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Time periodic optimal policy for operation of a water storage tank using the dynamic programming approach

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ABSTRACT

Operation of a water storage tank in a specific environment motivates mathematical studies on a discrete-time deterministic dynamic programming problem. The operator decides whether or not to open the valve releasing the water in the tank to a drip irrigation system, based on the information on the storage volume of the tank. Two cases of functional regularity, which are Lipschitz continuous and of bounded variations, are considered for the reward defining the performance index to be maximized. Firstly, it is shown that the value function inherits the Lipschitz continuity of the reward in the infinite time horizon problem with discounting. Then, time periodic value functions are discussed in terms of the fixed-point theorem. Discrete approximation of value functions is discussed as well, to conduct numerical experiments with a-posteriori error estimation applied to the real-world problem where the discount rate approaches to unity. It is found that a Skiba point appears as a threshold of valve opening for each day in an optimal policy for operation. Practically, setting a constant threshold throughout the period is quite reasonable and acceptable for the operator of the water storage tank to irrigate the farmland.

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1. Introduction

Water storage tanks are installed for different purposes such as irrigation, domestic water supply, and rainwater harvesting. One of the common functions of water storage tanks is to ensure the adequate provision of water to the user when the flux of available water does not meet the fluctuation of demand. Although operational practices for water resources management in the real-world are dominantly based on empirical knowledge, theories of inventory optimization are applicable to rationalizing such practices as well as seeking better policies, see e.g. [7,8,14,22,23]. In dynamic contexts, formulation of an optimal control problem is the universal and rigorous mathematical approach to deduce an optimal policy based on the principle of DP, where the Bellman equation governs the value function which is the supremum or the infimum of a performance index [3].

The notion of viscosity solution is ubiquitous in mathematically dealing with DP for continuous-time decision processes on Borel spaces. Existence, uniqueness, and stability of a viscosity solution representing the value function have been intensively discussed in standard textbooks including Crandall et al. [11], Fleming and Soner [15], and Bardi and Capuzzo-Dolcetta [1]. Those results support successful application of DP to practical problems in a variety of study areas. Kesri [19] proposed an algorithm to compute the value functions in optimal control problems, which are possibly applicable to resource management in general. In water resources management, Unami and Mohawesh [25] proved unique existence of a viscosity solution to the Hamilton–Jacobi–Bellman (HJB) equation appearing in a stochastic DP problem for reservoir operation. Ieda [18] dealt with the HJB quasi-variational inequality (QVI) associated with the combined impulse and stochastic optimal control problem over a finite time horizon, applying it to an optimal forest harvesting problem. Some types of control problems for discrete-time decision processes including the one discussed here have continuous-time counterparts which can be regarded as impulse control problems [4].

Most real-world problems in water resources management are stochastic in nature, and references to deterministic cases have been relatively few, as mentioned in Yakowitz [28]. Nevertheless, researches on operation of deterministic nonlinear systems are acquiring importance for comprehending complexity and singularity of phenomena. Crandall and Lions [10] established a general approach to the Hamilton-Jacobi (HJ) equations of first order appearing in deterministic DP problems. Grüne et al. [16] studied deterministic decision making problems for ecological management in a polluted shallow lake. For the same dynamical system to be controlled, Wagener [27] focused on indifference or Skiba points, for which there are two distinct optimal controls with equal value function achieved. Skiba points in water resources management are typically thresholds of opening valves or switching pumps [24]. Kossioris and Zohios [20] further investigated the shallow lake problem in the context of viscosity solution as well. Grüne and Semmler [17] developed computational methods for HJ equations involving Skiba points. Under some assumptions imposed on the system dynamics and the performance index, a unique bounded uniformly continuous viscosity solution exists for the HJ QVI [2]. El Farouq [12] worked on discrete approximations of such HJ QVIs. El Farouq [13] innovated HJ QVIs for degenerate problems. However, it seems that discontinuous solutions to HJ QVIs are least explored in the literature, although HJ equations have been well linked to first order quasilinear equations involving generalized solutions in the space of functions of bounded variations [9,21]. Bokanowski et al. [5] dealt with piecewise constant viscosity solutions to HJ equations with discontinuous initial data. Bokanowski et al. [6] presented high-order approximation schemes for solutions to stationary HJ equations in the space of bounded functions.

This paper discusses a discrete-time deterministic DP problem with a state variable defined on a closed interval in $\mathbb{R}$, being applied to time periodic optimal policy for operation of a water storage tank. The state variable represents the storage volume of the tank, while the control variable represents the operator's daily decision whether or not to open the valve releasing the water in the tank to a drip irrigation system. A discount rate is included in the performance index to obtain a time periodic value function solving the Bellman equation as well as the optimal policy. Weak assumptions imposed on the performance index implies generalized solution to the relevant Bellman equation.

The rest of the paper is structured as follows. The mathematical problem is stated in Section 2. In Section 3, several theorems are proven to characterize the solutions to the Bellman equation. Section 4 is devoted to the application and its results. Concluding remarks are provided in Section 5.

2. Mathematical statement of the problem

A discrete-time deterministic decision process with the time increment of 1 day is considered to mathematically model operation of the water storage tank. The storage volume X_t at the beginning of a given day t is the state variable ranging in a compact set K of non-negative real numbers

$$K = [0, V] \tag{1}$$

where V is the maximum storage volume of the tank. The decision variable u indicating the on/off state of the valve to release water from the tank to the water user is constrained in the set $U = \{0,1\}$ of admissible controls.

2.1 Mathematical preliminaries

The set of bounded functions defined on K is denoted by $\mathcal{B}(K)$, which is a Banach space equipped with the uniform norm

$$\|v\|_\infty = \sup_{x \in K} |v(x)| \tag{2}$$

for $v = v(x) \in \mathcal{B}(K)$. The set of continuous functions defined on K is denoted by $C(K)$, which is a subspace of $\mathcal{B}(K)$. The L^1 norm of a function $v = v(x) \in \mathcal{B}(K)$ is defined by

$$\|v\|_{L^1} = \int_0^V |v(x)| \mathrm{d}x . \tag{3}$$

The total variation $\mathrm{TV}(v)$ of $v \in \mathcal{B}(K)$ is defined by

$$\mathrm{TV}(v) = \sup_{P \in \mathcal{P}} \sum_{i=0}^{i<n_P} |v(x_{i+1}) - v(x_i)| \tag{4}$$

where $\mathcal{P}$ is the set of partitions

$$\mathcal{P} = \left\{ P = \left\{ x_0, \cdots, x_{n_P} \right\} \middle| x_i \in K, x_i \le x_{i+1}, \text{ for } 0 \le i < n_P \right\} . \tag{5}$$

A function $v \in \mathcal{B}(K)$ is said to be of bounded variation on K if and only if $\mathrm{TV}(v)$ is finite. The set of functions of bounded variations defined on K is denoted by $\mathrm{BV}(K)$, which is a Banach space equipped with the norm

$$\|v\|_{\mathrm{BV}} = \|v\|_{L^1} + \mathrm{TV}(v) \tag{6}$$

for $v = v(x) \in \mathrm{BV}(K)$. For a Lipschitz continuous $v \in \mathcal{B}(K)$,

$$\mathrm{TV}(v) \le |K| \, \mathrm{Lip}(v) = V \, \mathrm{Lip}(v) \tag{7}$$

where $|K|$ is the measure of *K*, and $\mathrm{Lip}(v)$ represents the Lipschitz constant of *v*.

2.2 Dynamics of water balance in the tank

The water balance in the tank is governed by

$$\begin{cases} X_{t^+} = \min\left(X_t + S(t), V\right) \\ X_{t+1} = \max\left(X_{t^+} - uR\left(X_{t^+}\right), 0\right) \end{cases} \tag{8}$$

where V is the maximum storage volume of the tank, $S(t)$ is the volume of water supplied to the tank on the day t and is time periodic, X_{t^+} is the storage volume of the tank immediately before making decision on opening the valve on the day t, and $R(x)$ is the volume of water released for the water user as a result of opening the valve when the storage volume of the tank is *x*. The transition between two states is deterministic and summarized as

$$\xi(t,x,k) = X_{t+1}, \text{ provided that } X_t = x \text{ and } u = u_k. \tag{9}$$

We impose the following assumptions on the functional properties of this $\xi(t,x,k)$.

Assumption 1. $\xi(t,x,k)$ is non-decreasing with respect to *x*, that is,

$$\xi(t,x,k) \le \xi(t,y,k), \text{ if } x \le y. \tag{10}$$

Assumption 2. $\xi(t,x,k)$ as a function of *x* is Lipschitz continuous, that is, there exists a Lipschitz constant $L^{\xi}_{t,k} = \mathrm{Lip}(\xi(t,x,k)) \ge 0$ for each *t* and *k* such that

$$\left|\xi(t,x,k) - \xi(t,y,k)\right| \le L^{\xi}_{t,k} |x-y|, \quad \forall x,y \in K. \tag{11}$$

Assumption 3. Furthermore, the Lipschitz constants $L^{\xi}_{t,k}$ are bounded by 1, that is,

$$L^{\xi} = \max_{t,k} L^{\xi}_{t,k} \le 1. \tag{12}$$

2.3 Formulation of DP problem

Decision making is dominated by the reward $f(t,x,k)$ received when the decision u_k is chosen for the state x. A policy π is a map $k=\pi(t,x)$ of those choices. The total reward received during the period from the current time $t=s<0$ until the terminal time $t=0$ with a positive discount rate $\delta<1$ for $X_s=x$ is the performance index denoted by

$$\Phi^{\pi}(s,x)=\sum_{t=s}^{t<0}\delta^{t-s}f\left(t,X_t,\pi\left(t,X_t\right)\right). \tag{13}$$

A policy π^* is said to be optimal if

$$\Phi^{\pi^*}(s,x)=\max_{\pi}\Phi^{\pi}(s,x) \tag{14}$$

for any time s and any state x. From the principle of DP, the Bellman equation

$$\Phi^{\pi^*}(s,x)=\max_{k}\left\{f(s,x,k)+\delta\Phi^{\pi^*}\left(s+1,\xi(s,x,k)\right)\right\} \tag{15}$$

is obtained and solved with terminal condition

$$\Phi^{\pi^*}(0,x)=0,\quad \forall x\in\Omega, \tag{16}$$

to determine the value function $\Phi^{\pi^*}(s,x)$, which henceforth shall be denoted by $\Phi^s=\Phi^s(x)$ as a function defined on *K* for each *s*. The optimal policy π^* may not be unique. It is assumed that the reward is bounded as $f(t,x,k)\le f^{\max}$ for any *t*, *x*, and *k*, and then the value function is obviously evaluated as

$$0\le\Phi^s\le\sum_{t=s}^{t<0}\delta^{t-s}f^{\max}=\frac{1-\delta^{-s}}{1-\delta}f^{\max}<\frac{1}{1-\delta}f^{\max}. \tag{17}$$

Regarding the functional regularity of $f(t,x,k)$, however, two cases are considered.

Case 1. $f(t,x,k)$ as a function of *x* is Lipschitz continuous on *K* with a Lipschitz constant $L^f_{t,k}=\mathrm{Lip}\left(f(t,x,k)\right)\ge 0$ for each *t* and *k*. The maximum of $L^f_{t,k}$ is denoted by L^f.

Case 2. $f(t,x,k)$ as a function of *x* is of bounded variations on *K* for each *t* and *k*.

3. Solutions of the Bellman equation

For each day *t*, the Bellman equation (15) prescribes a mapping $v^t = \varphi^t(v^{t+1})$ from $\mathcal{B}(K)$ to $\mathcal{B}(K)$ as

$$v^t(x) = \max_k \left\{ f(t,x,k) + \delta v^{t+1}\left(\xi(t,x,k)\right) \right\} \tag{18}$$

where $v^t(x)$ represent the value of v^t at $x \in K$. For each *k*, subsets $K_k(v^t)$ of the set K are defined as

$$K_k(v^t) = \left\{ x \middle| v^t(x) = f(t,x,k) + \delta v^{t+1}\left(\xi(t,x,k)\right) \right\}. \tag{19}$$

3.1 Properties of the mapping φ^t

Several properties of the mapping φ^t are stated as follows.

Theorem 1. *In Case 1, if* v^0 *is Lipschitz continuous, then* v^t *are also Lipschitz continuous for all* $t < 0$.

Proof. Assume that there exists $L^s = \mathrm{Lip}(v^s) \geq 0$ for all $s > t$, such that

$$\left|v^s(x) - v^s(y)\right| \leq L^s \left|x - y\right|, \quad \forall x, y \in K. \tag{20}$$

Without loss of generality, $v^t(x) \geq v^t(y)$ is assumed. Let $v^t(x) = f(t,x,j) + \delta v^{t+1}\left(\xi(t,x,j)\right)$. Then,

$$\begin{aligned}\left|v^t(x)-v^t(y)\right| &\le v^t(x)-v^t(y) \le v^t(x)-\left(f(t,y,j)+\delta v^{t+1}\left(\xi(t,y,j)\right)\right)\\ &= f(t,x,j)-f(t,y,j)+\delta\left(v^{t+1}\left(\xi(t,x,j)\right)-v^{t+1}\left(\xi(t,y,j)\right)\right)\\ &\le \left|f(t,x,j)-f(t,y,j)\right|+\delta\left|v^{t+1}\left(\xi(t,x,j)\right)-v^{t+1}\left(\xi(t,y,j)\right)\right|\\ &\le L^f_{t,j}\left|x-y\right|+\delta L^{t+1}\left|\xi(t,x,j)-\xi(t,y,j)\right|\\ &\le L^f_{t,j}\left|x-y\right|+\delta L^{t+1}L^{\xi}_{t,j}\left|x-y\right|\\ &=\left(L^f_{t,j}+\delta L^{t+1}L^{\xi}_{t,j}\right)\left|x-y\right|\end{aligned}. \tag{21}$$

Therefore, v^t is Lipschitz continuous with a Lipschitz constant $L^t = \mathrm{Lip}\left(v^t\right) \ge 0$ evaluated as

$$L^t \le L^f \sum_{s=t}^{s<0}\left(\delta L^{\xi}\right)^{-t+s}+\left(\delta L^{\xi}\right)^{-t} L^0 \le \frac{L^f}{1-\delta L^{\xi}}+\left(\delta L^{\xi}\right)^{-t} L^0 \le \frac{L^f}{1-\delta L^{\xi}}+L^0 . \tag{22}$$

□

In Case 2, however, $M^t = \mathrm{TV}\left(v^t\right)$ for all $t<0$ are not uniformly bounded in general. We consider a special case, which appears in the application described later, such that

$$\left[0,\theta^t\right]\subset K_0\left(v^t\right),\quad \left[\theta^t,V\right]\subset K_1\left(v^t\right) \tag{23}$$

where θ^t is a Skiba point at each *t*. Let $\left\{x_i^-,x_i^+\right\}=\left\{x_i,x_{i+1}\right\}$ such that

$$v^t\left(x_i^-\right)\le v^t\left(x_i^+\right)= f\left(t,x_i^+,j\right)+\delta v^{t+1}\left(\xi\left(t,x_i^+,j\right)\right) \tag{24}$$

for each i of each P. Then,

$$\begin{aligned}&\left|v^t\left(x_{i+1}\right)-v^t\left(x_i\right)\right|\\ &= v^t\left(x_i^+\right)-v^t\left(x_i^-\right)\\ &\le v^t\left(x_i^+\right)-\left(f\left(t,x_i^-,j\right)+\delta v^{t+1}\left(\xi\left(t,x_i^-,j\right)\right)\right)\\ &= f\left(t,x_i^+,j\right)-f\left(t,x_i^-,j\right)+\delta\left(v^{t+1}\left(\xi\left(t,x_i^+,j\right)\right)-v^{t+1}\left(\xi\left(t,x_i^-,j\right)\right)\right)\\ &\le\left|f\left(t,x_i^+,j\right)-f\left(t,x_i^-,j\right)\right|+\delta\left|v^{t+1}\left(\xi_i^+\right)-v^{t+1}\left(\xi_i^-\right)\right|\end{aligned} \tag{25}$$

where $\xi_i^{\pm}=\xi\left(t,x_i^{\pm},j\right)$, from which it is possible to constitute two other partitions $\left\{\xi_0^0,\cdots,\xi_{n_P^S}^0\right\}$ and $\left\{\xi_{n_P^S}^1,\cdots,\xi_{n_P}^1\right\}$ with a natural number n_P^S such that $\left\{\xi_i^j,\xi_{i+1}^j\right\}=\left\{\xi_i^+,\xi_i^-\right\}$,

$\xi_0^0 \le \cdots \le \xi_{n_P^S}^0$, and $\xi_{n_P^S}^1 \le \cdots \le \xi_{n_P}^1$, because Assumption 1 guarantees conservation of the orders of x_i and ξ_i^j as far as j is unchanged. Then, $\mathrm{TV}\left(v^t\right)$ is evaluated as

$$\begin{aligned}
&\sup_{P\in\mathcal{P}} \sum_{i=0}^{i<n_P} \left| v^t\left(x_{i+1}\right) - v^t\left(x_i\right) \right| \\
&\le \sup_{P\in\mathcal{P}} \sum_{i=0}^{i<n_P} \left(\left| f\left(t, x_i^+, j\right) - f\left(t, x_i^-, j\right) \right| + \delta \left| v^{t+1}\left(\xi_i^+\right) - v^{t+1}\left(\xi_i^-\right) \right| \right) \\
&\le \sup_{P\in\mathcal{P}} \left(\sum_{i=0}^{i<n_P^S} \left| f\left(t, x_i^+, 0\right) - f\left(t, x_i^-, 0\right) \right| + \sum_{i=n_P^S}^{i<n_P} \left| f\left(t, x_i^+, 1\right) - f\left(t, x_i^-, 1\right) \right| \right). \\
&+ \delta \sup_{P\in\mathcal{P}} \left(\sum_{i=0}^{i<n_P^S} \left| v^{t+1}\left(\xi_i^+\right) - v^{t+1}\left(\xi_i^-\right) \right| + \sum_{i=n_P^S}^{i<n_P} \left| v^{t+1}\left(\xi_i^+\right) - v^{t+1}\left(\xi_i^-\right) \right| \right) \\
&\le \mathrm{TV}\left(f\left(t, x, 0\right)\right) + \mathrm{TV}\left(f\left(t, x, 1\right)\right) + \delta C_r^{t+1} M^{t+1}
\end{aligned} \tag{26}$$

where C_r^s for $s > t$ is a non-negative constant not greater than 2. Provided that all C_r^t are bounded by a constant $C_r^{\max} \le 1$, all M^t are bounded as

$$\begin{aligned}
M^t &\le W^f \sum_{s=t}^{s<0} \left(\delta C_r^{\max}\right)^{-t+s} + \left(\delta C_r^{\max}\right)^{-t} M^0 \\
&\le \frac{W^f}{1 - \delta C_r^{\max}} + \left(\delta C_r^{\max}\right)^{-t} M^0 \\
&\le \frac{W^f}{1 - \delta C_r^{\max}} + M^0
\end{aligned} \tag{27}$$

where $W^f = \max_t \left(\mathrm{TV}\left(f\left(t, x, 0\right)\right) + \mathrm{TV}\left(f\left(t, x, 1\right)\right)\right)$.

Contraction is a common property of φ^t both for Case 1 and Case 2.

Theorem 2. φ^t *is a contraction mapping with respect to the uniform norm.*

Proof. Define two sequences $\left\{v_0^t\right\}$ and $\left\{v_1^t\right\}$ for $t \le 0$ in $\mathcal{B}\left(K\right)$ using $v_j^t = \varphi^t\left(v_j^{t+1}\right)$ with any $v_j^0 \in \mathcal{B}\left(K\right)$ for $j \in \left\{0,1\right\}$. For each x, $j \in \left\{0,1\right\}$ and $k_j^t \in U$ are chosen so that

$$v_j^t(x) = f\left(t, x, k_j^t\right) + \delta v_j^{t+1}\left(\xi\left(t, x, k_j^t\right)\right) \geq v_\mu^t(x) \tag{28}$$

where $\mu = \mathrm{mod}(j+1, 2)$ and let

$$w_\mu^t(x) = f\left(t, x, k_j^t\right) + \delta v_\mu^{t+1}\left(\xi\left(t, x, k_j^t\right)\right). \tag{29}$$

This leads to the evaluation

$$\begin{aligned} \left|v_0^t(x) - v_1^t(x)\right| &= v_j^t(x) - v_\mu^t(x) \\ &\leq v_j^t(x) - w_\mu^t(x) \\ &= \delta\left(v_j^{t+1}\left(\xi\left(t, x, k_j^t\right)\right) - v_\mu^{t+1}\left(\xi\left(t, x, k_j^t\right)\right)\right) \\ &\leq \delta\left|v_j^{t+1}\left(\xi\left(t, x, k_j^t\right)\right) - v_\mu^{t+1}\left(\xi\left(t, x, k_j^t\right)\right)\right| \\ &\leq \delta\left\|v_0^{t+1} - v_1^{t+1}\right\|_\infty \end{aligned} \tag{30}$$

and then

$$\left\|v_0^t - v_1^t\right\|_\infty = \left\|\varphi^t\left(v_0^{t+1}\right) - \varphi^t\left(v_1^{t+1}\right)\right\|_\infty \leq \delta\left\|v_0^{t+1} - v_1^{t+1}\right\|_\infty \tag{31}$$

implying that φ^t is a contraction mapping with respect to the uniform norm. □

Remark 1. When $S(t)$ and consequently φ^t are P-periodic, the fixed-point theorem is applicable to the stationary contraction mappings $\psi^{s,P} = \prod_{t=s}^{t<s+P} \varphi^t$ because

$$\begin{aligned} \left\|v_0^s - v_1^s\right\|_\infty &= \left\|\psi^{s,P}\left(v_0^{s+P}\right) - \psi^{s,P}\left(v_1^{s+P}\right)\right\|_\infty = \left\|\prod_{t=s}^{t<s+P} \varphi^t\left(v_0^{s+P}\right) - \prod_{t=s}^{t<s+P} \varphi^t\left(v_1^{s+P}\right)\right\|_\infty \\ &\leq \delta^P\left\|v_0^{s+P} - v_1^{s+P}\right\|_\infty \end{aligned} \tag{32}$$

as a result of Theorem 2. This guarantees unique existence of a P-periodic function $\Psi^s = \Psi^s(x)$ as the fix point

$$\Psi^s = \Psi^{s-P} = \psi^{s,P}\left(\Psi^s\right) \tag{33}$$

for all $s \leq 0$ in $\mathcal{B}(K)$.

In analogy to the discussion on functional regularity of v^t, φ^t becomes a contraction mapping with respect to the L^1 norm when $f(t,x,k)$ and $\xi(t,x,k)$ are well behaved. Indeed, the same assumption as in (23) with another non-negative constant C_c^s, which may not be finite, leads to the evaluation

$$\begin{aligned}
\left\|\varphi^t\left(v_0^{t+1}\right)-\varphi^t\left(v_1^{t+1}\right)\right\|_{L^1} &= \left\|v_0^t - v_1^t\right\|_{L^1} = \int_0^V \left|v_0^t(x)-v_1^t(x)\right|\mathrm{d}x = \int_0^V \left(v_j^t(x)-v_\mu^t(x)\right)\mathrm{d}x \\
&\leq \int_0^V \left(v_j^t(x)-w_\mu^t(x)\right)\mathrm{d}x \\
&= \delta\int_0^V \left(v_j^{t+1}\left(\xi\left(t,x,k_j^t\right)\right)-v_\mu^{t+1}\left(\xi\left(t,x,k_j^t\right)\right)\right)\mathrm{d}x \\
&\leq \delta\int_0^{\theta^t} \left|v_0^{t+1}\left(\xi(t,x,0)\right)-v_1^{t+1}\left(\xi(t,x,0)\right)\right|\mathrm{d}x \\
&\quad +\delta\int_{\theta^t}^V \left|v_0^{t+1}\left(\xi(t,x,1)\right)-v_1^{t+1}\left(\xi(t,x,1)\right)\right|\mathrm{d}x \\
&\leq \delta C_c^{t+1}\left\|v_0^{t+1}-v_1^{t+1}\right\|_{L^1}
\end{aligned} \tag{34}$$

where the same notations as in the proof of Theorem 2 are used.

3.2 Discrete approximation

An approximation method is presented in order to construct the P-periodic value function. We opt for the piecewise linear approximation $v_n^t \in C(K)$ of any $v^t \in \mathcal{B}(K)$ as

$$v_n^t(x) = \frac{(i+1)\Delta x - x}{\Delta x}v^t(i\Delta x) + \frac{x - i\Delta x}{\Delta x}v^t((i+1)\Delta x), \quad i\Delta x \leq x \leq (i+1)\Delta x \tag{35}$$

where the set K is discretized with a constant step $\Delta x = V/n$. Firstly, we assume Case 1 that $f(t,x,k)$ is Lipschitz continuous. Let ι_n denote the mapping $v_n^t = \iota_n(v^t)$ from $\mathcal{B}(K)$ to $C(K)$ for each n, and abbreviation $\varphi_n^t = \varphi^t \circ \iota_n$ is introduced. It is easy to show that $\left\|\iota_n(v)-\iota_n(w)\right\|_\infty \leq \|v-w\|_\infty$ for any $v,w \in \mathcal{B}(K)$, and therefore φ_n^t is a contraction mapping because of Theorem 2. There is error estimation

$$\left\|v-\iota_n(v)\right\|_\infty \leq \frac{\Delta x}{2}\mathrm{Lip}(v) \tag{36}$$

for Lipschitz continuous $v \in \mathcal{B}(K)$, as well as

$$\left\| v - \iota_n(v) \right\|_{L^1} \leq \frac{\Delta x}{2} \mathrm{TV}(v) \tag{37}$$

for $v \in \mathrm{BV}(K)$. The stationary mapping $\psi_n^{s,P} = \prod_{t=s}^{t<s+P} \varphi_n^t$ is contractive as well. Now, an approximation of a value function $v^s = \prod_{t=s}^{t<0} \varphi^t(v^0)$ for any $s<0$ is recursively calculated as $\overline{v}_n^s = \prod_{t=s}^{t<0} \varphi_n^t(v^0)$. With an analogous discussion in the proof of Theorem 1, $\overline{v}_n^s$ turns out to be Lipschitz continuous with the same Lipschitz constant $L^s \leq \frac{L^f}{1-\delta L^\xi} + L^0$, which does not depend on *n*. According to the Ascoli-Arzelà theorem, there exists a subsequence of $\{\overline{v}_n^s\}$ that converges uniformly to a Lipschitz continuous function with the same Lipschitz constant. Indeed, the discretization error is recursively estimated as

$$\begin{aligned}
\left\| v^s - \overline{v}_n^s \right\|_\infty &= \left\| v^s - \prod_{t=s}^{t<0} \varphi_n^t(v^0) \right\|_\infty \\
&\leq \delta \left\| v^{s+1} - \iota\left(\prod_{t=s+1}^{t<0} \varphi_n^t(v^0) \right) \right\|_\infty = \delta \left\| v^{s+1} - \overline{v}_n^{s+1} + \overline{v}_n^{s+1} - \iota(\overline{v}_n^{s+1}) \right\|_\infty \\
&\leq \delta \left(\left\| v^{s+1} - \overline{v}_n^{s+1} \right\|_\infty + \left\| \overline{v}_n^{s+1} - \iota(\overline{v}_n^{s+1}) \right\|_\infty \right) \\
&\leq \delta^{-1-s} \left\| v^{-1} - \overline{v}_n^{-1} \right\|_\infty + \sum_{t=s+1}^{t<0} \delta^{t-s} \left\| \overline{v}_n^t - \iota(\overline{v}_n^t) \right\|_\infty \\
&\leq \delta^{-s} \left\| v^0 - \iota_n(v^0) \right\|_\infty + \sum_{t=s+1}^{t<0} \delta^{t-s} \left\| \overline{v}_n^t - \iota(\overline{v}_n^t) \right\|_\infty
\end{aligned} \tag{38}$$

in the uniform norm. Using (36), the rightest hand of (38) is further evaluated as

$$\begin{aligned}
\delta^{-s} \left\| v^0 - \iota_n(v^0) \right\|_\infty + \sum_{t=s+1}^{t<0} \delta^{t-s} \left\| \overline{v}_n^t - \iota(\overline{v}_n^t) \right\|_\infty &\leq \frac{\Delta x}{2} \sum_{t=s}^{t<0} \delta^{t-s+1} L^{t+1} \\
&\leq \frac{\Delta x}{2} \frac{1-\delta^{-s}}{1-\delta} \delta \left(\frac{L^f}{1-\delta L^\xi} + L^0 \right) \\
&\leq \frac{\Delta x}{2(1-\delta)} \left(\frac{L^f}{1-\delta L^\xi} + L^0 \right)
\end{aligned} \tag{39}$$

resulting in uniform convergence of $\bar{v}_n^s$ to v^s as $n \to \infty$, and the order of convergence is Δx.

Let $\Phi^s = \Phi^s(x) = \prod_{t=s}^{t<0} \varphi^t(\Phi^0)$ and $\bar{\Phi}_n^s = \bar{\Phi}_n^s(x) = \prod_{t=s}^{t<0} \varphi_n^t(\Phi^0)$ with the common terminal condition $\Phi^0 = \bar{\Phi}_n^0(x) = \Phi^{\Pi^*}(0, x) = 0$ as in (16). The value function Φ^s is approximated by $\bar{\Phi}_n^s$ in the uniform norm as $n \to \infty$, because of (38) with (39). Remark 1 asserts that Φ^s and $\bar{\Phi}_n^s$ get close to the P-periodic value function Ψ^s and the fixed point of $\psi_n^{s,P}$, respectively, as $s \to -\infty$. Therefore, from

$$\left\|\Psi^s - \bar{\Phi}_n^s\right\|_\infty \le \left\|\Psi^s - \Phi^s\right\|_\infty + \left\|\Phi^s - \bar{\Phi}_n^s\right\|_\infty, \tag{40}$$

$\bar{\Phi}_n^s$ approaches to Ψ^s in the uniform norm as $s \to -\infty$ and $n \to \infty$.

Next, we attempt to weaken the functional regularity constraint of the reward $f(t, x, k)$ as in Case 2. Let $\Phi^s = \prod_{t=s}^{t<0} \varphi^t(\Phi^0)$ and $\bar{\Phi}_n^s = \prod_{t=s}^{t<0} \varphi_n^t(\Phi^0)$ for the reward $f(t, x, k)$ of bounded variations. As mentioned in Remark 1, there exists unique P-periodic value function $\Psi^s \in \mathcal{B}(K)$ for the reward $f(t, x, k)$. The evaluation

$$\left\|\Psi^s - \bar{\Phi}_n^s\right\|_{L^1} \le \left\|\Psi^s - \Phi^s\right\|_{L^1} + \left\|\Phi^s - \bar{\Phi}_n^s\right\|_{L^1} \le V\left\|\Psi^s - \Phi^s\right\|_\infty + \left\|\Phi^s - \bar{\Phi}_n^s\right\|_{L^1} \tag{41}$$

is valid if $C_r^{\max}$ and $C_c^{\max}$ are so small that $\mathrm{TV}(\Phi^s)$ is finite and φ^t is contractive. Indeed, the first term of the rightest hand side of (41) goes to zero as $s \to -\infty$ because of Remark 1, while the second term is further evaluated as

$$
\begin{aligned}
\left\|\Phi^s-\bar{\Phi}_n^s\right\|_{L^1} &\le \left(\delta C_c^{\max}\right)^{-s}\left\|\Phi^0-\iota_n\left(\Phi^0\right)\right\|_{L^1}+\sum_{t=s+1}^{t<0}\left(\delta C_c^{\max}\right)^{t-s}\left\|\bar{\Phi}_n^t-\iota\left(\bar{\Phi}_n^t\right)\right\|_{L^1} \\
&\le \frac{\Delta x}{2}\sum_{t=s}^{t<0}\left(\delta C_c^{\max}\right)^{t-s+1}\mathrm{TV}\left(\bar{\Phi}_n^t\right) \\
&\le \frac{\Delta x}{2}\frac{1-\left(\delta C_c^{\max}\right)^{-s}}{1-\delta C_c^{\max}}\delta C_c^{\max}\sup_{s\le t<0}\left(\mathrm{TV}\left(\bar{\Phi}_n^t\right)\right) \\
&\le \frac{\Delta x}{2\left(1-\delta C_c^{\max}\right)}\sup_{s\le t<0}\left(\mathrm{TV}\left(\bar{\Phi}_n^t\right)\right)
\end{aligned}
\tag{42}
$$

where $\sup_{s\le t<0}\left(\mathrm{TV}\left(\bar{\Phi}_n^t\right)\right)$ as $s\to-\infty$ shall be estimated from computational results.

Finally, we consider the performance index without discounting in Case 1. For positive $\delta<1$ and fixed n, there exists an integer $s_\varepsilon<0$ such that a pseudo periodic value function $\bar{\Psi}_n^s(\delta)$ satisfies

$$
\left\|\bar{\Psi}_n^s(\delta)-\psi_n^{s,P}\left(\bar{\Psi}_n^s(\delta)\right)\right\|_\infty<\varepsilon,\quad s\in\left\{s_\varepsilon-P,s_\varepsilon-P+1,\cdots,s_\varepsilon-1\right\}
\tag{43}
$$

for a small $\varepsilon>0$. The reward $f(t,x,k)$ is scaled down to the order of $1-\delta$ so that the upper bound of the Lipschitz constant L^s in (22) is fixed. Then, according to the Ascoli-Arzelà theorem, with $N\in\mathbb{N}$ there is a subsequence of $\left\{\bar{\Psi}_n^s\left(1-10^{-N}\right)\right\}$ uniformly converging to a Lipschitz continuous function $\bar{\Psi}_n^s(1)$, which further converges to a Lipschitz continuous function $\Psi^s(1)$ as $\varepsilon\to0$, $s\to-\infty$, and $n\to\infty$.

In the following section, we shall attempt to obtain pseudo periodic value functions with and without discounting via numerical experiments.

4. Application to the water storage tank in the real-world

We focus on a water storage tank which is a part of a micro irrigation scheme constructed in a harsh environment of the Jordan Rift Valley [26]. The micro irrigation scheme is equipped with a flash flood harvesting facility and a solar energy driven desalination plant. Freshwater produced by the desalination plant is stored in the tank connected to a drip irrigation system, which is the water user, with an on/off type valve. The rate of freshwater production is assumed to be deterministic and time-periodic. When the operator opens the valve, the stored water is released to the drip irrigation system by gravity. However, the release discharge exclusively depends on the storage volume in the tank because of hydrodynamics, while that discharge should be kept within a certain range to attain ideal infiltration in soil of the irrigated farmland.

4.1 Model parameters

The model parameters for the water storage tank are set as follows. The maximum storage volume V is 4.000 (m^3). From numerical simulation using a thermo-dynamical model verified with observed data, the volume of water supplied to the tank (m^3) on Julian day t is assumed to be deterministic as

$$S(t) = \begin{cases} 6.29808+48.9116\left|\sin\dfrac{\pi(t-6.92924)}{365}\right|^{2.50131} & \text{(common year)} \\ 6.29808+48.9116\left|\sin\dfrac{\pi(t-6.94823)}{366}\right|^{2.50131} & \text{(leap year)} \end{cases}, \tag{44}$$

which is 1461 days periodic, as shown in Fig. 1. From a hydraulic experiment, the release discharge $Q(x)$ (m^3/s) as a function of the storage volume x (m^3) is identified as

$$Q(x) = 2.690\times10^{-5} x^{2.882}. \tag{45}$$

The dynamics of storage volume y (m^3) in the tank while the valve is open is described as the initial value problem of the ordinary differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}\tau}=-Q(y),\quad y(0)=X_{t^+} \tag{46}$$

where τ (s) is the local time starting from the moment of valve opening on the day *t*. From (44) and (46) with (45), the transition to the state on the next day is determined as

$$\xi(t,X_t,k)=\begin{cases}X_{t^+} & \text{if } k=0\\ y(T) & \text{if } k=1\end{cases} \tag{47}$$

where *T* is the duration that the valve is kept open on the day *t*, which is set as 3600 (s). Fig. 2 shows $\xi(1285,x,0)$ (green line) and $\xi(1285,x,1)$ (blue line) with $y(T)$ (red line) when $X_{t^+}=x$, as $S(1285)$ is the maximum of $S(t)$, suggesting that $L^{\xi}_{1285,k}=1$. It is easy to prove $L^{\xi}=1$. The two cases of the reward $f(t,x,k)$ are considered here as

$$f(t,X_t,k)=\begin{cases}0 & \text{if } k=0\\ (1-\delta)\iota_n\left(F_l\left(X_{t^+}\right)\right) & \text{if } k=1\end{cases} \tag{48}$$

with the function $F_l(x)$ prescribed for each case as

$$F_1\left(X_{t^+}\right)=\frac{1}{1000}\int_0^T \chi_{I_Q}\left(Q\left(y(\tau)\right)\right)\mathrm{d}\tau \tag{49}$$

where χ is the indicator function, and I_Q is the interval of preferable release discharge, which is assumed to be $\left[10^{-3}/6,10^{-3}/2\right]$ (m^3/s), and

$$F_2(x)=\begin{cases}0 & \text{if } F_1(x)<3\\ 3 & \text{if } F_1(x)\geq 3\end{cases}. \tag{50}$$

Fig. 3 depicts $F_1(x)$ (red line) and $F_2(x)$ (thick black line). The Lipschitz constant L^f for Case 1 is not greater than $6(1-\delta)$, and W^f for Case 2 is equal to $6(1-\delta)$.

Fig. 1. The volume of water supplied to the tank changing with a period of 1461 days.

Fig. 2. The functions prescribing transition of states.

Fig. 3. The functions prescribing the reward.

4.2 Computational Results

Based on the discrete approximation described in Subsection 3.2, a series of numerical experiments was implemented for each case with $\delta = 1 - 10^{-N}$ for N = 1, 2, 3, 4, 5, and 6 on meshes of n = 400, 4000, and 40000. Red lines in Fig. 4 and Fig. 5 show the computed value functions $\bar{\Phi}_n^{mP}$ with negative integers m such that $\left\|\bar{\Phi}_n^{mP} - \bar{\Phi}_n^{(m-1)P}\right\|_\infty$ is less than the machine epsilon for Case 1 and Case 2, respectively. These can be regarded as the periodic value functions at t = 0. Green lines and blue lines show their candidates $f\left(mP, x, k\right) + \delta \bar{\Phi}_n^{mP+1}\left(\xi\left(mP, x, k\right)\right)$ with k = 0 and 1, respectively. The optimal policy $\pi^*\left(t, x\right)$ is delineated in Fig. 6 and Fig. 7, where $K_0\left(\bar{\Phi}_n^t\right)$ is blacked out in the t-x-plane of each panel, for Case 1 and Case 2, respectively.

Fig. 4. The periodic value functions at t = 0 for Case 1 with their candidates.

Fig. 5. The periodic value functions at t = 0 for Case 2 with their candidates.

Fig. 6. The time periodic optimal policy for Case 1.

Fig. 7. The time periodic optimal policy for Case 2.

From these figures, the following can be observed.

Case 1.

The discretization error for the time periodic value functions is evaluated as

$$
\begin{aligned}
\left\|\Psi^s-\bar{\Phi}_n^s\right\|_\infty &= \left\|\Psi^{s-P}-\bar{\Phi}_n^{s-P}\right\|_\infty \\
&\le \delta^P\left\|\Psi^s-\bar{\Phi}_n^s\right\|_\infty+\sum_{t=s-P+1}^{t<s}\delta^{t-s+P}\left\|\bar{\Phi}_n^t-\iota\left(\bar{\Phi}_n^t\right)\right\|_\infty \\
&\le \delta^P\left\|\Psi^s-\bar{\Phi}_n^s\right\|_\infty+\frac{1-\delta^P}{1-\delta}\delta\frac{\Delta x}{2}\max_{s-P<t<s}L^t \\
&\le \frac{\delta}{1-\delta}\frac{\Delta x}{2}\max_{s-P<t<s}L^t
\end{aligned}
\tag{51}
$$

from (38). According to the evaluation (22), the supremum of the Lipschitz constant L^t is $\frac{L^f}{1-\delta L^\xi}+L^0$, which is upper bounded by 6 assuming that $L^0=0$. This implies that the discretization error by (51) is significantly large under the prescribed values of $\delta=1-10^{-6}$ with $n=40000$ even. However, the computed value functions indicate far smaller L^t at the order of $1-\delta$ for each δ, most probably because $L_{t,0}^f=0$ and $L_{t,0}^\xi$ is close to 1 for smaller x where $\pi^*(t,x)=0$ and because $L_{t,1}^f$ is large and $L_{t,1}^\xi$ is small for larger x where $\pi^*(t,x)=1$, from the distribution of $\pi^*(t,x)$ which can be seen as $K_{\pi^*(t,x)}$ in Fig. 6. Therefore, the computational results for the three cases of $\delta\le 1-10^{-3}$ with $n=40000$ are convincing in terms of this a-posteriori error estimation. The two candidates of a value function are very close to each other in some parts of K. Nevertheless, a single Skiba point is found at each t for each δ, and its year periodic fluctuation exhibits different behavior with randomness for $\delta=1-10^{-3}$ in particular. Although the computed value functions may contain substantial discretization error, the optimal policy $\pi^*(t,x)$ seems converging to the one where the Skiba points define θ^t as the thresholds of valve opening with slight year periodic fluctuation. The thresholds for $\delta=1-10^{-6}$ range between 2.563 (m^3) in July and 2.696 (m^3) in January.

Case 2.

As no a-priori estimation for (41) is available, a-posteriori estimation as in Case 1 is replicated for the discretization error for the time periodic value functions in Case 2. As can be seen in Fig. 7, multiple Skiba points appear for some *t*, but almost same clear thresholds of valve opening are identified for all δ and *n*. Regarding those thresholds as θ^t, $C_r^{\max}$ and $C_c^{\max}$ inferred from Fig. 2 and Fig. 5 are small enough to conclude that the computational results for the three cases of $\delta \le 1-10^{-3}$ with $n = 40000$ are still convincing. There may be effects of truncation errors in the cases of $\delta \ge 1-10^{-4}$. The time periodic value function Ψ^s seems to be of step type for the convincing case of $\delta = 1-10^{-3}$, and it is inferred that Ψ^s for $\delta = 1$ has less points of discontinuity. The behaviors of the thresholds for all δ and *n* are similar to those of the inferred thresholds for $\delta = 1$ in Case 1. The thresholds for $\delta = 1-10^{-6}$ range between 2.665 (m^3) in July and 2.714 (m^3) in January.

5. Conclusions

Motivated by determining time periodic optimal policy for operation of the water storage tank, the discrete-time deterministic DP problem has been intensively examined. Firstly, regularity of value functions was identified for the infinite time horizon problem with discounted reward. Then, time periodic value functions were discussed in terms of the fixed-point theorem. The numerical experiments were conducted for the real-world application, based on the discrete approximation of value functions. Although the computational results for the value functions without discounting are not convincing enough because of discretization and truncation errors, an operation policy setting a constant threshold of valve opening at 2.6 (m^3) or 2.7 (m^3) throughout the year will be close to the optimal in the long run. With the policy using the constant threshold 2.6 (m^3) of valve opening, for example, the intervals of drip irrigation events are 16 days during the hot summer seasons and 106 days during the wet winter seasons. This is quite reasonable and acceptable for the operator of the water storage tank to irrigate the farmland.

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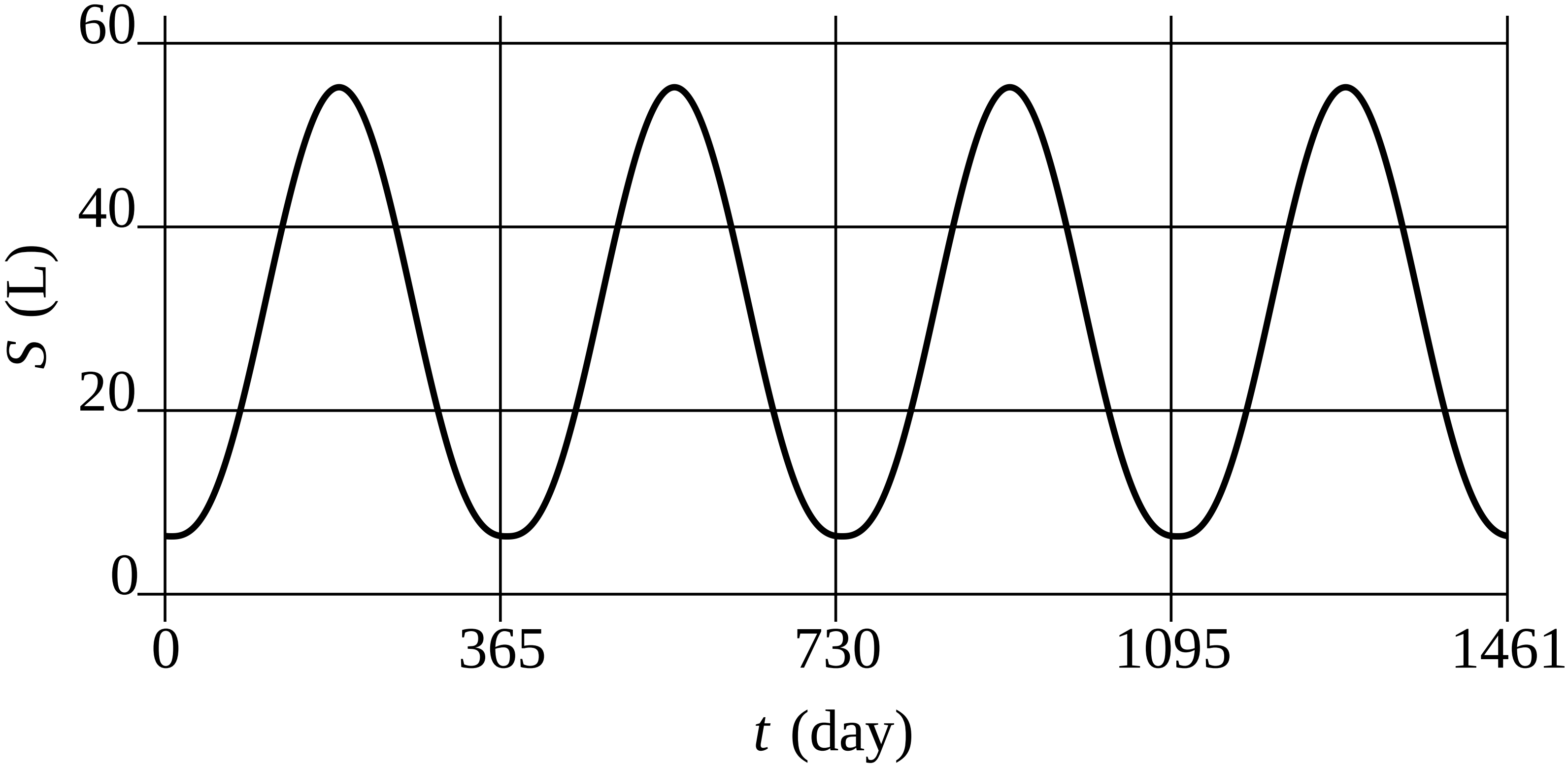
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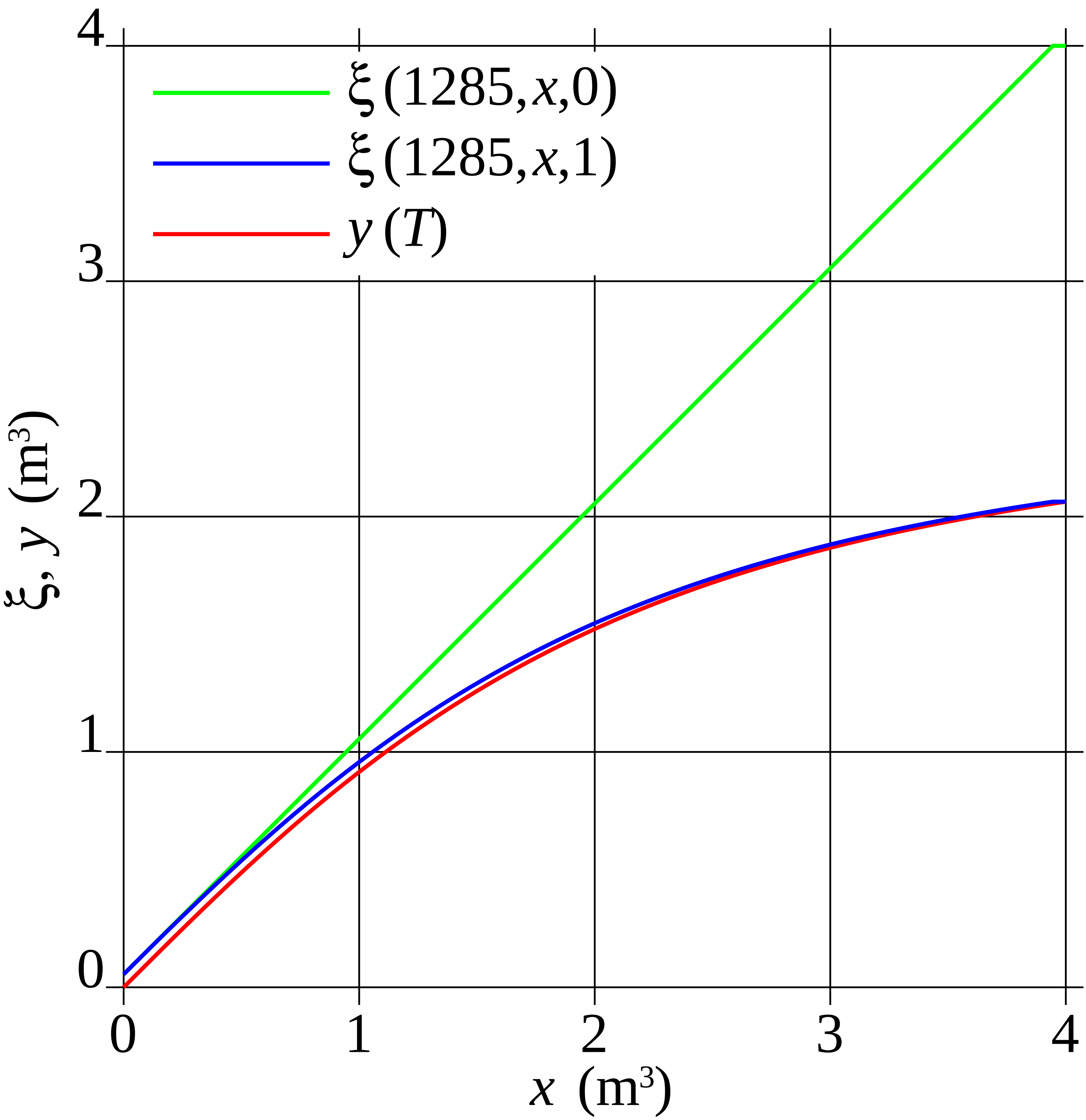
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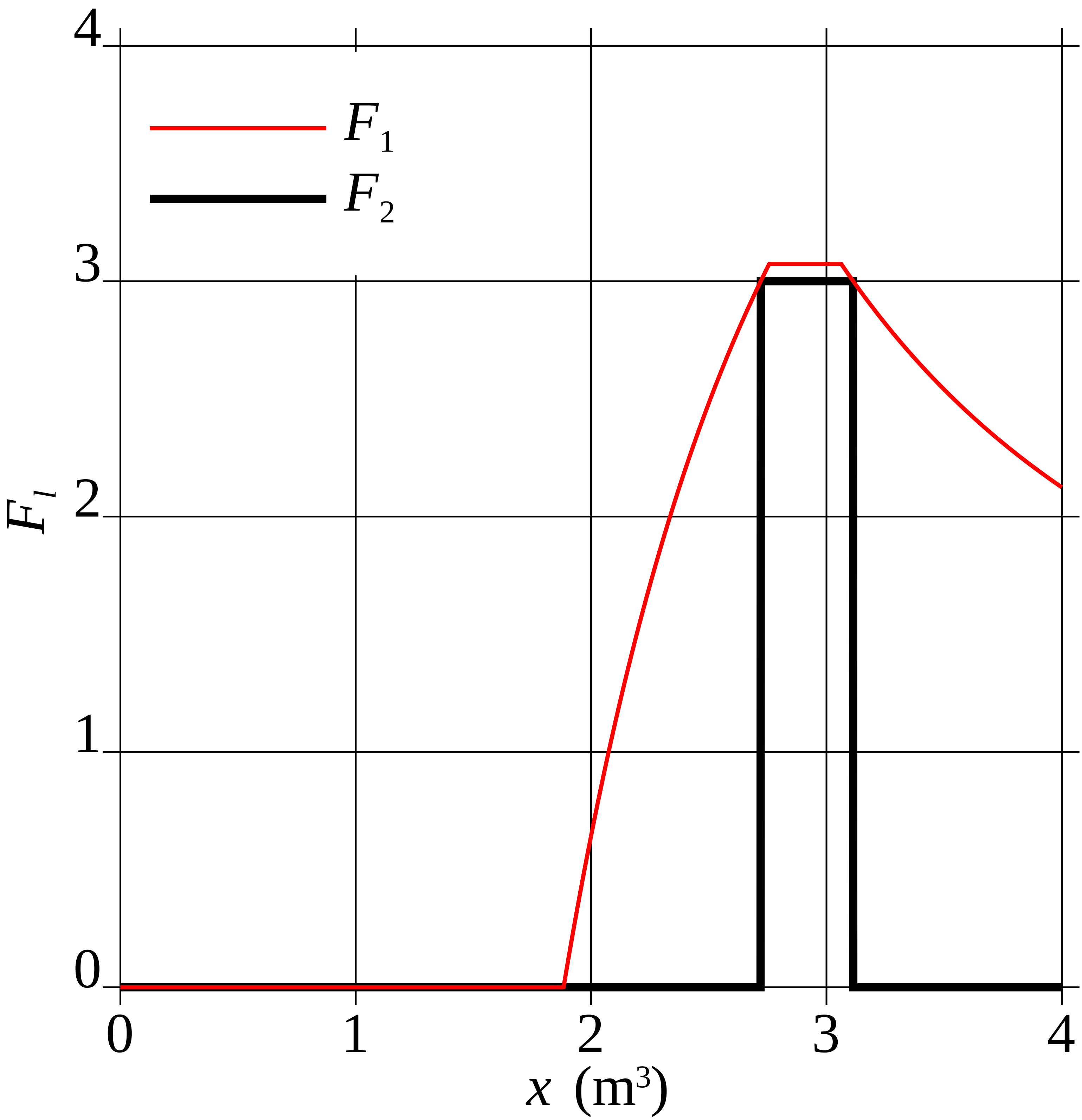
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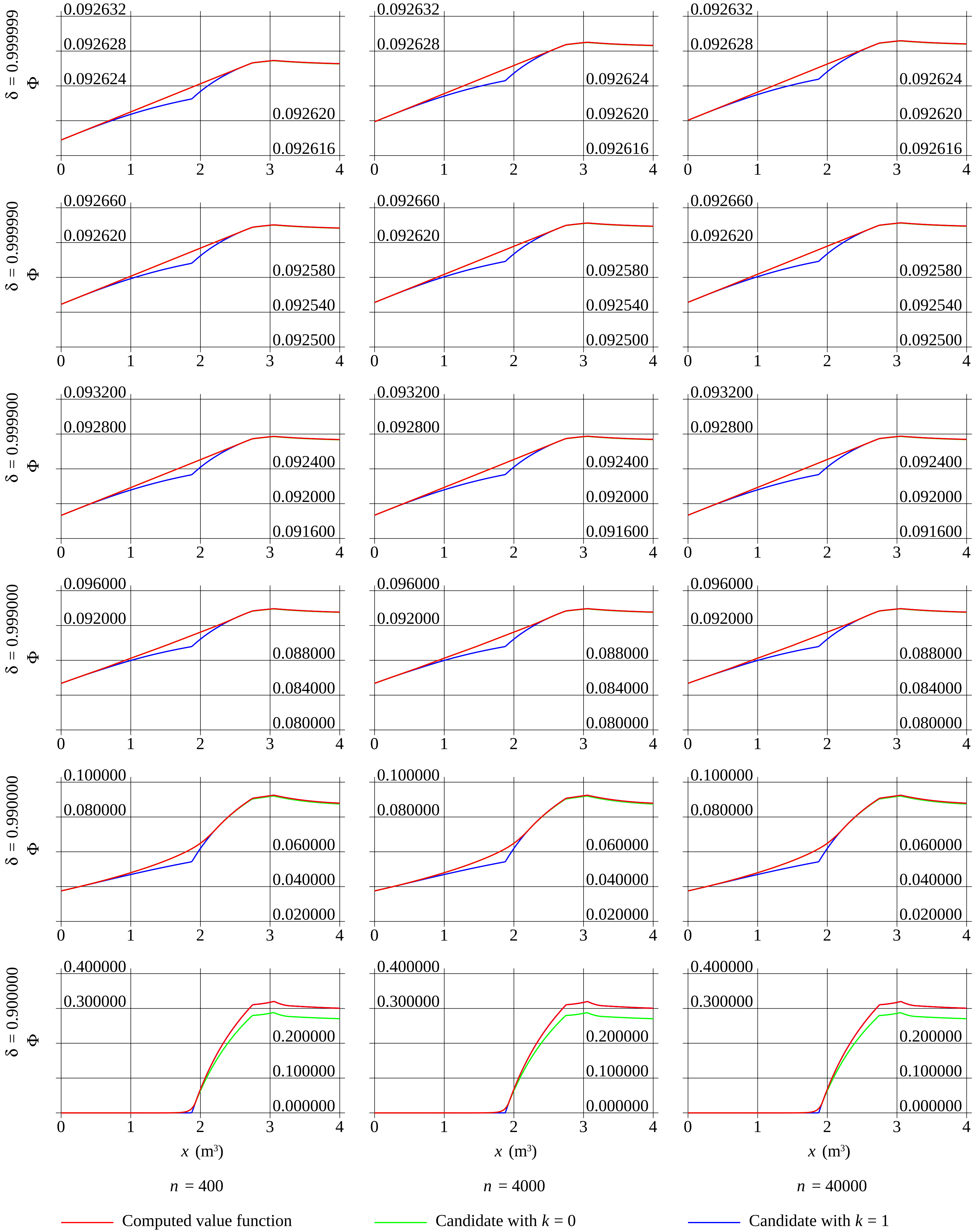

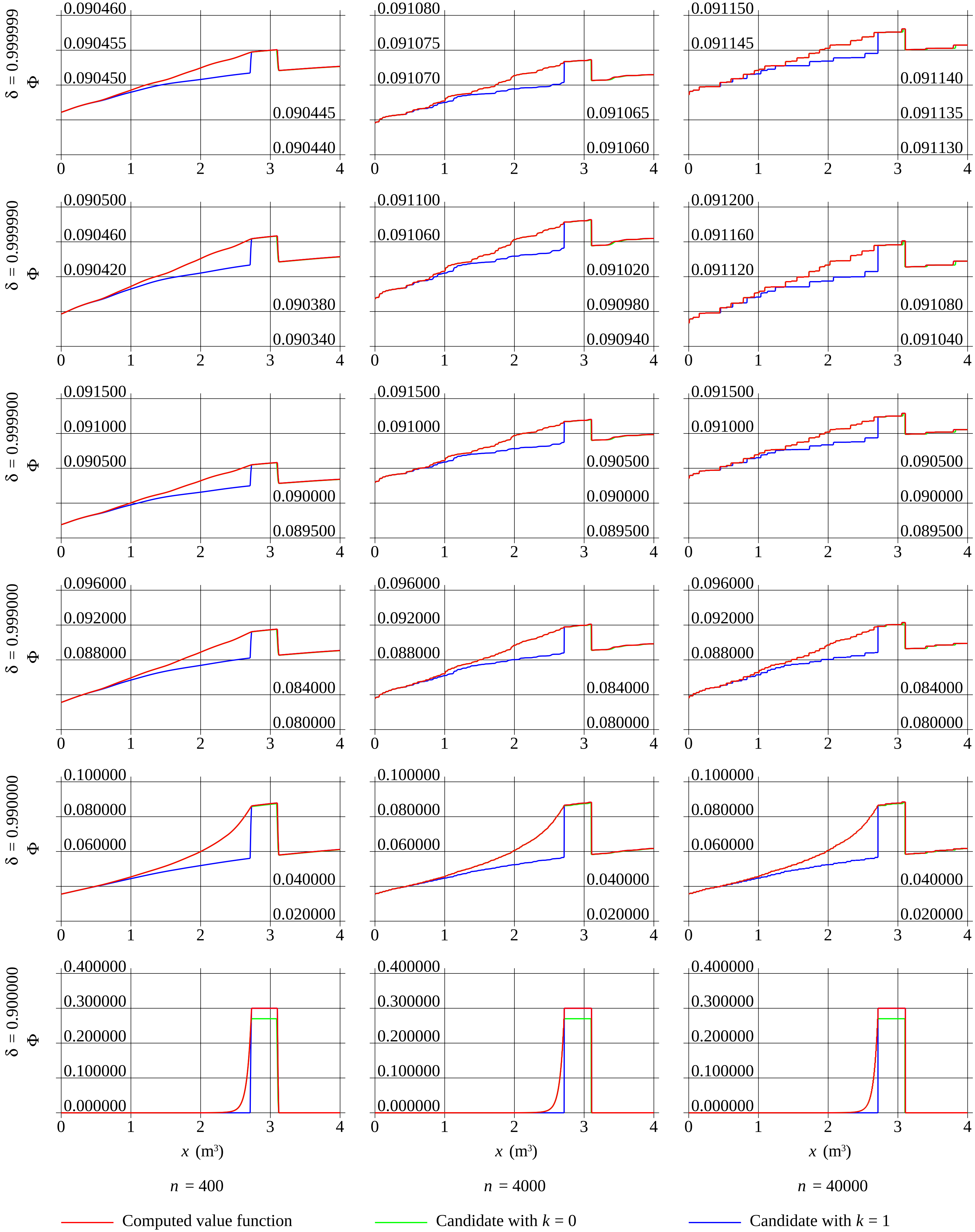
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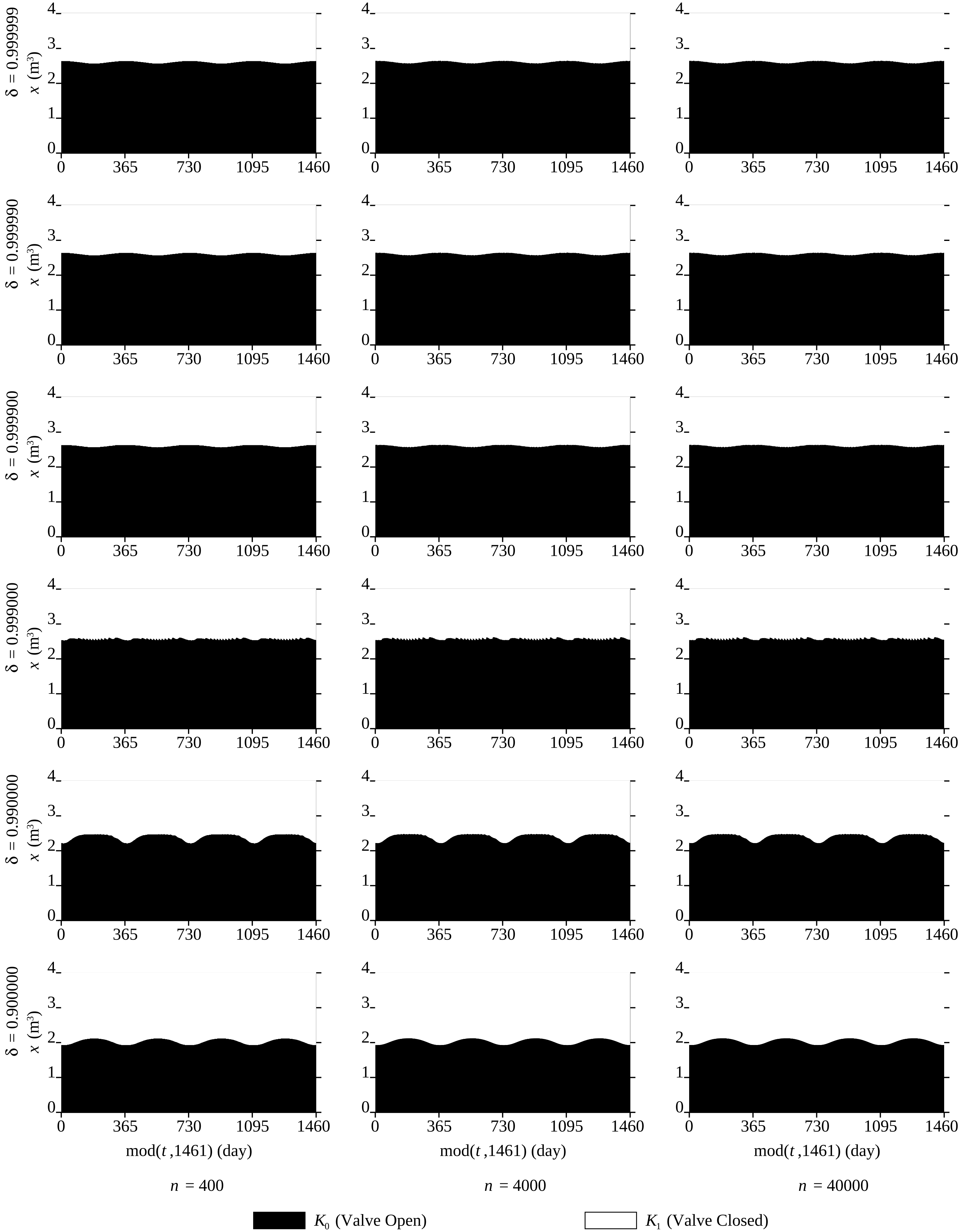
Candidate with k = 0
Candidate with k = 1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3
4
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365
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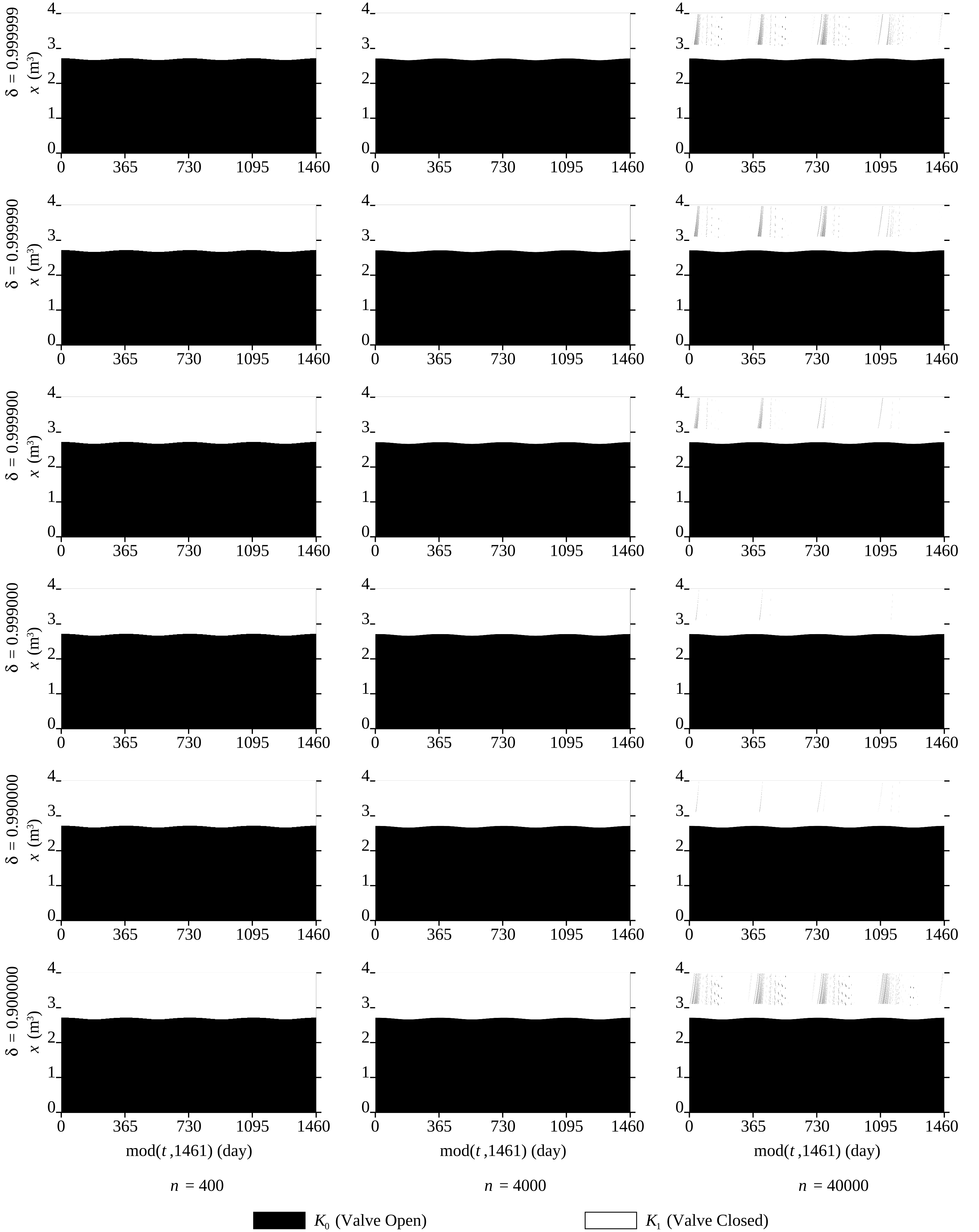
K0 (Valve Open)
K1 (Valve Closed)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mod(t ,1461) (day)
n = 400
n = 4000
n = 40000
K₀ (Valve Open)
K₁ (Valve Closed)