ABSTRACT

Title of dissertation:	ON EFFECTIVE EQUIDISTRIBUTION OF ERGODIC AVERAGES FOR HIGHER STEP NILFLOWS AND HIGHER RANK ACTIONS
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In the first part of the thesis, we prove the bounds of ergodic averages for nilflows on general higher step nilmanifolds. It is well known that a flow is not renormalizable on higher step nilpotent Lie algebras and it is not possible to adapt the methods of analysis in moduli space. Instead, we develop the technique called 'scaling method' for operators behave like renormalization. On the space of Lie algebra satisfying the "transverse" condition, we obtain the speed of convergence of ergodic averages with polynomial type of exponent regarding the structure of nilpotent Lie algebras.

In the second part, we introduce the results on higher rank actions on Heisenberg nilmanifolds. We develop the method for higher rank action by following the work of A.Bufetov and G.Forni. We construct the finitely-additive measure called *Bufetov functional* and obtain the prove deviation of ergodic averages on Heisenberg nilmanifolds. As an application, we prove the limit theorem of normalized ergodic integrals.

ON EFFECTIVE EQUIDISTRIBUTION OF ERGODIC AVERAGES FOR HIGHER STEP NILFLOWS AND HIGHER RANK ACTIONS

by

Minsung Kim

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Advisory Committee:
Professor Giovanni Forni, Chair/Advisor
Professor Dmitry Dolgopyat
Professor Adam Kanigowski
Professor Rodrigo Treviño
Professor William Gasarch (Dean's representative)

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Success is not final, failure is not fatal: it is the courage to continue that counts. - Winston Churchill.

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Table of Contents

Chapter 1: Introduction

In the first chapter, we introduce brief research history and results regarding nilmanifolds that the author have worked during the graduate studies.

1.1 Deviation of ergodic averages in parabolic dynamics

A dynamical system (X, T) or (X, φ_t) is typically defined by a transformation (discrete) or flow (continuous) on some phase space X. A dynamical system is divided in three categories (hyperbolic, elliptic and parabolic) according to the speed of divergence of close orbits. We say that a flow is called *parabolic* if the rate of divergence of nearby orbits is at most polynomial in time t. While there is a well-developed theory of hyperbolic and elliptic systems, there is not much general known theory which describes the typical behavior of parabolic flows. A basic feature in dynamical system (or a conservative system) may have is ergodicity, which means that from the probabilistic point of view the dynamics cannot be decomposed into invariant pieces. More precisely, if flow φ_t is ergodic, by definition of Birkhoff, for almost every $x \in X$,

$$\frac{1}{T}\int_0^T f \circ \phi_X^t(x)dt \longrightarrow \int_X f d\mu, \quad T \to \infty. \tag{1.1}$$

It is also called the orbit of $x \in X$ is *equidistributed* if (1.1) holds.

A fundamental problem in smooth ergodic theory is to establish **quantitative** estimates (called *effective equidistribution*) on the asymptotic behavior of ergodic integrals of smooth functions.

Theorem A. *Let flow φ_t be ergodic (uniquely) ergodic flow on X. For almost all (all) $x \in X$, $f \in C^\infty(X)$ and for fixed $T > 0$, there exists $\alpha \in (0,1)$ such that*

$$\left| \frac{1}{T} \int_0^T f \circ \phi_X^t(x) dt - \int_X f d\mu \right| \leq C T^{-\alpha} \|f\| .$$

A lemma proved by M. Ratner states that polynomial decay of correlations implies polynomial speed of convergence of ergodic averages for horocycle flow on a compact surface of constant negative curvature for the large set of $x \in X$.

This result was improved by M. Burger [Bur90] for all $x \in X$ if X is the unit cotangent bundle of a compact Riemannian surface of constant negative curvature. However, until the '90s not much was known about asymptotic behavior of other parabolic flows except some polynomial bounds in the horocycle flows.

In the mid '90s, there was a breakthrough in parabolic systems. The first discovery in this direction was achieved by Anton Zorich [Zor97]. A. Zorich discovered, by computer experiments on interval exchange transformations, new power laws for the ergodic integrals of generic non-exact Hamiltonian flows on higher genus surfaces. In Zorich's later work and in a joint paper authored by M. Kontsevich [KZ97], they were able to explain conjecturally most of the discoveries by relating them to the ergodic theory of Teichmüller flows on moduli spaces of Abelian differentials. (Zorich found that classes of return orbits on homology groups exhibit unbounded polynomial deviations with Lyapunov exponents, and later, in joint work with M. Kontsevich, they conjectured this phenomenon extends to ergodic integrals of smooth functions.)

Around this time, Giovanni Forni solved cohomological equations [For97] for area-preserving flows in dynamical approach and discovered that Zorich's phenomena are highly related to the obstructions of solving cohomological equations. Later, Kontsevitch-Zorich's conjecture was answered in his work [For02]: the deviation of ergodic averages was explained with the power of Lyapunov exponent of KZ-cocycle. (Cf. Zorich's survey [Zor].) Subsequently, L.Flaminio and G.Forni proved the similar phenomena in renormalizable parabolic systems could be explained: Horocycle flows on compact surfaces of constant negative curvature [FF03] and Heisenberg nilflows [FF06] via methods for representation theory.

1.1.1 Cohomological equations and renormalization

In this section, we will briefly see how the cohomological equation is related to deviation of ergodic averages.

Definition 1.1. *Let M be a compact manifold and define $\phi^t_X(x)$ be the flow generated by the vector field X for $x \in M$. Cohomological equation for flow is $L_X u = f$ where L_X is Lie derivative.*

We call f *coboundary* if there exists a function u satisfying $L_X u = f$ and call u *transfer function.* To solve the equation for u given f, we confront clear obstructions in cohomological equation. If distribution D is X-invariant, then $L_X D = D(L_X)$ and it is necessary to set $D(f) = 0$. As a toy model, irrational rotations on torus, there exists a unique obstruction (measure) and we solve the cohomological equations $Xu = f$ with Diophantine conditions. Then, it is easily computed by Fourier analysis

$$\left| \frac{1}{T} \int_0^T f \circ \phi_t^X(x) dt - \int_M f d\mu \right| \leq C_f. \tag{1.2}$$

Back to Zorich's conjecture, Forni's observation was from splitting ergodic averages by *invariant distributions* (D) and its *remainders* (R). We identify a curve with a current $\gamma = D + R$, where $\gamma_T(f) = \frac{1}{T} \int_0^T f \circ \phi_t^X(x) dt$ and decompose as a sum of distributional obstructions (*Invariant distributions*) and its remainder. However, in general, there exist countably many obstructions and it requires extra estimations for invariant distributions. Let us restate theorem A.

Theorem B. *A smooth flow φ_t^X on a finite dimensional manifold M has deviation spectrum $\lambda_1 > \cdots > \lambda_i > \cdots > 0$ with multiplicities $m_1, \cdots, m_i \in \mathbb{Z}^+$ if there exists a system $\{D_{ij} \in \mathbb{Z}^+, 1 \leq j \leq m_i\}$ of linearly independent X-invariant distributions such that, for almost all (or all) $x \in M$ and $f \in C^\infty(X)$ and $T > 0$,*

$$\int_0^T f \circ \varphi_t^X(x) dt = \sum_{i \in \mathbb{N}} \sum_{j}^{m_i} C_{ij}(x, T) D_{ij}(f) T^{\lambda_j} + R(x, T)(f).$$

Proof of theorem B is based on **renormalization** dynamics. Renormalization flow is defined on a moduli space by the family of flows Φ_{X_τ}, deforming under the action of the group diffeomorphism induced by automorphisms τ of the Lie algebra.

If the automorphism or frame τ is recurrent in the moduli space, then Φ_{X_τ} is called a *renormalization flow*. Roughly speaking, under such renormalization, it is possible to decompose ergodic averages in a self-similar way, and this enables one to control the behavior of recurrent orbits by deviation exponents as shown in Theorem B.

1.1.2 Beyond renormalization

What if parabolic systems do not allow the existence of renormalizable flows? General higher step nilflows are such examples. These flows are non-renormalizable due to the absence of recurrence on the moduli space, i.e we can not apply the methods in [FF06]. Instead, there was the first breakthrough in this direction for Quasi-abelian nilpotent Lie algebras (see [FF14]). Roughly speaking, their new approach called *scaling method* introduces how to choose scaling of the vector fields, which behaves like renormalization flow but we only approximate the estimate the deviation of ergodic averages by the size of scaled invariant distributions.

One of our goals in chapter 3 of this thesis is to explain how to extend this scaling methods to a class of general higher step nilmanifolds. We seek the classifications of higher step nilmanifolds that scaling methods can be applied. In our work, we introduce new condition of Lie algebras called *transversality*. This condition describes the general condition which admits displacement of the return orbit. This condition contains filiform and triangular nilmanifolds, however, our method is not applicable for *any* nilmanifolds due to lack of dimensions for return orbits on transverse nilmanifolds. Still, it presents a guidance for other non-renormalizable parabolic flows.

1.2 Deviation of ergodic averages of nilflows and Weyl sums

Finally, we have collected all the essential ingredients to introduce our first result of this thesis. By a generalization of by B. Green and T. Tao, all orbits of Diophantine flows on any nilmanifold become equidistributed at polynomial speed.

Theorem C ([GT12]). *If the projected linear toral flow has a Diophantine frequency, then there exist a constant $C > 0$ and an exponent $\delta \in (0,1)$ such that, for all Lipschitz function f on M and for all $(x,T) \in M \times \mathbb{R}^+$ we have*

$$\left| \frac{1}{T} \int_0^T f \circ \phi_X^t(x) dt \right| \leq C T^{-\delta} \|f\|_{Lip}$$

Their approach is an extension of Weyl's method, and it is expected to have exponential term $\delta(\simeq 1/2^{k-1})$, but it is far from optimal. Again, as introduced in the previous section, G. Forni and L. Flaminio's method in invariant distribution was extended to nilflows to obtain the theorem A with a specific exponent.

- In [FF06], Heisenberg nilflows have exponent $\delta = 1/2 + \epsilon$. (Optimal)
- In [FF14], higher step Quasi-abelian (Filiform) has δ which decays quadratically (for almost all point $x \in M$).

Given a real polynomial $P(x) = a_k x^k + \cdots + a_1 x + a_0$, *Weyl sums* are defined as $W_N := \sum_{n=1}^{N-1} e^{2\pi i P(n)}$. In particular, with choice of test function f, by the use of first return map of nilflows to transverse sections

$$\sum_{n=1}^{N-1} e^{2\pi i P(n)} = \int_0^N f \circ \phi_X^t(x) dt + O(1).$$

The study of Weyl sums has a long history that goes back to foundational works of Hardy, Littlewood, and Weyl. For quadratic polynomial P, by Fiedler-Körner-Zurkat [FJK77], the known bound is optimal. In higher polynomial, T.D Wooley's result [Woo16] proves (quadratic) polynomial type for **all** coefficients.[1] We also refer [BDG16] for proving Vinogradov's mean value theorem new approach from decoupling argument.

1.2.1 Comments on higher step nilflows.

Our results stated in this part are contained in chapter 3 of this thesis.

On general nilmanifolds, the renormalization of the flow fails due to a lack of enough Lie algebra automorphism. Instead, based on the theory of unitary representations for the nilpotent Lie group (Kirillov theory), it is possible to choose a proper scaling operator on the space of invariant distributions. The choice of scaling factor relies on the notion of degree, which is an order of polynomial in irreducible representations. Compared to the earlier work Flaminio-Forni's work on Quasi-abelian case [FF14], the main novelty of our results lies in our generalization of the scaling method to solve rescaled cohomological equation.

[1]Many problems in analytic number theory and its connections with dynamical system has been widely studied. In the sense of Weyl sum, the result of Flaminio-Forni is comparable to that of Wooley, but we still do not know if the bound of triangular is optimal or it can be improved since its structure is not comparable with well-known exponential sum.

The transversality condition enables the measure estimate for the return orbit. This condition is sufficient, and in principle, there are no obstructions to a generalisation to arbitrary nilflows with Diophantine frequencies and all points $x \in M$, except that this would require new approaches to estimation other than a Borel-Cantelli type argument. On the other hand, the necessity of the condition explains that the total number of elements in the basis cannot grow too fast as the step size gets larger: it grows almost linear in the number of steps and generators.

We can view these phenomena in the following way: if the growth of the number of elements in lower steps (generated by basis) is too large, then it lacks the dimensions to count the measure of return orbit on a transverse manifold. For instance, we observe this phenomenon in free nilpotent Lie algebras. Even a small number of generators create a large number of elements in the lower level under small steps of commutations, which behave in a completely different way than strictly triangular and Quasi-abelian.

Question 1. *Can we prove similar type of (with higher degree) polynomial upper bound for any nilmanifold?*

Further remarks and questions. One related result technique is applied to the bounds of twisted horocycle flows [FFT16], work of Flaminio-Forni-Tanis which improves the result of Tanis-Vishe [TV15]. This rescaling method was applied to obtain the rescaled cohomological equations.

Theorem D (Twisted ergodic integrals of flows)**.** *For all zero average functions f*

and for all $(x,T) \in M \times \mathbb{R}^+$, there exists a function P such that we have

$$\left|\frac{1}{T}\int_0^T e^{2\pi\iota\lambda t} f \circ \phi_t^X(x)dt - \int_M f d\mu\right| \leq C_f(\lambda)P(\lambda)T^{-\alpha} \|f\|_s . \tag{1.3}$$

It is proved by Avila and Forni [AF07] that typical translation flow is weakly mixing. Regarding its quantitative bound, Bufetov-Solomyak [BS18] proved an upper bound of twisted translation flows on genus 2 and prove spectral results. Their method is based on quantifying Veech's criterion [Vee84] and an argument so called Erdös - Kahane argument. Independently, Forni [For19] introduce the deviations of ergodic averages of *twisted* flows by developing the new idea of twisted cohomology, which motivates the improvement for higher genus case [BS19]. Later, very recent work of Treviño [Tre20] extends it to higher rank setting for self-similar tilings by adapting methods of Forni and Venkatesh [For19, Ven10].

* Twisted ergodic integrals of **nilflows** are ergodic integrals of product nilflows, hence they are covered by results on deviation of ergodic averages of nilflows. Thus, we can cast the following questions.

Question 2. *What is the spectral type of mixing time-changes of nilflows?*

Question 3. *Can we prove* ***quantitative*** *weakly mixing for higher rank actions on Heisneberg nilmanifolds?*

1.3 Deviation of ergodic averages - Bufetov's perspective

In this section, we introduce the concept of Bufetov functionals in parabolic dynamics and its applications to quantitative bound of time changes of higher rank abelian actions.

1.3.1 Limit theorem in higher rank actions on nilmanifolds.

Our results stated in this part are contained in chapter 4 of this thesis.

We recall that the *Central Limit Theorem* (CLT) holds for a flow (φ_t) on a probability space (M, μ) for a zero average function $f \in L^2(M, \mu)$ if in the sense of probability distributions

$$\frac{1}{\sqrt{T}} \int_0^T f \circ \phi_X^t(x) dt \to N(0, \sigma_f)$$

where $N(0, \sigma_f)$ is a Gaussian (normal) distribution on the real line of mean zero and variance $\sigma_f > 0$. The main result in this section is that the *failure* of CLT for several parabolic flows. Namely, the growth of the variance may follow a power-law with a exponent and limit distributions are always compactly supported.

Firstly, the asymptotic behavior and limiting distribution of ergodic averages of translation flow was studied by Alexander Bufetov in the series of works [Buf09, Buf10, Buf14]. He constructed finitely-additive Hölder measure that are known as *Bufetov functionals*. Deviation of ergodic averages can be proved in the sense of these measures, and it turned out that there is a duality between these functionals and invariant distributions of translation flow which plays a key role in the work of G. Forni.

Here are the list of dualities between invariant distributions and Bufetov functionals for parabolic flows/actions.

- Translation flow (Bufetov, [Buf14]) $\Longleftarrow$ (Forni, [For02])
- Interval exchange transformation (Klimenko, [Kli19]) $\Longleftarrow$ (Bufetov, [Buf14])
- Self-similar tiling (Bufetov-Solomyak [BS13])
- Horocycle flow (Bufetov-Forni, [BF14]) $\Longleftarrow$ (Flaminio-Forni, [FF03])
- Heisenberg nilflow (Forni-Kanigowski, [FK17]) $\Longleftarrow$ (Flaminio-Forni, [FF06])
- Higher rank actions (K, [Kim20b]) $\Longleftarrow$ (Cosentino-Flaminio, [CF15])

The construction of such functional was used to derive results on probabilistic behavior of ergodic averages of parabolic flows as stated above. Especially, it is not surprising that related result of limit distributions appeared by connections between Heisenberg group and theta series in several contexts. For Heisenberg flows, the first return map is obtained as a skew-translation, and its limit theorem of theta sums was studied by J. Griffin and J. Marklof [GM14] and generalized by Cellarosi-Marklof [CM16]. In analogy with higher rank actions, as an application of our result, we obtain a limit theorem for theta series on Siegel half spaces, introduced in the works of Götze and Gordin [GG04] and Marklof [Mar99].

Further remarks and questions. It is not known if it is possible to construct the Bufetov functionals on higher step nilmanifolds. On nilmanifolds with step $s > 2$, the moduli space is trivial and there is no known renormalization flows on any higher step nilmanifolds.

Question 4. *Can we construct Bufetov functional on higher step nilflows?*

1.3.2 Mixing of time changes of flows on nilmanifolds.

We have seen examples of parabolic flows such as horocycle, translation, and nilflows flows, but one can build new parabolic flows by perturbation. A *time-change* flow ϕ_t^V of a flow ϕ_t^X on M is defined as $\phi_t^V(x) := \phi^X_{\tau(x,t)}(x)$ for all $(x,t) \in M \times \mathbb{R}$, where τ is a cocycle over ϕ_t^V. This means that the flow moves along the same orbits at different speeds. If time change is measurably conjugated to original flow, then it is called *trivial* and time-changes are described by solutions of cohomological equations.

One interesting question is proving mixing[2] for (time-changes) parabolic flows. Historically, as a first example, mixing of all time changes of horocycle flow under mild differentiability conditions is proved by Marcus by shearing [Mar77]. *Shearing* means short segment transversal to the flow get sheared in the direction of the flow direction (or a direction which commutes with flow) by push-forward of flow. This curve is *asymptotically* approximated by the flow trajectories and this allows mixing by equidistribution of trajectories of flow. Later, its **quantitative** estimation was obtained by Forni-Ulcigrai [FU12] on time-changes of horocycle flows, which partially answers Katok-Thouvenot conjecture [KT06] about Lebesgue spectrum

[2] ϕ_t^V is mixing if $\mu(\phi_t^V(A) \cap B) \to \mu(A)\mu(B)$ for any $A, B \subset M$, as $t \to \infty$. Equivalently, for any $f, g \in C^0(M)$, $\int_M f \circ \phi_t^V \cdot g d\mu \to \int_M f d\mu \int_M g d\mu$.

type. A recent result of B. Fayad, G. Forni, and A. Kanigowski [FFK16] answered complete answer (**countable multiplicity** of Lebesgue spectrum of Kochergin flows and time-changes of horocycle flows) to the conjecture.

This argument was also studied in the context of nilflows. Nilflows are never mixing for the elliptic behavior on the base torus, but they are relatively mixing with polynomial speed. Mixing property of time changes of Heisenberg nilflows was firstly studied by Avila-Forni-Ulcigrai in [AFU11]. In this work, lack of parabolicity in the toral factor was removed by a reparametrization of non-trivial time changes. The result on nilmanifold was extended to higher step filiform nilmanifolds [Rav18, AFRU19]. A recent further direction is to prove multiple mixing following famous Rokhlin's question.

Question 5 (Rokhlin)**.** *Does 2-mixing imply higher order mixing? i.e*

$$\mu(\phi^X_{t_2-t_1}(A_1) \cap \phi^X_{t_3-t_2}(A_2) \cap \cdots \cap \phi^X_{t_{N+1}-t_N}(A_N)) \to \prod_{i=1}^{N} \mu(A_i),$$

as $t_{i+1} - t_i \to \infty$*.*

Recent results introduce new trends in strengthening mixing results: quantifying mixing, multiple mixing, and spectral properties[3]. One of the key techniques follows from **Ratner's property**. This enables the estimates of *quantitative shearings*, which measures the slow divergence of near orbits after translations. Proof of this property answers Mobius disjointness (or AOP). For further information, we leave the authors to check [FK20, KKPU19, FK16, FKPL18, FFKPL19, KPL20].

[3] It is recommended to refer general survey [KT06, KL20].

Further remarks and questions. Beyond the class of time-changes, parabolic flows can be constructed by twisting [Sim17]. New example of parabolic perturbations of unipotent flows was constructed by Ravotti [Rav19]. A very recent result of time-changes of unipotent flow (semisimple) on Lorentz group was introduced by Tang [Tan20].

Question 6. *Can time-changes of higher rank actions be mixing?*

Current work in the progress of the author indicates that we do not know the result of mixing can follow from classical shearing argument in abelian actions. Although it is possible to calculate push-forwards of vector fields, we still do not know how to apply equidistribution results. Here we finish the section by listing other questions for nilflows by G. Forni.

Question 7. *Does there exists a residual set of time changes of (Heisenberg) nilflows such that, for* ***all Diophantine nilflows****, mixing (non-trivial) time changes have polynomial decay of (multiple) correlations ? Are mixing time-changes (polynomially) mixing of higher order?*

Question 8. *Is any measurably non-trivial smooth time-change of a uniquely ergodic parabolic nilflows weakly mixing or mixing?*

Question 9. *What is the largest class of smooth time changes of (Heisenberg) nilflows such that there is a dichotomy between trivial time changes and (weakly) mixing time-changes ? How large is the set of mixing time-changes? Is it dense or generic ?*

The last question reminds us of the deviations of twisted integrals (1.3).

Question 10. *What is the spectral type of mixing time-changes of nilflows?*

1.4 Short notes on smooth conjugacy of nilflows

This section is devoted to my first major failure in the journey of research.

In section 1.1, we reviewed how the theory of solving cohomological equations is applied in estimating asymptotic of ergodic averages. Besides, the study of cohomological equations is a relevant part of the theory of (smooth) dynamical systems directly connected to basic questions such as the *triviality of time-changes for flows* and the *smooth conjugacy problem via linearization* as well. In this section, we would like to briefly introduce the main strategy and difficulties for proving the smooth conjugacy of nilflows.

Problem 1. *Can we prove local conjugacy of Heisenberg nilflows? I.e given nilflow ϕ_t^V, can we construct a diffeomorphism $h \in Diff^\infty(M)$ and vector field W close enough to V such that for all $x \in M$,*

$$h \circ \phi_t^V = \phi_t^W \circ h. \tag{1.4}$$

One well-known way of solving perturbation is to consider a linearized equation and repeat iterated estimation, a modification of Newton's rapidly converging iteration method. In Zehnder's paper [Zeh75], it is concerned with the solvability of the equation $F(f, u(f)) = 0$, for $u(f)$ whenever f is close to f_0 where $F(f_0, u(f_0)) = 0$

by (generalized) implicit function theorem. In general setting, $(D_2F)^{-1}$ may exist but unbounded (due to small-divisor). In such situations, the classical implicit function theorem does not apply, but we can approximate inverse operator under certain hypothesis of F. As a guiding principle, the paper presented the strategy, in the framework of families of linear spaces, graded Frechet spaces, but it did not specify the space about the topology and choice of proper norms, etc.

Heisenberg nilflow could be a good model to start since the Sobolev order is 1/2 (see [FF06]). We start by defining functional F by linearizations of (1.4) with additional *counter terms* which is necessary for vanishes of obstructions. This main issue was due to existence of infintiely many but **perturbed** obstructions in solving linearized equation F. In the iteration scheme, unfortunately, even very small size of perturbation, diffeomorphism h that is close to identity, accumulation of obstructions (**counter term**) grows too fast. That is, we could not construct choose proper Frechet space with suitable norm that controls the growth of perturbed obstructions.

Meanwhile, there were different approaches from Nikolaos Karaliolios around the same time. His method was adapted from the normal-form version of the K.A.M. theorem[4] by J. Moser in [Mos90]. He aimed to extend his methods applied on torus [Kar17, Kar18] to manifolds but it was not succesful either.

[4] In dynamical systems, it is an important question whether dynamical properties (e.g ergodicity) is preserved under small perturbations. It was firstly conjectured by Birkhoff and Hopf that a typical conservative dynamical system should be ergodic. In the mid of 1950s' after the important works of **Kolmogorov, Moser and Arnold**, the conjecture proposed turned out to be false in higher regularity. Nowadays, these results are known as *KAM theorem* and their idea suggests conditions for the existence and persistence of some region in the manifold with positive measure consisting of invariant tori.

Further remarks and questions. On higher rank actions, on the contrary, proving local conjugacy goes with different story. For parabolic actions, the rigidity phenomenon referred as *transversal local rigidity* is well studied.

In higher rank setting, the issue regarding infinite obstructions could be resolved by the methods called *higher rank trick*. This enables to reduce the case into all but finite obstructions. For general introduction, we recommend readers to check [KN11].

Question 11. *Can we prove transversal local rigidity of any nilmanifolds?*

For higher action on nilmanifolds, there are results for step 2 nilmanifolds [DK11]. Recent progress of Damjanovic and Tanis indicates local rigidity of higher rank actions on Heisenberg nilmanifolds [DT20]. However, on higher step nilmanifolds, it is not known since we have not obtained tame splittings. We finish this section with the following remark.

Question 12. *Can we prove local conjugacy of any other parabolic flows?*

In author's knowledge, the problem of setting up a KAM-type approach with Nash-Moser iteration for translation flows is harder. One good property of Heisenberg nilflows is that all invariant distributions have order $1/2$ (essentially). This is not true for translation flows because the order goes to ∞. However, the nilflow case is non-trivial as there are infinitely many obstructions. Unfortunately, the horocycle is the hardest as it combines the two types of difficulties.

1.5 Revisited deviation of ergodic averages - in the point of view of Hyperbolic dynamics.

We have seen two different approaches toward deviation of ergodic averages for parabolic dynamics in the previous section §1.1 and §1.3. Here we introduce the last new approach that recently developed: *transfer operator* methods in parabolic dynamics. A recent improvement in this direction is an adaptation of transfer operator techniques from hyperbolic dynamics. The method stems from the analysis of the transfer operator, firstly treated by Giulietti and Liverani [GL19]. They set up non-linear flows on a torus and prove asymptotics of ergodic averages in the sense of invariant distributions with eigenvalues of transfer operators called *Ruelle resonances.* Recent work of V. Baladi [Bal19] also proved that no deviation of ergodic averages for Giulietti and Liverani's flows.

Theorem E. *We say that map T has the Ruelle spectrum $(\lambda_i)_{i\in I}$ with Jordan block with dimension $(N_i)_{i\in I}$ on the space of functions C if, for any $f, g \in C$ and for any $\epsilon > 0$, there is an asymptotic expansions*

$$\int f \cdot g \circ T^n d\mu = \sum_{|\lambda_i|\geq\epsilon} \sum_{j\leq N_i} \lambda_i^n n^j c_{i,j}(f,g) + o(\epsilon^n)$$

Following Giulietti and Liverani's approach, when the flow is **renormalizable** by (partial) hyperbolic maps, it is proved under the following settings:

- A. Adam [Ada18] for horocycle flow on negative variable curvature.

- F. Faure, E. Lanneau, and S. Gouëzel [FGL19] for translation flow.
- L. Simonelli and O. Butterley [BS20] for Heisenberg nilflows.
- G. Forni [For20b] reinterpreted *Ruelle resonances* by studying cohomological equations (reverse way of all listed above)

Now, it is interesting to ask if we can find Ruelle resonances if flow is renormalizable continuously on its moduli space. i.e instead of map (transfer operator), we consider flows with renormalization cocycles called *transfer cocycle*.

Question 13. *Can we construct transfer cocycle that behaves renormalization translation flows?*

This question is indeed reminiscent of Forni's work [For02, For20a]. As discussed in section 1.1, the deviation of ergodic averages are determined by the Lyapunov spectrum of KZ-cocycles. It is natural to ask if there is a correspondance with Ruelle resonances for *transfer cocycle* and construct cocycles on inifnite dimensional function space. It is suggestive to follow constructions of cocycle of A. Bluementhal [Blu16], but it still remains to clarify relations with new anisotropic norm and Hodge-norm used in Forni's previous work. Likewise, it is also open whether we can construct a corresponding transfer cocycle in the Heisenberg flow setting.

Question 14. *Can we find asymptotics for the ergodic averages for recurrent for Heisenberg nilflows with expansions of resonances of transfer operators defined on an anisotropic space?*

Organization of this thesis.

This thesis contains the following works: Chapter 2 consists of backgrounds about nilmanifolds. Chapter 3 is about the work 'Effective equidistribution for generalized higher step nilflows' [Kim20a]. Chapter 4 contains result of the author about higher rank actions and limit theorem [Kim20b]. This research was partially supported by the NSF grant DMS 1600687.

Chapter 2: Background

In this section, we introduce basic definitions that will be used in the rest of chapters.

2.1 Structure on nilmanifolds

Let G be a connected and simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$, and let Γ be a lattice in G. The quotient $M = \Gamma\backslash G$ is then a compact nilmanifold on which G acts on the right by translations. Denote by $\mu = \mu_M$ the G-invariant probability measure on M.

Let $\mathfrak{g}$ be a d-step nilpotent real Lie algebra ($d \geq 2$) with a minimal set of generators $E^1 := \{\eta_1, \cdots, \eta_n\} \subset \mathfrak{g}$. For all $j \in \{1, \cdots d\}$, let $\mathfrak{g}_j$ be the descending central series of $\mathfrak{g}$:

$$\mathfrak{g}_1 = \mathfrak{g}, \mathfrak{g}_2 = [\mathfrak{g}, \mathfrak{g}], \cdots, \mathfrak{g}_j = [\mathfrak{g}_{j-1}, \mathfrak{g}], \cdots, \mathfrak{g}_d \subset \mathfrak{Z}(\mathfrak{g}) \tag{2.1}$$

where $\mathfrak{Z}(\mathfrak{g})$ is the center of $\mathfrak{g}$. Equivalently, the corresponding Lie subgroups $G^{(j)} =$

$\exp(\mathfrak{g}_j)$ form the descending central series of G

$$G = G^{(1)} \supset G^{(2)} \supset G^{(d-1)} \cdots \supset G^{(d)} = \{e_G\}.$$

For all j, the group $G^{(j+1)}$ is a closed normal subgroup of G, and there are natural epimorphisms $\pi^{(j)} : G \to G/G^{(j+1)}$. Then, the group $\Gamma^{(j+1)} := G^{(j+1)} \cap \Gamma$ is a lattice in $G^{(j+1)}$. Simlarly, we write a series of lattice

$$\Gamma = \Gamma^{(1)} \supset \Gamma^{(2)} \supset \Gamma^{(d-1)} \cdots \supset \Gamma^{(d)} = \{e_G\}.$$

Moreover, $\pi^{(j)}(\Gamma)$ is a lattice in $G/G^{(j+1)}$ and $M^{(j)} := \pi^{(j)}(G)/\pi^{(j)}(\Gamma)$ is a nilmanifold. It follows that

$$\pi^{(j)} : M = G/\Gamma \to M^{(j)}$$

whose fibers are the orbits of $G^{(j+1)}$ on G/Γ, homeomorphic to the nilmanifolds $G^{(j+1)}/(G^{(j+1)} \cap \Gamma)$.

The group $G^{ab} = G/[G, G]$ is abelian, connected and simply connected, hence isomorphic to $\mathbb{R}^{n+1}$ and $\Gamma^{ab} = \Gamma/[\Gamma, \Gamma]$ is a lattice in G^{ab}. We have a natural projection $pr_1 : M \to \mathbb{T}^{n+1}$, thus every nilmanifold M is a fiber bundle over a torus.

$$0 \to M^{(2)} \to M \xrightarrow{pr_1} \mathbb{T}^{n+1} \to 0. \tag{2.2}$$

Similarly, at every step $0 \leq i < d$, the action of $G^{(j)}$ on M induces a free action of the torus group, i.e $M^{(i)}$ is a bundle over $M^{(i+1)}$ with toral fiber.

Another fibration arises from the canonical homomorphism $G \to G/G' \approx \langle \exp \xi \rangle$ where G' is generated by codimension 1 ideal. For $\theta \in \mathbb{T}^1$, the fiber $M^a_\theta = {pr_2}^{-1}(\theta)$ is local section of the nilflow on M.

$$0 \to M^a_\theta \to M \xrightarrow{pr_2} \mathbb{T}^1 \to 0. \tag{2.3}$$

Let us recall the definition of Malcev basis.

Definition 2.1 (Malcev basis). *A Malcev basis for $\mathfrak{g}$ through the descending central series $\mathfrak{g}_j$ and strongly based at Γ is a basis $\eta^{(1)}_1, \cdots \eta^{(1)}_{n_1}, \cdots, \eta^{(d)}_1, \cdots \eta^{(d)}_{n_d}$ of $\mathfrak{g}$ satisfying the following.*

1. *If we set $E^j = \{\eta^{(j)}_1, \cdots \eta^{(j)}_{n_j}\}$, the elements of the set $E^j \cup E^{j+1} \cup \cdots \cup E^d$ form a basis of $\mathfrak{g}_j$. Denote $n_j = \dim(\mathfrak{g}_j) - \dim(\mathfrak{g}_{j+1})$ be the dimension of subspace E^j.*

2. *For each j, choice of the elements in order $\eta^{(j)}_1, \cdots \eta^{(j)}_{n_j}, \cdots \eta^{(k)}_1, \cdots \eta^{(k)}_{n_k}$ spans an ideal of $\mathfrak{g}_j$;*

3. *The lattice Γ is generated by*

$$\Gamma = \{\exp(\eta^{(1)}_1), \cdots, \exp(\eta^{(1)}_{n_1}), \cdots \exp(\eta^{(k)}_{n_k})\}.$$

2.2 Nilflows

On nilmanifold M, the nilflow ϕ_X^t generated by $X \in \mathbf{n}$ is the flow obtained by the restriction of this action to the one-parameter subgroup $(\exp tX)_{t\in\mathbb{R}}$ of G, with

$$\phi_X^t(x) = x\exp(tX), \quad x \in M,\ t \in \mathbb{R}.$$

The projection $\bar{X}$ of X is the generator of a linear flow $\psi_{\bar{X}} := \{\psi_X^t\}_{t\in\mathbb{R}}$ on $\mathbb{T}^{n+1} \approx \mathbb{R}^{n+1}\backslash\bar{\Gamma}$ defined by

$$\psi_{\bar{X}}^t(x_1, \cdots, x_{n+1}) = (x_1 + tv_1, \cdots, x_{n+1} + tv_{n+1}).$$

Then, canonical projections $pr_1 : M \to \mathbb{T}^{n+1}$ intertwines the flows ϕ_X^t and $\psi_{\bar{X}}^t$. We recall the following:

Theorem 2.2. *[AHG+63] The followings are equivalent.*

1. *The nilflow (ϕ_X^t) on M is ergodic.*
2. *The nilflow (ϕ_X^t) on M is uniquely ergodic.*
3. *The nilflow (ϕ_X^t) on M is minimal.*
4. *The projected flow $(\psi_{\bar{X}}^t)$ on $\overline{M} = G^{ab}/\Gamma^{ab} \simeq \mathbb{T}^{n+1}$ is an irrational linear flow.*

2.3 Examples

Definition 2.3. *Let* $\mathfrak{g} = \mathfrak{h}_3$ *be a Lie algebra with its basis* $\{X, Y, Z\}$ *satisfying following commutation relations* $[X, Y] = Z$*. Then it is called Heisenberg Lie algebra.*

In the matrix form, we write

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

There exists higher dimensional analogy for Heisenberg Lie algebra.

Definition 2.4. *Let* $\mathfrak{g} = \mathfrak{h}_{2d+1}$ *be a Lie algebra with its basis* $\{X_1 \cdots X_d, Y_1, \cdots Y_d, Z\}$ *satisfying following commutation relations* $[X_i, Y_i] = Z$*. Then, it is called* $(2d+1)$ *dimensional Heisenberg Lie algebra.*

In the matrix form, we write

$$\sum_{1 \le i \le d} (x_i X_i + y_i Y_i) + zZ \mapsto \begin{pmatrix} 0 & x_1 & x_2 & \cdots & x_d & z \\ 0 & 0 & \cdots & \cdots & \cdots & y_1 \\ \vdots & \vdots & \ddots & \ddots & \cdots & y_2 \\ \vdots & \vdots & 0 & 0 & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & y_d \end{pmatrix}. \tag{2.4}$$

Definition 2.5. *Let* $\mathfrak{g}_d$ *be a* $(d+1)$*-dimensional Lie algebra spanned by* $\{X, Y_1, \ldots, Y_d\}$ *with brackets given by* $[X, Y_i] = Y_{i+1}$ *for* $i < d$ *and* $[X, Y_d] = 0$*.* $\mathfrak{g}_d$ *is called Filiform*

(Quasi-abelian) Lie algebras.

We can represent $\mathfrak{g}_d$ in matrix form via the identification

$$a_0 X + \sum_{i=1}^{d} a_i Y_i \mapsto \begin{pmatrix} 0 & a_0 & 0 & 0 & \dots & 0 & a_d \\ 0 & 0 & a_0 & 0 & \dots & 0 & a_{d-1} \\ 0 & 0 & 0 & a_0 & \dots & 0 & a_{d-2} \\ & & & \ddots & & a_0 & \\ & & & \dots & & 0 & a_1 \\ & & & \dots & & & 0 \end{pmatrix}.$$

Note the following important property:

$$[\mathfrak{g}_d, Y_{j-1}] \subset span\{Y_j, \dots, Y_d\}. \tag{2.5}$$

Let G_d be the corresponding Lie group (given by exponentiating elements from $\mathfrak{g}_d$) and called *quasi-abelian filiform* Lie group. Notice that $d = 1$ corresponds to $\mathbb{R}^2$ and $d = 2$ corresponds to the Heisenberg group. Moreover, for all $1 \leq i, j \leq d$, we have $[Y_i, Y_j] = 0$. Note also that for $d > 1$, the center $Z(G_d)$ is one dimensional and spanned by Y_d.

Definition 2.6. *Let $\mathfrak{g}_d$ be a Lie algebra with basis $\{X_i^{(j)}\}$ for $1 \leq i \leq j, 1 \leq j \leq d$. By its identification with triangular matrix, $\mathfrak{g}_d$ contains d generators $\{X_i^{(1)}\}_{1 \leq i \leq d}$*

and one dimensional center $X_1^{(d)}$*. This Lie algebra* $\mathfrak{g}_d$ *is called Triangular.*

$$\sum_{1\le i\le j, 1\le j\le d} x_i^{(j)} X_i^{(j)} \mapsto \begin{pmatrix} 0 & x_1^{(1)} & \cdots & \cdots & x_1^{(d)} \\ 0 & 0 & x_2^{(1)} & x_i^{(j)} & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & x_d^{(1)} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.6}$$

Following matrix multiplication, it satisfies the commutation relations involved with generators:

$$[X_i^{(j)}, X_{i+j}^{(1)}] = X_i^{(j+1)} = [X_i^{(1)}, X_{i+1}^{(j)}]. \tag{2.7}$$

There are also other type of commutation relations that are not involved with generators, but we do not list them.

Chapter 3: On effective equidistribution of higher step nilflows.

In this chapter, we prove an estimate of the rate of convergence of ergodic averages for a class of nilflows on higher step of nilmanifolds under Diophantine conditions on the frequencies of their toral projections. Our main results present a special class of nilmanifolds satisfying *the transversality condition* (Definition 3.26). This shows the speed of ergodic average of nilflows with Diophantine conditions is polynomial for almost all points, as a function of step size and total number of elements of Lie algebras.

3.1 Main theorem

Our main result is the bound on the speed of convergence of ergodic averages along almost all orbits of Diophantine nilflows.

Let $\sigma = (\sigma_1, \cdots, \sigma_n) \in (0,1)^n$ be such that $\sigma_1 + \cdots + \sigma_n = 1$. For any $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n$, for any $N \in \mathbb{N}$ and every $\delta > 0$, let

$$R_\alpha(N, \delta) = \{r \in [-N, N] \cap \mathbb{Z} \mid |r\alpha|_1 \leq \delta^{\sigma_1}, \cdots, |r\alpha|_n \leq \delta^{\sigma_n}\}.$$

Definition 3.1 (Diophantine)**.** *For every $\nu > 1$, let $D_n(\bar{Y}, \sigma, \nu) \subset (\mathbb{R}\backslash\mathbb{Q})^n$ be the* *subset defined as follows: the vector* $\alpha \in D_n(\bar{Y}, \sigma, \nu)$ *if and only if there exists a constant* $C(\bar{Y}, \sigma, \alpha) > 0$ *such that, for all* $N \in \mathbb{N}$ *for all* $\delta > 0$,

$$\#R_\alpha(N, \delta) \le C(\bar{Y}, \sigma, \alpha) \max\{N^{1-\frac{1}{\nu}}, N\delta\}. \tag{3.1}$$

If $n = 1$, then it can be verified that above condition $D_1(\nu)$ contains the set of Diophantine irrational numbers of exponent $\nu \ge 1$ by an argument based on continued fraction. If $n \ge 2$, then the set $D_n(\nu)$ contains the set of simultaneous Diophantine vectors (see Lemma 3.42).

Theorem 3.2. *Let* $(\phi^t_{X_\alpha})$ *be a nilflow on a k-step nilmanifold* M *on* $n+1$ *generators such that the projected toral flow* $(\bar{\phi}^t_{X_\alpha})$ *is a linear flow with frequency vector* $\alpha := (1, \alpha_1, \cdots, \alpha_n) \in \mathbb{R} \times \mathbb{R}^n$. *Assume the Lie algebra satisfies the transversality condition and* $\alpha \in D_n(\nu)$ *for some* $1 \le \nu \le \frac{k}{2}$. *Then, there exists a Sobolev norm* $\|\cdot\|$ *on the space* $C^\infty(M)$ *of smooth function on* M *and for every* $\epsilon > 0$ *there exists a positive measurable function* $K_\epsilon \in L^p(M)$ *for all* $p \in [1, 2)$, *such that the following bound holds. For every smooth zero-average function* $f \in C^\infty(M)$, *for every* $T \ge 1$, *for almost all* $x \in M$,

$$\left| \frac{1}{T} \int_0^T f \circ \phi^t_{X_\alpha}(x) dt \right| \le K_\epsilon(x) T^{-\frac{1}{3S_{\mathbf{n}}(k)} + \epsilon} \|f\|$$

where $S_{\mathbf{n}}(k) := (n_1 - 1)(k-1) + n_2(k-2) + ... + n_{k-1}$ *where* n_i *is the dimension of basis* E^j *in Malcev basis.*

The above theorem is appreciated by its corollary on strictly triangular nilmanifold. Let $N_k^{(k)}$ denote a step k nilpotent Lie group on k generators. Up to isomorphism, $N_k^{(k)}$ is the group of upper triangular unipotent matrices

$$[x_1X_1, \cdots x_nX_n, \cdots y_i^{(j)}Y_i^{(j)} \cdots zZ] := \begin{pmatrix} 1 & x_1 & \cdots & \cdots & z \\ 0 & 1 & x_2 & y_i^{(j)} & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & x_n \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad x_i, y_i^{(j)}, z \in \mathbb{R} \tag{3.2}$$

with one dimensional center. This results proves that equidistribution at a polynomial speed with exponent which decays *cubically* as a function of number of steps.

Corollary 3.3. *Let $(\phi^t_{X_\alpha})$ be a nilflow on k-step strictly triangular nilmanifold M on k generators such that the projected toral flow $(\bar{\phi}^t_{X_\alpha})$ is a linear flow with frequency vector $\alpha := (1, \alpha_1, \cdots, \alpha_{k-1}) \in \mathbb{R} \times \mathbb{R}^{k-1}$. Under the condition that $\alpha \in D_n(\nu)$ for some $1 \leq \nu \leq \frac{k}{2}$, there exists a Sobolev norm $\|\cdot\|$ on the space $C^\infty(M_k^{(k)})$ of smooth function on $M_k^{(k)}$ and there exists a positive measurable function $K_\epsilon \in L^p(M_k^{(k)})$ for all $p \in [1, 2)$ and for every $\epsilon > 0$, such that the following bound holds. For every smooth zero-average function $f \in C^\infty(M_k^{(k)})$, for almost all $x \in M$ and for every $T \geq 1$,*

$$\left| \frac{1}{T} \int_0^T f \circ \phi^t_{X_\alpha}(x) dt \right| \leq K_\epsilon(x) T^{-\frac{1}{3(k-1)(k^2+k-3)} + \epsilon} \|f\|$$

We also establish the uniform bound for step-3 strictly triangular nilmanifold case. The result holds for *all points* by estimating the width with counting close return time directly under Roth-type Diophantine condition. The step-3 case (as well as filiform case, [For16]) is a good example to derive a simplified proof beyond the renormalization method, in contrast to the Heisenberg case [FF06, CF15].

Theorem 3.4. *Let (ϕ^t_X) be a nilflow on 3-step nilmanifold M on 3 generators such that the projected toral flow $(\bar{\phi}^t_X)$ is a linear flow with frequency vector $v := (1, \alpha, \beta)$ of Diophantine condition with exponent $\nu = 1 + \epsilon$ for all $\epsilon > 0$. For every $s > 26$, there exists a constant C_s such that for every zero-average function $f \in W^s(M)$, for all $(x, T) \in M \times \mathbb{R}$, we have*

$$\left| \frac{1}{T} \int_0^T f \circ \phi^t_X(x) dt \right| \leq C_s T^{-1/12+\epsilon} \left\| f \right\|_s .$$

In section 3.8, we present exponential mixing of hyperbolic nilautomorphism as a main application. Exponential mixing of ergodic automorphism and its applications to the Central Limit Theorem on compact nilmanifolds was proven by R. Spatzier and A. Gorodnik [GS14]. Their approach was based on the result of Green and Tao [GT12], and mixing follows from the equidistribution of the exponential map called box map satisfying certain Diophantine conditions. Our result also shows specific exponent of exponential mixing depending on the structure of nilmanifolds, which follows from equidistribution results and renormalization argument of hyperbolic automorphism. However, they are limited to special class of nilautomorphisms due to lack of hyperbolicity on the group of automorphisms on general nilpotent Lie algebras. (Cf. triangular step 3 with 3 generators.)

3.2 Structures

Let N be a connected and simply connected step k nilpotent Lie group. For any nilpotent Lie algebra $\mathbf{n}$, there exists a codimension 1 subalgebra $\mathfrak{I}$ where $\mathbf{n} = \mathbb{R}\xi \oplus \mathfrak{I}$. Then $\mathfrak{I}$ is an ideal and $[\mathbf{n}, \mathbf{n}] \subseteq \mathfrak{I}$. ([Hum12], Chapter 3, p.12, [GC90], Lemma 1.1.8). For convenience, we write dimension $a = \dim(\mathfrak{I}) = n_1 + \cdots + n_k$ and set $n = n_1$.

Definition 3.5. *An* adapted basis *of the Lie algebra* $\mathbf{n}$ *is an ordered basis* $(X, Y) := (X, Y_1, \cdots Y_a)$ *of* $\mathbf{n}$ *such that* $X \notin \mathfrak{I}$ *and* $Y := (Y_1, \cdots, Y_a)$ *is an basis of* $\mathfrak{I}$.

A strongly adapted *basis* $(X, Y) := (X, Y_1, \cdots Y_a)$ *is an adapted basis such that the following holds:*

1. *the system* $(X, Y_1, \cdots Y_n)$ *is a system of generators of* $\mathbf{n}$*, hence its projection is a basis of the Abelianisation* $\mathbf{n}/[\mathbf{n}, \mathbf{n}]$ *of the Lie algebra* $\mathbf{n}$*:*

2. *The system* $(Y_{n+1}, \cdots Y_a)$ *is a basis of the ideal* $[\mathbf{n}, \mathbf{n}]$.

Notation. Consider the set of indices

$$J := \{(i, j) \mid 1 \leq i \leq n_j, 1 \leq j \leq k\}$$

$$J^+ := \{(i, j) \mid 1 \leq i \leq n_1,\ j = 1\}$$

$$J^- := \{(i, j) \mid 1 \leq i \leq n_j,\ j > 1\}$$

$$J^{-2} := \{(i, j) \mid 1 \leq i \leq n_j,\ j > 2\}.$$

Recall that $\mathbf{n} = \mathbb{R}\xi \oplus \mathfrak{I}$. Let $\alpha := \alpha_i^{(j)} \in \mathbb{R}^J$ and $X := X_\alpha$ be the vector field on M defined

$$X_\alpha := \log[x^{-1}\exp(\sum_{(i,j)\in J}\alpha_i^{(j)}\eta_i^{(j)})], \quad x = \exp(\xi) \tag{3.3}$$

and equivalently we write

$$X_\alpha := -\xi + \sum_{(i,j)\in J}\alpha_i^{(j)}\eta_i^{(j)}. \tag{3.4}$$

For $\theta \in \mathbb{T}^1$ let $M_\theta^a = {pr_2}^{-1}(\theta)$ denote fiber over $\theta \in \mathbb{T}^1$ of the fibration pr_2. Transverse section M_θ^a of the nilflow $\{\phi_{X_\alpha}^t\}_{t\in\mathbb{R}}$, $\mathbf{s} = (s_i)_{i=1}^a \in \mathbb{R}^a$ corresponds to

$$\{\Gamma\exp(\theta\xi)\exp(\sum_{i=1}^a s_i\eta_i) \mid (s_i) \in \mathbb{R}^a\} = \{\Gamma\exp(e^{ad(\theta\xi)}\sum_{i=1}^a s_i\eta_i)\exp(\theta\xi) \mid (s_i) \in \mathbb{R}^a\}.$$

Lemma 3.6. *The flow $(\phi_{X_\alpha}^t)_{t\in\mathbb{R}}$ on M is isomorphic to the suspension of its first return map $\Phi_{\alpha,\theta} : M_\theta^a \to M_\theta^a$. For every $(i,j) \in J$, there exists a polynomial $p_{i,N}^{(j)}(\alpha,\mathbf{s})$ for $\mathbf{s} \in \mathbb{R}^a$ such that return map $\Phi_{\alpha,\theta}$ is given by the following:*

In the coordinate of $\mathbf{s} = (s_i^{(j)})$ for $\Gamma\exp(\theta\xi)\exp(\sum_{(i,j)\in J} s_i^{(j)}\eta_i^{(j)}) \in M_\theta^a$,

$$\Phi_{\alpha,\theta}(\mathbf{s}) = \Gamma\exp(\theta\xi)\exp(\sum_{(i,j)\in J}(s_i^{(j)} + \alpha_i^{(j)})\eta_i^{(j)} + [\sum_{(i,j)\in J} s_i^{(j)}\eta_i^{(j)}, X_\alpha] + \sum_{(i,j)\in J^{2-}} p_i^{(j)}(\alpha,s)\eta_i^{(j)}) \tag{3.5}$$

and for $r \in \mathbb{N}$,

$$\Phi^r_{\alpha,\theta}(\mathbf{s}) = \Gamma \exp(\theta\xi) \exp\Big(\sum_{(i,j)\in J} (s_i^{(j)} + r\alpha_i^{(j)})\eta_i^{(j)} + \Big[\sum_{(i,j)\in J} s_i^{(j)}\eta_i^{(j)}, rX_\alpha\Big] + \sum_{(i,j)\in J^{2-}} p_{i,r}^{(j)}(\alpha, s)\eta_i^{(j)}\Big). \quad (3.6)$$

Proof. By (3.3), we have

$$\begin{aligned}\exp\Big(\sum_{(i,j)\in J} s_i^{(j)}\eta_i^{(j)}\Big)\exp(X_\alpha) &= \exp\Big(\sum_{(i,j)\in J} s_i^{(j)}\eta_i^{(j)}\Big)x^{-1}\exp\Big(\sum_{(i,j)\in J}\alpha_i^{(j)}\eta_i^{(j)}\Big)\\ &= x^{-1}\exp\Big(e^{\mathrm{ad}(\xi)}\sum_{(i,j)\in J} s_i^{(j)}\eta_i^{(j)}\Big)\exp\Big(\sum_{(i,j)\in J}\alpha_i^{(j)}\eta_i^{(j)}\Big).\end{aligned}$$

By Baker-Campbell-Hausdorff formula, there exist polynomial $p_i^{(j)}(\alpha, s)$ with

$$\exp\Big(\sum_{(i,j)\in J} s_i^{(j)}\eta_i^{(j)}\Big)\exp(X_\alpha) = x^{-1}\exp\Big(\sum_{(i,j)\in J}(s_i^{(j)} + \alpha_i^{(j)})\eta_i^{(j)} + \Big[\sum_{(i,j)\in J} s_i^{(j)}\eta_i^{(j)}, X_\alpha\Big] + \sum_{(i,j)\in J^{2-}} p_i^{(j)}(\alpha, s)\eta_i^{(j)}\Big)$$

Since $x \in \Gamma$, we conclude

$$\begin{aligned}&\Gamma\exp(\theta\xi)\exp\Big(\sum_{(i,j)\in J} s_i^{(j)}\eta_i^{(j)}\Big)\exp(X_\alpha)\\ &= \Gamma\exp(\theta\xi)\exp\Big(\sum_{(i,j)\in J}(s_i^{(j)} + \alpha_i^{(j)})\eta_i^{(j)} + \Big[\sum_{(i,j)\in J} s_i^{(j)}\eta_i^{(j)}, X_\alpha\Big] + \sum_{(i,j)\in J^{2-}} p_i^{(j)}(\alpha, s)\eta_i^{(j)}\Big)\end{aligned}$$

The formula implies that $t = 1$ is a return time of the restriction of the flow to $M^a_\theta \subset M$. The formula for $r \in \mathbb{N}$ follows from induction. □

3.2.1 Kirillov theory and Classification

Kirillov's theory yields the complete classification of irreducible unitary representation of N. All the irreducible unitary representation of nilpotent Lie groups are parametrized by the coadjoint orbits $\mathcal{O} \subset \mathbf{n}^*$. A polarizing (or maximal subordinate) subalgebra for l is a maximal isotropic subspace $\mathbf{m} \subset \mathbf{n}$ for the l which is also a subalgebra of $\mathbf{n}$. It is well known that for any $l \in \mathbf{n}^*$ there exists a polarizing subalgebra $\mathbf{m}$ for a nilpotent Lie algebra $\mathbf{n}$. (See [GC90], Theorem 1.3.3) Let $\mathfrak{m}$ be polarizing subalgebra for a linear form $l \in \mathfrak{n}^*$. Then, the character $\chi_{l,\mathbf{m}} : \exp \mathbf{m} \to S^1$ is defined

$$\chi_{l,\mathbf{m}}(\exp Y) = e^{2\pi i l(Y)}.$$

To a pair $\Lambda = (l, \mathbf{m})$, we associate the the unitary representation

$$\pi_\Lambda = \mathrm{Ind}_{\exp \mathbf{m}}^N(\chi)$$

where induced representation σ is defined by

$$\sigma(x)f(g) = f(g \cdot x), \;\; x \in N, \;\; f \in H_{\pi_{l,\mathbf{m}}}.$$

These unitary representations are irreducible upto equivalence, and all unitary irreducible representations are obtained in this way. It is known that l and l' belong to the same coadjoint orbit if and only if $\pi_{l,\mathbf{m}}$ and $\pi_{l',\mathbf{m}'}$ are unitarily equivalent and $\pi_{l,\mathfrak{m}}$ is irreducible whenever $\mathbf{m}$ is maximal subordinate for l. For convenience, we abuse the notation: $\Lambda \in \mathcal{O} \iff l \in \mathcal{O}$ for $\Lambda = (l, \mathbf{m})$ and $\pi_\Lambda \simeq \pi_{\Lambda'}$ if l and l' are in same coadjoint orbit.

Since the action of N on M preserves the measure μ, we obtain a unitary representation π of N. The regular representation of $L^2(M)$ of N decomposes as a countable direct sum (or direct integral) of irreducible, unitary representation H_π, which occur with at most finite multiplicity

$$L^2(M, d\mu) = \bigoplus_{i \in \mathbb{N}} H_{\pi_i}. \tag{3.7}$$

The derived representation π_* of a unitary representation π of N on a Hilbert space H_π is the Lie algebra representation of $\mathbf{n}$ on H_π defined as follows. For every $X \in \mathbf{n}$,

$$\pi_*(X) = \lim_{t \to 0} (\pi(\exp tX) - I)/t. \tag{3.8}$$

We recall that a vector $v \in H_\pi$ is C^∞*-vectors* in H_π for representation π if the function $g \in N \mapsto \pi(g)v \in H_\pi$ is of class C^∞ as a function on N with values in a Hilbert space.

Lemma 3.7. *[FF07, Lemma 3.4] As a topological vector space*

$$C^\infty(H_\pi) = \mathcal{S}(\mathbb{R}, C^\infty(H'))$$

where $\mathcal{S}(\mathbb{R}, C^\infty(H'))$ is Schwartz space.

Suppose that $\mathbf{n} = \mathbb{R}X \oplus \mathfrak{I}$ and $N = \mathbb{R} \ltimes N'$ with a normal subgroup N' of

N. Let π' be a unitary irreducible representation of N' on a Hilbert space H'. Each irreducible representation H_π is unitarily equivalent to $L^2(\mathbb{R}, H', dx)$, and derived representation of π_* of the induced representation $\pi = \mathrm{Ind}_{N'}^{N}(\pi')$ has the following description.

For $f \in L^2(\mathbb{R}, H', dx)$, the group $\mathbb{R}$ acts by translations and its representation is polynomial in the variable x.

$$(\pi_*(Y)f)(x) = \iota P_Y(x)f(x) = \iota \sum_{j=0}^{k} \frac{1}{j!}(\Lambda \circ \mathrm{ad}_X^j Y)x^j f(x). \tag{3.9}$$

For any $Y \in \mathbf{n}$, we define its degree $d_Y \in \mathbb{N}$ with respect to the representation $\pi_*(Y)$ to be the degree of polynomial. Let $(d_1, \cdots, d_a)$ be the degrees of the elements $(Y_1, \cdots, Y_a)$ respectively. The degree of representation π is defined as the maximum of the degrees of the elements of any basis.

3.3 The cohomological equation

In this section, we prove a priori Sobolev estimate on the Green's operator for the cohomological equation $Xu = f$ of nilflow with generator X. We estimate bound of Green's operator on Sobolev norm and on scaling of invariant distributions.

3.3.1 Distributions and Sobolev space

Let $L^2(M)$ be the space of complex-valued, square integrable functions on M. Given ordered basis $\mathcal{F}$ of $\mathbf{n}$, the *transverse Laplace-Beltrami* operator is second-order differential operator defined by

$$\Delta_{\mathcal{F}} = -\sum_{i=1}^{a} Y_i^2,\ Y_i \in \mathfrak{I}.$$

For any $\sigma \geq 0$, let $|\cdot|_{\sigma,\mathcal{F}}$ be the transverse Sobolev norm defined as follows: for all functions $f \in C^\infty(M)$, let

$$|f|_{\sigma,\mathcal{F}} := \left\|(I+\Delta_{\mathcal{F}})^{\frac{\sigma}{2}} f\right\|_{L^2(M).}$$

Equivalently,

$$|f|_{\sigma,\mathcal{F}} = (\|f\|_2^2 + \sum_{1\leq m\leq\sigma} \|Y_{j_1}\cdots Y_{j_m} f\|_2^2)^{\frac{1}{2}}.$$

The completion of $C^\infty(M)$ with respect to the norm $|\cdot|_{\sigma,\mathcal{F}}$ is denoted $W^\sigma(M,\mathcal{F})$ and the distributional dual space (as a space of functional with values in H') to $W^\sigma(M)$ is denoted

$$W^{-\sigma}(M,\mathcal{F}) := (W^\sigma(M,\mathcal{F}))'.$$

We denote $C^\infty(H_\pi)$ the space of C^∞ vectors of the irreducible unitary representation π. Following notation in (3.7), let $W^\sigma(H_\pi) \subset H_\pi$ be the Sobolev space of vectors, endowed with the Hilbert space norm in the maximal domain of the essential self-adjoint operator $(I+\pi_*(\Delta_{\mathcal{F}}))^{\frac{\sigma}{2}}$. I.e, for every $f \in C^\infty(H_\pi)$ and $\sigma > 0$,

$$|f|_{\sigma,\mathcal{F}} := \left(\int_{\mathbb{R}} \left\|(1+\pi_*(\Delta_{\mathcal{F}}))^{\frac{\sigma}{2}} f(x)\right\|_{H'}^2 dx\right)^{1/2},$$

where $\pi_*(\Delta_{\mathcal{F}})$ is determined by derived representations.

Definition 3.8. *For any $X \in \mathbf{n}$, the space of X-invariant distributions for the representation π is defined as the space $\mathcal{I}_X(H_\pi)$ of all distributional solutions $D \in D'(H_\pi)$ of the equation $\pi_*(X)D = XD = 0$. Let*

$$\mathcal{I}_X^\sigma(H_\pi) := \mathcal{I}_X(H_\pi) \cap W^{-\sigma}(H_\pi)$$

be the subspace of invariant distributions of order at most σ on $\mathbb{R}^+$.

3.3.2 A priori estimates

The distributional obstruction to the existence of solutions of the cohomological equation

$$Xu = f, \quad f \in C^\infty(H_\pi)$$

in a irreducible unitary representation H_π is the normalized X-invariant distribution.

Definition 3.9. *For any $X \in \mathbf{n}$, the space of X-invariant distributions for the representation π is the space $\mathcal{I}_X(H_\pi)$ of all distributional solutions $D \in D'(H_\pi)$ of the equation $\pi_*(X)D = XD = 0$. Let*

$$\mathcal{I}_X^\sigma(H_\pi) := \mathcal{I}_X(H_\pi) \cap W^{-\sigma}(H_\pi)$$

be the subspace of invariant distributions of order at most $\sigma \in \mathbb{R}^+$.

By Lemma 3.5 of [FF07], each invariant distribution D has a Sobolev order equal to $1/2$, i.e $D \in W^{-\sigma}(H_\pi)$ for any $\sigma > 1/2$.

For all $\sigma > 1$, let $\mathcal{K}^\sigma(H_\pi) = \{f \in W^\sigma(H_\pi) \mid D(f) = 0 \in C^\infty(H_\pi),$ for any $D \in W^{-\sigma}(H_\pi)\}$ be the kernel of the X-invariant distribution on the Sobolev space $W^\sigma(H_\pi)$. The Green's operator $G_X : C^\infty(H_\pi) \to C^\infty(H_\pi)$ with

$$G_X f(t) = \int_{-\infty}^{t} f(s)ds$$

is well-defined on the kernel of distribution $\mathcal{K}^\infty(H_\pi)$ on $C^\infty(H_\pi)$. If $f \in \mathcal{K}^\sigma(H_\pi)$, then $\int_{\mathbb{R}} f(t)dt = 0 \in C^\infty(H')$ and

$$G_X f(t) = \int_{-\infty}^{t} f(s)ds = -\int_{t}^{\infty} f(s)ds \in C^\infty(\mathbb{R}, H').$$

Now we define generalized (complex-valued) invariant distribution on smooth vector $C^\infty(H_\pi)$.

Lemma 3.10. *The invariant distribution is generalized in the following sense. For every $f \in C^\infty(H_\pi) \subset L^2(\mathbb{R}, H')$, there exists a functional $\ell : C^\infty(H_\pi) \to \mathbb{C}$ such that*

$$D(f) = \int_{\mathbb{R}} \ell(f(t))dt. \tag{3.10}$$

Furthermore, $\ell \in W^{-s}(H_\pi)$ for $s > 1/2$ and

$$\int_{\mathbb{R}} \ell(f(t))dt = \ell\left(\int_{\mathbb{R}} f(t)dt\right). \tag{3.11}$$

Proof. We construct a linear functional ℓ in (3.10) as follows. Let $\chi \in C_0^\infty(\mathbb{R})$ be a smooth function with compact support with unit integral over $\mathbb{R}$. Given an invariant distribution $D \in \mathcal{I}_X(H_\pi)$, let us define $\ell(v) = D(f_v)$ for $f_v = \chi v$ and $v \in C^\infty(H')$.

Firstly, we prove that ℓ is well-defined. Let $\chi_1 \neq \chi_2 \in C_0^\infty(\mathbb{R})$ be functions with compact support such that $\int_{\mathbb{R}} \chi_1(t)dt = \int_{\mathbb{R}} \chi_2(t)dt = 1$. Note that there exists $\psi \in C_0^\infty(\mathbb{R})$ such that $\chi_1 - \chi_2 = \psi'$ with $\psi(t) = \int_{-\infty}^t (\chi_1(x) - \chi_2(x))dx$.

Then, we have

$$\chi_1(t)v - \chi_2(t)v = \frac{d}{dt}(\psi(t)v) = \pi_*(X)(\psi(t)v) \in C^\infty(H_\pi),$$

and $\chi_1(t)v - \chi_2(t)v$ is a X-coboundary for every $v \in C^\infty(H')$. Hence, $D(\chi_1(t)v - \chi_2(t)v) = 0$, which implies that $D(\chi_1(t)v) = D(\chi_2(t)v)$. Therefore, $\ell(v)$ does not depend on the choice of χ and the functional ℓ is well-defined.

Next, we verify that ℓ is a distribution on $C^\infty(H')$. Assume that $(v_n)_{n\in\mathbb{R}}$ is a sequence converging to v in $C^\infty(H')$. Then,

$$|\ell(v_n) - \ell(v)| = |D(\chi(t)v_n) - D(\chi(t)v)| = |D(\chi(t)(v_n - v))|.$$

For $s > 1/2$,

$$|D(\chi(t)(v_n - v))| \leq \|D\|_{-s} \|\chi(t)(v_n - v)\|_{W^s(H')}$$

In the representation, $\pi_*(Y_i) = \sum p_i(t)$ acts multiplication of polynomial $p_i(t)$ on $L^2(\mathbb{R}, H')$. By definition of Sobolev norm in representation, there exists a non-zero constant $C := C(\chi, p_1, \cdots p_s) = \max_{\substack{t\in\mathbb{R}, j_1+\cdots j_d = s \\ 0\leq j_i \leq s, 1\leq i\leq d\leq s}} \{\chi(t)p_1^{j_1}(t)\cdots p_d^{j_d}(t)\}$ such that

$$\|\chi(t)(v_n - v)\|_{W^s(H_\pi)} \leq C \|v_n - v\|_{W^s(H')}.$$

Therefore, ℓ is continuous on $C^\infty(H')$ and we prove the statement. Furthermore,

$$\|\ell\|_{-s} := \sup_{\|v\|=1} \frac{|\ell(v)|}{\|v\|_{H'}} \le C\,\|D\|_{-s}\,,$$

and $\ell \in W^{-s}(H')$.

To prove equality (3.11), for any $f(t) \in C^\infty(H_\pi)$, we observe $f(t) - \chi(t)\left(\int_{\mathbb{R}} f(t)dt\right)$ has zero averages. Since invariant distribution D is invariant under translation, we obtain $D(f - \chi(t)\left(\int_{\mathbb{R}} f(x)dx\right)) = 0$. Hence,

$$\int \ell(f(t))dt = D(f) = D\left(\chi(t)\int_{\mathbb{R}} f(x)dx\right) = \ell\left(\int_{\mathbb{R}} f(t)dt\right).$$

□

Let $\mathcal{O}$ be any coadjoint orbit of maximal rank. For all $(X, Y) \in \mathbf{n} \times \mathbf{n}_{k-1}$ and $\Lambda \in \mathcal{O}$, the skew-symmetric bilinear form

$$B_\Lambda(X, Y) = \Lambda([X, Y]).$$

Let

$$\begin{aligned} &\delta_{\mathcal{O}}(X, Y) := |B_\Lambda(X, Y)|, \text{ for any } \Lambda \in \mathcal{O} \\ &\delta_{\mathcal{O}}(X) := \max\{\delta_{\mathcal{O}}(X, Y) \mid Y \in \mathbf{n}_{k-1} \text{ and } \|Y\| = 1\}. \end{aligned} \tag{3.12}$$

Here we quote known estimates:

Lemma 3.11 (Lemma 2.5, [FF07]). *Let $X \in \mathbf{n}$ and $Y \in \mathbf{n}_{k-1}$ be any operator such that $B_l(X, Y) \neq 0$. There exists a codimension 1 ideal $\mathbf{n}' \subset \mathbf{n}$ with $X \notin \mathbf{n}'$*

and a unitary irreducible representation π *with the following properties. The derived representation* π_* *of the Lie algebra* $\mathbf{n}$ *satisfies*

$$\pi_*(X) = \frac{d}{dx}, \quad \pi_*(Y) = 2\pi\iota B_l(X,Y)xId_{H'} \text{ on } L^2(\mathbb{R}, H', dx).$$

Theorem 3.12 (Theorem 3.6, [FF07])**.** *Let* $\delta_{\mathcal{O}} > 0$*, and let* π *be an irreducible representation of* $\mathbf{n}$ *on a Hilbert space* H_π*. If* $f \in W^s(H_\pi)$*,* $s > 1$ *and* $D(f) = 0$ *for all* $D \in \mathcal{I}_X(H_\pi)$*, then* $G_X f \in W^r(H_\pi)$*, for all* $r < (s-1)/k$ *and there exists a constant* $C := C(X, k, r, s)$*, such that*

$$|G_X f|_{r,\mathcal{F}} \leq C \max\{1, \delta_{\mathcal{O}}(X)^{-(k-1)r-1)}\}|f|_{s,\mathcal{F}}.$$

3.3.3 Rescaling method

Definition 3.13. *The deformation space of a* k*-step nilmanifold* M *is the space* $T(M)$ *of all adapted bases of the Lie algebra* $\mathbf{n}$ *of the group* N*.*

The renormalization dynamics is defined as the action of diagonal subgroup of the Lie group on the deformation space. Let $\rho := (\rho_1, \cdots, \rho_a) \in (\mathbb{R}^+)^a$ be any vector with rescaling condition $\sum_{i=1}^a \rho_i = 1$. Then, there exist a one-parameter subgroup $\{A_t^\rho\}$ of the Lie group of $SL(a+1, \mathbb{R})$ defined as follows:

$$A_t^\rho(X, \cdots, Y_i, \cdots) = (e^t X, \cdots, e^{-\rho_i t} Y_i, \cdots). \tag{3.13}$$

The renormalization group $\{A_t^\rho\}$ preserves the set of all generalized Jordan basis.

However, it is not a group of automorphism of the Lie algebra. Therefore, the dynamics induced by the renormalization group on the deformation space is trivial (It has no recurrent orbits).

Definition 3.14. *Given any adapted basis* $\mathcal{F} = (X, Y_i)$, rescaled basis $\mathcal{F}(t) = (X(t), Y_i(t)) = \{e^t X, \cdots, e^{-\rho_i t} Y_i, \cdots\}$ *of* $\mathcal{F}$ *is a basis of Lie algebra* $\mathbf{n}$ *satisfying* (3.13).

Let $(d_1, \cdots, d_i)$ be the degrees of the elements $(Y_1, \cdots, Y_i)$ respectively. For any $\rho = (\rho_1, \cdots, \rho_a) \in \mathbb{R}^a$, let

$$\lambda_{\mathcal{F}}(\rho) := \min_{i:d_i \neq 0} \left(\frac{\rho_i}{d_i} \right) \tag{3.14}$$

Definition 3.15. *Scaling factor* ρ_i *is called Homogeneous if growth of scaling factor* ρ_i *is proportional to degree of element* Y_i*. That is, under homogeneous scaling,* $\lambda_{\mathcal{F}}(\rho) = \frac{\rho_i}{d_i}$ *for all* i*.*

For all $i = 1, \cdots, a$, denote

$$\Lambda_i^{(j)}(\mathcal{F}) := (\Lambda \circ \mathrm{ad}^j(X))(Y_i) \tag{3.15}$$

be the coefficients appearing in (3.9) and set

$$|\Lambda(\mathcal{F})| := \sup_{(i,j):1 \leq i \leq a, 0 \leq j \leq d_i} \left| \frac{\Lambda_i^{(j)}(\mathcal{F})}{j!} \right|. \tag{3.16}$$

Let $\mathfrak{U}(\mathbf{n})$ be the enveloping algebras of $\mathbf{n}$. The generator δ is the derivation on

$\mathfrak{U}(\mathbf{n}')$ obtained by extending the derivation $\mathrm{ad}(X)$ of $\mathbf{n}'$ to $\mathfrak{U}(\mathbf{n}')$. From nilpotency of $\mathbf{n}$ it follows that for any $L \in \mathfrak{U}(\mathbf{n}')$ there exists a first integer $[L]$ such that $\delta^{[L]+1}L = 0$.

Lemma 3.16. *For each element $L \in \mathfrak{I}$ with degree $[L] = i$, there exists $Q_j \in \mathfrak{U}(\mathbf{n})$ such that $\pi_*(L) = \sum_{j=0}^{i} \frac{1}{j!}\pi_*(Q_j)x^j$ and $[Q_j] = [L] + 1 - j$.*

Proof. Firstly, we fix elements X and Y as stated in the Lemma 3.11. For convenience, we normalize the constant of $\pi_*(Y)$ by 1. That is, there exist $X, Y \in \mathbf{n}$ such that

$$\pi_*(X) = \frac{d}{dx}, \quad \pi_*(Y) = x.$$

Now, we will replace the expansion of $\pi_*(L) := \sum_{j=0}^{i} \frac{1}{j!}\Lambda_L^{(j)}(\mathcal{F})x^j$ by choosing elements Q_i in enveloping algebra $\mathfrak{U}(\mathbf{n})$.

For the coefficient of top degree, denote $Q_i = \frac{1}{i!}\mathrm{ad}_X^i(L) \in \mathbf{n}$. For degree $i-1$, we set $Q_{i-1} = \mathrm{ad}_X^{i-1}(L) - Q_iY \in \mathfrak{U}(\mathbf{n})$ such that

$$\pi_*(Q_{i-1}) = \pi_*(\mathrm{ad}_X^{i-1}(L)) - \pi_*(Q_i)\pi_*(Y) = \Lambda_L^{(i-1)}(\mathcal{F}).$$

Repeating this process up to degree 0, there exist $\exists Q_l \in \mathfrak{U}(\mathbf{n})$ such that for $0 < l < i$

$$\pi_*(Q_l) = \pi_*(\mathrm{ad}_X^l(L)) - \frac{1}{l!}\sum_{j=l+1}^{i} \pi_*(Q_j)\pi_*(Y)^{j-l}$$

and

$$\pi_*(Q_0) = \pi_*(L) - \frac{1}{l!}\sum_{l=1}^{i} \pi_*(Q_l)\pi_*(Y)^l = \Lambda_L^{(j)}(\mathcal{F}).$$

□

Recall that for two self-adjoint operators A and B, $A \geq B$ if

$$\langle Au, u\rangle \geq \langle Bu, u\rangle, \quad u \in H.$$

Also, $A^2 \geq B^2 \iff \|Au\| \geq \|Bu\|$.

Lemma 3.17. *For any $r \geq 1$ and $a \geq 1$, there exists $C(a,r) > 0$ such that*

$$\Delta(t)^{2r} \leq C(a,r) \sum_{i=1}^{a} Y_i(t)^{4r}. \tag{3.17}$$

Proof. We prove it by induction. If $r = 1$,

$$\Delta(t)^2 = (\sum_{i=1} Y_i(t)^2)^2 = \sum_{i=1} Y_i(t)^4 + \sum_{i \neq j} Y_i(t)^2 Y_j(t)^2.$$

Note that

$$Y_i(t)^2 Y_j(t)^2 + Y_j(t)^2 Y_i(t)^2 \leq Y_i(t)^4 + Y_j(t)^4.$$

Then,

$$\Delta(t)^2 \leq (a+1) \sum_{i=1} Y_i(t)^4.$$

Assume that the statement holds for large r. Then, there exists $C_1(a,r)$

$$\begin{aligned}\Delta(t)^{2(r+1)} &\leq C_1(a,r) \left(\sum_{i=1}^{a} Y_i(t)^{4r}\right) \Delta(t)^2 \\ &\leq C_1(a,r)(a+1) \left(\sum_{i=1}^{a} Y_i(t)^{4r}\right) \sum_{i=1} Y_i(t)^4.\end{aligned}$$

Similarly, we have

$$Y_i(t)^{4r}Y_j(t)^4 + Y_j(t)^4Y_i(t)^{4r} \leq Y_i(t)^{4(r+1)} + Y_j(t)^{4(r+1)}.$$

Therefore, setting $C_2(a,r) = C_1(a,r)(a+1)(a+2)$,

$$\Delta(t)^{2(r+1)} \leq C_2(a,r)\sum_{i=1}^{a} Y_i(t)^{4(r+1)}.$$

We finish the proof. $\square$

For cohomological equation $X(t)u = f$, denote its Green's operator $G_{X(t)}$. The following theorem states an estimate for rescaled version of theorem 3.12.

Theorem 3.18. *Let $s > 2r(k+1)+1/2$. For any $f \in \mathcal{K}^s(M)$, there exists $C_{r,k,s} > 0$ such that the following holds: for all $t \in \mathbb{R}$,*

$$|G_{X(t)}f|_{r,\mathcal{F}(t)} \leq C_{r,k,s}e^{-(1-\lambda_{\mathcal{F}})t}\max\{1,(2\pi\delta_{\mathcal{O}})^{-2r(k+1)}\}|f|_{s,\mathcal{F}(t)}.$$

Proof. Firstly, we prove the bound of Green's operator with Sobolev norm with a fixed operator $Y(t)$. By Cauchy-Schwarz inequality,

$$\begin{aligned}
&\left\|Y(t)^l G_{X(t)}f\right\|^2 \\
&\leq \int_0^\infty \left(|2\pi\delta_{\mathcal{O}}(t)x|^l \int_x^\infty e^{-t}\left\|f(s)\right\|_{H'} ds\right)^2 dx + \int_{-\infty}^0 \left(|2\pi\delta_{\mathcal{O}}(t)x|^l \int_{-\infty}^x e^{-t}\left\|f(s)\right\|_{H'} ds\right)^2 dx \\
&\leq \int_0^\infty |2\pi\delta_{\mathcal{O}}(t)x|^{2l}\int_x^\infty e^{-2t}\left\|f(s)\right\|_{H'}^2 dsdx + \int_{-\infty}^0 |2\pi\delta_{\mathcal{O}}(t)x|^{2l}\int_{-\infty}^x e^{-2t}\left\|f(s)\right\|_{H'}^2 dsdx.
\end{aligned}$$

For all $\alpha > 1$, we set

$$C_{\alpha,l}^2 = \int_0^\infty (2\pi x)^{2l} \left(\int_x^\infty (1 + (4\pi^2 s^2))^{-l+\alpha} ds \right) dx + \int_{-\infty}^0 (2\pi x)^{2l} \left(\int_{-\infty}^x (1 + (4\pi^2 s^2))^{-l+\alpha} ds \right) dx < \infty. \tag{3.18}$$

By Hölder's inequality and change of variables,

$$\left\| Y(t)^l G_{X(t)} f \right\| \le C_{\alpha,l} e^{-(1-\lambda_{\mathcal{F}})t} \left(\frac{1}{2\pi\delta_{\mathcal{O}}} \right)^l \left\| (I - Y(t)^2)^{\frac{l+\alpha}{2}} f \right\|. \tag{3.19}$$

Now, let $L(t)$ be a rescaled element of $L \in \mathcal{F}$. By Lemma 3.16, there exists $Q_i(t) \in \mathfrak{U}(\mathbf{n})$ such that

$$\pi_*(L(t)) = \sum_{j=0}^{[L]} \frac{1}{j!} \pi_*(Q_j(t)) \pi_*(Y(t))^j. \tag{3.20}$$

Then, for Green's operator $G_{X(t)}$, we obtain

$$\begin{aligned} \pi_*(L(t)) G_{X(t)}(f) &= \sum_{j=0}^{[L]} \frac{1}{j!} \pi_*(Q_j(t)) \pi_*(Y(t))^j G_{X(t)}(f) \\ &= \sum_{j=0}^{[L]} \frac{1}{j!} \pi_*(Y(t))^j G_{X(t)}(\pi_*(Q_j(t)) f). \end{aligned} \tag{3.21}$$

Combining (3.19) and (3.21),

$$\begin{aligned} \left\| L(t) G_{X(t)}(f) \right\| &\le \sum_{j=0}^{[L]} \frac{1}{j!} \left\| Y(t)^j G_{X(t)}(Q_j(t) f) \right\| \\ &\le \sum_{j=0}^{[L]} C_{j,\alpha} e^{-(1-\lambda_{\mathcal{F}})t} \left(\frac{1}{2\pi\delta_{\mathcal{O}}} \right)^{j+1} \left\| (I - Y(t)^2)^{\frac{j+\alpha}{2}} (Q_j(t) f) \right\|. \end{aligned}$$

Note that by binomial formula, there exists $R_j(t) \in \mathfrak{U}(\mathbf{n})$ such that

$$\pi_*(L(t))^{2r} = \left(\sum_{j=0}^{[L]} \frac{1}{j!} Y(t)^j Q_j(t)\right)^{2r} := \sum_{j=0}^{2r[L]} Y(t)^j R_j(t). \tag{3.22}$$

Especially, $R_j(t)$ is product of $Q_i's$ and $[R_j(t)] = 2r([L]+1) - j$.

Specifically, here we assume that $L(t) = (Y_i(t))^{4r}$. Then, by (3.22),

$$\begin{aligned} \left\|Y_i(t)^{4r} G_{X(t)}(f)\right\| &\leq \sum_{j=0}^{4r[Y_i]} C_{j,\alpha} e^{-(1-\rho_Y)t} \left(\frac{1}{2\pi\delta_{\mathcal{O}}}\right)^{j+1} \left\|(I - Y(t)^2)^{\frac{j+\alpha}{2}} R_j(t) f\right\| \\ &\leq C(\alpha, r) e^{-(1-\rho_Y)t} \max\{1, {\delta_{\mathcal{O}}}^{-4r[Y_i]+1}\} |f|_{\alpha+4r([Y_i]+1), \mathcal{F}(t)}. \end{aligned} \tag{3.23}$$

Therefore, combining with Lemma 3.17

$$\begin{aligned} \left\|\pi_*(\Delta(t)^{2r}) G_{X(t)}(f)\right\| &\leq C(a, r) \sum_{i=1}^{a} \left\|\pi_*(Y_i(t)^{4r}) G_{X(t)}(f)\right\| \\ &\leq C(a, r, \alpha) \sum_{i=1}^{a} e^{-(1-\rho_Y)t} \max\{1, {\delta_{\mathcal{O}}}^{-4r([Y_i]+1)}\} |f|_{\alpha+4r([Y_i]+1), \mathcal{F}(t)} \\ &\leq C(a, r, \alpha) e^{-(1-\rho_Y)t} \max\{1, {\delta_{\mathcal{O}}}^{-4r(k+1)}\} |f|_{\alpha+4r(k+1), \mathcal{F}(t)}. \end{aligned}$$

Since $[\Delta^r] \leq 2kr$, there exists $C' = C'(a, \alpha, r, l, X) > 0$ such that

$$|G_{X(t)} f|_{2r, \mathcal{F}(t)} \leq C' e^{-(1-\rho_Y)t} \max\{1, (2\pi\delta_{\mathcal{O}})^{-4r(k+1)}\} |f|_{4(k+1)r+\alpha, \mathcal{F}(t)}.$$

By interpolation, for all $s > 2r(k+1) + 1/2$, there exists a constant $C_{r,s} := C_{r,s}(k, X) > 0$ such that

$$|G_{X(t)}f|_{r,\mathcal{F}(t)} \leq C_{r,s}e^{-(1-\rho_Y)t}\max\{1,(2\pi\delta_{\mathcal{O}})^{-2r(k+1)}\}|f|_{s,\mathcal{F}(t)}.$$

□

3.3.4 Scaling of invariant distribution

In this section, we introduce the Lyapunov norm and compare bounds between Sobolev dual norm and Sobolev Lyapunov norm of invariant distribution in every irreducible, unitary representation.

For all $t \in \mathbb{R}$ and $\lambda := \lambda_{\mathcal{F}}(\rho)$ defined in (3.14), let the operator $U_t : L^2(\mathbb{R}, H') \to L^2(\mathbb{R}, H')$ be the unitary operator defined as follows:

$$(U_t f)(x) = e^{-\frac{\lambda}{2}t}f(e^{-\lambda t}x). \tag{3.24}$$

We consider the comparison of the norm estimate on the scaling of invariant distributions

$$|D|_{-r,\mathcal{F}(t)} = \sup_{f\in W^r}\{|D(f)| : \|f\|_{r,\mathcal{F}(t)} = 1\}$$

in terms of $|D|_{-r,\mathcal{F}}$.

Theorem 3.19. *For $r \geq 1$ and $s > r(k+1)$, there exists a constant $C_{r,s} > 0$ such that for all $t \in \mathbb{R}$, the following bound holds:*

$$\|U_t f\|_{r,\mathcal{F}(t)} \leq C_{r,s}\|f\|_{s,\mathcal{F}}.$$

Proof. Assume the same hypothesis for $L \in \mathcal{F}$ and $L(t)$ in the proof of Theorem 3.18. By Lemma 3.16, there exists $(i-j+1)$th order $Q_j \in \mathfrak{U}(\mathbf{n})$ with

$$U_t^{-1} L(t) U_t = x^i Q_i + e^{-\lambda t} x^{i-1} Q_{i-1} + e^{-2\lambda t} x^{i-2} Q_{i-2} + \cdots + e^{-i\lambda t} Q_o.$$

Then, there exists $C_w > 0$ such that

$$\begin{aligned} \left\| U_t^{-1} L(t) U_t f \right\| &\leq \sum_{j=0}^{i} \left\| e^{-(i-j)\lambda t} x^j Q_j f \right\| \\ &\leq C_w \max_{0 \leq j \leq i} \left\| Y^j Q_j f \right\| \\ &\leq C_w |f|_{[L]+1,\mathcal{F}}. \end{aligned}$$

Since $[L] \leq k$, by unitarity

$$|U_t f|_{1,\mathcal{F}(t)} \leq C_1 |f|_{k+1,\mathcal{F}}. \tag{3.25}$$

Hence, for some $s > r(k+1)$,

$$|U_t f|_{r,\mathcal{F}(t)} \leq C_{r,s} |f|_{s,\mathcal{F}}. \tag{3.26}$$

□

Theorem 3.20. *For $r \geq 1, s > r(k+1)$, there exists a constant $C_{r,s} > 0$ such that* *for all λ, the invariant distribution defined in (3.10) satisfies*

$$|D|_{-s,\mathcal{F}} \leq C_{r,s} e^{-\frac{\lambda}{2}t} |D|_{-r,\mathcal{F}(t)}.$$

Proof. Recall the functional ℓ defined in the Lemma 3.10. For $f \in C^\infty(H_\pi)$,

$$\begin{aligned} D(U_t f) &= \int_{\mathbb{R}} \ell(e^{-\frac{\lambda}{2}t} f(e^{-\lambda t}x))dx \\ &= \int_{\mathbb{R}} e^{-\frac{\lambda}{2}t} \ell(f(e^{-\lambda t}x))dx \\ &= e^{\frac{\lambda}{2}t} \int_{\mathbb{R}} \ell(f(y))dy \\ &= e^{\frac{\lambda}{2}t} D(f). \end{aligned}$$

Then by unitarity,

$$|D|_{-r,\mathcal{F}(t)} = \sup_{f \neq 0} \frac{|D(f)|}{|f|_{r,\mathcal{F}(t)}} = \sup_{f \neq 0} \frac{|D(U_t f)|}{|U_t f|_{r,\mathcal{F}(t)}} \geq \sup_{f \neq 0} \frac{e^{\frac{\lambda}{2}t}|D(f)|}{C_{r,s}|f|_{s,\mathcal{F}}} = e^{\frac{\lambda}{2}t}|D|_{-s,\mathcal{F}}$$

and

$$|D|_{-s,\mathcal{F}} \leq C_{s,r} e^{-\frac{\lambda}{2}t} |D|_{-r,\mathcal{F}(t)}.$$

□

Definition 3.21 (Lyapunov norm)**.** *For any basis $\mathcal{F}$ and all $\sigma > 1/2$, define Lyapunov norm*

$$\|D\|_{-\sigma,\mathcal{F}} := \inf_{\tau \geq 0} e^{-\frac{\lambda_{\mathcal{F}}(\rho)}{2}\tau} |D|_{-\sigma,\mathcal{F}(\tau)}. \tag{3.27}$$

The following lemma is immediately from the definition of the norm.

Lemma 3.22. *For all $t \geq 0$, we have*

$$\|D\|_{-\sigma,\mathcal{F}} \leq e^{-\frac{\lambda_{\mathcal{F}}(\rho)}{2}t} \|D\|_{-\sigma,\mathcal{F}(t)} \,.$$

Proof. By definition of the norm,

$$\begin{aligned}\|D\|_{-\sigma,\mathcal{F}} &= \inf_{\tau\geq 0} e^{-\frac{\lambda_{\mathcal{F}}(\rho)}{2}\tau} |D|_{-\sigma,\mathcal{F}(\tau)} \\ &= e^{-\frac{\lambda_{\mathcal{F}}(\rho)}{2}t} \inf_{\tau+t\geq 0} e^{-\frac{\lambda_{\mathcal{F}}(\rho)}{2}\tau} |D|_{-\sigma,\mathcal{F}(t+\tau)} \leq e^{-\frac{\lambda_{\mathcal{F}}(\rho)}{2}t} \|D\|_{-\sigma,\mathcal{F}(t)} \,.\end{aligned}$$

□

We conclude this section by introducing useful inequality that follows from the theorem 3.20,

$$C_{r,s}^{-1} |D|_{-s,\mathcal{F}} \leq \|D\|_{-r,\mathcal{F}} \leq |D|_{-r,\mathcal{F}}. \tag{3.28}$$

3.4 A Sobolev trace theorem

In this section, we prove a Sobolev trace theorem for nilpotent orbits. According to this theorem, uniform norm of an ergodic integral is bounded in terms of the average width of the orbit segment times the transverse Sobolev norms of the function, with respect to a given basis of the Lie algebra.

3.4.1 Sobolev a priori bounds

Assume $\mathcal{F}(t) = (X(t), Y(t))$ is rescaled basis. For any $x \in M$, let $\phi_{x,t} : \mathbb{R} \times \mathbb{R}^a \to M$ be the local embedding defined by

$$\phi_{x,t}(\tau, \mathbf{s}) = x \exp(\tau X(t)) \prod_{i=1}^{a} \exp(s_i Y_i(t)), \ \mathbf{s} = (s_i)_{i=1}^{a}$$

Lemma 3.23. *For any $x \in M$, $t \geq 0$, and $f \in C^\infty(M)$, we have*

$$\partial_{s_i} f \circ \phi_{x,t}(\tau, \boldsymbol{s}) = S_i f \circ \phi_{x,t}(\tau, \boldsymbol{s}), \quad S_i = Y_i(t) + \sum_{l>i}^{a} q_l(s,t) Y_l(t) \in \mathbf{n}$$

where q is polynomial in s of degree at most $k-1$ and $|q_l(s,t)| \leq |q_l(s,0)|$ for all $t \geq 0$.

Proof. Let $\mathbf{s} + h_i$ denote sequence with $(\mathbf{s}+h)_i = s_i + h$ and $(\mathbf{s}+h_i)_j = s_j$, if $i \neq j$.

$$\partial_{s_i} f \circ \phi_{x,t}(\tau, \mathbf{s}) = \lim_{h \to 0} \frac{f \circ \phi_{x,t}(\tau, \mathbf{s}+h_i) - f \circ \phi_{x,t}(\tau, \mathbf{s})}{h}$$

and we plan to rewrite $f \circ \phi_{x,t}(\tau, \mathbf{s}+h_i)$ in suitable way to differentiate.

For fixed i and $j > i$,

$$\exp((s_i + h)Y_i(t))\exp(s_jY_j(t))$$
$$= \exp(s_iY_i(t))\exp(hY_i(t))\exp(s_jY_j(t))$$
$$= \exp(s_iY_i(t))\exp(e^{ad(hY_i(t))}s_jY_j(t))\exp(hY_i(t))$$
$$= \exp(s_iY_i(t))\exp(s_jY_j(t))\exp(\sum_{n=1}^{\infty}\frac{1}{n!}ad^n_{hY_i(t)}s_jY_j(t))\exp(hY_i(t))$$

By Campbell-Hausdorff formula, we set

$$= \exp(s_iY_j(t))\exp(s_iY_j(t))\exp(h(Y_i(t) + [Y_i(t), s_jY_j(t)]) + O(h^2))$$

Choose $j = i + 1$ and observe that all the terms of h are on right side. Iteratively, we will repeat this process from $j = i+1$ to a until all the terms of h pushed back. That is, we conclude

$$\begin{aligned}\phi_{x,t}(\tau, \mathbf{s} + h_i) = \phi_{x,t}(\tau, \mathbf{s})\exp(h(Y_i(t) &+ [Y_i(t), s_{i+1}Y_{i+1}(t)] \\ &+ [[Y_i(t), s_{i+1}Y_{i+1}(t)], s_{i+2}Y_{i+2}(t)] + \cdots + [Y_i(t), \cdots], s_aY_a(t)]\cdots]) \\ &+ O(h^2)).\end{aligned}$$

For convenience, we write coefficient function $q_l(s, t)$ in polynomial degree at most k for s such that

$$\phi_{x,t}(\tau, \mathbf{s} + h_i) = \phi_{x,t}(\tau, \mathbf{s})\exp(h(Y_i(t) + \sum_{l>i}^{a} q_l(s, t)Y_l(t)) + o(h^2)).$$

It concludes the proof by choosing $S_i = Y_i(t) + \sum_{l>i}^{a} q_l(s,t)Y_l(t)$. Commutation in rescaled elements $[Y_i(t), s_jY_j(t)] = s_j e^{-\rho t}Y_k(t)$ also implies $q_l(s,t)$ include exponential term with negative exponent such that it decreases on $t \geq 0$. $\square$

Let $\triangle_{\mathbb{R}^a}$ be the Laplacian operator on $\mathbb{R}^a$ given by $\triangle_{\mathbb{R}^a} = -\sum_{i=1}^{a} \frac{\partial^2}{\partial s_i^2}$. Given an open set $O \subset \mathbb{R}^a$ containing origin, let $\mathcal{R}_O$ be the family of all a-dimensional symmetric rectangles $R \subset [-\frac{1}{2}, \frac{1}{2}]^a \cap O$ that are centered at origin. The *inner width* of the set $O \subset \mathbb{R}^a$ is the positive number $w(O) = \sup\{\mathrm{Leb}(R) \mid R \in \mathcal{R}_O\}$, where Leb is Lebesgue measure on R. The width function of a set $\Omega \subset \mathbb{R} \times \mathbb{R}^a$ containing the line $\mathbb{R} \times \{0\}$ is the function $w_\Omega : \mathbb{R} \to [0,1]$ defined as follows:

$$w_\Omega(\tau) := w(\{\mathbf{s} \in \mathbb{R}^a \mid (\tau, \mathbf{s}) \in \Omega\}), \ \forall \tau \in \mathbb{R}$$

Consider the family $\mathcal{O}_{x,t,T}$ of open sets $\Omega \subset \mathbb{R} \times \mathbb{R}^a$ satisfying two conditions:

$$[0,T] \times \{0\} \subset \Omega \subset \mathbb{R} \times [-\frac{1}{2}, \frac{1}{2}]^a$$

and $\phi_{x,t}$ is injective on the open set $\Omega \subset \mathbb{R}^a$. The *average width* of the orbit segment of rescaled nilflow $\{\phi_{x,t}(\tau, 0) \mid 0 \leq t \leq T\}$

$$w_{\mathcal{F}(t)}(x,T) = \sup_{\Omega \in \mathcal{O}_{x,t,T}} \left(\frac{1}{T} \int_0^T \frac{ds}{w_\Omega(s)} \right)^{-1}. \tag{3.29}$$

is positive number. The following lemma is derived from standard Sobolev embedding theorem under rescaling argument.

Lemma 3.24. *[FF14, Lemma 3.7] Let $I \subset \mathbb{R}$ be an interval, and let $\Omega \subset \mathbb{R} \times \mathbb{R}^a$ be a Borel set containing the segment $I \times \{0\} \subset \mathbb{R} \times \mathbb{R}^a$. For every $\sigma > a/2$, there is a constant $C_s > 0$ such that for all functions $F \in C^\infty(\Omega)$ and all $\tau \in I$, we have*

$$\left(\int_I |F(\tau, 0)| d\tau \right)^2 \leq C_\sigma \left(\int_I \frac{d\tau}{w_\Omega(\tau)} \right) \int_\Omega |(I - \triangle_{\mathbb{R}^a})^{\frac{\sigma}{2}} F(\tau, \mathbf{s})| d\tau d\mathbf{s}.$$

For the rest of section, we prove generalized version of theorem 5.2 of [FFT16]. The following theorem indicates the bound of ergodic average of scaled nilflow $\phi^\tau_{X(t)}$ with width function on general nilmanifolds.

Theorem 3.25. *For all $\sigma > a/2$, there is a constant $C_\sigma > 0$ such that the following holds.*

$$\left| \frac{1}{T} \int_0^T f \circ \phi^\tau_{X(t)}(x) d\tau \right| \leq C_\sigma T^{-\frac{1}{2}} w_{\mathcal{F}(t)}(x, T)^{-\frac{1}{2}} |f|_{\sigma, \mathcal{F}(t)}$$

Proof. Recall that for any self-adjoint operators A and B,

$$(A + B)^2 \leq 2(A^2 + B^2).$$

Since $|s_i| \leq \frac{1}{2}$ and $t \geq 1$, by Lemma 3.23, each q_j is bounded in $\mathbf{s}$ and t. Then, by essentially skew-adjointness of $Y_i(t)$, there exists a large $C > 1$ with

$$\begin{aligned} -S_i^2 &= -(Y_i(t) + \sum_{l>i}^a q_l(s,t) Y_l(t))^2 \\ &\leq -C \sum_{j=i}^a Y_j(t)^2. \end{aligned}$$

Operators on both sides are essentially self-adjoint,

$$(I - \sum_{i=1}^{a} S_i^2)^{\frac{\sigma}{2}} \leq C^{\frac{\sigma}{2}}(I - \sum_{i=1}^{a} Y_i(t)^2)^{\frac{\sigma}{2}}.$$

Thus, there is a constant $C_\sigma > 0$ such that

$$\left\|(I - \triangle_{\mathbb{R}^a})^{\frac{\sigma}{2}} f \circ \phi_{x,t}\right\|_{L^2(\Omega)}^2 \leq C_\sigma \left\|(I - \Delta_{\mathcal{F}(t)})^{\frac{\sigma}{2}} f\right\|_{L^2(M)}^2. \tag{3.30}$$

By Lemma 3.24, we can see that for $\sigma > a/2$, setting $F(\tau, 0) = f \circ \phi_{X(t)}^\tau(x)$

$$\begin{aligned}
\left|\frac{1}{T}\int_0^T f \circ \phi_{X(t)}^\tau(x) dt\right|^2 &= \left(\frac{1}{T}\int_0^T |F(\tau, 0) d\tau|^2\right)^2 \\
&\leq C_\sigma \frac{1}{T}\left(\frac{1}{T}\int_0^T \frac{ds}{w_\Omega(s)}\right)\int_\Omega |(I - \triangle_{\mathbb{R}^a})^{\frac{\sigma}{2}} F(\tau, \mathbf{s})| d\tau d\mathbf{s} \\
&\leq C_\sigma T^{-1} w_{\mathcal{F}(t)}(x, T)^{-1} \left\|(I - \Delta_{\mathcal{F}(t)})^{\frac{\sigma}{2}} f\right\|_{L^2(M)}^2.
\end{aligned}$$

□

3.5 Average width estimate

In this section we prove estimates on the average width of orbits of nilflows. Let X_α be the vector field on M defined in (3.3). Recall the formula (3.4)

$$X_\alpha := \xi + \sum_{(i,j) \in J} \alpha_i^{(j)} \eta_i^{(j)}.$$

3.5.1 Almost periodic points

Let us introduce special type of condition for the Lie algebra $\mathbf{n}$ required for width estimate.

Definition 3.26. *The nilpotent Lie algebra* $\mathbf{n}$ *satisfies* transversality condition *if there exists basis* (X_α, Y) *of* $\mathbf{n}$ *such that*

$$\langle \mathfrak{G}_\alpha \rangle + Ran(ad_{X_\alpha}) + C_{\mathfrak{I}}(X_\alpha) = \mathbf{n} \tag{3.31}$$

where $\mathfrak{G}_\alpha = (X_\alpha, Y^{(1)})$ *is a set of generator,* $Ran(ad_{X_\alpha}) = \{Y \in \mathfrak{I} \mid Y = ad_{X_\alpha}(W),\ W \in \mathfrak{I}\}$ *and* $C_{\mathfrak{I}}(X_\alpha) = \{Y \in \mathfrak{I} \mid [Y, X_\alpha] = 0\}$ *is centralizer.*

It is clear that the set of generators are neither included in the range of ad_{X_α}, nor in the centralizer $C_{\mathfrak{I}}(X_\alpha)$. We will restrict $\mathbf{n}$ satisfying the condition (3.31) in the rest of sections.

Remark 3.27. *The transversality condition implies that displacement (or distance between* x *and* $\Phi^r_{\alpha,\theta}(x)$*), induced by return map* $\Phi_{\alpha,\theta}$*, should intersect the set of centralizer transversally. I.e the measure of the set of close return orbit in transverse manifold* M^a_θ *should not be invariant under the action of flow. This condition is crucial in estimating the almost periodic orbit* (3.43) *under rescaling of basis in the Lemma 3.33.*

Recall that M^a_θ denotes the fiber at $\theta \in \mathbb{T}^1$ of the fibration $pr_2 : M \to \mathbb{T}^1$. $\Phi_{\alpha,\theta}$ denote the first return map of nilflow $\{\phi^t_{X_\alpha}\}$ to the transverse section M^a_θ and

$\Phi^r_{\alpha,\theta}$ denote r-th iterate of the map $\Phi_{\alpha,\theta}$. Let G denote nilpotent Lie group with its lattice Γ defining $M^a_\theta = \Gamma\backslash G$. G acts on M^a_θ by right action and action of G extends on $M^a_\theta \times M^a_\theta$.

Define a map $\psi^{(r)}_{\alpha,\theta} : M^a_\theta \to M^a_\theta \times M^a_\theta$ given by $\psi^{(r)}_{\alpha,\theta}(x) = (x, \Phi^r_{\alpha,\theta}(x))$. By its definition, the map $\Phi^r_{\alpha,\theta}$ commutes with the action of the centralizer $C_G = \exp(C_{\mathfrak{J}}(X_\alpha)) \subset G$ and its action on product $M^a_\theta \times M^a_\theta$ commutes with $\psi^{(r)}_{\alpha,\theta}$. That is, for $c \in C_G$ and $x = \Gamma g$,

$$\psi^{(r)}_{\alpha,\theta}(xc) = (xc, \Phi^r_{\alpha,\theta}(xc)) = (xc, \Phi^r_{\alpha,\theta}(x)c) = \psi^{(r)}_{\alpha,\theta}(x)c. \tag{3.32}$$

Then quotient map is well-defined on

$$\Psi^{(r)}_{\alpha,\theta} := M^a_\theta / C_G \longrightarrow M^a_\theta \times M^a_\theta / C_G. \tag{3.33}$$

Setting. (i) In $M^a_\theta \times M^a_\theta$, we set diagonal $\Delta = \{(x,x) \mid x \in M^a_\theta\}$ which is isomorphic to M^a_θ by identifying (x,x) with $x \in M^a_\theta$. Given $(x,x) \in \Delta$, tangent space of diagonal is $T_{(x,x)}\Delta := \{(v,v) \mid v \in T_x M^a_\theta\}$ and its normal space is defined as $(T_{(x,x)}\Delta)^\perp = \{(v,-v) \mid v \in T_x M^a_\theta\} = T_{(x,x)}\Delta^\perp$. In total space $M^a_\theta \times M^a_\theta$, its tangent space splits by

$$T_{(x,x)}(M^a_\theta \times M^a_\theta) = T_{(x,x)}\Delta \oplus (T_{(x,x)}\Delta)^\perp.$$

For any $w_1, w_2 \in T_x M^a_\theta$,

$$(w_1, w_2) = 1/2(w_1 + w_2, w_1 + w_2) + 1/2(w_1 - w_2, -(w_1 - w_2)). \qquad (3.34)$$

(ii) Given $x = \Gamma h_1, y = \Gamma h_2 \in M^a_\theta$, define a set $\Delta_{(x,y)} = \{(xg, yg) \mid g \in G\} \subset M^a_\theta \times M^a_\theta$ for $(xg, yg) = (\Gamma h_1 g, \Gamma h_2 g)$ and $\Delta^\perp_{(x,y)} = \{(xg, yg^{-1}) \mid g \in G\}$ that contains (x, y). For $\psi^{(r)}_{\alpha,\theta}(x) = (x, \Phi^r_{\alpha,\theta}(x))$, its tangent space in $M^a_\theta \times M^a_\theta$ is decomposed

$$T_{(x,\Phi^r_{\alpha,\theta}(x))}(M^a_\theta \times M^a_\theta) = T_{(x,\Phi^r_{\alpha,\theta}(x))}\Delta_{(x,\Phi^r_{\alpha,\theta}(x))} \oplus (T_{(x,\Phi^r_{\alpha,\theta}(x))}\Delta_{(x,\Phi^r_{\alpha,\theta}(x))})^\perp.$$

Then tangent space of diagonal is $T_{(x,\Phi^r_{\alpha,\theta}(x))}\Delta_{(x,\Phi^r_{\alpha,\theta}(x))} = \{(v, d_x\Phi^r_{\alpha,\theta}(v)) \mid v \in T_x M^a_\theta\}$ and its normal space is identified as

$$(T_{(x,\Phi^r_{\alpha,\theta}(x))}\Delta_{(x,\Phi^r_{\alpha,\theta}(x))})^\perp = T_{(x,\Phi^r_{\alpha,\theta}(x))}\Delta^\perp_{(x,\Phi^r_{\alpha,\theta}(x))}.$$

By identification in (3.34), for $w_1 = v$ and $w_2 = -d_x\Phi^r_{\alpha,\theta}(v)$, we write

$$(T_{(x,\Phi^r_{\alpha,\theta}(x))}\Delta_{(x,\Phi^r_{\alpha,\theta}(x))})^\perp = \{(1/2(v - d_x\Phi^r_{\alpha,\theta}(v)), -1/2(v - d_x\Phi^r_{\alpha,\theta}(v)) \mid v \in T_x M^a_\theta\}. \qquad (3.35)$$

(iii) Now define orthogonal projection $\pi : M^a_\theta \times M^a_\theta \to M^a_\theta \times M^a_\theta$ along the direction of diagonal. That is, for $(x, y) \in M^a_\theta \times M^a_\theta$, there exists (x', y') such that

$\pi(x, y) = (x', y') \in \Delta_{(x,y)} \cap \Delta^{\perp}_{(x,x)}$. Then,

$$T_{\pi(x,y)}\Delta_{\pi(x,y)} = T_{(x,y)}\Delta_{(x,y)}, \quad T_{\pi(x,y)}\Delta^{\perp}_{\pi(x,y)} = T_{(x,y)}\Delta^{\perp}_{(x,y)}. \tag{3.36}$$

Define a map $F^{(r)} : M^a_\theta \to M^a_\theta \times M^a_\theta$ given by $F^{(r)} = \pi \circ \psi^{(r)}_{\alpha,\theta}$. In the local coordinate, by identification (3.35) and (3.36),

$$d_x F^{(r)}(v) = (1/2(v - d_x\Phi^r_{\alpha,\theta}(v)), -1/2(v - d_x\Phi^r_{\alpha,\theta}(v)), \quad v \in T_x M^a_\theta. \tag{3.37}$$

By (4.3) and definition of $F^{(r)}$, we have $F^{(r)}(xc) = F^{(r)}(x)c$ for $c \in C_G$. Then for all $r \in \mathbb{Z}$, $F^{(r)}$ induces a quotient map $\tilde{F}^{(r)} : M^a_\theta/C_G \to M^a_\theta \times M^a_\theta/C_G$. From (3.37), the range of differential $D\tilde{F}^{(r)}$ is determined by $I - D\Phi^r_{\alpha,\theta}$.

In the next lemma, we verify the range of differential map $D\tilde{F}^{(r)}$.

Lemma 3.28. *For all $r \in \mathbb{Z}\backslash\{0\}$, range of $I - D\Phi^r_{\alpha,\theta}$ on $\mathfrak{I}/C_{\mathfrak{I}}(X_\alpha)$ coincides with Ran(ad_{X_α}) and Jacobian of $\tilde{F}^{(r)}$ is non-zero constant.*

Proof. Recall that $\Phi^r_{\alpha,\theta}$ is r-th return map on M^a_θ. We find differential in the direction of each Y^j_i for fixed i and j. For $x \in M$, set a curve $\gamma^x_{i,j}(t) = x\exp(tY^{(j)}_i)\exp(rX_\alpha)$. Note that

$$\begin{aligned}\exp(tY^{(j)}_i)\exp(rX_\alpha) &= \exp(rX_\alpha)\exp(-rX_\alpha)\exp(tY^{(j)}_i)\exp(rX_\alpha)\\ &= \exp(rX_\alpha)\exp(e^{-r(\mathrm{ad}_{X_\alpha})}(tY^{(j)}_i))\end{aligned}$$

and

$$\frac{d}{dt}\gamma^x_{i,j}(t)\mid_{t=0}= e^{-r(\mathrm{ad}_{X_\alpha})}(Y^{(j)}_i).$$

By definition, $\frac{\partial\Phi^r_{\alpha,\theta}}{\partial s^{(j)}_i}(x) = \dfrac{d}{dt}(\gamma^x_{i,j}(t))\mid_{t=0}$ and we have $I - D\Phi^r_{\alpha,\theta} = I - \sum_{(i,j)\in J}\frac{\partial\Phi^r_{\alpha,\theta}}{\partial s^{(j)}_i}$.

Then,

$$(I - D\Phi^r_{\alpha,\theta})(\sum_{(i,j)\in J} s^{(j)}_i Y^{(j)}_i) = [r(\mathrm{ad}_{X_\alpha})(\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\mathrm{ad}_{X_\alpha})^k)](\sum_{(i,j)\in J} s^{(j)}_i Y^{(j)}_i). \quad (3.38)$$

Therefore, range of $I - D\Phi^r_{\alpha,\theta}$ is contained in $\mathrm{Ran}(\mathrm{ad}_{X_\alpha})$.

Conversely, $\frac{1-e^{-\mathrm{ad}_{X_\alpha}}}{\mathrm{ad}_{X_\alpha}} = \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}(\mathrm{ad}_{X_\alpha})^k$ is invertible and

$$I - D\Phi^r_{\alpha,\theta}\left((\frac{1-e^{-\mathrm{ad}_{X_\alpha}}}{\mathrm{ad}_{X_\alpha}})^{-1}(\sum_{(i,j)\in J} s^{(j)}_i Y^{(j)}_i)\right) = r(\mathrm{ad}_{X_\alpha})(\sum_{(i,j)\in J} s^{(j)}_i Y^{(j)}_i). \quad (3.39)$$

Therefore, we conclude that range of $I - D\Phi^r_{\alpha,\theta}$ is $\mathrm{Ran}(\mathrm{ad}_{X_\alpha})$.

If $\sum_{(i,j)\in J} s^{(j)}_i Y^{(j)}_i \in C_{\mathfrak{I}}(X_\alpha)$, then $(I-D\Phi^r_{\alpha,\theta})(\sum_{(i,j)\in J} s^{(j)}_i Y^{(j)}_i) = 0$ and kernel of $I - D\Phi^r_{\alpha,\theta}$ is $C_{\mathfrak{I}}(X_\alpha)$. I.e, $I - D\Phi^r_{\alpha,\theta}$ is bijective on $\mathfrak{I}/C_{\mathfrak{I}}(X_\alpha)$. Thus, by (3.38) Jacobian of $I - D\Phi^r_{\alpha,\theta}$ is non-zero constant and it concludes the statement.

□

Setting (continued). *(iv)* Set submanifold $\mathcal{S} \subset M^a_\theta \times M^a_\theta$ that consists of diagonal Δ and coordinates of generators in normal (transverese) directions. Denote its quotient $\mathcal{S}_C = \mathcal{S}/C_G \subset M^a_\theta \times M^a_\theta/C_G$. Then, following Lemma 4.25, we obtain transversality of $\tilde{F}^{(r)}$ to $\mathcal{S}_C$. For every $p \in (\tilde{F}^{(r)})^{-1}(\mathcal{S}_C)$, the transversality holds on tangent space:

$$T_{\tilde{F}^{(r)}(p)}\mathcal{S}_C + D\tilde{F}^{(r)}(T_pM_\theta^a/C_G) = T_{\tilde{F}^{(r)}(p)}(M_\theta^a \times M_\theta^a/C_G). \tag{3.40}$$

(v) Denote Lebesgue measure $\mathcal{L}^{a+1}(= vol_M)$ on nilmanifold M and conditional measure $\mathcal{L}_\theta^a(= vol_{M_\theta^a})$ on transverse manifold M_θ^a. On quotient space $M_{\theta,C}^a := M_\theta^a/C_{G_\theta}$, we write measure $\mathcal{L}_\theta^c(= vol_{M_\theta^a/C_G})$. Similarly, we set conditional measure $\mu_\theta^a(= vol_{M_\theta^a\times M_\theta^a})$ on product manifold and $\mu_\theta^c(= vol_{M_\theta^a\times M_\theta^a/C_G})$ on its quotient space.

Denote image of $F^{(r)}$ by $M_{\theta,r}^a := F^{(r)}(M_\theta^a) \subset M_\theta^a \times M_\theta^a$ and $M_{\theta,r,C}^a := \tilde{F}^{(r)}(M_{\theta,C}^a)$. We write its conditional Lebesgue measure $\mu_{\theta,r}^a := \mu_\theta^a|_{M_{\theta,r}^a}$ and $\mu_{\theta,r}^c := \mu_\theta^c|_{M_{\theta,r,c}^a}$ respectively.

For any open set $U_{\mathcal{S}_C} \subset M_\theta^a \times M_\theta^a/C_G$, we write push-forward measure $(\tilde{F}^{(r)})_*\mathcal{L}_\theta^c$

$$\begin{aligned}(\tilde{F}^{(r)})_*\mathcal{L}_\theta^c(U_{\mathcal{S}_C}\cap M_{\theta,r,C}^a) &= \mathcal{L}_\theta^c((\tilde{F}^{(r)})^{-1}(U_{\mathcal{S}_C}\cap M_{\theta,r,C}^a)) \\ &= \int_{U_{\mathcal{S}_C}} \sum_{x\in(\tilde{F}^{(r)})^{-1}(\{z\}),z\in U_{\mathcal{S}_C}} \frac{1}{Jac(\tilde{F}^{(r)}(x))} dvol_{M_{\theta,r,c}^a}(z).\end{aligned}$$

By compactness of M_θ^a (or M_θ^a/C_G), it is finite. By Lemma 4.25, Jacobian of $\tilde{F}^{(r)}$ is constant and $(\tilde{F}^{(r)})_*\mathcal{L}_\theta^c = \mu_{\theta,r}^c$ is Lebesgue.

By invariance of action of centralizer, for any neighborhood $U_\mathcal{S} \in M_\theta^a \times M_\theta^a$ with $U_{\mathcal{S}_C} = U_\mathcal{S}/C_G$,

$$\mu_{\theta,r}^c(U_{\mathcal{S}_C}\cap M_{\theta,r,C}^a) = \mu_{\theta,r}^a(U_\mathcal{S}\cap M_{\theta,r}^a) \tag{3.41}$$

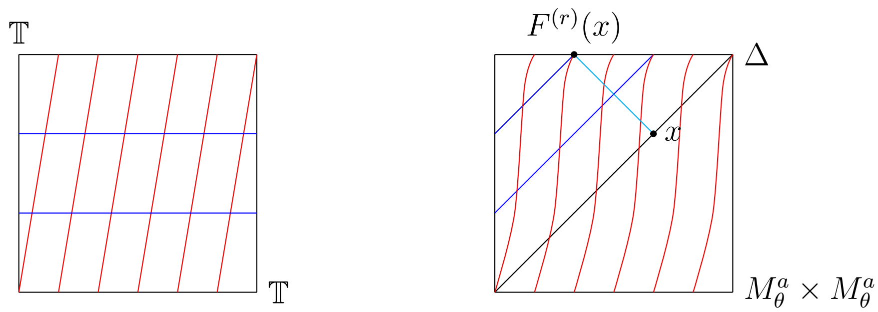

Figure 3.1: Illustration of displacement $F^{(r)}$ in product $M_\theta^a \times M_\theta^a$ and comparison with uniform expanding map.

and by definition of conditional measure,

$$\mu_{\theta,r}^a(U_{\mathcal{S}} \cap M_{\theta,r}^a) = \mu_\theta^a(U_{\mathcal{S}}). \tag{3.42}$$

Let d be a distance function in $M_\theta^a \times M_\theta^a$ and we abuse notation d for induced distance on $M_\theta^a \times M_\theta^a / C_G$. Set $U_\delta = \{z \in M_\theta^a \times M_\theta^a \mid d(z, \mathcal{S}) < \delta\}$ be a δ-tubular neighborhood of $\mathcal{S}$ and $U_{\delta,C} = \{z \in M_\theta^a \times M_\theta^a / C_G \mid d(z, \mathcal{S}_C) < \delta\}$ be its quotient.

Define *almost-periodic set* (set of r-th close return) on the diagonal

$$AP^r(\mathcal{U}_\delta) := \{x \in M_\theta^a \mid d(F^{(r)}(x), \mathcal{S}) < \delta\}. \tag{3.43}$$

Since $F^{(r)}$ commutes with C_G, $AP^r(\mathcal{U}_\delta)/C_G = \{x \in M_\theta^a/C_G \mid d(\tilde{F}^{(r)}(x), \mathcal{S}_C) < \delta\}$ and $\mathcal{L}_\theta^a(AP^r(\mathcal{U}_\delta)) = \mathcal{L}_\theta^c(AP^r(\mathcal{U}_\delta)/C_G)$.

The following volume estimate of *almost-periodic set* holds.

Lemma 3.29. *Let $U_{\delta,C}$ be any tubular neighborhood of $\mathcal{S}_C$ in $M_\theta^a \times M_\theta^a / C_G$. For* *all $r \in \mathbb{Z}\backslash\{0\}$, the conditional measure $vol_{M_\theta^a}$ of $AP^r(\mathcal{U}_\delta)$ is given as follows:*

$$\mathcal{L}_\theta^a(AP^r(\mathcal{U}_\delta)) = \mu_{\theta,r}^c(U_{\delta,C} \cap M_{\theta,r,C}^a).$$

Proof. By previous setting (v), it suffices to prove $\mathcal{L}_\theta^c(AP^r(\mathcal{U}_\delta)/C_G) = \mu_{\theta,r}^c(U_\delta \cap M_{\theta,r,C}^a)$. Note that $(\tilde{F}^{(r)})^{-1}(AP^r(\mathcal{U}_\delta)/C_G) = \{x \in M_\theta^a/C_G \mid d(z, \mathcal{S}_C) < \delta\}$ if $z = \tilde{F}^r(x)$ for some $x \in M_\theta^a/C_G$, otherwise it is an empty set.

Then, $(\tilde{F}^{(r)})^{-1}(AP^r(\mathcal{U}_\delta)/C_G) = (U_{\delta,C} \cap M_{\theta,r,C}^a)$. Thus, by definition of push-forward measure, the equality holds. □

Recall that $\tilde{F}^{(r)} : M_{\theta,C}^a \to M_{\theta,r,C}^a$ has non-zero constant Jacobian if $r \neq 0$ by Lemma 4.25 and it is a local diffeomorphism. Thus, by transversality of $\tilde{F}^{(r)}$, in a small tubular neighborhood U, $\tilde{F}^{(r)}$ is covering.

Lemma 3.30. *For any $z \in U \cap M_{\theta,r,C}^a$, there exist finite number of pre-images of $\tilde{F}^{(r)}$.*

Proof. If we suppose that $(\tilde{F}^{(r)})^{-1}(z)$ contains infinitely many different points, then since the manifold M_θ^a is compact (and $M_{\theta,C}^a$ is compact), there exists a sequence of pairwise different points $x_i \in (\tilde{F}^{(r)})^{-1}(z)$, which converges to x_0. We have $(\tilde{F}^{(r)})(x_0) = z$ and by inverse function theorem, the point x_0 has a neighborhood U' in which $\tilde{F}^{(r)}$ is a homeomorphism. In particular, $U'\backslash\{x_0\} \cap (\tilde{F}^{(r)})^{-1}(z) = \emptyset$, which leads a contradiction. □

Set $N_r(z) = \#\{x \in M_{\theta,C}^a \mid \tilde{F}^{(r)}(x) = z\}$ the number of pre-images of $\tilde{F}^{(r)}$. The number $N_r(z)$ is independent of choice of $z \in U \cap M^a_{\theta,r,C}$ since Jacobian is constant and degree of map is invariant (see [DFN12, §3]).

Now we introduce the volume estimate of δ-neighborhood $U_{\delta,C}$.

Proposition 3.31. *The following volume estimate holds: for any* $r \neq 0$*, there exists* $C := C(M^a_\theta) > 0$ *such that*

$$\mu^c_{\theta,r}(U_{\delta,C} \cap M^a_{\theta,r,C}) < C\delta.$$

Proof. Let $U \subset M^a_\theta \times M^a_\theta / C_G$ be a tubular neighborhood of $\mathcal{S}_C$ that contains $U_{\delta,C}$ with the following condition:

$$vol_{M^a_\theta \times M^a_\theta / C_G}(U_{\delta,C}) = \delta vol_{M^a_\theta \times M^a_\theta / C_G}(U). \tag{3.44}$$

If $U \cap M^a_{\theta,r,C} = \emptyset$, then there is nothing to prove since $U_{\delta,C} \cap M^a_{\theta,r,C} = \emptyset$. Assume $z \in U \cap M^a_{\theta,r,C}$ and let $\{J_k\}_{k\geq 1}$ be connected components of $(\tilde{F}^{(r)})^{-1}(U \cap M^a_{\theta,r,C})$. We firstly claim that $\tilde{F}^{(r)}|_{J_k}$ is injective.

Given $z \in U \cap M^a_{\theta,r,C}$, assume that there exist $x_1 \neq x_2 \in J_k$ for some k such that $z = \tilde{F}^{(r)}|_{J_k}(x_1) = \tilde{F}^{(r)}|_{J_k}(x_2)$. Let $\gamma : [0,1] \to J_k$ be a path that connects $\gamma(0) = x_1$ and $\gamma(1) = x_2$. Set the lift of path $\tilde{\gamma} = \tilde{F}^{(r)}|_{J_k} \circ \gamma : [0,1] \to U$. Then $\tilde{\gamma}(0) = \tilde{\gamma}(1) = z$ and $\tilde{\gamma}$ is a loop in U. Since U is simply connected, $\tilde{\gamma}$ is contractible and there exists a homotopy of path $g_s : [0,1] \to U$ such that $g_0 = \tilde{\gamma}$ is homotopic to a constant loop $g_1 = c$ by fixing two end points $\tilde{F}^{(r)}|_{J_k}(x_1) = \tilde{F}^{(r)}|_{J_k}(x_2) = z$ for $s \in [0,1]$.

Note that $\tilde{F}^{(r)}|_{J_k}^{-1} \circ g_s$ is a lift of homotopy g_s, and lift of g_0 is $\gamma = \tilde{F}^{(r)}|_{J_k}^{-1}(\tilde{\gamma})$ with fixed end points x_1 and x_2. By continuity of homotopy, g_s also keeps the same end points x_1 and x_2 fixed for all $s \in [0,1]$. Since g_1 is constant loop and its lift should be a single point, γ is homotopic to a constant. Since end points of γ is fixed, it has to be a constant but it leads a contradiction. Therefore, we have $x_1 = x_2$.

Set $\tilde{F}_k^{(r)} = \tilde{F}^{(r)}|_{J_k}$. Then by injectivity of $\tilde{F}_k^{(r)}$, we obtain

$$(\tilde{F}^{(r)})^{-1}(U) = (\tilde{F}^{(r)})^{-1}(U \cap M^a_{\theta,r,C}) = \bigcup_{k=1}^{N_r} J_k.$$

Furthermore, we obtain the following equality:

$$vol_{M^a_\theta/C_G}(J_k) = \frac{vol_{M^a_{\theta,r,C}}(U \cap M^a_{\theta,r,C})}{Jac(\tilde{F}_k^{(r)})} = \frac{vol_{M^a_{\theta,r,C}}(U \cap M^a_{\theta,r,C})}{Jac(\tilde{F}^{(r)})}. \tag{3.45}$$

Since volume of M^a_θ/C_G is a finite,

$$N_r\Big(\frac{vol_{M^a_\theta \times M^a_\theta/C_G}(U)}{Jac(\tilde{F}^{(r)})}\Big) = N_r\Big(\frac{vol_{M^a_{\theta,r,C}}(U \cap M^a_{\theta,r,C})}{Jac(\tilde{F}^{(r)})}\Big) \tag{3.46}$$

$$= \sum_{k=1}^{N_r} vol_{M^a_\theta/C_G}(J_k) < \infty. \tag{3.47}$$

Assume that $(\tilde{F}^{(r)})^{-1}(U_{\delta,C}) = \bigcup_{k=1}^{N_r}(\tilde{F}_k^{(r)})^{-1}(U_{\delta,C})$. Then by (3.45) and definition of conditional measure,

$$
\begin{aligned}
vol_{M_\theta^a/C_G}((\tilde{F}^{(r)})^{-1}(U_{\delta,C})) &= \sum_{k=1}^{N_r} vol_{M_\theta^a/C_G}((\tilde{F}_k^{(r)})^{-1}(U_{\delta,C})) \\
&= N_r\Big(\frac{vol_{M^a_{\theta,r,C}}(U_{\delta,C}\cap M^a_{\theta,r,C})}{Jac(\tilde{F}^{(r)})}\Big) \\
&= N_r\Big(\frac{vol_{M_\theta^a\times M_\theta^a/C_G}(U_{\delta,C})}{Jac(\tilde{F}^{(r)})}\Big).
\end{aligned}
$$

By previous last equality with condition (3.44),

$$
vol_{M_\theta^a/C_G}((\tilde{F}^{(r)})^{-1}(U_{\delta,C})) = N_r\Big(\frac{\delta vol_{M_\theta^a\times M_\theta^a/C_G}(U)}{Jac(\tilde{F}^{(r)})}\Big). \tag{3.48}
$$

Therefore, combining (3.46) and (3.48), there exists $C > 0$ such that

$$
\mu^c_{\theta,r}(U_{\delta,C}) = (\tilde{F}^{(r)})_* vol_{M_\theta^a/C_G}(U_{\delta,C}) = vol_{M_\theta^a/C_G}((\tilde{F}^{(r)})^{-1}(U_{\delta,C})) < C\delta.
$$

□

Definition 3.32. *For any basis $Y = \{Y_1, \cdots Y_a\}$ of codimension 1 ideal $\mathfrak{I}$ of $\mathbf{n}$, let I be the supremum of all constant $I' \in (0, \frac{1}{2})$ such that for any $x \in M$ the map*

$$
\phi_x^Y : (s_1, \cdots, s_a) \mapsto x\exp(\sum_{i=1}^{a} s_i Y_i) \in M \tag{3.49}
$$

is local embedding (injective) on the domain

$$
\{\boldsymbol{s} \in \mathbb{R}^a \mid |s_i| < I' \text{ for all } i = 1, \cdots, a\}.
$$

For any $x, x' \in M$, set local distance d_* (measured locally in the Lie algebra) on transvere section M^a_θ along Y_i direction by $d_{Y_i}(x, x') = |s_i|$ if there is $\mathbf{s} := (s_1, \cdots, s_a) \in [-I/2, I/2]^a$ such that

$$x' = x \exp(\sum_{i=1}^{a} s_i Y_i),$$

otherwise $d_{Y_i}(x, x') = I$.

Recall the projection map $pr_1 : M \to \mathbb{T}^{n+1}$ onto the base torus. On transverse manifold, for all $\theta \in \mathbb{T}^1$, let $pr_\theta : M^a_\theta \to \mathbb{T}^n$ be the restriction to M^a_θ. Then,

$$d_{Y_i}(pr_\theta(\Phi^r_{\alpha,\theta}(x)), pr_\theta(x)) = r\alpha_i, \;\; 1 \leq i \leq n.$$

We note distance $d_{Y_i}(\Phi^r_{\alpha,\theta}(x), x)$ on the generators does not depend on choice of x.

For any $L \geq 1$, $r \in \mathbb{Z}$, $x \in M^a_\theta$ and given scaling factor $\rho = (\rho_1, \cdots, \rho_a) \in [0, 1)^a$, we define

$$\begin{aligned} \epsilon_{r,L} &:= \max_{1 \leq i \leq n} \min\{I, L^{\rho_i} d_{Y_i}(\Phi^r_{\alpha,\theta}(x), x)\} \\ \delta_{r,L}(x) &:= \max_{n \leq i \leq a} \min\{I, L^{\rho_i} d_{Y_i}(\Phi^r_{\alpha,\theta}(x), x)\}. \end{aligned} \tag{3.50}$$

The condition $\epsilon_{r,L} < \epsilon < I$ and $\delta' < \delta_{r,L}(x) < \delta < I$ are equivalent to saying

$$\Phi^r_{\alpha,\theta}(x) = x \exp(\sum_{i=1}^{a} s_i Y_i) \tag{3.51}$$

for some vectors $\mathbf{s} := (s_1, \cdots, s_a) \in [-I/2, I/2]^a$ such that

$$|s_i| < \epsilon L^{-\rho_i}, \text{ for all } i \in \{1, \cdots, n\}$$

$$|s_i| < \delta L^{-\rho_i}, \text{ for all } i \in \{n+1, \cdots, a\}$$

$$|s_j| > \delta' L^{-\rho_j}, \text{ for some } j \in \{n+1, \cdots, a\}.$$

For every $r \in \mathbb{Z}\backslash\{0\}$ and $j \geq 0$, let $AP^r_{j,L} \subset M$ be sets defined as follows

$$AP^r_{j,L} = \begin{cases} \emptyset & \text{if } \epsilon_{r,L} > \frac{I}{2}; \\ (\delta_{r,L})^{-1}\left((2^{-(j+1)}I, 2^{-j}I]\right) & \text{otherwise.} \end{cases} \tag{3.52}$$

In the next lemma, Lebesgue measure of almost-periodic points set $AP^r_{j,L}$ on M is estimated by the volume of δ-neighborhood $\mathcal{U}_\delta$.

Lemma 3.33. *For all $r \in \mathbb{Z}\backslash\{0\}$, $j \in \mathbb{N}$, $L \geq 1$, the $(a+1)$ dimensional Lebesgue measure of the set $AP^r_{j,L}$ can be estimated as follows: there exists $C > 0$ such that*

$$\mathcal{L}^{a+1}(AP^r_{j,L}) \leq \frac{CI^{a-n}}{2^{j(a-n)}} L^{-\sum_{i=n+1}^a \rho_i}.$$

Proof. Without loss of generality, we assume that $AP^r_{j,L} \neq \emptyset$.

By Tonelli's theorem,

$$\mathcal{L}^{a+1}(AP^r_{j,L}) = \int_0^1 \mathcal{L}^a_\theta(AP^r_{j,L} \cap M^a_\theta) d\theta. \tag{3.53}$$

Recall the definition $AP^r(\mathcal{U}_\delta)$ in (3.43). Choose $\delta = \frac{I^{a-n}}{2^{j(a-n)}} L^{-\sum_{i=n+1}^a \rho_i}$ and set

$\mathcal{U}_\delta^{L,j} := \mathcal{U}_\delta$. Then we claim that $AP_{j,L}^r \cap M_\theta^a \subset AP^r(\mathcal{U}_\delta^{L,j})$.

For all $r > 0$ and $\theta \in \mathbb{T}^1$, if $x \in AP_{j,L}^r \cap M_\theta^a$ then $d_{Y_i}(\Phi_{\alpha,\theta}^r(x), x) \leq 2^{-j} I L^{-\rho_i}$ for all $i = n+1, \cdots, a$.

By identification of $x \in M_\theta^a$ to (x, x) in diagonal $\Delta \subset M_\theta^a \times M_\theta^a$, local distance $d_{Y_i}(\Phi_{\alpha,\theta}^r(x), x)$ is identified by the distance function d in the product $M_\theta^a \times M_\theta^a$. Thus, $x \in AP_{j,L}^r \cap M_\theta^a$ implies that $d(F^{(r)}(x), \mathcal{S}) < \delta$. That is, $AP_{j,L}^r \cap M_\theta^a \subset AP^r(\mathcal{U}_\delta^{L,j})$. By Lemma 3.29, the volume estimate follows

$$\mathcal{L}_\theta^a(AP_{j,L}^r \cap M_\theta^a) \leq \mathcal{L}_\theta^a(AP^r(\mathcal{U}_\delta^{L,j})) = \mu_{\theta,r}^c(\mathcal{U}_{\delta,C}^{L,j} \cap M_{\theta,r,C}^a).$$

Finally, by Proposition 3.31,

$$\mu_{\theta,r}^c(\mathcal{U}_{\delta,C}^{L,j} \cap M_{\theta,r,C}^a) \leq \frac{CI^{a-n}}{2^{j(a-n)}} L^{-\sum_{i=n+1}^a \rho_i}.$$

Thus, proof follows from formula (3.53). □

3.5.2 Expected width bounds.

We prove a bound on the average width of a orbit on nilmanifold with respect to scaled basis. This section follows in the same way of [FF14, §5.2]. For completion of the proof, we repeat the similar arguments in nilmanifolds under transverse conditions.

For $L \geq 1, r \in \mathbb{Z} \backslash \{0\}$, let us consider the function

$$h_{r,L} = \sum_{j=1}^{\infty} \min\{2^{j(a-n)}, (\frac{2}{\epsilon_{r,l}})^n\} \chi_{AP^r_{j,L}}. \tag{3.54}$$

Define cut-off function $J_{r,L} \in \mathbb{N}$ by the formula:

$$J_{r,L} := \max\{j \in \mathbb{N} \mid 2^{j(a-n)} \leq (\frac{2}{\epsilon_{r,l}})^n\}. \tag{3.55}$$

The function $h_{r,L}$ is

$$h_{r,L} = \sum_{j=1}^{J_{r,L}} 2^{j(a-n)} \chi_{AP^r_{j,L}} + \sum_{j > J_{r,L}} (\frac{2}{\epsilon_{r,l}})^n \chi_{AP^r_{j,L}}. \tag{3.56}$$

For every $L \geq 1$, let $\mathcal{F}^{(L)}_\alpha$ be the rescaled strongly adapted basis

$$\mathcal{F}^{(L)}_\alpha = (X^{(L)}_\alpha, Y^{(L)}_1, \cdots, Y^{(L)}_a) = (LX_\alpha, L^{-\rho_1} Y_1, \cdots, L^{-\rho_a} Y_a). \tag{3.57}$$

For $(x, T) \in M \times \mathbb{R}$, let $w_{\mathcal{F}^{(L)}_\alpha}(x, T)$ denote the average width of the orbit segment

$$\gamma^T_{X^{(L)}_\alpha}(x) := \{\phi^t_{X^{(L)}_\alpha}(x) \mid 0 \leq t \leq T\}.$$

We prove a bound for the average width of the orbit arc in terms of the following function

$$H^T_L := 1 + \sum_{|r|=1}^{[TL]} h_{r,L}. \tag{3.58}$$

Definition 3.34. *For $t \in [0, T]$, we define a set of points $\Omega(t) \subset \{t\} \times \mathbb{R}^a$ as follows:*

Case 1. *If* $\phi^t_{X^{(L)}_\alpha}(x) \notin \bigcup_{|r|=1}^{[TL]} \bigcup_{j>0} AP^r_{j,L}$, *let* $\Omega(t)$ *be the set of all points* $(t, s_1, \cdots, s_a)$ *such that*

$$|s_i| < I/4, \quad i \in \{1, \cdots, a\}.$$

If $\phi^t_{X^{(L)}_\alpha}(x) \in \bigcup_{|r|=1}^{[TL]} \bigcup_{j>0} AP^r_{j,L}$, *then we consider two subcases.*

Case 2-1. *if* $\phi^t_{X^{(L)}_\alpha}(x) \in \bigcup_{|r|=1}^{[TL]} \bigcup_{j>J_{r,L}} AP^r_{j,L}$, *let* $\Omega(t)$ *be the set of all points* $(t, \boldsymbol{s})$ *such that*

$$|s_i| < \frac{1}{4} \min_{1 \le |r| \le [TL]} \min_{j > J_{r,L}} \{\epsilon_{r,L} : \phi^t_{X^{(L)}_\alpha}(x) \in AP^r_{j,L}\}, \text{ for } i \in \{1, \cdots, n\}$$
$$|s_i| < \frac{I}{4} \quad \text{for } i \in \{n+1, \cdots, a\}$$

Case 2-2. *if* $\phi^t_{X^{(L)}_\alpha}(x) \in \bigcup_{|r|=1}^{[TL]} \bigcup_{j \le J_{r,L}} AP^r_{j,L} \backslash \bigcup_{|r|=1}^{[TL]} \bigcup_{j>J_{r,L}} AP^r_{j,L}$, *let* l *be the largest integer such that*

$$\phi^t_{X^{(L)}_\alpha}(x) \in \bigcup_{|r|=1}^{[TL]} \bigcup_{l \le j \le J_{r,L}} AP^r_{j,L} \backslash \bigcup_{|r|=1}^{[TL]} \bigcup_{j > J_{r,L}} AP^r_{j,L},$$

and let $\Omega(t)$ *be the set of all points* $(t, \boldsymbol{s})$ *such that*

$$|s_i| < \frac{I}{4}, \quad \text{for } i \in \{1, \cdots, n\}$$
$$|s_i| < \frac{I}{4} \frac{1}{2^{l+1}}, \quad \text{for } i \in \{n+1, \cdots, a\}.$$

We set

$$\Omega := \bigcup_{t\in[0,T]} \Omega(t) \subset [0,T]\times[-I/4,I/4]^a \subset [0,T]\times\mathbb{R}^a.$$

Lemma 3.35. *The restriction to* Ω *of the map*

$$(t,\boldsymbol{s}) \in \Omega \mapsto x\exp(tX_\alpha^{(L)})\exp(\sum_{i=1}^a s_i Y_i^{(L)}) \tag{3.59}$$

is injective.

Proof. For every $t\in[0,T]$, we define a set $\Omega(t)\subset\{t\}\times\mathbb{R}^a$ as follows:

$$\Omega := \bigcup_{t\in[0,T]} \Omega(t) \subset \mathbb{R}^a.$$

Then we set

$$\phi^t_{X_\alpha^{(L)}}(x)\exp(\sum_{i=1}^a s_i Y_i^{(L)}) = \phi^{t'}_{X_\alpha^{(L)}}(x)\exp(\sum_{i=1}^a s_i' Y_i^{(L)}). \tag{3.60}$$

Let us assume $t'\geq t$. By considering the projection on the base torus, we have the following identity:

$$\begin{aligned}(t,s_1,\cdots,s_n) \bmod \mathbb{Z}^{n+1} &= pr_1(\phi^t_{X_\alpha^{(L)}}(x))\\ &= pr_1(\phi^{t'}_{X_\alpha^{(L)}}(x)) = (t',s_1',\cdots,s_n') \bmod \mathbb{Z}^{n+1},\end{aligned} \tag{3.61}$$

which implies $t\equiv t'$ modulo $\mathbb{Z}$. As $\phi^{tL}_{X_\alpha} = \phi^t_{X_\alpha^{(L)}}$, the number $r_0 = t'-t$ is a non negative integer satisfying $r_0 \le TL$; hence $r_0 \le [TL]$.

If $r_0 = 0$, then $t' = t$ and $s'_i = s_i$. Then injectivity is obtained by definition of I. Assume that $r_0 \ne 0$. Let $p, q \in M^a_\theta$ and then we have

$$p := \phi^t_{X^{(L)}_\alpha}(x), \quad q := \phi^{t'}_{X^{(L)}_\alpha}(x) \Longrightarrow q = \Phi^{r_0}_{\alpha,\theta}(p).$$

From identity (3.60) we have

$$\begin{aligned} q &= p \exp(\sum_{i=1}^a s_i Y_i^{(L)}) \exp(-\sum_{i=1}^a s'_i Y_i^{(L)}) \\ &= p \exp(\sum_{i=1}^a (s'_i - s_i + P_i(s_i, s'_i)) L^{-\rho_i} Y_i) \end{aligned} \tag{3.62}$$

where P_i is polynomial expression following from Baker-Cambell-Hausdorff formula.

Note that $P_i = 0$ if $i = 1, \cdots, n$ and $|P_i| \le \sum_{l=1}^\infty 1/2|s_l s'_l|^l$ for $i > n$. Since $|s_i|, |s'_i| \le \frac{I}{4} \ll 1$,

$$q = p \exp(\sum_{i=1}^a (s'_i - s_i + \epsilon_i) L^{-\rho_i} Y_i), \quad \text{for some } \epsilon_i \in [0, I_\epsilon)$$

where $I_\epsilon = \sum_{l=1} (\frac{I}{4})^l = \frac{I}{4-I} < I/3$. Thus for all $i \in \{1, \cdots, a\}$,

$$L^{\rho_i} d_{Y_i}(p, \Phi^{r_0}_{\alpha,\theta}(p)) = L^{\rho_i} |(s'_i - s_i + \epsilon_i) L^{-\rho_i}| \le |s'_i| + |s_i| + |\epsilon_i|$$

and

$$\epsilon_{r_0,L} = \max_{1 \le i \le n} L^{\rho_i} d_{Y_i}(p, \Phi^{r_0}_{\alpha,\theta}(p)) \le \max_{1 \le i \le n} |s_i| + |s'_i| + |\epsilon_i| \le \frac{5}{6} I.$$

For the same reason, from formula (3.62) we also obtain that

$$\delta_{r_0,L}(p) = \delta_{-r_0,L}(q) < I/2.$$

By defining $j_0 \in \mathbb{N}$ as the unique non-negative integer such that

$$\frac{I}{2^{j_0+1}} \leq \delta_{r_0,L}(p) \leq \frac{I}{2^{j_0}}$$

and by the definition 3.32, we have $p \in AP^{r_0}_{j_0,L}$ and $q \in AP^{-r_0}_{j_0,L}$.

If $j_0 > J_{r_0,L} = J_{-r_0,L}$, then $p, q \in \bigcup_{|r|=1}^{[TL]} \bigcup_{j \geq J_{r,L}}$. It follows that the sets $\Omega(t)$ and $\Omega(t')$ are both defined on case 2-1. Hence,

$$\epsilon_{r_0,L} \leq \max_{1 \leq i \leq n} |s_i| + |s'_i| + |\epsilon_i| \leq \frac{5}{6}\epsilon_{r_0,L}$$

which is a contradiction.

If the map in formula (3.59) fails injective at points (t, s) and (t', s') with $t \geq t'$, then there are integers $r_0 \in [1, TL], j_0 \in [1, J(|r_0|)]$ and $\theta \in \mathbb{T}^1$ such that the points p and q satisfy

$$q = \Phi^{r_0}_{\alpha,\theta}(p), \quad p, q \notin \bigcup_{|r|=1}^{[TL]} \bigcup_{j > J_{r,L}} AP^r_{j,L}.$$

In this case, the sets $\Omega(t)$ and $\Omega(t')$ are both defined according to case (2-2). Let l_1 and l_2 as the largest integers such that

$$p \in \bigcup_{|r|=1}^{[TL]} \bigcup_{l_1 \leq j \leq J_{r,L}} AP^r_{j,L} \quad \text{and} \quad q \in \bigcup_{|r|=1}^{[TL]} \bigcup_{l_2 \leq j \leq J_{r,L}} AP^r_{j,L}.$$

On case (2-2), we have

$$|s_i| < \frac{I}{4}\frac{I}{2^{l_1+1}}, \quad |s_i'| < \frac{I}{4}\frac{I}{2^{l_2+1}}, \text{ for all } i \in \{n+1, \cdots, a\},$$

which also leads contradiction because $l_1, l_2 > j_0$ deduce the contradiction

$$\frac{I}{2^{j_0+1}} \leq \delta_{r_0,L}(p) \leq \max_{i\geq n+1} |s_i| + |s_i'| + |\epsilon_i| < \frac{I}{2^{l_1+1}} + \frac{I}{2^{l_2+1}} \leq \frac{I}{2}\frac{1}{2^{j_0+1}}.$$

Hence, the injectivity is proved. □

We reprove the Lemma 5.5 in [FF14] in the general settings (under transversality conditions) combining with Lemma 3.35.

Lemma 3.36. *For all $x \in M$ and for all $T, L \geq 1$ we have*

$$\frac{1}{w_{\mathcal{F}_\alpha^{(L)}}(x,T)} \leq \left(\frac{2}{I(Y)}\right)^a \frac{1}{T}\int_0^T H_L^T \circ \phi^t_{X_\alpha^{(L)}}(x)dt.$$

Proof. The width function w_Ω of the set Ω is given by the following:

$$w_\Omega(t) = \begin{cases} (\frac{I}{2})^a & \text{case 1} \\ (\frac{I}{2})^a(\frac{\min\{\epsilon_{r,L}\}}{2})^n & \text{case 2-1} \\ (\frac{I}{2})^a 2^{-(a-n)(l+1)} & \text{case 2-2,} \end{cases} \tag{3.63}$$

and it implies that

$$\frac{1}{w_\Omega(t)} \leq \begin{cases} (\frac{2}{I})^{a-n} & \text{case 1} \\ (\frac{2}{I})^{a-n} \sum_{|r|=1}^{[TL]} \sum_{j>J_{r,L}} \frac{2^n \chi_{AP^r_{j,L}}(\phi^t_{X^{(L)}_\alpha}(x))}{(\epsilon_{r,L})^n} & \text{case 2-1} \\ (\frac{2}{I})^{a} \sum_{|r|=1}^{[TL]} \sum_{j>J_{r,L}} 2^{(j+1)(a-n)} \chi_{AP^r_{j,L}}(\phi^t_{X^{(L)}_\alpha}(x)) & \text{case 2-2.} \end{cases} \tag{3.64}$$

By the definition of the function H^T_L in formula (3.58), we have

$$\frac{1}{w_\Omega(t)} \leq \left(\frac{2}{I}\right)^a H^T_L \circ \phi^t_{X^{(L)}_\alpha}(x), \text{ for all } t \in [0, T]. \tag{3.65}$$

From the definition (3.29) of the average width of the orbit segment $\{x \exp(tX^{(L)}_\alpha) \mid 0 \leq t \leq T\}$, we have the estimate

$$\frac{1}{w_{\mathcal{F}^{(L)}_\alpha}(x,T)} \leq \frac{1}{T}\int_0^T \frac{dt}{w_\Omega(t)} \leq \left(\frac{2}{I}\right)^a \frac{1}{T}\int_0^T H^T_L \circ \phi^t_{X^{(L)}_\alpha}(x) dt.$$

□

Lemma 3.37. *For all $r \in \mathbb{Z}\backslash\{0\}$ and for all $L \geq 1$, the following estimate holds:*

$$\left|\int_M h_{r,L}(x)dx\right| \leq CI(Y)^{a-n}(1 + J_{r,L})L^{-\sum_{i=n+1}^a \rho_i}$$

Proof. It follows from the Lemma 3.33 that for $r \neq 0$ and for all $j \geq 0$, the Lebesgue measure of the set $AP^r_{j,L}$ satisfies the following bound:

$$\mathcal{L}^{a+1}(AP^r_{j,L}) \leq \frac{CI^{a-n}}{2^{j(a-n)}} L^{-\sum_{i=n+1}^{a} \rho_i}. \tag{3.66}$$

From the formula (3.56), it follows that

$$\int_M h_{r,L}(x)dx \leq 1 + \sum_{j=1}^{J_{r,L}} 2^{j(a-n)} \mathcal{L}^{a+1}(AP^r_{j,L}) + \sum_{j>J_{r,L}} \frac{2^n \mathcal{L}^{a+1}(AP^r_{j,L})}{(\epsilon_{r,L})^n}.$$

By estimate in the formula (3.66), we immediately have that

$$\sum_{j=1}^{J_{r,L}} 2^{j(a-n)} \mathcal{L}^{a+1}(AP^r_{j,L}) \leq CI^{a-n} J_{r,L} L^{-\sum_{i=n+1}^{a} \rho_i}.$$

By the definition of the cut-off in formula (3.55) we have the bound

$$\frac{2^{n-(J_{r,L}+1)(a-n)}}{(\epsilon_{r,L})^n} \leq 1,$$

and by an estimate on a geometric sum

$$\sum_{j>J_{r,L}} \frac{2^n \mathcal{L}^{a+1}(AP^r_{j,L})}{(\epsilon_{r,L})^n} \leq \frac{2^{n-(J_{r,L}+1)(a-n)}}{(\epsilon_{r,L})^n} CI^{a-n} L^{-\sum_{i=n+1}^{a} \rho_i}$$
$$\leq CI^{a-n} L^{-\sum_{i=n+1}^{a} \rho_i}.$$

□

3.5.3 Diophantine estimation

In this section we review the concept of simultaneous Diophantine condition. The bounds on the expected average width is estimated under Diophantine conditions.

Definition 3.38. *For any basis* $\bar{Y} := \{\bar{Y}_1, \cdots, \bar{Y}_n\} \subset \mathbb{R}^n$*, let* $\bar{I} := \bar{I}(\bar{Y})$ *be the supremum of all constants* $\bar{I}' > 0$ *such that the map*

$$(s_1, \cdots, s_n) \to \exp(\sum_{i=1}^{n} s_i \bar{Y}_i) \in \mathbb{T}^n$$

is a local embedding on the domain $\{\boldsymbol{s} \in \mathbb{R}^n \mid |s_i| < \bar{I}' \text{ for all } i = 1, \cdots, n\}$.

For any $\theta \in \mathbb{R}^n$, let $[\theta] \in \mathbb{T}^n$ its projection onto the torus $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$ and let

$$|\theta|_1 = |s_1|, \cdots, |\theta|_i = |s_i|, \cdots, |\theta|_n = |s_n|,$$

if there is $\mathbf{s} := (s_1, \cdots, s_n) \in [-\bar{I}/2, \bar{I}/2]^n$ such that

$$[\theta] = \exp(\sum_{i=1}^{n} s_i \bar{Y}_i) \in \mathbb{T}^n;$$

otherwise we set $|\theta|_1 = \cdots = |\theta|_n = \bar{I}$.

Definition 3.39. *A vector* $\alpha \in \mathbb{R}^n \backslash \mathbb{Q}^n$ *is simultaneously Diophantine of exponent*

$\nu \geq 1$, say $\alpha \in DC_{n,\nu}$ if there exists a constant $c(\alpha) > 0$ such that, for all $r \in \mathbb{N}\backslash\{0\}$,

$$\min_i \|r\alpha_i\| = d(r\alpha, \mathbb{Z}^n) = \|r\alpha\| \geq \frac{c(\alpha)}{r^{\frac{\nu}{n}}}.$$

Definition 3.40. *Let $\sigma = (\sigma_1, \cdots, \sigma_n) \in (0,1)^n$ be such that $\sigma_1 + \cdots + \sigma_n = 1$. For any $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n$, for any $N \in \mathbb{N}$ and every $\delta > 0$, let*

$$R_\alpha(N, \delta) = \{r \in [-N, N] \cap \mathbb{Z} \mid |r\alpha|_1 \leq \delta^{\sigma_1}, \cdots, |r\alpha|_n \leq \delta^{\sigma_n}\}.$$

For every $\nu > 1$, let $D_n(\bar{Y}, \sigma, \nu) \subset (\mathbb{R}\backslash\mathbb{Q})^n$ be the subset defined as follows: the vector $\alpha \in D_n(\bar{Y}, \sigma, \nu)$ if and only if there exists a constant $C(\bar{Y}, \sigma, \alpha) > 0$ such that, for all $N \in \mathbb{N}$ for all $\delta > 0$,

$$\#R_\alpha(N, \delta) \leq C(\bar{Y}, \sigma, \alpha) \max\{N^{1-\frac{1}{\nu}}, N\delta\}. \tag{3.67}$$

The Diophantine condition implies a standard simultaneous Diophantine condition. We quote following Lemmas proved in [FF14, Lemma 5.9, 5.12].

Lemma 3.41. *Let $\alpha \in D_n$. For all $r \in \mathbb{Z}\backslash\{0\}$, we have*

$$\max\{|r\alpha|_1, \cdots, |r\alpha|_n\} \geq \min\{\frac{\bar{I}^2}{4}, \frac{1}{[1 + C(\bar{Y}, \sigma, \alpha)]^{2\nu}}\}\frac{1}{|r|^\nu}.$$

Lemma 3.42. *For all bases $\bar{Y} \subset \mathbb{R}^n$, for all $\sigma = (\sigma_1, \cdots, \sigma_n) \in (0,1)^n$ such that $\sigma_1 + \cdots + \sigma_n = 1$ and let $m(\sigma) = \min\{\sigma_1, \cdots \sigma_n\}$ and $M(\sigma) = \max\{\sigma_1, \cdots \sigma_n\}$. For* *all* $\nu \geq 1$*, the inclusion*

$$DC_{n,\nu} \subset D_n(\bar{Y}, \sigma, \nu)$$

holds under the assumption that

$$\mu \leq \min\{\nu, [\frac{M(\sigma)}{\nu} + 1 - \frac{1}{n}]^{-1}, [\frac{1}{\nu} + (1 - \frac{2}{n})(1 - \frac{m(\sigma)}{M(\sigma)})]^{-1}\}.$$

The set $D_n(\bar{Y}, \sigma, \nu)$ *has full measure if*

$$\frac{1}{\nu} < \min\{[M(\sigma)n]^{-1}, 1 - (1 - \frac{2}{n})(1 - \frac{m(\sigma)}{M(\sigma)})\}. \tag{3.68}$$

In dimension one, the vector space has unique basis up to scaling. The following result is immediate.

Lemma 3.43. *For all* $\nu \geq 1$ *the following identity holds:*

$$DC_{1,\nu} = D_1(\nu).$$

Let $\mathcal{F}_\alpha := (X_\alpha, Y)$ be a basis and let $\bar{Y} = \{\bar{Y}_1, \cdots, \bar{Y}_n\} \in \mathbb{R}$ denote the projection of the basis of codimension 1 ideal $\mathfrak{I}$ onto the Abelianized Lie algebra $\bar{\mathbf{n}} := \mathbf{n}/[\mathbf{n}, \mathbf{n}] \approx \mathbb{R}^n$. For $\rho = (\rho_1, \cdots, \rho_a) \in [0, 1)^a$, we write a vector of scaling exponents

$$\bar{\rho} = (\rho_1, \cdots, \rho_n), \quad |\bar{\rho}| = \rho_1 + \cdots + \rho_n.$$

Let $\alpha_1 = (\alpha_1^{(1)}, \cdots, \alpha_n^{(1)}) \in D_n(E, \bar{\rho}/|\bar{\rho}|, \nu)$. For brevity, let $C(\bar{Y}, \bar{\rho}/|\bar{\rho}|, \alpha_1)$

denote the constant in the Diophantine condition introduced in Definition 3.40 and let

$$C(\alpha_1) = 1 + C(\bar{Y}, \bar{\rho}/|\bar{\rho}|, \alpha). \tag{3.69}$$

We prove the upper bound on the cut-off function in the formula (3.55). Let $I = I(Y)$ and $\bar{I} = \bar{I}(\bar{Y})$ be the positive constant introduced in the Definition 3.32 and 3.38. We observe that $I \leq \bar{I}$ since the basis $\bar{Y}$ is the projection of the basis $Y \subset \mathbf{n}'$ and the canonical projection commutes with exponential map. Then the following logarithmic upper bound holds.

Lemma 3.44. *For every $\rho \in [0,1)^a$, for every $\nu \leq 1/|\bar{\rho}|$ and for every $\alpha \in D_n(\bar{Y}, \bar{\rho}/|\bar{\rho}|, \alpha)$, there exists a constant $K > 0$ such that, for all $T \geq 1$ and for all $r \in \mathbb{Z}\backslash\{0\}$, the following bound holds:*

$$J_{r,L} \leq K\{1 + \log^+[I(Y)^{-1}] + \log C(\alpha_1)\}(1 + \log|r|).$$

Proof. By Lemma 3.41 and by the definition of $\epsilon_{r,L}$ in formula (3.50), it follows that, for all $T > 0, L \geq 1$ and for all $r \in \mathbb{Z}\backslash\{0\}$, we have

$$\epsilon_{r,L} \geq \max_{1\leq i\leq n} \min\{I, |r\alpha_1|_i\} \geq \min\{I, \frac{\bar{I}^2}{4}, \frac{1}{[1 + C(\alpha_1)]^{2\nu}}\}\frac{1}{|r|^\nu}.$$

It follows by the above bound and by the definition of the cut-off function (3.55),

$$J_{r,L} \leq \frac{n}{a-n}(3\log 2 + 3\log^+(1/I) + 2\nu\log[1 + C(\alpha_1)] + \nu\log|r|),$$

□

Assume that there exists $\nu \in 1/|\bar{\rho}|$ such that $\alpha_1 \in D_n(E, \bar{\rho}/|\bar{\rho}|, \nu)$. For brevity, we introduce the following notation:

$$\mathcal{H}(Y, \rho, \alpha) = 1 + I(Y)^{a-n} C(\alpha_1)\{1 + \log^+[I(Y)^{-1}] + \log C(\alpha_1)\}. \tag{3.70}$$

Theorem 3.45. *For every $\rho \in [0,1)^a$, for every $\nu \leq 1/|\bar{\rho}|$ such that $\alpha_1 = \alpha_i^{(1)} \in D_n(\bar{Y}, \bar{\rho}, \nu)$ there exists a constant $K' > 0$ such that, for all $T > 0$ and for all $L \geq 1$, the following bounds holds:*

$$\left| \int_M H_L^T(x) dx \right| \leq K' \mathcal{H}(Y, \rho, \alpha)(1+T)(1 + \log^+ T + \log L) L^{1 - \sum_{i=1}^a \rho_i}.$$

Proof. By the definition of H_L^T in the formula (3.58), the statement follows from the Lemma 3.37 and Lemma 3.44. In fact, for all $r \in \mathbb{Z} \backslash \{0\}$ and all $j \geq 0$, by definition (3.52) the set $AP_{j,L}^r$ is nonempty only if $\epsilon_{r,L} < \frac{I}{2}$. Since $\nu \leq 1/|\bar{\rho}|$, it follows from the definition of the Diophantine class D_n

$$\#\{r \in [-TL, TL] \cap \mathbb{Z} \backslash \{0\} \mid AP_{j,L}^r \neq \emptyset\} \leq C(E, \sigma, \alpha)(1+T) L^{1-|\bar{\rho}|}.$$

Hence, the statement follows from the Lemma 3.37 and 3.44. □

3.5.4 Width estimates along orbit segments

We introduce a definition of good points, that is, points on the nilmanifold for which we can prove bounds on the width of sufficiently many orbit segments to derive by our method bounds on ergodic averages.

Definition 3.46. *For any increasing sequence* (T_i) *of positive real numbers, let* $h_i \in [1,2]$ *denote the ratio* $\log T_i/[\log T_i]$ *for every* $T_i \geq 1$*. Let's say* $N_i = [\log T_i]$ *and* $T_{j,i} = e^{jh_i}$ *for integer* $j \in [0, N_i]$.

Let $\zeta > 0$ *and* $w > 0$*. A point* $x \in M$ *is* (w, T_i, ζ)*-good for the basis* $\mathcal{F}_\alpha$ *if having set* $y_i = \phi^{T_i}_{X_\alpha}(x)$*, for all* $i \in \mathbb{N}$ *and for all* $0 \leq j \leq N_i$*, we have*

$$w_{\mathcal{F}(T_{j,i})}(x, 1) \geq w/T_i^\zeta, \quad w_{\mathcal{F}(T_{j,i})}(y_i, 1) \geq w/T_i^\zeta.$$

Lemma 3.47. *Let* $\zeta > 0$ *be fixed and let* (T_i) *be an increasing sequence of positive real numbers satisfying the condition*

$$\Sigma((T_i), \zeta) := \sum_{i \in \mathbb{N}} (\log T_i)^2 (T_i)^{-\zeta} < \infty. \tag{3.71}$$

Let $\rho \in [0,1)$ *with* $\sum \rho_i = 1$*. Then the Lebesgue measure of the complement of the set* $\mathcal{G}(w, (T_i), \zeta)$ *of* $(w, (T_i), \zeta)-$*good points is bounded above. That is,* $\exists K > 0$ *such that*

$$meas(\mathcal{G}(w, (T_i), \zeta)^c) \leq K\Sigma((T_i), \zeta)[1/I(Y)]^a \mathcal{H}(Y, \rho, \alpha) w$$

Proof. For all $i \in \mathbb{N}$ and for all $j = 0, \cdots, N_i$, let

$$\mathfrak{S}_{j,i} = \{z \in M : w_{\mathcal{F}_\alpha(T_{j,i})}(z, 1) < T_i^\zeta / w\}.$$

By definition we have

$$\mathcal{G}(w, (T_i), \zeta)^c = \bigcup_{i \in \mathbb{N}} \bigcup_{j=0}^{N_i} (\mathfrak{S}_{j,i} \cup \phi_{X_\alpha}^{-T_i}(\mathfrak{S}_{j,i})). \tag{3.72}$$

By Lemma 3.36 for all $z \in \mathfrak{S}_{j,i}$ we have

$$(I/2)^a T_i^\zeta / w < \int_0^1 H_{T_{j,i}}^1 \circ \phi_{X_\alpha}^\tau(z) d\tau = \frac{1}{T_{j,i}} \int_0^{T_{j,i}} H_{T_{j,i}}^1 \circ \phi_{X_\alpha}^\tau(z) d\tau.$$

It follows that

$$\mathfrak{S}_{j,i} \subset \mathfrak{S}(j,i) := \left\{ z \in M : \sup_{J>0} \frac{1}{J} \int_0^J H_{T_{j,i}}^1 \circ \phi_{X_\alpha}^\tau(z) d\tau > (I/2)^a T_i^\zeta / w \right\}.$$

By the maximal ergodic theorem, the Lebesgue measure meas$[\mathfrak{S}_{j,i}]$ of the set $\mathfrak{S}(j,i)$ satisfies the inequality

$$meas[\mathfrak{S}_{j,i}] \leq (2/I)^a (w/T_i^\zeta) \int_M H_{T_{j,i}}^1 z dz.$$

Let $\mathcal{H} = \mathcal{H}(Y, \rho, \nu)$ denote the constant defined in the formula (3.70). By theorem 3.45, since by hypothesis $\nu \leq 1/|\bar{\rho}|$ and $\alpha \in D_n(\bar{\rho}/|\bar{\rho}|, \nu)$, there exists a constant

$K'(a,n,\nu) > 0$ such that the following bound holds:

$$\left|\int_M H^1_{T_{j,i}}(z)dz\right| \leq K'\mathcal{H}(1+\log T_{j,i}).$$

Hence, by the definition of the $T_{j,i}$, we have

$$N_i \leq \log T_i \leq N_i + 1, \quad \log T_{j,i} \leq 2j. \tag{3.73}$$

Thus, for some constant K'', we have

$$meas[\mathfrak{S}_{j,i}] \leq K''(2/I)^a \mathcal{H} w(1+j) T_i^{-\zeta}.$$

By (3.73), for some constant $K''' > 0$,

$$meas(\bigcup_{j=0}^{N_i} \mathfrak{S}_{j,i} \cup \phi_{X_\alpha}^{-T_i}(\mathfrak{S}_{j,i})) \leq K'''(2/I)^a \mathcal{H} w(\log T_i)^2 T_i^{-\zeta}.$$

By sub-additivity of the Lebesgue measure, we derive the bound

$$meas(\bigcup_{i\in\mathbb{N}}\bigcup_{j=0}^{N_i} \mathfrak{S}_{j,i} \cup \phi_{X_\alpha}^{-T_i}(\mathfrak{S}_{j,i})) \leq K'''\Sigma((T_i),\zeta)\mathcal{H}w.$$

By formula (3.72), the above estimate concludes the proof. □

3.6 Bounds on ergodic average

We shall introduce assumptions on coadjoint orbits $\mathcal{O} \subset \mathbf{n}^*$.

Definition 3.48. *A linear form* $\Lambda \in \mathcal{O}$ *is* integral *if the coefficients* $\Lambda(\eta_i^{(m)}), (i, m) \in J$ *are integer multiples of* 2π. *Denote* $\widehat{M}$ *the set of coadjoint orbits of* $\mathcal{O}$ *of integral linear forms* Λ.

There exist coadjoint orbits $\mathcal{O} \subset \mathbf{n}^*$ that correspond to unitary representations which do not factor through the quotient $N/\exp \mathbf{n}_k$, $\mathbf{n}_k \subset Z(\mathbf{n})$. Such coadjoint orbits and unitary representation are called *maximal.* (See [FF07, Lemma 2.3])

Definition 3.49. *Given a coadjoint orbit* $\mathcal{O} \in \widehat{M_0}$ *and a linear functional* $\Lambda \in \mathcal{O}$, *let us denote* $\mathcal{F}_{\alpha,\Lambda}$ *the completed basis* $\mathcal{F}_{\alpha,\Lambda} = (X_\alpha, Y_\Lambda)$. *For all* $t \in \mathbb{R}$, *we write scaled basis* $\mathcal{F}_{\alpha,\Lambda}(t)$ *by*

$$\mathcal{F}_{\alpha,\Lambda}(t) = (X_\alpha(t), Y_\Lambda(t)) = A_t^\rho(X_\alpha, Y_\Lambda).$$

Let $\widehat{M_0}$ be subset of all coadjoint orbits of forms Λ such that $\Lambda(\eta_i^{(m)}) \neq 0$ for $m = k$. This space has maximal rank and $\Lambda(\eta_i^{(m)}) \neq 0, \forall (i, m) \in J$. For any $\mathcal{O} \in \widehat{M_0}$, let $H_{\mathcal{O}}$ denote the primary subspace of $L^2(M)$ which is a direct sum of sub-representations equivalent to $\mathrm{Ind}_{N'}^{N}(\Lambda)$. For adapted basis $\mathcal{F}$, set

$$W^r(H_{\mathcal{O}}, \mathcal{F}) = H_{\mathcal{O}} \cap W^r(M, \mathcal{F}).$$

3.6.1 Coboundary estimates for rescaled basis

Recall definition of *degree* of $Y_i^{(m)}$ and

$$d_i^{(m)} = \begin{cases} k-m, & \text{for all } 1 \le m \le k-1 \\ 0 & m = k. \end{cases}$$

For any linear functional Λ, the degree of the representation π_Λ only depends on its coadjoint orbits. We denote scaling vector $\rho \in (\mathbb{R}^+)^J$ such that

$$\sum_{(i,j)\in J} \rho_i^{(j)} = 1 \quad \text{and} \quad \rho_i^{(j)} = 0$$

for any $Y_i^{(j)}$ with $deg(Y) = 0$.

Assume that the number of basis of $\mathbf{n}$ with degree $k-m$ is n_m. Define

$$S_{\mathbf{n}}(k) := (n_1 - 1)(k-1) + n_2(k-2) + ... + n_{k-1} \tag{3.74}$$

and

$$\delta(\rho) := \min_{\substack{1\le m\le k-1 \\ 1\le i,j\le n_m}} \{\rho_i^{(m)} - \rho_j^{(m+1)},\ \rho_i^{(m)} - \rho_i^{(m+1)}\}. \tag{3.75}$$

Lemma 3.50. *We have that $\delta(\rho) \le \lambda(\rho)$. The above inequalities are strict unless one has homogeneous scaling*

$$\rho_i^{(j)} = \frac{d_j}{S_{\mathbf{n}}} \text{ for } j \le k-1.$$

Lemma 3.51. *There exists a constant $C > 0$ such that, for all $r \in \mathbb{R}^+$ and for any function $f \in W^r(H_{\mathcal{O}})$, we have*

$$\sum_{(m,i)\in J} |[X_\alpha(t), Y_i^{(m)}(t)]f|_{r,\mathcal{F}_{\alpha,\Lambda}(t)} \leq Ce^{t(1-\delta(\rho))}|f|_{r+1,\mathcal{F}_{\alpha,\Lambda}(t)}.$$

Proof. For all $(m,i) \in J$, we have

$$[X_\alpha(t), Y_i^{(m)}(t)] = \sum_{l\geq 1} c_l^{(m+1)} e^{t(1-\rho_i^{(m)}+\rho_l^{(m+1)})} Y_l^{(m+1)}(t).$$

We note that $c_l^{(j)} = 0$ for $j = k$ and for some l, which is determined by commutation relation. Setting $C = \max_{(i,j)\in J^+}\{|c_i^{(j)}|\}$,

$$\sum_{(m,i)\in J} |[X_\alpha(t), Y_i^{(m)}(t)]f|_{r,\mathcal{F}_{\alpha,\Lambda}(t)} \leq Ce^{t(1-\delta(\rho))} \sum_{(m,i)\in J^+} |Y_i^{(m)}(t)f|_{r,\mathcal{F}_{\alpha,\Lambda}(t)}.$$

□

For $x \in M$, let γ_x be the *Birkhoff average operator* $\gamma_x^T(f) = \frac{1}{T}\int_0^T f \circ \phi_{X_\alpha}^t(x)dt$ and consider the decomposition of the restriction of the linear functional γ_x to $W_0^r(H_{\mathcal{O}}, \mathcal{F}_{\alpha,\Lambda}(t))$ as an orthogonal sum $\gamma_x = D(t) + R(t) \in W_0^{-r}(H_{\mathcal{O}}, \mathcal{F}_{\alpha,\Lambda}(t))$ of X_α-invariant distribution $D(t)$ and an orthogonal complement $R(t)$.

Theorem 3.52. *Let $r > 2(k+1)(a/2+1)+1/2$. For $g \in W^r(H_{\mathcal{O}}, \mathcal{F}_{\alpha,\Lambda}(t))$ and for* *all* $t \geq 0$*, there exists a constant* $C_r^{(1)}$ *such that*

$$|R(t)(g)| \leq C_r^{(1)} e^{(1-\delta(\rho)-(1-\lambda))t} \max\{1, (2\pi\delta_{\mathcal{O}})^{-kr-1}\} \\ \times T^{-1} \left(w_{\mathcal{F}_{\alpha,\Lambda}(t)}(x,1)^{-\frac{1}{2}} + w_{\mathcal{F}_{\alpha,\Lambda}(t)}(\phi^T_{X_\alpha}(x),1)^{-\frac{1}{2}} \right) |g|_{r,\mathcal{F}_{\alpha,\Lambda}(t)}. \quad (3.76)$$

Proof. Fix $t \geq 0$ and set $D = D(t), R = R(t)$ for convenience. Let $g \in W^r_{\alpha,\Lambda}(H_{\mathcal{O}}, \mathcal{F}(t))$. We write $g = g_D + g_R$, where g_R is the kernel of X_α-invariant distributions and g_D is orthogonal to g_R in W^r. Then, g_R is a coboundary and $R(g_D) = 0$. Let $f = G^{X_\alpha(t)}_{X_\alpha,\Lambda}$. From $|D(g_R)| = 0$,

$$|R(g)| = |R(g_D + g_R)| = |R(g_R)| = |\gamma_x(g_R) - D(g_R)| = |\gamma_x(g_R)|. \quad (3.77)$$

By the Gottschalk-Hedlund argument,

$$\begin{aligned} |\gamma_x(g_R)| &= \left| \frac{1}{T} \int_0^T g \circ \phi^s_{X_\alpha}(x) ds \right| \\ &= \frac{1}{T} \left| f \circ \phi^T_{X_\alpha}(x) - f(x) \right| \\ &\leq \frac{1}{T} (|f(x)| + |f \circ \phi^T_{X_\alpha}(x)|) \end{aligned} \quad (3.78)$$

By Theorem 3.25 and Lemma 3.51, for any $\tau > a/2 + 1$, there exists a positive constant C_r such that for any $z \in M$

$$|f(z)| \leq \frac{C_r}{w_{\mathcal{F}_{\alpha,\Lambda}(t)}(z,1)^{\frac{1}{2}}} \left(Ce^{(1-\delta(\rho))t} |f|_{\tau,\mathcal{F}_{\alpha,\Lambda}(t)} + |g|_{\tau-1,\mathcal{F}_{\alpha,\Lambda}(t)} \right). \quad (3.79)$$

By Theorem 3.18, if $r > 2(k+1)\tau + 1/2$, then

$$|f|_{\tau,\mathcal{F}_{\alpha,\Lambda}(t)} \leq C_{r,k,\tau} e^{-(1-\lambda)t} \max\{1, (2\pi\delta_{\mathcal{O}})^{-kr-1}\} |g_R|_{r,\mathcal{F}_{\alpha,\Lambda}(t)}.$$

By orthogonality, we have $|g_R|_{r,\mathcal{F}_{\alpha,\Lambda}(t)} \leq |g|_{r,\mathcal{F}_{\alpha,\Lambda}(t)}$. □

Corollary 3.53. *For every $r > 2(k+1)(a/2+1)+1/2$, there is a constant $C_r^{(2)}$ such that the following holds for every $\mathcal{O} \in \widehat{I}_0$ and every $x \in M$. Then,*

$$|R|_{-r,\mathcal{F}_{\alpha,\Lambda}} \leq C_r^{(2)} [1/I(Y_\Lambda)]^{a/2} \max\{1, \delta_{\mathcal{O}}^{-(k-1)r-1}\} (1 + \|\Lambda\|_{\mathcal{F}_{\alpha,\Lambda}})^{r-1} T^{-1}.$$

Proof. For all $x \in M$, we have $w_{\mathcal{F}_{\alpha,\Lambda}}(x,1) \geq \left(\frac{I(Y_\Lambda)}{2}\right)^a$. It follows from Theorem 3.52 applied to the orthogonal decomposition of $\gamma_x = D(0) + R(0)$. □

3.6.2 Bounds on ergodic averages in an irreducible subrepresentation.

In this section, we derive the bounds on ergodic averages of nilflows for function in a single irreducible sub-representation. For brevity, let us set

$$C_r(\mathcal{O}) = \max\{1, (2\pi\delta_{\mathcal{O}})^{-kr-1}\}. \tag{3.80}$$

Proposition 3.54. *Let $r > 2(k+1)(a/2+1)+1/2$. Let (T_i) be an increasing sequence of positive real numbers ≥ 1 and let $0 < w < I(Y)^a$. Let $\zeta > 0$. There exists a constant $C_r(\rho)$ such that for every $\mathcal{G}(w,(T_i),\zeta)$-good points $x \in M$ and all*

$f \in W^r(H_{\mathcal{O}}, \mathcal{F})$, *we have*

$$\left|\frac{1}{T_i}\int_0^{T_i} f \circ \phi^t_{X_\alpha}(x)dt\right| \leq C_r(\rho)C_r(\mathcal{O})w^{-1/2}T_i^{-\delta(\rho)+\zeta+\lambda/2}|f|_{r,\mathcal{F}_\alpha}. \tag{3.81}$$

Proof. By group action of scaling (3.13), a sequence of frame is chosen $\mathcal{F}(t_j) = A_\rho^{t_j}\mathcal{F}$ with other scaling factors $\rho_i t_j$ on elements of Lie algebras Y_i. Then, as j increases from 0 to N, the scaling parameter t_j becomes larger, while the scaled length of the arc becomes shorter approaching to 1. Let $\phi^s_{X_j}(x)$ denote the flow of the scaled vector field $e^{t_j}X = X(t_j)$.

For each $j = 0, \cdots, N$, let $\gamma = D_j + R_j$ be the orthogonal decomposition of γ in the Hilbert space $W^{-r}(H_\pi, \mathcal{F}(t_j))$ into X_α-invariant distribution D_j and an orthogonal complement R_j. For convenience, we denote by $|\cdot|_{r,j}$ and $\|\cdot\|_{r,j}$ respectively, the transversal Sobolev norm $|\cdot|_{r,\mathcal{F}(t_j)}$ and Lyapunov Sobolev norm $\|\cdot\|_{r,\mathcal{F}(t_j)}$ relative to the rescaled basis $\mathcal{F}(t_j)$.

Let us set $N_i = [\log T_i]$ and $t_{j,i} := T_{j,i} = \log T_i^{j/N_i}$ for integer $j \in [0, N_i]$. We observe $N_i < \log T_i < N_i + 1$. For simplicity, we will omit index $i \in \mathbb{N}$ and set $T = T_i$, $N = N_i$ for a while within the proof and lemmas of this subsection.

Our goal is the estimate $|\gamma|_{-r,\mathcal{F}_\alpha} = |\gamma|_{-r,0}$ (the norm of distribution of unscaled basis). By triangle inequality and Corollary 3.53,

$$\begin{aligned}|\gamma|_{-r,0} &\leq |D_0|_{-r,0} + |R_0|_{-r,0} \\ &\leq |D_0|_{-r,0} + C_r^{(2)}[1/I(Y)]^{a/2}C_r(\Lambda_{\mathcal{O}})T^{-1}.\end{aligned} \tag{3.82}$$

We now estimate $|D_0|_{-r,0}$. By definition of the Lyapunov norm and its bound (3.28), for $-s < -r < 0$,

$$|D_0|_{-s,0} \leq C_{r,s} \|D_0\|_{-r,0} . \tag{3.83}$$

Since $D_j + R_j = D_{j-1} + R_{j-1}$, observe $D_{j-1} = D_j + R'_j$, where R'_j denotes the orthogonal projection of R_j, in the space $W^{-r}(H_{\mathcal{O}}, \mathcal{F}(t_{j-1}))$, on the space of invariant distribution. By definition of Lyapunov norm,

$$\begin{aligned}\|D_{j-1}\|_{-r,j-1} &\leq \|D_j\|_{-r,j-1} + \left\|R'_j\right\|_{-r,j-1} \\ &\leq \|D_j\|_{-r,j-1} + |R'_j|_{-r,j-1} \\ &\leq \|D_j\|_{-r,j-1} + |R_j|_{-r,j-1}.\end{aligned}$$

By Lemma 3.57, equivalence of norm gives

$$\|D_{j-1}\|_{-r,j-1} \leq \|D_j\|_{-r,j-1} + C|R_j|_{-r,j}. \tag{3.84}$$

By Lemma 3.22, for any X_α-invariant distribution D and for all $t_j \geq t_{j-1}$,

$$\|D\|_{-r,\mathcal{F}(t_{j-1})} \leq e^{-\lambda(\rho)(t_j - t_{j-1})/2} \|D\|_{-r,\mathcal{F}(t_j)} .$$

Since $\mathcal{F}(t_j) = A_\rho^{t_j - t_{j-1}} \mathcal{F}(t_{j-1})$ and $t_j - t_{j-1} = \log T/N$ implies

$$\|D_j\|_{-r,j-1} \leq T^{-\lambda(\rho)/2N} \|D_j\|_{-r,j} .$$

From (3.84) we conclude by induction

$$\|D_0\|_{-r,0} \leq T^{-\lambda(\rho)/2} \left(\|D_N\|_{-r,N} + C \sum_{l=0}^{N-1} T^{(l+1)\lambda(\rho)/2N} |R_{N-l}|_{-r,N-l} \right). \tag{3.85}$$

By Lemma 3.55 and 3.56,

$$\|D_0\|_{-r,0} \leq C_r^1(\rho) C_r(\mathcal{O}) w^{-1/2} T^{1-\delta(\rho)+\zeta/2-(1-\lambda)-\lambda(\rho)/2}. \tag{3.86}$$

From (3.82) and the above, we conclude that there exists a constant $C_r(\rho)$ such that

$$|\gamma|_{-r,\mathcal{F}} \leq C_r(\rho) C_r(\mathcal{O}) w^{-1/2} T^{-\delta(\rho)+\zeta/2+\lambda/2}.$$

□

Here we introduce the proof of supplementary lemmas.

Lemma 3.55. *For any r, there exists a constant $C_r > 0$ such that, for all good points $x \in \mathcal{G}(w, (T_i), \zeta)$, we have*

$$\|D_N\|_{-r,N} \leq C_r T^{\zeta/2} / w^{1/2}.$$

Proof. Recall from definition 3.46, for $x \in \mathcal{G}(w, (T_i), \zeta)$ and $y_i = \phi^{T_i}_{X_\alpha}(x)$ for all $i \in \mathbb{N}$

$$\frac{1}{w_{\mathcal{F}(t_j)}(x, 1)} \leq T_i^\zeta / w \quad \text{and} \quad \frac{1}{w_{\mathcal{F}(t_j)}(y_i, 1)} \leq T_i^\zeta / w \tag{3.87}$$

where $t_j = t_{j,i}$.

By definition of norm, we obtain $\|D_N\|_{-r,N} \leq |D_N|_{-r,N} \leq |\gamma|_{-r,N}$. The orbit segment $(\phi^t_{X_\alpha}(x))_{0\leq t\leq T}$ coincides with the orbit segment $(\phi^\tau_{X_\alpha(t_N)}(x))_{0\leq \tau\leq 1}$ of length 1 since $X_\alpha(t_N) = X_\alpha(\log T) = TX_\alpha$. By Theorem 3.25,

$$|\gamma|_{\mathcal{F}(t_N),-r} \leq C_r w_{\mathcal{F}(t_N)}(x,1)^{-1/2}.$$

By the inequality (3.87),

$$w_{\mathcal{F}(t_N)}(x,1)^{-1/2} \leq T^{\zeta/2}/w^{1/2}.$$

□

Lemma 3.56. *For every $r > 2(k+1)(a/2+1)+1/2$, there is a constant C such that for all good points $x \in \mathcal{G}(w,(T_i),\zeta)$, we have*

$$\sum_{l=0}^{N-1} T^{(l+1)\rho_Y/2N}|R_{N-l}|_{-r,N-l} \leq C_r^{(1)}(\rho)C_r(\mathcal{O})w^{-1/2}T^{1-\delta(\rho)-(1-\lambda)+\zeta/2}. \tag{3.88}$$

Proof. The orbit segment $(\phi^t_{X_\alpha}(x))_{0\leq t\leq T}$ has length $T^{l/N}$ with respect to the generator $X_\alpha(t_{N-l}) = X_\alpha((1-l/N)\log T) = T^{1-l/N}X_\alpha$. Thus, by Theorem 3.52 with $e^{(1-\delta(\rho))t_{N-l}} = T^{(1-l/N)(1-\delta(\rho))}$. Then,

$$\begin{aligned}|R_{N-l}|_{-r,N-l} &\leq C_r^{(1)}C_r(\mathcal{O})T^{(1-l/N)(1-\delta(\rho)-(1-\lambda))-l/N)}\left(\frac{1}{w_{\mathcal{F}(t_{N-l})}(x,1)^{\frac{1}{2}}}+\frac{1}{w_{\mathcal{F}(t_{N-l})}(y,1)^{\frac{1}{2}}}\right)\\ &\leq 2C_r^{(1)}C_r(\mathcal{O})w^{-1/2}T^{(1-l/N)(\lambda-\delta(\rho))-l/N+\zeta/2}.\end{aligned}$$

Let $C = 2C_r^{(1)}C_r(\mathcal{O})w^{-1/2}$. Remember that $N_i = [\log T_i]$ and $N_i \le \log T_i \le N_i + 1$, hence ${T_i}^{1/(N_i+1)} \le e \le {T_i}^{1/N_i}$. Recall that we set $T = T_i$, then

$$\begin{aligned}
&\sum_{l=0}^{N-1} T^{(l+1)\rho_Y/2N} |R_{N-l}|_{-r,N-l} \\
&\le CT^{1-\delta(\rho)-(1-\lambda)+\zeta/2} \sum_{l=0}^{N-1} T^{(l+1)\rho_Y/2N} T^{-l/N(1-\delta(\rho)-(1-\lambda))-l/N} \\
&\le CT^{1-\delta(\rho)-(1-\lambda)+\zeta/2+\rho_Y/2N} \sum_{l=0}^{N-1} T^{-l/N(2-\delta(\rho)-(1-\lambda)-\rho_Y/2)} \\
&\le e^r CT^{1-\delta(\rho)-(1-\lambda)+\zeta/2} \sum_{l=0}^{\infty} e^{-l(1+\lambda-\delta(\rho)-\rho_Y/2)}
\end{aligned}$$

By Lemma 3.50, we have $1 + \lambda - \delta(\rho) - \rho_Y/2 \ge 1 - \rho_Y/2 > 1/2$, thus geometric series converges. □

Lemma 3.57. *There exists a constant $C := C(r) > 0$ such that, for all $j = 0, \cdots, N$,*

$$C^{-1}|\cdot|_{-r,j} \le |\cdot|_{-r,j-1} \le C|\cdot|_{-r,j}.$$

Proof. From (3.73), $t_j - t_{j-1} \le 2$ and observe $\mathcal{F}(t_j) = A^{t_j - t_{j-1}}\mathcal{F}(t_{j-1})$. Passing from the frame $\mathcal{F}(t_{j-1})$ to $\mathcal{F}_t$, it can be verified that distortion of the corresponding transversal Sobolev norm is uniformly bounded. □

Let $\sigma = (\sigma_1, \cdots, \sigma_n) \in (0,1)^n$ be such that $\sigma_1 + \cdots + \sigma_n = 1$. For the simplicity, we choose $\sigma_i = 1/n$ from now on. Recall the definition 3.39 or Lemma 3.42 it implies that under certain exponent, $D_n(\sigma, \nu)$ contains simultaneous Diophantine condition. Let

$$\widetilde{M}_0 = \bigcup_{\mathcal{O} \in \hat{M}_0} \{\Lambda \in \mathcal{O} \mid \Lambda \text{ integral}\}$$

be collection of maximal integral coadjoint orbits.

Recall from (3.74), we assume

$$S_{\mathbf{n}}(k) := (n_1 - 1)(k-1) + n_2(k-2) + ... + n_{k-1}.$$

Theorem 3.58. *For any $\Lambda \in \widetilde{M}_0$, let $\nu \in [1, 1 + (k/2-1)\frac{1}{n}]$. Then, for any $r > (k+1)(a/2+1) + 1/2$, there exists a constant $C(\sigma, \nu)$ satisfying the following. For every $\epsilon > 0$, there exists a constant $K_\epsilon(\sigma, \nu) > 0$ such that, for every $\alpha_1 = (\alpha_1^{(1)}, \cdots, \alpha_n^{(1)}) \in D_n(\sigma, \nu)$ and for every $w \in (0, I(Y)^a]$ there exists a measurable set $\mathcal{G}_\Lambda(\sigma, \epsilon, w)$ satisfying the estimate*

$$meas(\mathcal{G}_\Lambda(\sigma, \epsilon, w)^c) \leq K_\epsilon(\sigma, \nu)(\frac{w}{I(Y_\Lambda)^a})\mathcal{H}(Y_\Lambda, \rho, \alpha). \tag{3.89}$$

For every $x \in \mathcal{G}_\Lambda(\sigma, \epsilon, w)$, for every $f \in W^r(H_{\mathcal{O}}, \mathcal{F})$ and $T \geq 1$ we have

$$\left|\frac{1}{T}\int_0^T f \circ \phi^t_{X_\alpha}(x)dt\right| \leq \frac{C_r(\sigma, \nu)C_r(\Lambda)}{w^{\frac{1}{2}}} T^{-(1-\epsilon)\frac{1}{3S_{\mathbf{n}}(k)}} |f|_{r, \mathcal{F}_{\alpha, \Lambda}}.$$

Proof. If the coadjoint orbit $\mathcal{O}$ is integral and maximal with full rank, then we can see that the optimal exponent will be attained by the following scaling. Let $\rho = (\rho_i^{(m)})$ be the vector given by homogeneous scaling:

$$\rho_i^{(j)} = \frac{d_j}{S_{\mathbf{n}}} \text{ for } i \leq k.$$

Let us set $\zeta = 2\delta(\rho)/3 - \lambda/3$. Let $\epsilon > 0$, for all $i \in \mathbb{N}$, let us set $T_i = i^{(1+\epsilon)\zeta^{-1}}$.

Then, there exists a constant $K_\epsilon(\rho) > 0$ such that

$$\Sigma(w, (T_i), \zeta) \leq \sum_i (\log T_i)^2 T_i^{-\zeta} \leq K_\epsilon(\rho).$$

Let $\mathcal{G} = \mathcal{G}_\Lambda(\sigma, \epsilon, w) = \mathcal{G}(w, (T_i), \zeta)$ be the set of $(w, (T_i), \zeta)$-good points for the basis $\mathcal{F}_\alpha$. The estimate in the formula (3.89) follows from the Lemma 3.47 and definition of good points. By Proposition 3.54, for all $x \in \mathcal{G}$ and for every $f \in W^r(H_\mathcal{O}, \mathcal{F})$, the estimate (3.81) holds true. Let $T \in [T_i, T_{i+1}]$. Then

$$\int_0^T f \circ \phi^t_{X_\alpha}(x) dt = \int_0^{T_i} f \circ \phi^t_{X_\alpha} dt + \int_{T_i}^T f \circ \phi^t_{X_\alpha}(x) dt = (I) + (II).$$

Let $C = C_r(\rho) C_r(\mathcal{O})/w^{1/2}$. The first term is estimated by the formula (3.81):

$$(I) \leq C T^{1-\delta(\rho)+\zeta/2+\lambda/2} |f|_{r, \mathcal{F}_{\alpha,\Lambda}} = C T^{1-2\delta(\rho)/3+\lambda/3} |f|_{r, \mathcal{F}_{\alpha,\Lambda}}.$$

For the second term, let us set $\gamma = (1+\epsilon)\zeta^{-1}$ and observe that $\gamma^{-1} = \zeta(1+\epsilon)^{-1} \geq (1-\epsilon)\zeta$. We have

$$\begin{aligned}(II) \leq (T - T_i) \|f\|_\infty &\leq \beta 2^{\gamma-1} T^{1-\gamma^{-1}} \|f\|_\infty \\ &\leq C'(\rho) T^{1-(1-\epsilon)(-2\delta(\rho)/3+\lambda/3)} |f|_{r, \mathcal{F}_{\alpha,\Lambda}}.\end{aligned}$$

By the estimates on the terms (I) and (II), the proof is completed.

□

Remark 3.59. *If $\mathcal{O}$ is integral but not maximal, then the restriction of Λ factors* *through an irreducible representation of the $k-1$ step nilpotent group $N/\exp n_k'$. Then, $\mathbf{n}/\mathbf{n}_k$ is polarizing subalgebra for subrepresentation and it reduces to the case of maximal integral. Since the growth rate is determined by the scaling factors and the exponent λ is determined by the step size and number of elements, the highest exponent is obtained by integral maximal full rank case.*

3.6.3 General bounds on ergodic averages.

Finally, in order to solve cohomological equation on nilmanifold, we glue the solutions constructed in every irreducible sub-representation of N. The main idea is to increase extra regularity of the Sobolev norm to obtain the estimates that are uniformly bounded across all irreducible subrepresentation.

Definition 3.60. *For every $\mathcal{O} \in \widehat{M_0}$, we define*

$$|\mathcal{O}| = \max_{\eta_i \in \mathbf{n}_k} |\Lambda(\eta_i^{(k)})|.$$

Note that $|\mathcal{O}|$ does not depend on the choice of Λ and $|\mathcal{O}| \neq 0$ by maximality. We specifically choose an element $\eta_*^{(k)}$ whose degree k such that

$$|\mathcal{O}| = |\Lambda(\eta_*^{(k)})|.$$

Lemma 3.61. *For every $\mathcal{O} \in \widehat{M_0}$ and for every $\Lambda \in \mathcal{O}$, we have*

$$I(Y_\Lambda)^{-a}\mathcal{H}(Y_\Lambda) \leq C(\alpha_1)(1 + \log C(\alpha_1))2^{a+1}.$$

Proof. The return time of the flow X_α to any orbit of the codimension one subgroup $N' \subset N$ is 1. Hence, by Definition 3.32, we have $I(Y_\Lambda) = 1/2$ for the basis. By (3.69), we have and $C(\alpha_1) \geq 1$. Then, from the definition of the constant $H(Y, \rho, \alpha)$, we obtain

$$\begin{aligned} I(Y_\Lambda)^{-a}\mathcal{H}(Y) &\leq I(Y_\Lambda)^{-a} + I(Y)^{-n}C(\alpha_1)\left(1 + \log^+[I(Y)^{-1}] + \log C(\alpha_1)\right) \\ &\leq C(\alpha_1)(1 + \log C(\alpha_1))\left(I(Y_\Lambda)^{-a} + I(Y_\Lambda)^{-n}\log^+[I(Y)^{-1}]\right) \\ &\leq 2C(\alpha_1)(1 + \log C(\alpha_1))I(Y_\Lambda)^{-a}. \end{aligned}$$

□

Corollary 3.62. *For every $\mathcal{O} \in \widehat{M}_0, \Lambda \in \mathcal{O}, w > 0$ and $\epsilon > 0$, let*

$$w_\Lambda = w|\Lambda(\mathcal{F})|^{-2a-\epsilon}. \tag{3.90}$$

Then, for every $w > 0$ and $\epsilon > 0$ the set

$$\mathcal{G}(\sigma, \epsilon, w) = \bigcap_{\Lambda \in \widehat{M}_0} \mathcal{G}_\Lambda(\sigma, \epsilon, w_\Lambda)$$

has measure greater than $1 - Cw\epsilon^{-1}$, with $C = 2^{-a+1}K_\epsilon(\sigma, \nu)C(\alpha_1)(1 + \log C(\alpha_1))$. Furthermore, if $\epsilon' < \epsilon$ we have $\mathcal{G}(\sigma, \epsilon, w) \subset \mathcal{G}(\sigma, \epsilon', w)$.

Proof. Recall that $|\Lambda(\mathcal{F})|$ is integral multiples of 2π. By Lemma 3.61, inequaltiy

(3.89) and definition of w_Λ, we have

$$meas(\mathcal{G}_\Lambda(\sigma,\epsilon,w_\Lambda)^c) \leq K_\epsilon(\sigma,\nu)(\frac{w_\Lambda}{I(Y)^a})\mathcal{H}(Y_\Lambda,\rho,\alpha)$$
$$\leq C'|\Lambda(\mathcal{F})|^{-2a-\epsilon}w,$$

where $C' = 2^{a+1}K_\epsilon(\sigma,\nu)C(\alpha_1)(1+\log C(\alpha_1))$. Since the $|\Lambda(\mathcal{F})| = 2\pi l$ is bounded by $(2l)^{a-1}$,

$$\sum_{\Lambda\in\tilde{M}_0} meas(\mathcal{G}_\Lambda(\sigma,\epsilon,w_\Lambda)^c) \leq 2^{-2a}wC'\sum_{l>0}\sum_{\Lambda\in\tilde{M}_0:|\Lambda|=2\pi l} l^{-a-\epsilon}$$
$$\leq Cw\sum_{l>0} l^{-1-\epsilon} < Cw\epsilon^{-1}.$$

The last statement on the monotonicity of the set follows from the analogous statement in Theorem 3.58. □

In every coadjoint orbit, we will make a particular choice of a linear form to accomplish the estimates of the bound for each irreducible sub-representation in terms of higher norms.

Definition 3.63. *For every $\mathcal{O} \in \widehat{M_0}$, we define $\Lambda_\mathcal{O}$ as the unique integral linear form $\Lambda \in \mathcal{O}$ such that*

$$0 \leq \Lambda(\eta_*^{(k-1)}) < |\mathcal{O}|.$$

The existence and uniqueness of $\Lambda_{\mathcal{O}}$ *follows from*

$$\Lambda \circ Ad(\exp(tX_\alpha))(\eta_*^{(k-1)}) = \Lambda(\eta_*^{(k-1)}) + t|\mathcal{O}|,$$

and the form $\Lambda \circ Ad(\exp(tX_\alpha)$ *is integral for all integer values of* $t \in \mathbb{R}$*.*

Lemma 3.64. *There exists a constant* $C(\Gamma) > 0$ *such that the following holds on the primary subspace* $C^\infty(H_{\mathcal{O}})$ *the following holds:*

$$|\Lambda_{\mathcal{O}}(\mathcal{F})| Id \leq C(\Gamma)(1 + \Delta_{\mathcal{F}})^{k/2}.$$

Proof. Let $x_0 = -\Lambda_{\mathcal{O}}(\eta_*^{(k-1)})/|\mathcal{O}|$. Then there exists a unique $\Lambda' \in \mathcal{O}$ such that $\Lambda'(\eta_*^{(k-1)}) = 0$ given by $\Lambda' = \Lambda \circ \mathrm{Ad}(e^{x_0 X_\alpha})$. The element $W \in \mathfrak{I}$ is represented in the representation as multiplication operators by the polynomials (3.9),

$$P(\Lambda, W)(x) = \Lambda(\mathrm{Ad}(e^{xX_\alpha})W). \tag{3.91}$$

By the definition of the linear form, the identity $[X_\alpha, \eta_*^{(k-1)}] = \eta_*^{(k)}$ implies

$$P(\Lambda', \eta_*^{(k-1)})(x) = |\mathcal{O}|x.$$

From (3.91), we have

$$\sum_j \frac{(-x)^j}{j!} P(\Lambda', \mathrm{ad}(X_\alpha)^j W) = \Lambda'(W), \quad \text{for all } W \in \mathfrak{I}.$$

Then we obtain

$$\begin{aligned}
\Lambda'(W) &= \sum_j \frac{(-x)^j}{j!} P(\Lambda', \mathrm{ad}(X_\alpha)^j W) \\
&= \sum_j \frac{(-1)^j}{j!} \left(\frac{P(\Lambda', \eta_*^{(k-1)})}{|\mathcal{O}|} \right)^j P(\Lambda', \mathrm{ad}(X_\alpha)^j W) \\
&= |\mathcal{O}|^{1-k} \sum_j \frac{(-1)^j}{j!} P(\Lambda', \eta_*^{(k-1)}) P(\Lambda', \eta_*^{(k)})^{k-1-j} P(\Lambda', \mathrm{ad}(X_\alpha)^j W).
\end{aligned}$$

For any $\Lambda \in \mathbf{n}^*$ the transversal Laplacian for a basis $\mathcal{F}$ in the representation π_Λ is the operator of multiplication by the polynomial and derivative operators

$$\Delta_{\Lambda,\mathcal{F}} = \sum_{W \in \mathcal{F}} \pi_\Lambda^{X_\alpha}(W)^2 = \sum_{W \in \mathcal{F}} P(\Lambda, W)^2.$$

Hence,

$$|P(\Lambda', \eta_j^{(m)})| \leq (1 + \Delta_{\Lambda',\mathcal{F}})^{1/2}.$$

By above identity in formula, the constant operators $\Lambda'(\eta_j^{(m)})$ are given by polynomial and derivative expressions of degree k in the operators $P(\Lambda', \eta_j^{(m)})$ we obtain the estimate

$$|\Lambda'(\mathcal{F})| \mathrm{Id} \leq C_1(\Gamma)(1 + \Delta_{\Lambda',\mathcal{F}})^{k/2}.$$

Since the representation $\pi_{\Lambda'}$ and $\pi_{\Lambda_\mathcal{O}}$ are unitarily intertwined by the translation operator by x_0, and since constant operators commute with translations, we also have

$$|\Lambda'(\mathcal{F})| \mathrm{Id} \leq C_1(\Gamma)(1 + \Delta_{\Lambda_\mathcal{O},\mathcal{F}})^{k/2}.$$

Since x_0 is bounded by a constant depending only step size k, the norms of the linear maps $\mathrm{Ad}(\exp(\pm x_0 X_\alpha))$ are bounded by a constant depending only on k. Therefore, $|\Lambda_{\mathcal{O}}(\mathcal{F})| \leq C_2(k)|\Lambda'(\mathcal{F})|$ and the statements of the lemma follows. □

Corollary 3.65. *There exists a constant $C'(\Gamma)$ such that for all $\mathcal{O} \in \widehat{M}_0$ and for any sufficiently smooth function $f \in H_{\mathcal{O}}$,*

$$C_r(\Lambda_{\mathcal{O}})|f|_{r,\mathcal{F}_{\alpha,\Lambda}} \leq C'(\Gamma) w^{-1/2} |f|_{r+l,\mathcal{F}_\alpha}$$

where $l = (kr+1)k/2 + ak$.

Proof. From the definition (3.80) we have $C_r(\Lambda_{\mathcal{O}}) = (1+|\Lambda_{\mathcal{O}}(\mathcal{F})|)^{l_1}$ with $l_1 = kr+1$. By the formula (3.90),

$$C_r(\Lambda_{\mathcal{O}}) w_\Lambda^{-1/2} \leq w^{-1/2}(1 + |\Lambda_{\mathcal{O}}(\mathcal{F})|)^{l_2}$$

with $l_2 = l_1 + 2a$. By Lemma 3.64 we have

$$(1 + |\Lambda_{\mathcal{O}}(\mathcal{F})|)^{l_2} \leq C'(\Gamma)(1 + \Delta_{\mathcal{F}})^{l_2 k/2}.$$

□

Proposition 3.66. *Let $r > (k+1)(3a/4+2)+1/2$. Let $\sigma = (1/n, \cdots, 1/n) \in (0,1)^n$ be a positive vector. Let us assume that $\nu \in [1, 1+(k/2-1)\frac{1}{n}]$ and let $\alpha \in D_n(\sigma, \nu)$.*

For every $\epsilon > 0$ and $w > 0$, there exists a measurable set $\mathcal{G}(\sigma, \epsilon, w)$ satisfying

$$meas(\mathcal{G}(\sigma, \epsilon, w)^c) \leq Cw\epsilon^{-1} \text{ with } C = 2^{-a+1} K_\epsilon(\sigma, \nu) C(\alpha_1)(1 + \log C(\alpha_1)),$$

such that for every $x \in \mathcal{G}(\sigma, \epsilon, w)$, for every $f \in W^r(M)$ and every $T \geq 1$ we have

$$\left| \frac{1}{T} \int_0^T f \circ \phi_{X_\alpha}^t(x) dt \right| \leq C w^{-1/2} T^{-(1-\epsilon)\frac{1}{3S_{\mathbf{n}}(k)}} |f|_{r, \mathcal{F}_\alpha}. \tag{3.92}$$

Proof. Let $\tau := r - ak/2 > (a+2)(k+1) + 1/2$. Let $f \in W^\tau(M, \mathcal{F})$ and let $f = \sum_{\mathcal{O} \in \widehat{M}_0} f_\mathcal{O}$ be its orthogonal decomposition onto the primary subspace $H_\mathcal{O}$. For each $\mathcal{O} \in \widehat{M}_0$, the constant $w_\mathcal{O}$ is given and the set

$$\mathcal{G}(\sigma, \epsilon, w) = \bigcap_{\mathcal{O} \in \widehat{M}_0} \mathcal{G}_\Lambda(\sigma, \epsilon, w_\mathcal{O})$$

has measure greater than $1 - Cw\epsilon^{-1}$ as proved in Corollary 3.62.

If $x \in \mathcal{G}(\sigma, \epsilon, w)$, then by Theorem 3.58 and Corollary 3.65, the following estimate holds true for every $\mathcal{O} \in \widehat{M}_0$ and all $T \geq 1$:

$$\left| \frac{1}{T} \int_0^T f_\mathcal{O} \circ \phi_{X_\alpha}^t(x) dt \right| \leq C_r(\sigma, \nu) w^{-1/2} T^{-(1-\epsilon)\frac{1}{3S_{\mathbf{n}}(k)}} |f_\mathcal{O}|_{r, \mathcal{F}_\alpha}.$$

For any $\tau > 0$ and any $\epsilon' > 0$, by Lemma 3.64 and orthogonal splitting of $H_\mathcal{O}$

we have

$$\left|\sum_{\mathcal{O}\in\widehat{M}_0} |f_{\mathcal{O}}|_\tau\right|^2 \leq \sum_{\mathcal{O}\in\widehat{M}_0} (1+|\Lambda_{\mathcal{O}}(\mathcal{F}_\alpha)|)^{-a-\epsilon'} \sum_{\mathcal{O}\in\widehat{M}_0} (1+|\Lambda_{\mathcal{O}}(\mathcal{F}_\alpha)|)^{a+\epsilon'} |f_{\mathcal{O}}|^2_{\tau,\mathcal{F}_\alpha}$$

$$\leq C(a)|f|^2_{\tau+(a+\epsilon')k/2,\mathcal{F}_\alpha}$$

and the theorem follows after renaming the constant. □

Proof of Theorem 3.2. Under same hypothesis of proposition 3.66, for $i \in \mathbb{N}$ let $w_i = 1/2^i C$ and $\mathcal{G}_i = \mathcal{G}(\sigma, \epsilon, w_i)$. Set $K_\epsilon(x) = 1/{w_i}^{1/2}$ if $x \in \mathcal{G}_i \backslash \mathcal{G}_{i-1}$. By proposition 3.66, the set $\mathcal{G}_i$ are increasing and satisfy $meas(\mathcal{G}_i^c) \leq 1/2^i\epsilon$. Hence, the set $\mathcal{G}(\sigma,\epsilon) = \bigcup_{i\in\mathbb{N}} \mathcal{G}_i$ has full measure and the function K is in $L^p(M)$ for every $p \in [1, 2)$. □

Proof of Corollary 3.3. For step-k strictly triangular nilpotent Lie algebra $\mathbf{n}$ has dimension $\frac{1}{2}k(k+1)$ with 1 dimensional center. If the coadjoint orbit $\mathcal{O}$ is integral and maximal, then the optimal exponent will be attained by the formula (3.74). Let $\rho = (\cdots, \rho_i^{(m)}, \cdots)$ be the rescaling factor given :

$$S_{\mathbf{n}}(k) = [(k-1)^2 + \sum_{n=1}^{k-2} n(n+1)] = (k-1)(k^2+k-3),$$

and we choose homogeneous scaling

$$\rho_i^{(j)} = \frac{d_i^{(j)}}{S_{\mathbf{n}}(k)} = \frac{k-j}{(k-1)(k^2+k-3)} \text{ for } i \leq j.$$

Then, we can verify that $\lambda(\rho) = \delta(\rho) = \frac{1}{(k-1)(k^2+k-3)}$. By inductive argument with rescaling again, the exponent is obtained which proves Corollary 3.3. $\square$

3.7 Uniform bound of the average width of step 3 case.

In this section, we prove Theorem 3.4 on the effective equidistribution of nilflow on strictly triangular step 3 nilmanifold. On its structure, it is possible to derive uniform bound under Roth-type Diophantine condition due to linear divergence of of orbit. This argument is based on counting principles of close return times which substitute the necessity of good point.

3.7.1 Average Width Function

Let N be a step 3 nilpotent Lie group on 3 generators introduced in (3.2). We denote its Lie algebra $\mathbf{n}$ with its basis $\{X_1, X_2, X_3, Y_1, Y_2, Z\}$ satisfying following commutation relations

$$[X_1, X_2] = Y_1, \ [X_2, X_3] = Y_2, \ [X_1, Y_2] = [Y_1, X_3] = Z. \tag{3.93}$$

As introduced in section 2, $\{\phi_V^t\}_{t\in\mathbb{R}}$ is a measure preserving flow generated by $V := X_1 + \alpha X_2 + \beta X_3$ and $(1, \alpha)$ satisfies standard simultaneous Diophantine condition (3.39).

By definition of the average width (3.63), for any $t \geq 0$ and for any $(x, T) \in M \times [1, +\infty)$ we construct an open set $\Omega_t(x, T) \subset \mathbb{R}^6$ which contains the segment

$\{(s, 0, \cdots, 0) \mid 0 \leq s \leq T\}$ such that the map

$$\begin{aligned}&\phi_x(s, x_2, x_3, y_1, y_2, z)\\&= \Gamma x \exp(se^t V) \exp(e^{-\frac{1}{3}t} x_2 X_2 + e^{-\frac{1}{3}t} x_3 X_3 + e^{-\frac{1}{6}t} y_1 Y_1 + e^{-\frac{1}{6}t} y_2 Y_2 + zZ)\end{aligned}$$

is injective on $\Omega_t(x, T)$. Injectivity fails if and only if there exists vectors

$$(s, x_2, x_3, y_1, y_2, z) \neq (s', x_2', x_3', y_1', y_2', z')$$

such that

$$\begin{aligned}&\Gamma x \exp(s' e^t V) \exp(e^{-\frac{1}{3}t} x_2' X_2 + e^{-\frac{1}{3}t} x_3' X_3 + e^{-\frac{1}{6}t} y_1' Y_1 + e^{-\frac{1}{6}t} y_2' Y_2 + z' Z) \\&= \Gamma x \exp(se^t V) \exp(e^{-\frac{1}{3}t} x_2 X_2 + e^{-\frac{1}{3}t} x_3 X_3 + e^{-\frac{1}{6}t} y_1 Y_1 + e^{-\frac{1}{6}t} y_2 Y_2 + zZ).\end{aligned} \tag{3.94}$$

Let us denote $r = s' - s$ and $\tilde{x_i} = x_i{}' - x_i, \tilde{y_i} = y_i{}' - y_i$ and $\tilde{z} = z' - z$. Let $c_\Gamma > 0$ denote the distance from the identity of the smallest non-zero element of the lattice Γ.

Lemma 3.67. *Under equality (3.94), we obtain followings:*

$$\begin{aligned}&\tilde{x_2}(t, s) = \tilde{x_2}, \quad \tilde{x_3}(t, s) = \tilde{x_3}, \quad \tilde{y_1}(t, s) = \tilde{y_1} + e^{\frac{5}{6}t} s \tilde{x_2}\\&\tilde{y_2}(t, s) = \tilde{y_2} + \alpha e^{\frac{5}{6}t} s \tilde{x_3} + 1/2 e^{-1/2t}(x_2 x_3' - x_2' x_3).\end{aligned}$$

From definition 3.32, denote $c_\Gamma (= I) > 0$ denote the supremum of all constants

$c_\Gamma \in (0, 1/2)$ such that for all the map $\phi_x(0, \mathbf{s})$ is local embedding on the domain

$$\{\mathbf{s} \in \mathbb{R}^5 \mid |x_i|, |y_i| < c_\Gamma \text{ for all i}\}.$$

Let us assume that $|x_i|', |x_i|, |y_i|, |y_i'| \leq c_\gamma/4$ and $\tilde{x}_i, \tilde{y}_i, \tilde{z} \in [-\frac{c_\Gamma}{2}, \frac{c_\Gamma}{2}]$. Projecting on base torus,

$$\exp(re^t \bar{V}) \exp(e^{-\frac{1}{3}t} \tilde{x}_2 \bar{X}_2 + e^{-\frac{1}{3}t} \tilde{x}_3 \bar{X}_3) \in \overline{\Gamma}. \tag{3.95}$$

we obtain that re^t is return time for the projected toral linear flow at distance at most distance $e^{-t/3} c_\Gamma/2$.

Definition 3.68. *Let $R_t(x, T)$ denote the set of $r \in [-T, T]$ such that the equation (3.95) on projected torus has a solution $\tilde{x}_2, \tilde{x}_3 \in [-\frac{c_\Gamma}{2}, \frac{c_\Gamma}{2}]$.*

By construction for every $r \in R_t(x, T)$, the solution $\tilde{x}_i := \tilde{x}_i(r)$ of the identity in formula (3.94) is unique. Recall that $w_{\Omega_t(r)}(s)$ be the (inner) width function for $s \in [0, T]$.

Lemma 3.69. *The following average-width estimation holds.*

$$\frac{1}{T} \int_0^T \frac{ds}{w_{\Omega_t(r)}(s)} \leq \frac{1024}{c_\Gamma^2} \frac{C_\alpha}{e^{\frac{2}{3}t} \|(\tilde{x}_2(r), \tilde{x}_3(r))\|}. \tag{3.96}$$

Proof. Given $r \in R_t(x, T)$, let $\mathcal{S}(r)$ be the set of $s \in [0, T]$ such that there exists a solution of identity which fails injectivity. Here we approximate concrete width estimates with counting principles. By its definition, $\mathcal{S}(r)$ is a union of intervals I^* of length at most $\max\{c_\Gamma|\tilde{x}_2(r)|^{-1}e^{-5t/6}/2, c_\Gamma|\alpha\tilde{x}_3(r)|^{-1}e^{-5t/6}/2\}$. To count the number of such intervals, we will choose certain points where the distance is minimized. As long as $|\tilde{x}_2(r)| \geq e^{-5t/6}$, there exists solution s^* of the equation $\tilde{y_1}(t,s) = \tilde{y_1} + e^{\frac{5}{6}t}s\tilde{x}_2$. The same holds for $|\tilde{x}_3(r)|$.

Let $\mathcal{S}^*(r)$ be the set of all such solutions. Its cardinality can be estimated by counting points.

Claim.

$$\#\mathcal{S}^*(r) \leq c_\Gamma^{-1} C_\alpha \left\|(\tilde{x}_2(r), \tilde{x}_3(r))\right\| e^{\frac{1}{6}t}T. \tag{3.97}$$

Proof. Let's say s^* is almost crossing point on the manifold M which means the distance between orbit and its return is minimized.

$$s^* = \min_s \max\{|\tilde{y_1}(t,s)|, |\tilde{y_2}(t,s)|\}$$

with

$$\tilde{y_1}(t,s) = \tilde{y_1} + e^{\frac{5}{6}t}s\tilde{x}_2$$

$$\tilde{y_2}(t,s) = \tilde{y_2} + \alpha e^{\frac{5}{6}t}s\tilde{x}_3 + 1/2e^{-\frac{1}{2}t}(x_2x_3' - x_2'x_3)$$

If either distance $|\tilde{y_1}(t,s)|$ or $|\tilde{y_2}(t,s)|$ dominates another, then it reduces to simply finding a solution to single equation. For other case, we assume $|\tilde{y_1}(t,s)| = |\tilde{y_2}(t,s)|$. We distinguish following two cases. In either case, restrict on either $\tilde{y_1}(r) = 0$ or $\tilde{y_2}(r) = 0$ in specific subspace for convenience.

If $\tilde{y_1}(t,s) = \tilde{y_2}(t,s)$, then assuming $\tilde{y_1}(r) = 0$, we obtain

$$s = \frac{e^{-\frac{5}{6}t}(\tilde{y_2} + 1/2e^{-\frac{1}{2}t}(x_2x_3' - x_2'x_3))}{\tilde{x}_2(r) - \alpha\tilde{x}_3(r)}$$

If $\tilde{y_1}(t,s) = -\tilde{y_2}(t,s)$, then

$$s = \frac{e^{-\frac{5}{6}t}(\tilde{y_2} + 1/2e^{-\frac{1}{2}t}(x_2x_3' - x_2'x_3))}{\tilde{x}_2(r) + \alpha\tilde{x}_3(r)}$$

From bound $|\tilde{y_2} + 1/2e^{-\frac{1}{2}t}(x_2x_3' - x_2'x_3)| \geq |2/c_\gamma - 16/c_\gamma^2|$, we can count

$$\#\mathcal{S}^*(r) \leq \begin{cases} (2c_\gamma - 16)/c_\Gamma^{-2}|\tilde{x}_2(r) - \alpha\tilde{x}_3(r)|e^{\frac{1}{6}t}T & \text{if } \tilde{x}_2(r)\tilde{x}_3(r) < 0 \\ (2c_\gamma - 16)/c_\Gamma^{-2}|\tilde{x}_2(r) + \alpha\tilde{x}_3(r)|e^{\frac{1}{6}t}T & \text{if } \tilde{x}_2(r)\tilde{x}_3(r) > 0 \end{cases}$$

thus, we are done. □

Since $\#\mathcal{S}^*(r)$ counts specific subspace on whole components, it suffices to conclude the number of interval has same bounds with $\#\mathcal{S}^*(r)$.

Define the set $\Omega_t(x,T)(r) \subset [0,T] \times \mathbb{R}^5$ as follows. Let

$$\Omega_t(r) := \{(s, x_2, x_3, y_1, y_2, z) \mid \max\{|x_2|, |x_3|, |y_1|, |y_2|\} < \delta_r(t,s), |z| < c_\Gamma/16\}.$$

and

$$\Omega_t(x,T) = \bigcap_{r \in R_t(x,T)} \Omega_t(r).$$

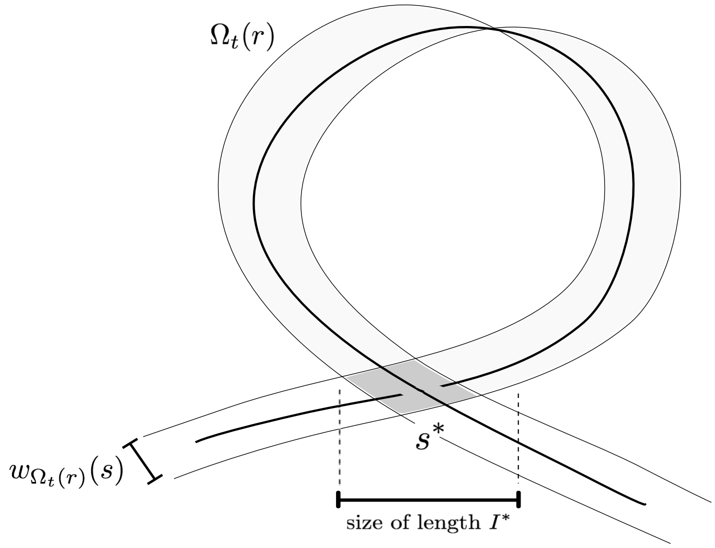

Figure 3.2: Illustration of width function and related quantities

Under above construction, the map ϕ_x is injective on $\Omega_t(x,T)$. The open set $\Omega_t(r) \cap \Omega_t(-r)$ are narrowed near both endpoints of the return time r so that their images in M have no self-intersections given by return times r and $-r$.

For every $r \in R_t(x,T)$ and every $s \in [0,T]$, we define function

$$\delta_r(t,s) = \begin{cases} \frac{1}{16}\,\|(\tilde{x}_2(r),\tilde{x}_3(r))\|\,|(s-s^*)e^{\frac{5t}{6}}| & \text{for s} \in I^* \text{ with } |s-s^*| \geq e^{-\frac{5}{6}t} \\ \frac{1}{16}\,\|(\tilde{x}_2(r),\tilde{x}_3(r))\| & \text{for s} \in I^* \text{ with } |s-s^*| \leq e^{-\frac{5}{6}t} \\ \frac{c_\Gamma}{16} & \text{for all s} \in [0,T]\backslash\mathcal{S}(r) \end{cases}$$

associated with $\Omega_t(r)$.

By the definition of inner width and by construction of the set $\Omega_t(r)$ we have that

$$w_{\Omega_t(r)}(s) = \delta_r(t,s)^2,\ \forall s \in [0,T]$$

from which it follows that for every subinterval $I^* \subset S(r)$ we have (using definition of δ_r)

$$\int_{I^*} \frac{ds}{w_{\Omega_t(r)}(s)} \leq \frac{512}{e^{\frac{5t}{6}} \left\|(\tilde{x}_2(r), \tilde{x}_3(r))\right\|^2}.$$

By the upper bound on the length of interval I^* and on the cardinality of the set $\mathcal{S}^*(r)$ we finally derive the conclusion. $\square$

Recall from Definition 3.39, we choose simultaneously Diophantine number $\alpha \in \mathbb{R}^2 \backslash \mathbb{Q}^2$ of exponent $\nu \geq 1$.

Lemma 3.70. *Given Diophantine condition of exponent $\nu \geq 1$, there exists a constant $C(\alpha) > 0$ such that all solutions of formula (3.95) satisfy the following lower bound*

$$\|(\tilde{x}_2, \tilde{x}_3)\|_{\mathbb{Z}^2} \geq C_\alpha e^{(\frac{1}{3} - \frac{\nu}{2})t} r^{-\frac{\nu}{2}}.$$

Proof. From projected identity (3.95) on base 3-torus,

$$(re^t, re^t\alpha + e^{-t/3}\tilde{x}_2, re^t\beta + e^{-t/3}\tilde{x}_3) \in \mathbb{Z}^3$$

If it holds, we set $re^t = q \in \mathbb{Z}$ and there exists $(p_1, p_2) \in \mathbb{Z}^2$ such that $p_1 - q\alpha = e^{-t/3}\tilde{x}_2,\ p_2 - q\beta = e^{-t/3}\tilde{x}_3$. By the Diophantine condition, there exists a constant $C(\alpha)$ such that

$$\begin{aligned}\|(\tilde{x}_2, \tilde{x}_3)\|_{\mathbb{Z}^2} &= e^{\frac{1}{3}t} \|(p_1 - q\alpha, p_2 - q\beta)\|_{\mathbb{Z}^2} \\ &= e^{\frac{1}{3}t} \|(q\alpha, q\beta)\|_{\mathbb{Z}^2} \\ &= e^{\frac{1}{3}t} \|q(\alpha)\|_{\mathbb{Z}^2} \\ &\geq C_\alpha e^{\frac{1}{3}t} q^{-\frac{\nu}{2}}\end{aligned}$$

which proves the statement. □

For every $n \in \mathbb{N}$, let $R_t^{(n)}(x,T) \subset R_t(x,T)$ characterized by

$$\max(\tilde{x}_2(r), \tilde{x}_3(r)) \in (\frac{c_\Gamma}{2^{n+1}}, \frac{c_\Gamma}{2^n}].$$

Lemma 3.71. *If the frequency of the projected linear flow satisfy Diophantine condition of exponent $\nu = \sqrt{2} + \epsilon$, for all $\epsilon > 0$, then there exists $C_\epsilon > 0$ such that*

$$\# R_t^{(n)}(x,T) \leq C_\epsilon(\bar{V}) T \frac{c_\Gamma}{2^n} e^{\frac{2}{3}t + \frac{\epsilon t}{2}} \tag{3.98}$$

Proof. Under a Diophantine condition of exponent $\nu \geq 1$, from inequality (3.67) and definition of $R_t^{(n)}(x,T)$, we have following:

$$\# R_t^{(n)}(x,T) \leq C_\nu(V) \max\{(Te^t)^{1-\frac{1}{\nu}}, Te^t \frac{c_\Gamma}{2^n} e^{-\frac{5t}{6}}\} \tag{3.99}$$

It suffices to show $(Te^t)^{1-\frac{1}{\nu}}$ is less than or equal to the desired bound. From

Lemma 3.70,

$$(Te^t)^{1-\frac{1}{\nu}}/\,\|(\tilde{x}_2,\tilde{x}_3)\| \le (Te^t)^{1-\frac{1}{\nu}} C_\alpha e^{(-\frac{1}{3}+\frac{\nu}{2})t} r^{\frac{\nu}{2}}$$
$$\le T^{1+\frac{\nu}{2}-\frac{1}{\nu}} C_\alpha e^{(\frac{2}{3}+\frac{\nu}{2}-\frac{1}{\nu})t}.$$

Limiting $\nu \to \sqrt{2}$,

$$(Te^t)^{1-\frac{1}{\nu}}/\,\|(\tilde{x}_2,\tilde{x}_3)\| \le C_\alpha T e^{(\frac{2}{3}+\frac{\epsilon}{2})t}.$$

Approximating $\|(x_2,x_3)\| \sim 1/2^n$,

$$(Te^t)^{1-\frac{1}{\nu}} \le C_\epsilon(\bar{V}) T \frac{c_\Gamma}{2^n} e^{(\frac{2}{3}+\frac{\epsilon}{2})t}.$$

□

By combining counting return time and width estimates, we obtain uniform bound.

Proposition 3.72. *There exists a constant*

$$\frac{1}{T}\int_0^T \frac{ds}{w_{\Omega_t(r)}(s)} \le C_\epsilon(V) e^{\epsilon t}.$$

Proof. By inequality (3.96) and lemma 3.71

$$\frac{1}{T}\int_0^T \frac{ds}{w_{\Omega_t(r)}(s)} \leq \sum_{r\in R_t}\left(\frac{1024}{c_\Gamma^2}\frac{C_\alpha}{e^{\frac{2}{3}t}\,\|(\tilde{x}_2(r),\tilde{x}_3(r))\|}\right)$$

$$\leq C_\epsilon(V)e^{\epsilon t}.$$

□

Corollary 3.73. *For every nilflow generated with V such that projected flow on $\mathbb{T}^3$ is Diophantine condition of $\nu \in [1, \sqrt{2}+\epsilon]$. For every $\epsilon > 0$, there exists a constant $C_\epsilon(V) > 0$ such that, for all $t \geq 0$ and for all $(x,T) \in M \times \mathbb{R}$ we have*

$$w_{\mathcal{F}(t)}(x,T) \geq C_\epsilon(V)^{-1}e^{-\epsilon t}.$$

Proof of Theorem 3.4. By Corollary 3.73, it goes without quoting Good points technique and Lyapunov norm. Improved bound of R in Theorem 3.52 can be obtained.

$$|R(g)|_{-r} \leq C_r(1+\delta_{\mathcal{O}}^{-1})^{r-2}T^{-1} \tag{3.100}$$

We revisit backward iteration scheme introduced in proof of Theorem 3.54. We know that

$$\begin{aligned}|\gamma|_{-r,0} &\leq |D_0|_{-r,0} + |R_0|_{-r,0} \\ &\leq |D_0|_{-r,0} + C_r(1+\delta_{\mathcal{O}}^{-1})^{r-2}T^{-1}\end{aligned} \tag{3.101}$$

and

$$|D_0|_{-r,0} \leq |D_N|_{-r,0} + \sum_{j=1}^{N} |R'_{j-1}|_{-r,0}. \tag{3.102}$$

Changing the length to 1 and by uniform width bound from corollary 3.73,

$$|D_N|_{-r,0} \leq C_r T^{-1/12} |D_N|_{-s,\mathcal{F}(t_N)} \leq C w_{\mathcal{F}(t_N)}(x,1)^{-1/2} \leq C_\epsilon T^\epsilon \tag{3.103}$$

Then, by inductive argument resembling (3.85),

$$|D_0|_{-s,0} \leq C_{r,s} T^{-1/12} \left(|D_N|_{-r,\mathcal{F}(t_N)} + \sum_{j=1}^{N} C_j T_j^{1/12} |R_{j-1}|_{-r,\mathcal{F}(t_{j-1})} \right) \tag{3.104}$$

Therefore

$$|\gamma|_{-r,0} \leq C_r(\mathcal{O}) T^{-1/12+\epsilon}.$$

Finally, we glue all the function on irreducible representation $H_{\mathcal{O}}$, which only increases the regularity accordingly. □

3.8 Application : Mixing of nilautomorphism.

In this section, as a further application of main equidistribution results, we can verify the explicit rate of exponential mixing of hyperbolic automorphism relying on renormalization argument.

Let $\mathfrak{F}_{2,3} = \{X_1, X_2, Y_1, Z_1, Z_2\}$ be step 3 free nilpotent Lie algebra with two generators with commutation relations

$$[X_1, X_2] = Y_1, \quad [X_1, Y_1] = Z_1, \quad [X_2, Y_1] = Z_2.$$

The group of automorphism on Lie algebras induces automorphism on the nilmanifold

$$Aut(\mathbf{n}) = \left\{ \begin{bmatrix} A & & \\ & 1 & \\ & & A \end{bmatrix}, \ A \in SL(2, \mathbb{Z}) \right\}$$

and we consider a hyperbolic automorphism T with an eigenvalue $\lambda > 1$ with corresponding eigenvector $V = X_1 + \alpha X_2$ satisfying Diophantine condition $(1, \alpha)$ on base torus $\mathbb{T}^2$. By direct computation, the following renormalization holds :

$$T \circ \exp(tV) = \exp(t\lambda V) \circ T.$$

Theorem 3.74. *Let (ϕ_V^t) be a nilflow on 3-step nilmanifold $M = \mathfrak{F}_{2,3}/\Gamma$ such that the projected toral flow $(\bar{\phi}_V^t)$ is a linear flow with frequency vector $v := (1, \alpha)$ in Roth-type Diophantine condition (with exponent $\nu = 1 + \epsilon$ for all $\epsilon > 0$). For every $s > 12$, there exists a constant C_s such that for every zero-average function*

$f \in W^s(M)$, *for all* $(x,T) \in M \times \mathbb{R}$, *we have*

$$\left| \frac{1}{T} \int_0^T f \circ \phi_V^t(x) dt \right| \leq C_s T^{-1/6+\epsilon} \|f\|_s . \tag{3.105}$$

The detailed computation follows from the section 3.7 and it is similar to the case of step 3 filiform [For16]. The only difference with filiform is that it has an extra element in center which is redundant in actual calculation on width, only raising required regularity of zero-average function.

Proposition 3.75. *Hyperbolic nilautomorphism* T *is exponential mixing.*

Proof. Let $f, g \in C^1(M)$ be smooth. Define $\langle f, g\rangle = \int_M fg d\mu$. Since Haar measure is invariant under ϕ_V^t,

$$\langle f \circ T^n, g\rangle = \int_0^1 \langle f \circ T^n \circ \phi_V^t, g \circ \phi_V^t\rangle dt.$$

Integration by parts,

$$\langle f \circ T^n, g\rangle = \langle \int_0^1 f \circ T^n \circ \phi_V^t dt, g \circ \phi_V^t\rangle \tag{3.106}$$

$$- \int_0^1 \langle \int_0^t f \circ T^n \circ \phi_V^s ds, Vg \circ \phi_V^t\rangle dt. \tag{3.107}$$

Therefore,

$$\langle f \circ T^n, g\rangle = (\|g\|_\infty + \|Vg\|_\infty) \int_M \sup_{s\in[0,1]} \left| \int_0^s f \circ T^n \circ \phi_V^t dt \right| d\mu. \tag{3.108}$$

By renormalizing the flow, $T^n \circ \phi_V^t = \phi_V^{\lambda^n t} \circ T^n$ and

$$\begin{aligned}\int_0^s f \circ T^n \circ \phi_V^t(x)dt &= \int_0^s f \circ \phi_V^{\lambda^n t} \circ T^n(x)dt \\ &= \frac{1}{\lambda^n}\int_0^{\lambda^n s} f \circ \phi_V^t \circ T^n(x)dt.\end{aligned}$$

Therefore, by the result of equidistribution (3.105),

$$\langle f \circ T^n, g\rangle \leq \lambda^{(-1/6+\epsilon)n} \|f\|_s (\|g\|_\infty + \|Vg\|_\infty) \to 0. \tag{3.109}$$

□

3.9 Appendix: Free group type of step 5 with 3 generators.

In this appendix, we introduce specific example of nilpotent Lie algebra which goes beyond our approach introduced in the section 3.5. In this example, we will show the failure of transversality condition.

Let $\mathfrak{F}_n$ be free nilpotent Lie algebra with n generators and $(\mathfrak{F}_n)_{k+1}$ be $k+1$th subalgebra in central series, following notation in (2.1). Denote $\mathfrak{F}_{n,k} := \mathfrak{F}_n/(\mathfrak{F}_n)_{k+1}$ quotient of free algebra with n generators $\mathfrak{F}_n$ and it is finite dimensional.

Definition 3.76. *Let* $\mathbf{n}$ *be nilpotent Lie algebra satisfying* generalized transversality condition *if there exists basis* (X_α, Y_Λ) *of* $\mathbf{n}$ *for each irreducible representation* $\pi_\Lambda^{X_\alpha}$ *such that*

$$\langle \mathfrak{G}_\alpha \rangle + Ran(ad_{X_\alpha}) + C_{\mathfrak{I}}(\pi_\Lambda^{X_\alpha}) = \mathbf{n} \tag{3.110}$$

where $C_{\mathfrak{I}}(\pi_\Lambda^{X_\alpha}) = \{Y \in \mathfrak{I} \mid \Lambda([Y, X_\alpha]) = 0\}$.

Generalized transversality condition implies existence of completed basis for each irreducible representation $\pi_\Lambda^{X_\alpha}$ of non-zero degree. That is, given adapted basis $\mathcal{F} = (X, Y_1, \cdots, Y_a)$, there exists reduced system $\bar{\mathcal{F}} = (X, Y_1', \cdots, Y_{a'}')$ satisfying transversality condition (3.31) and $\pi_\Lambda^X(Y_m') = 0$ for all $a' \leq m \leq a$.

Now we will investigate an example that fails transversality condition as well as that in the sense of representation. Let $\mathcal{F} = (X, Y_i^{(j)})$ be basis of $\mathfrak{F}_{5,3}$ with generators $\{X_1, X_2, X_3\}$ with the following relations:

$$X_1 \quad X_2 \quad X_3$$

$$Y_1 \quad Y_2 \quad Y_3$$

$$Z_1 \quad Z_2 \quad \cdots \quad Z_8 \quad Z_9$$

with

$$[X_1, X_2] = Y_1, \quad [X_2, X_3] = Y_2, \quad [X_1, X_3] = Y_3$$

$$[X_1, Y_1] = Z_1, \quad [X_1, Y_2] = Z_2, \quad [X_1, Y_3] = Z_3$$

$$[X_2, Y_1] = Z_4, \quad [X_2, Y_2] = Z_5, \quad [X_2, Y_3] = Z_6$$

$$[X_3, Y_1] = Z_7, \quad [X_3, Y_2] = Z_8, \quad [X_3, Y_3] = Z_9$$

and rest of elements are generated commutation relations with these. In general, we write elements $Y_j^{(i)} \in \mathbf{n}_i \backslash \mathbf{n}_{i+1}$ and $Y_i^{(5)} \in \mathfrak{Z}(\mathbf{n})$ for all i. By Jacobi-identity

$$[X_1, [X_2, X_3]] + [X_2, [X_3, X_1]] + [X_3, [X_1, X_2]] = 0 \iff Z_2 - Z_6 + Z_7 = 0.$$

For fixed α_i and β_i, let

$$V = X_1 + \alpha_2 X_2 + \alpha_3 X_3 + \beta_1 Y_1 + \beta_2 Y_2 + \beta_3 Y_3$$

and set $\mathfrak{I}$ ideal of $\mathfrak{F}_{5,3}$ codimension 1, not containing V.

Proposition 3.77. *$\mathfrak{F}_{5,3}$ does not satisfy generalized transversality condition for some irreducible representation.*

Proof. To find centralizer in Lie algebra, for $a_i, b_i \in \mathbb{R}$, set

$$[V, X] = 0 \iff X = a_1 X_1 + a_2 X_2 + a_3 X_3 + b_1 Y_1 + b_2 Y_2 + b_3 Y_3 + c_1 Z_1 + \cdots + c_8 Z_8.$$

Then, it contains

$$(a_2 - \alpha_2 a_1) Y_1 + (\alpha_2 a_3 - \alpha_3 a_2) Y_2 + (a_3 - \alpha_3 a_1) Y_3$$
$$+ (b_1 - \beta_1 a_1) Z_1 + (b_2 - \beta_2 a_1) Z_2 + (b_3 - \beta_3 a_1) Z_3 + \cdots = 0.$$

By linear independence, all the coefficients vanish and it remains

$$a_1 X_1 + a_2 X_2 + a_3 X_3 = a_1 (X_1 + \alpha_2 X_2 + \alpha_3 X_3)$$

$$b_1Y_1 + b_2Y_2 + b_3Y_3 = a_1(\beta_1Y_1 + \beta_2Y_2 + \beta_3Y_3)$$

Therefore, there is no non-trivial element in $C_{\mathfrak{Z}}(V) \cap \mathbf{n}_2 \backslash \mathbf{n}_3$. Since range of ad_V has rank 2, this model does not satisfy transversality condition in the Lie algebra level.

Now, we verify *generalized transversality condition* is not satisfied on some irreducible representation. By Schur's lemma, an irreducible representation π^V_Λ acts as a constant on center.

Assume $\pi_*(W_i) = s_i I \neq 0$ for some $W_i \in \mathfrak{Z}(\mathbf{n})$. Then, it is possible to choose element $L_i \in \mathbf{n}_2 \backslash \mathbf{n}_3$ such that

$$\begin{cases} \pi_*([V, L_1]) = (a_1t^2 + a_2t + a_3) \\ \pi_*([V, L_2]) = (b_1t^2 + b_2t + b_3) \\ \pi_*([V, L_3]) = (c_1t^2 + c_2t + c_3) \end{cases}$$

with (a_i, b_i, c_i) are non-proportional for each i, and

$$\pi_*(\mathrm{ad}^3_V(L_i)) = \pi_*(W_i) \neq 0.$$

However, on given irreducible representation, any linear combination of L_1, L_2

and L_3 does not give any trivial relation. If $s_1L_1 + s_2L_2 + s_3L_3 \in C_{\mathfrak{J}}(\pi_\Lambda^V)$, then

$$\begin{aligned}
&\pi_*([V, s_1L_1 + s_2L_2 + s_3L_3]) \\
&= s_1(a_1t^2 + a_2t + a_3) + s_2(b_1t^2 + b_2t + b_3) + s_3(c_1t^2 + c_2t + c_3) \\
&= (s_1a_1 + s_2b_1 + s_3c_1)t^2 + (s_1a_2 + s_2b_2 + s_3c_2)t + (s_1a_3 + s_2b_3 + s_3c_3) = 0.
\end{aligned}$$

The system of equations has trivial solution ($t = 0$) by linear independence of each coefficients. Then, there does not exist any element of $\mathbf{n}_2 \backslash \mathbf{n}_3$ that has degree 0. However, range of ad_V has rank 2 and generalized transversality condition cannot be satisfied in this example. $\square$

Chapter 4: Higher rank actions on Heisenberg nilmanifolds

In this chapter, our main results are on limit distributions of higher rank abelian actions. We firstly introduce the Bufetov functional for higher rank abelian actions under bounded type of Diophantine conditions. Our main argument is based on the renormalization argument for higher rank actions by induction argument. This is a key idea used in Cosentino and Flaminio in [CF15], but we extend their constructions to rectangular shape and derive the deviation of ergodic averages of higher rank actions. Likewise, this explains the duality between Bufetov functionals and invariant currents appeared in [CF15]. The crucial part is handling the estimation of deviation ergodic averages on (stretched) rectangles, and this enables to derive our main theorems.

As a corollary, we can prove there exists a limit distributions of (normalized) ergodic integrals of abelian actions with variance 1. More specifically, for almost all limit of normalized ergodic integrals of converges in distribution to a nondegenerate compactly supported measure on the real line, which is certain form of Bufetov functionals. This generalizes a limit theorem for theta series on Siegel half spaces, which introduced in the works of Götze and Gordin [GG04] and Marklof [Mar99]. (See [Tol78,MM07,MNN07] for general introduction and nilflow case [GM14,CM16].)

4.1 Main theorem

One of the main objects of this section is a space of finitely-additive measures defined on the space of all squares on Heisenberg manifold M. We state our results beginning with an overview of Bufetov functional.

Definition 4.1. *For* $(m, \boldsymbol{T}) \in M \times \mathbb{R}^d_+$, *denote the* standard rectangle *for action* P,

$$\Gamma^X_{\boldsymbol{T}}(m) = \{\mathsf{P}^{d,\alpha}_{\boldsymbol{t}}(m) \mid \boldsymbol{t} \in U(\boldsymbol{T}) = [0, \boldsymbol{T}^{(1)}] \times \cdots \times [0, \boldsymbol{T}^{(d)}]\}. \tag{4.1}$$

Let $\mathsf{Q}^{d,Y}_y := \exp(y_1 Y_1 + \cdots + y_d Y_d)$, $y = (y_1, \cdots, y_d) \in \mathbb{R}^d$ be the action generated by elements Y_i of standard basis. Set $\phi^Z_z := \exp(zZ)$ is the flow generated by central element Z.

Definition 4.2. *Let* $\mathfrak{R}$ *be the collection of the* generalized rectangles *in* M. *For each* $1 \leq d \leq g$ *and* $\boldsymbol{t} = (t_1, \cdots, t_d)$,

$$\mathfrak{R} := \bigcup_{1 \leq i \leq d} \bigcup_{(y,z) \in \mathbb{R}^d \times \mathbb{R}} \bigcup_{(m, \boldsymbol{T}) \in M \times \mathbb{R}^d_+} \{(\phi^Z_{t_i z}) \circ \mathsf{Q}^{d,Y}_y \circ \mathsf{P}^{d,\alpha}_{\boldsymbol{t}}(m) \mid \boldsymbol{t} \in U(\boldsymbol{T})\}.$$

Theorem 4.3. *For any irreducible representation* H, *there exists a measure* $\hat{\beta}_H(\Gamma) \in \mathbb{C}$ *for every rectangle* $\Gamma \in \mathfrak{R}$, *such that the following holds:*

1. (Additive property) *For any decomposition of disjoint rectangles* $\Gamma = \bigcup_{i=1}^n \Gamma_i$ *or those intersections have zero measure,*

$$\hat{\beta}_H(\alpha, \Gamma) = \sum_{i=1}^n \hat{\beta}_H(\alpha, \Gamma_i).$$

2. (Scaling property) *For* $\boldsymbol{t} \in \mathbb{R}^d$,

$$\hat{\beta}_H(r_{\boldsymbol{t}}[\alpha], \Gamma) = e^{-(t_1 + \cdots t_d)/2} \hat{\beta}_H(\alpha, \Gamma).$$

3. (Invariance property) *For any action* $\mathsf{Q}^{d,Y}_{\tau}$ *generated by* Y_i*'s and* $\tau \in \mathbb{R}^d_+$,

$$\hat{\beta}_H(\alpha, (\mathsf{Q}^{d,Y}_{\tau})_* \Gamma) = \hat{\beta}_H(\alpha, \Gamma).$$

4. (Bounded proprety) *For any rectangle* $\Gamma \in \mathfrak{R}$, *there exists a constant* $C(\Gamma) > 0$ *such that for* $\hat{X} = \hat{X}_1 \wedge \cdots \wedge \hat{X}_d$,

$$|\hat{\beta}_H(\alpha, \Gamma)| \leq C(\Gamma)(\int_{\Gamma} |\hat{X}|)^{d/2}.$$

For arbitrary rectangle $U_{\mathbf{T}} = [0, \mathbf{T}^{(1)}] \times \cdots \times [0, \mathbf{T}^{(d)}]$, pick $\mathbf{T}'^{(i)} \in [0, \mathbf{T}^{(i)}]$ for each i to decompose $U_{\mathbf{T}}$ into 2^d sub-rectangles. We write $\mathbf{P}(\mathbf{T}')$ collection of 2^d vertices $v = (v^{(1)}, v^{(2)}, \cdots, v^{(d)})$ where $v^{(i)} \in \{0, \mathbf{T}'^{(i)}\}$. Let $U_{\mathbf{T},v}$ be a rectangle whose sides $\boldsymbol{I}_v = (I^{(1)}, I^{(2)}, \cdots, I^{(d)}) \in \mathbb{R}^d_+$ where

$$I^{(l)} = \begin{cases} T^{(l)} - T'^{(l)} & \text{if} \quad v^{(l)} = T'^{(l)} \\ T'^{(l)} & \text{if} \quad v^{(l)} = 0. \end{cases}$$

Then, we have $U_{\mathbf{T}} = \bigcup_{v \in \mathbf{P}(\mathbf{T}')} U_{\mathbf{T},v}$.

Theorem 4.4. *Let us denote* $\beta_H(\alpha, m, \boldsymbol{T}) := \hat{\beta}_H(\alpha, \Gamma^X_{\boldsymbol{T}}(m))$. *The function* β_H

satisfies following properties:

1. (Cocycle property) *For all* $(m, \boldsymbol{T}_1, \boldsymbol{T}_2) \in M \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\beta_H(\alpha, m, \boldsymbol{T}_1 + \boldsymbol{T}_2) = \sum_{v \in \mathbf{P}(\boldsymbol{T}_1)} \beta_H(\alpha, \mathsf{P}_v^{d,\alpha}(m), \boldsymbol{I}_v).$$

2. (Scaling property) *For all* $m \in M$,

$$\beta_H(r_{\boldsymbol{t}}\alpha, m, \boldsymbol{T}) = e^{(t_1 + \cdots t_d)/2} \beta_H(\alpha, m, \boldsymbol{T}).$$

3. (Bounded proprety) *Let us denote largest length of side* $T_{max} = \max_i \boldsymbol{T}^{(i)}$. *Then, there exists a constant* $C_H > 0$ *such that*

$$\beta_H(\alpha, m, \boldsymbol{T}) \leq C_H T_{max}^{d/2}.$$

4. (Orthogonality) *For all* $[\alpha] \in DC$ *and all* $\boldsymbol{T} \in \mathbb{R}^d$, *bounded function* $\beta_H(\alpha, \cdot, \boldsymbol{T})$ *belongs to the irreducible component, i.e,*

$$\beta_H(\alpha, \cdot, \boldsymbol{T}) \in H \subset L^2(M).$$

By representation theory introduced (4.4), for any $f \in W^s(M)$ has a decomposition

$$f = \sum_H f_H$$

and we define *Bufetov cocycle* associated to f (or form ω) as the sum

$$\beta^f(\alpha, m, \boldsymbol{T}) = \sum_H D_\alpha^H(f)\beta_H(\alpha, m, \boldsymbol{T}). \tag{4.2}$$

Remark 4.5. *For convenience, identification of distributions of form ω and function f were used. The formula (4.2) yields a duality between the space of basic (closed) currents and invariant distributions. In the similar setting for horocycle flow, refer [BF14, Cor 1.2, p.10].*

Given a Jordan region U and a point $m \in M$, set $P_U^{d,\alpha} m$ the Birkhoff sums associated to some $m \in M$ for the action $\mathsf{P}_x^{d,\alpha}$ given by

$$\left\langle P_U^{d,\alpha} m, \omega \right\rangle := \int_U f(\mathsf{P}_x^{d,\alpha} m) dx_1 \cdots dx_d$$

for any degree $\mathfrak{p}$-form $\omega = f\hat{X}_1^\alpha \wedge \cdots \wedge \hat{X}_d^\alpha$, with $f \in C_0^\infty(M)$ (smooth function with zero averages).

Let the family of random variable

$$E_{T_n}(f) := \frac{1}{vol(U(T_n))^{1/2}} \left\langle P_{U(T_n)}^{d,\alpha}(\cdot), \omega_f \right\rangle,$$

and we are interested in asymptotic behavior of the probability distributions of $E_{T_n}(f)$. Our goal is to understand the asymptotics of E_{T_n}.

Theorem 4.6. *For every closed form $\omega_f \in \Lambda^d \mathfrak{p} \otimes W^s(M)$ with $s > s_{d,g} = d(d + 11)/4 + g + 1/2$, which is not a coboundary, the limit distribution of the family* *of random variables $E_{T_n}(f)$ exists, and for almost all frequency α, it has compact support on the real line.*

4.2 Preliminary

We review definitions about Heisenberg manifold and its moduli space.

4.2.1 Heisenberg manifold

Let H^g be standard $2g+1$ dimensional Heisenberg group and set $\Gamma := \mathbb{Z}^g \times \mathbb{Z}^g \times \frac{1}{2}\mathbb{Z}$ a discrete and co-compact subgroup of H^g. We shall call it standard lattice of H^g and the quotient $M := \mathsf{H}^g/\Gamma$ will be called *Heisenberg manifold.* Lie algebra $\mathfrak{h}^g = Lie(\mathsf{H}^g)$ of H^g is equipped with basis $(X_1, \cdots, X_g, Y_1, \cdots Y_g, Z)$ satisfying canonical commutation relation

$$[X_i, Y_j] = \delta_{ij} Z. \tag{4.3}$$

For $1 \leq d \leq g$, let $\mathsf{P}^d < \mathsf{H}^g$ be the subgroup with Lie algebra $\mathfrak{p}$ generated by $(X_1, \cdots, X_d)$ and for any $\alpha \in Sp_{2g}(\mathbb{R})$, set $(X_i^\alpha, Y_i^\alpha, Z) = \alpha^{-1}(X_i, Y_i, Z)$ for $1 \leq i \leq d$. We define parametrization of the subgroup $\alpha^{-1}(\mathsf{P}^d)$

$$\mathsf{P}_x^{d,\alpha} := \exp(x_1 X_1^\alpha + \cdots + x_d X_d^\alpha), \quad x = (x_1, \cdots, x_d) \in \mathbb{R}^d.$$

By central extension of $\mathbb{R}^{2g}$ by $\mathbb{R}$, we have an exact sequence

$$0 \to Z(H^g) \to H^g \to \mathbb{R}^{2g} \to 0.$$

The natural projection map $pr : M \to H^g/(\Gamma Z(H^g))$ maps M onto a $2g$-dimensional torus $\mathbb{T}^{2g} := \mathbb{R}^{2g}/\mathbb{Z}^{2g}$.

4.2.1.1 Moduli space

The group of automorphisms of H^g that are trivial on the center is $Aut_0(\mathsf{H}^g) = Sp(2g, \mathbb{R}) \ltimes \mathbb{R}^{2g}$. Since dynamical properties of actions are invariant under inner automorphism, we restrict our interest to $Sp(2g, \mathbb{R})$. We call *moduli space* of the standard Heisenbeg manifold the quotient $\mathfrak{M}_g = Sp(2g, \mathbb{R})/Sp(2g, \mathbb{Z})$. We regard $Sp(2g, \mathbb{R})$ as the deformation space of the the standard Heisenbeg manifold M and $\mathfrak{M}_g$ as the moduli space of M.

Siegel modular variety is double coset space $\Sigma_g = K_g \backslash Sp_{2g}(\mathbb{R})/Sp_{2g}(\mathbb{Z})$ where K_g is maximal compact subgroup $Sp_{2g}(\mathbb{R}) \cap SO_{2g}(\mathbb{R})$ of $Sp_{2g}(\mathbb{R})$.

For $\alpha \in Sp_{2g}(\mathbb{R})$, we denote $[\alpha] := \alpha Sp_{2g}(\mathbb{Z})$ its projection on the moduli space $\mathfrak{M}_g$ and write $[[\alpha]] := K_g \alpha Sp_{2g}(\mathbb{Z})$ the projection of α to the Siegel modular variety Σ_g. Double coset $K_g \backslash Sp_{2g}(\mathbb{R})/1_{2g}$ is identifed to the Siegel upper half space $\mathfrak{H}_g := \{Z \in Sym_g(\mathbb{C}) \mid \Im(Z) > 0\}$. *Siegel upper half space* of genus g is complex manifold of symmetric complex $g \times g$ matrices $Z = X + iY$ with positive definite symmetric imaginary part $\Im(Z) = Y$ and arbitrary real part X. We note $\Sigma_g \approx Sp_{2g}(\mathbb{Z}) \backslash \mathfrak{H}_g$.

4.2.1.2 Sobolev bundles

Given basis (V_i) of Lie algebra, let $\Delta = -\sum V_i^2$ denote Laplacian via the standard basis. Similarly, denote Δ_α Laplacian defined by the basis $(\alpha^{-1})_* V_i$. For any $s \in \mathbb{R}$ and any C^∞ function $f \in L^2(M)$,

$$\|f\|_{\alpha,s} = \langle f, (1+\Delta_\alpha)^s f \rangle^{1/2}.$$

Let $W^s_\alpha(M)$ be the completion of $C^\infty(M)$ with above norm and denote $W^{-s}_\alpha(M)$ its dual space. Extending it to the exterior algebra, define the Sobolev spaces $\Lambda^d \mathfrak{p} \otimes W^s(M)$ of cochains of degree d, and use the same notations for the norms.

The group $Sp_{2g}(\mathbb{Z})$ acts on the right on the trivial bundles

$$Sp_{2g}(\mathbb{R}) \times W^s(M) \to Sp_{2g}(\mathbb{R}).$$

We obtain the quotient flat bundle of Sobolev spaces over the moduli space:

$$(Sp_{2g}(\mathbb{R}) \times W^s(M))/Sp_{2g}(\mathbb{Z}) \to \mathfrak{M}_g = Sp_{2g}(\mathbb{R})/Sp_{2g}(\mathbb{Z})$$

the fiber over $[\alpha] \in \mathfrak{M}_g$ is locally identified with the space $W^s_\alpha(M)$.

$Sp_{2g}(\mathbb{Z})$ acts on the right on the trivial bundles by

$$(\alpha, \varphi) \mapsto (\alpha, \varphi)\gamma = (\alpha\gamma, \gamma^* \varphi), \quad \gamma \in Sp_{2g}(\mathbb{Z}).$$

By invariance of $Sp_{2g}(\mathbb{Z})$ action, we denote the class (α, φ) by $[\alpha, \varphi]$ and write

$Sp_{2g}(\mathbb{Z})$-invariant Sobolev norm

$$\|(\alpha, f)\|_s := \|f\|_{\alpha,s} .$$

4.3 Analysis on Heisenberg manifolds

In this section, we will recall definitions of currents, representation and renormalization on moduli space.

4.3.1 Invariant currents

We denote the bundle of $\mathfrak{p}$-forms of degree j of Sobolev order s by $A^j(\mathfrak{p}, \mathfrak{M}^s)$. Similarly, there is a flat bundle of distribution $A_j(\mathfrak{p}, \mathfrak{M}^{-s})$ whose fiber over $[\alpha]$ is locally identified with the space $W_\alpha^{-s}(M)$ normed by $\|\cdot\|_{\alpha,-s}$.

In the following, we set $\omega^{d,\alpha} = dX_1^\alpha \wedge \cdots \wedge dX_d^\alpha$ a top dimensional $\mathfrak{p}$-form and identify d dimensional currents $\mathcal{D}$ with distributions, for any $f \in C^\infty(M)$

$$\langle \mathcal{D}, f \rangle := \langle \mathcal{D}, f\omega^{d,\alpha} \rangle.$$

Definition 4.7. *For $s > 0$, we denote $D \in Z_d(\mathfrak{p}, W^{-s}(M))$ a closed P-invariant currents of dimension d and Sobolev order s. Then, from formal identities, $\langle D, X_i^\alpha(f) \rangle = 0$ for all test function f and $i \in [1, d]$.*

By [CF15, Prop 3.13], for any $s > d/2$ with $d = \dim \mathsf{P}$, denote $I_d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$ the space of P-invariant currents of Sobolev order s, which coincides with the space of closed currents of dimension d.

- It is one dimensional space if $\dim \mathsf{P} = g$, or an infinite-dimensional space if $\dim \mathsf{P} < g$. We have $I_d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g)) \subset W^{-d/2-\epsilon}(\mathbb{R}^g)$ for all $\epsilon > 0$.

Let $\omega \in \Lambda^d \mathfrak{p}' \otimes W^s(\mathbb{R}^g)$ with $s > (d+1)/2$. Then, ω admits a primitive Ω if and only if $T(\omega) = 0$ for all $T \in I_d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g))$. We may have $\Omega \in \Lambda^{d-1}\mathfrak{p}' \otimes W^t(\mathbb{R}^g)$ for any $t < s - (d+1)/2$.

4.3.2 Representation

We write Hilbert sum decomposition

$$L^2(M) = \bigoplus_{n \in \mathbb{Z}} H_n \tag{4.4}$$

into closed H^g-invariant subspaces. For some fixed $K > 0$, we write $f = \sum_{n \in \mathbb{Z}} f_n \in L^2(M)$, $f_n \in H_n$ where

$$H_n = \{f \in L^2(M) \mid \exp(tZ)f = \exp(2\pi \imath n K t) f\}.$$

We also have $W^s(M) = \bigoplus_i W^s(H_i)$ of $W^s(M)$ into closed H^g-invariant subspaces $W^s(H_i) = W^s(M) \cap H_i$. The center $Z(H^g)$ has spectrum $2\pi\mathbb{Z}\backslash\{0\}$ the space splits as Hilburt sum of H^g-module H_i, which is equivalent to irreducible representation π.

Theorem 4.8. *[Stone-Von Neumann] For $\alpha = (X_i, Y_i, Z)$, the unitary irreducible representation π of the Heisenberg group of non-zero central parameter K, is unitar-*

ily equivalent to Schrödinger representation. For infinitesimal representation with parameter n for $k = 1, 2, \cdots, g$

$$D\pi(X_k) = \frac{\partial}{\partial x_k}, \quad D\pi(Y_k) = 2\pi\iota nKx_k, \quad D\pi(Z) = 2\pi\iota nK.$$

4.3.3 Best Sobolev constant

The *Sobolev embedding theorem* implies that for any $\alpha \in Sp(2g, \mathbb{R})$ and $s > g + 1/2$, there exists a constant $B_s(\alpha)$ such that for any $f \in W^s_\alpha(M)$,

$$\|f\|_\infty \leq B_s(\alpha) \|f\|_{s,\alpha}.$$

The *best Sobolev constant* is defined as the function on the group of automorphism

$$B_s(\alpha) = \sup_{f \in W^s_\alpha(M)} \frac{\|f\|_\infty}{\|f\|_{s,\alpha}}. \tag{4.5}$$

By Proposition 4.8 of [CF15], there exists a universal constant $C(s) > 0$ such that the *best Sobolev constant* satisfies the estimate

$$B_s([[\alpha]]) \leq C(s) \cdot (Hgt[[\alpha]])^{1/4}. \tag{4.6}$$

From the Sobolev embedding theorem and the definition of the best Sobolev constant, we have the following bound.

Lemma 4.9. *[CF15, Lemma 5.5] For any Jordan region $U \subset \mathbb{R}^d$ with Lebesgue* *measure $|U|$, for any $s > g+1/2$ and all $m \in M$,*

$$\left\|[\alpha, (P_U^{d,\alpha} m)]\right\|_{-s} \le B_s(\alpha)|U|.$$

4.3.4 Renormalization

Denote diagonal matrix $\delta_i = diag(d_1, \cdots, d_g)$ with $d_i = 1$, $d_k = 0$ if $k \neq i$. Then, for each $1 \le i \le g$, we denote $\hat{\delta}_i = \begin{bmatrix} \delta_i & 0 \\ 0 & -\delta_i \end{bmatrix} \in \mathfrak{sp}_{2g}$.

Any such $\hat{\delta}_i$ generate a one-parameter subgroup of automorphism $r_i^t = e^{t\hat{\delta}_i}$. We denote (rank d) renormalization flow $r_{\mathbf{t}} := r_{i_1}^{t_1} \cdots r_{i_d}^{t_d}$ for $\mathbf{t} = (t_1, \cdots, t_d)$, and

$$r_i^t[\alpha, \omega] = [r_i^t \alpha, \omega], \quad r_i^t[\alpha, \mathcal{D}] = [r_i^t \alpha, \mathcal{D}].$$

Let $U_t : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be unitary operator for $\mathbf{t} = (t_1, \cdots, t_d)$,

$$U_t f(x) = e^{-(t_1 + \cdots + t_d)/2} f(e^{t_1} x_1, \cdots e^{t_d} x_d). \tag{4.7}$$

That is, for invariant currents D_α^H,

$$D_{r_t(\alpha)}^H = e^{(t_1 + \cdots + t_d)/2} D_\alpha^H.$$

Then, the action of $\mathbb{R}^d$ defined by their parametrization is

$$\mathsf{P}_x^{d,(r_1^{t_1} \ldots r_d^{t_d} \alpha)} = \mathsf{P}_{(e^{-t_1} x_1, \ldots, e^{-t_d} x_d)}^{d,\alpha} \tag{4.8}$$

and the Birkhoff sum satisfy identities

$$P_U^{d,(r_1^{t_1}\dots r_d^{t_d}\alpha)}m = e^{(t_1+\dots+t_d)/2}P^{d,\alpha}_{(e^{-t_1},\dots,e^{-t_d})U}m. \tag{4.9}$$

4.3.5 Diophantine condition

Definition 4.10. *The height of a point Z in Siegel upper half space $\mathfrak{H}_g$ is the positive number*

$$hgt(Z) := det\Im(Z).$$

The height function $Hgt : \Sigma_g \to \mathbb{R}^+$ to be the maximal height of a $Sp_{2g}(\mathbb{Z})$ orbit. That is, for the class of $[Z] \in \Sigma_g$,

$$Hgt([Z]) := \max_{\gamma \in Sp_{2g}(\mathbb{Z})} hgt(\gamma(Z)).$$

Let $\exp \boldsymbol{t}\hat{\delta}(d)$ be the subgroup of $Sp_{2g}(\mathbb{R})$ defined by $\exp(\boldsymbol{t}\hat{\delta}(d))X_i = e^{t_i}X_i$, for $i = 1, \cdots, d$, and $\exp(\boldsymbol{t}\hat{\delta}(d))X_i = X_i$ for $i = d+1, \cdots g$. We also denote $r_{\boldsymbol{t}} = \exp \boldsymbol{t}\hat{\delta}(d)$.

Lemma 4.11. *[CF15, Lemma 4.9] For any $[\alpha] \in \mathfrak{M}_g$ and any $t \geq 0$,*

$$Hgt([[\exp(t\hat{\delta}(d))\alpha]]) \leq (det(e^{t\delta}))^2 Hgt([[\alpha]]). \tag{4.10}$$

Definition 4.12. *[CF15, Definition 4.10] We say that an automorphism $\alpha \in Sp_{2g}(\mathbb{R})$ or a point $[\alpha] \in \mathfrak{M}_g$ is $\hat{\delta}(d)$-*Diophantine *of type σ if there exists a $\sigma > 0$*

and a constant $C > 0$ such that

$$Hgt([[\exp(-\boldsymbol{t}\hat{\delta}(d))\alpha]]) \leq CHgt([[\exp(-\boldsymbol{t}\hat{\delta}(d))]])^{(1-\sigma)}Hgt([[\alpha]]),\ \forall \boldsymbol{t} \in \mathbb{R}^d_+. \quad (4.11)$$

This states that $\alpha \in Sp_{2g}(\mathbb{R})$ satisfies a $\hat{\delta}(d)$-Diophantine if the height of the projection of $\exp(-\boldsymbol{t}\hat{\delta}(d))\alpha$ *in the Siegel modular variety Σ_g is bounded by $e^{2(t_1+\cdots t_d)(1-\sigma)}$.*

- $[\alpha] \in \mathfrak{M}_g$ *satisfies a* $\hat{\delta}$-Roth condition *if for any $\epsilon > 0$ there exists a constant $C > 0$ such that*

$$Hgt([[\exp(-\boldsymbol{t}\hat{\delta}(d))\alpha]]) \leq CHgt([[\exp(-\boldsymbol{t}\hat{\delta}(d))]])^{\epsilon}Hgt([[\alpha]]),\ \forall \boldsymbol{t} \in \mathbb{R}^d_+. \quad (4.12)$$

That is, $\hat{\delta}(d)$-Diophantine of type $0 < \sigma < 1$.

- $[\alpha]$ *is of* bounded type *if there exists a constant $C > 0$ such that*

$$Hgt([[\exp(-\boldsymbol{t}\hat{\delta}(d))]]) \leq C,\ \forall \boldsymbol{t} \in \mathbb{R}^d_+. \quad (4.13)$$

For $1 \leq d \leq g$, according to Margulis-Kleinblock [KM99], a generalization of Khinchin-Sullivan logarithm law for geodesic excursion [Sul82] holds.

Definition 4.13. *Let $X = G/\Lambda$ be a homogeneous space equipped with the probability Haar measure μ. A function $\phi : X \to \mathbb{R}$ is said k-DL (distance like) for some exponent $k > 0$ if it is uniformly continuous and if there exist constants $C_1, C_2 > 0$*

such that

$$C_1 e^{-kz} \leq \mu(\{x \in X \mid \phi(x) \geq z\}) \leq C_2 e^{-kz}, \quad \forall z \in \mathbb{R}.$$

Theorem 1.9 of [KM99] states the following.

Proposition 4.14. *Let G be a connected semisimple Lie group without compact factors, μ its normalized Haar measure, $\Lambda \subset G$ an irreducible lattice, $\mathfrak{a}$ a Cartan subalgebra of the Lie algebra of G. Let $\mathfrak{d}_+$ be a nonempty open cone in a d-dimensional subalgebra $\mathfrak{d}$ of $\mathfrak{a}$. If $\phi : G/\Lambda \to \mathbb{R}$ is a k-DL function for some $k > 0$, then for μ-almost all $x \in G/\Lambda$ one has*

$$\limsup_{\mathbf{z} \in \mathfrak{d}_+, \mathbf{z} \to \infty} \frac{\phi(\exp(\mathbf{z})x)}{\log \|\mathbf{z}\|} = \frac{d}{k}$$

By Lemma 4.7 of [CF15], logarithm of Height function is DL-function with exponent $k = \frac{g+1}{2}$ on the Siegel variety Σ_g (and induces on $\mathfrak{M}_g = Sp_{2g}(\mathbb{R})/Sp_{2g}(\mathbb{Z})$). Hence, we obtain the following proposition.

Proposition 4.15. *Under the assumption $X = \mathfrak{M}_g$ of Proposition 4.14, for $s > g + 1/2$, there exists a full measure set $\Omega_g(\hat{\delta})$ and for all $[\alpha] \in \Omega_g(\hat{\delta}) \subset \mathfrak{M}_g$*

$$\limsup_{\boldsymbol{t} \to \infty} \frac{\log Hgt([[\exp(-\boldsymbol{t}\hat{\delta}(d))\alpha]])}{\log \|\boldsymbol{t}\|} \leq \frac{2d}{g+1}. \tag{4.14}$$

Any such $[\alpha]$ satisfies a $\hat{\delta}$-Roth condition (4.12).

For any $L > 0$ and $1 \leq d \leq g$, let DC(L) denote the set of $[\alpha] \in \mathfrak{M}_g$ such that

$$\int_0^\infty \cdots \int_0^\infty e^{-(t_1 + \cdots t_d)/2} \text{Hgt}([[r_{-\mathbf{t}}\alpha]])^{1/4} dt_1 \cdots dt_d \leq L. \tag{4.15}$$

Let DC denote the union of the sets DC(L) over all $L > 0$. It follows immediately that the set $DC \subset \mathfrak{M}_g$ has full Haar volume.

4.4 Constructions of the functionals

For an irreducible representation H, there exists *basic current* B_α^H associated to D_α^H. The current is *basic* in the sense that for all $j \in \{i_1, \cdots, i_d\}$,

$$\iota_{X_j} B_\alpha^H = L_{X_j} B_\alpha^H = 0.$$

The basic current B_α^H is defined as $B_\alpha^H = D_\alpha^H \eta_X$. The formula implies that for every d-form ξ,

$$B_\alpha^H(\xi) = D_\alpha^H(\frac{\eta_X \wedge \xi}{\omega})$$

where $\eta_X := \iota_{X_{i_1}} \cdots \iota_{X_{i_d}} \omega$ and ω is an invariant volume form.

The basic current B_α^H belongs to a dual Sobolev space of currents. We write any smooth d-form $\xi = \sum \xi^{(i)} \hat{X}_i$, where $\hat{X}_i \in \Lambda^d \mathfrak{p}'$. It follows that the space of smooth d-form is identified to the product of $C^\infty(M)$ by isomorphism $\xi \longrightarrow \xi^{(i)}$. By isomorphism, we define Sobolev space of currents $\Omega_\alpha^s(M)$ and their dual spaces of currents $\Omega_\alpha^{-s}(M)$.

By Sobolev embedding theorem, for every rectangle Γ, the current $\Gamma \in \Omega_\alpha^{-s}(M)$

for $s > (2d+1)/2$. Then, all basic currents $B^H_\alpha \in \Omega^{-s}_\alpha(M)$ for all $s > d/2$ since $D^H_\alpha \in \Omega^{-s}_\alpha(M)$ for all $s > d/2$.

4.4.1 Constructions of the functionals

For any exponent $s > d/2$, Hilbert bundle induces an orthogonal decomposition

$$A_d(\mathfrak{p}, \mathfrak{M}^{-s}) = Z_d(\mathfrak{p}, \mathfrak{M}^{-s}) \oplus R_d(\mathfrak{p}, \mathfrak{M}^{-s})$$

where $R_d(\mathfrak{p}, \mathfrak{M}^{-s}) = Z_d(\mathfrak{p}, \mathfrak{M}^{-s})^\perp$. Denote by I^{-s} and R^{-s} the corresponding orthogonal projection operator and by I^{-s}_α and R^{-s}_α the restrictions to the fiber over $[\alpha] \in \mathfrak{M}$ for $\alpha \in Sp(2g, \mathbb{R})$. In particular, for the Birkhoff averages $D = P^{d,\alpha}_U m$, we call $I^{-s}_\alpha(D) = I^{-s}[\alpha, D]$ boundary term and $R^{-s}_\alpha(D) = R^{-s}[\alpha, D]$ remainder term respectively. Consider the orthogonal projection

$$D = I^{-s}_{r_{-\boldsymbol{t}}[\alpha]}(D) + R^{-s}_{r_{-\boldsymbol{t}}[\alpha]}(D). \tag{4.16}$$

For fixed α, let $\Pi^{-s}_H : A_d(\mathfrak{p}, W^{-s}_\alpha(M)) \to A_d(\mathfrak{p}, W^{-s}_\alpha(H))$ denote the orthogonal projection on a single irreducible unitary representation. We further decompose projection operators with

$$\Pi^{-s}_H = \mathscr{B}^{-s}_\alpha(\Gamma) B^{-s,H}_\alpha + R^{-s,H}_\alpha$$

where $\mathscr{B}^{-s}_{H,\alpha} : A_d(\mathfrak{p}, W^{-s}_\alpha(M)) \to \mathbb{C}$ denote the orthogonal component map of P-invariant currents (closed), supported on a single irreducible unitary representation.

The *Bufetov functionals* on rectangles $\Gamma \in \mathfrak{R}$ are defined for all $\alpha \in DC$ as follows.

Lemma 4.16. *Let $\alpha \in DC(L)$. For $s > s_d = d(d+11)/4 + g + 1/2$, the limit*

$$\hat{\beta}_H(\alpha, \Gamma) = \lim_{t_d \to \infty} \cdots \lim_{t_1 \to \infty} e^{-(t_1 + \cdots t_d)/2} \mathscr{B}^{-s}_{H, r_{\boldsymbol{t}}[\alpha]}(\Gamma)$$

exists and define a finitely-additive finite measure on the standard rectangle (4.1) $\Gamma := \Gamma^X_{\boldsymbol{T}}(m)$ for $m \in M$. There exists constant $C(s, \Gamma) > 0$ such that the following estimate holds:

$$|\Pi^{-s}_{H,\alpha}(\Gamma) - \hat{\beta}(\alpha, \Gamma) B^H_\alpha|_{\alpha, -s} \leq C(s, \Gamma)(1 + L). \tag{4.17}$$

Proof. For simplicity, we omit dependence of H. For every $\mathbf{t} \in \mathbb{R}^d$, we have the following orthogonal splitting:

$$\Pi^{-s}_{H,\alpha}(\Gamma) = \mathscr{B}^{-s}_{\alpha,\mathbf{t}}(\Gamma) B_{\alpha,\mathbf{t}} + R_{\alpha,\mathbf{t}},$$

where

$$\mathscr{B}^{-s}_{\alpha,\mathbf{t}} := \mathscr{B}^{-s}_{H, r_{\mathbf{t}}[\alpha]}, \; B_{\alpha,\mathbf{t}} := B^{-s,H}_{r_{\mathbf{t}}[\alpha]}, \; R_{\alpha,\mathbf{t}} := R^{-s,H}_{r_{\mathbf{t}}[\alpha]}.$$

For any $\mathbf{h} \in \mathbb{R}^d$, we have

$$\mathscr{B}^{-s}_{\alpha,\mathbf{t}+\mathbf{h}}(\Gamma) B_{\alpha,\mathbf{t}+\mathbf{h}} + R_{\alpha,\mathbf{t}+\mathbf{h}} = \mathscr{B}^{-s}_{\alpha,\mathbf{t}}(\Gamma) B_{\alpha,\mathbf{t}} + R_{\alpha,\mathbf{t}}.$$

By reparametrization (4.8), $B_{\mathbf{t}+\mathbf{h}} = e^{-(h_1+\cdots h_d)/2}B_{\mathbf{t}}$,

$$\mathscr{B}^{-s}_{\alpha,\mathbf{t}+\mathbf{h}}(\Gamma) = e^{(h_1+\cdots h_d)/2}\mathscr{B}^{-s}_{\alpha,\mathbf{t}}(\Gamma) + \mathscr{B}^{-s}_{\alpha,\mathbf{t}+\mathbf{h}}(R_{\alpha,\mathbf{t}}) \tag{4.18}$$

and it follows that

$$\mathscr{B}^{-s}_{\alpha,\mathbf{t}+\mathbf{h}}(\Gamma) = e^{h_1/2}\mathscr{B}^{-s}_{\alpha,t_1,t_2+h_2,\cdots,t_d+h_d}(\Gamma) + \mathscr{B}^{-s}_{\alpha,\mathbf{t}+\mathbf{h}}(R_{\alpha,\mathbf{t}}). \tag{4.19}$$

By differentiating at $h_1 = 0$,

$$\frac{d}{dt_1}\mathscr{B}^{-s}_{\alpha,t_1,t_2+h_2,\cdots,t_d+h_d}(\Gamma) = \frac{1}{2}\mathscr{B}^{-s}_{\alpha,t_1,t_2+h_2,\cdots,t_d+h_d}(\Gamma) + [\frac{d}{dh_1}\mathscr{B}^{-s}_{\alpha,\mathbf{t}+\mathbf{h}}(R_{\alpha,\mathbf{t}})]_{h_1=0}. \tag{4.20}$$

Therefore, we solve the following first order ODE

$$\frac{d}{dt_1}\mathscr{B}^{-s}_{\alpha,t_1,t_2+h_2,\cdots,t_d+h_d}(\Gamma) = \frac{1}{2}\mathscr{B}^{-s}_{\alpha,t_1,t_2+h_2,\cdots,t_d+h_d}(\Gamma) + \mathcal{K}^{(1)}_{\alpha,\boldsymbol{t}}(\Gamma)$$

where

$$\mathcal{K}^{(1)}_{\alpha,\boldsymbol{t}}(\Gamma) = [\frac{d}{dh_1}\mathscr{B}^{-s}_{\alpha,\mathbf{t}+\mathbf{h}}(R_{\alpha,\mathbf{t}})]_{h_1=0}.$$

Then, the solution of the differential equation is

$$\begin{aligned}\mathscr{B}^{-s}_{\alpha,t_1,t_2+h_2,\cdots t_d+h_d}(\Gamma) &= e^{t_1/2}[\mathscr{B}^{-s}_{\alpha,0,t_2+h_2,\cdots t_d+h_d}(\Gamma) + \int_0^{t_1} e^{-\tau_1/2}\mathcal{K}^{(1)}_{\alpha,\tau}(\Gamma)d\tau_1] \\ &= e^{t_1/2}\mathscr{B}^{-s}_{\alpha,0,t_2+h_2,\cdots t_d+h_d}(\Gamma) + \int_0^{t_1} e^{(t_1-\tau_1)/2}\mathcal{K}^{(1)}_{\alpha,\tau}(\Gamma)d\tau_1.\end{aligned}$$

Note by reparametrization

$$e^{t_1/2}\mathscr{B}^{-s}_{\alpha,0,t_2+h_2,\cdots t_d+h_d}(\Gamma) = e^{h_2/2}\mathscr{B}^{-s}_{\alpha,t_1,t_2,t_3+h_3\cdots t_d+h_d}(\Gamma)$$

and it is possible to differentiate the previous equation with respect to h_2 again. Then

$$\frac{d}{dt_2}\mathscr{B}^{-s}_{\alpha,t_1,t_2,\cdots t_d+h_d}(\Gamma) = \frac{1}{2}\mathscr{B}^{-s}_{\alpha,t_1,t_2,t_3+h_3,\cdots t_d+h_d} + \int_0^{t_1} e^{-\tau_1/2}\mathcal{K}^{(2)}_{\alpha,\tau}(\Gamma)d\tau_1. \tag{4.21}$$

where $\mathcal{K}^{(2)}_{\alpha,\tau}(\Gamma) = \frac{d}{dh_2}\mathcal{K}^{(1)}_{\alpha,\tau}(\Gamma)$.

Then, the solution of (4.21) is

$$\begin{aligned}
&\mathscr{B}^{-s}_{\alpha,t_1,t_2,\cdots t_d+h_d}(\Gamma)\\
&= e^{t_2/2}[\mathscr{B}^{-s}_{\alpha,t_1,0,t_3+h_3,\cdots t_d+h_d} + \int_0^{t_2} e^{-\tau_2/2}\int_0^{t_1} e^{(t_1-\tau_1)/2}\mathcal{K}^{(2)}_{\alpha,\tau}(\Gamma)d\tau_1 d\tau_2]\\
&= e^{h_3/2}\mathscr{B}^{-s}_{\alpha,t_1,t_2,t_3,\cdots t_d+h_d} + \int_0^{t_2} e^{(t_2-\tau_2)/2}\int_0^{t_1} e^{(t_1-\tau_1)/2}\mathcal{K}^{(2)}_{\alpha,\tau}(\Gamma)d\tau_1 d\tau_2.
\end{aligned}$$

Inductively, we solve first order ODE repeatedly and obtain the following solution

$$\mathscr{B}^{-s}_{\alpha,\boldsymbol{t}}(\Gamma) = e^{(t_1+\cdots t_d)/2}\Big(\mathscr{B}^{-s}_{\alpha,0} + \int_0^{t_d}\cdots\int_0^{t_1} e^{-(\tau_1+\cdots\tau_d)/2}\mathcal{K}^{(d)}_{\alpha,\tau}(\Gamma)d\tau_1\cdots d\tau_d\Big) \tag{4.22}$$

where

$$\mathcal{K}^{(d)}_{\alpha,\tau}(\Gamma) = [\frac{d}{dh_d}\cdots\frac{d}{dh_1}\mathscr{B}^{-s}_{\alpha,\mathbf{t}+\mathbf{h}}(R_{\alpha,\mathbf{t}})]_{h_d\cdots h_1=0}.$$

Let $\langle \cdot, \cdot \rangle_t$ denote the inner product in Hilbert space $\Omega^{-s}_{r_{\text{-}\mathbf{t}}[\alpha]}$. From the intertwining formula,

$$\begin{aligned}\mathscr{B}^{-s}_{\alpha,\boldsymbol{t}+\boldsymbol{h}}(R_{\alpha,\boldsymbol{t}}) &= \langle R_{\alpha,\boldsymbol{t}}, \frac{B_{\alpha,\boldsymbol{t}+\boldsymbol{h}}}{|B_{\alpha,\boldsymbol{t}+\boldsymbol{h}}|^2_{\boldsymbol{t}+\boldsymbol{h}}} \rangle_{\alpha,\boldsymbol{t}+\boldsymbol{h}} \\ &= \langle R_{\alpha,\boldsymbol{t}} \circ U_{-\boldsymbol{h}}, \frac{B_{\alpha,\boldsymbol{t}+\boldsymbol{h}} \circ U_{-\boldsymbol{h}}}{|B_{\alpha,\boldsymbol{t}+\boldsymbol{h}}|^2_{\boldsymbol{t}+\boldsymbol{h}}} \rangle \\ &= \langle R_{\alpha,\boldsymbol{t}} \circ U_{-\boldsymbol{h}}, \frac{B_{\alpha,\boldsymbol{t}}}{|B_{\alpha,\boldsymbol{t}}|^2_t} \rangle = \mathscr{B}^{-s}_{\alpha,\boldsymbol{t}}(R_{\alpha,\boldsymbol{t}} \circ U_{-\boldsymbol{h}}).\end{aligned}$$

In the sense of distribution,

$$\begin{aligned}\frac{d}{dh_d} \cdots \frac{d}{dh_1}(R_{\alpha,\boldsymbol{t}} \circ U_{-\boldsymbol{h}}) &= -R_{\alpha,\boldsymbol{t}} \circ (\frac{d}{2} + \sum_{i=1}^{d} X_i(t)) \circ U_{-\boldsymbol{h}} \\ &= [(\sum_{i=1}^{d} X_i(t) - \frac{d}{2}) R_{\alpha,\boldsymbol{t}}] \circ U_{-\boldsymbol{h}}\end{aligned}$$

and we compute derivative term of (4.20) in representation,

$$[\frac{d}{dh_d} \cdots \frac{d}{dh_1}(\mathscr{B}^{-s}_{\alpha,\boldsymbol{t}+\boldsymbol{h}}(R_{\alpha,t}))]_{\boldsymbol{h}=0} = -\mathscr{B}^{-s}_{\alpha,\boldsymbol{t}}((\sum_{i=1}^{d} X_i(t) - \frac{d}{2}) R_{\alpha,\boldsymbol{t}})).$$

Set $\mathcal{K}_{\alpha,t}(\Gamma) = |R_{\alpha,\boldsymbol{t}}|_{r_{-\boldsymbol{t}}\alpha,-(s+1)}$ with a bounded non-negative function, then by Lemma 4.20, we claim that

$$|\mathscr{B}^{-s}_{\alpha,\boldsymbol{t}}((\sum_{i=1}^{d} X_i(t) - \frac{d}{2}) R_{\alpha,\boldsymbol{t}}))| \leq \mathcal{K}_{\alpha,\tau}(\Gamma) \leq C(s,\Gamma) Hgt([r_{-\boldsymbol{t}}\alpha])^{1/4}.$$

Therefore, the solution of equation (4.22) exists under Diophantine condition

(4.15) and the following holds:

$$\lim_{t_d\to\infty}\cdots\lim_{t_1\to\infty} e^{-(t_1+\cdots t_d)/2}\mathscr{B}^{-s}_{\alpha,t}(\Gamma) = \hat{\beta}_H(\alpha,\Gamma).$$

Moreover, by continuity, the complex number

$$\hat{\beta}_H(\alpha,\Gamma) = \mathscr{B}^{-s}_{\alpha,0} + \int_0^\infty \cdots \int_0^\infty e^{-(\tau_1+\cdots\tau_d)/2}\mathcal{K}_{\alpha,\tau}(\Gamma)d\tau_1\cdots d\tau_d$$

depends on $\alpha \in DC(L)$. Since we have

$$\Pi^{-s}_{H,\alpha}(\Gamma) - \hat{\beta}(\alpha,\Gamma)B^H_\alpha = R_0 - \left(\int_0^\infty \cdots \int_0^\infty e^{-(\tau_1+\cdots\tau_d)/2}\mathcal{K}_{\alpha,\tau}(\Gamma)d\tau_1\cdots d\tau_d\right) B^H_\alpha,$$

by Diophantine condition again,

$$|\Pi^{-s}_{H,\alpha}(\Gamma) - \hat{\beta}(\alpha,\Gamma)B^H_\alpha|_{\alpha,-s} \le C(s,\Gamma)(1+L).$$

□

4.4.2 Remainder estimates

In this subsection, we obtain estimate for remainder term which is used in Lemma 4.16. Firstly, we prove the bound of Birkhoff sum of rectangles.

Lemma 4.17. *[CF15, Lemma 5.7] Let $s > d/2+2$. There exists a constant*

$C = C(s) > 0$ such that for all $t_i \geq 0$ for $1 \leq i \leq d$, we have

$$\begin{aligned}\left\|I^{-s}[\alpha, D]\right\|_{-s} &\leq e^{-(t_1+\cdots+t_d)/2} \left\|I^{-s}[r_1^{-t_1} \cdots r_d^{-t_d}\alpha, D]\right\|_{-s} \\ &\quad + C_1|t_1 + \cdots + t_d| \int_0^1 e^{-u(t_1+\cdots+t_d)/2} \left\|R^{-s}[r_1^{-ut_1} \cdots r_d^{-ut_d}\alpha, D\right\|_{-(s-2)} du.\end{aligned}$$

By Stokes' theorem, we have the following remainder estimate.

Lemma 4.18. *[CF15, Lemma 5.6] For any non-negative $s' < s - (d+1)/2$ and Jordan region $U \subset \mathbb{R}^d$, there exists $C = C(d, g, s, s') > 0$ such that*

$$\left\|R^{-s}[\alpha, (P_U^{d,\alpha} m)]\right\|_{-s} \leq C \left\|[\alpha, \partial(P_U^{d,\alpha} m)]\right\|_{-s'}.$$

Here we prove quantitative bound of Birkhoff averages of higher rank actions on rectangle. (Cf. [CF15, theorem 5.10]).

Theorem 4.19. *For $s > s_d$, there exists a constant $C(s, d) > 0$ such that the following holds. For any $t_i > 0$, $m \in M$ and $U_d(t) = [0, e^{t_1}] \times \cdots \times [0, e^{t_d}]$, we have*

$$\begin{aligned}\left\|[\alpha, (P_{U_d(t)}^{d,\alpha} m)]\right\|_{-s} \leq C \sum_{k=0}^{d} \sum_{1 \leq i_1 < \cdots < i_k \leq d} \int_0^{t_{i_k}} \cdots \int_0^{t_{i_1}} \exp\left(\frac{1}{2}\sum_{l=1}^{d} t_l - \frac{1}{2}\sum_{l=1}^{k} u_{i_l}\right) \\ \times Hgt\left(\left[\left[\prod_{1 \leq j \leq d} r_j^{-t_j} \prod_{l=1}^{k} r_{i_l}^{u_{i_l}} \alpha\right]\right]\right)^{1/4} du_{i_1} \cdots du_{i_k}.\end{aligned} \tag{4.23}$$

Proof. We prove by induction. For $d = 1$, it follows from the theorem 5.8 in [CF15]. We assume that the result holds for $d - 1$. Decompose the current as a sum of boundary and remainder term as in (4.16).

Step 1. We estimate the boundary term. By Lemma 4.17, renormalize terms with $r^u = r_1^u \cdots r_d^u$. Then, we have

$$\begin{aligned}
&\left\| I^{-s}[\alpha, (P^{d,\alpha}_{U_d(t)} m)] \right\|_{-s} \leq e^{-(t_1+\cdots+t_d)/2} \left\| I^{-s}[r_1^{-t_1} \cdots r_d^{-t_d}\alpha, (P^{d,\alpha}_{U_d(t)} m)] \right\|_{-s} \\
&+ C_1(s) \int_0^{t_1+\cdots+t_d} e^{-ud/2} \left\| R^{-s}[r^{-u}\alpha, (P^{d,\alpha}_{U_d(t)} m)] \right\|_{-(s-2)} du \qquad (4.24) \\
&:= (I) + (II)
\end{aligned}$$

By renormalization (4.9) and Lemma 4.9 for unit volume,

$$\begin{aligned}
\left\| I^{-s}[r_1^{-t_1} \cdots r_d^{-t_d}\alpha, (P^{d,\alpha}_{U_d(t)} m)] \right\|_{-s} &= e^{t_1+\cdots+t_d} \left\| I^{-s}[r_1^{-t_1} \cdots r_d^{-t_d}\alpha, (P^{d, r_1^{-t_1} \cdots r_d^{-t_d}\alpha}_{U_d(0)} m)] \right\|_{-s} \\
&\leq C_2 e^{t_1+\cdots+t_d} Hgt([[r_1^{-t_1} \cdots r_d^{-t_d}\alpha]])^{1/4}
\end{aligned}$$

Hence,

$$I \leq C_2 e^{(t_1+\cdots t_d)/2} Hgt([[r_1^{-t_1} \cdots r_d^{-t_d}\alpha]])^{1/4},$$

where the sum corresponds to the first term ($k = 0$) in the statement.

Step 2. To estimate (II),

$$\begin{aligned}
\left\| R^{-s}[r^{-u}\alpha, (P^{d,\alpha}_{U_d(t)} m)] \right\|_{-(s-2)} &= \left\| e^{ud} R^{-s}[r^{-u}\alpha, (P^{d,r^{-u}\alpha}_{U_d(t-u)} m)] \right\|_{-(s-2)} \\
&\leq C_3(s, s') e^{ud} \left\| [r^{-u}\alpha, \partial(P^{d,r^{-u}\alpha}_{U_d(t-u)} m)] \right\|_{-s'}.
\end{aligned} \qquad (4.25)$$

The boundary $\partial(\mathsf{P}^{d,r^{-t}\alpha}_{U_d})$ is the sum of $2d$ currents of dimension $d-1$. These currents are Birkhoff sums of d face subgroups obtained from $\mathsf{P}^{d,r^{-t}\alpha}_{U_d}$ by omitting one of the base vector fields X_i. It is reduced to $(d-1)$ dimensional shape obtained from $U_d(t-u) := [0, e^{t_1-u}] \times \cdots [0, e^{t_d-u}]$. For each $1 \leq j \leq d$, there are Birkhoff sums along $d-1$ dimensional cubes. By induction hypothesis, we add all the $d-1$ dimensional cubes by adding all the terms along j:

$$\left\|[r^{-u}\alpha, (P^{d-1,r^{-u}\alpha}_{U_{d-1}(t-u)}m)]\right\|_{-s'} \leq C_4(s', d-1)\sum_{j=1}^{d}\sum_{k=0}^{d-1}\sum_{\substack{1\leq i_1<\cdots<i_k\leq d\\ i_l\neq j,\forall l}}\int_0^{t_{i_k-u}}\cdots\int_0^{t_{i_1-u}}$$
$$\exp(\frac{1}{2}\sum_{\substack{l=1\\ l\neq j}}^{d}(t_l-u)-\frac{1}{2}\sum_{l=1}^{k}u_{i_l})Hgt([[\prod_{\substack{1\leq l\leq d\\ l\neq j}} r_l^{-(t_l-u)}\prod_{l=1}^{k}r_{i_l}^{u_{i_l}}(r^{-u}\alpha)]])^{1/4}du_{i_1}\cdots du_{i_k}. \tag{4.26}$$

Combining (4.24) and (4.25), we obtain the estimate for (II).

$$(II) \leq C_5(s', d-1)\sum_{j=1}^{d}\sum_{k=0}^{d-1}\sum_{\substack{1\leq i_1<\cdots<i_k\leq d\\ i_l\neq j}}\int_0^{t_1+\cdots t_d}\int_0^{t_{i_k-u}}\cdots\int_0^{t_{i_1-u}}du_{i_1}\cdots du_{i_k}du$$
$$\times\exp(\frac{1}{2}\sum_{\substack{l=1\\ l\neq j}}^{d}t_l-\frac{1}{2}u-\frac{1}{2}\sum_{l=1}^{k}u_{i_l})Hgt([[\prod_{1\leq l\leq d}r_l^{-t_l}\prod_{l=1}^{k}r_{i_l}^{u_{i_l}}(r_j^{-u+t_j}\alpha)]])^{1/4}.$$

Applying the change of variable $u_j = t_j - u$, we obtain

$$(II) \leq C_6(s', d-1)\sum_{j=1}^{d}\sum_{k=1}^{d-1}\sum_{\substack{1\leq i_1<\cdots<i_k\leq d\\ i_l\neq j}}\int_{-(t_1+\cdots t_d)+t_j}^{t_j}\int_0^{t_{i_k-u}}\cdots\int_0^{t_{i_1-u}}du_{i_1}\cdots du_{i_k}du_j$$
$$\times\exp(\frac{1}{2}(t_1+\cdots t_d)-\frac{1}{2}u_j-\frac{1}{2}\sum_{l=1}^{k}u_{i_l})Hgt([[\prod_{1\leq l\leq d}r_l^{-t_l}\prod_{l=1}^{k}r_{i_l}^{u_{i_l}}(r_j^{-u_j}\alpha)]])^{1/4}.$$

Simplifying multi-summation above, (with $-(t_1 + \cdots t_d) + t_j \le 0$)

$$(II) \le C_7(s', d) \sum_{k=1}^{d} \sum_{1 \le i_1 < \cdots < i_k \le d} \int_0^{t_{i_k}} \cdots \int_0^{t_{i_1}} du_{i_1} \cdots du_{i_k}$$
$$\times \exp(\frac{1}{2}(t_1 + \cdots t_d) - \frac{1}{2}\sum_{l=1}^{k} u_{i_l}) Hgt([[\prod_{1 \le l \le d} r_l^{-t_l} \prod_{l=1}^{k} r_{i_l}^{u_{i_l}} \alpha]])^{1/4}.$$

Step 3. (Remainder estimate). The remainder term is obtained from Lemma 4.18 (Stokes' theorem). Following step 2, estimate of remainder reduces to that of $d-1$ form. Combining with the step 1, we have the following

$$\left\| R^{-s}[\alpha, \partial(P_{U_d}^{d,\alpha} m)] \right\|_{-s} \le C(s) \sum_{i=1}^{d-1} \left\| I^{-s}[\alpha, (P_{U_i}^{i,\alpha} m)] \right\|_{-s} + \left\| R^{-s}[\alpha, (P_{U_1}^{1,\alpha} m)] \right\|_{-s} \tag{4.27}$$

where U_i is i-dimensional rectangle. Sum of the boundary terms are absorbed in the bound of $(I) + (II)$. For 1-dimensional remainder with interval Γ_T, the boundary is a 0-dimensional current. Then,

$$\langle \partial(P_{U_T}^{1,\alpha} m), f \rangle = f(\mathsf{P}_{U_T}^{1,\alpha} m) - f(m).$$

Hence, by Sobolev embedding theorem and by definition of Sobolev constant (4.5) and (4.6),

$$\left\| R^{-s}[\alpha, \partial(P_{U_T}^{1,\alpha} m)] \right\|_{-s} \le 2B_{s'}([[\alpha]]) \le C(s) Hgt([[\alpha]])^{1/4}.$$

Then, by inequality (4.10)

$$Hgt([[\alpha]])^{1/4} \leq Ce^{(t_1+\cdots t_d)/2} Hgt([[r_1^{-t_1} \cdots r_d^{-t_d} \alpha]])^{1/4}.$$

This implies that remainder term produces one more term like the bound of (I). Therefore, the theorem follows from combining all the terms $(I), (II)$, and remainder. □

Now we prove the estimate for constructing Bufetov functionals.

Lemma 4.20. *Let $s > s_d$. There exists a constant $C(s) > 0$ such that for any rectangle $U_\Gamma = [0, e^{\Gamma_1}] \times \cdots \times [0, e^{\Gamma_d}]$,*

$$\mathcal{K}_{\alpha,t}(\Gamma) \leq C(s, \Gamma) Hgt([r_{-\boldsymbol{t}}\alpha])^{1/4}.$$

Proof. Recall that $\mathcal{K}_{\alpha,t}(\Gamma) = \left\| R^{-s}[r_{-\boldsymbol{t}}[\alpha], (P_{U_\Gamma}^{d,r_{-\boldsymbol{t}}\alpha} m)] \right\|_{-(s+1)}$ and by Lemma 4.18, it is equivalent to prove to find the bound of $d-1$ currents. By theorem 4.19, we obtain the remainder estimate.

$$\mathcal{K}_{\alpha,t}(\Gamma) \leq C \sum_{k=0}^{d-1} \sum_{1\leq i_1 < \cdots < i_k \leq d-1} \int_0^{\Gamma_{i_k}} \cdots \int_0^{\Gamma_{i_1}} \exp(\frac{1}{2}\sum_{l=1}^{d-1} \Gamma_l - \frac{1}{2}\sum_{l=1}^{k} u_{i_l})$$
$$\times Hgt([[\prod_{1\leq j\leq d-1} r_j^{-\Gamma_j} \prod_{l=1}^{k} r_{i_l}^{u_{i_l}} (r_{-\boldsymbol{t}}\alpha)]])^{1/4} du_{i_1} \cdots du_{i_k}.$$

It follows from (4.10) that for $0 \le k \le d-1$,

$$Hgt([[\prod_{1\le j\le d-1} r_j^{-\Gamma_j} \prod_{l=1}^{k} r_{i_l}^{u_{i_l}}(r_{-\boldsymbol{t}}\alpha)]])^{1/4} \le e^{\frac{1}{2}(\sum_{l=1}^{k} u_{i_l} - \sum_{l=1}^{d} \Gamma_l)} Hgt([[r_{-\boldsymbol{t}}\alpha]])^{1/4}.$$

Then, we obtain

$$\left\| R^{-s}[r_{-\boldsymbol{t}}\alpha, (P_{U_\Gamma}^{d,r_{-\boldsymbol{t}}\alpha} m)] \right\|_{-s} \le C\Big(\sum_{k=0}^{d-1} \sum_{1\le i_1<\cdots<i_k\le d-1} \prod_{l=1}^{k} \Gamma_{i_l}\Big) Hgt([[r_{-\boldsymbol{t}}\alpha]])^{1/4}. \quad (4.28)$$

Setting $C(s,\Gamma) = C\Big(\sum_{k=0}^{d-1} \sum_{1\le i_1<\cdots<i_k\le d-1} \prod_{l=1}^{k} \Gamma_{i_l}\Big)$, we obtain the conclusion. □

4.4.3 Extensions of domain

Now we extend the domain of Bufetov functional defined on standard rectangle $\Gamma_{\mathbf{T}}^X$ to the class $\mathfrak{R}$.

Lemma 4.21. *Bufetov functional defined on standard rectangle $\Gamma_{\boldsymbol{T}}^X$ extends to the class $(\mathsf{Q}_y^{d,Y})_*\Gamma_{\boldsymbol{T}}^X$ for any $y \in \mathbb{R}^d$.*

Proof. First, we prove that Bufetov functional exists and invariant under the action $\mathsf{Q}_y^{d,Y}$. It suffices to verify that Bufetov functional is invariant under the rank 1 action $\mathsf{Q}_\tau^{1,Y}$ for $\tau \in \mathbb{R}$.

Given a standard rectangle Γ, set $\Gamma_{\mathsf{Q}} := (\mathsf{Q}_\tau^{1,Y})_*\Gamma$. Let $D(\Gamma, \Gamma_{\mathsf{Q}})$ be the $(d+1)$ dimensional space spanned by the trajectories of the action of $\mathsf{Q}_\tau^{1,Y}$ projecting Γ onto Γ_{Q}. Then $D(\Gamma, \Gamma_{\mathsf{Q}})$ is union of all orbits I of action $\mathsf{Q}_\tau^{1,Y}$ such that the boundary of I, d-dimensional faces, is contained in $\Gamma \cup \Gamma_{\mathsf{Q}}$, and interior of I is disjoint from $\Gamma \cup \Gamma_{\mathsf{Q}}$.

By definition, denote $r_t := r_i^t$. Then, $r_{-t}(\Gamma)$ and $r_{-t}(\Gamma_{\mathsf{Q}})$ are respectively the support of the currents $r_t^*\Gamma$ and $r_t^*\Gamma_{\mathsf{Q}}$. Thus, we have the following identity

$$r_t^* D(\Gamma, \Gamma_{\mathsf{Q}}) = D(r_{-t}(\Gamma), r_{-t}(\Gamma_{\mathsf{Q}})).$$

Since the currents $\partial D(\Gamma, \Gamma_{\mathsf{Q}}) - (\Gamma - \Gamma_{\mathsf{Q}})$ is composed of orbits of the action of $\mathsf{Q}_\tau^{1,Y}$, it follows that

$$\partial[r_t^* D(\Gamma, \Gamma_{\mathsf{Q}})] - (r_t^*\Gamma - r_t^*\Gamma_{\mathsf{Q}}) = r_t^*[\partial D(\Gamma, \Gamma_{\mathsf{Q}}) - (\Gamma - \Gamma_{\mathsf{Q}})] \to 0. \tag{4.29}$$

Now, we turn to prove the volume of $D(r_{-t}(\Gamma), r_{-t}(\Gamma_{\mathsf{Q}}))$ is uniformly bounded for all $t > 0$. For any $p \in \Gamma$, set $\tau(p)$ be length of the arc lying in $D := D(\Gamma, \Gamma_{\mathsf{Q}})$, and set $\tau_\Gamma := \sup\{\tau(p) \mid p \in \Gamma\} < \infty$. We write

$$vol_{d+1}(D) = \int_\Gamma \tau dvol_d.$$

Since $vol_d(r_{-t}\Gamma) \leq e^t vol_d(\Gamma)$,

$$vol_{d+1}(r_{-t}D) = \int_{r_{-t}\Gamma} \tau dvol_d \leq \tau_\Gamma e^{-t} vol_d(r_{-t}\Gamma) \leq \tau_\Gamma vol_d(\Gamma) < \infty. \tag{4.30}$$

Note that current $(\mathsf{Q}_\tau^{1,Y})_*\Gamma - \Gamma$ is equal to the boundary of a $(d+1)$ dimensional current D. By arguments in remainder estimate (or Sobolev embedding theorem),

$$|(\mathsf{Q}_\tau^{1,Y})_*\Gamma - \Gamma|_{r^{-t}\alpha, -s} \leq C_s \tau B_s([[r^{-t}\alpha]]) \leq C_s \tau \mathrm{Hgt}[[r^{-t}\alpha]]^{1/4} \tag{4.31}$$

is finite for all $t > 0$.

Then, by (4.29), (4.30) and existence of Bufetov functional (from Diophantine condition), the last inequality holds

$$|\mathscr{B}^{-s}_{\alpha,t}((\mathsf{Q}^{1,Y}_{\tau})_*\Gamma) - \mathscr{B}^{-s}_{\alpha,t}(\Gamma)|_{\alpha,-s} < \infty.$$

Therefore, by definition of Bufetov functional in the Lemma 4.16, it is extended and invariant under the action of $\mathsf{Q}^{1,Y}_{\tau}$. □

Similarly, Bufetov functional defined on the standard rectangles is extended to $(\phi^{Z}_{t_i z}) \circ \mathsf{P}^{d,\alpha}_{\boldsymbol{t}}(m)$. We postpone its proof by Lemma 4.38. Since the flow generated by Z commutes with other actions P and Q, combining with the invariance under the action Q from Lemma 4.21, we extend the domain of Bufetov functional to the class $\mathfrak{R}$.

Proof of theorem 4.3. Additive property follows from linearity of projections and limit. It is immediate to derive *scaling property* from the definition.

Bounded property. By scaling property,

$$\hat{\beta}(\alpha, \Gamma) = e^{dt/2}\hat{\beta}_H(r^t[\alpha], \Gamma).$$

Choose $t = \log(\int_\Gamma |\hat{X}|)$ and $\hat{X} = \hat{X}_1 \wedge \cdots \wedge \hat{X}_d$, then uniform bound of Bufetov functional on bounded size of rectangles,

$$|\hat{\beta}(\alpha,\Gamma)| \leq C(\Gamma)(\int_\Gamma |\hat{X}|)^{d/2}.$$

Invariance property follows directly from the Lemma 4.21. □

4.4.4 Bound of functionals

We define excursion function

$$\begin{aligned} E_{\mathfrak{M}}(\alpha,\mathbf{T}) &:= \int_0^{\log T^{(d)}} \cdots \int_0^{\log T^{(1)}} e^{-(t_1+\cdots t_d)/2}\mathrm{Hgt}([[r_{\mathbf{t}-\log\mathbf{T}}\alpha]])^{1/4} dt_1\cdots dt_d \\ &= \prod_{i=1}^d (T^{(i)})^{1/2} \int_0^{\log T^{(d)}} \cdots \int_0^{\log T^{(1)}} e^{(t_1+\cdots t_d)/2}\mathrm{Hgt}([[r_{\mathbf{t}}\alpha]])^{1/4} dt_1\cdots dt_d. \end{aligned}$$

Denote $\boldsymbol{tT} = (t_1 T^{(1)},\cdots,t_d T^{(d)})$ and $\mathbf{t} = (t_1,\cdots,t_d)$.

Lemma 4.22. *For any Diophantine* $[\alpha] \in DC(L)$ *and for any* $f \in W^s(M)$ *for* $s > s_d + 1/2$*, the Bufetov functional* β^f *is defined by a uniformly convergent series.*

$$|\beta^f(\alpha,m,\boldsymbol{tT})| \leq C_s(L + \prod_{i=1}^d (T^{(i)})^{1/2}(1+\prod_{i=1}^d t_i + E_{\mathfrak{M}}(\alpha,\mathbf{T}))\,\|\omega\|_{\alpha,\mathbf{s}}$$

for $\omega = f\omega^{d,\alpha} \in \Lambda^d\mathfrak{p}\otimes W^s(M)$.

Proof. It follows from Lemma 4.16 that there exists a constant $C > 0$ such that

$$|\beta_H(\alpha,m,\mathbf{t})| \leq C(1 + L + \prod_{i=1}^d t_i).$$

By exact scaling property,

$$\beta_H(\alpha, m, \boldsymbol{tT}) = \prod_{i=1}^{d} (T^{(i)})^{1/2} \beta_H(r_{\log \mathbf{T}}[\alpha], m, \mathbf{t}).$$

By Diophantine condition (4.15), whenever $\alpha \in DC(L)$ then $r_{\log \mathbf{T}}[\alpha] \in DC(L_T)$ with

$$L_T \leq L \prod_{i=1}^{d} (T^{(i)})^{-1/2} + E_{\mathfrak{M}}(\alpha, \mathbf{T}).$$

Thus for all $(m, \mathbf{t}) \in M \times \mathbb{R}^d$,

$$|\beta_H(r_{\log \mathbf{T}}[\alpha], m, \mathbf{t})| \leq C(1 + L_T + \prod_{i=1}^{d} t_i).$$

It follows that for all $s > 1/2$, we have

$$|\beta^f(\alpha, m, \boldsymbol{tT})| \leq C_s \prod_{i=1}^{d} (T^{(i)})^{1/2} (1 + L + \prod_{i=1}^{d} t_i) \sum_{n \in \mathbb{Z}} \|\omega_n\|_{\alpha,s}$$
$$\leq C_s \prod_{i=1}^{d} (T^{(i)})^{1/2} (1 + L_T + \prod_{i=1}^{d} t_i) (\sum_{n \in \mathbb{Z}} (1 + n^2)^{-s'})^{-1/2} (\sum_{n \in \mathbb{Z}} \left\| (1 - Z^2)^{s'/2} \omega_n \right\|_{\alpha,s}^2)^{1/2}.$$

Therefore, for all $s' > s_d$, there exists a constant $C_{s,s'} > 0$ such that

$$|\beta^f(\alpha, m, \mathbf{tT})| \leq C_{r,r'} \prod_{i=1}^{d} (T^{(i)})^{1/2} (1 + L_T + \prod_{i=1}^{d} t_i) \|\omega\|_{\alpha,s+s'}.$$

□

This lemma implies that all properties of the Bufetov functionals associated to a single irreducible component β_H can be extended to the Bufetov functionals β^f

for any $f \in W^s(M)$. From this, we can derive bounded property for the cocycles $\beta(\alpha, m, \mathbf{T})$ with respect to m along the orbits of actions in time $\mathbf{T} \in \mathbb{R}^d$ respectively.

Corollary 4.23. *For all $s > s_d + 1/2$, there exists a constant $C_s > 0$ such that for almost all frequency α and for all $f \in W^s(M)$ and for all $(m, \boldsymbol{T}) \in M \times \mathbb{R}^d$, we have*

$$|\left\langle P^{d,\alpha}_{U(\boldsymbol{T})} m, \omega \right\rangle - \beta^f(\alpha, m, \boldsymbol{T})| \leq C_s(1+L) \|\omega\|_{\alpha,s}. \tag{4.32}$$

for $U(\boldsymbol{T}) = [0, T^{(1)}] \times \cdots [0, T^{(d)}]$ and $\omega = f\omega^{d,\alpha} \in \Lambda^d \mathfrak{p} \otimes W^s(M)$.

Proof. By Lemma 4.16 and 4.22, asymptotic formula (4.17) on each irreducible provides proof of Corollary 4.23. □

4.5 Limit distributions

In this section, we prove Theorem 4.6, limit distribution of Birkhoff sums of higher rank actions on squares.

4.5.1 Limiting distributions

Lemma 4.24. *There exists a continuous modular function $\theta_H : Aut_0(\mathsf{H}^g) \to H \subset L^2(M)$ such that*

$$\lim_{|U(\boldsymbol{T})| \to \infty} \left\| \frac{1}{vol(U(\boldsymbol{T}))^{1/2}} \left\langle P^{d,\alpha}_{U(\boldsymbol{T})}(\cdot), \omega_f \right\rangle - \theta_H(r_{\log \boldsymbol{T}}[\alpha]) D^H_\alpha(f) \right\|_{L^2(M)} = 0.$$

The family $\{\theta_H(\alpha) \mid \alpha \in Aut_0(\mathsf{H}^g)\}$ has a constant norm in $L^2(M)$.

Proof. By Fourier transform, the space of smooth vectors and Sobolev space $W^s(H)$ is represented as Schwartz-type space $\mathscr{S}^s(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} |(1 + \sum_i \frac{\partial^2}{\partial u_i^2} + \sum_i u_i^2)^{s/2} \hat{f}(u)|^2 du < \infty.$$

Let $\boldsymbol{t}, \boldsymbol{u} \in \mathbb{R}^d$. Then we claim for any $f \in \mathscr{S}^s(\mathbb{R}^d)$, there exists $\theta[\alpha] \in L^2(\mathbb{R}^d)$ such that

$$\lim_{|U(\boldsymbol{T})| \to \infty} \left\| \frac{1}{vol(U(\boldsymbol{T}))^{1/2}} \int_{U(\boldsymbol{T})} f(\boldsymbol{u} + \boldsymbol{t}) d\boldsymbol{t} - \theta_H(r_{\log \boldsymbol{T}}[\alpha]) Leb(f) \right\|_{L^2(\mathbb{R}^d, d\boldsymbol{u})} = 0.$$

Equivalently,

$$\lim_{|U(\boldsymbol{T})| \to \infty} \left\| \frac{1}{vol(U(\boldsymbol{T}))^{1/2}} \int_0^{T^{(d)}} \cdots \int_0^{T^{(1)}} e^{i\boldsymbol{t} \cdot \hat{\boldsymbol{u}}} \hat{f}(\hat{\boldsymbol{u}}) dt - \hat{\theta}_H(r_{\log \boldsymbol{T}}[\alpha]) \hat{f}(0) \right\|_{L^2(\mathbb{R}^d, d\hat{u})} = 0.$$

For $\chi \in L^2(\mathbb{R}^d, d\hat{u})$, we denote

$$\chi_j(\hat{u}) = \frac{e^{i\hat{u}_j} - 1}{i\hat{u}_j}, \ \chi(\hat{u}) = \prod_{j=1}^d \chi_j(\hat{u}).$$

Let $\hat{\theta}[\alpha](\hat{u}) := \chi(\hat{u})$ for all $\hat{u} \in \mathbb{R}^d$. Then, by intertwining formula, for $\mathbf{T} \in \mathbb{R}^d$ and $\mathbf{u} \in \mathbb{R}^d$,

$$U_{\mathbf{T}}(f)(\hat{u}) = \prod_{i=1}^d (T^{(i)})^{1/2} f(\boldsymbol{T}\hat{\boldsymbol{u}}), \text{ for } \boldsymbol{T}\hat{\boldsymbol{u}} = (T^{(1)}\hat{u}_1, \cdots, T^{(d)}\hat{u}_d).$$

Then, for all $\alpha \in A$,

$$\hat{\theta}(r_{\log \boldsymbol{T}}[\alpha])(\hat{u}) = U_{\mathbf{T}}(\chi)(\hat{u}) = \prod_{i=1}^{d} (T^{(i)})^{1/2} \chi(\boldsymbol{T}\hat{\boldsymbol{u}}).$$

The function $\theta[\alpha]$ is defined by Fourier inverse transform

$$\|\theta_H(\alpha)\|_H = \|\theta(\alpha)\|_{L^2(\mathbb{R}^d)} = \left\|\hat{\theta}(\alpha)\right\|_{L^2(\mathbb{R}^d)} = \|\chi(\hat{u})\|_{L^2(\mathbb{R}^d, d\hat{u})} = C > 0.$$

By integration,

$$\begin{aligned}\int_0^{T^{(d)}} \cdots \int_0^{T^{(1)}} e^{i\boldsymbol{t}\cdot\hat{\boldsymbol{u}}} \hat{f}(\hat{u}) dt &= (\prod_{i=1}^{d} T^{(i)}) \chi(\boldsymbol{T}\hat{\boldsymbol{u}}) \hat{f}(\hat{u}) \\ &= (\prod_{i=1}^{d} T^{(i)}) \chi(\boldsymbol{T}\hat{\boldsymbol{u}})(\hat{f}(\hat{u}) - \hat{f}(0)) + (\prod_{i=1}^{d} T^{(i)})^{1/2} \hat{\theta}(r_{\log \boldsymbol{T}}[\alpha])(\hat{u}) \hat{f}(0).\end{aligned}$$

We note that $\prod_{i=1}^d T^{(i)} = vol(U(\boldsymbol{T}))$. Then the claim reduces to the following:

$$\limsup_{vol(U(T)) \to \infty} \left\| vol(U(\boldsymbol{T}))^{1/2} \chi(\boldsymbol{T}\hat{\boldsymbol{u}})(\hat{f}(\hat{u}) - \hat{f}(0)) \right\|_{L^2(\mathbb{R}^d)} = 0.$$

If $f \in \mathscr{S}^s(\mathbb{R}^g)$ with $s > d/2$, function $\hat{f} \in C^0(\mathbb{R}^d)$ and bounded. Thus, by Dominated convergence theorem,

$$\left\| vol(U(\boldsymbol{T}))^{1/2} \chi(\boldsymbol{T}\hat{\boldsymbol{u}})(\hat{f}(\hat{u}) - \hat{f}(0)) \right\|_{L^2(\mathbb{R}^d, d\hat{u})} = \left\| \chi(\nu)(\hat{f}(\frac{\nu}{\boldsymbol{T}}) - \hat{f}(0)) \right\|_{L^2(\mathbb{R}^d, d\nu)} \to 0.$$

□

Corollary 4.25. *There exists a constant $C > 0$ such that for any $s > d/2$, for any*

$\alpha \in A$ and $f \in W^s(H)$, we have

$$\lim_{|U(\boldsymbol{T})| \to \infty} \frac{1}{vol(U(\boldsymbol{T}))^{1/2}} \left\| \left\langle P^{d,\alpha}_{U(\boldsymbol{T})} m, \omega_f \right\rangle \right\|_{L^2(M)} = C|D^H_\alpha(f)|.$$

From Corollary 4.25, we derive the following limit result for the L^2 norm of Bufetov functionals.

Corollary 4.26. *For irreducible component H and $\alpha \in DC$, there exists $C > 0$ such that*

$$\lim_{|U(\boldsymbol{T})| \to \infty} \frac{1}{vol(U(\boldsymbol{T}))^{1/2}} \|\beta_H(\alpha, \cdot, \boldsymbol{T})\|_{L^2(M)} = C.$$

Proof. By the normalization of invariant distribution in Sobolev space $W^s(M)$, there exists a function $f^H_\alpha \in W^s(H)$ such that $D_\alpha(f^H_\alpha) = \left\| f^H_\alpha \right\|_s = 1$. For all $\alpha \in DC(L)$, by asymptotic formula (4.32),

$$\left| \left\langle P^{d,\alpha}_{U(\boldsymbol{T})} m, \omega \right\rangle - \beta^f(\alpha, m, \boldsymbol{T}) \right| \leq C_s(1 + L).$$

Therefore, L^2-estimate follows from Corollary 4.25. □

A relation between the Bufetov functional and the modular function θ_H is established below.

Corollary 4.27. *For any $L > 0$ and invariant probability measure supported on $DC(L) \subset \mathfrak{M}_g$,*

$$\beta_H(\alpha, \cdot, 1) = \theta_H([\alpha]), \quad \text{for } \mu\text{-almost all } [\alpha] \in \mathfrak{M}_g.$$

Proof. By Theorem 4.23 and Lemma 4.24, there exists a constant $C > 0$ such that for all $\alpha \in supp(\mu) \subset DC(L)$, for all $T > 0$ we have

$$\lim_{|U(\boldsymbol{T})|\to\infty} \|\beta_H(r_{\log \boldsymbol{T}}[\alpha], \cdot, 1) - \theta_H(r_{\log \boldsymbol{T}}[\alpha])\|_{L^2(M)} \le \frac{C_\mu}{vol(U(\boldsymbol{T}))^{1/2}}. \tag{4.33}$$

By Luzin's theorem, for any $\delta > 0$ there exists a compact subset $E(\delta) \subset \mathfrak{M}$ such that we have the measure bound $\mu(\mathfrak{M} \backslash E(\delta)) < \delta$ and the function $\beta_H(\alpha, \cdot, 1) \in L^2(M)$ depends continuously on $[\alpha] \in E(\delta)$. By Poincare recurrence, there is a full measure subset $E'(\delta) \subset E(\delta)$ of $\mathbb{R}^d$-action.

For every $\alpha_0 \in E'(\delta)$, there is diverging sequence (t_n) such that $\{r^{t_n}(\alpha_0)\} \subset E(\delta)$ and $\lim_{n\to\infty} r^{t_n}(\alpha_0) = (\alpha_0)$. By continuity of θ_H and β_H at $[\alpha_0]$, we have

$$\begin{aligned} &\|\beta_H([\alpha_0], \cdot, 1) - \theta_H([\alpha_0])\|_{L^2(M)} \\ &\qquad = \lim_{n\to\infty} \|\beta_H(r_{\log \boldsymbol{T_n}}[\alpha_0], \cdot, 1) - \theta_H(r_{\log \boldsymbol{T_n}}[\alpha_0])\|_{L^2(M)} = 0. \end{aligned} \tag{4.34}$$

Thus, we have $\beta_H([\alpha], \cdot, 1) = \theta_H([\alpha])$ for $[\alpha] \in E'(\delta)$. It follows that the set of equality fails has less than any $\delta > 0$, thus the identity holds for almost all $[\alpha]$. $\square$

For all $\alpha \in Aut_0(\mathsf{H}^g)$, general smooth function $f \in W^s(M)$ for $s > s_d + 1/2$, f decompose an infinite sum, and the functional θ^f is defined by a convergent series.

$$\theta^f(\alpha) = \sum_H D^H_\alpha(f)\theta_H(\alpha). \tag{4.35}$$

The following result is an extention to general asymptotic theorem from Corollary 4.27.

Theorem 4.28. *For all $\alpha \in Aut_0(\mathsf{H}^g)$, and for all $f \in W^s(M)$ for $s > s_d + 1/2$,*

$$\lim_{n\to\infty} \left\| \frac{1}{vol(U(T_n))^{1/2}} \left\langle P^{d,\alpha}_{U(T_n)} m, \omega_f \right\rangle - \theta^f(r_{\log \boldsymbol{T_n}} \alpha) \right\|_{L^2(M)} = 0.$$

4.5.2 Proof of Theorem 4.6

By theorem 4.28, we summarize our results on limit distributions for higher rank actions.

Theorem 4.29. *Let (T_n) be any sequence such that*

$$\lim_{n\to\infty} r_{\log \boldsymbol{T_n}}[\alpha] = \alpha_\infty \in \mathfrak{M}_g.$$

For every closed form $\omega_f \in \Lambda^d \mathfrak{p} \otimes W^s(M)$ with $s > s_d + 1/2$, which is not a coboundary, the limit distribution of the family of random variables

$$E_{T_n}(f) := \frac{1}{vol(U(T_n))^{1/2}} \left\langle P^{d,\alpha}_{U(T_n)}(\cdot), \omega_f \right\rangle$$

exists and is equal to the distribution of the function $\theta^f(\alpha_\infty) = \beta(\alpha, \cdot, 1) \in L^2(M)$. If $\alpha_\infty \in DC$, then $\theta^f(\alpha_\infty)$ is bounded function on M, and the limit distribution has compact support.

Proof of theorem 4.6. Since $\alpha_\infty \in \mathfrak{M}_g$, the existence of limit follows from

Corollary 4.27 and Lemma 4.28. □

A relation with Birkhoff sum and theta sum was introduced in [CF15, §5.3], and as an applications, we derive limit theorem of theta sums.

Corollary 4.30. *Let $\mathscr{Q}[x] = x^\top \mathscr{Q} x$ be the quadratic forms defined by $g \times g$ real matrix $\mathscr{Q}$, $\alpha = \begin{pmatrix} I & 0 \\ \mathscr{Q} & I \end{pmatrix} \in Sp_{2g}(\mathbb{R})$, $\ell(x) = \ell^\top x$ be the linear form defined by $l \in \mathbb{R}^g$. Then, Theta sum*

$$\Theta(\mathscr{Q}, l; N) = N^{-g/2} \sum_{n \in \mathbb{Z}^g \cap [0,N]} e(\mathscr{Q}[n] + \ell(n))$$

has limit distribution and it has compact support.

4.6 L^2-lower bounds

In this section we prove bounds for the square mean of ergodic integrals along the leaves of foliations of the torus into circles transverse to central direction.

4.6.1 Structure of return map

Let $\mathbb{T}^{g+1}_\Gamma$ denote $(g+1)$-dimensional torus with standard frame (X_i, Y_i, Z) with

$$\mathbb{T}^{g+1}_\Gamma := \{\Gamma \exp(\sum_{i=1}^{g} y_i Y_i + zZ) \mid (y_i, z) \in \mathbb{R} \times \mathbb{R}\}.$$

It is convenient to work with the polarized Heisenberg group. Set $H^g_{pol} \approx \mathbb{R}^g \times$

$\mathbb{R}^g \times \mathbb{R}$ equipped with the group law $(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + yx')$.

Definition 4.31. *Reduced standard Heisenberg group H^g_{red} is defined by quotient $H^g_{pol}/(\{0\} \times \{0\} \times \frac{1}{2}\mathbb{Z}) \approx \mathbb{R}^g \times \mathbb{R}^g \times \mathbb{R}/\frac{1}{2}\mathbb{Z}$. Reduced standard latice Γ^g_{red} is $\mathbb{Z}^g \times \mathbb{Z}^g \times \{0\}$ and the quotient H^g_{red}/Γ^g_{red} is isomorhic to standard Heisenberg manifold H^g/Γ.*

Now, we consider return map of $\mathsf{P}^{d,\alpha}$ on $\mathbb{T}^{g+1}_\Gamma$. For $x = (x_1, \cdots, x_g) \in \mathbb{R}^g$,

$$\exp(x_1 X^\alpha_1 + \cdots + x_g X^\alpha_g) = (x_\alpha, x_\beta, w \cdot x), \text{ for some } x_\alpha, x_\beta \in \mathbb{R}^d.$$

In H^g_{red},

$$\exp(x_1 X^\alpha_1 + \cdots + x_g X^\alpha_g) \cdot (0, y, z) = (x_\alpha, y + x_\beta, z + w \cdot x).$$

Then, given $(n, m, 0) \in \Gamma^g_{red}$,

$$\exp(x_1 X^\alpha_1 + \cdots + x_g X^\alpha_g) \cdot (0, y, z) \cdot (n, m, 0) = \exp(x'_1 X^\alpha_1 + \cdots + x'_g X^\alpha_g) \cdot (0, y', z') \quad (4.36)$$

if and only if $x'_\alpha = x_\alpha + n$, $y' = y + (x_\beta - x'_\beta) + m$ and $z' = z + (w - w') \cdot x + n(y + x_\beta)$.

Assume $\langle X^\alpha_i, X_j \rangle \neq 0$ for all i, j, and we write first return time $t_{Ret} = (t_{Ret,1}, \cdots t_{Ret,g})$ for $\mathsf{P}^{d,\alpha}$ on transverse torus $\mathbb{T}^{g+1}_\Gamma$. We denote domain for return time $U(t_{Ret}) = [0, t_{Ret,1}] \times \cdots [0, t_{Ret,g}]$. Return map of action $\mathsf{P}^{d,\alpha}$ on $\mathbb{T}^{g+1}_\Gamma$ has a form of skew-shift

$$A_{\rho,\tau}(y,z) = (y+\rho, z+v\cdot y+\tau_i) \quad \text{on } \mathbb{R}^g/\mathbb{Z}^g \times \mathbb{R}/K^{-1}\mathbb{Z}. \tag{4.37}$$

From computation of each rank 1 action, for each $1 \le i \le d$, it is a composition of commuting linear skew-shift

$$A_{i,\rho,\tau}(y,z) = (y+\rho_i, z+v_i\cdot y+\tau_i) \quad \text{on } \mathbb{R}^g/\mathbb{Z}^g \times \mathbb{R}/K^{-1}\mathbb{Z} \tag{4.38}$$

for some constant $\rho_i, \sigma_i \in \mathbb{R}^g$ and $v_i \in \mathbb{R}^g$. For each $j \ne k$,

$$A_{j,\rho,\tau} \circ A_{k,\rho,\tau} = A_{k,\rho,\tau} \circ A_{j,\rho,\tau}.$$

Given pair $(\mathbf{m}, n) \in \mathbb{Z}^g_{K|n|} \times \mathbb{Z}$, let $H_{(\mathbf{m},n)}$ denote the corresponding factor and $C^\infty(H_{(m,n)})$ be subspace of smooth function on $H_{(m,n)}$. Denote $\{e_{\mathbf{m},n} \mid (\mathbf{m}, n) \in \mathbb{Z}^g_{|n|} \times \mathbb{Z}\}$ the basis of characters of $\mathbb{T}^{g+1}_{\Gamma}$ and for all $(y,z) \in \mathbb{T}^g \times \mathbb{T}$,

$$e_{\mathbf{m},n}(y,z) := \exp[2\pi\iota(\mathbf{m}\cdot y + nKz)].$$

For each $A_{i,\rho,\sigma}$ and $v_i = (v_{i1}, \cdots v_{id})$, the orbit can be identified with the following dual orbit

$$\begin{aligned}\mathcal{O}_{A_i}(\boldsymbol{m}, n) &= \{(\boldsymbol{m} + (nj_i)v_i, n),\ j_i \in \mathbb{Z}\} \\ &= \{(m_1 + (nv_{i1})j_i, \cdots, m_d + (nv_{id})j_i, n),\ j_i \in \mathbb{Z}\}.\end{aligned}$$

If $n = 0$, the orbit $[(\mathbf{m}, 0)] \subset \mathbb{Z}^g \times \mathbb{Z}$ of $(m, 0)$ is reduced to a single element. If $n \neq 0$, then the dual orbit $[(\mathbf{m}, n)] \subset \mathbb{Z}^{g+1}$ of $(\mathbf{m}, n)$ for higher rank actions is described as follows:

$$\mathcal{O}_A(\boldsymbol{m}, n) = \{(m_k + n \sum_{i=1}^{d} (v_{ik} j_i), n)_{1 \leq k \leq d} : j = (j_1, \cdots j_d) \in \mathbb{Z}^d\}.$$

It follows that every A-orbit for rank $\mathbb{R}^d$-action (or A^j-orbit) can be labeled uniquely by a pair $(\mathbf{m}, n) \in \mathbb{Z}^g_{|n|} \times \mathbb{Z} \backslash \{0\}$ with $\mathbf{m} = (m_1, \cdots, m_g)$. Thus, the subspace of functions with non-zero central character can be split as a direct sum of components $H_{(m,n)}$ with $\mathbf{m} \in \mathbb{Z}^g, n \in \mathbb{Z} \backslash \{0\}$. Then,

$$L^2(\mathbb{T}^{g+1}_{\mathsf{\Gamma}}) = \bigoplus_{\omega \in \mathcal{O}_A} H_\omega.$$

Now we proceed to the cosideration of the higher cohomology problem which appears in the space of Fourier coefficients.

4.6.2 Higher cohomology for $\mathbb{Z}^d$-action of skew-shifts

We consider a $\mathbb{Z}^d$ action of return map $\mathsf{P}^{d,\alpha}$ on torus $\mathbb{T}^{g+1}_{\mathsf{\Gamma}}$. By identification of cochain complex on torus, it is equivalent to consider the following cohomological equation for degree d form ω,

$$\omega = d\Omega \iff \varphi(x, t) = D\Phi(x, t), \quad x \in \mathbb{T}^g, \ t \in \mathbb{Z}^d. \tag{4.39}$$

We restrict our interest of d-cocycle $\varphi : \mathbb{T}^{g+1}_\Gamma \times \mathbb{Z}^d \to \mathbb{R}$ with $\Phi : \mathbb{T}^{g+1}_\Gamma \to \mathbb{R}^d$, $\Phi = (\Phi_1, \cdots \Phi_d)$ and D is coboundary operator $D\Phi = \sum_{i=1}^d (-1)^{i+1}\Delta_i\Phi_i$ where $\Delta_i\Phi_i = \Phi_i \circ A_{i,\rho,\tau} - \Phi_i$. The following proposition is the generalization to the argument of [KK95, Prop 2.2]. Let us denote $A^j = A^{j_1}_{1,\rho,\tau} \circ \cdots \circ A^{j_d}_{d,\rho,\tau}$.

Proposition 4.32. *A cocycle $\hat{\varphi}$ satisfies cohomological equation (4.39) if and only if $\sum_{j\in\mathbb{Z}^d} \hat{\varphi}_{(\mathbf{m},n)} \circ A^j = 0$ for $j = (j_1, \cdots, j_d) \in \mathbb{Z}^d$.*

Proof. We consider dual equation

$$\hat{\varphi} = D\hat{\Phi}. \tag{4.40}$$

Let us denote the following notation:

$$(\delta_i\hat{\varphi})(m_1, \cdots, m_d) = \delta(m_i)\hat{\varphi}(m_1, \cdots, m_d), \text{ and } \quad \delta(0) = 1, \text{ otherwise } 0.$$

$$(\Sigma_i\hat{\varphi})(m_1, \cdots, m_d) = \sum_{j=-\infty}^{\infty} \hat{\varphi} \circ (A_1^{m_1} \cdots A_i^j \cdots A_d^{m_d})$$

$$(\Sigma_i^+\hat{\varphi})(m_1, \cdots, m_d) = \sum_{j=m_i}^{\infty} \hat{\varphi} \circ (A_1^{m_1} \cdots A_i^j \cdots A_d^{m_d})$$

$$(\Sigma_i^-\hat{\varphi})(m_1, \cdots, m_d) = -\sum_{j=-\infty}^{m_i-1} \hat{\varphi} \circ (A_1^{m_1} \cdots A_i^j \cdots A_d^{m_d})$$

It is clear that $\Sigma_i^- - \Sigma_i^+ = \Sigma_i$ and $\Sigma_i^+\hat{\varphi} = \Sigma_i^-\hat{\varphi}$ if and only if $\Sigma_i\hat{\varphi} = 0$.

Note that

$$\Sigma_i^+\Delta_i = \Sigma_i^-\Delta_i = id, \quad \Delta_i\Sigma_i^+ = \Delta_i\Sigma_i^- = id. \tag{4.41}$$

By direct calculation of Fourier coefficient, $\Sigma_i(\hat{\varphi}-\delta_i\Sigma_i\hat{\varphi})=0$. Let $\hat{\Phi}_i(\hat{\varphi})=\Sigma_i^-(\hat{\varphi}-\delta_i\Sigma_i\hat{\varphi})$, then $\hat{\Phi}_i(\hat{\varphi})$ vanishes at ∞. By (4.41),

$$\hat{\varphi}-\delta_i\Sigma_i\hat{\varphi}=\Delta_i\hat{\Phi}_i(\hat{\varphi}).$$

We can proceed this by induction.

$$\begin{aligned}
\hat{\varphi}-\Sigma_{1,\cdots,d}\hat{\varphi} &= \sum_{i=1}^{d}(\delta_1\cdots\delta_{i-1}\Sigma_1\cdots\Sigma_{i-1}\hat{\varphi}-\delta_1\cdots\delta_i\Sigma_1\cdots\Sigma_i\hat{\varphi}) \\
&= \sum_{i=1}^{d}(\delta_1\cdots\delta_{i-1}\Sigma_1\cdots\Sigma_{i-1}\hat{\varphi}-\delta_i\Sigma_i(\Sigma_1\cdots\Sigma_{i-1}\hat{\varphi})) \\
&= \sum_{i=1}^{d}(-1)^{i+1}\Delta_i\hat{\Phi}_i(\hat{\varphi})
\end{aligned}$$

where

$$\hat{\Phi}_i(\hat{\varphi})=(-1)^{i+1}\Sigma_i^-\delta_1\cdots\delta_{i-1}(\Sigma_1\cdots\Sigma_{i-1}\hat{\varphi}-\delta_i\Sigma_i(\Sigma_1\cdots\Sigma_{i-1}\hat{\varphi}))$$

and $\hat{\Phi}_i(\hat{\varphi})$ vanishes at ∞. Thus, $\hat{\Phi}_i$ is a solution of (4.40) if and only if $\Sigma_1\cdots\Sigma_d\hat{\varphi}=0$. □

For fixed $(\mathbf{m},n)\in\mathbb{Z}^g\times\mathbb{Z}$, we denote obstruction of cohomological equation restricted to the orbit of $(\mathbf{m},n)$ by $\mathcal{D}_{m,n}(\varphi)=\sum_{j\in\mathbb{Z}^d}\hat{\varphi}_{(\mathbf{m},n)}\circ A^j$.

Lemma 4.33. *There exists a distributional obstruction to the existence of a smooth solution $\varphi\in C^\infty(H_{(\mathbf{m},n)})$ of the cohomological equation (4.39).*

A generator of the space of invariant distribution $\mathcal{D}_{m,n}$ has form of

$$\mathcal{D}_{\mathbf{m},n}(e_{a,b}) := \begin{cases} e^{-2\pi\iota\sum_{i=1}^{d}[(\mathbf{m}\cdot\rho_i+nK\tau_i)j_i+nK\tau_i\binom{j_i}{2}]} & \text{if } (a,b)=(m_k+K\sum_{i=1}^{d}(v_{ik}j_i),n)_{1\le k\le g} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. From previous observation, there exists obstruction

$$\mathcal{D}_{\mathbf{m},n}(\varphi) = \sum_{j\in\mathbb{Z}^d}\int_{\mathbb{T}_\Gamma^{g+1}} \varphi(x,y)\overline{e_{\mathbf{m},n}\circ A^j_{\rho,\tau}}dxdy. \tag{4.42}$$

By direct computation, for fixed $j=(j_1,\cdots j_d)$,

$$e_{\mathbf{m},n}\circ A^j_{\rho,\tau}(y,z) = \prod_{i=1}^{d}\left(e^{2\pi\iota[(\mathbf{m}\cdot\rho_i+nK\tau_i)j_i+nK\tau_i\binom{j_i}{2}]}\right)\left(e^{2\pi\iota(\mathbf{m}\cdot y+K(z+n\sum_{k=1}^{d}(v_{ik}j_i)y_k))}\right).$$

Then, we choose $\varphi=e_{a,b}$ for $(a,b)=(m_k+K\sum_{i=1}^{d}(v_{ik}j_i),n)_{1\le k\le g}$ in the non-trivial orbit $(n\neq 0)$,

$$\mathcal{D}_{\mathbf{m},n}(e_{a,b}) = e^{-2\pi\iota\sum_{i=1}^{d}[(\mathbf{m}\cdot\rho_i+nK\tau_i)j_i+nK\tau_i\binom{j_i}{2}]}.$$

□

We conclude this section by introducing the theory of unitary representations.

$$L^2(M) = \bigoplus_{n\in\mathbb{Z}} H_n := \bigoplus_{n\in\mathbb{Z}}\bigoplus_{i=1}^{\mu(n)} H_{i,n} \tag{4.43}$$

where $H_n = \bigoplus_{i=1}^{\mu(n)} H_{i,n}$ is irreducible representation with central parameter n and $\mu(n)$ is countable by Howe-Richardson multiplicity formula. By direct calculation, we obtain $\mu(n) = |n|^d$ for all $n \neq 0$.

For P-action, the space of invariant currents $I_d(\mathfrak{p}, \mathscr{S}(\mathbb{R}^g)) \subset W^{-s}(\mathbb{R}^g)$ for $s > d/2 + \epsilon$ for all $\epsilon > 0$. That is, by normalization of invariant distributions in the Sobolev space, for any irreducible components $H = H_n$ and α, there exists a non-unique function f_α^H such that

$$D_\alpha(f_\alpha^H) = \left\| f_\alpha^H \right\|_s = 1. \tag{4.44}$$

4.6.3 Changes of coordinates

For any frame $(X_i^\alpha, Y_i^\alpha, Z)_{i=1}^g$, denote transverse cylinder for any $m \in M$,

$$\mathscr{C}_{\alpha,m} := \{m \exp(\sum_{i=1}^g y_i' Y_i^\alpha + z' Z) \mid (y', z') \in U(t_{Ret}^{-1}) \times \mathbb{T}\}.$$

Let $\Phi_{\alpha,m} : \mathbb{T}_\Gamma^{g+1} \to \mathscr{C}_{\alpha,m}$ denote the maps: for any $\xi \in \mathbb{T}_\Gamma^{g+1}$, let $\xi' \in \mathscr{C}_{\alpha,m}$ denote first intersection of the orbit $\{\mathsf{P}_t^{d,\alpha}(\xi) \mid t \in \mathbb{R}_+^d\}$ with transverse cylinder $\mathscr{C}_{\alpha,m}$. Then, there exists first return time to cylinder $t(\xi) = (t_1(\xi), \cdots, t_d(\xi)) \in \mathbb{R}_+^d$ such that

$$\xi' = \Phi_{\alpha,m}(\xi) = \mathsf{P}_{t(\xi)}^{d,\alpha}(\xi), \quad \forall \xi \in \mathbb{T}_\Gamma^{g+1}.$$

Let (y, z) and (y', z') denote the coordinates on $\mathbb{T}_\Gamma^{g+1}$ and $\mathscr{C}_{\alpha,m}$ given by the exponential map respectively,

$$(y,z) \to \xi_{y,z} := \Gamma \exp(\sum_{i=1}^{g} y_i Y_i + zZ), \quad (y',z') \to m \exp(\sum_{i=1}^{g} y_i' Y_i^{\alpha} + z'Z).$$

For $1 \le i, j \le g$ and matrix $A = (a_{ij}), B = (b_{ij}), C = (c_{ij}), D = (d_{ij})$, set

$$\alpha := \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in Sp(2g, \mathbb{R}),$$

satisfying $A^t D - C^t B = I_{2g}$, $C^t A = A^t C$, $D^t B = B^t D$, and $\det(A) \neq 0$.

Recall that $X_i^{\alpha} = \sum_j a_{ij} X_j + b_{ij} Y_j + w_i Z$ and $Y_i^{\alpha} = \sum_j c_{ij} X_j + d_{ij} Y_j + v_i Z$ with $det(A) \neq 0$.

Let $x = \Gamma \exp(\sum_{i=1}^{d} y_{x,i} Y_i + z_x Z) \exp(\sum_{i=1}^{d} t_{x,i} X_i)$, for some $(y_x, z_x) \in \mathbb{T}^d \times \mathbb{R}/K\mathbb{Z}$ and $t_x = (t_{x,i}) \in [0,1)^d$. Then, the map $\Phi_{\alpha,x} : \mathbb{T}_{\Gamma}^{g+1} \to \mathscr{C}_{\alpha,m}$ is defined by $\Phi_{\alpha,x}(y,z) = (y',z')$ where

$$\begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_g' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1g} \\ a_{21} & a_{22} & \cdots & a_{2g} \\ \vdots & \vdots & \vdots & \vdots \\ a_{g1} & a_{g2} & \cdots & a_{gg} \end{bmatrix} \begin{bmatrix} y_1 - y_{x,1} \\ y_2 - y_{x,2} \\ \vdots \\ y_g - y_{x,g} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1g} \\ b_{21} & \cdots & b_{2g} \\ \vdots & \vdots & \vdots \\ b_{g1} & \cdots & b_{gg} \end{bmatrix} \begin{bmatrix} t_{x,1} \\ t_{x,2} \\ \vdots \\ t_{x,g} \end{bmatrix}, \qquad (4.45)$$

and $z' = z + P(\alpha, x, y)$ for some degree 4 polynomial P.

Therefore, the map $\Phi_{\alpha,x}$ is invertible with

$$\Phi_{\alpha,x}^{*}(dy_1' \wedge \cdots dy_g' \wedge dz') = \frac{1}{det(A)} dy_1 \wedge \cdots dy_g \wedge dz.$$

Since $A^tD - C^tB = I_{2g}$, by direct computation we obtain return time, we have

$$\begin{bmatrix} t_1(\xi) \\ t_2(\xi) \\ \vdots \\ t_g(\xi) \end{bmatrix} = \begin{bmatrix} d_{11} & \cdots & d_{1g} \\ d_{21} & \cdots & d_{2g} \\ \vdots & \vdots & \vdots \\ d_{d1} & \cdots & d_{gg} \end{bmatrix} \begin{bmatrix} t_{x,1} \\ t_{x,2} \\ \vdots \\ t_{x,d} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1g} \\ c_{21} & c_{22} & \cdots & c_{2g} \\ \vdots & \vdots & \vdots & \vdots \\ c_{g1} & c_{g2} & \cdots & c_{gg} \end{bmatrix} \begin{bmatrix} y_1 - y_{x,1} \\ y_2 - y_{x,2} \\ \vdots \\ y_g - y_{x,g} \end{bmatrix}. \qquad (4.46)$$

Then,

$$\|t(\xi)\| \leq \max_i |t_i(\xi)|^g \leq \max_i |\sum_{j=1}^{g} d_{ij}t_{x,i} + c_{ij}(y_i - y_{x,i})|^g \leq \max_i \|Y_i^\alpha\|^g .$$

4.6.4 L^2-lower bound of functional.

We will prove bounds for the square mean of integrals along foliations of the torus $\mathbb{T}_\Gamma^{g+1}$ into torus $\{\xi \exp(\sum_{i=1}^g y_iY_i) \mid y_i \in \mathbb{T}\}_{\xi \in \mathbb{T}_\Gamma^{g+1}}$.

Lemma 4.34. *There exists a constant $C > 0$ such that for all $\alpha = (X_i^\alpha, Y_i^\alpha, Z)$, and for every irreducible component H of central parameter $n \neq 0$, there exists a function f_H such that*

$$|f_H|_{L^\infty(H)} \leq Cvol(U(t_{Ret}))^{-1}|\mathcal{D}_\alpha^H(f_H)|,$$

$$|f_H|_{\alpha,s} \leq Cvol(U(t_{Ret}))^{-1}|\mathcal{D}_\alpha^H(f_H)|(1 + \frac{T(t_{Ret})}{vol(U(t_{Ret}))}\|Y\|)^s(1+n^2)^{s/2}$$

where $\|Y\| := \max_{1\leq i\leq g}\|Y_i^\alpha\|$ and $T(t_{Ret}) = \sum_{i=1}^g t_{Ret,i}$.

On rectangular domain $U(\boldsymbol{T})$, for all $m \in \mathbb{T}_\Gamma^{g+1}$ and $T^{(i)} \in \mathbb{Z}_{t_{Ret,i}}$,

$$\left\| \left\langle P^{d,\alpha}_{U(\boldsymbol{T})}(\mathsf{Q}^{g,Y}_y m), \omega_H \right\rangle \right\|_{L^2(\mathbb{T}^g, dy)} = |\mathcal{D}^H_\alpha(f_H)| \left(\frac{vol(U(\boldsymbol{T}))}{vol(U(t_{Ret}))} \right)^{1/2}. \tag{4.47}$$

In addition, whenever $H \perp H' \subset L^2(M)$ the functions

$$\left\langle P^{d,\alpha}_{U(\boldsymbol{T})}(\mathsf{Q}^{g,Y}_y m), \omega_H \right\rangle \quad and \quad \left\langle P^{d,\alpha}_{U(\boldsymbol{T})}(\mathsf{Q}^{g,Y}_y m), \omega_{H'} \right\rangle$$

are orthogonal in $L^2(\mathbb{T}^g, dy)$.

Proof. As explained in §5.1, the space $L^2(\mathbb{T}_\Gamma^{d+1})$ decompose as a direct sum of irreducible subspaces invariant under the action of each $A_{j,\rho,\sigma}$. It follows that the subspace of functions with non-zero central character can be split as direct sum of components $H_{(\mathbf{m},n)}$ with $(\mathbf{m}, n) \in \mathbb{Z}^g_{|n|} \times \mathbb{Z}\backslash\{0\}$ with $\mathbf{m} = (m_1, \cdots, m_g)$. For $F \in H_{(\mathbf{m},n)}$, the function is characterized by Fourier expansion

$$F = \sum_{j \in \mathbb{Z}^d} F_j e_{A^j(\boldsymbol{m},n)} = \sum_{j \in \mathbb{Z}^d} F_j e_{(m_k + K \sum_{i=1}^d (v_{ik} j_i), n)}.$$

Then, by Lemma 4.33,

$$\mathcal{D}_{(\mathbf{m},n)}(e_{A^j(\boldsymbol{m},n)}) = e^{-2\pi\iota \sum_{i=1}^d [(m\cdot\rho_i + nK\tau_i) j_i + nK\tau_i \binom{j_i}{2})]}. \tag{4.48}$$

For any irreducible representation $H := H_n$ with central parameter $n \neq 0$, there exists $\mathbf{m} \in \mathbb{Z}^d_{|n|}$ such that the operator I_α maps the space H onto $H_{(\mathbf{m},n)}$. The operator $I_\alpha : L^2(M) \to L^2(\mathbb{T}_\Gamma^{g+1})$ is defined

$$f \to I_\alpha(f) := \int_{U(t_{Ret})} f \circ \mathsf{P}_x^{d,\alpha}(\cdot) dx. \tag{4.49}$$

Then, operator I_α is surjective linear map of $L^2(M)$ onto $L^2(\mathbb{T}_\Gamma^{g+1})$ with right inverse defined as follows:

Let $\chi \in C_0^\infty(0,1)^g$ be any function of jointly integrable with integral 1. For any $F \in L^2(\mathbb{T}_\Gamma^{g+1})$, let $R_\alpha^\chi(F) \in L^2(M)$ be the function defined by

$$R_\alpha^\chi(F)(\mathsf{P}_v^{d,\alpha}(m)) = \frac{1}{vol(U(t_{Ret}))}\chi(\frac{v}{t_{Ret}})F(m), \ (m,v) \in \mathbb{T}_\Gamma^{g+1} \times U(t_{Ret}).$$

Then, it follows from the definition that there exists a constant $C_\chi > 0$ such that

$$\begin{aligned} |R_\alpha^\chi(F)|_{\alpha,s} &\leq C_\chi vol(U(t_{Ret}))^{-1}(1 + \sum_{i=1}^{g} t_{Ret,i}^{-1} \|Y_i^\alpha\|)^s \|F\|_{W^s(\mathbb{T}_\Gamma^{g+1})} \\ &\leq C_\chi vol(U(t_{Ret}))^{-1}(1 + \frac{T(t_{Ret})}{vol(U(t_{Ret}))} \|Y\|)^s \|F\|_{W^s(\mathbb{T}_\Gamma^{g+1})} . \end{aligned}$$

Choose $f_H := R_\alpha^\chi(e_{m,n}) \in C^\infty(H)$ such that $I_\alpha(f_H) = e_{m,n}$ and

$$\int_{U(t_{Ret})} f_H \circ \mathsf{P}_t^{d,\alpha}(y,z) dt = e_{m,n}(y,z), \text{ for } (y,z) \in \mathbb{T}_\Gamma^{g+1}. \tag{4.50}$$

By (4.44) and (4.48), we have $|\mathcal{D}_H(f_H)| = |\mathcal{D}_{(m,n)}(e_{m,n})| = 1$. Therefore, it follows that

$$|f_H|_{L^\infty(H)} \leq C_\chi vol(U(t_{Ret}))^{-1}$$

$$|f_H|_{\alpha,s} \leq C vol(U(t_{Ret}))^{-1} |D^H_\alpha(f_H)| (1 + \frac{T(t_{Ret})}{vol(U(t_{Ret}))} \|Y\|)^s (1+n^2)^{s/2}.$$

Moreover, since $\{e_{m,n} \circ A^j_{\rho,\tau}\}_{j\in\mathbb{Z}^d} \subset L^2(\mathbb{T}^g_\Gamma, dy)$ is orthonormal, we verify

$$\begin{aligned} \left\| \left\langle P^{d,\alpha}_{U(\boldsymbol{T})}(\mathsf{Q}^{g,Y}_y m), \omega_H \right\rangle \right\|_{L^2(\mathbb{T}^g, dy)} &= \left\| \sum_{j_d=0}^{[\frac{T^{(d)}}{t_{Ret,d}}]-1} \cdots \sum_{j_1=0}^{[\frac{T^{(1)}}{t_{Ret,1}}]-1} e_{m,n} \circ A^j_{\rho,\tau} \right\|_{L^2(\mathbb{T}^g, dy)} \\ &= \left(\frac{vol(U(\boldsymbol{T}))}{vol(U(t_{Ret}))} \right)^{1/2}. \end{aligned}$$

□

For any infinite dimensional vector $\mathbf{c} := (c_{i,n}) \in l^2$, let $\beta_{\mathbf{c}}$ denote Bufetov functional

$$\beta_{\mathbf{c}} = \sum_{n\in\mathbb{Z}} \sum_{i=1}^{\mu(n)} c_{i,n} \beta^{i,n}.$$

For any $\mathbf{c} := (c_{i,n})$, let $|\mathbf{c}|_s$ denote the norm defined as

$$|\mathbf{c}|_s^2 = \sum_{n\in\mathbb{Z}\setminus\{0\}} \sum_{i=1}^{\mu(n)} (1 + K^2 n^2)^s |c_{i,n}|^2.$$

From Corollary 4.26,

$$\|\beta_{\mathbf{c}}(\alpha, \cdot, \mathbf{T})\|^2_{L^2(M)} \leq C^2 |\mathbf{c}|^2_{l^2} vol(U(\boldsymbol{T})).$$

Lemma 4.35. *For any $s > s_d + 1/2$, there exists a constant $C_s > 0$ such that for* *all* $\alpha \in DC(L)$*, for all* $\boldsymbol{c} \in l^2$*, for all* $z \in \mathbb{T}$ *and all* $T > 0$*,*

$$\left| \|\beta_{\boldsymbol{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \boldsymbol{T})\|_{L^2(\mathbb{T}^g,dy)} - \left(\frac{vol(U(\boldsymbol{T}))}{vol(U(t_{Ret}))}\right)^{1/2} |\boldsymbol{c}|_0 \right|$$
$$\leq C_s(vol(U(t_{Ret})) + vol(U(t_{Ret}))^{-1})(1+L)(1 + \frac{T(t_{Ret})}{vol(U(t_{Ret}))}\|Y\|)^s |\boldsymbol{c}|_s. \quad (4.51)$$

Proof. By (4.43), for every $n \neq 0$, there exists a function $f_{i,n} \in C^\infty(H)$ with $D(f_{i,n}) = 1$. Let $f_{\mathbf{c}} = \sum_{n\in\mathbb{Z}} \sum_{i=1}^{\mu(n)} c_{i,n} f_{i,n}$, by adding functions on all irreducibles. Then, by Lemma 4.34,

$$|f_{\mathbf{c}}|_{L^\infty(M)} \leq C|\mathbf{c}|_{l^1} \quad (4.52)$$

$$|f_{\mathbf{c}}|_{\alpha,s} \leq C vol(U(t_{Ret}))^{-1}(1 + \frac{T(t_{Ret})}{vol(U(t_{Ret}))}\|Y\|)^s |\mathbf{c}|_s. \quad (4.53)$$

By orthogonality,

$$\left\| \left\langle P^{d,\alpha}_{U(\boldsymbol{T})} \circ \mathsf{Q}^{g,Y}_y, \omega_{\mathbf{c}} \right\rangle \right\|_{L^2(\mathbb{T}^g,dy)} = \left(\frac{vol(U(\boldsymbol{T}))}{vol(U(t_{Ret}))}\right)^{1/2} |\mathbf{c}|_0.$$

From the estimation for each $f_{i,n}$ in Lemma 4.34, for every $z \in \mathbb{T}$ and all $T > 0$, we have

$$\left\| \left\langle P^{d,\alpha}_{U(\boldsymbol{T})}(\Phi_{\alpha,x}(\xi_{y,z})), \omega_{\mathbf{c}} \right\rangle - \left\langle P^{d,\alpha}_{U(\boldsymbol{T})}(\xi_{y,z}), \omega_{\mathbf{c}} \right\rangle \right\|_{L^2(\mathbb{T}^g,dy)} \leq 2|f_{\mathbf{c}}|_{L^\infty(M)} \|Y\|.$$

Let $T_{\alpha,i} = t_{Ret,i}([T/t_{Ret,i}] + 1)$ and $U(t_\alpha) = [0, T_{\alpha,1}] \times \cdots \times [0, T_{\alpha,g}]$. Then,

$$\left\| \left\langle P^{d,\alpha}_{U(\boldsymbol{T})}(\xi_{y,z}), \omega_{\mathbf{c}} \right\rangle - \left\langle P^{d,\alpha}_{U(t_\alpha)}(\xi_{y,z}), \omega_{\mathbf{c}} \right\rangle \right\|_{L^2(\mathbb{T}^g, dy)} \le vol(U(t_{Ret}))|f_{\mathbf{c}}|_{L^\infty(M)}.$$

Therefore, for some constant $C' > 0$ such that

$$\left| \left\| \left\langle P^{d,\alpha}_{U(\boldsymbol{T})}(\Phi_{\alpha,x}(\xi_{y,z})), \omega_{\mathbf{c}} \right\rangle \right\|_{L^2(\mathbb{T}^g, dy)} - \left(\frac{vol(U(\boldsymbol{T}))}{vol(U(t_{Ret}))} \right)^{1/2} |\mathbf{c}|_0 \right| \le C' vol(U(t_{Ret}))|\mathbf{c}|_{l^1}.$$

For all $s > s_d + 1/2$, by asymptotic property of Theorem 4.23, there exists constant $C_s > 0$ such that

$$\left| \left\langle P^{d,\alpha}_{U(\boldsymbol{T})} m, \omega \right\rangle - \beta_H(\alpha, m, \boldsymbol{T}) D^H_\alpha(f_H) \right| \le C_s(1 + L) \left\| f \right\|_{\alpha,s}.$$

Applying $\beta_{\mathbf{c}} = \beta^{f_{\mathbf{c}}}$ and combining bounds on the function $f_{\mathbf{c}}$ with (4.52),

$$\begin{aligned}
&\left| \|\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \boldsymbol{T})\|_{L^2(\mathbb{T}^g, dy)} - \left(\frac{vol(U(\boldsymbol{T}))}{vol(U(t_{Ret}))} \right)^{1/2} |\mathbf{c}|_0 \right| \\
&\le C' vol(U(t_{Ret}))|\mathbf{c}|_{l^1} + C_s vol(U(t_{Ret}))^{-1}(1 + L)|f_{\mathbf{c}}|_{\alpha,s} \\
&\le C'_s(vol(U(t_{Ret})) + vol(U(t_{Ret}))^{-1})(1 + L)(1 + \frac{T(t_{Ret})}{vol(U(t_{Ret}))} \|Y\|)^s |\mathbf{c}|_s.
\end{aligned}$$

Therefore, we derive the estimates in the statement. □

4.7 Analyticity of functionals

In this section we will prove that for all $\alpha \in DC$, the Bufetov functionals on any square are real analytic.

Definition 4.36. *For every* $t \in \mathbb{R}$, $1 \leq i \leq d$, *and* $m \in M$, *the* stretched (in direction of Z) rectangle *is denoted by*

$$[\Gamma_T]^Z_{i,t}(m) := \{(\phi^Z_{ts_i}) \circ \mathsf{P}^{d,\alpha}_{\boldsymbol{s}}(m) \mid \boldsymbol{s} \in U(\boldsymbol{T})\}. \tag{4.54}$$

Recall definition 4.1 for standard rectangles. For $\mathbf{s} = (s_1, \cdots, s_d) \in \mathbb{R}^d$, let us denote $\Gamma_{\mathbf{T}}(\mathbf{s}) := (\gamma_1(s_1), \cdots, \gamma_d(s_d))$ for $\gamma_i(s_i) = \exp(s_i X_i)$. Similarly, we also write

$$[\Gamma_T]^Z_{i,t}(\mathbf{s}) := (\gamma_1(s_1), \cdots, \gamma^Z_{i,t}(s_i), \cdots \gamma_d(s_d)) \tag{4.55}$$

where $\gamma^Z_{i,t}(s_i) := \phi^Z_{ts_i}(\gamma_i(s_i))$ is a stretched curve.

Definition 4.37. *The* restriction $\Gamma_{T,i,s}$ *of the rectangle* $\Gamma_{\boldsymbol{T}}$ *is defined on restricted domain* $U_{T,i,s} = [0, T^{(1)}] \times \cdots \times [0, s] \cdots \times [0, T^{(d)}]$ *for* $s \leq T^{(i)}$ *as following.*

$$\Gamma_{T,i,s}(\boldsymbol{s}) := \Gamma_{\boldsymbol{T}}(\boldsymbol{s}), \;\; \boldsymbol{s} \in U_{T,i,s}.$$

Recall the orthogonal property on a irreducible component H (central parameter $n \in \mathbb{Z}\backslash\{0\}$). For any $(m, \mathbf{T}) \in M \times \mathbb{R}^d_+$ and $t \in \mathbb{R}$,

$$\beta_H(\alpha, \phi^Z_t(m), \mathbf{T}) = e^{2\pi\iota Knt}\beta_H(\alpha, m, \mathbf{T}). \tag{4.56}$$

We obtain the following lemma for stretched rectangle by applying orthogonal property.

Lemma 4.38. *For fixed elements (X_i, Y_i, Z) satisfying commutation relation (4.3), the following formula for rank 1 action holds:*

$$\hat{\beta}_H(\alpha, [\Gamma_T]^Z_{i,t}) = e^{2\pi\iota tnKT^{(i)}}\hat{\beta}_H(\alpha, \Gamma_T) - 2\pi\iota nKt \int_0^{T^{(i)}} e^{2\pi\iota nKts_i}\hat{\beta}_H(\alpha, \Gamma_{T,i,s})ds_i.$$

Proof. Let $\alpha = (X_i, Y_i, Z)$ and ω be d-form supported on a single irreducible representation H. We obtain following the formula for stretches of curve $\gamma^Z_{i,t}$ (see [FK17, §4, Lemma 9.1]),

$$\frac{d\gamma^Z_{i,t}}{ds_i} = D\phi^Z_{ts_i}(\frac{d\gamma_i}{ds_i}) + tZ \circ \gamma^Z_{i,t}.$$

It follows that pairing is given by

$$\begin{aligned}\langle [\Gamma_{\mathbf{T}}]^Z_{i,t}, \omega\rangle &= \int_{U(\boldsymbol{T})} \omega(\frac{d\gamma_1}{ds_1}(s_1), \cdots, \frac{d\gamma^Z_{i,t}}{ds_i}(s_i), \cdots, \frac{d\gamma_d}{ds_d}(s_d))d\mathbf{s} \\ &= \int_{U(\boldsymbol{T})} e^{2\pi\iota nKts_i}[\omega(\frac{d\gamma_1}{ds_1}(s_1), \cdots, \frac{d\gamma_d}{ds_d}(s_d))] + \iota_Z\omega \circ [\Gamma_{\mathbf{T}}]^Z_{i,t}(\mathbf{s})d\mathbf{s}\end{aligned}$$

Denote $d-1$ dimensional triangle $U_{d-1}(\boldsymbol{T})$ with $U(\boldsymbol{T}) = U_{d-1}(\boldsymbol{T}) \times [0, T^{(i)}]$. Integration by parts for a fixed i-th integral gives

$$\int_{U(\boldsymbol{T})} e^{2\pi\iota nKts_i}[\omega(\frac{d\gamma_1}{ds_1}(s_1),\cdots,\frac{d\gamma_d}{ds_d}(s_d))]d\mathbf{s}$$
$$= e^{2\pi\iota nKtT^{(i)}}\int_{U(\boldsymbol{T})}[\omega(\frac{d\gamma_1}{ds_1}(s_1),\cdots,\frac{d\gamma_d}{ds_d}(s_d))]d\mathbf{s}$$
$$- 2\pi\iota nKt\int_0^{T^{(i)}} e^{2\pi\iota nKts_i}\int_{U_{d-1}(\boldsymbol{T})}\left(\int_0^{s_i}[\omega(\frac{d\gamma_1}{ds_1}(s_1),\cdots,\frac{d\gamma_i}{ds_i}(r)\cdots,\frac{d\gamma_d}{ds_d}(s_d))]dr\right)d\mathbf{s}.$$

Then, we have the following formula

$$\langle[\Gamma_{\mathbf{T}}]^Z_{i,t},\omega\rangle = e^{2\pi\iota nKtT^{(i)}}\langle[\Gamma_{\mathbf{T}}],\omega\rangle - 2\pi\iota nKt\int_0^{T^{(i)}} e^{2\pi\iota nKts_i}\langle\Gamma_{T,i,s},\omega\rangle ds_i$$
$$+\int_{U(\boldsymbol{T})}(\iota_Z\omega\circ[\Gamma_{\mathbf{T}}]^Z_{i,t})(\mathbf{s})d\mathbf{s}.$$

Since the action of $\mathsf{P}^{d,X}_{\mathbf{t}}$ for $\mathbf{t}\in\mathbb{R}^d$ is identity on the center Z,

$$\lim_{t_d\to\infty}\cdots\lim_{t_1\to\infty} e^{-(t_1+\cdots t_d)/2}\int_{U(\boldsymbol{T})}(\iota_Z(\mathsf{P}^{d,X}_{\mathbf{t}})^*\omega\circ[\Gamma_{\mathbf{T}}]^Z_{i,t})(\mathbf{s})d\mathbf{s} = 0.$$

Thus, it follows by definition of Bufetov functional, the statement holds. □

Here we define a restricted vector $\mathbf{T}_{i,s}$ of $\mathbf{T} = (T^{(1)},\cdots,T^{(d)})\in\mathbb{R}^d$. For fixed i, pick $s_i\in[0,T^{(i)}]$ such that $\mathbf{T}_{i,s}\in\mathbb{R}^d$ is a vector with its coordinates

$$T^{(j)}_{i,s} = \begin{cases} T^{(j)} & \text{if} \quad j\neq i \\ s_i & \text{if} \quad j = i. \end{cases}$$

Similarly, $\mathbf{T}_{i_1,\cdots,i_k,s}$ is a vector with $i_1,\cdots i_k$ coordinates replaced by $s_{i_1},\cdots,s_{i_k}$.

Lemma 4.39. *Let $y = (y_1, \cdots, y_d) \in \mathbb{R}^d$. The following equality holds for each rank-1 action.*

$$\beta_H(\alpha, \phi^{Y_i}_{y_i}(m), \boldsymbol{T}) = \\ e^{-2\pi\iota y_i nKT^{(i)}}\beta_H(\alpha, m, \boldsymbol{T}) + 2\pi\iota nKy_i \int_0^{T^{(i)}} e^{-2\pi\iota y_i nKs_i}\beta_H(\alpha, m, \boldsymbol{T}_{i,s})ds_i. \quad (4.57)$$

Proof. By definition (4.1), (4.55) and commutation relation (4.3), it follows that

$$\phi^{Y_i}_{y_i}(\Gamma^X_{\mathbf{T}}(m)) = [\Gamma^X_{\mathbf{T}}(\phi^{Y_i}_{y_i}(m))]^Z_{i,t}.$$

By the invariance property of Bufetov functional and Lemma 4.38,

$$\begin{aligned}
&\beta_H(\alpha, m, \mathbf{T}) = \hat{\beta}_H(\alpha, \phi^{Y_i}_{y_i}(\Gamma^X_T(m)) \\
&= e^{2\pi\iota y_i nKT^{(i)}}\hat{\beta}_H(\alpha, \Gamma^X_{\mathbf{T}}(\phi^{Y_i}_{y_i}(m)) - 2\pi\iota nKy_i \int_0^{T^{(i)}} e^{2\pi\iota nKy_i s_i}\hat{\beta}_H(\alpha, \Gamma^X_{T,i,s}(\phi^{Y_i}_{s_i}(m))ds_i \\
&= e^{2\pi\iota y_i nKT^{(i)}}\beta_H(\alpha, \phi^{Y_i}_{y_i}(m), \mathbf{T}) - 2\pi\iota nKy_i \int_0^{T^{(i)}} e^{2\pi\iota nKy_i s_i}\beta_H(\alpha, \phi^{Y_i}_{s_i}(m), \mathbf{T}_{i,s})ds_i.
\end{aligned}$$

Then statement follows immediately. □

We extend previous lemma for higher rank actions by induction.

Lemma 4.40. *The following equality holds for rank-d action.*

$$\begin{aligned}
&\beta_H(\alpha, \mathsf{Q}_y^{d,Y}(m), \boldsymbol{T}) = e^{-2\pi\iota \sum_{j=1}^d y_j nKT^{(j)}} \beta_H(\alpha, m, \boldsymbol{T}) \\
&+ \sum_{k=1}^{d} \sum_{1 \leq i_1 < \cdots < i_k \leq d} \prod_{j=1}^{k} (2\pi\iota nKy_{i_j}) e^{-2\pi\iota nK(\sum_{l \notin \{i_1, \cdots, i_k\}} y_l T^{(l)})} \\
&\times \int_0^{T^{(i_1)}} \cdots \int_0^{T^{(i_k)}} e^{-2\pi\iota nK(y_{i_1}s_{i_1} + \cdots + y_{i_k}s_{i_k})} \beta_H(\alpha, m, \boldsymbol{T}_{i_1, \cdots, i_k, s}) ds_{i_k} \cdots ds_{i_1}
\end{aligned}$$

Proof. Assume inductive hypothesis works for rank $d-1$. For convenience, we write

$$\mathsf{Q}_y^{d,Y}(m) = \phi_{y_d}^{Y_d} \circ \mathsf{Q}_{y'}^{d-1,Y}(m) \text{ for } y' \in \mathbb{R}^{d-1} \text{ and } y = (y', y_d) \in \mathbb{R}^d.$$

By applying Lemma 4.39,

$$\begin{aligned}
\beta_H(\alpha, \mathsf{Q}_y^{d,Y}(m), \mathbf{T}) &= e^{-2\pi\iota y_d nKT^{(d)}} \beta_H(\alpha, \mathsf{Q}_{y'}^{d-1,Y}(m), \mathbf{T}) \\
&\quad + 2\pi\iota nKy_d \int_0^{T^{(d)}} e^{-2\pi\iota y_d nKs_d} \beta_H(\alpha, \mathsf{Q}_{y'}^{d-1,Y}(m), \mathbf{T}_{d,s}) ds_d \qquad (4.58) \\
&:= I + II
\end{aligned}$$

Firstly, by induction hypothesis

$$\begin{aligned}
&I = e^{-2\pi\iota \sum_{j=1}^{d-1} y_j nKT^{(j)}} \beta_H(\alpha, m, \mathbf{T}) \\
&+ \sum_{k=1}^{d-1} \sum_{1 \leq i_1 < \cdots < i_k \leq d-1} \prod_{j=1}^{k} (2\pi\iota nKy_{i_j}) e^{-2\pi\iota nK(\sum_{l \notin \{i_1, \cdots, i_k\}} y_l T^{(l)} + y_d T^{(d)})} \\
&\times \int_0^{T^{(i_1)}} \cdots \int_0^{T^{(i_k)}} e^{-2\pi\iota nK(y_{i_1}s_{i_1} + \cdots + y_{i_k}s_{i_k})} \beta_H(\alpha, m, \mathbf{T}_{i_1, \cdots, i_k, s}) ds_{i_k} \cdots ds_{i_1}
\end{aligned}$$

which contains 0 to $d-1$th iterated integrals containing $e^{-2\pi\iota nKy_dT^{(d)}}$ outside of iterated integrals.

For the second part, we apply induction hypothesis again for restricted rectangle $\mathbf{T}_{d,s}$. Then,

$$
\begin{aligned}
II = {} & 2\pi\iota nKy_d \int_0^{T^{(d)}} e^{-2\pi\iota y_d nKs_d} \big[e^{-2\pi\iota \sum_{j=1}^{d-1} y_j nKT^{(j)}} \beta_H(\alpha, m, \mathbf{T}_{d,s})\big] ds_d \\
& + \sum_{k=1}^{d-1} \sum_{1 \le i_1 < \cdots < i_k \le d-1} (2\pi\iota nKy_d) \prod_{j=1}^{k} (2\pi\iota nKy_{i_j}) e^{-2\pi\iota nK(\sum_{l \notin \{i_1,\cdots,i_k\}} y_l T^{(l)})} \\
& \times \int_0^{T^{(d)}} \left(\int_0^{T^{(i_1)}} \cdots \int_0^{T^{(i_k)}} ds_{i_k} \cdots ds_{i_1} \right) ds_d \\
& \times e^{-2\pi\iota nK(y_{i_1}s_{i_1} + \cdots + y_{i_k}s_{i_k} + y_d s_d)} \beta_H(\alpha, m, \mathbf{T}_{i_1,\cdots,i_k,d,s}).
\end{aligned}
$$

The term II consist of 1 to d-th iterated integrals containing $e^{-2\pi\iota nKy_d s_d}$ inside of iterated integrals. Thus, combining these two terms, we prove the statement. □

For any $R > 0$, the *analytic norm* defined for all $\mathbf{c} \in l^2$ as

$$\|\mathbf{c}\|_{\omega,R} = \sum_{n \neq 0} \sum_{i=1}^{\mu(n)} e^{nR} |c_{i,n}|.$$

Let Ω_R denote the subspace of $\mathbf{c} \in \ell^2$ such that $\|c\|_{\omega,R}$ is finite.

Lemma 4.41. *For $\boldsymbol{c} \in \Omega_R$, any $\alpha \in DC(L)$ and $\boldsymbol{T} \in \mathbb{R}^d_+$, the function*

$$\beta_{\boldsymbol{c}}(\alpha, \mathsf{Q}_y^{d,Y} \circ \phi_z^Z(m), \boldsymbol{T}),\ (y,z) \in \mathbb{R}^d \times \mathbb{T}$$

extends to a holomorphic function in the domain

$$D_{R,T} := \{(y,z) \in \mathbb{C}^d \times \mathbb{C}/\mathbb{Z} \mid \sum_{i=1}^{d} |Im(y_i)|T^{(i)} + |Im(z)| < \frac{R}{2\pi K}\}. \tag{4.59}$$

The following bound holds: for any $R' < R$ there exists a constant $C > 0$ such that, for all $(y,z) \in D_{R',T}$ we have

$$|\beta_{\boldsymbol{c}}(\alpha, \mathsf{Q}_y^{d,Y} \circ \phi_z^Z(m), \boldsymbol{T})|$$
$$\leq C_{R,R'} \left\|c\right\|_{\omega,R} (L + vol(U(\boldsymbol{T}))^{1/2}(1 + E_M(a,\boldsymbol{T}))(1 + K\sum_{i=1}^{d} |Im(y_i)|T^{(i)})$$

Proof. By Lemma 4.40 and (4.56),

$$\beta_{\mathbf{c}}(\alpha, \mathsf{Q}_y^{d,Y} \circ \phi_z^Z(m), \mathbf{T}) = e^{(z - 2\pi\iota\sum_{j=1}^{d} y_j nKT^{(j)})}\beta_H(\alpha, m, \mathbf{T})$$
$$+ \sum_{k=1}^{d} \sum_{1 \leq i_1 < \cdots < i_k \leq d} \prod_{j=1}^{k} (2\pi\iota nKy_{i_j}) e^{-2\pi\iota nK(\sum_{l \notin \{i_1,\cdots,i_k\}} y_l T^{(l)})}$$
$$\times e^{2\pi\iota nKz} \int_0^{T^{(i_1)}} \cdots \int_0^{T^{(i_k)}} e^{-2\pi\iota nK(y_{i_1}s_{i_1} + \cdots + y_{i_k}s_{i_k})}\beta_H(\alpha, m, \mathbf{T}_{i_1,\cdots,i_k,s}) ds_{i_k} \cdots ds_{i_1}$$

As a consequence, by Lemma 4.22 for each variable $(y_i, z) \in \mathbb{C} \times \mathbb{C}/\mathbb{Z}$, Then for the rank d action, by induction, for $(y,z) \in \mathbb{C}^d \times \mathbb{C}/\mathbb{Z}$ we have

$$|\beta_{\mathbf{c}}(\alpha, \mathsf{Q}_y^{d,Y} \circ \phi_z^Z(x), \mathbf{T})|$$
$$\leq (L + vol(U(\boldsymbol{T}))^{1/2}(1 + E_M(a, \boldsymbol{T}))[C_1 \sum_{n\neq 0}\sum_{i=1}^{\mu(n)} e^{nR}|c_{i,n}|e^{2\pi|Im(z-\sum_{i=1}^d T^{(i)}y_i)|nK}$$
$$+ \sum_{k=1}^{d} C_k \left(\sum_{1\leq i_1 < \cdots < i_k \leq d} \prod_{j=1}^{k}(|Im(y_{i_j})|T^{(i_j)}) \sum_{n\neq 0}\sum_{i=1}^{\mu(n)} n|c_{i,n}|e^{2\pi(|Im(z)|+\sum_{j=1}^k T^{(i_j)}|Im(y_{i_j})|)nK}\right)].$$

Therefore, the function $\beta_{\mathbf{c}}(\alpha, \mathsf{Q}_y^{d,Y} \circ \phi_z^Z(m), \mathbf{T})$ is bounded by a series of holomorphic functions on $\mathbb{C}^d \times \mathbb{C}/\mathbb{Z}$ and it converges uniformly on compact subsets of domain $D_{R,T}$. □

4.8 Measure estimation for bounded-type

In this section, we prove measure estimation of Bufetov functional under bounded-type case (4.13). This result is generalization of §11 of [FK17].

Let $\mathcal{O}_r$ denote thee space of holomorphic functions on the ball $B_{\mathbb{C}}(0,r) \subset \mathbb{C}^n$. We recall the *Chebyshev degree*, the best constant $d_f(r)$ stated in the following theorem and estimation of *valency*.

Theorem 4.42. *[Bru99, Thm 1.9] For any $f \in \mathcal{O}_r$, there is a constant $d := d_f(r) > 0$ such that for any convex set $D \subset B_{\mathbb{R}}(0,1) := B_{\mathbb{C}}(0,1) \cap \mathbb{R}^n$, for any measurable subset $U \subset D$*

$$\sup_D |f| \leq \left(\frac{4nLeb(D)}{Leb(U)}\right)^d \sup_U |f|.$$

Let $\mathcal{L}_t$ denote the set of one-dimensional complex affine spaces $L \subset \mathbb{C}^n$ such that $L \cap B_{\mathbb{C}}(0,t) \neq \emptyset$.

Definition 4.43. *[Bru99, Def 1.6] Let $f \in \mathcal{O}_r$. The number*

$$\nu_f(t) := \sup\{\textit{valency of } f \mid L \cap B_{\mathbb{C}}(0,t) \neq \emptyset\}$$

is called the valency of f in $B_{\mathbb{C}}(0,t)$.

By [Bru99, Prop 1.7], for any $f \in \mathcal{O}_r$, and the valency $\nu_f(t)$ is finite and any $t \in [1,r)$ there is a constant $c := c(r) > 0$ such that

$$d_f(r) \leq c\nu_f(\frac{1+r}{2}). \tag{4.60}$$

Lemma 4.44. *Let $L > 0$ and $\mathscr{B} \subset DC(L)$ be a bounded subset. Given $R > 0$, for all $c \in \Omega_R$ and all $\boldsymbol{T}^{(i)} > 0$, let $\mathcal{F}(c, \boldsymbol{T})$ denote the family of real analytic functions of the variable $y \in [0,1)^d$ defined as*

$$\mathcal{F}(c, \boldsymbol{T}) := \{\beta_{\boldsymbol{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \boldsymbol{T}) \mid (\alpha, x, z) \in \mathscr{B} \times M \times \mathbb{T}\}.$$

There exists $\boldsymbol{T}_{\mathscr{B}} := (\boldsymbol{T}_{\mathscr{B}}^{(i)})$ and $\rho_{\mathscr{B}} > 0$, such that for every $(R, \boldsymbol{T})$ such that $R/\boldsymbol{T}^{(i)} \geq \rho_{\mathscr{B}}$ and $\boldsymbol{T}^{(i)} \geq \boldsymbol{T}_{\mathscr{B}}^{(i)}$, and for all $\boldsymbol{c} \in \Omega_R \backslash \{0\}$, we have

$$\sup_{f \in \mathcal{F}(c,T)} \nu_f < \infty.$$

Proof. Since $\mathscr{B} \subset \mathfrak{M}$ is bounded, for each time $t_i \in \mathbb{R}$ and $1 \leq i \leq g$,

$$0 < t_{i,\mathscr{B}}^{min} = \min_i \inf_{\alpha \in \mathscr{B}} t_{Ret,i,\alpha} \leq \max_i \sup_{\alpha \in \mathscr{B}} t_{Ret,i,\alpha} = t_{\mathscr{B}}^{max} < \infty.$$

For any $\alpha \in \mathscr{B}$ and $x \in M$, the map $\Phi_{\alpha,x} : [0,1)^d \times \mathbb{T} \to \prod_{i=1}^d [0, t_{\alpha,i}) \times \mathbb{T}$ in (4.45) extends to a complex analytic diffeomorphism $\hat{\Phi}_{\alpha,x} : \mathbb{C}^d \times \mathbb{C}/\mathbb{Z} \to \mathbb{C}^d \times \mathbb{C}/\mathbb{Z}$. By Lemma 4.41, it follows that for fixed $z \in \mathbb{T}$, real analytic function $\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \boldsymbol{T})$ extends to a holomorphic function defined on a region

$$H_{\alpha,m,R,t} := \{y \in \mathbb{C}^d \mid \sum_{i=1}^d |Im(y_i)| \leq h_{\alpha,m,R,t}\}.$$

By boundedness of the set $\mathscr{B} \subset \mathfrak{M}$, it follows that

$$\inf_{(\alpha,x) \in \mathscr{B} \times M} h_{\alpha,m,R,t} := h_{R,T} > 0.$$

We remark that the function $h_{\alpha,m,R,t}$ and its lower bound $h_{R,T}$ can be obtained from the formula (4.45) for the polynomial $\Phi_{\alpha,x}$ and the definition of the domain $D_{R,T}$ in formula (4.59).

For every $r > 1$, there exists $\rho_{\mathscr{B}} > 1$ such that, for every R and $\mathbf{T}$ with $R/\mathbf{T}^{(i)} > \rho_{\mathscr{B}}$, then as a function of $y \in \mathbb{T}^d$

$$\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \mathbf{T}) \in \mathcal{O}_r.$$

Then, by Lemma 4.41, the family $\mathcal{F}(c, \mathbf{T})$ is uniformly bounded and normal.

By Lemma 4.35, for sufficiently large pair $\mathbf{T}$, no sequence from $\mathcal{F}(c, \mathbf{T})$ can converge to a constant. By Lemma 10.3 of [FK17], for any normal family $\mathcal{F} \subset \mathcal{O}_R$ such that no functions is constant along a one-dimensional complex line, hence the statement follows. □

We derive measure estimates of Bufetov functionals on the rectangular domain.

Lemma 4.45. *Let $\alpha \in DC$ such that the forward orbit of $\mathbb{R}^d$-action $\{r_{\boldsymbol{t}}[\alpha]\}_{\boldsymbol{t}\in\mathbb{R}^d_+}$ is contained in a compact set of M. There exist $R, C, \delta > 0$ and $T_0 \in \mathbb{R}^d_+$ such that, for every $\boldsymbol{c} \in \Omega_R \backslash \{0\}$, $T \geq T_0$ and for every $\epsilon > 0$, we have*

$$vol(\{m \in M \mid |\beta_{\boldsymbol{c}}(\alpha, m, \boldsymbol{T})| \leq \epsilon vol(U(\boldsymbol{T}))^{1/2}\}) \leq C\epsilon^{\delta}.$$

Proof. Since $\alpha \in DC$ and the orbit $\{r_{\boldsymbol{t}}[\alpha]\}_{\boldsymbol{t}\in\mathbb{R}_+}$ is contained in a compact set, there exists $L > 0$ such that $r_{\mathbf{t}}(\alpha) \in DC(L)$ for all $\boldsymbol{t} \in \mathbb{R}^d_+$. Then, we choose $\mathbf{T}_0 \in \mathbb{R}^d$ from conclusion of Lemma 4.44. By scaling property of Bufetov functionals,

$$\beta_{\mathbf{c}}(\alpha, m, \boldsymbol{T}) = \left(\frac{vol(U(\boldsymbol{T}))}{vol(U(\boldsymbol{T_0}))}\right)^{1/2} \beta_{\mathbf{c}}(g_{\log(\boldsymbol{T}/\boldsymbol{T_0})}[\alpha], m, \boldsymbol{T}_0).$$

By Fubini's theorem, it suffices to estimate

$$Leb(\{y \in [0,1]^d \mid |\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \boldsymbol{T}_0)| \leq \epsilon\}).$$

Let $\delta^{-1} = c(r)\sup_{f\in\mathcal{F}(c,T_0)} v_f(\frac{1+r}{2}) < \infty$. Since by Lemma 4.35, we have

$$\inf_{(\alpha,x,z)\in\mathscr{B}\times M\times\mathbb{T}} \sup_{y\in[0,1]^d} |\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \boldsymbol{T}_0)| > 0.$$

By theorem 4.42, for unit ball $D = B_{\mathbb{R}}(0,1)$ and $U = \{y \in [0,1]^d \mid |\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \boldsymbol{T}_0)| \leq \epsilon\}$ and bound in formula (4.60), there exists a constant $C > 0$ and $\delta > 0$ such that for all $\epsilon > 0$ and $(\alpha, x, z) \in \mathscr{B} \times M \times \mathbb{T}$,

$$Leb(\{y \in [0,1]^d \mid |\beta_{\mathbf{c}}(\alpha, \Phi_{\alpha,x}(\xi_{y,z}), \boldsymbol{T}_0)| \leq \epsilon\}) \leq C\epsilon^{\delta}.$$

Then statement follows from Fubini theorem. □

Corollary 4.46. *Let α be as in the previous Lemma 4.45. There exist $R, C, \delta > 0$ and $T_0 \in \mathbb{R}^d_+$ such that, for every $\boldsymbol{c} \in \Omega_R \backslash \{0\}$, $T \geq T_0$ and for every $\epsilon > 0$, we have*

$$vol(\{x \in M \mid \left|\langle P^{d,\alpha}_{U(\boldsymbol{T})} m, \omega_{\boldsymbol{c}}\rangle\right| \leq \epsilon vol(U(\boldsymbol{T}))^{1/2}\}) \leq C\epsilon^{\delta}.$$

Remeciement

If you are lucky enough to have lived in Paris as a young man, then wherever you go for the rest of your life, it stays with you, for Paris is a moveable feast.

– Ernest Hemingway, A Moveable Feast.

Quand je suis venu à Paris le premier fois de ma vie, j'ai rêvé que je pourrai habiter ici un jour. Paris étais très belle, romantique, animée. Chaque moment est plein de joie et la vie de Paris semblait belle et rose à un jeune homme. Étonnamment, ce rêve est devenu réalité subitement. J'ai visité l'IMJ-PRG en doctorant pendant l'annee 2017 - 2018 et 2019 printemps. Heureusement, il y a beaucoup de moments magnifiques et gentilles personnes qui m'ont aidé et encouragé. La vie à paris a définitivement changé mon point de vue sur la vie et m'a appris la leçon: *Soyez heureux et profitez de ce que j'ai donné. C'est la vie.* Je pourrai trouver des millions de raisons de revenir à Paris a l'avenir.

Avant tout, je suis honoré et reconnaissant envers mon directeur *Giovanni Forni* pour me pourvoir de moments précieux. Je suis très reconnaissant envers Anton Zorich et Carlos Matheus pour ses chaleureuses salutations et support pendant tout cette période.

L'un des meilleurs moments de ma vie à Paris a été de participer un cour en *École Normale Supérieure.* J'ai pris RER B à la gare de luxembourg et j'aimais marcher à travers les petites ruelles pour les écoles dans la fraîcheur tous les matins. Ensuite, j'arrive en classe et m'assieds en attendant que la classe commence. C'était un moment paisible et heureux, un jour de mars. Sûrement, je remecie Anna Erschler de m'avoir appris la marche aléatoire et mon camarade de classe intelligent

Emmanuel Rauzy. Merci à Antoine Julia, Juan-Pablo et Yi Pan pour me aider cours en ENS aussi.

Remontons avant ma vie en *Sophie Germain, Paris 7.* Quand j'atterrissais on Paris, je ne connaissais rien de parler français. Il faisait très froid, pauvre et seul et il a fallu du temps pour que le printemp arrive. Je suis redevable à mes amis, sans qui je n'aurais jamais résolu petits et grands problèms en ma nouvelle vie.

C'étais mon plasir de recontrer gens tous les jours et bavarder ensemble. Je tiens à remecier particulierement les amis du **bureau 652**. (Élie Goudout, Kevin Massard, Emmanuel Rauzy). Anna Rosine et Rodolfo David ont partagé le même bureau à court terme et on a plasir ensemble. Vous me manquez. J'étais très heureux de participer soutenance de E.L.I.E en Avril et content de recontrer la généruse famille de **Dr. Goûter**.

Je remecie memberes de séminar etudiant systèmes dynamiques: Rodolfo Gutiérrez, Yi Pan, Hao Wu, Frank Trujillo, Davi Obata, et Mauricio Poletti. Pas seulement math, ils m'aident beaucoup de discutter. C'est aussi amusant et agréable de avoir dejeuné et gouté avec mes amies à Sophie Germain: Wille, Omar, R.E.D.A (SF3 garçon), Corentin, Sacha (Roi des clés), Charles, Jérémie (Piano garçon), Romain, Colin, Jerôme (Roi de la bibliothèque), Théophile, Pierre (Echecs garçons), Alex (Chanteur), Marie, Mario, Léa, Nicolas, Andreas et Amandine pour tout leur compréhension et gentillnesse. Merci à Poone, Paula, Macarena, Siarhei, Irene, Yanbo et Nhung Nguyen aussi.

Tous les samedis, on jouait au basket: Hao, Ivan, Ratko, Grégoire (mon honour IV), Jieao, Rui et Rahman pour joue ensemble. Par la concurrence et la coopération, nous pourrions être améliorés. Nous sommes une bonne équipe!! C'était aussi un grand moment de profiter de la victoire de la France en coupe du monde 2018 avec Anthony Sanchez. J'apprécie son esprit positif et m'a amené au festival. (Allez les bleus!) J'espere que nous pourrons nous recontre encore quelque part.

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Enfin et sourtout, je remecie profondément ma famille de leur soutien constant. Ils étaient vraiment heureux pour me revenir a Paris autant que j'étais. Je remecie Elie, Kevin, et Grégoire encore pour corriger le brouillon du document.

Tous les matins et tard le soir, je vois des gens courir sur le parc pendant les mois d'hiver. Sur le chemin de l'école, je vois des gens lire les livres en foule. N'importe jeunes, vieux, hommes ou femmes, ils cherchent passionnément le but de leur vie.

Petites ruelles brunes, parc avec arbres verts, les lumières de la nuit noire de Paris vont me manquer pendant un moment.

Paris est une fête.

Juin 2019

À Paris.

Mon enfant, ma sœur,
Songe à la douceur
D'aller là-bas vivre ensemble !
Aimer à loisir,
Aimer et mourir
Au pays qui te ressemble !
Les soleils mouillés
De ces ciels brouillés
Pour mon esprit ont les charmes
Si mystérieux
De tes traîtrès yeux,
Brillant à travers leurs larmes.

Là, tout n'est qu'ordre et beauté,
Luxe, calme et volupté.

Des meubles luisants,
Polis par les ans,
Décoreraient notre chambre ;
Les plus rares fleurs
Mêlant leurs odeurs
Aux vagues senteurs de l'ambre,
Les riches plafonds,
Les miroirs profonds,
La splendeur orientale,
Tout y parlerait
À l'âme en secret
Sa douce langue natale.

Là, tout n'est qu'ordre et beauté,
Luxe, calme et volupté.

Vois sur ces canaux
Dormir ces vaisseaux
Dont l'humeur est vagabonde ;
C'est pour assouvir
Ton moindre désir
Qu'ils viennent du bout du monde.
– Les soleils couchants
Revêtent les champs,
Les canaux, la ville entière,
D'hyacinthe et d'or ;
Le monde s'endort
Dans une chaude lumière.
Là, tout n'est qu'ordre et beauté,
Luxe, calme et volupté.

L'Invitation au voyage, Les Fleurs du Mal (1857). Charles Baudelaire.

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