic polynomial has degree $2^{n+1} - 4$ for a $n$-RRR mechanism; this conjecture has been verified for $n$ from 3 to 6. We also conjecture that this polynomial has the lowest possible degree. This was confirmed to be the case for the 3-RRR robot, for which we found special architectures (not reported here due to space constraints) that lead to 12 distinct solutions for the DKP[2].

## 4 Example

We consider a 5-RRR robot with geometric parameters presented in Tab. 1, which correspond to a physical prototype (Fig. 5). We found that the robot has three distinct configurations, which are shown in Fig. 4.

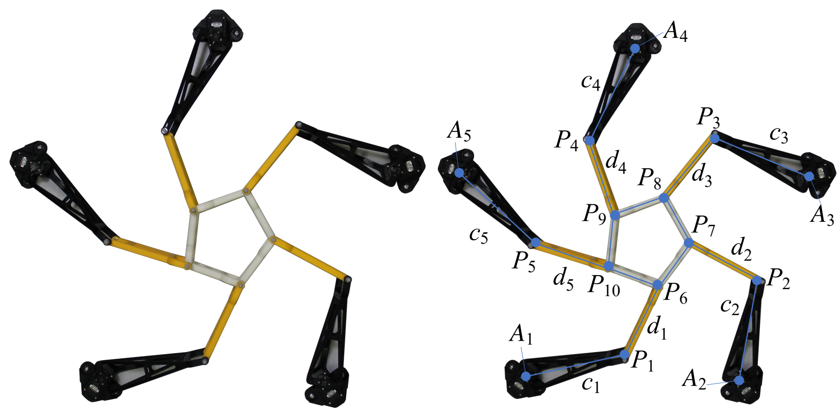
In the multimedia attachment available at https://youtu.be/j6D5lJSZPMo it is possible to see the following motions: the EE rigid translation and rotation, the movement of the mechanism between two out of the 32 possible solutions of the IKP (for a given EE pose) and the use of the platform to grip and move objects.

[2] The 3-RRR is known to have at most 6 distinct solutions [10]; however, in our analysis for generality we also consider solutions where the EE is rotated outside the plane of motion, which doubles the number of solutions. The former case can still be analyzed through bilateration.




(a) (b) (c)

Fig. 4: The three possible configurations for the 5-RRR robot at hand.




(a) (b)

Fig. 5: (a) 5-RRR prototype built at IRI; (b) its corresponding schematic.

## 5 Conclusions

In this work, a class of planar parallel robots have been proposed and studied in terms of position analysis. In order to solve the DKP, the bilateration method has been explored and applied, developing an iterative procedure that can be generalized as the number of kinematic chains varies. A numerical example is then presented, where the bilateration method has been helpful to solve the DKP in a reasonable time (about ten minutes). Finally, some tests performed on a prototype are presented in the final section. Directions for future work include:

- proving the conjecture about the degree of the characteristic polynomial defined in Sec. 3, and verify that this degree is the lowest possible;
- expanding our procedure to general planar PRCPs. For instance, $n$-RPR or $n$-PRR robots could be easily studied with the same methods considered in this paper.

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