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Recursion Relations on the Power Series Expansion of the Universal Weierstrass Sigma Function

By

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Abstract

The main aim of this paper is an exposition of the theory of Buchstaber and Leykin on the heat equations for the multivariate sigma functions. We treat only the elliptic curve case, but keeping the most general elliptic curve equation, which may be useful for number theoretic applications.

Introduction

Let $(\omega',\,\omega'')$ be a fixed basis of a lattice in $\mathbb{C}$ satisfying $\omega''/\omega' > 0$. Let $\wp(u)$ be the Weierstrass function associated to this lattice, which satisfies the equation

$$\wp'(u)^2 = 4\wp(u)^3 - g_2\wp(u) - g_3, \tag{0.1}$$

where

$$g_2 = 60 \sum_{(n',n'')\neq(0,0)} \frac{1}{(n'\omega'+n''\omega'')^4}, \qquad g_3 = 140 \sum_{(n',n'')\neq(0,0)} \frac{1}{(n'\omega'+n''\omega'')^6}. \tag{0.2}$$

The Weierstrass function $\sigma(u)$ is defined by

$$\sigma(u) = u \exp\biggl(\int_0^u\int_0^u \Bigl(\tfrac{1}{u^2} - \wp(u)\Bigr) du du\biggr). \tag{0.3}$$

It is well-known that the power series expansion of the function $\sigma(u)$ around $u = 0$ has coefficients in $\mathbb{Q}[g_2, g_3]$. Weierstrass [17] gave, by a quite technical method of integration

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from (0.1), a system of equations satisfied by the function $\sigma(u)$, that is

$$\begin{aligned}(0.4)\qquad &\Big(4g_2\frac{\partial}{\partial g_2}+6g_3\frac{\partial}{\partial g_3} \quad -u\frac{\partial}{\partial u}-\frac{1}{2} \quad +\frac{3}{2}\Big)\,\sigma(u)=0,\\ &\Big(6g_3\frac{\partial}{\partial g_2}-\frac{1}{3}{g_2}^2\frac{\partial}{\partial g_3} \quad -\frac{1}{2}\frac{\partial^2}{\partial u^2}+\frac{1}{24}g_2u^2 \quad +0\Big)\,\sigma(u)=0.\end{aligned}$$

Here we have divided the left hand sides into three parts to make it easier to explain these equations later. We note that [11] has an overview of [17]. Using these equations, Weierstrass gave a recursion relation for the coefficients of the power series expansion $\sigma(u)$ around $u=0$. He also gave relations for its expansions around the other two-division points.

On the same year as [17], Frobenius-Stickelberger [6] gave a different method from [17] to the same result. Using

$$(0.5)\qquad \wp(u)=\frac{1}{u^2}+\frac{g_2}{20}u^2+\frac{g_3}{28}u^4+\frac{{g_2}^2}{1200}u^6+\cdots$$

and a corresponding expansion of the Weierstrass function $\zeta(u)$, they obtained the formulae

$$\begin{aligned}(0.6)\qquad &\omega'\frac{\partial g_2}{\partial\omega'}+\omega''\frac{\partial g_2}{\partial\omega''}=-4g_2, \qquad \omega'\frac{\partial g_3}{\partial\omega'}+\omega''\frac{\partial g_3}{\partial\omega''}=-6g_3,\\ &\eta'\frac{\partial g_2}{\partial\omega'}+\eta''\frac{\partial g_2}{\partial\omega''}=-6g_3, \qquad \eta'\frac{\partial g_3}{\partial\omega'}+\eta''\frac{\partial g_3}{\partial\omega''}=-\frac{1}{3}{g_2}^2,\end{aligned}$$

where $\eta'=\zeta(u+\omega')-\zeta(u)$, $\eta''=\zeta(u+\omega'')-\zeta(u)$ are independent of u. Then we have

$$(0.7)\quad \omega'\frac{\partial}{\partial\omega'}+\omega''\frac{\partial}{\partial\omega''}=-4g_2\frac{\partial}{\partial g_2}-6g_3\frac{\partial}{\partial g_3}, \quad \eta'\frac{\partial}{\partial\omega'}+\eta''\frac{\partial}{\partial\omega''}=-6g_3\frac{\partial}{\partial g_2}-\frac{1}{3}{g_2}^2\frac{\partial}{\partial g_3}.$$

Moreover, they obtained (p.318 in [7])

$$(0.8)\qquad \omega'\frac{\partial\eta'}{\partial\omega'}+\omega''\frac{\partial\eta'}{\partial\omega''}=\eta', \quad \omega'\frac{\partial\eta''}{\partial\omega'}+\omega''\frac{\partial\eta''}{\partial\omega''}=\eta''.$$

Consequently, on p.326 of [7], they give the same system of heat equations as in [17]. Since the Weierstrass sigma function is expressed by using Jabobi theta series and Dedekind eta function as

$$(0.9)\qquad \sigma(u)=-\boldsymbol{\eta}({\omega'}^{-1}\omega'')^{-3}\cdot\frac{\omega'}{2\pi}\cdot\exp\big(\tfrac{1}{2}u^2\eta'{\omega'}^{-1}\big)\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}({\omega'}^{-1}u|{\omega'}^{-1}\omega''),$$

the equations (0.4) must be given by rewriting the famous heat equation

$$(L-H)\,\vartheta\begin{bmatrix}b\\ a\end{bmatrix}(z,\tau)=0,$$

where $L=4\pi\boldsymbol{i}\frac{\partial}{\partial\tau}$ and $H=\frac{\partial^2}{\partial z^2}$. In order to do so, we need (0.7). We understand this situation as follows: the operation of the left hand side of (0.7) described in terms of ω', ω', η', η'' on (0.9) with (0.8) just corresponds to the operation of the right hand side of

(0.7) described in terms of g_2, g_3 to (0.3) through the expansion (0.5) of $\wp(u)$.

Since, unfortunately, there is no object in the higher genus case corresponding to the Eisenstein series (0.2), it is impossible to generalise the work of Frobenius and Stickelberger as well as that of Weierstrass'.

However, the pioneering work [3] by Buchstaber and Leykin gave a quite general method which is applicable for higher genus cases. Roughly speaking, their idea is that the left hand side of (0.7) is given by looking at the behaviour of operations of the right hand side of (0.7) to the first de Rham cohomology of the curve.

At the conference "Mathematical structures observed from the theory of integrable systems and its applications" organised by Shinsuke Iwao, held at RIMS on September 2018, one of the authors (Y.Ô.) gave a talk on the theory of Buchstaber-Leykin (BL)-theory, in higher genus cases. In this article we restrict ourselves to the genus one case, since a paper on the results in higher genus cases by the authors and coworkers is now submitted for publication ([5]). However, we treat an elliptic curve which is more general than the Weierstrass form (0.1) treated in [3] and [5], since some elliptic curves defined over certain fields do not have a model of this form. The more general form we consider is

$$\mathscr{E}\ :\ y^2 = (a_1x + a_3)y + (x^3 + a_2x^2 + a_4x + a_6), \tag{0.10}$$

and this form is sufficient to define any elliptic curve over any field. Here, take care that the choice of the opposite signs in front of a_1 and a_3 is not usual. The sigma function attached to (0.10) is expanded as

$$\begin{aligned}\sigma(u) = u &+ \big(a_2 + (\tfrac{a_1}{2})^2\big)\tfrac{u^3}{3!} + \big(a_2{}^2 + 2(\tfrac{a_1}{2})^2 a_2 + (\tfrac{a_1}{2})^4 + a_3a_1 + 2a_4\big)\tfrac{u^5}{5!}\\ &+ \big(a_2{}^3 + 3(\tfrac{a_1}{2})^2 a_2{}^2 + (3(\tfrac{a_1}{2})^4 + 3a_3a_1 + 6a_4)a_2\\ &\qquad + (\tfrac{a_1}{2})^6 + 6a_3(\tfrac{a_1}{2})^3 + 6a_4(\tfrac{a_1}{2})^2 + 6a_3{}^2 + 24a_6\big)\tfrac{u^7}{7!} + \cdots .\end{aligned} \tag{0.11}$$

Though this expansion is obtained by the usual transformation from the Weierstrass form to the universal form, BL-theory works well also for the universal case naively. That this is the first reason why we choose the universal case for this article. The authors wish to explain the essential idea of BL-theory by restricting to the genus one case. This is the second reason. Thirdly, in order to describe BL-theory, we need to define the sigma function concretely. However, the definition of the over all constant multiplicative factor in the front of the sigma function is slightly uncertain when we define it by using theta series, except for genus one and two cases.

The solution space of the system of heat equations given by BL-theory seems to be of one dimensional, as in Weierstrass' original case. However, the reason of this is not completely clear to the authors. We note that BL-theory possibly gives a good explanation why the constant multiple factor in the definition of the sigma in terms of theta series is a minus eighth root of the discriminant of the curve as is explained in

Lemma 4.17 of [2] (see also [5]).

To conclude, we outline the structure of this article. Section 2 is devoted to preliminaries. The left hand sides of (0.4) is divided into three parts as $L_i - H_i + \frac{1}{8}L_i(\log \varDelta)$ $(i = 0,\ 2)$. After explaining general heat equations in Section 3 we construct L_i in Subsections 4.1 and in 4.2, H_i and $L_i(\log \varDelta)$ in Subsection 4.3. We give a final form in Subsection 4.4.

One of the authors (Y.Ô.) had an opportunity to discuss this work with S. Matsutani. His valuable comments are deeply appreciated by the authors.

Contents

Notation Convention. As usual, we denote by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ the ring of rational integers, the field of rational numbers, the field of real numbers. and the field of complex numbers. We denote by $\mathrm{Mat}(n, R)$ be the ring of square matrices of size n over some ring R. For any matrix A which is operated by an operator L, we denote by $L(A)$ the matrix obtained by operating L for each entries in A. For operators L_1, $\cdots$, L_n which operate an element z, we define $(L_1, \cdots, L_n)(z) = (L_1(z), \cdots, L_n(z))$.

1 Preliminaries

In this section, we recall the fundamental setting from [12].

1.1 The most general elliptic curve

As mentioned in the introduction, we start at the most general elliptic curve defined by

$$\mathscr{E} : y^2 = (a_1x + a_3)y + x^3 + a_2x^2 + a_4x + a_6 \tag{1.1}$$

with the point ∞ at infinity which is the unity of its group structure. Note that the signs in front of a_1 and a_3 are non-standard.

We assume that the a_j are variables or indeterminates. When we discuss the Riemann surface corresponding to this curve, for instance on its periods et cetera, we assume that the a_j are independent complex variables. We use the following notation:

$$\begin{aligned} f(x,y) &= y^2 - (a_1x + a_3)y - (x^3 + a_2x^2 + a_4x + a_6), \\ f_1(x,y) &= \tfrac{\partial}{\partial x} f(x,y) = -a_1y - (3x^2 + 2a_2x + a_4), \\ f_2(x,y) &= \tfrac{\partial}{\partial y} f(x,y) = 2y - (a_1x + a_3). \end{aligned} \tag{1.2}$$

We define the weight, which is denoted by wt, as $\mathrm{wt}(x) = -2$, $\mathrm{wt}(y) = -3$, $\mathrm{wt}(a_j) = -j$. Then all objects and equations in this paper are of homogeneous weight. Let

$$\omega_1(x,y) = \frac{dx}{f_y(x,y)} = \frac{dx}{2y - a_1x - a_3}.$$

This is the standard basis of the space of holomorphic differential forms (the differential forms of the first kind). The differential forms ω_1 and the form

$$\eta_1(x,y) = \frac{-xdx}{2y - a_1x - a_3}$$

of the second kind form a natural symplectic basis of the first de Rham cohomology explained later. Using $t = x/y$, it is easy to get (see [12] for example)

$$\begin{aligned} \omega_1(x,y) &= \frac{dx}{2y - a_1x - a_3} = \frac{\frac{dx}{dt}dt}{2y - a_1x - a_3} \in (1 + t\,\mathbb{Z}[\tfrac{1}{2}, \boldsymbol{a}][[t]])dt, \\ \omega_1(x,y) &= -\frac{dy}{f_x(x,y)} \in (1 + t\,\mathbb{Z}[\tfrac{1}{3}, \boldsymbol{a}][[t]])dt. \end{aligned} \tag{1.3}$$

Therefore, we have

$$\begin{aligned} \omega_1(x,y) &= (1 - a_1t + (a_2 + a_1{}^2)t^2 + \cdots)dt \in (1 + t\,\mathbb{Z}[\boldsymbol{a}][[t]])dt, \\ \eta_1(x,y) &= (-t^{-2} + a_3t - (a_4 + 2a_1a_3)t^2 - \cdots)dt \in (-t^{-2} + t\,\mathbb{Z}[\boldsymbol{a}][[t]])dt. \end{aligned} \tag{1.4}$$

These expansions are used to compute the symplectic product in the first de Rham cohomology later.

1.2 The Legendre relation

While in this paper we treat the curve $\mathscr{E}$ mainly defined over $\mathbb{C}$, it is natural to be regarded as a scheme over $\operatorname{Spec}\mathbb{Q}[\boldsymbol{a}]$. However, for example, though the period matrix Ω defined later can not be treated algebraically, the power series expansion of $\sigma(u)$ we finally obtain is an expansion over the ring $\mathbb{Q}[\boldsymbol{a}]$. Supposing $d\,\mathbb{Q}[\boldsymbol{a}]=0$, we define

$$H^1_{\mathrm{dR}}(\mathscr{E}/\mathbb{Q}[\boldsymbol{a}]) = \frac{\big\{\,h(x,y)dx/f_2(x,y)\,\big|\,h(x,y)\in\mathbb{Q}[\boldsymbol{a},x,y]/(f)\big\}}{d\big(\mathbb{Q}[\boldsymbol{a},x,y]/(f)\big)}, \tag{1.5}$$

where d is the standard derivation. which is a $\mathbb{Q}[\boldsymbol{a}]$-module of rank 2 generated by ω_1 and η_1. In this paper we frequently switch between regarding the a_js as indeterminates or complex numbers.

In Lemma 3.1 and following, we see that $H^1_{\mathrm{dR}}(\mathscr{E}/\mathbb{Q}[\boldsymbol{a}])$ is a $\mathbb{Q}[\boldsymbol{a},\frac{\partial}{\partial\boldsymbol{a}}]$-module, where $\frac{\partial}{\partial\boldsymbol{a}}$ stands for the set of $\frac{\partial}{\partial a_j}$s, which are explained in and after 3.1. This module is canonically isomorphic to

$$\big\{\,h(x,y)\,dx\,\big|\,h(x,y)\in\operatorname{Frac}\big(\mathbb{Q}[\boldsymbol{a},x,y]/(f)\big)\big\}/d\big(\operatorname{Frac}\big(\mathbb{Q}[\boldsymbol{a},x,y)]/(f)\big)\big),$$

where Frac stands for taking fractional field, though this fact is not used in this paper. We shall denote the module (the first de Rham cohomology) by $H^1_{\mathrm{dR}}(\mathscr{E}/\mathbb{Q}[\boldsymbol{a}])$ through out this paper (see [10] or [13]). We choose a symplectic basis α, β of the first homology group $H_1(\mathscr{E},\mathbb{Z})$ with intersection products $\alpha{\cdot}\beta=-\beta{\cdot}\alpha=1$, $\alpha{\cdot}\alpha=\beta{\cdot}\beta=0$. The regular polygon obtained from the Riemann $\mathscr{E}$ with respect to the cutting loops α, β is denoted by $\mathscr{E}^{\circ}$. Then we have the natural alternating non-degenerate bilinear form

$$H^1_{\mathrm{dR}}(\mathscr{E}/\mathbb{Q}[\boldsymbol{a}])\times H^1_{\mathrm{dR}}(\mathscr{E}/\mathbb{Q}[\boldsymbol{a}])\ \longrightarrow\ \mathbb{Q}[\boldsymbol{a}],\qquad (\omega,\,\eta)\ \longmapsto\ \omega\star\eta=\operatorname*{Res}_{P\in\mathscr{E}^{\circ}}\Big(\int_{\infty}^{P}\omega\Big)\eta(P).$$

The forms ω_1 and η_1 in (1.3) form a symplectic basis of $H^1_{\mathrm{dR}}(\mathscr{E}/\mathbb{Q}[\boldsymbol{a}])$ with respect to this bilinear form. Indeed, the products for these forms are given by using (1.4) as

$$\begin{aligned}\omega_1\star\eta_1&=\operatorname{Res}\Big(\int_0^t\omega_1(t)\Big)\eta_1(t)=\operatorname{Res}\big((t+\cdots)(-t^{-2}+a_3t+\cdots)dt\big)=-1,\\ \eta_1\star\omega_1&=\operatorname{Res}\Big(\int_{t_0}^t\eta_1(t)\Big)\omega_1(t)=\operatorname{Res}\big((t^{-1}+\cdots)(1+\cdots)\big)=1,\end{aligned}\tag{1.6}$$

and likewise

$$\omega_1\star\omega_1=0,\eta_1\star\eta_1=0.$$

We introduce the periods

$$\omega'=\int_{\alpha}\omega_1(x,y),\quad \omega''=\int_{\beta}\omega_1(x,y)$$

by using those objects defined above. Then, from our choice of α and β and ω_1 and η_1 we see the imaginary part of ω''/ω' is positive (Riemann's inequality). Then the

$\mathbb{Z}$-module

$$\Lambda = \omega'\mathbb{Z} + \omega''\mathbb{Z} \tag{1.7}$$

gives a lattice in $\mathbb{C}$. Moreover, we introduce the periods

$$\eta' = \int_\alpha \eta_1(x,y), \quad \eta'' = \int_\beta \eta_1(x,y),$$

and the matrix of periods defined by

$$\Omega = \begin{bmatrix} \omega' \; \omega'' \\ \eta' \; \eta'' \end{bmatrix}. \tag{1.8}$$

Since (ω_1, η_1) is a symplectic basis, we see that

$$-1 = \operatorname{Res}\Big(\int_0^t \omega_1(t)\Big)\eta_1(t) = \int_{\partial\mathscr{E}_\circ}\Big(\int_0^t \omega_1(t)\Big)\eta_1(t) = \frac{1}{2\pi\boldsymbol{i}}(\omega'\eta'' - \omega''\eta''),$$

and we have the Legendre relation

$$-\det(\Omega) = \omega''\eta' - \omega'\eta'' = 2\pi\boldsymbol{i}. \tag{1.9}$$

For any $u \in \mathbb{C}$, we define u', $u'' \in \mathbb{R}$ by $u = \omega' u' + \omega'' u''$. Likewise, for $\ell \in \Lambda$, we write $\ell = \omega'\ell' + \omega''\ell''$. Let ∞ and I_1 be the initial point and the tail point, respectively, of the side α of $\mathscr{E}^\circ$. Then the Riemann constant vector of $\mathscr{E}$ is given by

$$\delta_1 = -\tfrac{1}{2}\tau - \int_\infty^{I_1} \hat\omega_1 + \int_\alpha \Big(\int_\infty^P \hat\omega_1\Big)\hat\omega_1(P), \quad \delta = \omega'\,\delta_1,$$

where $\tau = {\omega'}^{-1}\omega''$, $\hat\omega_1 = {\omega'}^{-1}\omega_1$. Then δ_1 is independent of the values of the a_js and is given by (see [13], pp.76–80, for instance)

$$\delta = \delta'\omega' + \delta''\omega'', \quad [\delta' \; \delta''] = [\tfrac{1}{2}, \tfrac{1}{2}].$$

This δ gives the theta characteristic of the theta function by which the sigma function is defined (see (2.2) of the next section). Using the above notation, we define a linear form $L(\ ,\ )$ on $\mathbb{C}\times\mathbb{C}$ by

$$L(u,v) = u\,(v'\eta' + v''\eta''). \tag{1.10}$$

This is $\mathbb{C}$-linear with respect to the first variable, and $\mathbb{R}$-linear with respect to the second one. Moreover, we define, for any $\ell \in \Lambda$,

$$\chi(\ell) = \exp\big(2\pi\boldsymbol{i}(\delta'\ell'' - \delta'\ell' + \tfrac{1}{2}\ell'\ell'')\big) = \exp\big(\pi\boldsymbol{i}(\ell' + \ell'' + \ell'\ell'')\big) \in \{1, -1\}. \tag{1.11}$$

Remark 1.12. The difference of twice the genus of a given curve over $\mathbb{C}$ and the number of parameters of some deformations of the curve is called the *modality* of the deformation. Our curve $\mathscr{E}$ as a deformation parameterised by b_4 and b_6 of (2.6) below is of modality 0. For more detailed discussion, see [5]. Modality is very important when we investigate the behaviour of the heat equations of multivariate sigma functions.

2 The sigma function

2.1 Analytic construction of the sigma function

In this section, we recall the definition of $\sigma(u)$ and its properties.

Definition 2.1. We denote the Dedekind eta function by $\boldsymbol{\eta}(\tau)$, namely

$$\boldsymbol{\eta}(\tau) = e^{\frac{\pi \boldsymbol{i}\tau}{12}} \prod_{n=1}^{\infty}(1 - e^{2\pi \boldsymbol{i} n\tau}), \quad (\tau \in \mathbb{C},\ \mathrm{Im}(\tau) > 0).$$

The *sigma function* $\sigma(u)$ on $\mathbb{C}$ associated with the curve $\mathscr{E}$ is defined by[1]

$$\sigma(u) = -\boldsymbol{\eta}({\omega'}^{-1}\omega'')^{-3} \cdot \frac{\omega'}{2\pi} \cdot \exp\left(\tfrac{1}{2}u^2\eta'{\omega'}^{-1}\right)\vartheta\begin{bmatrix}\delta''\\ \delta'\end{bmatrix}({\omega'}^{-1}u \big| {\omega'}^{-1}\omega''). \tag{2.2}$$

This function $\sigma(u)$ has the following properties.

Proposition 2.3. *Suppose that the* $\{a_j\}$ *are constants in* $\mathbb{C}$ *and* Δ *is not zero. Then* $\sigma(u)$ *satisfies the following:*
(1) $\sigma(u)$ *is an entire function on* $\mathbb{C}$;
(2) $\sigma(u)$ *is an odd function, namely,* $\sigma(-u) = -\sigma(u)$;
(3) $\sigma(u + \ell) = \chi(\ell)\,\sigma(u) \exp L(u + \frac{1}{2}\ell, \ell)$ *for any* $u \in \mathbb{C}$ *and* $\ell \in \Lambda$, *where* Λ, L, *and* χ *are defined in* (1.7), (1.10), *and* (1.11), *respectively*;
(4) $\sigma(u) = 0 \iff u \in \Lambda$;
(5) *The power series expansion of* $\sigma(u)$ *around* $u = 0$ *is of the form* $\sigma(u) = u + O(u^3)$, *belongs to* $\mathbb{Q}[\boldsymbol{a}][[u]]$, *and absolutely converges for all* u.

Proof. (1) This is obvious from the definition (2.2). (2) This follows from the fact that the theta series in (2.2) is known to be an odd function. (3) This is shown by using (1.9) and the translational formula of the theta series (see [13] Lemma 6.47(i), for example). (4) The zeroes of the theta series in (1.9) are all simple and only at Λ which is well-known (see [15], pp.167–168). (5) The reader will find more general proof in [10]. However, we shall prove this property as follows. We define

$$\wp(u) := -\frac{d^2}{du^2}\log\sigma(u). \tag{2.4}$$

By 2.3(1), (2), (3), and (4), we see $\wp(u)$ has only poles of order 2 at any points Λ. For any $u \in \mathbb{C}$, there exists a unique point $(x, y) \in \mathscr{E}$ such that $u = \int_\infty^{(x,y)} \omega_1$. Hence, we have a function $u \mapsto x$, which is denoted by $x(u)$, has the same poles as $\wp(u)$ and its Laurent expansion at the origin is of the form $\frac{1}{u^2} + \cdots$. Therefore, $\wp(u) = x(u)$. Using $du = \omega_1$ and (1.4), we see that $x(u)$ has a Laurent expansion around $u = 0$ with all the coefficients in $\mathbb{Q}[\boldsymbol{a}]$ (see [12] for details). Integrating this, we see $\sigma(u) \in \mathbb{Q}[\boldsymbol{a}][[u]]$. □

[1] Here, $\vartheta\begin{bmatrix}b\\ a\end{bmatrix}(z|\tau) = \sum_{n\in\mathbb{Z}} \exp\left[2\pi\boldsymbol{i}\{\tfrac{1}{2}(n+b)^2\tau + (n+b)(z+a)\}\right]$

Remark 2.5.

(1) The properties 2.3 (1), (2), (3), (4), for example, characterise $\sigma(u)$.

(2) The expansion $\sigma(u)$ at $u=0$ is Hurwitz integral over $\mathbb{Z}[\frac{1}{2}a_1, a_2, a_3, a_4, a_6]$, and the expansion of $\sigma(u)^2$ at $u=0$ is Hurwitz integral over $\mathbb{Z}[\boldsymbol{a}]$. We refer the reader the paper [14] for the definition of Hurwitz integrality and these properties. However, our result (a recursion relation of the coefficients) shows that the expansion of $\sigma(u)$ is Hurwitz integral over $\mathbb{Z}[\frac{1}{2}, \frac{1}{3}, \boldsymbol{a}]$. This was remarked by Buchstaber.

2.2 Remark on the Weierstrass form

We recall the relation of our functions to Weierstrass' original ones from [12]. As usual, letting

$$y = Y - \tfrac{1}{2}(a_1x + a_3), \quad x = X - \tfrac{1}{12}({a_1}^2 + 4a_2),$$

and

$$\begin{aligned} b_4 &= -\big(\tfrac{1}{48}{a_1}^4 + \tfrac{1}{6}a_2{a_1}^2 - \tfrac{1}{2}a_3a_1\tfrac{1}{3}{a_2}^2 - a_4\big),\\ b_6 &= -\big(- \tfrac{1}{864}{a_1}^6 - \tfrac{1}{72}a_2{a_1}^4 + \tfrac{1}{24}a_3{a_1}^3\\ &\quad (-\tfrac{1}{18}{a_2}^2 + \tfrac{1}{12}a_4){a_1}^2 + \tfrac{1}{6}a_2a_3a_1 - \tfrac{1}{4}{a_3}^2 - \tfrac{2}{27}{a_2}^3 + \tfrac{1}{3}a_4a_2 - a_6\big), \end{aligned} \tag{2.6}$$

the equation $f(x,y)=0$ is transformed to $Y^2 - (X^3 + b_4X + b_6) = 0$. We denote the Weierstrass $\wp$- and σ-functions with respect to $g_2 = -4b_4$, $g_3 = -4b_6$ by $\wp_{\mathrm{W}}(u)$ and $\sigma_{\mathrm{W}}(u)$, respectively, we have relations

$$\wp(u) = \wp_{\mathrm{W}}(u) - \tfrac{{a_1}^2+4a_2}{12}, \qquad \sigma(u) = \sigma_{\mathrm{W}}(u)\cdot \exp\big(\tfrac{{a_1}^2+4a_2}{24}u^2\big) \tag{2.7}$$

with (2.4) and (2.2). Since $\wp'(u) = 2y + a_1x + a_3$, we have also that

$$\wp'(u)^2 = 4\wp(u)^3 + ({a_1}^2 + 4a_2)\wp(u)^2 + (4a_4 + 2a_1a_3)\wp(u) + (4a_6 + {a_3}^2).$$

2.3 The discriminant Δ

The discriminant Δ of $\mathscr{E}$ is one of the simplest polynomials of the coefficients a_js of its defining equation $f(x,y)=0$ such that $\Delta \neq 0$ if and only if $\mathscr{E}$ is smooth (non-singular). It is uniquely determined up to a sign (see [16] pp.42-47, for instance). It is given by

$$\begin{aligned} \Delta = &-a_6{a_1}^6 + a_3a_4{a_1}^5 + ((-{a_3}^2 - 12a_6)a_2 + {a_4}^2){a_1}^4\\ &+ (8a_3a_4a_2 + {a_3}^3 + 36a_6a_3){a_1}^3 + ((-8{a_3}^2 - 48a_6){a_2}^2 + 8{a_4}^2a_2\\ &+ (-30{a_3}^2 + 72a_6)a_4){a_1}^2 + (16a_3a_4{a_2}^2 + (36{a_3}^3 + 144a_6a_3)a_2 - 96a_3{a_4}^2)a_1\\ &+ (-16{a_3}^2 - 64a_6){a_2}^3 + 16{a_4}^2{a_2}^2a + (72{a_3}^2 + 288a_6)a_4a_2\\ &- 64{a_4}^3 - 27{a_3}^4 - 216a_6{a_3}^2 - 432{a_6}^2. \end{aligned}$$

which is of homogeneous weight $\mathrm{wt}(\Delta) = -12$. It relates to the Dedekind eta function as

$$\Delta = \left(\frac{2\pi}{\omega'}\right)^{12} \boldsymbol{\eta}({\omega'}^{-1}\omega'')^{24}, \quad \boldsymbol{\eta}({\omega'}^{-1}\omega'')^{-3} = \left(\frac{2\pi}{\omega'}\right)^{\frac{3}{2}} \Delta^{-\frac{1}{8}}.$$

It is well-known that (see [15], p.176, line. 20)

$$2\pi\cdot\boldsymbol{\eta}({\omega'}^{-1}\omega'')^3 = \frac{d}{dz}\,\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z|{\omega'}^{-1}\omega'')\Big|_{z=0} = \omega'\frac{d}{du}\,\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}({\omega'}^{-1}u|{\omega'}^{-1}\omega'')\Big|_{u=0}. \tag{2.8}$$

Therefore, the expansion around $u = 0$ of the right hand side of (2.2) is of the form $u + \cdots$. Moreover, we have

Lemma 2.9. (2.2) The function $\sigma(u)$ is independent of the choice of a symplectic base α, β of $H_1(\mathscr{E}, \mathbb{Z})$.

Proof. See [12] or 6.47 (ii) in [13]. □

3 Theory of heat equations

3.1 Operators acting on the first de Rham cohomology

This and the following sections are devoted to explaining the theory of Buchstaber and Leykin [3], on the differentiation of Abelian functions with respect to their parameters, as clearly as we can. That generalises the work of Frobenius and Stickelberger [7], discussed above, on the elliptic case of this problem.

In higher genus case, we do not have any facts corresponding to (0.5) and (0.2). We explain the method given by [3] which implies (0.7).

We firstly investigate a general operator in $\mathbb{Q}[\boldsymbol{a}][\partial_{\boldsymbol{a}}]$. Here, the symbol $\partial_{\boldsymbol{a}}$ means the set $\{\frac{\partial}{\partial a_j}\}$. However, we shall explain the operator $\frac{\partial}{\partial a_j}$. To do so, the following Lemma is required. Let K be a field and R be a function field of one variable with K the field of coefficients. We will apply this lemma for $K = \mathbb{Q}(\boldsymbol{a})$. Take a transcendental element ξ in R over K and fix it. Let D_ξ be the unique derivation on R satisfying

$$D_\xi(\xi) = 0 \quad \text{and} \quad D_\xi(\omega) = D_\xi\Big(\frac{\omega}{d\xi}\Big)d\xi \quad \text{for any } \omega \in R\,d\xi.$$

Let ζ be another transcendental element in R over K. Then we have the following relation between D_ξ and D_ζ, which is know from Chevalley.

Lemma 3.1. (Chevalley [4]) *For any $w \in R$, we have*

$$D_\xi(wd\xi) - D_\zeta(wd\xi) = d(-wD_\zeta\xi).$$

Proof. See Lemma 1 in [8], which is included in [5] also. □

By Lemma 3.1, any element in $\mathbb{Q}[\boldsymbol{a}][\partial_{\boldsymbol{a}}]$ operates linearly on the space $H^1_{\mathrm{dR}}(\mathscr{E}/\mathbb{Q}[\boldsymbol{a}])$ of (1.5). Let L be a first order operator, namely, an element in $\bigoplus_j \mathbb{Q}[\boldsymbol{a}]\frac{\partial}{\partial a_j}$. We define

$\Gamma^L \in \mathrm{Mat}(2, \mathbb{Q}[\boldsymbol{a}])$ as the transpose of the representation matrix with respect to the basis $\boldsymbol{\omega} = (\omega_1, \omega_{-1})$ for the following action of L:

$$L(\boldsymbol{\omega}) = \boldsymbol{\omega}\, {}^t(\Gamma^L). \tag{3.2}$$

In [3], the matrix ${}^t(\Gamma^L)$ defined by (3.2) is called the *Gauss-Manin connection* for the vector field L. Accordingly, by integrating (3.2) along α and β in $H_1(\mathscr{E}, \mathbb{Z})$, we get the natural action

$$L(\varOmega) = \Gamma^L\, \varOmega \quad \text{of } L \text{ on } \varOmega = \begin{bmatrix} \omega' & \omega'' \\ \eta' & \eta'' \end{bmatrix}. \tag{3.3}$$

This is none other than the generalisation (0.7) of Frobenius-Stickelberger. Of course, since $\varOmega$ is the period matrix of a symplectic basis, these elements must satisfy

$$ {}^t\varOmega J \varOmega = 2\pi i J, \quad \text{where } J = \begin{bmatrix} & 1_g \\ -1_g & \end{bmatrix} \tag{3.4}$$

by (1.6). By this equation, we see how L operates on the field $\mathbb{Q}(\omega', \omega'')$, as follows. Operating L on both sides of (3.4), we see that the matrix Γ^L satisfies

$$ {}^t\Gamma^L J + J\Gamma^L = 0, \quad \text{i.e. } {}^t(\Gamma^L J) = \Gamma^L J. \tag{3.5}$$

Thus we may write

$$-\Gamma^L J = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}, \quad \Gamma^L = \begin{bmatrix} -\beta & \alpha \\ -\gamma & \beta \end{bmatrix} \in \mathrm{Mat}(2, \mathbb{Q}[\boldsymbol{a}]). \tag{3.6}$$

In order to be able to generalise our calculations to other curves, we define here g to be the genus of the curve, which is equal to 1 in the simple example considered in this paper. Now in an overload of notation, the α and β above are $g \times g$ matrices, scalars in the current $g = 1$ case, unconnected with the symplectic basis α, β described in Subsection 1.2.

Remark 3.7. We use a different notation from pp.273–274 of [3], for $D(x, y, \lambda)$, Ω, Γ, and β. Our $\boldsymbol{\omega}$ equals to $D(x, y, \lambda)$ by changing the sign on the latter half entries. The others are naturally modified according to this difference and taking transposes. We will give a detailed comparison of our notation with theirs in [5].

Conversely, starting from a matrix

$$\Gamma = \begin{bmatrix} -\beta & \alpha \\ -\gamma & \beta \end{bmatrix} \in \mathfrak{sp}\big(2, \mathbb{Q}[\boldsymbol{a}]\big) = \mathfrak{sl}\big(2, \mathbb{Q}[\boldsymbol{a}]\big),$$

we get uniquely an operator $L \in \bigoplus_j \mathbb{Q}[\boldsymbol{a}] \frac{\partial}{\partial a_j}$ such that $\Gamma^L = \Gamma$. This is a natural generalisation of the situation investigated by Frobenius-Stickelberger [7].

3.2 The primary heat equation

We shall proceed with the heat equations satisfied by the sigma function. We first recall the equation

$$(L-H)\,\vartheta\begin{bmatrix} b \\ a \end{bmatrix}(z,\tau)=0,$$

where $L=4\pi\boldsymbol{i}\frac{\partial}{\partial\tau}$ and $H=\frac{\partial^2}{\partial z^2}$. However, we remark that the same equation holds for each individual term $\exp 2\pi\boldsymbol{i}\big(\frac{1}{2}\tau(n+b)^2+(n+b)(z+a)\big)$ of the sum in the theta series. Our task is to convert the equation above to one satisfied by the sigma function.

We take again an arbitrary $L\in\bigoplus_j\mathbb{Q}[\boldsymbol{a}]\frac{\partial}{\partial a_j}$ as in the previous section. Then we have its associated symmetric matrix $-\Gamma^L J=\begin{bmatrix}\alpha & \beta\\ \beta & \gamma\end{bmatrix}$. Using this matrix, we define the following differential operator H^L of order 2 :

$$H^L=\tfrac{1}{2}\,[\,\tfrac{d}{du}\quad u\,]\begin{bmatrix}\alpha & \beta\\ \beta & \gamma\end{bmatrix}\begin{bmatrix}\frac{d}{du}\\ u\end{bmatrix}+\tfrac{1}{2}\,\mathrm{Tr}\,\beta=\tfrac{1}{2}\,\alpha\,\tfrac{d^2}{du}+\beta\,u\tfrac{d}{du}+\tfrac{1}{2}\,\gamma\,u^2+\tfrac{1}{2}\,\beta, \tag{3.8}$$

where in the genus 1 case, $\mathrm{Tr}\,\beta=\beta$. This notation is used to generalise to the higher genus case as required. Taking any fixed $b=b'\omega'+b''\omega''\in\mathbb{C}$, we define

$$G(b,u,\varOmega)=\left(\frac{2\pi}{\omega'}\right)^{\frac{1}{2}}\exp\big(-\tfrac{1}{2}u\eta'\omega'^{-1}u\big)\exp\Big(2\pi\boldsymbol{i}\big(\tfrac{1}{2}b''\omega'^{-1}\omega''b''+b''(\omega'^{-1}u+b')\big)\Big). \tag{3.9}$$

Then we have the following equation satisfied by this function.

Theorem 3.10. *Let L be an element in $\bigoplus_j\mathbb{Q}[\boldsymbol{a}]\frac{\partial}{\partial a_j}$ and H^L be defined as above. Then, for the function $G(b,u,\varOmega)$ in* (3.9)*, we have*

$$(L-H^L)\,G(b,u,\varOmega)=0. \tag{3.11}$$

Proof. By a straightforward calculation using the Legendre relation (3.4), we see that both $L\log G(b,u,\varOmega)$ and $H^L\log G(b,u,\varOmega)$ give

$$\tfrac{1}{2}u\omega'^{-1}\eta'\alpha\eta'\omega'^{-1}u-2\pi\boldsymbol{i}\,b''\omega'^{-1}\alpha\eta'\omega'^{-1}u-2\pi^2\,b''\omega'^{-1}\alpha\omega'^{-1}b''$$
$$-\tfrac{1}{2}\alpha\eta'\omega'^{-1}+2\pi\boldsymbol{i}\,b''\omega'^{-1}\beta u-u\omega'^{-1}\eta'\beta u+\tfrac{1}{2}\mathrm{tr}\beta+\tfrac{1}{2}u\gamma u.$$

Hence the desired equation. □

Since the equation (3.11) is the starting point of the following theory, we call this the *primary heat equation.*

Corollary 3.12. *Using functions of the form* (3.9)*, we let*

$$\rho(u)=\rho(u,\varOmega)=\sum_b G(b,u,\varOmega), \tag{3.13}$$

where b runs through the elements of any set $\subset\mathbb{C}$ such that the sum converges absolutely. Then we have

$$(L-H^L)\,\rho(u,\varOmega)=0$$

for any L satisfying (3.5).

Proof. Since both of L and H^L are independent of b, each term of $\rho(u)$ satisfies (3.11). □

3.3 The algebraic heat operators

Definition 3.14. Assume that $\mathscr{E}$ is non-singular. Let Ω be the usual period matrix defined by (1.8), and δ be its Riemann constant. The function defined by

$$\tilde{\sigma}(u) = \tilde{\sigma}(u, \Omega) = \frac{\omega'}{2\pi} \cdot \exp\left(\tfrac{1}{2}u^2\eta'\omega'^{-1}\right)\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(\omega'^{-1}u|\omega'^{-1}\omega'')$$

is written as

$$\tilde{\sigma}(u) = \sum_{n\in\mathbb{Z}} G({}^t[\delta'\ n + \delta''], u, \Omega).$$

Since the imaginary part of $\omega'^{-1}\omega''$ is positive, this series converges absolutely.

This is a special case of $\rho(u)$ of (3.13), which means that

$$(L - H^L)\,\tilde{\sigma}(u) = 0. \tag{3.15}$$

In 3.12, since both L and H^L are independent of b, we see that there are infinitely many linearly independent entire functions $\rho(u)$ on $\mathbb{C}$ satisfying $(L-H^L)\rho(u) = 0$. Moreover, we have infinitely many linearly independent operators $L - H^L$ which annihilate $\tilde{\sigma}(u)$. We shall find a nice set of L's whose operation characterises $\sigma(u)$.

However, because our aim is to find a method to calculate the power series expansion of the sigma function, which is its algebraic aspect, we need an in-depth discussion. For our purpose,
(A1) we need to find operators in the ring $\mathbb{Q}[\boldsymbol{a}][\partial_{\boldsymbol{a}}, \frac{d}{du}]$ which annihilate $\sigma(u)$, and
(A2) we require that any function annihilated by all such operators belongs to $\mathbb{Q}[\boldsymbol{a}][[u]]$.
If we ignore the choice of an eighth root for Δ, we have

$$\sigma(u) = -\Delta^{\frac{1}{8}}\tilde{\sigma}(u).$$

Now, by replacing $-\Delta^{\frac{1}{8}}$ by other arbitrary function Ξ of a_j's, we shall construct a heat equation satisfied by $\Xi\tilde{\sigma}(u)$. Since L is a derivation with respect to the a_js but H^L is a differential operator with respect to u, we see, for such function Ξ depending only on the a_js, that

$$L\,\Xi\tilde{\sigma}(u) = (L\Xi)\,\tilde{\sigma}(u) + \Xi(L\tilde{\sigma}(u)), \qquad H^L\,\Xi\tilde{\sigma}(u) = \Xi(H^L\tilde{\sigma}(u)).$$

Therefore, $\Xi\tilde{\sigma}(u)$ satisfies

$$(L - H^L)(\Xi\tilde{\sigma}(u)) = \tfrac{L\Xi}{\Xi}\,\Xi\,\tilde{\sigma}(u) = (L\log\Xi)\Xi\,\tilde{\sigma}(u). \tag{3.16}$$

If $\Xi\,\tilde{\sigma}(u)$ is the correct sigma function, the left hand side of the above is in $\mathbb{Q}[\boldsymbol{a}][u]]$, So, if we suppose that the correct $\sigma(u)$ is equal to $-\Delta^{-\frac{1}{8}}\tilde{\sigma}(u)$, then we should narrow down the choice of L to one such that

$$L\log\Delta = \tfrac{L(\Delta)}{\Delta} \in \mathbb{Q}[\boldsymbol{a}].$$

4 Heat equation for the universal sigma function

We explain two methods for calculating Δ explicitly, which work for curves of higher genera as well. These methods also provide operators tangent to the variety defined by $\Delta = 0$, too. Throughout this section, we suppose that all the a_js are variables or indeterminates.

4.1 The first method to compute the discriminant

We recall the first method to compute the discriminant Δ from Section 2.3 of [3]. We take 1 and x as a basis of the $\mathbb{Q}[\boldsymbol{a}]$-module $\mathbb{Q}[\boldsymbol{a}][x,y]/(f_1,f_2)$. We use the rather overloaded notation $M(x,y) = [1\ x]$ as used in [3] and [5] and is a symbol extending to higher genus cases[2]. Moreover, we use $\check{M}(x,y) = [x\ 1]$, the one with reversed order. We define a matrix $T = [\,T_{ij}\,]_{i,j=1,2}$ with $\mathrm{wt}(T_{ij}) = -2(i+j)$ by the equalities

$$(4.1)\qquad -6\,f(x,y)\,M_i(x,y) \equiv \sum_{j=1}^{2} T_{ij}\,M_{2+1-j}(x,y) \mod (f_1,f_2)$$

$$(\ \text{i.e.}\quad -6\,f(x,y)\,M(x,y) \equiv T\,\check{M}(x,y) \mod (f_1,f_2)\ \ ),$$

where $f_1 = \frac{\partial}{\partial x}f(x,y)$, $f_2 = \frac{\partial}{\partial y}f(x,y)$. The factor -6 ensures simplification of the final form of heat equations and the signs of the first row and column of T are negative. These T_{ij} are uniquely determined because $\mathbb{Q}[\boldsymbol{a}][x,y]/(f_1,f_2)$ is a $\mathbb{Q}[\boldsymbol{a}]$-module of rank 2 spanned by the entries in $M(x,y)$ and we can reduce the order of x in the left hand side by using $f_1(x,y) = \cdots - 3x^2 + \cdots$ and that of y by using $f_2(x,y) = 2y + \cdots$. Then we see $\Delta = \det(T)$ which is shown in 4.8, though this is checked directly.

4.2 The second method to compute the discriminant and L-operators

We compute the discriminant by another method described in Section 4.2 of [2]. This method seems much simpler than the first method in 4.1.

Let introduce $H = H\big((x,y),(z,w)\big)$ which is defined by

$$H = \frac{1}{2}\begin{vmatrix} \frac{f_1(x,y)-f_1(z,w)}{x-z} & \frac{f_2(x,y)-f_2(z,w)}{x-z} \\[4pt] \frac{f_1(z,y)-f_1(x,w)}{y-w} & \frac{f_2(z,y)-f_2(x,w)}{y-w} \end{vmatrix}.$$

[2] In earlier version of [5] it is denoted as $M(x,y) = {}^t[1\ x]$.

For any $g \in \mathbb{Q}[\boldsymbol{a}][x,y]$, its Hessian is defined by

$$\operatorname{Hess} g = \begin{vmatrix} \frac{\partial^2}{\partial x^2} g & \frac{\partial^2}{\partial x \partial y} g \\ \frac{\partial^2}{\partial y \partial x} g & \frac{\partial^2}{\partial y^2} g \end{vmatrix}.$$

Lemma 4.2. (Buchstaber-Leykin [2], p.64) *Let x, y, z, w be indeterminates and I be the ideal in $\mathbb{Q}[\boldsymbol{a}][x,y,z,w]$ generated by $f_1(x,y)$, $f_2(x,y)$, $f_1(z,w)$, and $f_2(z,w)$. The determinant H has the following properties.*
(1) $H\big((x,y),(x,y)\big) \in \mathbb{Q}[\boldsymbol{a}][x,y,z,w]$.
(2) $H\big((x,y),(x,y)\big) = \operatorname{Hess} f(x,y)$.
(3) $H\big((x,y),(z,w)\big) = H\big((z,w),(x,y)\big)$.
(4) *For any $F\big((x,y),(z,w)\big) \in \mathbb{Q}[\boldsymbol{a}][x,y,z,w]$, we have*

$$H\big((z,w),(x,y)\big)\, F\big((x,y),(z,w)\big) \equiv H\big((x,y),(z,w)\big)\, F\big((z,w),(x,y)\big) \mod I.$$

Proof. We can check these properties directly as we are treating only the genus one case. (1) We have

$$H = \frac{1}{2}\begin{vmatrix} -\frac{a_1(y-w)}{x-z} - 3(x+z) - 2a_2 & \frac{2(y-w)}{x-z} - a_1 \\ -a_1(y+w) - \frac{3(z^2-x^2)+2a_2(z-x)}{y-w} & 2 - \frac{a_1(z-x)}{y-w} \end{vmatrix} = -6x - 6z - {a_1}^2 - 4a_2,$$

and this is a polynomial. (2) $H\big((x,y),(x,y)\big) = \operatorname{Hess}(f) = -12x - {a_1}^2 - 4a_2$ as desired. (3) is obvious from the definition. (4) Rewriting the difference of two sides of the equation, we see that $F\big((x,y),(z,w)\big) - F\big((z,w),(x,y)\big)$ removes the denominators $x-z$, $y-w$ in the definition of H and that it belongs to I. □

We will use, instead of T in (4.1), the matrix $V = [V_{ij}]$ with entries in $\mathbb{Q}[\boldsymbol{a}]$ which is defined by the equation

$$M(x,y)\, V\, {}^t M(z,w) = f(x,y)\, H$$

in the ring $\mathbb{Q}[x,y,z,w] \,/\, \big(f_1(x,y), f_2(x,y), f_1(z,w), f_2(z,w)\big)$. We see V is symmetric because of 4.2. The result is as follows

$$V_{11} = \tfrac{1}{12}(-{a_1}^4 - 8{a_1}^2 a_2 + 24 a_1 a_3 - 16{a_2}^2 + 48 a_4),$$
$$V_{12} = V_{21} = \tfrac{1}{12}(-{a_1}^3 a_3 - 2{a_1}^2 a_4 - 4 a_1 a_2 a_3 - 8 a_2 a_4 + 18{a_3}^2 + 72 a_6),$$
$$V_{22} = \tfrac{1}{12}(-{a_1}^2 {a_3}^2 + 12{a_1}^2 a_6 - 16 a_1 a_3 a_4 + 12 a_2 {a_3}^2 + 48 a_2 a_6 - 16{a_4}^2).$$

Remark 4.3. Replacing $\mathscr{E}$ by any hyperelliptic curve, we have a general form of V_{ij} (see ([5]).

We introduce an matrix $[\, H_{ij} \,]$ in order to see the coincidence of $\det(T)$ and $\det(V)$:

$$H\big((x,y),(z,w)\big) = \check{M}(x,y)\, [\, H_{ij} \,]\, {}^t\check{M}(z,w). \tag{4.4}$$

Lemma 4.5. *The matrix* $[\,H_{ij}\,]$ *is of the form*

$$[\,H_{ij}\,] = \begin{bmatrix} -4a_2 & -6 \\ -6 & \end{bmatrix}. \tag{4.6}$$

Proof. Straightforward. □

Lemma 4.7. $\det(V) = \det(T)$.

Proof. Since

$$\begin{aligned} f(x,y)\,H\big((x,y),(z,w)\big) &= f(x,y)\,\check{M}(z,w)\,[\,H_{jk}\,]\,{}^t\check{M}(x,y) \\ &= M(z,w)\,[\,-\tfrac{1}{2\cdot 3}\,T_{ij}\,][\,H_{jk}\,]\,{}^t\check{M}(x,y), \end{aligned}$$

we see that V equals $-\frac{1}{2\cdot 3}T[H_{jk}]$ with sorted rows in reverse order. Since $[H_{jk}]$ is a skew-upper-triangular matrix of the form (4.6), we have the desired equality. □

Lemma 4.8. *The determinant* $\det(T) = \det(V)$ *is a constant multiple of* $\varDelta$.

Proof. We assume $\boldsymbol{a} \subset \mathbb{C}$. Take the map given by multiplication by $f(x,y)$

$$\cdot\, f(x,y)\;:\;\mathbb{Q}[\boldsymbol{a}][x,y]/(f_1,f_2) \longrightarrow \mathbb{Q}[\boldsymbol{a}][x,y]/(f_1,f_2).$$

The vector $M(x,y)$ consists of the elements of a basis of $\mathbb{Q}[\boldsymbol{a}][x,y]/(f_1,f_2)$ as a $\mathbb{Q}[\boldsymbol{a}]$-module. The matrix T is no other than the representation matrix with respect to this basis. So $\det(T) = 0$ if and only if the rank[3] of the co-kernel $\mathbb{Q}[\boldsymbol{a}][x,y]/(f,f_1,f_2)$ of the map is positive. This is exactly the case that the ideal (f,f_1,f_2) does not contain $1 \in \mathbb{Q}[\boldsymbol{a}][x,y]$. By Theorem 5.4 i) in [9] (Hilbert's Nullstellensatz), we see this is equivalent to saying that there exists $(x,y) \in \mathbb{C}^2$ such that

$$f(x,y) = f_1(x,y) = f_2(x,y) = 0.$$

Therefore the discriminant $\varDelta$ is a factor of $\det(T)$. Moreover, since the matrix T reflects precisely the rank of $\mathbb{Q}[\boldsymbol{a}][x,y]/(f,f_1,f_2)$, $\det(T)$ must be a factor of the discriminant. □

Now we are ready to define the following operators:

Definition 4.9. *We define*

$$L_0 = V_{11}\frac{\partial}{\partial a_4} + V_{12}\frac{\partial}{\partial a_6}\quad [0], \qquad L_2 = V_{12}\frac{\partial}{\partial a_4} + V_{22}\frac{\partial}{\partial a_6}\quad [-2], \tag{4.10}$$

where the numbers in brackets signify the weight of the corresponding operators.

These operators are indeed tangent to the discriminant variety $\varDelta = 0$ which is ensured by (4.12) in the next subsection.

[3]the Tjurina number at $\boldsymbol{a}$.

4.3 The representation matrices and and tangentiality of L-operators

The operators in (4.10) give linear transforms of $H^1_{\mathrm{dR}}(\mathscr{E}/\mathbb{Q}[\boldsymbol{a}])$ whose representation matrices with respect to $\boldsymbol{\omega} = (\omega_1, \eta_1)$ are denoted simply by

$$\Gamma_0 = \Gamma^{L_0}, \quad \Gamma_2 = \Gamma^{L_2}.$$

The calculation procedure to get Γ_0 and Γ_2 is as follows. We suppose $\frac{\partial x}{\partial a_i} = 0$ for $i = 0$ and 2. Then we have

$$2y\frac{\partial y}{\partial a_6} + (a_1x + a_3)\frac{\partial y}{\partial a_6} = 1, \quad 2y\frac{\partial y}{\partial a_4} + (a_1x + a_3)\frac{\partial y}{\partial a_4} = x$$

by $f = 0$. Hence

$$\begin{aligned}
\frac{\partial}{\partial a_6}\omega_1 &= \frac{\partial}{\partial a_6}\frac{1}{2y + a_1x + a_3}dx = \frac{-1}{(2y + a_1x + a_3)^2}\cdot 2\frac{\partial y}{\partial a_6}dx = \frac{-2}{(2y + a_1x + a_3)^3}dx,\\
\frac{\partial}{\partial a_4}\omega_1 &= \frac{\partial}{\partial a_4}\frac{1}{2y + a_1x + a_3}dx = \frac{-1}{(2y + a_1x + a_3)^2}\cdot 2\frac{\partial y}{\partial a_4}dx = \frac{-2x}{(2y + a_1x + a_3)^3}dx,\\
\frac{\partial}{\partial a_4}\eta_1 &= \frac{\partial}{\partial a_4}\frac{x}{2y + a_1x + a_3}dx = \frac{-x}{(2y + a_1x + a_3)^2}\cdot 2\frac{\partial y}{\partial a_4}dx = \frac{-2x^2}{(2y + a_1x + a_3)^3}dx.
\end{aligned}$$

By using $f = 0$ at any time, we have

$$\begin{aligned}
d\big(\tfrac{1}{f_2}\big) &= (\tfrac{1}{2}a_3a_1 + a_4)\tfrac{\partial}{\partial a_6}\omega_1 + (\tfrac{1}{2}{a_1}^2 + 2a_2)\tfrac{\partial}{\partial a_6}\omega_1 + 3\tfrac{\partial}{\partial a_6}\eta_1,\\
d\big(\tfrac{x}{f_2}\big) &= (-\tfrac{3}{4}{a_3}^2 - 3a_6)\tfrac{\partial}{\partial a_6}\omega_1 + (-a_3a_1 - 2a_4)\tfrac{\partial}{\partial a_4}\omega_1 + (-\tfrac{1}{4}{a_1}^2 - a_2)\tfrac{\partial}{\partial a_4}\eta_1 - \tfrac{1}{2}\omega_1,\\
d\big(\tfrac{x^2}{f_2}\big) &= ((\tfrac{1}{16}{a_3}^2+\tfrac{1}{4}a_6){a_1}^2+\tfrac{1}{4}a_2{a_3}^2+a_6a_2))\tfrac{\partial}{\partial a_6}\omega_1\\
&\quad + (\tfrac{1}{8}a_3{a_1}^3+\tfrac{1}{4}a_4{a_1}^2+\tfrac{1}{2}a_2a_3a_1-\tfrac{3}{4}{a_3}^2+a_4a_2-3a_6)\tfrac{\partial}{\partial a_4}\omega_1\\
&\qquad + (\tfrac{1}{16}{a_1}^4 + \tfrac{1}{2}a_2{a_1}^2 - a_3a_1 + {a_2}^2 - 2a_4)\tfrac{\partial}{\partial a_4}\eta_1 + (\tfrac{1}{8}{a_1}^2 + \tfrac{1}{2}a_2)\omega_1 + \tfrac{1}{2}\eta_1.
\end{aligned}$$

We denote by D the determinant of the 3×3-matrix consists of the coefficients with respect to $\frac{\partial}{\partial a_6}\omega_1$, $\frac{\partial}{\partial a_4}\omega_1$, and $\frac{\partial}{\partial a_4}\eta_1$ of the above. Solving this modulo the exact forms, we have

$$\begin{aligned}
D\cdot\tfrac{\partial}{\partial a_6}\omega_1 &= (\tfrac{1}{16}a_3{a_1}^3 + \tfrac{1}{8}a_4{a_1}^2 + \tfrac{1}{4}a_2a_3a_1 - \tfrac{9}{8}{a_3}^2 + \tfrac{1}{2}a_4a_2 - 9/2a_6)\omega_1\\
&\quad + (\tfrac{1}{16}{a_1}^4 + \tfrac{1}{2}a_2{a_1}^2 - \tfrac{3}{2}a_3a_1 + {a_2}^2 - 3a_4)\eta_1,\\
D\cdot\tfrac{\partial}{\partial a_4}\omega_1 &= \big(-\tfrac{1}{16}{a_3}^2 + \tfrac{3}{4}a_6){a_1}^2 - a_4a_3a_1 + \tfrac{3}{4}a_2{a_3}^2 + (3a_6a_2 - {a_4}^2)\omega_1\\
&\quad + (-\tfrac{1}{16}a_3{a_1}^3 - \tfrac{1}{8}a_4{a_1}^2 - \tfrac{1}{4}a_2a_3a_1 + \tfrac{9}{8}{a_3}^2 - \tfrac{1}{2}a_4a_2 + \tfrac{9}{2}a_6\big)\eta_1,\\
D\cdot\tfrac{\partial}{\partial a_4}\omega_1(=\tfrac{\partial}{\partial a_6}\eta_1) &= \big(-\tfrac{1}{8}a_6{a_1}^4 + \tfrac{1}{8}a_4a_3{a_1}^3 - \tfrac{1}{8}a_2{a_3}^2 + (-a_6a_2 + \tfrac{1}{8}{a_4}^2){a_1}^2 + \tfrac{3}{16}a_1{a_3}^3\\
&\quad +(\tfrac{1}{2}a_4a_2 + \tfrac{3}{4}a_6)a_1a_3 + (-\tfrac{1}{2}{a_2}^2 + \tfrac{3}{8}a_4){a_3}^2 - 2a_6{a_2}^2 + \tfrac{1}{2}{a_4}^2a_2 + \tfrac{3}{2}a_6a_4\big)\omega_1\\
&\qquad + \big((\tfrac{1}{16}{a_3}^2 - \tfrac{3}{4}a_6){a_1}^2 + a_4a_3a_1 - \tfrac{3}{4}a_2{a_3}^2 - 3a_6a_2 + {a_4}^2\big)\eta_1,
\end{aligned}$$

where all these equalities are regarded modulo the exact forms.

Substituting these into (4.10), we see that the determinant D is completely cancelled out, and we have (simply!)

$$L_0(\omega_1,\,\eta_1)=(\omega_1,\,\eta_1)\,{}^t\Gamma_0,\quad L_2(\omega_1,\,\eta_1)=(\omega_1,\,\eta_1)\,{}^t\Gamma_2,$$

where

$$\Gamma_0=\begin{bmatrix} & -1\\ \frac{1}{6}{a_1}^2+\frac{2}{3}a_2 & 1\end{bmatrix},\quad \Gamma_2=\begin{bmatrix} & 1\\ \frac{1}{6}a_3a_1+\frac{1}{3}a_4 & \end{bmatrix}. \tag{4.11}$$

We have the following important

Proposition 4.12. (Buchstaber-Leykin) *Let* $L=M(x,y)\,{}^t[\,L_0\ L_2\,]$. *We have*

$$L(\varDelta)=\mathrm{Hess}f\cdot\varDelta \tag{4.13}$$

in the ring $\mathbb{Q}[\boldsymbol{a}][x,y]/(f_1,f_2)$, *which is explicitly written as*

$$(L_0,\ L_2)(\varDelta)=[\,12,\ {a_1}^2+4a_2\,]\,\varDelta.$$

Proof. There is no proof in [3] and we do not give a proof here. See [5] for a proof for the hyperelliptic case due to S. Yasuda. □

Corollary 4.14. *The operators* L_0 *and* L_2 *are tangent to the discriminant* $\varDelta$, *i.e.* $(L_0\varDelta)/\varDelta$, $(L_2\varDelta)/\varDelta\in\mathbb{Q}[\boldsymbol{a}]$.

Proof. Obvious from 4.12. □

Remark 4.15. If $a_1=a_2=a_3=0$, namely, in the case of the Weierstrass form, any operator $D\in\mathbb{Q}[\boldsymbol{a}][\frac{\partial}{\partial\boldsymbol{a}}]$ which is tangent to $\varDelta$ is a linear combination of L_0 and L_2. In our situation, we have more linearly independent tangent operators

$$\begin{gathered} E=a_1\tfrac{\partial}{\partial a_1}+2a_2\tfrac{\partial}{\partial a_2}+3a_3\tfrac{\partial}{\partial a_3}+4a_4\tfrac{\partial}{\partial a_4}+6a_6\tfrac{\partial}{\partial a_6}\quad (\textit{Euler's vector field}),\\ V_{i1}\tfrac{\partial}{\partial a_1}+V_{i2}\tfrac{\partial}{\partial a_3},\quad V_{i1}\tfrac{\partial}{\partial a_2}+V_{i2}\tfrac{\partial}{\partial a_4},\quad V_{i1}\tfrac{\partial}{\partial a_4}+V_{i2}\tfrac{\partial}{\partial a_6}\quad (i=1,\ 2). \end{gathered} \tag{4.16}$$

The operator E is indeed tangent to $\varDelta=0$ because $\varDelta$ is of homogeneous weight. We will use E as well. However, the sigma function for our case is determined only by L_0 and L_2. We refer the reader to Theorem A8 in [1] concerning this remark.

4.4 Final forms of heat operators

We denote simply $H_0=H^{L_0}$, $H_2=H^{L_2}$. Letting $\Gamma_i=\begin{bmatrix}-\beta_i & \alpha_i\\ -\gamma_i & \beta_i\end{bmatrix}$ $(i=0,\,2)$, we have

$$H_i=\tfrac{1}{2}\alpha_i\tfrac{\partial}{\partial u^2}+\beta_i u\tfrac{\partial}{\partial u}+\tfrac{1}{2}\gamma_i u^2+\tfrac{1}{2}\mathrm{Tr}\,\beta_i.$$

We have explicitly these and $\frac{1}{8}L_i(\log\varDelta)$ as follows:

$$\begin{aligned} &H_0=0+u\tfrac{\partial}{\partial u}-\tfrac{1}{12}({a_1}^2+4a_2)u^2+\tfrac{1}{2}, && \tfrac{1}{8}L_0(\log\varDelta)=\tfrac{3}{4},\\ &H_2=\tfrac{1}{2}\tfrac{\partial^2}{\partial u^2}+0-\tfrac{1}{12}(a_3a_1+2a_4)u^2+0, && \tfrac{1}{8}L_2(\log\varDelta)=\tfrac{1}{8}({a_1}^2+4a_2). \end{aligned} \tag{4.17}$$

Then our refined BL-theory states that $L_0-H_0+\frac{1}{8}L_0(\log\varDelta)$ and $L_2-H_2+\frac{1}{8}L_2(\log\varDelta)$ annihilate $\sigma(u)$, which is the first statement of the following theorem.

Now, (3.16) leads to the following.

Theorem 4.18. (Universal version of BL-theory) *One has*

$$\big(L_0 - H_0 + \tfrac{1}{8}L_0(\log\varDelta)\big)\,\sigma(u) = 0, \qquad \big(L_2 - H_2 + \tfrac{1}{8}L_2(\log\varDelta)\big)\,\sigma(u) = 0.$$

Moreover, for an unknown function $\varphi(u)$, any solution of

$$\big(L_0 - H_0 + \tfrac{1}{8}L_0(\log\varDelta)\big)\,\varphi(u) = 0, \qquad \big(L_2 - H_2 + \tfrac{1}{8}L_2(\log\varDelta)\big)\,\varphi(u) = 0$$

is a absolute-constant multiple of $\sigma(u)$.

Proof. The former part is seen by (3.15) and (3.16). On the latter part, after rewriting these equations to relations on the expansion coefficients, we solve them in Section 5 and see that the solution is of 1-dimensional. □

Remark 4.19. (1) It is not obvious to the authors why $L_0 - H_0 + \frac{1}{8}L_0(\log\varDelta)$ and $L_2 - H_2 + \frac{1}{8}L_2(\log\varDelta)$ determines $\sigma(u)$. The authors do not know a fundamental reason why the coefficients of power series expansion of $\sigma(u)$ have a recursion relation.
(2) We see that our L_0 has extra terms coming from $\exp\big(\frac{{a_1}^2+4a_2}{24}u^2\big)$ of (2.7) in comparison with Weierstrass' original L_0 (i.e. to L_0 of the case $a_1 = a_2 = a_3 = 0$).
(3) If we use Euler's vector field E in (4.16) instead of L_0, we get a simpler heat equation, which states that the expansion of $\sigma(u)$ is homogeneous of weight 1.

We compute the heat operator $E - H^E + \frac{1}{8}E\log\varDelta$ for the calculation in Section 5. By (1.4), we see that

$$\begin{aligned} E(\omega_1) + \omega_1 &= \big(1 + 2a_1t + 3(a_2 + {a_1}^2)t^2 + \cdots\big)dt = d\big(t + a_1t^2 + (a_2 + {a_1}^2)t^3 + \cdots\big) \\ &= d\Big(t\,\frac{\omega_1}{dt}\Big) = d\Big(\frac{x}{y}\cdot\frac{\frac{dx}{dt}}{2y - a_1x - a_3}\Big) = d\Bigg(\frac{x}{y}\cdot\frac{1}{2y - a_1x - a_3}\,\frac{1}{\frac{d}{dx}\big(\frac{x}{y}\big)}\Bigg), \\ E(\eta_1) - \eta_1 &= (t^{-2} + 2a_3t - 3(a_4 + 2a_1a_3)t^2 + \cdots)dt \\ &= d(-t^{-1} + a_3t^2 - 3(a_4 + 2a_1a_3)t^3 + \cdots) = d\Big(t\,\frac{\eta_1}{dt}\Big) = \cdots, \end{aligned}$$

which are exact forms. Obviously, $E\log\varDelta = 12$. We then have $\Gamma^E = \begin{bmatrix} -1 & \\ & 1 \end{bmatrix}$, and

$$\begin{aligned} E - H^E + \tfrac{1}{8}E\log\varDelta &= E - u\tfrac{\partial}{\partial u} - \tfrac{1}{2} + \tfrac{12}{8} \\ &= a_1\tfrac{\partial}{\partial a_1} + 2a_2\tfrac{\partial}{\partial a_2} + 3a_3\tfrac{\partial}{\partial a_3} + 4a_4\tfrac{\partial}{\partial a_4} + 6a_6\tfrac{\partial}{\partial a_6} - u\tfrac{\partial}{\partial u} + 1. \end{aligned} \tag{4.20}$$

5 Recursion Relations

By (4.20), we see the expansion of $\sigma(u)$ is of the form

$$\sigma(u) = \sum_{k-n_1-2n_2-3n_3-4n_4-6n_6=1} b(n_1,n_2,n_3,n_4,n_6)\,{a_1}^{n_1}{a_2}^{n_2}{a_3}^{n_3}{a_4}^{n_4}{a_6}^{n_6}\frac{u^k}{k!},$$

where $b(n_1, n_2, n_3, n_4, n_6) \in \mathbb{Q}$. We denote by b_{L_0} and b_{H_0} the coefficients of

$$a_1{}^{n_1} a_2{}^{n_2} a_3{}^{n_3} a_4{}^{n_4} a_6{}^{n_6} \frac{u^k}{k!}$$

with $k = 1 + n_1 + 2n_2 + 3n_3 + 4n_4 + 6n_6$ in $L_0\big(\sigma(u)\big)$ and $(H_0 - \frac{1}{8}L_0 \log \varDelta)\big(\sigma(u)\big)$, respectively, and by b_{L_2} and b_{H_2} the coefficients of

$$a_1{}^{n_1} a_2{}^{n_2} a_3{}^{n_3} a_4{}^{n_4} a_6{}^{n_6} \frac{u^{k-2}}{(k-2)!}$$

in $L_2\big(\sigma(u)\big)$ and $(H_2 - \frac{1}{8}L_2 \log \varDelta)\big(\sigma(u)\big)$, respectively. which are explicitly expressed as in the last page. Therefore, we have two recursion relations

$$b_{L_0} - b_{H_0} = 0, \qquad b_{L_2} - b_{H_2} = 0.$$

Definition 5.1. We fix integers n_1, n_2, n_3, n_4, n_6. Then, we define the *relative weight* (with respect to these integers) of the symbol

$$b(n_1 + k_1,\, n_2 + k_2,\, n_3 + k_3,\, n_4 + k_4,\, n_6 + k_6)$$

to be $-(k_1 + 2k_2 + 3k_3 + 4k_4 + 6k_6)$.

It is easy to see the relation $b_{L_2} - b_{H_2} = 0$ gives indeed a recursion relation by looking at the relative weights for all symbols appeared in this relation. Namely, there exists a relative weight 0 symbol which is $b(n_1,\, n_2,\, n_3, n_4,\, n_6)$ and the other symbols are of lower weight. The explicit formulae are given below.

Remark 5.2. If we set $a_1 = a_2 = a_3 = 0$, the relations $b_{L_i} - b_{H_i} = 0$ $(i = 0,\, 2)$ coincide with the relations (0.4) of Weierstrass ([17]) (see [11] also).

Finally, we shall describe the actual calculation. Using the recursion relation $b_{L_2} - b_{H_2} = 0$, we have the coefficients, as in the following order,

$b(0,0,0,0,0) = 1$, $b(0,0,0,1,0) = 2$, $b(0,0,0,0,1) = 24$, $b(2,0,0,0,0) = \frac{1}{2}$,
$b(4,0,0,0,0) = \frac{1}{16}$, $b(0,1,0,0,0) = 1$, $b(2,1,0,0,0) = \frac{1}{2}$, $b(4,1,0,0,0) = \frac{3}{16}$,
$b(6,0,0,0,0) = \frac{1}{64}$, $b(0,0,2,0,0) = 6$, $b(1,0,1,0,0) = 1$, $b(3,0,1,0,0) = \frac{3}{4}$, $\cdots$,

and we get the expansion (0.11).

6 Some problems

We pick up some questions on the direction of this investigation for higher genus curves.
Q1. For a given general curve, describe the power series expansion of the sigma function explicitly around any point on the Abelian image of the curve, as Weierstrass gave expansions around the 2-division points on any elliptic curve.
Q2. Give a closed formula for the coefficients of the expansion of the sigma function around the origin for any curve, as S. Yasuda [18] obtained for the Weierstrass sigma function.

The final forms of the recursion equations are as follows (the RHS's show the corresponding terms in L_i and $H_i - \frac{1}{8}L_i \log \Delta$) :

$$
\begin{aligned}
b_{L_0} = \tfrac{1}{12}\big(& -(n_4{+}1)\cdot b(n_1-4, n_2, n_3, n_4+1, n_6)\cdot 16 && \cdots\ (-a_1{}^4 \\
& -8(n_4{+}1)\cdot b(n_1-2, n_2-1, n_3, n_4+1, n_6)\cdot 4 && \cdots\ -8a_1{}^2a_2 \\
& +24(n_4{+}1)\cdot b(n_1-1, n_2, n_3-1, n_4+1, n_6)\cdot 2 && \cdots\ +24a_1a_3 \\
& -16(n_4{+}1)\cdot b(n_1, n_2-2, n_3, n_4+1, n_6) && \cdots\ -16a_2{}^2 \\
& +48\, n_4\cdot b(n_1, n_2, n_3, n_4, n_6) && \cdots\ +48a_4)\tfrac{\partial}{\partial a_4} \\
& -(n_6{+}1)\cdot b(n_1-3, n_2, n_3-1, n_4, n_6+1)\cdot 8 && \cdots\ +(-a_1{}^3a_3 \\
& -2(n_6{+}1)\cdot b(n_1-2, n_2, n_3, n_4-1, n_6+1)\cdot 4 && \cdots\ -2a_1{}^2a_4 \\
& -4(n_6{+}1)\cdot b(n_1-1, n_2-1, n_3-1, n_4, n_6+1)\cdot 2 && \cdots\ -4a_1a_2a_3 \\
& -8(n_6{+}1)\cdot b(n_1, n_2-1, n_3, n_4-1, n_6+1) && \cdots\ -8a_2a_4 \\
& +18(n_6{+}1)\cdot b(n_1, n_2, n_3-2, n_4, n_6+1) && \cdots\ +18a_3{}^2 \\
& +72\, n_6\cdot b(n_1, n_2, n_3, n_4, n_6)\big), && \cdots\ +72a_6)\tfrac{\partial}{\partial a_6},
\end{aligned}
$$

$$
\begin{aligned}
b_{H_0} = {} & b(n_1, n_2, n_3, n_4, n_6)\cdot(k-1) && \cdots\ u\tfrac{\partial}{\partial u} \\
& -\tfrac{1}{12}\cdot b(n_1-2, n_2, n_3, n_4, n_6)\cdot(k-1)(k-2)\cdot 4 && \cdots\ +(-\tfrac{1}{12}a_1{}^2 \\
& +\tfrac{1}{3}\cdot b(n_1, n_2-1, n_3, n_4, n_6)\cdot(k-1)(k-2) && \cdots\ +\tfrac{1}{3}a_2)u^2 \\
& +\tfrac{1}{2}\cdot b(n_1, n_2, n_3, n_4, n_6) && \cdots\ +\tfrac{1}{2} \\
& -\tfrac{3}{4}\cdot b(n_1, n_2, n_3, n_4, n_6), && \cdots\ +\tfrac{1}{8}(-12),
\end{aligned}
$$

$$
\begin{aligned}
b_{L_2} = \tfrac{1}{12}\big(& -(n_4{+}1)\cdot b(n_1-3, n_2, n_3-1, n_4+1, n_6)\cdot 8 && \cdots\ \tfrac{1}{12}\big(-a_1{}^3a_3 \\
& -2\, n_4\cdot b(n_1-2, n_2, n_3, n_4, n_6)\cdot 4 && \cdots\ -2a_1{}^2a_4 \\
& -4(n_4{+}1)\cdot b(n_1-1, n_2-1, n_3-1, n_4+1, n_6)\cdot 2 && \cdots\ -4a_1a_2a_3 \\
& -8\, n_4\cdot b(n_1, n_2-1, n_3, n_4, n_6) && \cdots\ -8a_2a_4 \\
& +18(n_4{+}1)\cdot b(n_1, n_2, n_3-2, n_4+1, n_6) && \cdots\ +18a_3{}^2 \\
& +72(n_4{+}1)\cdot b(n_1, n_2, n_3, n_4+1, n_6-1) && \cdots\ +72a_6\big)\tfrac{\partial}{\partial a_4} \\
& -1(n_6{+}1)\cdot b(n_1-2, n_2, n_3-2, n_4, n_6+1)\cdot 4 && \cdots\ +\tfrac{1}{12}\big(-a_1{}^2a_3{}^2 \\
& +12\, n_6\cdot b(n_1-2, n_2, n_3, n_4, n_6)\cdot 4 && \cdots\ +12a_1{}^2a_6 \\
& -16(n_6{+}1)\cdot b(n_1-1, n_2, n_3-1, n_4-1, n_6+1)\cdot 2 && \cdots\ -16a_1a_3a_4 \\
& +12(n_6{+}1)\cdot b(n_1, n_2-1, n_3-2, n_4, n_6+1) && \cdots\ +12a_2a_3{}^2 \\
& +48\, n_6\cdot b(n_1, n_2-1, n_3, n_4, n_6) && \cdots\ +48a_2a_6 \\
& -16(n_6{+}1)\cdot b(n_1, n_2, n_3, n_4-2, n_6+1)\big), && \cdots\ -16a_4{}^2\big)\tfrac{\partial}{\partial a_6},
\end{aligned}
$$

$$
\begin{aligned}
b_{H_2} = {} & \tfrac{1}{2}\cdot b(n_1, n_2, n_3, n_4, n_6) && \cdots\ \tfrac{1}{2}\tfrac{\partial^2}{\partial u^2} \\
& -\tfrac{1}{12}\cdot b(n_1-1, n_2, n_3-1, n_4, n_6)\cdot(k-1)(k-2)\cdot 2 && \cdots\ +(-\tfrac{1}{12}a_3a_1 \\
& -\tfrac{1}{6}\cdot b(n_1, n_2, n_3, n_4-1, n_6)\cdot(k-1)(k-2) && \cdots\ -\tfrac{1}{6}a_4)u^2 \\
& -\tfrac{1}{8}\cdot b(n_1-2, n_2, n_3, n_4, n_6)\cdot 4 && \cdots\ -\tfrac{1}{8}a_1{}^2 \\
& -\tfrac{1}{2}\cdot b(n_1, n_2-1, n_3, n_4, n_6). && \cdots\ -\tfrac{1}{2}a_2.
\end{aligned}
$$

A script for computing the expansion of the $\sigma(u)$ by using the recursion relation $b_{L_2} - b_{H_2} = 0$ for `pari/GP` is available at `http://www2.meijo-u.ac.jp/~yonishi`.

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