$T\bar{T}$-deformation and Liouville gravity

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Abstract

We consider a gravitational perturbation of the Jackiw-Teitelboim (JT) gravity with an arbitrary dilaton potential and study the condition under which the quadratic action can be seen as a $T\bar{T}$-deformation of the matter action. As a special case, the flat-space JT gravity discussed by Dubovsky et al. [arXiv:1706.06604] is included. Another interesting example is a hyperbolic dilaton potential. This case is equivalent to a classical Liouville gravity with a negative cosmological constant and then a $T\bar{T}$-deformation of the matter action with a finite coupling is realized as a gravitational perturbation on AdS$_2$.

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1. Introduction

An intriguing subject is to understand $T\bar{T}$-deformations of 2D quantum field theory (QFT). An infinitesimal $T\bar{T}$-deformation is triggered by a composite operator $\alpha \det\left(T^{[0]}_{\mu\nu}\right)$, where $T^{[0]}_{\mu\nu}$ is the energy-momentum tensor of the original system and α is a constant parameter of dimension (length)^2. Hence this is an irrelevant perturbation of QFT. It has been elucidated in a seminal paper by Sasha Zamolodchikov [1] that this determinant operator is well-defined in two dimensions and the expectation values of the operator for the arbitrary (non-degenerate) energy eigenstates exhibit the factorization property under some basic assumptions. The finite-version of $T\bar{T}$-deformation is described by the following $T\bar{T}$-flow equation [2,3][1]:

$$\frac{\mathrm{d}\mathcal{L}^{[\alpha]}}{\mathrm{d}\alpha} = -\frac{1}{4}\det\left(T^{[\alpha]}_{\mu\nu}\right). \tag{1.1}$$

Note here the energy-momentum tensor is for the deformed Lagrangian $\mathcal{L}^{[\alpha]}$. For a nice review on the $T\bar{T}$-deformation, see [5].

Another interesting aspect of $T\bar{T}$-deformation is an intimate connection to a 2D dilaton gravity (which is often called the Jackiw-Teitelboim (JT) gravity [6,7][2]). In the work by Dubovsky et al. [10], it has been shown that a gravitational perturbation in the flat-space JT gravity can be seen as a $T\bar{T}$-deformation of the matter action with a finite coupling.[3] Then, a generalization of the work [10] in flat space to AdS$_2$ and dS$_2$ has been discussed in [12]. However, the discussion in [12] is restricted to the conformal matter case and hence only the infinitesimal $T\bar{T}$-deformation has been discussed.

In this paper, we will explain how to remove this conformal matter condition. The point is to replace the dilaton potential utilized in [13] with the hyperbolic one considered in [14]. In particular, the hyperbolic dilaton potential model is equivalent to a classical Liouville gravity with a negative cosmological constant [15,16]. Hence, in other words, a gravitational perturbation in the classical Liouville gravity[4] can be seen as a $T\bar{T}$-deformation of the matter action with a finite coupling defined on AdS$_2$.

This paper is organized as follows. In section 2, we study a gravitational perturbation in the JT gravity with an arbitrary dilaton potential. The quadratic action can be regarded as an infinitesimal $T\bar{T}$-deformation of the matter action with the conformal matter condition. This section contains a brief review of the work [12]. In section 3, we introduce the classical Liouville gravity with a negative cosmological constant and show that its gravitational perturbation can be seen as

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[1]   The closed form of $T\bar{T}$-deformation is discussed in [4].

[2]   For reviews on 2D dilaton gravity, see [8,9].

[3]   For the finite $T\bar{T}$-deformation in the flat space JT gravity, see [11].

[4]   Our analysis here is at the classical level. For the relation between $T\bar{T}$-deformation and non-critical string at the quantum level, see [11,17,18].

a $T\bar{T}$-deformation without taking the conformal matter condition. Section 4 is devoted to conclusion and discussion. Appendix A explains how to derive a gravitationally dressed S-matrix. Appendix B introduces the general vacuum solution in the Liouville gravity with a cosmological constant.

2. JT gravity with conformal matter

In this section, we will consider a gravitational perturbation of the JT gravity with an arbitrary dilaton potential and derive the quadratic action. As a result, we can figure out the condition under which the gravitational perturbation can be seen as a $T\bar{T}$-deformation of the matter action. For example, for the constant dilaton potential, it can be seen as a $T\bar{T}$-deformation as shown in [10]. When the dilaton potential is given by a sum of the constant and a cosmological constant as utilized in [13], the conformal matter condition is necessary [12]. That is, an infinitesimal $T\bar{T}$-deformation is realized on AdS$_2$ or dS$_2$.

2.1. Our setup and notation

The classical action of the JT gravity in the Lorentzian signature is given by the sum of the dilaton gravity action $S_{\mathrm{dg}}[g_{\mu\nu}, \phi]$ ($g_{\mu\nu}$: metric, ϕ: dilaton) and the matter action $S_m[\psi, g_{\mu\nu}, \phi]$ (ψ: matter):

$$S[g_{\mu\nu}, \phi, \psi] = S_{\mathrm{dg}}[g_{\mu\nu}, \phi] + S_m[\psi, g_{\mu\nu}, \phi], \tag{2.1}$$

$$S_{\mathrm{dg}}[g_{\mu\nu}, \phi] = \frac{1}{2\kappa} \int \mathrm{d}^2x \sqrt{-g}\, [\phi\, R - U(\phi)]. \tag{2.2}$$

Here $x^\mu = (x^0, x^1) = (t, x)$ and $\kappa \equiv 8\pi G_N$, where G_N is 2D Newton constant. The dilaton potential $U(\phi)$ is an arbitrary scalar function now. The matter action S_m may include a nontrivial dilaton coupling in general.

The equations of motion of this system are given by

$$R - U'(\phi) + \frac{2\kappa}{\sqrt{-g}} \frac{\delta S_m}{\delta \phi} = 0, \tag{2.3}$$

$$\frac{1}{2} g_{\mu\nu} U(\phi) - \left(\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \nabla^2 \phi \right) = \kappa\, T_{\mu\nu}, \tag{2.4}$$

and the one for the matter field. Here we have used the identity $R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R$ for the second equation. The energy-momentum tensor $T_{\mu\nu}$ for the matter field ψ is defined as

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}}. \tag{2.5}$$

When the matter action depends on the dilaton, $T_{\mu\nu}$ also depends on the dilaton.

The trace of (2.4) is given by

$$\nabla^2 \phi + U(\phi) = \kappa\, g^{\mu\nu} T_{\mu\nu} \equiv \kappa T. \tag{2.6}$$

Here T is a trace of the energy-momentum tensor $T^{\mu\nu}$. By using (2.6), the Einstein equation (2.4) can be rewritten as

$$-\nabla_\mu \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} U(\phi) = \kappa \left(T_{\mu\nu} - g_{\mu\nu} T \right). \tag{2.7}$$

The vacuum solution. For later convenience, let us discuss the vacuum solution (i.e., $T_{\mu\nu} = 0$). When the matter action S_m is turned off, the equations of motion (2.3) and (2.4) reduced to

$$\bar{R} - U'(\bar{\phi}) = 0 \,, \tag{2.8}$$

$$-\bar{\nabla}_\mu \bar{\nabla}_\nu \bar{\phi} + \bar{g}_{\mu\nu} \bar{\nabla}^2 \bar{\phi} + \frac{1}{2} \bar{g}_{\mu\nu} U(\bar{\phi}) = 0 \,. \tag{2.9}$$

Here we have denoted the vacuum solution as $\bar{g}_{\mu\nu}$ and $\bar{\phi}$, and the covariant derivative $\bar{\nabla}$ is defined with $\bar{g}_{\mu\nu}$. The trace of (2.9) is given by

$$\bar{\nabla}^2 \bar{\phi} + U(\bar{\phi}) = 0 \,. \tag{2.10}$$

Since the dilaton potential $U(\phi)$ can be deleted from (2.9) by using (2.10), the resulting expression is

$$\bar{\nabla}_\mu \bar{\nabla}_\nu \bar{\phi} = \frac{1}{2} \bar{g}_{\mu\nu} \bar{\nabla}^2 \bar{\phi} \,. \tag{2.11}$$

Given the explicit form of $U(\phi)$, the vacuum solution is also determined.

2.2. The quadratic action

Next, we shall consider a gravitational perturbation around the vacuum solution,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \,, \qquad \phi = \bar{\phi} + \sigma \,, \qquad \psi = 0 + \psi \,, \tag{2.12}$$

where $h_{\mu\nu}$ and σ are fluctuations of metric and dilaton, respectively, and ψ itself is regarded as a fluctuation around the trivial background $\bar{\psi} = 0$. In the following, the indices in the perturbations are raised, lowered, and contracted with the background metric $\bar{g}_{\mu\nu}$, say $h^{\mu\nu} \equiv \bar{g}^{\mu\rho} \bar{g}^{\nu\sigma} h_{\rho\sigma}$.

Let us expand the classical action $S[g_{\mu\nu}, \phi, \psi]$ in (2.1) in terms of the fluctuations (2.12):

$$\begin{aligned}
S[g_{\mu\nu}, \phi, \psi] &= S^{(0)} + S^{(1)} + S^{(2)} + \cdots \\
&= S_{\text{dg}}^{(0)}[\bar{g}_{\mu\nu}, \bar{\phi}] + S_{\text{dg}}^{(1)}[\bar{g}_{\mu\nu}, \bar{\phi}; h_{\mu\nu}, \sigma] + S_{\text{dg}}^{(2)}[\bar{g}_{\mu\nu}, \bar{\phi}; h_{\mu\nu}, \sigma] \\
&\quad + S_{\text{m}}^{(1)}[\bar{g}_{\mu\nu}; \psi] + S_{\text{m}}^{(2)}[\bar{g}_{\mu\nu}; \psi, h_{\mu\nu}] + \cdots \,,
\end{aligned} \tag{2.13}$$

where the superscript of $S^{(n)}$ denotes the order of fluctuation. The zeroth order part $S_{\text{dg}}^{(0)}$ is the classical value of S_{dg} with the vacuum configuration. The first order action $S_{\text{dg}}^{(1)}$ should vanish since the vacuum solution satisfies the equations of motion with $\bar{\psi} = 0$. For the matter sector, $S_{\text{m}}^{(1)}$ describes the matter field action on the classical background $\bar{g}_{\mu\nu}$.

By expanding (2.3) and (2.7), the equations of motion for $h_{\mu\nu}$ and σ can be obtained as

$$\bar{\nabla}^\mu \bar{\nabla}^\nu h_{\mu\nu} - \bar{\nabla}^2 h - \frac{1}{2} U'(\bar{\phi}) h - U''(\bar{\phi})\sigma + \frac{2\kappa}{\sqrt{-\bar{g}}} \frac{\delta S_m^{(1)}}{\delta \phi}(\bar{\phi}) = 0 \,, \tag{2.14}$$

$$\begin{aligned}
&\bar{\nabla}_\mu \bar{\nabla}_\nu \sigma + \frac{1}{2} \bar{g}_{\mu\nu} U'(\bar{\phi})\,\sigma \\
&= -\kappa\,(T_{\mu\nu}^{(0)} - \bar{g}_{\mu\nu} T^{(0)}) - \frac{1}{2} U(\bar{\phi})\, h_{\mu\nu} + \frac{1}{2} \bar{\nabla}^\rho \bar{\phi}\,(\bar{\nabla}_\mu h_{\rho\nu} + \bar{\nabla}_\nu h_{\rho\mu} - \bar{\nabla}_\rho h_{\mu\nu}) \,,
\end{aligned} \tag{2.15}$$

where $T^{(0)}{}_{\mu\nu}$ is the energy-momentum tensor for the matter theory described by $S_{\text{m}}^{(1)}$.

Suppose that $h_{\mu\nu}$ takes the following form [12]:

$$h_{\mu\nu} = -2\kappa(T^{(0)}_{\mu\nu} - \bar{g}_{\mu\nu} T^{(0)})k\,, \tag{2.16}$$

where k is a constant parameter with dimension (length)^2 while κ is dimensionless. This is a covariant version of the one employed in the flat-space JT case [10].

As a result, the quadratic action $S^{(2)}$ is simplified as

$$S^{(2)} = \int \mathrm{d}^2x\,\sqrt{-\bar{g}}\left[\frac{1}{4\kappa}U''(\bar{\phi})\,\sigma^2 - \kappa\left(k - \frac{k^2}{4}U(\bar{\phi})\right)\left(T^{(0)}_{\mu\nu}T^{(0)\,\mu\nu} - T^{(0)\,2}\right)\right]. \tag{2.17}$$

The second term is proportional to the $T\bar{T}$ operator, though the coefficient depends on the background dilaton $\bar{\phi}$ and in general has space-time coordinate dependence.

Therefore, if $U''(\bar{\phi}) = 0$ and the metric fluctuation $h_{\mu\nu}$ satisfies the ansatz (2.16), the quadratic action can be regarded as a $T\bar{T}$ deformation of the original matter action, up to the background dilaton dependence. However, we still need to check the existence of σ as a solution to the equations of motion.

In the following, let us see two examples, 1) the flat-space JT gravity and 2) the Almheiri-Polchinski (AP) model.

1) The flat-space JT gravity. Let us first revisit the case of the flat-space JT gravity [10]. This case is realized by taking a constant dilaton potential

$$U(\phi) = \Lambda \quad : \text{constant}. \tag{2.18}$$

In the Cartesian coordinates, a vacuum solution is obtained as

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1, +1)\,, \qquad \bar{\phi} = \frac{\Lambda}{4}(t^2 - x^2)\,. \tag{2.19}$$

Note here that the dilaton is non-trivial but the background metric is still 2D Minkowski space-time.

The metric ansatz (2.16) satisfies (2.14) only if the matter action $S^{(1)}_m$ does not depend on the dilaton

$$\frac{\delta S^{(1)}_m}{\delta\phi} = 0\,. \tag{2.20}$$

The solution for σ is explicitly written down as a non-local solution [12]

$$\sigma(t, x) = a_1 + a_2\,t + a_3\,x + \sigma_{\text{non-local}}(t, x)\,, \tag{2.21}$$

where $a_{1,2,3}$ are arbitrary constants, and $\sigma_{\text{non-local}}$ is a non-local part of σ given by

$$\sigma_{\text{non-local}}(t, x) = \frac{\kappa}{2}\left[k\Lambda\int_0^x \mathrm{d}x'\,x'\,T^{(0)}_{tt}(t, x') + k\Lambda\int_0^t \mathrm{d}t'\,t'\,T^{(0)}_{xx}(t', x)\right.$$

$$\left. + 2(k\Lambda - 1)\int_0^t \mathrm{d}t'\int_0^x \mathrm{d}x'\,T^{(0)}_{tx}(t', x')\right.$$

$$+ (k\Lambda - 2) \left(\int_{t_1}^{t} \mathrm{d}t' \int_{t_2}^{t'} \mathrm{d}t'' \, T_{xx}^{(0)}(t'', 0) + \int_{x_1}^{x} \mathrm{d}x' \int_{x_2}^{x'} \mathrm{d}x'' \, T_{tt}^{(0)}(0, x'') \right) \Bigg] .$$
(2.22)

By using this expression of σ, one can introduce dynamical coordinates explicitly and evaluate the gravitationally dressed S-matrix [10] (for the detail, see Appendix A).

2) The Almheiri-Polchinski model. A bit non-trivial example satisfying the condition $U''(\bar{\phi}) = 0$ is the Almheiri-Polchinski (AP) model [13] specified by the following dilaton potential:

$$U(\phi) = \Lambda - \frac{2}{L^2}\phi \,,$$
(2.23)

where L is the AdS radius. In comparison to the flat-space JT case, a negative cosmological constant is additionally included.

In the conformal gauge, the metric is parametrized as

$$\mathrm{d}^2 s = \bar{g}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = -2\mathrm{e}^{2\bar{\omega}}\mathrm{d}x^+ \mathrm{d}x^- \,,$$
(2.24)

where the light-cone coordinates are defined as

$$x^\pm \equiv \frac{1}{\sqrt{2}}(t \pm x) \,.$$
(2.25)

The general vacuum solution is given by [13]

$$\mathrm{e}^{2\bar{\omega}} = \frac{2L^2}{(x^+ - x^-)^2} \,, \qquad \bar{\phi} = \frac{\Lambda L^2}{2} + \frac{a + b(x^+ + x^-) + c\,x^+x^-}{x^+ - x^-} \,,$$
(2.26)

where a, b and c are arbitrary constants. In the following discussion, we will consider only constant dilaton case with $a = b = c = 0$ so as to drop off the coordinate dependence of the dilaton background.

By imposing the metric ansatz (2.16), the equation of motion (2.14) is rewritten as

$$\frac{T^{(0)}\,k}{L^2} + \frac{1}{\sqrt{-\bar{g}}}\frac{\delta S_m^{(1)}}{\delta\phi}\left(\bar{\phi}\right) = 0 \,.$$
(2.27)

A solution is to employ a conformal matter which does not couple to the dilaton [12]:

$$T^{(0)} = 0, \qquad \frac{\delta S_m^{(1)}}{\delta\phi} = 0 \,.$$
(2.28)

Notably, this conformal matter condition is not a unique solution and there may remain another possibility to take a particular dilaton dependence of the matter action. We will discuss this issue in the next section.

Let us solve the equation of motion for σ in (2.15). Due to the conformal matter condition and the conservation law for the energy-momentum tensor, $T_{++}^{(0)}$ and $T_{--}^{(0)}$ are holomorphic and anti-holomorphic functions, respectively. Each component of (2.15) is evaluated as

$$\mathrm{e}^{2\bar{\omega}}\partial_+ \left(\mathrm{e}^{-2\bar{\omega}}\partial_+\sigma\right) = -\,\kappa\,T_{++}^{(0)}(x^+) \,,$$
$$\mathrm{e}^{2\bar{\omega}}\partial_- \left(\mathrm{e}^{-2\bar{\omega}}\partial_-\sigma\right) = -\,\kappa\,T_{--}^{(0)}(x^-) \,,$$
$$\partial_+\partial_-\sigma + \frac{2\sigma}{(x^+ - x^-)^2} = 0 \,.$$
(2.29)

A solution to the equations in (2.29) is given by [12]

$$\sigma(x^+, x^-) \equiv \frac{I_0(x^+, x^-) + I^+(x^+, x^-) - I^-(x^+, x^-)}{x^+ - x^-}.\qquad(2.30)$$

Here I_0 is the sourceless solution,

$$I_0(x^+, x^-) \equiv A + B\,(x^+ + x^-) + C\,x^+ x^-, \qquad A,\ B,\ C: \text{arbitrary real consts.},\quad(2.31)$$

and $I^\pm(x^+, x^-)$ correspond to the non-local parts of dilaton:

$$I^\pm(x^+, x^-) \equiv \kappa \int\limits_{u^\pm}^{x^\pm} \mathrm{d}s\,(s - x^+)(s - x^-)\,T_{\pm\pm}^{(0)}(s),\qquad(2.32)$$

where $u^\pm$ are arbitrary constants.

We should emphasize that in comparison to the flat-space JT gravity, the conformal matter condition $T^{(0)} = 0$ is necessary here. Because of the conformal matter condition, a general deformed system $\mathcal{L}^{(\alpha)}$ cannot be taken as the original matter action $S_m^{(1)}$. Hence a finite $T\bar{T}$-deformation cannot also be considered, though an infinitesimal $T\bar{T}$-deformation of a conformal field. This is a summary of the work [12].

Obviously, it is a significant issue to consider how to remove this conformal matter condition in the case of AdS$_2$. In the next section, we will present another example supporting a non-conformal matter.

3. Liouville gravity and $T\bar{T}$-deformation

So far, we have considered at most a linear potential like (2.23) in order to solve the condition $U''(\phi) = 0$. Note however that the condition we have to solve is that $U''(\bar\phi) = 0$ and the argument is the background dilaton $\bar\phi$ rather than ϕ. Hence it is enough to consider the behavior of the dilaton potential around the vacuum solution and it is possible to take account of more complicated dilaton potentials.

As such an example, we will consider a classical Liouville gravity with a negative cosmological constant.[5] Remarkably, the quadratic action is recast into a $T\bar{T}$-deformation of the original matter action with a finite coupling (i.e., the conformal matter condition is not necessary).

3.1. Classical Liouville gravity

The classical action of the Liouville gravity with a negative cosmological constant is

$$S = \frac{1}{2\kappa}\int \mathrm{d}^2x\,\sqrt{-G}\left(\phi R_{(G)} - \frac{2\eta}{L^2}(\nabla_{(G)}\phi)^2 - \frac{1}{2\eta}\mathrm{e}^{2\eta\left(\Lambda - \frac{2}{L^2}\phi\right)} + \frac{1}{2\eta}\right) + S_m\left[\psi,\,G_{\mu\nu}\right],\qquad(3.1)$$

where $R_{(G)}$ and $\nabla_{(G)}$ are the Ricci scalar and covariant derivative, respectively, defined with the metric $G_{\mu\nu}$. Then η is a new constant parameter with dimension (length)^2. When η is negative ($\eta < 0$), the kinetic term of the dilaton has the wrong sign and the potential of (3.1) is not bounded from below. Hence, we take η to be positive as a natural choice,

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[5] The classical Liouville gravity can be derived from pure Einstein gravity in $2 + \epsilon$ dimensions [19].

$$\eta > 0 \,. \tag{3.2}$$

In comparison to the action (2.2), the Liouville gravity action (3.1) has the dilaton kinetic term. Hence, in order to employ the argument in Section 2, we have to remove the dilaton kinetic term by performing an appropriate Weyl transformation.

Let us consider the following Weyl transformation depending on the dilaton [16],

$$G_{\mu\nu} = e^{-\eta\left(\Lambda - \frac{2}{L^2}\phi\right)} g_{\mu\nu} \,. \tag{3.3}$$

In the frame with $g_{\mu\nu}$, the kinetic term has been removed as follows:

$$S = \frac{1}{2\kappa} \int d^2x \sqrt{-g} \left(\phi R - \frac{1}{\eta}\sinh\left[\eta\left(\Lambda - \frac{2\phi}{L^2}\right)\right]\right) + S_m \left[\psi, \, e^{-\eta\left(\Lambda - \frac{2}{L^2}\phi\right)} g_{\mu\nu}\right] \,. \tag{3.4}$$

Thus the dilaton potential $U(\phi)$ in (2.2) is identified with the following hyperbolic potential:

$$U(\phi) = \frac{1}{\eta}\sinh\left[\eta\left(\Lambda - \frac{2\phi}{L^2}\right)\right] \,. \tag{3.5}$$

Note here that the matter action S_m now depends on the dilaton explicitly through the Weyl factor of the metric $g_{\mu\nu}$.

Originally, this hyperbolic-type potential was introduced in [14] so as to support Yang-Baxter deformations [20–22] of AdS$_2$, where η corresponds to the deformation parameter. In the $\eta \to 0$ limit, the AP model (2.23) is reproduced.

It is known that the AdS$_2$ metric with a constant dilaton is one of the vacuum solutions (for the general solution, see Appendix B). In the conformal gauge (2.24), this solution is given by

$$e^{2\bar{\omega}} = \frac{2L^2}{(x^+ - x^-)^2} \,, \qquad \bar{\phi} = \frac{\Lambda L^2}{2} \,. \tag{3.6}$$

In the following, we will consider fluctuations around this vacuum solution. For this constant dilaton background, one can show that

$$U(\bar{\phi}) = 0 \,, \qquad U'(\bar{\phi}) = -\frac{2}{L^2} \,, \qquad U''(\bar{\phi}) = 0 \,. \tag{3.7}$$

Thus this hyperbolic dilaton potential (3.5) indeed satisfies the condition $U''(\bar{\phi}) = 0$.

3.2. The quadratic action

Let us then consider the quadratic action for the hyperbolic potential (3.5). By supposing the ansatz (2.16), the equation in (2.14) is simplified as

$$\frac{2\kappa}{L^2} T^{(0)} k + \frac{2\kappa}{\sqrt{-\bar{g}}} \frac{\delta S_m^{(1)}}{\delta\phi}(\bar{\phi}) = 0 \,. \tag{3.8}$$

The dilaton dependence in the matter action has been determined in (3.4), and the second term in (3.8) is replaced by the trace of the energy-momentum tensor as follows:

$$\frac{\delta S_m^{(1)}}{\delta\phi}(\bar{\phi}) = -\frac{2\eta}{L^2} g^{\mu\nu}\frac{\delta S_m^{(1)}}{\delta g^{\mu\nu}} = \frac{\eta}{L^2}\sqrt{-\bar{g}} \, T^{(0)} \,. \tag{3.9}$$

As a result, (3.8) reduces to a simple equation,

$$\frac{2\kappa}{L^2} (k + \eta) \, T^{(0)} = 0 \,. \tag{3.10}$$

A possible solution is the conformal matter case $T^{(0)} = 0$. Then the matter action S_m is invariant under the Weyl transformation and its dilaton dependence disappears. Hence, it is the same as the AP model case discussed in Section 2.

Unless the matter action is conformal, k is directly connected to η like

$$k = -\eta \,. \tag{3.11}$$

Originally, k has been introduced as an arbitrary constant of the metric ansatz (2.16). In comparison to the flat-space JT case where k is completely free, in the present case k is determined completely by the initial set-up of the Liouville action.

With the condition (3.11), the quadratic action (2.17) leads to the form of $T\bar{T}$-deformation on the AdS$_2$ background,

$$S^{(2)} = \kappa \, \eta \int \mathrm{d}^2 x \, \sqrt{-\bar{g}} \, \left(T^{(0)}_{\mu\nu} T^{(0)\mu\nu} - T^{(0)2} \right) \,. \tag{3.12}$$

Note here that the deformation is measured by $\kappa \, \eta$. It is significant to see the signature of the deformation because it is sensitive to the physics of the deformed theory. Recall that both κ and η are positive. Hence the deformation (3.12) corresponds to the negative sign in the convention of [26]. Then the deformed theory should have a UV cut-off (at least) in the flat-space limit, because the energy becomes complex in the UV region. Hence the above result would have an intimate connection with the cut-off AdS geometry [23,24] or the random boundary geometry [25].

On the other hand, a negative η corresponds to a positive-sign $T\bar{T}$-deformation. Then the deformed theory does not have the UV cut-off. However, if η is negative, then the potential of the dilaton is not bounded from below and the dilaton becomes unstable. This case may be interpreted as a quantum Liouville theory and then be related to the Little String Theory scenario proposed in [26].

3.3. The explicit solution of σ

The remaining task is to derive a non-trivial solution to the equations of motion (2.14) and (2.15). For this purpose, let us start from considering some properties of the energy-momentum tensor.

The energy momentum tensor $T^{(0)}_{\mu\nu}$ should satisfy the conservation law.

$$\bar{\nabla}^\mu T^{(0)}_{\mu\nu} = 0 \,. \tag{3.13}$$

In the conformal gauge (2.24), the components of the conservation law are given by

$$\partial_- T^{(0)}_{++} = -\partial_+ T^{(0)}_{+-} - \frac{2}{x^+ - x^-} T^{(0)}_{+-} \,, \qquad \partial_+ T^{(0)}_{--} = -\partial_- T^{(0)}_{+-} + \frac{2}{x^+ - x^-} T^{(0)}_{+-} \,. \tag{3.14}$$

The trace of the energy-momentum tensor $T^{(0)}{}_{+-}$ is not zero and gives rise to a no-trivial contribution. Moreover, in comparison to the conformal matter case, the (++) component of the energy-momentum tensor $T^{(0)}{}_{++}$ is no longer a holomorphic function and it depends on x^- as well. This is also the same for $T^{(0)}_{--}$.

Thus, the equations in (2.29) are rewritten as

$$e^{2\bar{\omega}}\partial_+ \left(e^{-2\bar{\omega}}\partial_+\sigma\right) = -\kappa\, T_{++}^{(0)}(x^+, x^-),$$

$$e^{2\bar{\omega}}\partial_- \left(e^{-2\bar{\omega}}\partial_-\sigma\right) = -\kappa\, T_{--}^{(0)}(x^+, x^-),$$

$$\partial_+\partial_-\sigma + \frac{2}{(x^+ - x^-)^2}\sigma = \kappa\, T_{+-}^{(0)}(x^+, x^-). \tag{3.15}$$

It is useful to introduce a scalar function $M(x^+, x^-)$ as

$$\sigma = \frac{M(x^+, x^-)}{x^+ - x^-}. \tag{3.16}$$

The equations in (3.15) are further rewritten as

$$\partial_+^2 M = -\kappa\,(x^+ - x^-)\,T_{++}^{(0)}(x^+, x^-),$$

$$\partial_-^2 M = -\kappa\,(x^+ - x^-)\,T_{--}^{(0)}(x^+, x^-),$$

$$(x^+ - x^-)\,\partial_+\partial_- M + \partial_+ M - \partial_- M = 2\kappa\,(x^+ - x^-)\,T_{+-}^{(0)}(x^+, x^-). \tag{3.17}$$

The solution is given by

$$M(x^+, x^-) = I_0(x^+, x^-) + \mathcal{I}^+(x^+, x^-) - \mathcal{I}^-(x^+, x^-), \tag{3.18}$$

where $I_0(x^+, x^-)$ is the sourceless solution given in (2.31). $\mathcal{I}^+(x^+, x^-)$ and $\mathcal{I}^-(x^+, x^-)$ are defined as

$$\mathcal{I}^+(x^+, x^-) \equiv \frac{\kappa}{2}\int_{u^+}^{x^+} \mathrm{d}s\,(s - x^+)(s - x^-)\,T_{++}^{(0)}(s, x^-), \tag{3.19}$$

$$\mathcal{I}^-(x^+, x^-) \equiv \frac{\kappa}{2}\int_{u^-}^{x^-} \mathrm{d}s\,(s - x^+)(s - x^-)\,T_{--}^{(0)}(x^+, s). \tag{3.20}$$

This solution resembles the one in the AP case (2.32). However, the energy-momentum tensor is not (anti-)holomorphic, hence be careful for calculating partial derivatives of $\mathcal{I}^\pm$.

It would be instructive to demonstrate, for example, the calculation of the partial derivative of $\mathcal{I}^-$:

$$\partial_+\mathcal{I}^-(x^+, x^-)$$

$$= \frac{\kappa}{2}\int_{u^-}^{x^-} \mathrm{d}s\left[-(s - x^-)T_{--}^{(0)}(x^+, s) + (s - x^+)(s - x^-)\partial_+ T_{--}^{(0)}(x^+, s)\right]$$

$$= \frac{\kappa}{2}\int_{u^-}^{x^-} \mathrm{d}s\left[-(s - x^-)T_{--}^{(0)}(x^+, s) - (s - x^+)(s - x^-)\partial_s T_{+-}^{(0)}(x^+, s) - 2(s - x^-)T_{+-}^{(0)}(x^+, s)\right]$$

$$= \frac{\kappa}{2}\int_{u^-}^{x^-} \mathrm{d}s\left[-(s - x^-)T_{--}^{(0)}(x^+, s) - (x^+ - x^-)T_{+-}^{(0)}(x^+, s)\right]. \tag{3.21}$$

From the second line to the third line, we have used the conservation law (3.14) and also assumed that the boundary terms vanish. Similarly, one can also evaluate the second-order derivative as follows:

$$\partial_+^2 \mathcal{I}^-(x^+, x^-)$$

$$= \frac{\kappa}{2} \int_{u^-}^{x^-} ds \left[-(s - x^-)\partial_+ T_{--}^{(0)}(x^+, s) - T_{+-}^{(0)}(x^+, s) - (x^+ - x^-)\partial_+ T_{+-}^{(0)}(x^+, s) \right]$$

$$= \frac{\kappa}{2} \int_{u^-}^{x^-} ds \left[-(s - x^-)\partial_s T_{+-}^{(0)}(x^+, s) - 2\frac{(s - x^-)}{x^+ - s} T_{+-}^{(0)}(s, x^-) \right.$$

$$\left. - T_{+-}^{(0)}(x^+, s) - (x^+ - x^-)\partial_+ T_{+-}^{(0)}(x^+, s) \right]$$

$$= \frac{\kappa}{2} \int_{u^-}^{x^-} ds \, (x^+ - x^-) \left[-\frac{2}{x^+ - s} T_{+-}^{(0)}(x^+, s) - \partial_+ T_{+-}^{(0)}(x^+, s) \right]$$

$$= \frac{\kappa}{2}(x^+ - x^-) T_{++}^{(0)}(x^+, x^-),$$

$$\partial_+ \partial_- \mathcal{I}^-(x^+, x^-) = \frac{\kappa}{2} \int_{u^-}^{x^-} ds \left[T_{--}^{(0)}(x^+, s) - T_{+-}^{(0)}(x^+, s) \right]. \tag{3.22}$$

Thus, one can directly confirm the solution (3.18) satisfies the equations in (3.17).

4. Conclusion and discussion

In this paper, we have revisited gravitational perturbations of the JT gravity and discussed the condition under which those can be seen as $T\bar{T}$-deformations. In addition to the known examples like the flat-space JT gravity and the AP model, as a novel example, we have studied the Liouville gravity with a negative cosmological constant. The conformal matter condition is necessary for the AP model but not for the Liouville gravity.

The Liouville gravity can also be seen as a Yang-Baxter deformation of the AP model. Then the parameter measuring the Yang-Baxter deformation is connected with the one of $T\bar{T}$-deformation, and also controls the behavior of the Liouville potential and the stability of the dilaton field. When the Liouville potential is bounded from below, the $T\bar{T}$-deformation is the negative sign $T\bar{T}$-deformation [23]. The positive sign case may be related to a non-critical string approach [11,17,18]. It is also interesting to study a quantum aspect of our result by following [27].

It is an open problem to consider our result in the context of NAdS$_2$/NCFT$_1$ [13,28–30]. As discussed in [14], the Yang-Baxter deformation breaks the $SL(2)$ symmetry and changes the UV behavior of the AdS$_2$ geometry. In particular, a singularity surface emerges at the middle of the bulk as a holographic screen and such a geometry would also be related to the cut-off AdS geometry proposed in [23]. It is nice to study the boundary behavior of our non-local solution of the dilaton to figure out the boundary dual for the Liouville gravity.

Finally, it is known that a single-trace $T\bar{T}$-deformation is related to a Yang-Baxter deformation in the context of AdS$_3$/CFT$_2$ [31–35]. It is interesting to understand a relation between this fact and our result via the dimensional reduction. It is also nice to consider a supersymmetric version of our analysis by following [36–40].

Declaration of competing interest

Acknowledgements

Appendix A. Derivation of the gravitationally dressed S-matrix

We shall derive here a gravitational dressing factor of the S-matrix. This was originally derived in [10] in the light-cone coordinates without the explicit solution of σ. It is instructive to reproduce the factor by using our exact solution of σ with the Cartesian coordinates.

Introducing the dynamical coordinates.   Let us first introduce the dynamical coordinates X^μ defined as

$$X^\mu \equiv -\frac{2}{\Lambda}\partial^\mu\phi = x^\mu + Y^\mu\,, \qquad Y^\mu \equiv -\frac{2}{\Lambda}\partial^\mu\sigma\,. \tag{A.1}$$

The components of Y^μ are explicitly given by

$$Y^t(t,x) = \frac{2}{\Lambda}a_2 + \kappa\,k\left[x\,T_{tx}^{(0)}(t,x) + t\,T_{xx}^{(0)}(t,x)\right]$$
$$+ \frac{\kappa}{\Lambda}(k\Lambda - 2)\left(\int_0^x dx'\,T_{tx}^{(0)}(t,x') + \int_{t_2}^t dt'\,T_{xx}^{(0)}(t',0)\right), \tag{A.2}$$

$$Y^x(t,x) = -\frac{2}{\Lambda}a_3 - \kappa\,k\left[x\,T_{tt}^{(0)}(t,x) + t\,T_{tx}^{(0)}(t,x)\right]$$
$$- \frac{\kappa}{\Lambda}(k\Lambda - 2)\left(\int_0^t dt'\,T_{tx}^{(0)}(t',x) + \int_{x_2}^x dx'\,T_{tt}^{(0)}(0,x')\right), \tag{A.3}$$

where the indices have been lowered in the right-hand side. Then Y^μ satisfies

$$\partial_\mu Y^\nu = -\frac{2}{\Lambda}\partial_\mu\partial^\nu\sigma = \frac{2\kappa}{\Lambda}\left((1 - k\Lambda) - \frac{k\Lambda}{2}x^\rho\partial_\rho\right)(T^{(0)\,\nu}_{\quad\mu} - \delta_\mu^\nu T^{(0)}), \tag{A.4}$$

where we have used (2.4).

In the standard manner, the conserved charge is given by

$$P_\mu \equiv \int_{-\infty}^{\infty} dx \, T_{t\mu}^{(0)}(t, x). \tag{A.5}$$

The total energy P_t and momentum P_x are given by, respectively,

$$P_t = \int_{-\infty}^{\infty} dx \, T_{tt}^{(0)}(t, x), \qquad P_x = \int_{-\infty}^{\infty} dx \, T_{tx}^{(0)}(t, x). \tag{A.6}$$

The conservation law of the energy momentum tensor is given as

$$\partial^\mu T_{\mu\nu}^{(0)} = 0. \tag{A.7}$$

In the Cartesian coordinates, it is expressed as

$$\partial_t T_{tt}^{(0)} = \partial_x T_{tx}^{(0)}, \qquad \partial_t T_{tx}^{(0)} = \partial_x T_{xx}^{(0)}. \tag{A.8}$$

Using the relations in (A.8) and the invariance under the parity-transformation,

$$T_{tx}^{(0)}(t, \infty) = T_{tx}^{(0)}(t, -\infty), \qquad T_{xx}^{(0)}(t, \infty) = T_{xx}^{(0)}(t, -\infty), \tag{A.9}$$

the conservation of the charges P_μ is shown as follows;

$$\partial_t P_t = \int_{-\infty}^{\infty} dx \, \partial_x T_{tx}^{(0)}(t, x) = 0, \qquad \partial_t P_x = \int_{-\infty}^{\infty} dx \, \partial_x T_{xx}^{(0)}(t, x) = 0. \tag{A.10}$$

Note here that Y^μ still contains four arbitrary parameters a_2, a_3, t_2 and x_2. In order to fix the expression of Y^μ definitely, we need to impose some boundary conditions for Y^μ. Then, as a result, (A.2) and (A.3) can be expressed in terms of the conserved charges P_μ.

Let us first impose a boundary condition for the energy momentum tensor as follows:

$$x \, T_{\mu\nu}^{(0)}(t, x) \to 0 \qquad (x \to \pm\infty). \tag{A.11}$$

By using the conservation of $T_{\mu\nu}^{(0)}$ in (A.8), one can obtain the following relations:

$$\int_{t_2}^{t} dt' \, T_{xx}^{(0)}(t', 0) = \int_{-\infty}^{0} dx' \, T_{tx}^{(0)}(t, x') - \int_{-\infty}^{0} dx' \, T_{tx}^{(0)}(t_2, x') + \int_{t_2}^{t} dt' \, T_{xx}^{(0)}(t', -\infty), \tag{A.12}$$

$$\int_{0}^{t} dt' \, T_{tx}^{(0)}(t', x) = \int_{-\infty}^{x} dx' \, T_{tt}^{(0)}(t, x') - \int_{-\infty}^{x} dx' \, T_{tt}^{(0)}(0, x') + \int_{0}^{t} dt' \, T_{tx}^{(0)}(t', -\infty). \tag{A.13}$$

Then Y^μ can be rewritten as

$$\begin{aligned}
Y^t(t, x) = & \frac{2}{\Lambda} a_2 + \kappa \, k \left[x \, T_{tx}^{(0)}(t, x) + t \, T_{xx}^{(0)}(t, x) \right] \\
& + \frac{\kappa}{\Lambda}(k\Lambda - 2) \left(\int_{-\infty}^{x} dx' \, T_{tx}^{(0)}(t, x') - \int_{-\infty}^{0} dx' \, T_{tx}^{(0)}(t_2, x') \right),
\end{aligned} \tag{A.14}$$

$$Y^x(t, x) = -\frac{2}{\Lambda} a_3 - \kappa \, k \left[x \, T_{tt}^{(0)}(t, x) + t \, T_{tx}^{(0)}(t, x) \right]$$
$$- \frac{\kappa}{\Lambda}(k\lambda - 2) \left(\int_{-\infty}^{x} dx' \, T_{tt}^{(0)}(t, x') - \int_{-\infty}^{x_2} dx' \, T_{tt}^{(0)}(0, x') \right). \tag{A.15}$$

Now the unknown constants a_2 and a_3 are determined by using the boundary condition (A.11) as follows:

$$a_2 = \frac{\kappa}{2}(k\Lambda - 2) \int_{-\infty}^{0} dx' \, T_{tx}^{(0)}(t_2, x') + \frac{\Lambda}{2} Y_{(-)}^t, \tag{A.16}$$

$$a_3 = -\frac{\kappa}{2}(k\Lambda - 2) \int_{-\infty}^{x_2} dx' \, T_{tt}^{(0)}(0, x') - \frac{\Lambda}{2} Y_{(-)}^x, \tag{A.17}$$

where we have defined $Y_{(\pm)}^\mu \equiv Y^\mu|_{x \to \pm\infty}$. Using these expressions of $a_{2,3}$, we find that

$$Y^t(t, x) = Y_{(-)}^t + \kappa \, k \left[x \, T_{tx}^{(0)}(t, x) + t \, T_{xx}^{(0)}(t, x) \right] + \frac{\kappa}{\Lambda}(k\Lambda - 2) \int_{-\infty}^{x} dx' \, T_{tx}^{(0)}(t, x'), \tag{A.18}$$

$$Y^x(t, x) = Y_{(-)}^x - \kappa \, k \left[x \, T_{tt}^{(0)}(t, x) + t \, T_{tx}^{(0)}(t, x) \right] - \frac{\kappa}{\Lambda}(k\Lambda - 2) \int_{-\infty}^{x} dx' \, T_{tt}^{(0)}(t, x'). \tag{A.19}$$

Taking $x \to \infty$ and using (A.6) leads to the following relations:

$$Y_{(+)}^t - Y_{(-)}^t = \frac{\kappa}{\Lambda}(k\Lambda - 2) P_x, \qquad Y_{(+)}^x - Y_{(-)}^x = -\frac{\kappa}{\Lambda}(k\Lambda - 2) P_t. \tag{A.20}$$

By employing a parity symmetric prescription, we obtain that

$$Y_{(\pm)}^t = \mp\frac{\kappa}{2\Lambda}(k\Lambda - 2) P_x, \qquad Y_{(\pm)}^x = \pm\frac{\kappa}{2\Lambda}(k\Lambda - 2) P_t. \tag{A.21}$$

It is useful to introduce a new quantity $\widetilde{P}_\mu$ defined as

$$\widetilde{P}_\mu \equiv 2 \int_{-\infty}^{x} dx \, T_{t\mu}^{(0)}(t, x) - P_\mu. \tag{A.22}$$

In the spacial infinity region $x \to \pm\infty$, $\widetilde{P}_\mu$ becomes the conserved charge $\widetilde{P}_\mu \to \pm P_\mu$.

Finally, the dynamical coordinates in (A.1) are expressed in terms of $T_{\mu\nu}^{(0)}$ as follows:

$$X^\mu = x^\mu - \kappa \, k \, (T^{(0)\mu}{}_\nu - \delta_\nu^\mu T^{(0)}) x^\nu - \frac{\kappa}{2\Lambda} (k \, \Lambda - 2) \epsilon^{\mu\nu} \widetilde{P}_\nu. \tag{A.23}$$

Here $\epsilon^{\mu\nu}$ is an antisymmetric tensor normalized as $\epsilon^{tx} = -1$. For simplicity, we will set $k = 0$ in the following discussion. Then, the metric fluctuation $h_{\mu\nu}$ and the quadratic action vanish while σ does not. The dynamical coordinates in (A.23) are simplified as

$$X^\mu = x^\mu + \frac{\kappa}{\Lambda} \epsilon^{\mu\nu} \widetilde{P}_\nu. \tag{A.24}$$

This corresponds to the one obtained in [10].

The gravitationally dressed S-matrix. A significant implication of the dynamical coordinates in (A.23) is the gravitationally dressed S-matrix [10].

Let us consider a scattering process in a scalar field theory. Here the detail of the interaction potential is not necessary. In the infinite past $t \to -\infty$, N_{in} particles are prepared and each of them has a momentum $p_{(i)}$. Then the asymptotic field (in-field) is given by

$$\psi = \int \frac{\mathrm{d}p}{\sqrt{2E}} \frac{1}{2\pi} \left[a_{\text{in}}^\dagger(p) \, \mathrm{e}^{-ip_\mu x^\mu} + \text{h.c.} \right] . \tag{A.25}$$

It is known that a $T\bar{T}$-deformed QFT on the undeformed background is equivalent to the undeformed QFT with the dynamical coordinates [10,41–43]. This statement means that the deformation effect for the asymptotic state can be evaluated by replacing the original coordinates x^μ with the dynamical ones X^μ.

As a result, a creation operator $a_{\text{in}}^\dagger$ gets an extra-phase factor $\mathrm{e}^{ip_\mu Y^\mu}$ and a dressed creation operator can be defined as

$$A_{\text{in}}^\dagger(p) \equiv a_{\text{in}}^\dagger(p) \, \mathrm{e}^{ip_\mu Y^\mu} . \tag{A.26}$$

By employing this dressed operator $A_{\text{in}}^\dagger(p)$ (instead of $a_{\text{in}}^\dagger(p)$), the associated dressed in-state can be defined as

$$\begin{aligned}
\big|\{p_{(i)}\}, \text{ in}\big\rangle_{\text{dressed}} &\equiv \prod_{i=1}^{N_{\text{in}}} A_{\text{in}}^\dagger(p_{(i)}) \, |0\rangle \\
&= \exp\left(i \sum_{i=1}^{N_{\text{in}}} p_{(i)_\mu} Y^\mu(x_{(i)}) \right) \big|\{p_{(i)}\}, \text{ in}\big\rangle .
\end{aligned} \tag{A.27}$$

In the infinite past, $Y^\mu(x_{(i)})$ can be evaluated as follows:

$$\begin{aligned}
Y^\mu(x_{(i)}) &= \frac{\kappa}{\Lambda} \epsilon^{\mu\nu} \left[2 \int_{-\infty}^{x_{(i)}} \mathrm{d}x' \, T_{t\nu}^{(0)}(t, x') - P_\nu \right] \\
&= \frac{\kappa}{\Lambda} \epsilon^{\mu\nu} \left[2 \left(\frac{1}{2} p_{(i)_\nu} + \sum_{j<i} p_{(j)_\nu} \right) - \sum_{i=1}^{N_{in}} p_{(i)_\nu} \right] \\
&= \frac{\kappa}{\Lambda} \epsilon^{\mu\nu} \left(p_{(i)_\nu} + \sum_{j<i} p_{(j)_\nu} - \sum_{j>i} p_{(j)_\nu} \right) .
\end{aligned} \tag{A.28}$$

From the first line to the second line, we have assumed the mid-point prescription. Finally, one can write the dressed state in terms of $p_{(i)}$.

$$\big|\{p_{(i)}\}, \text{ in}\big\rangle_{\text{dressed}} = \exp\left(2i \frac{\kappa}{\Lambda} \sum_{i=1}^{N_{\text{in}}} \sum_{i<j} \epsilon^{\mu\nu} p_{(i)_\mu} p_{(j)_\nu} \right) \big|\{p_{(i)}\}, \text{ in}\big\rangle . \tag{A.29}$$

The phase factor in front of the original in-state is nothing but the gravitational dressing factor. Similarly, the phase factor for the out-state can be evaluated and then the dressed S-matrix can be derived as shown in [10].

Appendix B. The vacuum solution in the deformed AP model

Here, we introduce the general vacuum solution in the Yang-Baxter deformed AP model (3.5). This is a short review of the result obtained in [14].

In the conformal gauge (2.24), the vacuum equations of motion (2.8) and (2.9) can be decomposed into a copy of two Liouville equations and a constraint condition as follows:

$$2\,\partial_+\partial_-\bar\omega_1 + \frac{1}{L^2}e^{2\bar\omega_1} = 0\,, \tag{B.1}$$

$$2\,\partial_+\partial_-\bar\omega_2 + \frac{1}{L^2}e^{2\bar\omega_2} = 0\,, \tag{B.2}$$

$$-e^{2\bar\omega}\partial_\pm\left(e^{-2\bar\omega}\partial_\pm\bar\phi\right) = 0\,. \tag{B.3}$$

Here $\bar\omega_{1,2}$ are defined as

$$\bar\omega_1 \equiv \bar\omega + \eta\left(\frac{2}{L^2}\bar\phi - \Lambda\right)\,, \qquad \bar\omega_2 \equiv \bar\omega - \eta\left(\frac{2}{L^2}\bar\phi - \Lambda\right)\,. \tag{B.4}$$

The general solutions for each of the Liouville equations (B.1) and (B.2) are given by, respectively,

$$e^{2\bar\omega_1} = \frac{2\,L^2\,\partial_+X_1^+\partial_-X_1^-}{\left(X_1^+ - X_1^-\right)^2}\,, \qquad e^{2\bar\omega_2} = \frac{2\,L^2\,\partial_+X_2^+\partial_-X_2^-}{\left(X_2^+ - X_2^-\right)^2}\,. \tag{B.5}$$

Here $X_{1,2}^+$ ($X_{1,2}^-$) are arbitrary holomorphic (anti-holomorphic) functions. Moreover, $X_{1,2}^\pm$ must satisfy the condition (B.3). Since (B.3) can be rewritten by using the Schwarzian derivative Sch{ , },

$$\text{Sch}\{X_1^\pm,\,x^\pm\} - \text{Sch}\{X_2^\pm,\,x^\pm\} = 0\,, \tag{B.6}$$

$X_1^\pm$ and $X_2^\pm$ are the same functions up to an $SL(2)$ transformation. Finally, the general vacuum solution in terms of $\bar\omega$ and $\bar\phi$ are given by $\bar\omega_{1,2}$ as

$$e^{2\bar\omega} = e^{\bar\omega_1 + \bar\omega_2}\,, \qquad \bar\phi = \frac{\Lambda\,L^2}{2} + \frac{L^2}{4\eta}(\bar\omega_1 - \bar\omega_2)\,. \tag{B.7}$$

Example.   As an example, let us consider the following parametrization:

$$X_1^+(x^+) = \frac{(1-\eta\beta)x^+ - 2\eta\alpha}{-2\eta\gamma x^+ + (1+\eta\beta)}\,, \qquad X_1^-(x^-) = x^-\,,$$

$$X_2^+(x^+) = \frac{(1+\eta\beta)x^+ + 2\eta\alpha}{2\eta\gamma x^+ + (1-\eta\beta)}\,, \qquad X_2^-(x^-) = x^-\,, \tag{B.8}$$

where α, β and γ are real constants. This vacuum solution describes the Yang-Baxter deformations of AdS$_2$ as follows:

$$e^{2\bar\omega} = \frac{2L^2\,[1 - \eta^2(\beta^2 + 4\alpha\gamma)]}{(x^+ - x^-)^2 - \eta^2\left(2\alpha + \beta(x^+ + x^-) - 2\gamma x^+x^-\right)^2}\,,$$

$$\bar\phi = \frac{\Lambda\,L^2}{2} + \frac{L^2}{4\eta}\log\left|\frac{x^+ - x^- + \eta\,(2\alpha + \beta(x^+ + x^-) - 2\gamma x^+x^-)}{x^+ - x^- - \eta\,(2\alpha + \beta(x^+ + x^-) - 2\gamma x^+x^-)}\right|\,. \tag{B.9}$$

By taking the undeformed limit $\eta \to 0$, this solution reduces to the general solution in the original AP model (2.26).

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