Classifying the irreducible components of moduli stacks of torsion-free sheaves on K3 surfaces and an application to Brill-Noether theory

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Introduction

Purpose

Irr decomp of **moduli stacks of torsion-free sheaves** of rk 2 on K3 surfaces of $\rho = 1$
&
irr decomp of **Brill-Noether(BN) locus** on Hilbert schs of pts.

※ Moduli stacks can parametrize **unstable sheaves**.

Previous research

- The case of ruled surfaces

→ C.Walter (1995)

- Stratification of moduli stacks

→ V.Hoskins (2018) or T.L.Gomez, I.Sols and A.Zamora (2015)

Mukai vector

X : Proj K3 surf$/\mathbb{C}$ of $\rho = 1$, $E \in \mathrm{Coh}(X)$

1. $\boldsymbol{v(E) := (\mathrm{rk}(E), c_1(E), \mathrm{ch}_2(E) + \mathrm{rk}(E))} \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$
2. $\boldsymbol{\langle v, w \rangle := -[v]_0[w]_2 + [v]_1[w]_1 - [v]_2[w]_0} \in \mathbb{Z}$
   , where $v := ([v]_0, [v]_1, [v]_2) \in \mathbb{Z} \oplus \mathrm{Pic}(X) \oplus \mathbb{Z}$

Moduli stacks

3. $\boldsymbol{\mathscr{M}(v)}$ **:**
   - Ob. **flat family $\mathscr{E}/U$ paramet torsion-free sheaves w/ Mukai vector v**
   - Mor. $(\varphi, \alpha) : \mathscr{E}/U \to \mathscr{E}'/U'$
     $(\varphi : U \to U'$: mor of schs, $\alpha : \mathscr{E} \to (\mathrm{id}_X \times \varphi)^* \mathscr{E}'$: iso)
4. $\mathscr{M}^{\mathrm{HN}}_{(v_1, v_2)}(v) := \left\{ E \in \mathscr{M}(v) \;\middle|\; \begin{array}{l} {}^{\exists}(0 \subset E_1 \subset E) : \text{HN-filtration} \\ \text{s.t. } v(E_1) = v_1, v(E/E_1) = v_2 \end{array} \right\}$
5. $\mathscr{M}^{\mathrm{ss}}(v) := \{ E \in \mathscr{M}(v) \mid E : \text{semistable} \}$

Irreducible decomposition of $\mathscr{M}(v)$

Main Theorem 1

v_0 : primitive Mukai vector
$v := mv_0$ $(m \in \mathbb{Z})$
Assume $[v]_0 = 2$ & v satisfies one of (a) $\sim$ (c)

$$(a) : \langle v, v \rangle > 0, \qquad (b) : \langle v, v \rangle = 0, -2 \text{ and } v \text{ is primitive },$$
$$(c) : \langle v, v \rangle < -2 \text{ and } \langle v_0, v_0 \rangle \neq -2.$$

Then,

$$\mathscr{M}(v) = \begin{cases} \overline{\mathscr{M}^{ss}(v)} \cup \bigcup_{\langle v_1, v_2 \rangle \leq 1} \overline{\mathscr{M}^{HN}_{(v_1, v_2)}(v)} & \text{(a), (b)} \\ \bigcup \overline{\mathscr{M}^{HN}_{(v_1, v_2)}(v)} & \text{(c)} \end{cases}$$

Irreducible decomposition of $\mathscr{M}(v)$

Remark
For proof of Thm 1,

- theory of stratification via HN-filt
- theory of moduli sps of sheaves on K3 surfs by K.Yoshioka

are important.

$\rightsquigarrow$ We get the relation between the stratas by calculating dims etc..

Application to BN theory

Definition (BN locus of Hilbert schs of pts)

D : eff div on X
$N \in \mathbb{N}$ s.t. $N \leq h^0(\mathscr{O}(D))$

$$\boldsymbol{W^i_N(D) := \{Z \in \mathrm{Hilb}^N(X) \mid h^1(\mathscr{I}_Z(D)) \geq i+1\}}$$

Remark

$$H^0(\mathscr{I}_Z(D)) - \{0\}/\mathbb{C}^* = \text{eff divs lin equiv to } D \text{ passing through } Z.$$

For general $Z \in \mathrm{Hilb}^N(X)$,

$$h^0(\mathscr{I}_Z(D)) = h^0(\mathscr{O}_X(D)) - \ell(\mathscr{O}_Z) = \text{expected dimension.}$$

But, for $Z \in W^i_N(D)$,

$$h^0(\mathscr{I}_Z(D)) > h^0(\mathscr{O}_X(D)) - \ell(\mathscr{O}_Z).$$

※ $h^0(\mathscr{I}_Z(D)) = h^0(\mathscr{O}_X(D)) - \ell(\mathscr{O}_Z) + h^1(\mathscr{I}_Z(D))$.

Application to BN theory

Main Theorem 2

$D := nH$, $v := (2, nH, \frac{n^2H^2}{2} - N + 2)$, where H : amp gen of $\mathrm{Pic}(X)$.
If $\langle v, v\rangle > 0$, there exists the following 1 to 1 corresp

$$\left\{\textbf{the irr comps of } \boldsymbol{W^0_N(D)}\right\}$$

$$\textbf{1 to 1} \updownarrow$$

$$\left\{\overline{\mathscr{M}^{\mathbf{HN}}_{(v_1,v_2)}(v)} \;\middle|\; (v_1, v_2) \textbf{ satisfying } (*)\right\} \cup \left\{\overline{\mathscr{M}^{ss}(v)}\right\}$$

$$\subsetneq \{\textbf{the irr comps of } \mathscr{M}(v)\} .$$

$$[v_1]_1, [v_2]_1 \neq 0 : \text{effective}, \langle v_1, v_2\rangle \leq 1, [v_2]_2 \geq -1 \qquad (*)$$

Remark

- If Z : general in $W^0_N(D)$, the corresp is given by ext'n
$$0 \to \mathscr{O}_X \to E \to \mathscr{I}_Z(D) \to 0.$$
- Thm 2 $\rightsquigarrow$ (non-emptyness,) the dims and the num of irr comps of $W^0_N(D)$.

Thank you for listening !